样本均值与样本方差相互独立的充要条件

定理 设总体 X 的分布函数 $F(x)$ 具有二阶矩, 即 $EX = \mu < \infty, DX = \sigma^2 < \infty$, 若 $(X_1, X_2, \cdots, X_n)$ 为来自总体 X 的一个样本, 则样本均值 $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ 与样本方差 $S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2$ 相互独立的充要条件是总体 $X \sim N(\mu, \sigma^2)$.[a]

证明. 充分性可参见一般教材, 此处从略, 下面证明必要性. 设总体 X 的特征函数为 $\alpha(t)$, $\bar{X}$ 的特征函数为 $\varphi_1(t_1)$, S^2 的特征函数为 $\varphi_2(t_2)$, $(\bar{X}, S^2)$ 的特征函数为 $\varphi(t_1, t_2)$. 因为 $\bar{X}$ 与 S^2 相互独立, 故 $\varphi(t_1, t_2) = \varphi_1(t_1)\varphi_2(t_2)$, 而由特征函数的性质知:

$$\varphi_1(t_1) = \left(\alpha\left(\frac{t_1}{n}\right)\right)^n, \frac{\varphi_2'(0)}{i} = ES^2 = \sigma^2, \quad \text{i.e.} \quad \varphi_2'(0) = i\sigma^2.$$

故

$$\left.\frac{\partial\varphi(t_1, t_2)}{\partial t_2}\right|_{t_2=0} = \varphi_1(t_1)\varphi_2'(0) = i\sigma^2\left(\alpha\left(\frac{t_1}{n}\right)\right)^n. \tag{1}$$

另一方面, $\varphi(t_1, t_2) = E\left[\mathrm{e}^{i(t_1\bar{X} + t_2 S^2)}\right]$, 故

$$\begin{aligned}
\left.\frac{\partial\varphi(t_1, t_2)}{\partial t_2}\right|_{t_2=0} &= E\left(iS^2\mathrm{e}^{i(t_1\bar{X} + t_2 S^2)}\right)\Big|_{t_2=0} = E\left(iS^2\mathrm{e}^{it_1\bar{X}}\right) \\
&= iE\left[\left(\frac{1}{n}\sum_{j=1}^{n} X_j^2 - \bar{X}^2\right)\mathrm{e}^{it_1\bar{X}}\right] \\
&= \frac{i}{n}\sum_{j=1}^{n} E\left(X_j^2\mathrm{e}^{i\frac{t_1}{n}\sum_{j=1}^{n} X_j}\right) - iE\left(\bar{X}^2\mathrm{e}^{it_1\bar{X}}\right).
\end{aligned}$$

又 $(X_1, X_2, \cdots, X_n)$ 的联合分布函数为

$$F^*(x_1, x_2, \cdots, x_n) = F(x_1)F(x_2)\cdots F(x_n).$$

故由 Riemann-Stieltjes 积分的性质有

$$\begin{aligned}
E\left(\bar{X}^2\mathrm{e}^{it_1\bar{X}}\right) &= \frac{1}{n^2}\left[\left(\sum_{j=1}^{n} X_j^2 + \sum_{j\neq k} X_j X_k\right)\mathrm{e}^{i\frac{t_1}{n}\sum_{j=1}^{n} X_j}\right] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} E\left(X_j^2\mathrm{e}^{i\frac{t_1}{n}\sum_{j=1}^{n} X_j}\right) + \frac{1}{n^2}\sum_{j\neq k} E\left(X_j X_k\mathrm{e}^{i\frac{t_1}{n}\sum_{j=1}^{n} X_j}\right) \\
&= \left(\alpha\left(\frac{t_1}{n}\right)\right)^{n-1}\frac{1}{n^2}\sum_{j=1}^{n}\int_{-\infty}^{\infty} x^2\mathrm{e}^{i\frac{t_1}{n}x}\mathrm{d}F(x) + \left(\alpha\left(\frac{t_1}{n}\right)\right)^{n-2}\frac{1}{n^2}\left[\int_{-\infty}^{\infty} x_k\mathrm{e}^{i\frac{t_1}{n}x_k}\mathrm{d}F(x_k)\right] \\
&= \frac{1}{n}\left(\alpha\left(\frac{t_1}{n}\right)\right)^{n-1}\int_{-\infty}^{\infty} x^2\mathrm{e}^{i\frac{t_1}{n}x}\mathrm{d}F(x) + \frac{n-1}{n}\left(\alpha\left(\frac{t_1}{n}\right)\right)^{n-2}\left[\int_{-\infty}^{\infty} x\mathrm{e}^{i\frac{t_1}{n}x}\mathrm{d}F(x)\right]^2.
\end{aligned}$$

