Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
May 1, 2007

- 4954: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (a, b) that satisfy

  $$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} = \frac{111+i}{111-i}$$

  with $a < b$.

- 4955: *Proposed by Kenneth Korbin, New York, NY.*

  Between 100 and 200 pairs of red sox are mixed together with between 100 and 200 pairs of blue sox. If three sox are selected at random, then the probability that all three are the same color is 0.25. How many pairs of sox were there altogether?

- 4956: *Proposed by Kenneth Korbin, New York, NY.*

  A circle with radius $3\sqrt{2}$ is inscribed in a trapezoid having legs with lengths of 10 and 11. Find the lengths of the bases.

- 4957: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = 1, a_1 = 2, a_2 = 1$ and for all $n \geq 3$, $a_n^3 = a_{n-1}a_{n-2}a_{n-3}$. Find $\lim\limits_{n\to\infty} a_n$.

- 4958: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $f : [a, b] \to R$ ($0 < a < b$) be a continuous function on $[a, b]$ and derivable in (a, b).

Prove that there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{1}{c\sqrt{ab}} \cdot \frac{\ln(ab/c^2)}{\ln(c/a) \cdot \ln(c/b)}.$$

- 4959: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

  Find all numbers $N = ab$, were $a, b = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, such that

$$[S(N)]^2 = S(N^2),$$

where S(N)=a+b is the sum of the digits. For example:

$$\begin{array}{rcll} N & = & 12 & N^2 = 144 \\ S(N) & = & 3 & S(N^2) = 9 \quad \text{and } [\text{S(N)}]^2 = \text{S(N}^2\text{)}. \end{array}$$

Solutions

- 4918: *Proposed by Kenneth Korbin, New York, NY.*

  Find the dimensions of an isosceles triangle that has integer length inradius and sides and which can be inscribed in a circle with diameter 50.

Solution by Paul M. Harms, North Newton, KS.

Put the circle on a coordinate system with center at $(0,0)$ and the vertex associated with the two equal sides at $(0,25)$. Also make the side opposite the $(0,25)$ vertex parallel to the x-axis. Using (x,y) as the vertex on the right side of the circle, we have $x^2 + y^2 = 25^2 = 625$. Let d be the length of the equal sides. Using the right triangle with vertices at $(0,25), (0,y)$, and (x,y) we have $(25-y)^2 + x^2 = d^2$.

Then $d^2 = (25-y)^2 + (25^2 - y^2) = 1250 - 50y$; the semi-perimeter $s = x + d$ and the inradius $r = \sqrt{\dfrac{x^2(d-x)}{d+x}}$. Using $x^2 + y^2 = 25^2$, we will check to see if $x = 24$ and $y = 7$ satisfies the problem. The number $d^2 = 900$, so $d = 30$. The inradius $r = \sqrt{\dfrac{24^2(6)}{54}} = 8$. Thus the isosceles triangle with side lengths $30, 30, 48$ and $r = 8$ satisfies the problem. If $x = 24$ and $y = -7$, then $d = 40$ and $r = 12$. The isosceles triangle with side lengths $40, 40, 48$ and $r = 12$ also satisfies the problem.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David Stone and John Hawkins, Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4919: *Proposed by Kenneth Korbin, New York, NY.*

  Let x be any even positive integer. Find the value of

$$\sum_{k=0}^{x/2} \binom{x-k}{k} 2^k.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX.

To simplify matters, let $x = 2n$ and

$$S(n) = \sum_{k=0}^{n} \binom{2n-k}{k} 2^k.$$

Since

$$\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i}$$

for $m \geq 2$ and $1 \leq i \leq m-1$, we have

$$\begin{aligned}
\binom{2n+4-k}{k} &= \binom{2n+3-k}{k-1} + \binom{2n+3-k}{k} \\
&= \binom{2n+3-k}{k-1} + \binom{2n+2-k}{k-1} + \binom{2n+2-k}{k} \\
&= \binom{2n+3-k}{k-1} + \binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2} + \binom{2n+2-k}{k} \\
&= \binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2}
\end{aligned}$$

for $n \geq 1$ and $2 \leq k \leq n+1$.

Therefore, for $n \geq 1$,

$$\begin{aligned}
S(n+2) &= \sum_{k=0}^{n+2} \binom{2n+4-k}{k} 2^k \\
&= 1 + (2n+3)\cdot 2 + \sum_{k=2}^{n+1} \binom{2n+4-k}{k} 2^k + 2^{n+2} \\
&= 1 + (2n+3)\cdot 2 + \sum_{k=2}^{n+1} \binom{2n+2-k}{k} 2^k + 2\sum_{k=2}^{n+1} \binom{2n+3-k}{k-1} 2^k \\
&\quad - \sum_{k=2}^{n+1} \binom{2n+2-k}{k-2} 2^k + 2^{n+2} \\
&= 4 + \sum_{k=0}^{n+1} \binom{2n+2-k}{k} 2^k + 2\sum_{k=1}^{n} \binom{2n+2-k}{k} 2^{k+1} - \sum_{k=0}^{n-1} \binom{2n-k}{k} 2^{k+2} + 2^{n+2} \\
&= S(n+1) + 4\sum_{k=0}^{n+1} \binom{2n+2-k}{k} 2^k - \sum_{k=0}^{n-1} \binom{2n-k}{k} 2^{k+2} - 2^{n+2} \\
&= 5S(n+1) - 4\sum_{k=0}^{n} \binom{2n-k}{k} 2^k \\
&= 5S(n+1) - 4S(n).
\end{aligned}$$

To solve for $S(n)$, we use the usual techniques for solving homogeneous linear difference equations with constant coefficients. If we look for a solution of the form $S(n) = t^n$, with $t \neq 0$, then

$$S(n+2) = 5S(n+1) - 4S(n)$$

becomes

$$t^2 = 5t - 4,$$

whose solutions are $t = 1, 4$. This implies that the general solution for $S(n)$ is

$$S(n) = A \cdot 4^n + B \cdot 1^n = A \cdot 4^n + B,$$

for some constants A and B. The initial conditions $S(1) = 3$ and $S(2) = 11$ yield $A = \dfrac{2}{3}$ and $B = \dfrac{1}{3}$. Hence,

$$S(n) = \frac{2}{3} \cdot 4^n + \frac{1}{3} = \frac{2^{2n+1}+1}{3}$$

for all $n \geq 1$. The final solution is

$$\sum_{k=0}^{x/2} \binom{x-k}{k} 2^k = \frac{2^{x+1}+1}{3}$$

for all even positive integers x.

Also solved by David E. Manes, Oneonta, NY, David Stone, John Hawkins, and Scott Kersey (jointly), Statesboro, GA, and the proposer.

- 4920: *Proposed by Stanley Rabinowitz, Chelmsford, MA.*

Find positive integers a, b, and c (each less than 12) such that

$$\sin\frac{a\pi}{24} + \sin\frac{b\pi}{24} = \sin\frac{c\pi}{24}.$$

Solution by John Boncek, Montgomery, AL.

Recall the standard trigonometric identity:

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y.$$

Let $x + y = \dfrac{a\pi}{24}$ and $x - y = \dfrac{b\pi}{24}$. Then

$$\sin\frac{a\pi}{24} + \sin\frac{b\pi}{24} = 2\sin\frac{(a+b)\pi}{48}\cos\frac{(a-b)\pi}{48}.$$

We can make the right hand side of this equation equal to $\sin\dfrac{c\pi}{24}$ if we let $a - b = 16$ and $a + b = 2c$, or in other words, by choosing a value for c and then taking $a = 8 + c$ and $b = c - 8$.

Since we want positive solutions, we start by taking $c = 9$. This gives us $a = 17$ and $b = 1$. Since $\sin\dfrac{17\pi}{24} = \sin\dfrac{7\pi}{24}$, replace $a = 17$ by $a = 7$ and we have a solution $a = 7, b = 1$ and $c = 9$.

By taking $c = 10$ and $c = 11$ and using the same analysis, we obtain two additional triples that solve the problem, namely: $a = 6, b = 2, c = 10$ and $a = 5, b = 3, c = 11$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, NY, NY; Peter, E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4921: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos^{2006} x + 2006\sin^2 x}{2006 + \sin^{2006} x + \cos^{2006} x} dx.$

Solution by Michael C. Faleski, Midland, MI.

Call this integral I. Now, substitute $\sin^2 x = 1 - \cos^2 x$ and add to the numerator $\sin^{2006} x - \sin^{2006} x$ to give

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{2006 + \sin^{2006} x + \cos^{2006} x - (2006\cos^2 x + \sin^{2006} x)}{2006 + \sin^{2006} x + \cos^{2006} x} dx \\ &= \int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{2006\cos^2 x + \sin^{2006} x}{2006 + \sin^{2006} x + \cos^{2006} x} dx. \end{aligned}$$

The second integral can be transformed with $u = \pi/2 - x$ to give

$$\int_0^{\pi/2} \frac{2006\cos^2 x + \sin^{2006} x}{2006 + \sin^{2006} x + \cos^{2006} x} dx = -\int_{\pi/2}^{0} \frac{\cos^{2006} u + 2006\sin^2 u}{2006 + \sin^{2006} u + \cos^{2006} u} du = I.$$

Hence, $I = \int_0^{\pi/2} dx - I \Longrightarrow 2I = \dfrac{\pi}{2} \Longrightarrow I = \dfrac{\pi}{4}$.

$$\int_0^{\pi/2} \frac{\cos^{2006} x + 2006\sin^2 x}{2006 + \sin^{2006} x + \cos^{2006} x} dx = \frac{\pi}{4}.$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Ovidiu Furdui, Kalamazoo, MI; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4922: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b be real numbers such that $0 < a < b$ and let $f : [a, b] \to R$ be a continuous function in $[a, b]$ and derivable in (a, b). Prove that there exists $c \in (a, b)$ such that

$$cf(c) = \frac{1}{\ln b - \ln a} \int_a^b f(t)\, dt.$$

Solution by David E. Manes, Oneonta, NY.

For each $x \in [a, b]$, define the function $F(x)$ so that $F(x) = \int_a^x f(t)dt$. Then $F(b) = \int_a^b f(t)dt, F(a) = 0$ and, by the Fundamental Theorem of Calculus, $F'(x) = f(x)$ for each $x \in (a, b)$.

Let $g(x) = \ln(x)$ be defined on $[a, b]$. Then both functions F and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $g'(x) = \dfrac{1}{x} \neq 0$

for each $x \in (a, b)$. By the Extended Mean-Value Theorem, there is at least one number $c \in (a, b)$ such that

$$\frac{F'(c)}{g'(c)} = \frac{F(b) - F(a)}{g(b) - g(a)} = \frac{\displaystyle\int_a^b f(t)dt}{\ln b - \ln a}.$$

Since $\dfrac{F'(c)}{g'(c)} = cf(c)$, the result follows.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4923: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Show that if $n \geq 6$ and is composite, then n divides $(n-2)!$.

  Solution by Brian D. Beasley, Clinton, SC.

  Let n be a composite integer with $n \geq 6$. We consider two cases:

  (i) Assume n is not the square of a prime. Then we may write $n = ab$ for integers a and b with $1 < a < b < n-1$. Thus a and b are distinct and are in $\{2, 3, \ldots, n-2\}$, so $n = ab$ divides $(n-2)!$.

  (ii) Assume $n = p^2$ for some odd prime p. Then $n - 2 = p^2 - 2 \geq 2p$, since $p > 2$. Hence both p and $2p$ are in $\{3, 4, \ldots, n-2\}$, so $n = p^2$ divides $(n-2)!$.

  Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Luke Drylie (student, Old Dominion U.), Chesapeake, VA; Kenneth Korbin, NY, NY; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4924: *Proposed by Kenneth Korbin, New York, NY.*

  Given $\displaystyle\sum_{N=1}^{\infty} \frac{F_N}{K^N} = 3$ where F_N is the N^{th} Fibonacci number. Find the value of the positive number K.

  Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

  The ratio test along with the fact that $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ implies $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{K^n}$ converges for $K > \dfrac{1+\sqrt{5}}{2}$. Then

$$\begin{aligned}
3 = \sum_{n=1}^{\infty} \frac{F_n}{K^n} &= \frac{1}{K} + \frac{1}{K^2} + \sum_{n=3}^{\infty} \frac{F_n}{K^n} \\
&= \frac{1}{K} + \frac{1}{K^2} + \sum_{n=3}^{\infty} \frac{F_{n-1} + F_{n-2}}{K^n} \\
&= \frac{1}{K} + \frac{1}{K^2} + \frac{1}{K} \sum_{n=3}^{\infty} \frac{F_{n-1}}{K^{n-1}} + \frac{1}{K^2} \sum_{n=3}^{\infty} \frac{F_{n-2}}{K^{n-2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{K}+\frac{1}{K^2}+\frac{1}{K}\left(3-\frac{1}{K}\right)+\frac{3}{K^2}\\
&= \frac{4}{K}+\frac{3}{K^2} \Rightarrow K=\frac{2+\sqrt{13}}{3}.
\end{aligned}
$$

Also solved by Brian D. Beasley, Clinton, SC; Sam Brotherton (student, Rockdale Magnet School For Science and Technology), Conyers, GA; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Luke Drylie (student, Old Dominion U.), Chesapeake, VA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Boris Rays (jointly), Portsmouth, VA & Chesapeake,VA (respectively); N. J. Kuenzi, Oshkosh, WI; Tom Leong, Scotrun, PA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4925: *Proposed by Kenneth Korbin, New York, NY.*

  In the expansion of

$$\frac{x^4}{(1-x)^3(1-x^2)}=x^4+3x^5+7x^6+13x^7+\cdots$$

find the coefficient of the term with x^{20} and with x^{21}.

Solution 1 by Brian D. Beasley, Clinton, SC.

We have

$$
\begin{aligned}
\frac{1}{(1-x)^3(1-x^2)} &= \frac{1}{(1-x)^4(1+x)}\\
&= (1-x+x^2-x^3+\cdots)(1+x+x^2+x^3+\cdots)^4\\
&= (1-x+x^2-x^3+\cdots)(1+2x+3x^2+4x^3+\cdots)^2\\
&= (1-x+x^2-x^3+\cdots)(1+4x+10x^2+20x^3+\cdots),
\end{aligned}
$$

where the coefficients of the second factor in the last line are the binomial coefficients $C(k,3)$ for $k=3,4,5,\ldots$. Hence, allowing for the x^4 in the original numerator, the desired coefficient of x^{20} is

$$\sum_{k=3}^{19} C(k,3)(-1)^{19-k}=525.$$

Similarly, the desired coefficient of x^{21} is

$$\sum_{k=3}^{20} C(k,3)(-1)^{20-k}=615.$$

Solution 2 by Tom Leong, Scotrun, PA.

Equivalently, we find the coefficients of x^{16} and x^{17} in

$$\frac{1}{(1-x)^3(1-x^2)}. \qquad (1)$$

We use the following well-known generating functions:

$$\begin{aligned}
\frac{1}{1-x^2} &= 1+x^2+x^4+x^6+\cdots \\
\frac{1}{(1-x)^{m+1}} &= \binom{m}{m}+\binom{m+1}{m}x+\binom{m+2}{m}x^2+\binom{m+3}{m}x^3+\cdots.
\end{aligned}$$

A decomposition of (1) is

$$\frac{1}{(1-x)^3(1-x^2)}=\frac{1}{2}\frac{1}{(1-x)^4}+\frac{1}{4}\frac{1}{(1-x)^3}+\frac{1}{8}\frac{1}{(1-x)^2}+\frac{1}{8}\frac{1}{(1-x)}.$$

Thus the coefficient of x^n is

$$\frac{1}{2}\binom{n+3}{3}+\frac{1}{4}\binom{n+2}{2}+\frac{1}{8}\binom{n+1}{1}+\frac{1}{8}=\frac{(n+2)(n+4)(2n+3)}{24} \quad \text{if } n \text{ is even}$$

or

$$\frac{1}{2}\binom{n+3}{3}+\frac{1}{4}\binom{n+2}{2}+\frac{1}{8}\binom{n+1}{1}=\frac{(n+1)(n+3)(2n+7)}{24} \quad \text{if } n \text{ is odd.}$$

So the coefficient of x^{16} is $\frac{18\cdot 20\cdots 35}{24}=525$ and the coefficient of x^{17} is $\frac{18\cdot 20\cdots 41}{24}=$ 615.

Solution 3 by Paul M. Harms, North Newton, KS.

When

$$-1<x<1, \quad \frac{1}{1-x}=1+x+x^2+\cdots.$$

Taking two derivatives, we obtain for

$$-1<x<1, \quad \frac{2}{(1-x)^3}=2+3(2)x+4(3)x^2+\cdots.$$

When

$$-1<x<1, \quad \frac{x^4}{1-x^2}=x^4+x^6+x^8+\cdots.$$

The series for $\frac{x^4}{(1-x)^3(1-x^2)}$ can be found by multiplying

$$\frac{1}{2}\cdot\frac{2}{(1-x)^3}\cdot\frac{x^4}{(1-x^2)}=\frac{1}{2}\Big[2+3(2)x+4(3)x^2+\cdots+18(17)x^{16}+19(18)x^{17}+\cdots\Big]\Big[x^4+x^6+x^8+\cdots\Big].$$

The coefficient of x^{20} is

$$\frac{1}{2}\Big[18(17)+16(15)+14(13)+\cdots 4(3)+2\Big]=525.$$

The coefficient of x^{21} is

$$\frac{1}{2}\Big[19(18)+17(16)+15(14)+\cdots 5(4)+3(2)\Big]=615.$$

Comment: Jahangeer Kholdi and Boris Rays noticed that the coefficients in $x^4+3x^5+7x^6+13x^7+22x^8+34x^9+50x^{10}+\cdots$, are the partial sums of the alternate triangular numbers. I.e., $1, 3, 1+6, 3+10, 1+6+15, 3+10+21, \cdots$, which leads to the coefficients of x^{20} and x^{21} being 525 and 615 respectively.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Jahangeer Kholdi and Boris Rays (jointly), Portsmouth, VA & Chesapeake,VA (respectively); Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4926: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Calculate
$$\sum_{n=1}^{\infty} \frac{nF_n^2}{3^n}$$
where F_n is the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution by David Stone and John Hawkins, Statesboro, GA.

By Binet's Formula, $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, where α and β are the solutions of the quadratic equation $x^2 - x - 1 = 0;\ \alpha = \dfrac{1+\sqrt{5}}{2},\ \beta = \dfrac{1-\sqrt{5}}{2}$.

Note that $a - b = \sqrt{5},\ \alpha \cdot \beta = -1,\ \alpha^2 + \beta^2 = 3,$ and $\alpha^6 + \beta^6 = 18$. Also recall from calculus that $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$. Thus we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{nF_n^2}{3^n} &= \sum_{n=1}^{\infty} \frac{n}{3^n} \frac{\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}}{5} \\
&= \sum_{n=1}^{\infty} \frac{n}{3^n} \frac{\alpha^{2n} - 2(-1)^n + \beta^{2n}}{5} \\
&= \frac{1}{5}\left\{ \sum_{n=1}^{\infty} n\left(\frac{\alpha^2}{3}\right)^n - 2\sum_{n=1}^{\infty} n\left(\frac{-1}{3}\right)^n + \sum_{n=1}^{\infty} n\left(\frac{\beta^2}{3}\right)^n \right\} \\
&= \frac{1}{5}\left\{ \frac{\frac{\alpha^2}{3}}{\left[1-\frac{\alpha^2}{3}\right]^2} - 2\frac{\frac{-1}{3}}{\left[1+\frac{1}{3}\right]^2} + \frac{\frac{\beta^2}{3}}{\left[1-\frac{\beta^2}{3}\right]^2} \right\}, \quad \text{valid because } \frac{\beta^2}{3} < \frac{\alpha^2}{3} < 1; \\
&= \frac{1}{5}\left\{ \frac{3\alpha^2}{[3-\alpha^2]^2} + \frac{3}{8} + \frac{3\beta^2}{[3-\beta^2]^2} \right\} \\
&= \frac{3}{5}\left\{ \frac{\alpha^2}{[\beta^2]^2} + \frac{1}{8} + \frac{\beta^2}{[\alpha^2]^2} \right\} \text{ because } \alpha^2 + \beta^2 = 3, \\
&= \frac{3}{5}\left\{ \frac{1}{8} + \frac{\alpha^6 + \beta^6}{\alpha^4\beta^4} \right\} \text{ by algebra,}
\end{aligned}$$

$$= \frac{3}{5}\left\{\frac{1}{8}+\frac{18}{1}\right\} = \frac{87}{8}.$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA, and the proposer.

- 4927: *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Barcelona, Spain.*

Let k be a positive integer and let

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{2k(2n+1)} \qquad \text{and} \qquad B = \sum_{n=0}^{\infty} (-1)^n \left\{ \sum_{m=0}^{2k} \frac{(-1)^m}{(4k+2)n+2m+1} \right\}.$$

Prove that $\dfrac{B}{A}$ is an even integer for all $k \geq 1$.

Solution by Tom Leong, Scotrun, PA.

Note that inside the curly braces in the expression for B, the terms of the (alternating) sum are the reciprocals of the consecutive odd numbers from $(4k+2)n+1$ to $(4k+2)n+(4k+1)$. As $n = 0, 1, 2, \ldots$, the reciprocal of every positive odd number appears exactly once in this sum (disregarding its sign). Thus

$$B = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{2k} \frac{(-1)^{m+n}}{(4k+2)n+2m+1} \right\} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1}$$

from which we find $\dfrac{B}{A} = 2k$. (In fact, it is well-known that $B = \pi/4$.)

Comment by Editor: This problem was incorrectly stated when it was initially posted in the May, 06 issue of SSM. The authors reformulated it, and the correct statement of the problem and its solution are listed above. The corrected version was also solved by **Paul M. Harms of North Newton, KS.**

- 4928: *Proposed by Yair Mulian, Beer-Sheva, Israel.*

Prove that for all natural numbers n

$$\int_0^1 \frac{2x^{2n+1}}{x^2-1}dx = \int_0^1 \frac{x^n}{x-1} + \frac{1}{x+1}dx.$$

Comment by Editor: The integrals in their present form do not exist, and I did not see this when I accepted this problem for publication. Some of the readers rewrote the problem in what they described as "its more common form;" i.e., to show that $\displaystyle\int_0^1 \frac{2x^{2n+1}}{x^2-1} - \left(\frac{x^n}{x-1} + \frac{1}{x+1}\right)dx = 0$. But I believe that one cannot legitimately recast the problem in this manner, because the $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ if, and only if, $f(x)$ and $g(x)$ is each integrable over these limits. So as I see it, the problem as it was originally stated is not solvable. Mea culpa, once again.

- 4929: *Proposed by Michael Brozinsky, Central Islip, NY.*

  An archaeological expedition uncovered 85 houses. The floor of each of these houses was a rectangular area covered by mn tiles where $m \leq n$. Each tile was a 1 unit by 1 unit square. The tiles in each house were all white, except for a (non-empty) square configuration of blue tiles. Among the 85 houses, all possible square configurations of blue tiles appeared once and only once. Find all possible values of m and n.

  Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

  Assume that each configuration of blue tiles is a $k \times k$ square. Since $m \leq n$ and each such configuration was non-empty, it follows that $k = 1, 2, \ldots, m$. For each value of k, there are $(m-k+1)(n-k+1)$ possible locations for the $k \times k$ configuration of blue tiles. Since each arrangement appeared once and only once among the 85 houses, we have

$$\begin{aligned} 85 &= \sum_{k=1}^{m} (m-k+1)(n-k+1) \\ &= \sum_{k=1}^{m} (m+1)(n+1) - (m+n+2)\sum_{k=1}^{m} k + \sum_{k=1}^{m} k^2 \\ &= m(m+1)(n+1) - (m+n+2)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6} \\ &= \frac{m(m+1)}{6}[3n-(m-1)] \end{aligned}$$

  or

$$m(m+1)[3n-(m-1)] = 510. \tag{1}$$

  This implies that m and $m+1$ must be consecutive factors of 510. By checking all 16 factors of 510, we see that the only possible values of m are $1, 2, 5$. If $m = 2$, (1) does not produce an integral solution for n. If $m = 1$ or 5, equation (1) yields $n = 85$ or 7 (respectively). Therefore, the only solutions are $(m, n) = (1, 85)$ or $(5, 7)$.

  Also solved by Tom Leong, Scotrun, PA; Paul M. Harms, North Newton, KS; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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Solutions to the problems stated in this issue should be posted before
June 1, 2007

- 4960: *Proposed by Kenneth Korbin, New York, NY.*

  Equilateral triangle ABC has an interior point P such that

  $$\overline{AP}=\sqrt{5},\ \overline{BP}=\sqrt{12},\ \text{and } \overline{\mathrm{CP}}=\sqrt{17}.$$

  Find the area of $\triangle APB$.

- 4961: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon is inscribed in a circle with diameter d. Find the area of the hexagon if its sides are $3, 3, 3, 4, 4$ and 4.

- 4962: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of quadrilateral $ABCD$ if the midpoints of the sides are the vertices of a square and if $AB=\sqrt{29}$ and $CD=\sqrt{65}$.

- 4963: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate

  $$\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{1}{3^{i+j}}.$$

- 4964: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let x, y be real numbers and we define the law of composition

  $$x\perp y = x\sqrt{1+y^2}+y\sqrt{1+x^2}.$$

Prove that $(R,+)$ and $(R,\perp)$ are isomorphic and solve the equation $x \perp a = b$.

- 4965: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let h_a, h_b, h_c be the heights of triangle ABC. Let P be any point inside $\triangle ABC$. Prove that

  $$(a)\quad \frac{h_a}{d_a}+\frac{h_b}{d_b}+\frac{h_c}{d_c}\geq 9, \quad (b)\quad \frac{d_a^2}{h_a^2}+\frac{d_b^2}{h_b^2}+\frac{d_c^2}{h_c^2}\geq \frac{1}{3},$$

  where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB respectively.

Solutions

- 4930: *Proposed by Kenneth Korbin, New York, NY.*

  Find an acute angle y such that $\cos(y)+\cos(3y)-\cos(5y)=\dfrac{\sqrt{7}}{2}$.

  Solution by Brian D. Beasley, Clinton, SC.

  Given an acute angle y, let $c=\cos(y)>0$. We use $\cos(3y)=4c^3-3c$ and $\cos(5y)=16c^5-20c^3+5c$ to transform the given equation into

  $$-16c^5+24c^3-7c=\frac{\sqrt{7}}{2}.$$

  Since this equation in turn is equivalent to

  $$32c^5-48c^3+14c+\sqrt{7}=(8c^3-4\sqrt{7}c^2+\sqrt{7})(4c^2+2\sqrt{7}c+1)=0,$$

  we need only determine the positive zeros of $f(x)=8x^3-4\sqrt{7}x^2+\sqrt{7}$. Applying $\cos(7y)=64c^7-112c^5+56c^3-7c$, we note that the six zeros of

  $$64x^6-112x^4+56x^2-7=f(x)(8x^3+4\sqrt{7}x^2-\sqrt{7})$$

  are $\cos(k\pi/14)$ for $k\in\{1,3,5,9,11,13\}$. We let $g(x)=8x^3+4\sqrt{7}x^2-\sqrt{7}$ and use $g'(x)=24x^2+8\sqrt{7}x$ to conclude that g is increasing on $(0,\infty)$, and hence has at most one positive zero. But $g(1/2)>0$, $\cos(\pi/14)>1/2$, and $\cos(3\pi/14)>1/2$, so $\cos(\pi/14)$ and $\cos(3\pi/14)$ must be zeros of $f(x)$ instead. Thus we may take $y=\pi/14$ or $y=3\pi/14$ in the original equation.

  Also solved by: Dionne Bailey, Elsie Campbell, and Charles Dimminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayete, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Satesboro, GA, and the proposer.

- 4931: *Proposed by Kenneth Korbin, New York, NY.*

  A Pythagorean triangle and an isosceles triangle with integer length sides both have the same length perimeter $P=864$. Find the dimensions of these triangles if they both have the same area too.

  Solution by David Stone and John Hawkins (jointly), Statesboro, GA.

  Surprisingly, there exists only one such pair of triangles: the (primitive) Pythagorean tiangle $(135,352,377)$ and the isosceles triangle $(366,366,132)$. Each has a perimeter 864 and area $23,760$.

  By Heron's Formula (or geometry), an isosceles triangle with given perimeter P and sides

(a, a, b) has area

$$A = \frac{b}{4}\sqrt{4a^2 - b^2} = \frac{P - 2a}{4}\sqrt{P(4a - P)}, \text{ where } \frac{P}{4} \le a \le \frac{P}{2}.$$

In our problem, $P = 864$. We can analyze possibilities to reduce the number of cases to check or we can use a calculator or computer to check all possibilities. In any case, there are only a few such triangles with integer length sides:

$$\left\{\begin{array}{ccc} a & b & A \\ 222 & 420 & 15,120 \\ 240 & 384 & 27,648 \\ 270 & 324 & 34,992 \\ 312 & 240 & 34,560 \\ 366 & 132 & 23,760 \end{array}\right\}$$

Now, if (a, b, c) is a Pythaorean triangle with given perimeter P and given area A, we can solve the equations

$$\begin{array}{rcl} P & = & a + b + c \\ c^2 & = & a^2 + b^2 \\ A & = & \dfrac{ab}{2} \end{array}$$

to obtain $a = \dfrac{(P^2 + 4A) \pm \sqrt{P^4 - 24P^2A + 16A^2}}{4P}, \quad b = \dfrac{2A}{a}, \ c = P - a - \dfrac{2A}{a}.$

We substitute $P = 864$ and the values for A from the above table. Only with $A = 23,760$ do we find a solutions $(135, 352, 377)$. (Note that the two large values of A each produce a negative under the radical because those values of A are too large to be hemmed up by a perimeter of 864, while the first two values of A produce right triangles with non-integer sides.)

Also solved by Brain D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Amihai Menuhin, Beer-Sheva, Israel, Harry Sedinger, St. Bonaventure, NY, and the proposer.

- 4932: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let ABC be a triangle with semi-perimeter s, in-radius r and circum-radius R. Prove that

$$\sqrt[3]{r^2} + \sqrt[3]{s^2} \le 2\sqrt[3]{2R^2}$$

and determine when equality holds.

Solution by the proposer.

From Euler's inequality for the triangle $2r \le R$, we have $r/R \le 1/2$ and

$$\left(\frac{r}{R}\right)^{2/3} \le \left(\frac{1}{2}\right)^{2/3} \tag{1}$$

Next, we will see that

$$\frac{s}{R} \le \frac{3\sqrt{3}}{2} \tag{2}$$

In fact, from Sine's Law
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$$
we have
$$\frac{a+b+c}{\sin A+\sin B+\sin C}=2R$$
or
$$\frac{s}{R}=\frac{a+b+c}{2R}=\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$$
as claimed. Notice that the last inequality is an immediate consequence of Jensen's inequality applied to the function $f(x)=\sin x$ that is concave in $[0,\pi]$.

Finally, from (1) and (2), we have
$$\left(\frac{r}{R}\right)^{2/3}+\left(\frac{s}{R}\right)^{2/3}\le\left(\frac{1}{2}\right)^{2/3}+\left(\frac{3\sqrt{3}}{2}\right)^{2/3}=2\sqrt[3]{2}$$
from which the statement immediately follows as desired. Note that equality holds when $\triangle ABC$ is equilateral, as immediately follows from (1) and (2).

- 4933: *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Let n be a positive integer. Prove that
$$\frac{1}{n}\sum_{k=1}^{n}k\binom{n}{k}^{1/2}\le\frac{1}{2}\sqrt{(n+1)2^n}.$$

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX .

By the Binomial Theorem,
$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x^k &= (1+x)^n\\
\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k &= \frac{d}{dx}(1+x)^n\\
\sum_{k=1}^{n}k\binom{n}{k}x^{k-1} &= n(1+x)^{n-1}\\
\sum_{k=1}^{n}k\binom{n}{k}x^{k} &= nx(1+x)^{n-1}\\
\frac{d}{dx}\sum_{k=1}^{n}k\binom{n}{k}x^{k} &= \frac{d}{dx}\left[nx(1+x)^{n-1}\right]\\
\sum_{k=1}^{n}k^2\binom{n}{k}x^{k-1} &= n(1+x)^{n-2}(nx+1) \qquad (1).
\end{aligned}$$
Evaluating (1) when $x=1$,
$$\begin{aligned}
\sum_{k=1}^{n}k^2\binom{n}{k} &= n(n+1)2^{n-2}\\
\frac{1}{n}\sum_{k=1}^{n}k^2\binom{n}{k} &= \frac{(n+1)2^n}{4} \qquad (2).
\end{aligned}$$

By the Root Mean Square Inequality and (2),

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} k\binom{n}{k}^{1/2} &\le \sqrt{\frac{\sum\limits_{k=1}^{n} k^2\binom{n}{k}}{n}} \\
&= \sqrt{\frac{(n+1)2^n}{4}} \\
&= \frac{1}{2}\sqrt{(n+1)2^n}.
\end{aligned}$$

Also solved by the proposer.

- 4934: *Proposed by Michael Brozinsky, Central Islip, NY.*

Mrs. Moriaty had two sets of twins who were always getting lost. She insisted that one set must chose an arbitrary non-horizontal chord of the circle $x^2 + y^2 = 4$ as long as the chord went through $(1, 0)$ and they were to remain at the opposite endpoints. The other set of twins was similarly instructed to choose an arbitrary non-vertical chord of the same circle as long as the chord went through $(0, 1)$ and they too were to remain at the opposite endpoints. The four kids escaped and went off on a tangent (to the circle, of course). All that is known is that the first set of twins met at some point and the second set met at another point. Mrs. Moriaty did not know where to look for them but Sherlock Holmes deduced that she should confine her search to two lines. What are their equations?

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

The equations of the two lines are $x = 4$ for the first set of twins and $y = 4$ for the second set of twins.

The vertical chord through the point (1,0) meets the circle at points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. The slopes of the tangent lines are $-\dfrac{1}{\sqrt{3}}$ and $\dfrac{1}{\sqrt{3}}$. So the equations of the tangent lines are

$$y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}} \quad \text{and} \quad y = \frac{1}{\sqrt{3}}x - \frac{4}{\sqrt{3}}.$$

These tangent lines meet at the point (4,0). Otherwise, a non-vertical (and non-horizontal) chord through the point (1,0) intersects the circle at points (a, b) and (c, d), $bd \neq 0$, $b \neq d$. The slopes of the tangent lines are $-\dfrac{a}{b}$ and $-\dfrac{c}{d}$. So the equations of the tangent lines are

$$y = -\frac{a}{b}x + \frac{4}{b} \quad \text{and} \quad y = -\frac{c}{d}x + \frac{4}{d}.$$

The x-coordinate of the point of intersection of the tangent lines is $\dfrac{4(d-b)}{ad-bc}$. And since the points $(a, b), (c, d)$ and (1,0) are on the chord, we have

$$\frac{b-0}{a-1} = \frac{d-0}{c-1}$$

or

$$d - b = ad - bc.$$

Therefore, the x-coordinate of the point of intersection of the tangent lines is 4.

Similar calculations apply to position of the second set of twins.

Also solve by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4935: *Proposed by Xuan Liang, Queens, NY and Michael Brozinsky, Central Islip, NY.*

Without using the converse of the Pythagorean Theorem nor the concepts of slope, similar triangles or trigonometry, show that the triangle with vertices $A(-1,0), B(m^2,0)$ and $C(0,m)$ is a right triangle.

Solution by Harry Sedinger, St. Bonaventure, NY.

Let $O=(0,0)$. The area of $\triangle ABC$ is

$$\begin{aligned}\frac{1}{2}\Big(|OB|\Big)\Big(|AC|\Big) &= \frac{1}{2}m(m^2+1)=\frac{1}{2}m\sqrt{m^2+1}\sqrt{m^2+1}\\ &= \frac{1}{2}\sqrt{m^4+m^2}\sqrt{m^2+1}=\frac{1}{2}\Big(|BC|\Big)\Big(|AB|\Big).\end{aligned}$$

Thus if AB is considered the base of $\triangle ABC$, its height is $|BC|$, so $AB\perp BC$ and $\triangle ABC$ is a right triangle.

Also solved by Charles Ashbacher, Cedar Rapis, IA; Brian D. Beasley, Clinton, SC; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; William Weirich (student Virginia Commonwealth University), Richmond, VA, and the proposers.

Editor's comment: Several readers used the distance formula or the law of cosines, or the dot product of vectors in their solutions; but to the best of my knowledge, these notions are obtained with the use of the Pythagorean Theorem.

Problems

Ted Eisenberg, Section Editor

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Solutions to the problems stated in this issue should be posted before July 1, 2007

- 4966: *Proposed by Kenneth Korbin, New York, NY.*

  Solve:
  $$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$
  with $0 < x < 1$.

- 4967: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with an interior point P such that $\overline{\mathrm{AP}}^2 + \overline{\mathrm{BP}}^2 = \overline{\mathrm{CP}}^2$, and with an exterior point Q such that $\overline{\mathrm{AQ}}^2 + \overline{\mathrm{BQ}}^2 = \overline{\mathrm{CQ}}^2$, where points C, P, and Q are in a line. Find the lengths of $\overline{\mathrm{AQ}}$ and $\overline{\mathrm{BQ}}$ if $\overline{\mathrm{AP}} = \sqrt{21}$ and $\overline{\mathrm{BP}} = \sqrt{28}$.

- 4968: *Proposed by Kenneth Korbin, New York, NY.*

  Find two quadruples of positive integers (a, b, c, d) such that
  $$\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i} = \frac{a-i}{a+i}\cdot\frac{b-i}{b+i}\cdot\frac{c-i}{c+i}\cdot\frac{d-i}{d+i}$$
  with $a < b < c < d$ and $i = \sqrt{-1}$.

- 4969: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that
  $$\frac{1}{a^2\left(\frac{1}{a}+\frac{1}{c}\right)} + \frac{1}{b^2\left(\frac{1}{b}+\frac{1}{a}\right)} + \frac{1}{c^2\left(\frac{1}{c}+\frac{1}{b}\right)} \geq \frac{3}{2}$$

- 4970: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $f : [0, 1] \longrightarrow \mathbf{R}$ be a contintuous convex function. Prove that

$$\frac{3}{4}\int_0^{1/5} f(t)dt + \frac{1}{8}\int_0^{2/5} f(t)dt \geq \frac{4}{5}\int_0^{1/4} f(t)dt.$$

- 4971: *Proposed by Howard Sporn, Great Neck, NY and Michael Brozinsky, Central Islip, NY.*

  Let $m \geq 2$ be a positive integer and let $1 \leq x < y$. Prove:

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Solutions

- 4936: *Proposed by Kenneth Korbin, New York, NY.*

  Find all prime numbers P and all positive integers a such that $P - 4 = a^4$.

Solution 1 by Daniel Copeland (student, Saint George's School), Spokane, WA.

$$\begin{aligned} P &= a^4 + 4 \\ &= (a^2+2)^2 - 4a^2 \\ &= (a^2 - 2a + 2)(a^2 + 2a + 2). \end{aligned}$$

Since P is a prime, one of the factors of P must be 1. Since a is a positive integer, $a^2 - 2a + 2 = 1$ which yields the only positive solution $a = 1, P = 5$.

Solution 2 by Timothy Bowen (student, Waynesburg College), Waynesburg, PA.

The only solution is $P = 5$ and $a = 1$.
Case 1: Integer a is an even integer. For $a = 2n$, note $P = a^4+4 = (2n)^4+4 = 4\cdot(4n^4+1)$. Clearly, P is a composite for all natural numbers n.

Case 2: Integer a is an odd integer. For $a = 2n+1$, note that $P = a^4+4 = (2n+1)^4+4 = (4n^2+8n+5)(4n^2+1)$. P is prime only for $n = 0$ (corresponding to $a = 1$ and $P = 5$). Otherwise, P is a composite number for all natural numbers n.

Solution 3 by Jahangeer Kholdi & Robert Anderson (jointly), Portsmouth, VA.