故

$$\left.\frac{\partial \varphi(t_1,t_2)}{t_2}\right|_{t_2=0} = \frac{n-1}{n} i \left\{ \left(\alpha\left(\frac{t_1}{n}\right)\right)^{n-1} \int_{-\infty}^{\infty} x^2 \mathrm{e}^{i\frac{t_1}{n}x} \mathrm{d}F(x) + \frac{n-1}{n}\left(\alpha\left(\frac{t_1}{n}\right)\right)^{n-2} \left[\int_{-\infty}^{\infty} x \mathrm{e}^{i\frac{t_1}{n}x} \mathrm{d}F(x)\right]^2 \right\} \tag{2}$$

由 (1) 和 (2) 式得

$$\alpha\left(\frac{t_1}{n}\right) \int_{-\infty}^{\infty} x^2 \mathrm{e}^{i\frac{t_1}{n}x} \mathrm{d}F(x) - \left[\int_{-\infty}^{\infty} x \mathrm{e}^{i\frac{t_1}{n}x} \mathrm{d}F(x)\right]^2 = \left(\alpha\left(\frac{t_1}{n}\right)\right)^2 \sigma^2. \tag{3}$$

又因为 $\alpha(t) = \int_{-\infty}^{\infty} \mathrm{e}^{itx} \mathrm{d}F(x)$, 故

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = i \int_{-\infty}^{\infty} x \mathrm{e}^{itx} \mathrm{d}F(x), \frac{\mathrm{d}^2\alpha(t)}{\mathrm{d}t^2} = i^2 \int_{-\infty}^{\infty} x^2 \mathrm{e}^{itx} \mathrm{d}F(x).$$

则由 (3) 式可知, 关于 $\alpha(t)$ 有

$$-\alpha \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 = \sigma^2\alpha^2. \tag{4}$$

下面解此微分方程. 令

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = p, \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = p\frac{\mathrm{d}p}{\mathrm{d}\alpha}.$$

故 (4) 化为一阶方程:

$$-\alpha p \frac{\mathrm{d}p}{\mathrm{d}\alpha} + p^2 = -\frac{\alpha}{2}\frac{\mathrm{d}p^2}{\mathrm{d}\alpha} + p^2 = \sigma^2\alpha^2.$$

令 $Q = p^2$, 将上述方程化为

$$-\frac{\alpha}{2}\frac{\mathrm{d}Q}{\mathrm{d}\alpha} + Q = \sigma^2\alpha^2.$$

对此一阶线性微分方程, 不难得到通解为 $Q = (-2\sigma^2 \ln\alpha + C_1)\alpha^2$. 于是可得

$$p = \alpha\sqrt{-2\sigma^2\ln\alpha + C_1} = \frac{\mathrm{d}\alpha}{\mathrm{d}t}.$$

再一步解得此方程通解为

$$-\frac{1}{\sigma^2}\sqrt{-2\sigma^2\ln\alpha + C_1} = t + C_2,$$

进而得

$$\ln\alpha = A + Bt - \frac{1}{2}\sigma^2 t^2, \alpha(t) = \mathrm{e}^{A+Bt-\frac{1}{2}\sigma^2t^2}.$$

又根据特征函数的性质有: $1 = \alpha(0) = \mathrm{e}^A$, $A = 0$, 再由 $\mu = \left.\frac{\alpha'(t)}{i}\right|_{t=0} = \frac{B}{i}$, 故 $B = i\mu$, 即总体 X 的特征函数为 $\alpha(t) = \mathrm{e}^{i\mu t - \frac{1}{2}\sigma^2t^2}$, 故 $X \sim N(\mu, \sigma^2)$, 证毕. □

[a]证明方法选自http://www.doc88.com/p-7156270513349.html