The only prime is $P = 5$ when $a = 1$. Consider $P = a^4 + 4$. If a is an even positive integer, then clearly P is even and hence a composite integer. Moreover, if a is a positive integer ending in digits $\{1, 3, 7 \text{ or } 9\}$, then P is a positive integer ending with the digit of 5. This also implies P is divisible by 5 and hence a composite. Lastly, assume $a = 10k+5$ where $k = 0$ or $k > 0$; that is a is a positive integer ending with a digit of 5. Then $P = (10k+5)^4 + 4$. But

$$P = (10k+5)^4 + 4 = (100k^2 + 80k + 17)(100k^2 + 120k + 37).$$

Hence, for all positive integers $a > 1$ the positive integer P is composite.

Also solved by Brian D. Beasley, Clinton, SC; Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY;

Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Vicki Schell, Pensacola, FL; R. P. Sealy, Sackville, New Brunswick, Canada; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins of Statesboro, GA jointly with Chris Caldwell of Martin, TN, and the proposer.

- **4937:** *Proposed by Kenneth Korbin, New York, NY.*

Find the smallest and the largest possible perimeter of all the triangles with integer-length sides which can be inscribed in a circle with diameter 1105.

Solution by Paul M. Harms, North Newton, KS.

Consider a radius line from the circle's center to one vertex of an inscribed triangle. Assume at this vertex one side has a length a and subtends a central angle of $2A$ and the other side making this vertex has a length b and subtends a central angle of $2B$.

Using the perpendicular bisector of chords, we have $\sin A = \dfrac{a/2}{1105/2} = \dfrac{a}{1105}$ and $\sin B = \dfrac{b}{1105}$. Also, the central angle of the third side is related to $2A+2B$ and the perpendicular bisector to the third side gives

$$\begin{aligned} \sin(A+B) = \frac{c}{1105} &= \sin A\cos B + \sin B\cos A \\ &= \frac{a}{1105}\frac{\sqrt{1105^2-b^2}}{1105} + \frac{b}{1105}\frac{\sqrt{1105^2-a^2}}{1105} \\ \text{Thus}\quad c &= \frac{1}{1105}\left(a\sqrt{1105^2-b^2} + b\sqrt{1105^2-a^2}\right). \end{aligned}$$

From this equation we find integers a and b which make integer square roots. Some numbers which do this are {47,1104 105, 1100, 169, 1092, etc. }. Checking the smaller numbers for the smallest perimeter we see that a triangle with side lengths {105,169,272} gives a perimeter of 546 which seems to be the smallest perimeter.

To find the largest perimeter we look for side lengths close to the lengths of an inscribed equilateral triangle. An inscribed equilateral triangle for this circle has side length close to 957. Integers such as $884, 943, 952, 975$, and 1001 make integer square roots in the equation for c. The maximum perimeter appears to be 2870 with a triangle of side lengths {943,952,975}.

Comment: David Stone and John Hawkins of Statesboro, GA used a slightly different approach in solving this problem. Letting the side lengths be a, b, and c and noting that the circumradius is 552.5 they obtained

$$\frac{1105}{2} = \frac{abc}{4\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}}$$

which can be rewritten as

$$\sqrt{a+b+c)(a+b-c)(a-b+c)(b+c-a)} = \frac{abc}{(2)(5)(13)(17)}.$$

They then used that part of the law of sines that connects in any triangle ABC, side length $a, \angle A$ and the circumradius R; $\dfrac{a}{\sin A} = 2R$. This allowed them to find that $c^2 =$

$a^2+b^2 \mp \dfrac{2ab\sqrt{1105^2-c^2}}{1105}$. Noting that the factors of a,b, and c had to include the primes 2,5,13 and 17 and that $1105^2 - c^2$ had to be a perfect square, (and similarly for $1105^2 - b^2$ and $1105^2 - a^2$) they put EXCEL to work and proved that {105, 272, 169} gives the smallest perimeter and that {952, 975, 943} gives the largest. All in all they found 101 triangles with integer side lengths that can be inscribed in a circle with diameter 1105.

Also solved by the proposer.

- 4938: *Proposed by Luis Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b and c be the sides of an acute triangle ABC. Prove that

$$\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2} \geq 6\left[\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right)\right]^{1/3}$$

Solution by proposers.

First, we claim that $a^2 \geq 2(b^2+c^2)\sin^2(A/2)$. In fact, the preceding inequality is equivalent to $a^2 \geq (b^2+c^2)(1-\cos A)$ and

$$\begin{aligned} a^2-(b^2+c^2)(1-\cos A) &= b^2+c^2-2bc\cos A-(b^2+c^2)+(b^2+c^2)\cos A \\ &= (b-c)^2\cos A \geq 0. \end{aligned}$$

Similar inequalities can be obtained for b and c. Multiplying them up, we have

$$a^2b^2c^2 \geq 8(a^2+b^2)(b^2+c^2)(c^2+a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2). \qquad (1)$$

On the other hand, from GM-HM inequality we have

$$\begin{aligned} \sin^2(A/2)\sin^2(B/2)\sin^2(C/2) &\geq \left(\frac{3}{1/\sin^2(A/2)+1/\sin^2(B/2)+1/\sin^2(C/2)}\right)^3 \\ &= \left(\frac{3}{\csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)}\right)^3. \end{aligned}$$

Substituting into the statement of the problem yields

$$\begin{aligned} \left(\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\right)^3 &\geq 216\left(\frac{a^2+b^2}{c^2}\right)\left(\frac{b^2+c^2}{a^2}\right)\left(\frac{c^2+a^2}{b^2}\right) \\ &= 216\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right). \end{aligned}$$

Notice that equality holds when $A = B = C = \pi/3$. That is, when $\triangle ABC$ is equilateral and we are done.

- 4939: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

For any positive integer n, prove that

$$\left\{4^n+\left[\sum_{k=0}^{n-1}3^{k+1/2}\binom{2n}{2k+1}\right]^2\right\}^{1/2}$$

is a whole number.

Solution by David E. Manes, Oneonta, NY.

Let $W = 4^n + \left[\sum_{k=0}^{n-1} 3^{k+1/2}\binom{2n}{2k+1}\right]^2$ and notice that it suffices to show that $\sqrt{W}$ is a whole number. Expanding $(\sqrt{3}+1)^{2n}$ and $(\sqrt{3}-1)^{2n}$ using the Binomial Theorem and subtracting the second expansion from the first, one obtains

$$\sum_{k=0}^{n-1} 3^{k+1/2}\binom{2n}{2k+1} = \frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2}.$$

Therefore,

$$\begin{aligned} W &= 4^n + \left[\frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2}\right]^2 \\ &= 4^n + \frac{(\sqrt{3}+1)^{4n} - 2^{2n+1} + (\sqrt{3}-1)^{4n}}{4} \\ &= \frac{2^{2n+2} + (\sqrt{3}+1)^{4n} - 2^{2n+1} + (\sqrt{3}-1)^{4n}}{4} \\ &= \frac{(\sqrt{3}+1)^{4n} + 2^{2n+1} + (\sqrt{3}-1)^{4n}}{4} \\ &= \left[\frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2}\right]^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \sqrt{W} &= \frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2} = \sum_{k=0}^{n}\binom{2n}{2k}(\sqrt{3})^{2k} \\ &= \sum_{k=0}^{n}\binom{2n}{2k}3^k, \text{ a whole number.} \end{aligned}$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul H. Harms, North Newton, KS, and the proposer.

- 4940: *Proposed by Michael Brozinsky, Central Islip, NY and Leo Levine, Queens, NY .*

Let $S = \{n \in N | n \geq 5\}$. Let $G(x)$ be the fractional part of x, i.e., $G(x) = x - [x]$ where $[x]$ is the greatest integer function. Characterize those elements T of S for which the function

$$f(n) = n^2\Big(G\Big(\frac{(n-2)!}{n}\Big)\Big) = n.$$

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

T is the set of primes in S. One form of Wilson's Theorem states: A necessary and sufficient condition that n be prime is that $(n-1)! \equiv -1(\text{mod n})$. But $(n-1)! = (n-1)(n-2)!$ with $n-1 \equiv -1(\text{mod n})$. Therefore $(n-2)! \equiv 1(\text{mod n})$ if, and only if, n is prime. Therefore

$$f(n) = n^2\Big(G\Big(\frac{(n-2)!}{n}\Big)\Big) = n^2 \cdot \frac{1}{n} = n \ \text{ if, and only if, } \ n \geq 5 \text{ is prime.}$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **4941:** *Proposed by Tom Leong, Brooklyn, NY.*

The numbers $1, 2, \cdots, 2006$ are randomly arranged around a circle.
(a) Show that we can select 1000 adjacent numbers consisting of 500 even and 500 odd numbers.
(b) Show that part (a) need not hold if the numbers were randomly arranged in a line.

Solution 1 by Paul Zorn, Northfield, MN.

Claim: Suppose we have 1003 0's and 1003 1's arranged in a circle, like a 2006-hour clock. Then there must be a stretch of length of 1000 containing 500 of each.

Proof: Call the clock positions $1, 2, \cdots, 2006$ as on an ordinary clock, and let $a(n)$ be 0 or 1, depending on what's at position n. Let $S(n) = a(n) + a(n+1) + \cdots + a(n+999)$, where addition in the arguments is mod 2006.

Note that $S(n)$ is just the number of 1's in the 1000-hour stretch starting at n, and we're done if $S(n) = 500$ for some n.

Now $S(n)$ has two key properties, both easy to show:
i) $S(n+1)$ differs from $S(n)$ by at most 1
ii) $S(1) + S(2) + S(3) + \cdots S(2006) = 1000\cdot$(sum of *all* the 1's around the circle) $=1000(1003)$.

From **i)** and **ii)** it follows that if $S(j) > 500$ and $S(k) < 500$ for some j and k, then $S(n) = 500$ for some n *between* j and k. So suppose, toward contradiction, that (say) $S(n) > 500$ for all n. Then

$$S(1) + S(2) + S(3) + \cdots + S(2006) > 2006 \cdot 501 = 1003(1002),$$

which contradicts ii) above.

Solution 2 by Harry Sedinger, St. Bonaventure, NY.

Denote the numbers going around the circle in a given direction as $n_1, n_2, \cdots, n_{206}$ where n_i and n_{i+1} are adjacent for each i and n_{2006} and n_1 are also adjacent. Let S_i be the set of 1,000 adjacent numbers going in the same direction and starting with n_i. Let $E(S_i)$ be the number of even numbers in S_i. It is easily seen that each number occurs in exactly 1000 such sets. Thus the sum S of occurring even numbers in all such sets is 1,003 (the number of even numbers) times 1000 which is equal to 1,003,000.

a) Suppose that $E(S_i) \neq 500$ for every i. Clearly $E(S_i)$ and $E(S_{i+1})$ differ by at most one, (as do $E(S_{2006})$ and $E(S_1)$), so either $E(S_i) \leq 499$ for every i or $E(S_i) \geq 501$ for every i. In the first case $S \leq 499 \cdot 2,006 < 1003,000$, a contradiction, and in the second case $S \geq 501 \cdot 2006 > 1,003,000$, also a contradiction. Hence $E(S_i) = 500$ for some k and the number of odd numbers in S_k is also 500.

b) It is easily seen that a) does not hold if the numbers are sequenced by 499 odd, followed by 499 even, followed by 499 odd, followed by 499 even, followed by 4 odd, and followed by 4 even.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Apologies Once Again

I inadvertently forgot to mention that David Stone and John Hawkins of Statesboro, GA jointly solved problems 4910 and 4911. But worse, in my comments on 4911 (Is is possible for the sums of the squares of the six trigonometric functions to equal one), I mentioned that only two of the 26 solutions that were submitted considered the problem with respect to complex arguments. (For real arguments the answer is no; but for complex arguments it is yes.) David and John's solution considered both arguments–which makes my omission of their name all the more embarrassing. So once again, mea-culpa.

Problems

Ted Eisenberg, Section Editor

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1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
September 1, 2007

- 4972: *Proposed by Kenneth Korbin, New York, NY.*

  Find the length of the side of equilateral triangle ABC if it has a cevian $\overline{\mathrm{CD}}$ such that

  $$\overline{\mathrm{AD}} = x, \quad \overline{\mathrm{BD}} = x + 1 \quad \overline{\mathrm{CD}} = \sqrt{y}$$

  where x and y are positive integers with $20 < x < 120$.

- 4973: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of trapezoid ABCD if it is inscribed in a circle with radius R=2, and if it has base $\overline{\mathrm{AB}} = 1$ and $\angle \mathrm{ACD} = 60^{\mathrm{o}}$.

- 4974: *Proposed by Kenneth Korbin, New York, NY.*

  A convex cyclic hexagon has sides $a, a, a, b, b,$ and b. Express the values of the circumradius and the area of the hexagon in terms of a and b.

- 4975: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Solve in R the following system of equations

  $$\left.\begin{array}{c} 2x_1 = 3x_2\,\sqrt{1+x_3^2} \\ 2x_2 = 3x_3\,\sqrt{1+x_4^2} \\ \ldots\ldots \\ 2x_n = 3x_1\,\sqrt{1+x_2^2} \end{array}\right\}$$

- 4976: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that
$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab}\geq 27.$$

- 4977: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $1<a<b$ be real numbers. Prove that for any $x_1, x_2, x_3 \in [a,b]$ there exist $c \in (a,b)$ such that
$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log c}.$$

Solutions

- 4942: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive integers a and b. Find the minimum and the maximum possible values of the sum $(a+b)$ if $\dfrac{ab-1}{a+b}=2007$.

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX.

If $\dfrac{ab-1}{a+b}=2007$, then
$$\begin{aligned} ab-1 &= 2007(a+b)\\ ab-2007a-2007b &= 1\\ ab-2007a-2007b+2007^2 &= 1+2007^2\\ (a-2007)(b-2007) &= 2\cdot 5^2\cdot 13\cdot 6197 \qquad (\mathbf{1}). \end{aligned}$$

Since **(1)** and the sum $(a+b)$ are symmetric in a and b, then we will assume that $a<b$. By the prime factorization in **(1),** there are exactly 12 distinct values for $(a-2007)$ and $(b-2007)$ which are summarized below.

$a-2007$	$b-2007$	a	b	$a+b$
1	$4,028,050$	$2,008$	$4,030,057$	$4,032,065$
2	$2,014,025$	$2,009$	$2,016,032$	$2,018,041$
5	$805,610$	$2,012$	$807,617$	$809,629$
10	$402,805$	$2,017$	$404,812$	$406,829$
13	$309,850$	$2,020$	$311,857$	$313,877$
25	$161,122$	$2,032$	$163,129$	$165,161$
26	$154,925$	$2,033$	$156,932$	$158,965$
50	$80,561$	$2,057$	$82,568$	$84,625$
65	$61,970$	$2,072$	$63,977$	$66,049$
130	$30,985$	$2,137$	$32,992$	$35,129$
325	$12,394$	$2,332$	$14,401$	$16,733$
650	$6,197$	$2,657$	$8,204$	$10,861$

Thus, the minimum value is $10,861$, and the maximum value is $4,032,065$.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; John Nord, Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4943: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with $\overline{AB} = 19$, $\overline{BC} = 8$, $\overline{CD} = 6$, and $\overline{AD} = 17$. Find the area of the quadrilateral if both $\overline{AC}$ and $\overline{BD}$ also have integer lengths.

  Solution by Brian D. Beasley, Clinton, SC.

  Let $x = \overline{AC}$ and $y = \overline{BD}$, where both x and y are positive integers. Let A_1 be the area of triangle ABC, A_2 be the area of triangle of ADC, A_3 be the area of triangle BAD, and A_4 be the area of triangle BCD. Then by Heron's formula, we have

$$A_1 = \sqrt{s(s-19)(s-8)(s-x)} \qquad A_2 = \sqrt{t(t-17)(t-6)(t-x)},$$

  where $s = (19 + 8 + x)/2$ and $t = (17 + 6 + x)/2$. Similarly,

$$A_3 = \sqrt{u(u-19)(u-17)(u-y)} \qquad A_4 = \sqrt{v(v-8)(v-6)(v-y)},$$

  where $u = (19 + 17 + y)/2$ and $v = (8 + 6 + y)/2$. Also, the lengths of the various triangle sides imply $x \in \{12, 13, \cdots, 22\}$ and $y \in \{3, 4, \cdots, 13\}$. We consider three cases for the area T of $ABCD$:

  Case 1: Assume $ABCD$ is convex. Then $T = A_1 + A_2 = A_3 + A_4$. But a search among the possible values for x and y yields no solutions in this case.

  Case 2: Assume $ABCD$ is not convex, with triangle BAD containing triangle BCD (i.e., C is interior to ABD). Then $T = A_1 + A_2 = A_3 - A_4$. Again, a search among the possible values for x and y yields no solutions in this case.

  Case 3: Assume $ABCD$ is not convex, with triangle ABC containing triangle ADC (i.e., D is interior to ABC). Then $T = A_1 - A_2 = A_3 + A_4$. In this case, a search among the possible values for x and y yields the unique solution $x = 22$ and $y = 4$; this produces $T = \sqrt{1815} = 11\sqrt{15}$.

  Due to the lengths of the quadrilateral, these are the only three cases for $ABCD$. Thus the unique value for its area is $11\sqrt{15}$.

  Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 4944: *Proposed by James Bush, Waynesburg, PA.*

  Independent random numbers a and b are generated from the interval $[-1, 1]$ to fill the matrix $A = \begin{pmatrix} a^2 & a^2 + b \\ a^2 - b & a^2 \end{pmatrix}$. Find the probability that the matrix A has two real eigenvalues.

  Solution by Paul M. Harms, North Newton, KS.

  The characteristic equation is $(a^2 - \lambda)^2 - (a^4 - b^2) = 0$. The solutions for λ are $a^2 + \sqrt{a^4 - b^2}$ and $a^2 - \sqrt{a^4 - b^2}$. There are two real eigenvalues when $a^4 - b^2 > 0$ or $a^2 > |b|$. The region in the ab coordinate system which satisfies the inequality is between the parabolas $b = a^2$ and $b = -a^2$ and inside the square where a and b are both in $[-1, 1]$. From the symmetry of the region we see that the probability is the area in the first quadrant between the a-axis and $b = a^2$ from $a = 0$ to $a = 1$. Integrating gives a probability of $\dfrac{1}{3}$.

  Also solved by Tom Leong, Scotrun, PA; John Nord, Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA; Boris Rays, Chesapeake, VA, and the proposer.

- 4945: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Prove that

$$17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+L_{k+2}^4\Big)^{1/2}=L_n^2+3L_{n+1}^2+5L_nL_{n+1}$$

where L_n is the n^{th} Lucas number defined by $L_0=2, L_1=1$ and for all $n\geq 2, L_n=L_{n-1}+L_{n-2}$.

Solution by Tom Leong, Scotrun, PA.

Using the identity $a^4+b^4+(a+b)^4=2(a^2+ab+b^2)^2$ we have

$$\begin{aligned}
17+\sqrt{2}\sum_{k=1}^{n}\left(L_k^4+L_{k+1}^4+L_{k+2}^4\right)^{1/2} &= 17+\sqrt{2}\sum_{k=1}^{n}\left(L_k^4+L_{k+1}^4+(L_k+L_{k+1})^4\right)^{1/2}\\
&= 17+2\sum_{k=1}^{n}\left(L_k^2+L_kL_{k+1}+L_{k+1}^2\right)\\
&= 17+\sum_{k=1}^{n}L_k^2+\sum_{k=1}^{n}L_{k+1}^2+\sum_{k=1}^{n}(L_k+L_{k+1})^2\\
&= 17+\sum_{k=1}^{n}L_k^2+\sum_{k=1}^{n}L_{k+1}^2+\sum_{k=1}^{n}L_{k+2}^2\\
&= 17+L_{n+2}^2+2L_{n+1}^2-L_2^2-2L_1^2+3\sum_{k=1}^{n}L_k^2\\
&= 17+(L_n+L_{n+1})^2+2L_{n+1}^2-3^2-2\cdot 1^2+3\sum_{k=1}^{n}L_k^2\\
&= L_n^2+3L_{n+1}^2+2L_nL_{n+1}+6+3\sum_{k=1}^{n}L_k^2\\
&= L_n^2+3L_{n+1}^2+2L_nL_{n+1}+6+3\left(L_nL_{n+1}-2\right)\\
&= L_n^2+3L_{n+1}^2+5L_nL_{n+1}
\end{aligned}$$

where we used the identity $\sum_{k=1}^{n}L_k^2=L_nL_{n+1}-2$ which is easily proved via induction.

Comment: Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie started off their solution with

$$2(L_k^4+L_{k+1}^4+L_{k+2}^4)=(L_k^2+L_{k+1}^2+L_{k+2}^2)^2$$

and noted that this is a special case of Candido's Identity $2(x^4+y^4+(x+y)^4)=(x^2+y^2+(x+y)^2)^2$, for which Roger Nelsen gave a proof without words in *Mathematics Magazine* (vol. 78,no. 2). Candido used this identity to establish that $2(F_n^4+F_{n+1}^4+F_{n+2}^4)=(F_n^2+F_{n+1}^2+F_{n+2}^2)$, where F_n denotes the n^{th} Fibonacci number.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS, and the proposer.

- 4946: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let z_1, z_2 be nonzero complex numbers. Prove that

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|\right)\geq 4.$$

Solution by David Stone and John Hawkins (jointly), Statesboro, GA.

We note that for $a, b > 0$,

$$\begin{array}{rcl} a^2-2ab+b^2 & = & (a-b)^2 \geq 0 \\ \text{so } a^2+2ab+b^2 & \geq & 4ab \\ \text{so } (a+b)(a+b) & \geq & 4ab \\ \text{so } \dfrac{(a+b)}{ab}(a+b) & \geq & 4 \\ \text{or } \left(\dfrac{1}{a}+\dfrac{1}{b}\right)(a+b) & \geq & 4 \end{array}$$

Therefore, **(1)** $\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)(|z_1|+|z_2|)\geq 4.$
For two complex numbers $w=a+bi$ and $v=c+di$, we have

$$\begin{array}{rcl} |(w-v)^2|+|(w+v)^2| & = & |w-v|^2+|w+v|^2=(a-c)^2+(b-d)^2+(a+c)^2+(b+d)^2 \\ & = & 2(a^2+b^2+c^2+d^2)=2(|w|^2+|v|^2) \end{array}$$

so, **(2)** $|(w-v)^2|+|(w+v)|^2=2(|w^2|+|v^2|)$.
Let w be such that $w^2=z_1$ and v be such that $v^2=z_2$. Substituting this into **(2),** we get $|w^2-2wv+v^2|+|w^2+2wv+v^2|=2(|z_1|+|z_2|)$, hence

$$\left|\frac{z_1+z_2}{2}-wv\right|+\left|\frac{z_1+z_2}{2}+wv\right|=|z_1|+|z_2|.$$

Since $(wv)^2=z_1z_2$, wv must equal $\sqrt{z_1z_2}$ or $-\sqrt{z_1z_2}$. Thus the preceding equation becomes

$$\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|=|z_1|+|z_2|.$$

Multiplying by $\frac{1}{|z_1|}+\frac{1}{|z_2|}$, we get

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|\right)=\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)(|z_1|+|z_2|)\geq 4$$

by inequality **(1).**

Also solved by Tom Leong Scotrun, PA, and the proposers.

- 4947: *Proposed by Tom Leong, Brooklyn, NY.*

Define a set S of positive integers to be *among composites* if for any positive integer n, there exists an $x \in S$ such that all of the $2n$ integers $x \pm 1, x \pm 2, \ldots, x \pm n$ are composite. Which of the following sets are among composites? (a) The set $\{a+dk | k \in N\}$ of terms of any given arithmetic progression with $a, d \in N, d > 0$. (b) The set of squares. (c) The set of primes. (d)* The set of factorials.

Remarks and solution by the proposer, (with a few slight changes made in the comments by the editor).

This proposal arose after working Richard L. Francis's problems 4904 and 4905; it can be considered a variation on the idea in problem 4904. My original intention was to propose parts (c) and (d) only; however, I couldn't solve part (d) and, after searching the MAA journals, I later found that the question posed by part (c) is not original at all. An article in (*The Two-Year College Mathematics Journal*, Vol. 12, No. 1, Jan 1981, p. 36) solves part (c). However it appears that the appealing result of part (c) is not well-known and the solution I offer differs from the published one. Parts (a) and (b), as far as I know, are original.

Solution. The sets in (a), (b) and (c) are all among composites. In the solutions below, let n be any positive integer.

(a) Choose $m \geq n$ and $m > d$. Clearly the consecutive integers $(3m)! + 2, (3m)! + 3, \ldots, (3m)! + 3m$ are all composite. Furthermore since $d \leq m - 1$, one of the integers $(3m)! + m + 2, (3m)! + m + 3, \ldots, (3m)! + 2m$ belongs to the arithmetic progression and we are done.

(b) By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes congruent to 1 mod 4. Let $p > n$ be prime with $p \equiv 1 \pmod 4$. From the theory of quadratic residues, we know -1 is a quadratic residue mod p, that is, there is a positive integer r such that $r^2 \equiv -1 \pmod p$. Also by Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. Put $x = [r(p-1)!]^2$. Then $x \pm 2, x \pm 3, \ldots, x \pm (p-1)$ are all composite. Furthermore, $x - 1 = [r(p-1)!]^2 - 1 = [r(p-1)! + 1][r(p-1)! - 1]$ is composite and $x \equiv r^2[(p-1)!]^2 \equiv -1(-1)^2 \equiv -1 \pmod p$, that is, $x+1$ is composite.

(c) Let $p > n+1$ be an odd prime. First note $p!$ and $(p-1)! - 1$ are relatively prime. Indeed, the prime divisors of $p!$ are all primes not exceeding p while none of those primes divide $(p-1)!-1$ (clearly primes less than p do not divide $(p-1)!-1$, while $(p-1)!-1 \equiv -2 \pmod p$ by Wilson's theorem). Appealing to Dirichlet's theorem again, there are infinitely many primes x of the form $x = kp! + (p-1)! - 1$. So $x-1, x-2, \ldots, x-(p-2)$ and $x+1, x+3, x+4, \ldots, x+p$ are all composite. By Wilson's theorem, $(p-1)!+1$ is divisible by p; hence $x+2$ is divisible by p, that is, composite.

Remarks. (b) In fact, it can similarly be shown that the set of nth powers for any positive integer n is among composites.

(d) For any prime p, let $x = (p-1)!$. Then $x \pm 2, x \pm 3, \ldots, x \pm (p-1)$ are all composite and by Wilson's theorem, $x+1$ is also composite. It remains: is $x - 1 = (p-1)! - 1$ composite? I don't know; however it's unlikely to be prime for all primes p.

Problems

Ted Eisenberg, Section Editor

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1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
December 15, 2007

- 4978: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with side $\overline{\text{AB}} = 9$ and with cevian $\overline{\text{CD}}$. Find the length of $\overline{\text{AD}}$ if $\triangle$ADC can be inscribed in a circle with diameter equal to 10.

- 4979: *Proposed by Kenneth Korbin, New York, NY.*

  Part I: Find two pairs of positive numbers (x, y) such that

  $$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{\sqrt{65}}{2},$$

  where x is an integer.

  Part II: Find four pairs of positive numbers (x, y) such that

  $$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{65}{2},$$

  where x is an integer.

- 4980: *J.P. Shiwalkar and M.N. Deshpande, Nagpur, India.*

  An unbiased coin is sequentially tossed until $(r+1)$ heads are obtained. The resulting sequence of heads (H) and tails (T) is observed in a linear array. Let the random variable X denote the number of double heads (HH's, where overlapping is allowed) in the resulting sequence. For example: Let $r = 6$ so the unbiased coin is tossed till 7 heads are obtained and suppose the resulting sequence of H's and T's is as follows:

  HHTTT**H**TTTT**HHH**TT**H**

Now in the above sequence, there are three double heads (HH's) at toss number $(1,2),(11,12)$ and $(12,13)$. So the random variable X takes the value 3 for the above observed sequence. In general, what is the expected value of X?

- 4981: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$5^x + 3^x + 2^x - 28x + 18 = 0.$$

- 4982: *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Barcelona, Spain.*
  Calculate

$$\lim_{n\to\infty} \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1} \frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1} \frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1} \frac{1}{i_1 i_2 \ldots i_n}\right).$$

- 4983: *Proposed by Ovidiu Furdui, Kalamazoo, MI.*
  Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\} dx,$$

  where $\{a\}$ is the *fractional part* of a.

Solutions

- 4948: *Proposed by Kenneth Korbin, New York, NY.*
  The sides of a triangle have lengths x_1, x_2, and x_3 respectively. Find the area of the triangle if

$$(x - x_1)(x - x_2)(x - x_3) = x^3 - 12x^2 + 47x - 60.$$

Solution by Jahangeer Kholdi and Robert Anderson (jointly), Portsmouth, VA.

The given equation implies that

$$\begin{aligned} x_1 + x_2 + x_3 &= 12 \\ x_1x_2 + x_1x_3 + x_2x_3 &= 47 \\ x_1x_2x_3 &= 60 \end{aligned}$$

from which by inspection, $x_1 = 3, x_2 = 4$ and $x_3 = 5$.

Editor's comment: At the time this problem was sent to the technical editor, the Journal was in a state of transition. A new editor- in-chief was coming on board and there was some question as to the future of the problem solving column. As such, I sent an advanced copy of the problem solving column to many of the regular contributors. In that advanced copy this polynomial was listed as $(x-x_1)(x-x_2)(x-x_3) = x^3-12x^2+47x-59$, and not with the constant term as listed above. Well, many of those who sent in solutions solved the problem in one of two ways: as above, obtaining the perimeter $x_1 + x_2 + x_3 = 12$; and then finding the area with Heron's formula. $A = \sqrt{6(6-x_1)(6-x_2)(6-x_3)}$.

Substituting 6 into $(x-x_1)(x-x_2)(x-x_3) = x^3-12x^2+47x-59$ gives $(6-x_1)(6-x_2)(6-x_3) = 7$. So, $A = \sqrt{(6)(7)} = \sqrt{42}$. But others noted that the equation $x^3-12x^2+47x-59$ has only one real root, and this gives the impossible situation of having a triangle with the lengths of two of its sides being complex numbers. The intention of the problem was that a solution should exist, and so the version of this problem that was posted on the internet had a constant term of -60. In the end I counted a solution as being correct if the solution path was correct, with special kudos going to those who recognized that the advanced copy version of this problem was not solvable.

Also solved by Brian D. Beasley, Clinton, SC; Mark Cassell (student, St. George's School), Spokane, WA; Pat Costello, Richmond, KY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student, St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken (two solutions as outlined above), Dayton, OH; John Nord (two solutions as outlined above), Spokane, WA; Boris Rays, Chesapeake, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **4949:** *Proposed by Kenneth Korbin, New York, NY.*

A convex pentagon is inscribed in a circle with diameter d. Find positive integers a, b, and d if the sides of the pentagon have lengths a, a, a, b, and b respectively and if $a > b$. Express the area of the pentagon in terms of a, b, and d.

Solution by David Stone and John Hawkins, Statesboro, GA.

Note, that any solution can be scaled upward by any integer factor to produce infinitely many similar solutions.

We have three isosceles triangles with base a and equal sides $\dfrac{d}{2}$, and two isosceles triangles with base b and equal sides $\dfrac{d}{2}$. Let α be the measure of the angle opposite base a, and let β be the measure of the angle opposite the base b. Then $3\alpha + 2\beta = 2\pi$.

For each triangle with base a, the perimeter is $d + a$, and Heron's formula gives

$$A_n = \sqrt{\left(\frac{d+a}{2}\right)\left(\frac{d-a}{2}\right)\left(\frac{a}{2}\right)} = \frac{a}{4}\sqrt{d^2-a^2}.$$

We can also use the Law of Cosines to express the cosine of α as $\cos\alpha = \dfrac{a^2 - 2\left(\dfrac{d}{2}\right)^2}{-2\left(\dfrac{d}{2}\right)^2} = \dfrac{d^2-2a^2}{d^2}$.

From the Pythagorean Identity, it follows that

$$\sin\alpha = \sqrt{1 - \left(\frac{d^2-2a^2}{d^2}\right)^2} = \frac{1}{d^2}\sqrt{d^4 - d^4 + 4a^2d^2 - 4a^4} = \frac{2a}{d^2}\sqrt{d^2-a^2}.$$

Because the triangle is isosceles, with equal sides forming the angle α, an altitude through angle α divides the triangle into two equal right triangles. Therefore, $\cos\dfrac{\alpha}{2} = \dfrac{1}{d}\sqrt{d^2-a^2}$ and $\sin\dfrac{\alpha}{2} = \dfrac{a}{d}$.

For the triangles with base b, we can similarly obtain $A_b = \frac{b}{4}\sqrt{d^2 - b^2}$ and $\cos\beta = \frac{d^2 - 2b^2}{d^2}$.

The area for the convex polygon is then

$$\begin{aligned} A_{polygon} &= 3A_a + 2A_b \\ &= \frac{3a}{4}\sqrt{d^2 - a^2} + \frac{b}{2}\sqrt{d^2 - b^2} \\ &= \frac{1}{4}\left(3a\sqrt{d^2 - a^2} + 2b\sqrt{d^2 - b^2}\right) \end{aligned}$$

in terms of a, b, and d.

Solving $3\alpha + 2\beta = 2\pi$, we find $\beta = \frac{2\pi - 3\alpha}{2} = \pi - \frac{3\alpha}{2}$.

Therefore,

$$\cos\beta = \cos\left(\pi - \frac{3\alpha}{2}\right) = -\cos\left(\frac{3\alpha}{2}\right) = -\cos\left(\alpha + \frac{\alpha}{2}\right) = -\cos\frac{\alpha}{2}\cos\alpha + \sin\frac{\alpha}{2}\sin\alpha.$$

Replacing the trig functions in this formula with the values computed above gives

$$\frac{d^2 - 2b^2}{d^2} = -\frac{\sqrt{d^2 - a^2}}{d}\left(\frac{d^2 - 2a^2}{d^2}\right) + \frac{a}{d}\left(\frac{2a}{d^2}\right)\sqrt{d^2 - a^2} = \frac{\sqrt{d^2 - a^2}}{d}\left(4a^2 - d^2\right).$$

Solving for b^2 in terms of a and d gives

$$b^2 = \frac{d^3 - \sqrt{d^2 - a^2}\left(4a^2 - d^2\right)}{2d}, \text{ or } b = \sqrt{\frac{d^3 - \sqrt{d^2 - a^2}\left(4a^2 - d^2\right)}{2d}}.$$

Note also that **(1)** $2b^2 = d^2 - \frac{\sqrt{d^2 - a^2}\left(4a^2 - d^2\right)}{d}$.

We can use this expression for b to compute the area of the polygon solely in terms of a and d.

$$A_{polygon} = \frac{3a}{4}\sqrt{d^2 - a^2} + \frac{b}{2}\sqrt{d^2 - b^2} = \frac{3a}{4}\sqrt{d^2 - a^2} + \frac{a|3d^2 - 4a^2|}{4d}.$$

To find specific values which satisfy the problem, we use equation **(1).**

If $d^2 - a^2 = m^2$, then **(1)** becomes **(2)** $2b^2 = d^2 - \frac{m\left(4a^2 - d^2\right)}{d} = d^2 - \frac{m\left(3a^2 - m^2\right)}{d}$**.**

Then(a, m, d) is a Pythagorean triple, and thus a scalar multiple of a primitive Pythagorean triple (A, B, C). Using the standard technique, this triple is generated by two parameters, s and t:

$$\begin{cases} A = 2st \\ B = s^2 - t^2 \\ C = s^2 + t^2 \end{cases},$$

where $s > t$, s and t are relatively prime and have opposite parity. There are the two possibilities, where k is some scalar:

$$a = kA = 2kst,\ m = kB = k\left(s^2 - t^2\right), \text{and } d = kC = k\left(s^2 + t^2\right)$$

or

$$m = kA = 2kst,\ a = kB = k\left(s^2 - t^2\right), \text{and } d = kC = d\left(s^2 + t^2\right).$$

We'll find solutions satisfying the first set of conditions, recognizing that this will probably not produce all solutions of the problem. Substituting these in **(2)**,we find

$$2b^2 = d^2 - \frac{m(3a^2 - m^2)}{d} \ = \ k\left(s^2 + t^2\right)^2 - \frac{k(s^2 - t^2)\left(3(2ks)^2 - k^2\left(s^2 - t^2\right)^2\right)}{k(s^2 + t^2)}.$$

Simplifying, we find that $b^2 = \dfrac{k^2 s^2 \left(s^2 - 3t^2\right)^2}{s^2 + t^2}$, and we want this b to be an integer.

The simplest possible choice is to let $k^2 = s^2 + t^2$ (so that (s, t, k) is itself a Pythagorean triple); this forces $b = s\left(s^2 - 3t^2\right)$. We then have

$$a = 2kst = 2st\sqrt{s^2 + t^2},\ \ m = \sqrt{s^2 + t^2}\left(s^2 - t^2\right),\ \ d = k(s^2 + t^2) = k^3 = \left(s^2 + t^2\right)^{3/2} \text{ and}$$

$$b = s\left(s^2 - 3t^2\right).$$

That is, if (s, t, k) is a Pythagorean triple with $s^2 - 3t^2 > 0$, we have

$$\begin{cases} a = 2kst \\ b = s\left(s^2 - 3t^2\right) \\ d = k^3. \end{cases}$$

The restriction that $a > b$ imposes further conditions on s and t (roughly, $s < 3.08t$).

Some results, due to Excel:

s	t	k	b	a	d	*Area*
12	5	13	828	1,560	2,197	1,024,576
15	8	17	495	4,080	4,913	3,396,630
35	12	37	27,755	31,080	50,653	604,785,405
80	39	89	146,960	555,360	704,969	85,620,163,980
140	51	149	1,651,580	2,127,720	3,307,949	2,530,718,023,785
117	44	125	922,077	1,287,000	1,953,125	829,590,714,707
168	95	193	193,032	6,160,560	7,189,057	6,053,649,964,950
208	105	233	2,119,312	10,177,440	12,649,337	25,719,674,553,300
187	84	205	2,580,787	6,440,280	8,615,125	14,516,270,565,027
252	115	277	6,004,908	16,054,920	21,253,933	86,507,377,177,725
209	120	241	100,529	12,088,560	13,997,521	21,678,178,927,350
247	96	265	8,240,167	12,567,360	18,609,625	77,495,769,561,288
352	135	377	24,368,608	35,830,080	53,582,633	647,598,434,135,400

Also solved by the proposer

- 4950: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that
$$\frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}} + \frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}} + \frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \geq 3.$$

Solution by Kee-Wai Lau, Hong Kong, China

Since
$$\begin{aligned} a+b &= \frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a^3}+\sqrt[4]{b^3}) \; + (\sqrt[4]{a}-\sqrt[4]{b})^2(\sqrt{a}+\sqrt[4]{a}\sqrt[4]{b}+\sqrt{b})}{2} \\ &\geq \frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a^3}+\sqrt[4]{b^3})}{2} \end{aligned}$$

with similar results for $b+c$ and $c+a$, so by the arithmetic mean-geometric mean inequality, we have
$$\begin{aligned} & \frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}} + \frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}} + \frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \\ \geq\; & \sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c} \\ \geq\; & 3\sqrt[12]{abc} \\ =\; & 3 \text{ as required.} \end{aligned}$$

Also solved by Michael Brozinsky (two solutions), Central Islip, NY; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX, and the proposer.

- 4951: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be the angles of an acute triangle ABC. Prove that
$$\pi \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}} \geq \alpha\sin\sqrt{\alpha}+\beta\sin\sqrt{\beta}+\gamma\sin\sqrt{\gamma}.$$

Solution by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX.

Since α, β, and γ are the angles of an acute triangle,
$$\alpha, \beta, \gamma \in (0, \frac{\pi}{2}) \text{ and } \frac{\alpha}{\pi}+\frac{\beta}{\pi}+\frac{\gamma}{\pi} = 1$$

Let $f(x) = \sin\sqrt{x}$ on $(0, \frac{\pi}{2})$. Then, since
$$f''(x) = -\frac{\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}}{4x^{3/2}} < 0$$

on $(0, \frac{\pi}{2})$, it follows that $f(x)$ is concave down on $(0, \frac{\pi}{2})$. Hence, by Jensen's Inequality and (1)
$$\frac{\alpha}{\pi}\sin\sqrt{\alpha}+\frac{\beta}{\pi}\sin\sqrt{\beta}+\frac{\gamma}{\pi}\sin\sqrt{\gamma} \quad\leq\quad \sin\sqrt{\frac{\alpha}{\pi}\cdot\alpha+\frac{\beta}{\pi}\cdot\beta+\frac{\gamma}{\pi}\cdot\gamma}$$

$$= \quad \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}},$$

with equality if and only if $\alpha=\beta=\gamma=\dfrac{\pi}{3}$.

Also solved by the proposer

- 4952: *Proposed by Michael Brozinsky, Central Islip, NY & Robert Holt, Scotch Plains, NJ.*

  An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

  Solution by Paul M. Harms, North Newton, KS

  Let $\begin{pmatrix} n \\ r \end{pmatrix}$ be the combination of n things taken r at a time. With k independent features there are $\begin{pmatrix} k \\ 1 \end{pmatrix} = k$ number of different "groups" containing one feature, $\begin{pmatrix} k \\ 2 \end{pmatrix}$ different "groups" containing two features, etc. To have the sum of independent features in a plot of three houses be divisible by three, there are four possibilities. **I.** Each house in a plot has one feature. **II.** Each house in a plot has two features. **III.** Each house in a plot has three features. **IV.** One house in a plot has one feature, another house has two features, and the third house has three features.

  The maximum number of distinct plots can be found by summing the number of plots for each of the four possibilities above. The sum is

  $$\begin{pmatrix} \binom{k}{1} \\ 3 \end{pmatrix} + \begin{pmatrix} \binom{k}{2} \\ 3 \end{pmatrix} + \begin{pmatrix} \binom{k}{3} \\ 3 \end{pmatrix} + \begin{pmatrix} k \\ 1 \end{pmatrix}\begin{pmatrix} k \\ 2 \end{pmatrix}\begin{pmatrix} k \\ 3 \end{pmatrix}$$

  This is the result for part **b).**

  For part **a),** first consider a house with one fixed feature. There are plots in possibilities I and IV. In possibility I the other two houses can have any combination of the other $(k-1)$ single features so there are $\begin{pmatrix} k-1 \\ 2 \end{pmatrix}$ plots. In possibility IV the number of plots with a house with one fixed feature is $\begin{pmatrix} k \\ 1 \end{pmatrix}\begin{pmatrix} k \\ 2 \end{pmatrix}\begin{pmatrix} k \\ 3 \end{pmatrix}$. The number of plots with houses with different features is the following: For a house with one fixed feature there are $\begin{pmatrix} k-1 \\ 2 \end{pmatrix} + \begin{pmatrix} k \\ 2 \end{pmatrix}\begin{pmatrix} k \\ 3 \end{pmatrix}$ plots. For a house with two fixed features there are $\binom{\binom{k}{2}-1}{2} + \binom{k}{1}\binom{k}{3}$ plots. For a house with three fixed features there are $\begin{pmatrix} \binom{k}{2}-1 \\ 2 \end{pmatrix} + \begin{pmatrix} k \\ 1 \end{pmatrix}\begin{pmatrix} k \\ 2 \end{pmatrix}$ plots.

  Also solved by the proposer.

- 4953: *Proposed by Tom Leong, Brooklyn, NY.*

  Let$\pi(x)$ denote the number of primes not exceeding x. Fix a positive integer n and define sequences by $a_1 = b_1 = n$ and

$$a_{k+1} = a_k - \pi(a_k) + n, \qquad b_{k+1} = \pi(b_k) + n + 1 \quad \text{for k} \geq 1.$$

  a) Show that $\lim_{k\to\infty} a_k$ is the n^{th} prime.

  b) Show that $\lim_{k\to\infty} b_k$ is the n^{th} composite.

  Solution by Paul M. Harms, North Newton, KS.

  Any positive integer m is less than the m^{th} prime since 1 is not a prime. In part **a)** with $a_1 = n$, we have $\pi(n)$ primes less than or equal to n. We need $n - \pi(n)$ more primes than n has in order to get to the n^{th} prime. Note that a_2 is greater than a_1 by $n - \pi(n)$. If **all** of the integers from $a_1 + 1$ to a_2 are prime, then a_2 is the n^{th} prime. If not all of the integers indicated in the last sentence are primes, we see that a_3 is greater than a_2 by the number of non-primes from $a_1 + 1$ to a_2. This is true in general from a_k to a_{k+1} since $a_{k+1} = a_k + (n - \pi(a_k))$. If a_k is not the n^{th} prime, then a_{k+1} will increase by the quantity of integers to get to the n^{th} prime provided all integers a_{k+1} will increase by the quantity of integers to get to the n^{th} prime provided all integers $a_k + 1$,to a_{k+1}. We see that the sequence increases until some $a_m = N$, the n^{th} prime. Then $a_{m+1} = a_m + (n - \pi(a_m)) = a_m + 0 = a_m$. In this same way it is seen that $a_k = a_m$ for all k greater that m. Thus the limit for the sequence in part **a)** is the n^{th} prime.

  For part **b)** note that n is less than the n^{th} composite. Since the integer 1 and integers $\pi(n)$ are not composite, the n^{th} composite must be at least $1+\pi(n)$ greater than n. With $b_1 = n$ we see that $b_2 = n + (1 + \pi(n))$. Then b_2 will be the n^{th} composte provided all integers $n + 1, n + 2, \cdots, n + 1 + \pi(n)$ are composites. If some of the integers in the last sentence are prime, then b_3 is greater than b_2 by the number of primes in the integers from $b_1 + 1$ to b_2. In general, b_{k+1} is greater than b_k by the number of primes in the integers from $b_{k-1} + 1$ to b_k and the sequence will be an increasing sequence until the n^{th} composite is reached. If $b_m = N$, the n^{th} composite, then all integers from $b_{m-1} + 1$ to b_m are composite. Then $\pi(b_{m-1}) = \pi(b_m)$ and $b_{m+1} = \pi(b_{m-1}) + 1 + n = b_m = N$. We see that $b_k = N$ for all k at least as great as m. Thus the limit of the sequence in part **b)** is the n^{th} composite.

  Also solved by David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4954: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (a, b) that satisfy

$$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} = \frac{111+i}{111-i}$$

  with $a < b$.

  Solution by David E. Manes, Oneonta, NY.

  The only solutions (a, b) with $a < b$ are $(112, 12433), (113, 6272), (172, 313)$, and $(212, 233)$.

  Expanding the given equation and clearing fractions, one obtains $[2(111)(a + b) - 2(ab - 1)]i = 0$. Therefore, $\dfrac{ab-1}{a+b} = 111$. Let $b = a + k$ for some positive integer k. Then the above equation reduces to a quadratic in a; namely $a^2 + (k-222)a - (111k+1) = 0$ with roots given by

$$a = \frac{(222-k) \pm \sqrt{k^2 + 49288}}{2}.$$

Since a is a positive integer, it follows that $k^2 + 49288 = n^2$ or

$$n^2 - k^2 = (n+k)(n-k) = 49288 = 2^3 \cdot 61 \cdot 101.$$

Therefore, $n+k$ and $n-k$ are positive divisors of 49288. The only such divisors yielding solutions are

$n+k$	$n-k$
24644	2
12322	4
404	122
244	202

Solving these equations simultaneously gives the following values for(n,k) :

$$(12323, 12321), (6163, 6159), (263, 141), \text{ and } (223, 21)$$

from which the above cited solutions for a and b are found.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Daniel Copeland (student at St. George's School), Spokane, WA; Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly; students at Taylor University), Upland, IN; Grant Evans (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Homeira Pajoohesh, David Stone, and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4955: *Proposed by Kenneth Korbin, New York, NY.*

Between 100 and 200 pairs of red sox are mixed together with between 100 and 200 pairs of blue sox. If three sox are selected at random, then the probability that all three are the same color is 0.25. How many pairs of sox were there altogether?

Solution by Brian D. Beasley, Clinton, SC.

Let R be the number of pairs of red sox and B be the number of pairs of blue sox. Then $200 \leq R + B \leq 400$ and

$$\frac{2R(2R-1)(2R-2) + 2B(2B-1)(2B-2)}{(2R+2B)(2R+2B-1)(2R+2B-2)} = \frac{1}{4}.$$

Thus $4[R(2R-1)(R-1) + B(2B-1)(B-1)] = (R+B)(2R+2B-1)(R+B-1)$, or equivalently

$$4(2R^2 + 2B^2 - R - B - 2RB)(R+B-1) = (2R^2 + 2B^2 - R - B + 4RB)(R+B-1).$$

This yields $6R^2 + 6B^2 - 3R - 3B - 12RB = 0$ and hence $2(R-B)^2 = R+B$. Letting $x = R - B$, we obtain $R = x^2 + \frac{1}{2}x$ and $B = x^2 - \frac{1}{2}x$, so x is even. In addition, the size of $R + B$ forces $|x| \in \{10, 12, 14\}$. A quick check shows that only $|x| = 12$ produces values for R and B between 100 and 200, giving the unique solution $\{R, B\} = \{138, 150\}$. Thus $R + B = 288$.

Also solved by Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS, and the proposer.

- 4956: *Proposed by Kenneth Korbin, New York, NY.*

  A circle with radius $3\sqrt{2}$ is inscribed in a trapezoid having legs with lengths of 10 and 11. Find the lengths of the bases.

  Solution by Eric Malm, Stanford, CA.

  There are two different solutions: one when the trapezoid is shaped like /O\, and the other when it is configured like /O/. In fact, by reflecting the right-hand half of the plane about the x-axis, we can interchange between these two cases. Anyway, in the first case, the lengths of the bases are $7-\sqrt{7}$ and $14+\sqrt{7}$, and in the second case they are $7+\sqrt{7}$ and $14-\sqrt{7}$.

  Also solved by Michael Brozinsky, Central Islip, NY; Daniel Copeland (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Nate Wynn (student at St. George's School), Spokane, WA, and the proposer.

- 4957: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = 1, a_1 = 2, a_2 = 1$ and for all $n \geq 3$, $a_n^3 = a_{n-1}a_{n-2}a_{n-3}$. Find $\lim\limits_{n\to\infty} a_n$.

  Solution by Michael Brozinsky, Central Islip, NY.

  If we write $a_n = 2^{b_n}$ we have $b_n = \dfrac{b_{n-1}+b_{n-2}+b_{n-3}}{3}$ where $b_0 = 0, b_1 = 1$, and $b_2 = 0$. The characteristic equation is

$$\begin{aligned} x^3 &= \frac{x^2}{3}+\frac{x}{3}+\frac{1}{3} \text{ with roots} \\ r_1 &= 1,\ r_2 = \frac{-1+i\sqrt{2}}{3},\ \text{and } r_3 = \frac{-1-i\sqrt{2}}{3}. \end{aligned}$$

  The generating function $f(n)$ for $\{b_n\}$ is (using the initial conditions) found to be

$$\begin{aligned} f(n) &= A + B\left(\frac{-1+i\sqrt{2}}{3}\right)^n + C\left(\frac{-1-i\sqrt{2}}{3}\right)^n \text{ where} \\ A &= \frac{1}{3},\ B = -\frac{1}{6}-\frac{5i\sqrt{2}}{12},\ \text{and } C = -\frac{1}{6}+\frac{5i\sqrt{2}}{12}. \end{aligned}$$

  Since $|r_2| = |r_3| = \dfrac{\sqrt{6}}{4} < 1$ we have the last two terms in the expression for $f(n)$ approach 0 as n approaches infinity, and hence $\lim\limits_{n\to\infty} b_n = \dfrac{1}{3}$ and so $\lim\limits_{n\to\infty} a_n = \sqrt[3]{2}$.

  Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays and Jahangeer Khold (jointly), Chesapeake, VA & Portsmouth, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 4958: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $f : [a, b] \to R$ $(0 < a < b)$ be a continuous function on $[a, b]$ and derivable in (a, b). Prove that there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{1}{c\sqrt{ab}} \cdot \frac{\ln(ab/c^2)}{\ln(c/a) \cdot \ln(c/b)}.$$

Solution by the proposer.
Consider the function $F : [a, b] \to R$ defined by

$$F(x) = (\ln x - \ln a)(\ln x - \ln b) \exp\left[\sqrt{ab}\, f(x)\right]$$

Since F is continuous function on $[a, b]$, derivable in (a, b) and $F(a) = F(b) = 0$, then by Rolle's theorem there exists $c \in (a, b)$ such that $F'(c) = 0$. We have

$$F'(x) = \left[\frac{1}{x}(\ln x - \ln b) + \frac{1}{x}(\ln x - \ln a)\right.$$

$$\left. + \sqrt{ab}(\ln x - \ln a)(\ln x - \ln b) f'(x)\right] \exp\left[\sqrt{ab}\, f(x)\right]$$

and

$$\frac{1}{c} \ln\left(\frac{c^2}{ab}\right) + \sqrt{ab} \ln\left(\frac{c}{a}\right) \ln\left(\frac{c}{b}\right) f'(c) = 0$$

From the preceding immediately follows

$$\sqrt{ab} \ln(c/a) \ln(c/b)\, f'(c) = \frac{1}{c} \ln(ab/c^2)$$

and we are done.

- 4959: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

Find all numbers $N = ab$, were $a, b = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, such that

$$[S(N)]^2 = S(N^2),$$

where S(N)=a+b is the sum of the digits. For example:

$$\begin{aligned} N &= 12 \qquad N^2 = 144 \\ S(N) &= 3 \qquad S(N^2) = 9 \quad \text{and } [\text{S(N)}]^2 = \text{S(N}^2\text{)}. \end{aligned}$$

Solution by Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly, students at Taylor University), Upland, IN.

We start by considering the possibilities that exist for N. Since there are 10 possibilities for a and for b, there are 100 possibilities for N. It would not be incorrect to check all 100 cases, however we need not do so.

We can eliminate the majority of these 100 cases without directly checking them. If we assume that $S(N) \geq 6$, then $[S(N)] \geq 36$, which means that for the property to hold, $S(N^2) \geq 36$ as well. This would require $N^2 \geq 9999$. However, this leads us to a contradiction because the largest possible value for N by our definition is $99, and \mathrm{N}^2$ in that case is only $N^2 = 99^2 = 9801 < 9999$. Therefore, we need not check any number N such $S(N) > 6$. More precisely, any number N in the intervals$[6, 9]; [15, 19]; [24, 29]; [33, 39]; [42, 49]; [51, 99]$ need not be checked. This leaves us with 21 cases that can easily be checked.

After checking each of these cases separately, we find that for 13 of them, the property $[S(N)]^2 = S(N^2)$ does in fact hold. These 13 solutions are

$$N = 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 30, 31.$$

We show the computation for $N = 31$ as an example:

$$\begin{array}{rcl} N & = & 31 \\ S(N) & = & 3+1=4 \\ [S(N)]^2 & = & 4^2 = 16 \\ [S(N)]^2 & = & S(N^2) = 16 \text{ for } N = 31. \end{array} \qquad \begin{array}{l} N^2 = 31^2 = 961 \\ S(N^2) = 9+6+1 = 16 \end{array}$$

The other 12 solutions can be checked similarly.

Also solved by Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Robert Anderson and Boris Rays (jointly), Portsmouth, Portsmouth, & Chesapeake, VA; Peter E. Liley, Lafayette, IN; Jim Moore, Seth Bird and Jonathan Schrock (jointly, students at Taylor University), Upland, IN; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Late Solutions

Late solutions by **David E. Manes of Oneonta, NY** were received for problems 4942 and 4944.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
January 15, 2008

- 4984: *Proposed by Kenneth Korbin, New York, NY.*

  Prove that
  $$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2009}+\sqrt{2011}}>\sqrt{120}.$$

- 4985: *Proposed by Kenneth Korbin, New York, NY.*

  A Heron triangle is one that has both integer length sides and integer area. Assume Heron triangle ABC is such that $\angle B = 2\angle A$ and with (a,b,c)=1.

  PartI : Find the dimensions of the triangle if side a = 25.
  PartII : Find the dimensions of the triangle if 100 < a < 200.

- 4986: *Michael Brozinsky, Central Islip, NY.*

  Show that if $0 < a < b$ and $c > 0$, that
  $$\sqrt{(a+c)^2+d^2}+\sqrt{(b-c)^2+d^2}\leq\sqrt{(a-c)^2+d^2}+\sqrt{(b+c)^2+d^2}.$$

- 4987: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be the sides of a triangle ABC with area S. Prove that
  $$(a^2+b^2)(b^2+c^2)(c^2+a^2)\leq 64S^3\csc 2A\csc 2B\csc 2C.$$

- 4988: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation
$$3^{x^2-x-z}+3^{y^2-y-x}+3^{z^2-z-y}=1.$$

- 4989: *Proposed by Tom Leong, Scotrun, PA.*

  The numbers $1,2,3,\cdots,2n$ are randomly arranged onto $2n$ distinct points on a circle. For a chord joining two of these points, define its *value* to be the absolute value of the difference of the numbers on its endpoints. Show that we can connect the $2n$ points in disjoint pairs with n chords such that no two chords intersect inside the circle and the sum of the values of the chords is exactly n^2.

Solutions

- 4960: *Proposed by Kenneth Korbin, New York, NY.*

  Equilateral triangle ABC has an interior point P such that
$$\overline{AP}=\sqrt{5},\ \overline{BP}=\sqrt{12},\ \text{and } \overline{\text{CP}}=\sqrt{17}.$$

  Find the area of $\triangle APB$.

  Solution by Scott H. Brown, Montgomery, AL.

  First rotate $\triangle ABC$ about point C through a counter clockwise angle of 60^o. This will create equilateral triangle CBB' and interior point P'. Since triangle ABC is equialteral and $m\angle ACB=60^o$, $\overline{AC}$ falls on $\overline{BC}$, and $\overline{CP'}=\sqrt{17}$, $\overline{B'P'}=\sqrt{12}$, $\overline{BP'}=\sqrt{5}$. Now $\triangle CPA\cong\triangle CP'B$ and $m\angle ACP=m\angle BCP'$, so $m\angle PCP'=60^o$.

  Second, draw $\overline{PP'}$, forming isosceles triangle PCP'. Since $m\angle PCP'=60^o$, triangle PCP' is equilateral. We find $\overline{PP'}=\sqrt{17}$, $\overline{PA}=\overline{P'B}=\sqrt{5}$ and $\overline{PB}=\sqrt{12}$. So triangle PBP' is a right triangle.

  Third, $m\angle APB'=120^o$ and $m\angle PBP'=90^o$. We find $m\angle PBA+m\angle P'BB'=30^o$. Since $m\angle P'BB'=m\angle PAB$, then by substitution, $m\angle PBA+m\angle PAB=30^o$. Thus $m\angle APB=150^o$.

  Finally, we find the area of triangle APB$=\dfrac{1}{2}(\sqrt{5})(\sqrt{12})\sin(150^o)=\dfrac{\sqrt{15}}{2}$ square units.

  (Reference: Challenging Problems in Geometry 2, Posamentier & Salkind, p. 39.)

  Also solved by Mark Cassell (student, Saint George's School), Spokane, WA; Matt DeLong, Upland, IN; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4961: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon is inscribed in a circle with diameter d. Find the area of the hexagon if its sides are $3,3,3,4,4$ and 4.

  Solution 1 by John Nord, Spokane, WA.

For cyclic quadrilateral ABCD with sides a, b, c, and d, two different formulations of the area are given, Brahmagupta's formula and Bretschneider's formula.

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s = \frac{a+b+c+d}{2} \quad (1)$$

$$A = \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4R} \text{ where R is the circumradius} \quad (2)$$

In order to employ the cyclic quadrilateral theorems, place a diagonal into the hexagon to obtain two inscribed quadrilaterals. The first has side lengths of 3,3,3, and x and the second has side lengths of 4,4,4 and x.

Equating (1) and (2) and solving for R yields

$$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}} \quad (3)$$

Both quadrilaterals are inscribed in the same circle so (3) can be used for both quadrilaterals and they can be set equal to each other. Solving for x is surprisingly simple and the area computations can be calculated using (1) directly. The area of the inscribed hexagon with sides 3,3,3,4,4, and 4 is $\dfrac{73\sqrt{3}}{4}$.

Solution 2 by Jonathan Schrock, Seth Bird, and Jim Moore (jointly, students at Taylor University), Upland, IN.

Since the hexagon is convex and cyclic, a radius of the circumscribing circle can be drawn to each vertex producing six isosceles triangles. The formula for the height of one of these triangles is $\frac{1}{2}\sqrt{4r^2-c^2}$ where c is the length of the base of the triangle and r is the radius of the circle. Since $2r = d$ (the diameter of the circle), the area of any one of these triangles will therefore be $\frac{c}{4}\sqrt{d^2-c^2}$. The total area of the hexagon is the sum of the areas of the triangles. There are three triangles for which $c = 3$ and three for which $c = 4$. So the total area of the hexagon in terms of d is $3\sqrt{d^2-16} + \frac{9}{4}\sqrt{d^2-9}$.

We can determine d by rearranging the hexagon so that the side lengths alternate as 3,4,3,4,3,4. This creates three congruent quadrilaterals. Consider just one of these quadrilaterals and label it ABCO, where A, B, and C lie on the circle and O is the center of the circle. Since the interior angle for a circle is 360^o and there are three quadrilaterals, $\angle AOC = 120^o$. By constructing a line from A to C we can see by the symmetry of the rearranged hexagon, that $\angle ABC = 120^o$. Using the law of cosines,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\left(\overline{AB}\right)\left(\overline{BC}\right)\cos(120^o),$$

which can be written as $\overline{AC}^2 = 3^2 + 4^2 - 2(3)(4)\cos(120^o)$. That is, $\overline{AC} = \sqrt{37}$. To determine d we use the law of cosines again. Here,

$$\overline{AC}^2 = \overline{AO}^2 + \overline{CO}^2 - 2\left(\overline{AO}\right)\left(\overline{CO}\right)\cos(120^o),$$

which can be written as $37 = \dfrac{d^2}{2} - \dfrac{d^2}{2}\cos(120^o)$. Solving for d gives $d = 2\sqrt{\dfrac{37}{3}}$. Substituting this value of d into the formula $3\sqrt{d^2-16} + \frac{9}{4}\sqrt{d^2-9}$ gives the area of the hexagon as $\dfrac{73\sqrt{3}}{4}$.

Comment by editor: David Stone and John Hawkins of Statesboro GA generalized the problem for any convex, cyclic hexagon with side lengths a, a, a, b, b, b (with $0 < a \leq b$) and with d as the diameter of the circumscribing circle. They showed that d is uniquely determined by the values of a and b, $d = \sqrt{\frac{4}{3}(a^2 + ab + b^2)}$. Then they asked the question: What fraction of the circle's area is covered by the hexagon? They found that in general, the fraction of the circle's area covered by the hexagon is:

$$\frac{\frac{\sqrt{3}}{4}(a^2 + 4ab + b^2)}{\frac{\pi}{3}(a^2 + ab + b^2)} = \frac{3\sqrt{3}(a^2 + 4ab + b^2)}{4\pi(a^2 + ab + b^2)} = \frac{3\sqrt{3}}{4\pi}\frac{(a+b)^2 + 2ab}{(a+b)^2 - ab} = \left(\frac{3\sqrt{3}}{4\pi}\right)\frac{1+2c}{1-c}$$

where $c = \dfrac{ab}{(a+b)^2}$.

They continued on by stating that in fact, c takes on the values $0 < c \leq 1/4$, thus forcing $1 < \dfrac{1+2c}{1-c} \leq 2$. So by appropriate choices of a and b, the hexagon can cover from $\dfrac{3\sqrt{3}}{4\pi} \approx 0.4135$ of the circle up to $\dfrac{3\sqrt{3}}{4\pi} \cdot 2 \approx 0.827$ of the circle. A regular hexagon, where $a = b$ and $c = 1/4$, would achieve the upper bound and cover the largest possible fraction of the circle.

For instance, we can force the hexagon to cover exactly one half the circle by making $\left(\dfrac{3\sqrt{3}}{4\pi}\right)\dfrac{1+2c}{1-c} = \dfrac{1}{2}$. This would require $c = \dfrac{2\pi - 3\sqrt{3}}{2\left(3\sqrt{3} + \pi\right)} \approx 0.0651875$. Setting this equal to $\dfrac{ab}{(a+b)^2}$, we find that $\dfrac{a}{b} = \dfrac{\left(6\sqrt{3} - \pi\right) \pm \sqrt{3(27 - \pi^2)}}{2\pi - 3\sqrt{3}}$.
That is, if $b = 13.2649868a$, the hexagon will cover half of the circle.

Also solved by Matt DeLong, Upland, IN; Peter E. Liley, Lafayette, IN; Mandy Isaacson, Julia Temple, and Adrienne Ramsay (jointly, students at Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA , and the proposer.

- 4962: *Proposed by Kenneth Korbin, New York, NY.*

Find the area of quadrilateral $ABCD$ if the midpoints of the sides are the vertices of a square and if $AB = \sqrt{29}$ and $CD = \sqrt{65}$.

Solution by proposer.

Conclude that $AC \perp BD$ and that $AC = BD$. Then, there are positive numbers (w, x, y, z) such that

$$\begin{aligned} w + x &= AC, \\ y + z &= BD, \\ w^2 + y^2 &= 29, \text{ and} \\ x^2 + z^2 &= 65. \end{aligned}$$

Then, $(w, x, y, z) = (\frac{11}{\sqrt{10}}, \frac{19}{\sqrt{10}}, \frac{13}{\sqrt{10}}, \frac{17}{\sqrt{10}})$ and $AC = BD = \dfrac{30}{\sqrt{10}}$. The area of the quadrilateral then equals $\frac{1}{2}(AC)(BD) = \dfrac{1}{2}\left(\dfrac{30}{\sqrt{10}}\right)\left(\dfrac{30}{\sqrt{10}}\right) = 45.$

Also solved by Peter E. Liley, Lafayette, IN, and by Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA.

- 4963: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate
$$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{1}{3^{i+j}}.$$

Solution 1 by Ken Korbin, New York, NY.

$$\begin{aligned}
\sum_{1\le i<j\le n} \frac{1}{3^{i+j}} &= \left(\frac{1}{3^3}+\frac{1}{3^4}\right)+\left(\frac{2}{3^5}+\frac{2}{3^6}\right)+\left(\frac{3}{3^7}+\frac{3}{3^8}\right)+\left(\frac{4}{3^9}+\frac{4}{3^{10}}\right)+\cdots \\
&= \frac{4}{3^4}+\frac{8}{3^6}+\frac{12}{3^8}+\frac{16}{3^{10}}+\cdots \\
&= \frac{4}{3^4}\left[1+\frac{2}{3^2}+\frac{3}{3^4}+\frac{4}{3^6}+\cdots\right] \\
&= \frac{4}{3^4}\left[1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right]^2 \\
&= \frac{4}{3^4}\left[\frac{1}{1-\dfrac{1}{3^2}}\right]^2 \\
&= \frac{4}{3^4}\left[\frac{9}{8}\right]^2 = \frac{1}{16}.
\end{aligned}$$

Solutions 2 and 3 by Pat Costello, Richmond, KY.

2) When $n=2$ we have $\dfrac{1}{3^{1+2}}$.

When $n=3$ we have $\dfrac{1}{3^{1+3}}+\dfrac{1}{3^{2+3}}$.

When $n=4$ we have $\dfrac{1}{3^{1+4}}+\dfrac{1}{3^{2+4}}+\dfrac{1}{3^{3+4}}$.

Adding down the columns we obtain:

$$\begin{aligned}
&\sum_{k=3}^{\infty}\frac{1}{3^k}+\sum_{k=5}^{\infty}\frac{1}{3^k}+\sum_{k=7}^{\infty}\frac{1}{3^k}+\cdots \\
&\qquad = \frac{(1/3)^3}{1-1/3}+\frac{(1/3)^5}{1-1/3}+\frac{(1/3)^7}{1-1/3}+\cdots \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\right)^3\left(1+(1/3)^2+(1/3)^4+\cdots\right) \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\right)^3\left(1+(1/9)+(1/9)^2+\cdots\right) \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\right)^3\left(\frac{1}{1-1/9}\right) = \frac{1}{16}.
\end{aligned}$$

3) Another way to see that the value is 1/16 is to write the limit as the double sum

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{i=2}^{n-1}\frac{1}{3^{n+i}} &= \sum_{n=2}^{\infty}\frac{1}{3^n}\sum_{i=2}^{n-1}\frac{1}{3^i} = \sum_{n=2}^{\infty}\frac{1}{3^n}\left(\frac{(1/3)-(1/3)^n}{1-(1/3)}\right)\\
&= \frac{3}{2}\sum_{n=2}^{\infty}\frac{1}{3^n}\Big((1/3)-(1/3)^n\Big)\\
&= \frac{3}{2}\left((1/3)\sum_{n=2}^{\infty}\frac{1}{3^n}-\sum_{n=2}^{\infty}\frac{1}{9^n}\right)\\
&= \frac{3}{2}\left((\frac{1}{3})\frac{1/9}{1-1/3}-\frac{1/(81)}{1-1/9}\right)\\
&= \frac{3}{2}\left(\frac{1}{18}-\frac{1}{72}\right)=\frac{1}{16}.
\end{aligned}$$

Also solved by Bethany Ballard, Nicole Gottier, Jessica Heil (jointly, students at Taylor University), Upland, IN; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4964: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let x, y be real numbers and we define the law of composition

$$x \perp y = x\sqrt{1+y^2}+y\sqrt{1+x^2}.$$

Prove that $(R,+)$ and $(R,\perp)$ are isomorphic and solve the equation $x \perp a = b$.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

Define $f:(R,+)\to(R,\perp)$ by $f(x)=\sinh x$.
Then f is one-to-one and onto, and

$$\begin{aligned}
f(a+b) &= \sinh(a+b)\\
&= \sinh a\cosh b+\cosh a\sinh b\\
&= \sinh a\sqrt{1+\sinh^2 b}+\sinh b\sqrt{1+\sinh^2 a}\\
&= f(a)\perp f(b)
\end{aligned}$$

Therefore $(R,+)$ and $(R,\perp)$ are isomorphic abelian groups.
Note that:

$$\begin{cases}
\textbf{i)}\ \ \mathrm{f}(0)=0 \text{ and that } \mathrm{f}(-\mathrm{a})=-\mathrm{f}(\mathrm{a}).\\
\textbf{ii)}\ \ \text{In } (\mathrm{R},\perp)\\
\qquad 0\perp a = 0\sqrt{1+a^2}+a\sqrt{1+0^2}=a \text{ and}\\
\qquad a\perp(-a)=a\sqrt{1+a^2}-a\sqrt{1+a^2}=0.
\end{cases}$$

If $x\perp a=b$, then $x=b\perp(-a)=b\sqrt{1+a^2}-a\sqrt{1+b^2}$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY, and the proposer.

- 4965: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let h_a, h_b, h_c be the heights of triangle ABC. Let P be any point inside $\triangle ABC$. Prove that

$$(a) \quad \frac{h_a}{d_a} + \frac{h_b}{d_b} + \frac{h_c}{d_c} \geq 9, \quad (b) \quad \frac{d_a^2}{h_a^2} + \frac{d_b^2}{h_b^2} + \frac{d_c^2}{h_c^2} \geq \frac{1}{3},$$

where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB respectively.

Solution to part (a) by Scott H. Brown, Montgomery, AL.

Suppose P is any point inside triangle ABC. Let AP, BP, and CP be the line segments whose distances from the vertices are x, y, and z respectively. Let AP, BP, and CP intersect the sides BC, CA, and AB, at points L, M, and N respectively. Denote PL, PM, and PN by u, v, and w respectively.
In reference [1] it is shown that

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} \geq 6, \qquad (1)$$

with equality holding only if P in the centroid of triangle ABC.

Considering the heights h_a, h_b, and h_c, and the distances respectively to the sides from P as d_a, d_b, and d_c in terms of u, v, w, x, y, and z gives:

$$\frac{h_a}{d_a} = \frac{x+u}{u}, \qquad \frac{h_b}{d_b} = \frac{y+v}{v}, \qquad \frac{h_c}{d_c} = \frac{z+w}{w}. \qquad (2)$$

Applying inequality (1) gives:

$$\frac{h_a}{d_a} + \frac{h_b}{d_b} + \frac{h_c}{d_c} \geq 9,$$

with equality holding only if P is the centroid of triangle ABC.

Reference [1]. Some Inequalities For A Triangle, L. Carlitz, American Mathematical Monthly, 1964, pp. 881-885.

Solution to part (b) by the proposers.

For the triangles BPC, APC, APB we have,

$$\begin{aligned}
[BPC] &= d_a \times \frac{BC}{2} = \frac{d_a}{h_a} \times \frac{h_a\, BC}{2} = \frac{d_a}{h_a} \times [ABC] \\
[APC] &= d_b \times \frac{AC}{2} = \frac{d_b}{h_b} \times \frac{h_b\, AC}{2} = \frac{d_b}{h_b} \times [ABC] \\
[APB] &= d_c \times \frac{AB}{2} = \frac{d_c}{h_c} \times \frac{h_c\, ABC}{2} = \frac{d_c}{h_c} \times [ABC]
\end{aligned}$$

Adding up the preceding expressions yields,

$$\left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} \right) [ABC] = [ABC]$$

and

$$\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = 1$$

Applying AM-QM inequality, we get

$$\sqrt{\frac{\frac{d_a^2}{h_a^2} + \frac{d_b^2}{h_b^2} + \frac{d_c^2}{h_c^2}}{3}} \geq \frac{1}{3}\left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c}\right) = \frac{1}{3}$$

from which the inequality claimed immediately follows. Finally, notice that equality holds when $d_a/h_a = d_b/h_b = d_c/h_c = 1/3$. That is, when $\triangle ABC$ is equilateral and P is its centroid.

- 4966: *Proposed by Kenneth Korbin, New York, NY.*

  Solve:

$$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$

with $0 < x < 1$.

Solution 1 by Elsie Campbell, Dionne Bailey, & Charles Diminnie, San Angelo, TX.

Let $x = \cos\theta$ where $\theta \in (0, \frac{\pi}{2})$. Then,

$$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$

becomes

$$\begin{aligned}
16\cos\theta + 30\sqrt{1-\cos^2\theta} &= 17\sqrt{1+\cos\theta} + 17\sqrt{1-\cos\theta} \\
&= 17\sqrt{2}\left(\sqrt{\frac{1+\cos\theta}{2}} + \sqrt{\frac{1-\cos\theta}{2}}\right) \\
&= 34\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right) \\
&= 34\left(\cos\frac{\pi}{4}\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\sin\frac{\theta}{2}\right) \\
&= 34\cos(\frac{\pi}{4} - \frac{\theta}{2}). \qquad (1)
\end{aligned}$$

Let $\cos\theta_0 = \frac{8}{17}$. Then by (1),

$$\begin{aligned}
\cos(\frac{\pi}{4} - \frac{\theta}{2}) &= \frac{8}{17}\cos\theta + \frac{15}{17}\sin\theta \\
&= \cos\theta_0\cos\theta + \sin\theta_0\sin\theta \\
&= \cos(\theta_0 - \theta).
\end{aligned}$$

Therefore,

$$\begin{array}{rrcl}
 & \theta_0 - \theta & = & \frac{\pi}{4} - \frac{\theta}{2} \\
\Rightarrow & \theta & = & 2\theta_0 - \frac{\pi}{2} \\
\Rightarrow & x & = & \frac{240}{289}
\end{array} \quad \text{or} \quad \begin{array}{rrcl}
 & \theta_0 - \theta & = & -(\frac{\pi}{4} - \frac{\theta}{2}) \\
\Rightarrow & \theta & = & \frac{2}{3}\theta_0 + \frac{\pi}{6} \\
\Rightarrow & x & = & \cos(\frac{2}{3}\cos^{-1}\frac{8}{17} + \frac{\pi}{6}).
\end{array}$$

Remark: This solution is an adaptation of the solution on pp.13-14 from *Mathematical Miniatures* by Savchev and Andreescu.

Solution 2 by Brian D. Beasley, Clinton, SC.

Since $0 < x < 1$, each side of the given equation will be positive, so we may square both sides without introducing any extraneous solutions. After simplifying, this yields

$$(480x - 289)\sqrt{1-x^2} = 161(2x^2 - 1).$$

For each side of this equation to have the same sign (or zero), we require $x \in (0, 289/480] \cup [\sqrt{2}/2, 1)$. We now square again, checking for actual as well as extraneous solutions. This produces

$$(1156x^3 - 867x + 240)(289x - 240) = 0,$$

so one potential solution is $x = 240/289$. The cubic formula yields three more, namely

$$x \in \{-\cos(\frac{1}{3}\cos^{-1}(\frac{240}{289})), \sin(\frac{1}{3}\sin^{-1}(\frac{240}{289})), \cos(\frac{1}{3}\cos^{-1}(-\frac{240}{289}))\}.$$

Of these four values, only two are in $x \in (0, 289/480] \cup [\sqrt{2}/2, 1)$:

$$x = \frac{240}{289} \quad \text{and} \quad x = \sin(\frac{1}{3}\sin^{-1}(\frac{240}{289})).$$

Addendum. The given equation generalizes nicely to

$$2ax + 2b\sqrt{1-x^2} = c\sqrt{1+x} + c\sqrt{1-x},$$

where $a^2 + b^2 = c^2$ with $a < b$. The technique outlined above produces

$$(4c^2x^3 - 3c^2x + 2ab)(c^2x - 2ab) = 0,$$

so one solution (which checks in the original equation) is $x = 2ab/c^2$. Another solution (does it always check in the original equation?) is $x = \sin(\frac{1}{3}\sin^{-1}(\frac{2ab}{c^2}))$, which is connected to the right triangle with side lengths $(b^2 - a^2, 2ab, c^2)$ in the following way:

If we let 3θ be the angle opposite the side of length $2ab$ in this triangle, then we have $2ab/c^2 = \sin(3\theta) = -4\sin^3\theta + 3\sin\theta$, which brings us right back to $4c^2x^3 - 3c^2x + 2ab = 0$ for $x = \sin\theta$.

Similarly, we may show that the other two solutions are $x = -\cos(\frac{1}{3}\cos^{-1}(\frac{2ab}{c^2}))$ and $x = \cos(\frac{1}{3}\cos^{-1}(-\frac{2ab}{c^2}))$; the first of these is never in $(0, 1)$, but will the second ever be a solution of the original equation?

Also solved by John Boncek, Montgomery, AL; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4967: *Proposed by Kenneth Korbin, New York, NY.*

Given equilateral triangle ABC with an interior point P such that $\overline{\text{AP}}^2 + \overline{\text{BP}}^2 = \overline{\text{CP}}^2$, and with an exterior point Q such that $\overline{\text{AQ}}^2 + \overline{\text{BQ}}^2 = \overline{\text{CQ}}^2$, where points C, P, and Q are in a line. Find the lengths of $\overline{\text{AQ}}$ and $\overline{\text{BQ}}$ if $\overline{\text{AP}} = \sqrt{21}$ and $\overline{\text{BP}} = \sqrt{28}$.

Solution by Paul M. Harms, North Newton, KS.

Put the equilateral triangle on a coordinate system with A at $(0, 0)$, B at $(a, \sqrt{3}a)$ and C at $(2a, 0)$ where $a > 0$. The point P is at the intersection of the circles

$$x^2 + y^2 \;=\; 21$$

$$\begin{aligned}(x-a)^2+(y-\sqrt{3}a)^2 &= 28 \text{ and}\\ (x-2a)^2+y^2 &= 28+21=49.\end{aligned}$$

Using $x^2+y^2=21$ in the last two circles we obtain

$$\begin{aligned}-2ax-2\sqrt{3}ay+4a^2=28-21 &= 7 \text{ and}\\ -4ax+4a^2 &= 49-21=28.\end{aligned}$$

From the last equation $x=\dfrac{a^2-7}{a}$ and, using the linear equation, we get $y=\dfrac{2a^2+7}{2\sqrt{3}a}$. Putting these x, y values into $x^2+y^2=21$ yields the quadratic in a^2, $16a^4-392a^2+637=0$. From this equation $a^2=22.75$ or $a^2=1.75$. From the distances given in the problem, a^2 must be 22.75. The coordinates of P are $x=3.3021$ and $y=3.1774$. The line through C and P is $y=-0.5094x+4.85965$.

Let Q have coordinates (x_1, y_1). An equation for $\overline{AQ}^2+\overline{BQ}^2=\overline{CQ}^2$ can be found using the coordinates $Q(x_1, y_1), A(0,0), B(4.7697, 8.2614)$, and $C(9.5394, 0)$. An equation is

$$x_1^2+y_1^2+(x_1-4.7697)^2+(y_1-8.2614)^2=(x_1-9.5394)^2+y_1^2.$$

Simplifying and replacing y_1 by $-0.5094x_1+4.85965$ yields the quadratic equation $1.2595x_1^2+13.0052x_1-56.6783=0$. In order that Q is exterior to the triangle we need the solution $x_1=-13.6277$. Then $y_1=-0.5094x_1+4.85965=11.8020$. The distance from A to Q is $\sqrt{325}=18.0278$ and the distance from B to Q is $\sqrt{351}=18.7350$.

Also solved by Zhonghong Jiang, New York, NY, and the proposer.

- **4968:** *Proposed by Kenneth Korbin, New York, NY.*

Find two quadruples of positive integers (a, b, c, d) such that

$$\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i}=\frac{a-i}{a+i}\cdot\frac{b-i}{b+i}\cdot\frac{c-i}{c+i}\cdot\frac{d-i}{d+i}$$

with $a<b<c<d$ and $i=\sqrt{-1}$.

Solution 1 by Brian D. Beasley, Clinton, SC.

We need $((a+i)(b+i)(c+i)(d+i))^2=((a-i)(b-i)(c-i)(d-i))^2$, so

$$(a+i)(b+i)(c+i)(d+i)=\pm(a-i)(b-i)(c-i)(d-i).$$

Then either

$$(ab-1)(c+d)+(a+b)(cd-1)=0 \quad\text{ or }\quad (ab-1)(cd-1)-(a+b)(c+d)=0.$$

But $(ab-1)(c+d)>0$ and $(a+b)(cd-1)>0$, so the first case cannot occur. In the second case, since $d=(ab+ac+bc-1)/(abc-a-b-c)>0$, we have $abc>a+b+c$. Then $d\ge 4$ implies

$$3\le c\le\frac{ab+4a+4b-1}{4ab-a-b-4},$$

where we note that $1\le a<b$ implies $4ab>a+b+4$. Thus $2\le b\le(7a+11)/(11a-7)$, so $a\le 5/3$. Thus $a=1$, which yields $b\in\{2,3,4\}$.

If $(a,b)=(1,2)$, then $d=(3c+1)/(c-3)$, so $c<d$ forces $c\in\{4,5,6\}$. Only $c\in\{4,5\}$ will yield integral values for d, producing the two solutions $(1,2,4,13)$ and $(1,2,5,8)$ for (a,b,c,d).

If $(a,b)=(1,3)$, then $d=(2c+1)/(c-2)$, so $3<c<d$ forces $c=4$. But this yields $d=9/2$.

If $(a,b)=(1,4)$, then $d=(5c+3)/(3c-5)$, but $4<c<d$ forces the contradiction $c\le 3$.

Hence the only two solutions for (a,b,c,d) are $(1,2,4,13)$ and $(1,2,5,8)$.

Solution 2 by Dionne Bailey, Elsie Campbell, & Charles Diminnie, San Angelo, TX.

By using the following properties of complex numbers,

$$(\overline{z_1 z_2})=\bar{z}_1\bar{z}_2,\quad \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2},\quad \bar{\bar{z}}=z,$$

we see that the left and right sides of the equation are conjugates and hence, the equation reduces to

$$Im\left(\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i}\right)=0. \tag{1}$$

If $z=(a+i)\,(b+i)\,(c+i)\,(d+i)=A+Bi$, then (1) becomes

$$Im\left(\frac{z}{\bar{z}}\right)=0,$$

which reduces to $AB=0$ or equivalently, $A=0$ or $B=0$. With some labor, we get

$$\begin{aligned}A &= 1-(ab+ac+ad+bc+bd+cd)+abcd\\ &= (ab-1)\,(cd-1)-(a+b)\,(c+d)\ \text{ and}\\ B &= (abc+abd+acd+bcd)-(a+b+c+d)\\ &= (a+d)\,(bc-1)+(b+c)\,(ad-1)\,.\end{aligned}$$

Therefore, a,b,c,d must satisfy

$$(ab-1)\,(cd-1)=(a+b)\,(c+d) \tag{2}$$

or

$$(a+d)\,(bc-1)+(b+c)\,(ad-1)=0. \tag{3}$$

Immediately, the condition $1\le a<b<c<d$ rules out equation (3) and we may restrict our attention to equation (2).

Since $c\ge 3$ and $d\ge 4$, we obtain

$$(cd-1)-(c+d)=(c-1)\,(d-1)-2>0$$

and hence,

$$c+d<cd-1.$$

Using this and the fact that $(ab-1)>0$, equation (2) implies that

$$(ab-1)\,(c+d)<(ab-1)\,(cd-1)=(a+b)\,(c+d)\,,$$

or

$$ab-1<a+b.$$

This in turn implies that

$$0\le (a-1)\,(b-1)<2.$$

Then, since $1 \leq a < b$, we must have $a = 1$ and equation (2) becomes

$$(b-1)(cd-1) = (b+1)(c+d). \tag{4}$$

Finally, $b \geq 2$ implies that

$$cd - 1 = \frac{b+1}{b-1}(c+d) = \left(1 + \frac{2}{b-1}\right)(c+d) \leq 3(c+d)$$

or

$$0 \leq (c-3)(d-3) \leq 10. \tag{5}$$

To complete the solution, we solve each of the 11 possibilities presented by (5) and then substitute back into (4) to solve for the remaining variable. It turns out that the only situation which yields feasible answers for b, c, d is the case where $(c-3)(d-3) = 10$. We show this case and two others to indicate the reasoning applied.

Case 1. If

$$(c-3)(d-3) = 0,$$

then since $1 = a < b < c < d$, we must have $c = 3$ and $b = 2$. When these are substituted into (4), we get

$$3d - 1 = 3(3+d)$$

which is impossible.

Case 2. If

$$(c-3)(d-3) = 6,$$

then since $c < d$, we must have $c-3 = 1$, $d-3 = 6$ or $c-3 = 2$, $d-3 = 3$. These yield $c = 4$, $d = 9$ or $c = 5$, $d = 6$. However, neither pair gives an integral answer for b when these are substituted into (4).

Case 3. If

$$(c-3)(d-3) = 10,$$

then since $c < d$, we must have $c-3 = 1$, $d-3 = 10$ or $c-3 = 2$, $d-3 = 5$. These yield $c = 4$, $d = 13$ or $c = 5$, $d = 8$. When substituted into (4), both pairs give the answer $b = 2$.

Therefore, the only solutions for which a, b, c, d are integers, with $1 \leq a < b < c < d$, are $(a, b, c, d) = (1, 2, 4, 13)$ or $(1, 2, 5, 8)$.

Also solved by Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

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- 4969: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^2\left(\frac{1}{a} + \frac{1}{c}\right)} + \frac{1}{b^2\left(\frac{1}{b} + \frac{1}{a}\right)} + \frac{1}{c^2\left(\frac{1}{c} + \frac{1}{b}\right)} \geq \frac{3}{2}$$

Solution by Kenneth Korbin, New York, NY.

Let $x = \dfrac{1}{a}, y = \dfrac{1}{b}, z = \dfrac{1}{c}$. Then, $K = \dfrac{x^2}{x+z} + \dfrac{y^2}{y+x} + \dfrac{z^2}{z+y}$.

Let $U_1 = \dfrac{x}{\sqrt{x+z}}, U_2 = \dfrac{y}{\sqrt{y+x}}, U_3 = \dfrac{z}{\sqrt{z+y}}$. Then, $K = (U_1)^2 + (U_2)^2 + (U_3)^2$.

Let $V_1 = \sqrt{x+z}, V_2 = \sqrt{y+x}, V_3 = \sqrt{z+y}$. Then, by the Cauchy inequality,

$$\begin{aligned} K &= (U_1)^2 + (U_2)^2 + (U_3)^2 \\ &\geq \frac{(U_1V_1 + U_2V_2 + U_3V_3)^2}{(V_1)^2 + (V_2)^2 + (V_3)^2} \\ &= \frac{(x+y+z+)^2}{2(x+y+z)} = \frac{x+y+z}{2} \end{aligned}$$

Then, by the AM-GM inequality,

$$\begin{aligned} K &\geq \frac{x+y+z}{2} \\ &\geq \frac{1}{2}(3)(\sqrt[3]{xyz}) \\ &= \frac{3}{2}(1) = \frac{3}{2}. \end{aligned}$$

Note: $abc = 1$ implies $xyz = 1$.

Comment by editor: John Boncek of Montgomery, AL noted that this problem is a variant of an exercise given in Andreescu and Enescu's *Mathematical Olympiad Treasures,* (Birkhauser, 2004, problem 6, page 108.)

Also solved by John Boncek; David E. Manes, Oneonta, NY, and the proposer.

- 4970: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let $f : [0,1] \longrightarrow \mathbf{R}$ be a contintuous convex function. Prove that

$$\frac{3}{4}\int_0^{1/5} f(t)dt + \frac{1}{8}\int_0^{2/5} f(t)dt \geq \frac{4}{5}\int_0^{1/4} f(t)dt.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

By the convexity of f we have

$$\frac{3}{4}f\left(\frac{s}{5}\right) + \frac{1}{4}f\left(\frac{2s}{5}\right) \geq f\left((\frac{3}{4})(\frac{s}{5}) + (\frac{1}{4})(\frac{2s}{5})\right) = f\left(\frac{s}{4}\right)$$

for $0 \leq s \leq 1$. Hence,

$$\frac{3}{4}\int_0^1 f\left(\frac{s}{5}\right)ds + \frac{1}{4}\int_0^1 f\left(\frac{2s}{5}\right)ds \ \geq \ \int_0^1 f\left(\frac{s}{4}\right)ds.$$

By substituting $s = 5t$ in the first integral, $s = \dfrac{5t}{2}$ in the second at the left and $s = 4t$ in the integral at the right, we obtain the inequality of the problem.

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

Note 1. Consider the behavior in the extreme case: if f is a linear function, then equality holds:

$$\frac{3}{4}\int_0^{1/5}(mt+b)dt+\frac{1}{8}\int_0^{2/5}(mt+b)dt=\frac{3}{4}\left[\frac{m}{2}\left(\frac{1}{5}\right)^2+b\frac{1}{5}\right]+\frac{1}{8}\left[\frac{m}{2}\left(\frac{2}{5}\right)^2+b\frac{2}{5}\right]=\frac{1}{40}m+\frac{1}{5}b,$$

and

$$\frac{4}{5}\int_0^{1/4}(mt+b)dt=\frac{4}{5}\left[\frac{m}{2}\left(\frac{1}{4}\right)^2+b\frac{1}{4}\right]=\frac{1}{40}m+\frac{1}{5}b.$$

We rewrite the inequality in an equivalent form by clearing fractions and splitting the integrals so that they are taken over non-overlapping intervals:

$$\frac{3}{4}\int_0^{1/5}f(t)dt+\frac{1}{8}\int_0^{2/5}f(t)dt\ge\frac{4}{5}\int_0^{1/4}f(t)dt \iff$$

$$30\int_0^{1/5}f(t)dt+5\left[\int_0^{1/5}f(t)dt+\int_{1/5}^{1/4}f(t)dt+\int_{1/4}^{2/5}f(t)dt\right]\ge 32\left[\int_0^{1/5}f(t)dt+\int_{1/5}^{1/4}f(t)dt\right] \iff$$

$$3\int_0^{1/5}f(t)dt+5\int_{1/4}^{2/5}f(t)dt\ge 27\int_{1/5}^{1/4}f(t)dt. \qquad (1)$$

So we see that the interval of interest, $\left[0,\frac{2}{5}\right]$, has been partitioned into three subintervals $\left[0,\frac{1}{5}\right]$, $\left[\frac{1}{5},\frac{1}{4}\right]$ and $\left[\frac{1}{4},\frac{2}{5}\right]$.

Consider the secant line through the two points $\left(\frac{1}{5},f\left(\frac{1}{5}\right)\right)$ and $\left(\frac{1}{4},f\left(\frac{1}{4}\right)\right)$. The linear function giving this line is $s(t)=20\left[f\left(\frac{1}{4}\right)-f\left(\frac{1}{5}\right)\right]t+\left[5f\left(\frac{1}{5}\right)-4f\left(\frac{1}{4}\right)\right]$. It is straightforward to use the convexity condition to show that this line lies above $f(t)$ on the middle interval$\left[\frac{1}{5}\right]$, and lies below $f(t)$ on the outside intervals $\left[0,\frac{1}{5}\right]$ and $\left[\frac{1}{4},\frac{2}{5}\right]$. That is

$$\begin{aligned} s(t) &\ \ge\ f(t) \text{ on } \left[\frac{1}{5},\frac{1}{4}\right] \text{ and} && (2)\\ s(t) &\ \le\ f(t) \text{ on } \left[0,\frac{1}{5}\right], \text{ and } \left[\frac{1}{4},\frac{2}{5}\right] && (3). \end{aligned}$$

Considering the sides of (1),

$$3\int_0^{1/5}f(t)dt+5\int_{1/4}^{2/5}f(t)dt\ge 3\int_0^{1/5}s(t)dt+5\int_{1/4}^{2/5}s(t)dt \text{ by (3).}$$

and

$$3\int_0^{1/5}s(t)dt+5\int_{1/4}^{2/5}s(t)dt=27\int_{1/5}^{1/4}s(t)dt \text{ by (Note 1),}$$

and

$$27\int_{1/5}^{1/4}s(t)dt\ge 27\int_{1/5}^{1/4}f(t)dt \text{ by (2).}$$

Therefore (1) is true.

Also solved by John Boncek, Montgomery, AL and the proposers.

- 4971: *Proposed by Howard Sporn, Great Neck, NY and Michael Brozinsky, Central Islip, NY.*

  Let $m \geq 2$ be a positive integer and let $1 \leq x < y$. Prove:

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Solution 1 by Brian D. Beasley, Clinton, SC.

We let $f(x) = x^m - (x-1)^m$ for $x \geq 1$ and show that f is strictly increasing on $[1, \infty)$. Since $f'(x) = mx^{m-1} - m(x-1)^{m-1}$, we have $f'(x) > 0$ if and only if $x^{m-1} > (x-1)^{m-1}$. Since $x \geq 1$ and $m \geq 2$, this latter inequality holds, so we are done.

Solution 2 by Matt DeLong, Upland, IN.

Let $X = x - 1$ and $Y = y - 1$. Then $0 \leq X < Y$, $x = X + 1$, and $y = Y + 1$. Expanding $(X+1)^m - X^m$ and $(Y+1)^m - Y^m$ we see that

$$(X+1)^m - X^m = mX^{m-1} + \frac{m(m-1)}{2}X^{m-2} + \cdots + mX + 1$$

and

$$(Y+1)^m - Y^m = mY^{m-1} + \frac{m(m-1)}{2}Y^{m-2} + \cdots + mY + 1.$$

Since $0 \leq X < Y$, we can compare these two sums term-by-term and conclude that each term involving Y is larger than the corresponding term involving X. Therefore,

$$(X+1)^m - X^m < (Y+1)^m - Y^m.$$

Since $x = X + 1$ and $y = Y + 1$, we have shown that

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

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Solution 3 by José Luis Díaz-Barrero, Barcelona, Spain.

We will argue by induction. The case when $m = 2$ trivially holds because $x^2 - (x-1)^2 = 2x - 1 < 2y - 1 = y^2 - (y-1)^2$. Suppose that

$$x^m - (x-1)^m < y^m - (y-1)^m$$

holds and we have to see that

$$x^{m+1} - (x-1)^{m+1} < y^{m+1} - (y-1)^{m+1}$$

holds. In fact, multiplying by $m+1$ both sides of $x^m - (x-1)^m < y^m - (y-1)^m$ yields

$$(m+1)(x^m - (x-1)^m) < (m+1)(y^m - (y-1)^m)$$

and

$$\int_1^x (m+1)(x^m - (x-1)^m)\,dx < \int_1^y (m+1)(y^m - (y-1)^m)\,dy$$

from which immediately follows

$$x^{m+1} - (x-1)^{m+1} < y^{m+1} - (y-1)^{m+1}$$

Therefore, by the PMI the statement is proved and we are done.

Solution 4 by Kenneth Korbin, New York, NY.

Let $m \geq 2$ be a positive integer, and let $1 \leq x < y$. Then,

$$\begin{array}{rcl} (y-1)^m & < & y^m, \text{ and} \\ (y-1)^{m-1}(x-1) & < & y^{m-1}(x), \text{ and} \\ (y-1)^{m-2}(x-1)^2 & < & y^{m-2}(x^2), \text{ and} \\ & . & \\ & . & \\ & . & \\ y^0 = 1 & \leq & x^m. \end{array}$$

Adding gives

$$\Big[(y-1)^m + (y-1)^{m-1}(x-1) + \cdots + 1\Big] < \Big[y^m + y^{m-1}x + y^{m-2}x^2 + \cdots + x^m\Big].$$

Multiplying both sides by $[(y-1)-(x-1)] = [y-x]$ gives

$$(y-1)^m - (x-1)^m < y^m - x^m.$$

Therefore

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; various teams of students at Taylor University in Upland, IN:

Bethany Ballard, Nicole Gottier, and Jessica Heil;
Mandy Isaacson, Julia Temple, and Adrienne Ramsay;
Jeremy Erickson, Matthew Russell, and Chad Mangum;
Seth Bird, Jim Moore, and Jonathan Schrock;

and the proposers.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
February 15, 2008

- 4990: *Proposed by Kenneth Korbin, New York, NY.*

  Solve
  $$40x + 42\sqrt{1-x^2} = 29\sqrt{1+x} + 29\sqrt{1-x}$$
  with $0 < x < 1$.

- 4991: *Proposed by Kenneth Korbin, New York, NY.*

  Find six triples of positive integers (a, b, c) such that
  $$\frac{9}{a} + \frac{a}{b} + \frac{b}{9} = c.$$

- 4992: *Proposed by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX.*

  A closed circular cone has integral values for its height and base radius. Find all possible values for its dimensions if its volume V and its total area (including its circular base) A satisfy $V = 2A$.

- 4993: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation
  $$126x^7 - 127x^6 + 1 = 0.$$

- 4994: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be three nonzero complex numbers lying on the circle $C = \{z \in \mathbf{C} : |z| = r\}$. Prove that the roots of the equation $az^2 + bz + c = 0$ lie in the ring shaped region $D = \left\{ z \in \mathbf{C} : \dfrac{1-\sqrt{5}}{2} \leq |z| \leq \dfrac{1+\sqrt{5}}{2} \right\}$.

- 4995: *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India.*

Let A be a triangular array $a_{i,j}$ where $i = 1, 2, \cdots,$ and $j = 0, 1, 2, \cdots, i$. Let

$$a_{1,0} = 1,\ a_{1,1} = 2,\ \text{and } a_{i,0} = T(i+1) - 2 \text{ for } i = 2, 3, 4, \cdots,$$

where $T(i+1) = (i+1)(i+2)/2$, the usual triangular numbers. Furthermore, let $a_{i,j+1} - a_{i,j} = j + 1$ for all j. Thus, the array will look like this:

$$\begin{array}{c} 1 \quad 2 \\ 4 \quad 5 \quad 7 \\ 8 \quad 9 \quad 11 \quad 14 \\ 13 \quad 14 \quad 16 \quad 19 \quad 23 \\ 19 \quad 20 \quad 22 \quad 25 \quad 29 \quad 34 \end{array}$$

Show that for every pair (i, j), $4a_{i,j} + 9$ is the sum of two perfect squares.

Solutions

- 4972:*Proposed by Kenneth Korbin, New York, NY.*

Find the length of the side of equilateral triangle ABC if it has a cevian $\overline{\text{CD}}$ such that

$$\overline{\text{AD}} = x, \quad \overline{\text{BD}} = x + 1 \quad \overline{\text{CD}} = \sqrt{y}$$

where x and y are positive integers with $20 < x < 120$.

Solution by Kee-Wai Lau, Hong Kong, China.

Applying the cosine formula to triangle CAD, we obtain

$$\overline{CD}^2 = \overline{AD}^2 + \overline{AC}^2 - 2\overline{AD} \cdot \overline{AC} \cos 60^o,$$

or

$$\begin{aligned} (\sqrt{y})^2 &= x^2 + (2x+1)^2 - 2x(2x+1)\cos 60^o \\ y &= 3x^2 + 3x + 1. \end{aligned}$$

For $20 < x < 120$, we find using a calculator that y is the square of a positive integer if $x = 104, y = 32761$. Hence the length of the side of equilateral triangle ABC is 209.

Comments:

1) Scott H. Brown, Montgomery, AL.

The list of pairs (x, y) that satisfy the equation $y = 3x^2 + 3x + 1$ is so large I will not attempt to name each pair...numerous triangles with the given conditions can be found.

2) David Stone and John Hawkins, Statesboro, GA.

The restriction on x seems artificial–every x produces a triangle. In fact, if we require the cevian length to be an integer, this becomes a Pell's Equation problem and we can generate nice solutions recursively in the usual fashion. The first few for $x,\ s = 2x+1,\ y = 3x^2+3x+1,\ \&$ cevian $= \sqrt{\text{y}}$ are:

7	15	169	13
104	209	32761	181
1455	2911	6355441	2521
20272	40545	1232922769	35113

Also solved by Peter E. Liley, Lafayette, IN, and the proposer.

- 4973: *Proposed by Kenneth Korbin, New York, NY.*

Find the area of trapezoid ABCD if it is inscribed in a circle with radius R=2, and if it has base $\overline{\text{AB}} = 1$ and $\angle\text{ACD} = 60^{\text{o}}$.

Solution by David E. Manes, Oneonta, NY.

The area A of the trapezoid is given by $A = \dfrac{3\sqrt{3}}{8}\left(15+\sqrt{5}\right)$.

Since the trapezoid is cyclic, it is isosceles so that $AD = BC$. Note that $\angle ACD = 60^o \Rightarrow \angle CAB = 60^o$ since alternate interior angles of a transversal intersecting two parallel lines are congruent. Therefore, from the law of sines in triangle ABC, $\dfrac{BC}{\sin 60^o} = 2R$ or $BC = 2\sqrt{3}$. Using the law of cosines in triangle ABC,

$$BC^2 = 1 + AC^2 - 2AC\cdot\cos 60^o,\ \text{ or } \text{AC}^2 - \text{AC} - 11 = 0.$$

Thus, AC is the positive root of this equation so that $AC = \dfrac{1+3\sqrt{5}}{2}$. Similarly, using the law of cosines in triangle ACD and recalling that $AD = BC$, one obtains

$$AD^2 = AC^2 + DC^2 - 2\cdot AC\cdot DC\cdot\cos 60^o$$

or $\text{DC}^2 - \left(\dfrac{1+3\sqrt{5}}{2}\right)DC + \dfrac{-1+3\sqrt{5}}{2} = 0$. Noting that $DC > 2$ and $\sqrt{6-2\sqrt{5}} = \sqrt{(1-\sqrt{5})^2} = \sqrt{5}-1$, it follows that $DC = 3\sqrt{5}-1$. Finally, let H be the point on line segment $\overline{DC}$ such that $\overline{AH}$ is perpendicular to $\overline{DC}$. Then the height h of the trapezoid is given by $h = AC\cdot\sin 60^o = \dfrac{\sqrt{3}}{4}\left(1+3\sqrt{5}\right)$. Hence,

$$A = \frac{1}{2}\left(AB+DC\right)\cdot h = \frac{1}{2}\left(1+3\sqrt{5}-1\right)\frac{\sqrt{3}}{4}\left(1+3\sqrt{5}\right) = \frac{3\sqrt{3}}{8}\left(15+\sqrt{5}\right).$$

Also solved by Robert Anderson, Gino Mizusawa, and Jahangeer Kholdi (jointly), Portsmouth, VA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Zhonghong Jiang, NY, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4974: *Proposed by Kenneth Korbin, New York, NY.*

  A convex cyclic hexagon has sides $a, a, a, b, b,$ and b. Express the values of the circumradius and the area of the hexagon in terms of a and b.

  Solution by Kee-Wai Lau, Hong Cong, China.

  We show that the circumradius R is $\sqrt{\dfrac{a^2+ab+b^2}{3}}$ and the area A of the hexagon is $\dfrac{\sqrt{3}(a^2+4ab+b^2)}{4}$.

  Denote the angle subtended by side a and side b at the center of the circumcircle respectively by θ and ϕ. Since $3\theta+3\phi=360^o$ so $\theta=120-\phi$ and

$$\cos\theta=\cos(120^o-\phi)=\frac{-\cos\phi+\sqrt{3}\sin\phi}{2}. \text{ Hence,}$$

$$(2\cos\theta+\cos\phi)^2=3(1-\cos^2\phi) \text{ or } 4\cos^2\theta+4\cos\theta\cos\phi+4\cos^2\phi-3=0.$$

  Now by the cosine formula $\cos\theta=\dfrac{2R^2-a^2}{2R^2}$ and $\cos\phi=\dfrac{2R^2-b^2}{2R^2}$.

  Therefore,

$$\begin{aligned}(2R^2-a^2)^2+(2R^2-a^2)(2R^2-b^2)+(2R^2-b^2)^2-3R^4 &= 0 \text{ or}\\ 9R^4-6(a^2+b^2)R^2+a^4+a^2b^2+b^4 &= 0.\end{aligned}$$

  Solving the equation we obtain $R^2=\dfrac{a^2+ab+b^2}{3}$ or $R^2=\dfrac{a^2-ab+b^2}{3}$. The latter result is rejected because if not, then for $a=b$, we have $\cos\theta=\cos\phi<0$ so that $\theta+\phi>180^o$, which is not true. Hence, $R=\sqrt{\dfrac{a^2+ab+b^2}{3}}$.

  To find A, we need to find the area of the triangles with sides R, R, a and R, R, b. The heights to bases a and b are respectively $\dfrac{\sqrt{4R^2-a^2}}{2}=\dfrac{\sqrt{3}(a+2b)}{6}$ and $\dfrac{\sqrt{4R^2-b^2}}{2}=\dfrac{\sqrt{3}(2a+b)}{6}$. Hence the area of the hexagon equals

$$3\left(\frac{\sqrt{3}a(a+2b)}{12}+\frac{\sqrt{3}b(2a+b)}{12}\right)=\frac{\sqrt{3}}{4}\left(a^2+4ab+b^2\right) \text{ as claimed.}$$

  Also solved by Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Zhonghong Jiang, NY, NY; David E. Manes, Oneonta, NY; M. N. Deshpande, Nagpur, India; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; Jonathan Schrock, Seth Bird, and Jim Moore (jointly, students at Taylor University), Upland, IN; David Wilson, Winston-Salem, NC, and the proposer.

- 4975: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Solve in R the following system of equations

$$\left.\begin{aligned} 2x_1 &= 3x_2\,\sqrt{1+x_3^2} \\ 2x_2 &= 3x_3\,\sqrt{1+x_4^2} \\ &\cdots\cdots \\ 2x_n &= 3x_1\,\sqrt{1+x_2^2} \end{aligned}\right\}$$

Solution by David Stone and John Hawkins, Statesboro, GA.

Squaring each equation and summing, we have

$$4(x_1^2+x_2^2+x_3^2+\cdots+x_n^2) = 9(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)+9(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+\cdots+x_{n-1}^2x_n^2).$$

So

$$0 = 5(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)+9(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+\cdots+x_{n-1}^2x_n^2).$$

Because these squares are non-negative and the sum is zero, each term on the right-hand side must indeed equal 0. Therefore $x_1 = x_2 = x_3 = \cdots = x_n = 0$.

Alternatively, we could multiply the equations to obtain

$$2^n x_1x_2x_3x_4\cdots x_n = 3^n x_1x_2x_3x_{4n}\sqrt{1+x_1^2}\sqrt{1+x_2^2}\cdots\sqrt{1+x_n^2}.$$

If all x_k are non-zero, we'll have $\left(\dfrac{2}{3}\right)^n = \sqrt{1+x_1^2}\sqrt{1+x_2^2}\cdots\sqrt{1+x_n^2}$. The term on the left is < 1, while each term on the right is > 1, so the product is > 1. Thus we have reached a contradiction, forcing all x_k to be zero.

Also solved by Bethany Ballard, Nicole Gottier, and Jessica Heil (jointly, students, Taylor University), Upland, IN; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Mandy Isaacson, Julia Temple, and Adrienne Ramsay (jointly, students, Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

- 4976: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers. Prove that

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \geq 27.$$

Solution by Matt DeLong, Upland, IN.

In fact, I will prove that the sum is at least 39. Rewrite the sum

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \text{ as}$$

$$\frac{a^2}{bc}+3\frac{b}{c}+9\frac{c}{b}+\frac{b^2}{ca}+3\frac{c}{a}+9\frac{a}{c}+\frac{c^2}{ab}+3\frac{a}{b}+9\frac{b}{a}.$$

Rearranging the terms gives

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)+6\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)$$

Now, repeatedly apply the Arithmetic Mean-Geometric Mean inequality.

$$\begin{aligned}
\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab} &\geq 3\left(\frac{a^2b^2c^2}{bccaab}\right)^{1/3}=3\\
\frac{b}{c}+\frac{c}{b} &\geq 2\left(\frac{bc}{cb}\right)^{1/2}=2\\
\frac{c}{a}+\frac{a}{c} &\geq 2\left(\frac{ac}{ca}\right)^{1/2}=2\\
\frac{a}{b}+\frac{b}{a} &\geq 2\left(\frac{ab}{ba}\right)^{1/2}=2\\
\frac{c}{b}+\frac{a}{c}+\frac{b}{a} &\geq 3\left(\frac{cab}{bca}\right)^{1/3}=3.
\end{aligned}$$

Thus, we have

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)+6\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)\geq 3+3(2+2+2)+6(3).$$

In other words

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab}\geq 39$$

.

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX; Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly, students, Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4977: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $1 < a < b$ be real numbers. Prove that for any $x_1, x_2, x_3 \in [a, b]$ there exist $c \in (a, b)$ such that

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log c}.$$

Solution by Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX .

Strictly speaking, the conclusion is incorrect as stated. If $a = x_1 = x_2 = x_3$, then

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log a}.$$

Similarly,

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log b}$$

when $b = x_1 = x_2 = x_3$.
The statement is true when $1 < a \leq x_1 \leq x_2 \leq x_3 \leq b$ with $x_1 \neq x_3$. Since

$$\frac{3}{\log x_1x_2x_3}=\frac{3}{\log x_1+\log x_2+\log x_3},$$

then
$$\frac{4}{\log x_3} < \frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{3}{\log x_1 x_2 x_3} < \frac{4}{\log x_1}.$$
By the Intermediate Value Theorem, there exists $c \in (a, b)$ such that
$$\frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{3}{\log x_1 x_2 x_3} = \frac{4}{\log c}.$$

Solution 2 by Paul M. Harms, North Newton, KS.

Assume $x_1 < x_3$ with x_2 in the interval $[x_1, x_3]$. For $x > 1$, we note that $f(x) = \log(x)$ and $g(x) = 1/\log(x)$ are both continuous, one-to-one, positive functions with $f(x)$ strictly increasing and $g(x)$ strictly decreasing.

Consider
$$\frac{3}{\log(x_1 x_2 x_3)} = \frac{1}{\frac{\log(x_1) + \log(x_2) + \log(x_3)}{3}}.$$
The denominator is the average of the 3 log values which means this average value is between the extremes $\log x_1$ and $\log x_3$. Since $f(x)$ is one-to-one and continuous there is a value x_4 where $x_1 < x_4 < x_3$ and $\log x_4 = \dfrac{(\log x_1 + \log x_2 + \log x_3)}{3}$ with $\log x_4$ between $\log x_1$ and $\log x_3$.

The equation in the problem can now be written
$$\frac{\frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{1}{\log x_4}}{4} = \frac{1}{\log c} \text{ or}$$
$$\frac{g(x_1) + g(x_2) + g(x_3) + g(x_4)}{4} = \frac{1}{\log c}.$$
The average of the four $g(x)$ values is between the extremes $g(x_1)$ and $g(x_3)$. Since $g(x)$ is continuous and one-to-one there is a value $x = c$ such that
$$g(c) = \frac{1}{\log c} = \frac{g(x_1) + g(x_2) + g(x_3) + g(x_4)}{4}$$
where $x_1 < c < x_3$ and, thus, $a < c < b$.

Note that if $x_1 = x_2 = x_3$, then we obtain $c = x_1 = x_2 = x_3$. If we want $a < c < b$, then it appears that we need to keep x_1, x_2 and x_3 away from a and b when these three x-values are equal to each other.

Also solved by Michael Brozinsky, Central Islip, NY; Matt DeLong, Upland, IN; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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March 15, 2008

- 4996: *Proposed by Kenneth Korbin, New York, NY.*

  Simplify:
  $$\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right).$$

- 4997: *Proposed by Kenneth Korbin, New York, NY.*

  Three different triangles with integer-length sides all have the same perimeter P and all have the same area K.
  Find the dimensions of these triangles if $K = 420$.

- 4998: *Proposed by Jyoti P. Shiwalkar & M.N. Deshpande, Nagpur, India.*

  Let $A = [a_{i,j}]$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$ be a triangular array satisfying the following conditions:

  $$\begin{aligned} 1)\ a_{i,1} &= L(i) \text{ for all } i \\ 2)\ a_{i,i} &= i \text{ for all } i \\ 3)\ a_{i,j} &= a_{i-1,j} + a_{i-2,j} + a_{i-1,j-1} - a_{i-2,j-1} \text{ for } 2 \le j \le (i-1). \end{aligned}$$

  If $T(i) = \sum_{j=1}^{i} a_{i,j}$ for all $i \ge 2$, then find a closed form for $T(i)$, where $L(i)$ are the Lucas numbers, $L(1) = 1, L(2) = 3$, and $L(i) = L(i-1) + L(i-2)$ for $i \ge 3$.

- 4999: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real triplets (x, y, z) such that

$$\begin{aligned} x+y+z &= 2 \\ 2^{x+y^2}+2^{y+z^2}+2^{z+x^2} &= 6\sqrt[9]{2} \end{aligned}$$

- 5000: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Of all the right triangles inscribed in the unit circle, which has the Morley triangle of greatest area?

- 5001: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Evaluate:

$$\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right) dx.$$

Solutions

- 4978: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with side $\overline{\text{AB}} = 9$ and with cevian $\overline{\text{CD}}$. Find the length of $\overline{\text{AD}}$ if $\triangle$ADC can be inscribed in a circle with diameter equal to 10.

  Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak, and Paula Koca (jointly), San Angelo, TX.

  Let $x = \overline{AD}$ and $y = \overline{CD}$. If A is the area of $\triangle ADC$, then

$$A = \frac{1}{2}(9)\, x \sin 60^o = \frac{9}{4}\sqrt{3}x.$$

  Since the circumradius of $\triangle ADC$ is 5, we have

$$5 = \frac{9xy}{4A} = \frac{y}{\sqrt{3}}$$

  and hence,

$$y = 5\sqrt{3}.$$

  Then, by the Law of Cosines,

$$75 = y^2 = x^2 + 81 - 2(9)\, x \cos 60^o = x^2 - 9x + 81$$

  which reduces to

$$x^2 - 9x + 6 = 0.$$

  Therefore, there are two possible solutions:

$$\overline{AD} = x = \frac{9 \pm \sqrt{57}}{2}.$$

  Also solved by Scott H. Brown, Montgomery, AL; Daniel Copeland, Portland, OR; M.N. Deshpande, Nagpur, India; Paul M. Harms, North

Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Xiezhang Li, David Stone & John Hawkins (jointly), Statesboro, GA; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles, McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4979: *Proposed by Kenneth Korbin, New York, NY.*

Part I: Find two pairs of positive numbers (x, y) such that

$$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{\sqrt{65}}{2},$$

where x is an integer.

Part II: Find four pairs of positive numbers (x, y) such that

$$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{65}{2},$$

where x is an integer.

Solution 1 by Brian D. Beasley, Clinton, SC.

(I) We need $0 < y \leq 1$, so requiring x to be an integer yields

$$x = \frac{\sqrt{65}}{2}\left(\sqrt{1+y}-\sqrt{1-y}\right) \in \{1, 2, 3, 4, 5\}.$$

We solve for y to obtain $y = 2x\sqrt{65-x^2}/65$. Substituting $x \in \{1, 2, 3, 4, 5\}$ yields five solutions for (x, y), with two of these also having y rational, namely

$$(x, y) = (1, 16/65) \quad \text{and} \quad (x, y) = (4, 56/65).$$

(II) We again need $0 < y \leq 1$, so requiring x to be an integer yields

$$x = \frac{65}{2}\left(\sqrt{1+y}-\sqrt{1-y}\right) \in \{1, 2, \ldots, 45\}.$$

We solve for y to obtain $y = 2x\sqrt{4225-x^2}/4225$. Substituting $x \in \{1, 2, \ldots, 45\}$ yields 45 solutions for (x, y), with four of these also having y rational, namely

$$(x, y) = (16, 2016/4225); \quad (x, y) = (25, 120/169);$$
$$(x, y) = (33, 3696/4225); \quad (x, y) = (39, 24/25).$$

Solution 2 by James Colin Hill, Cambridge, MA.

Part I: The given equation yields $4x^2 = 130(1+\sqrt{1-y^2})$. Let $y = \cos\theta$. Then

$$\sin\theta = \frac{4x^2}{130} - 1.$$

For $x \in Z^+$, we find several solutions, including the following (rational) pair:

$$\begin{aligned} x &= 1,\ y = 16/65 \\ x &= 4,\ y = 56/64. \end{aligned}$$

Part II: The given equation yields $\sin\theta = \dfrac{4x^2}{8450} - 1$, where $y = \cos\theta$ as before. For $x \in Z^+$, we find many solutions, including the following (rational) four:

$$x = 16,\ y = 2016/4225$$

$$\begin{aligned} x &= 25,\ y = 120/169 \\ x &= 33,\ y = 3696/4225 \\ x &= 39,\ y = 24/25 \end{aligned}$$

Also solved by John Boncek, Montgomery, AL; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, David C. Wilson, Winston-Salem, NC, and the proposer.

- 4980: *J.P. Shiwalkar and M.N. Deshpande, Nagpur, India.*

  An unbiased coin is sequentially tossed until $(r+1)$ heads are obtained. The resulting sequence of heads (H) and tails (T) is observed in a linear array. Let the random variable X denote the number of double heads (HH's, where overlapping is allowed) in the resulting sequence. For example: Let $r = 6$ so the unbiased coin is tossed till 7 heads are obtained and suppose the resulting sequence of H's and T's is as follows:

  HHTTT**H**TTTT**HHH**TT**H**

  Now in the above sequence, there are three double heads (HH's) at toss number $(1,2), (11,12)$ and $(12,13)$. So the random variable X takes the value 3 for the above observed sequence.

  In general, what is the expected value of X?

  Solution by N. J. Kuenzi, Oshkosh,WI.

  Let $X(r)$ be the number of double heads (HH) in the resulting sequence.
  First consider the case $r = 1$. Since the resulting sequence of heads (H) and tails (T) ends in either TH or HH, $P[X(1) = 0] = \dfrac{1}{2}$ and $P[X(1) = 1] = \dfrac{1}{2}$. So $E[X(1)] = \dfrac{1}{2}$.

  Next let $r > 1$, an unbiased coin is tossed until $(r+1)$ heads are obtained. If the resulting sequence of $H's$ and $T's$ ends in TH then $X(r) = X(r-1)$. And if the resulting sequence of $H's$ and $T's$ ends in HH then $X(r) = X(r-1)$. So

  $$P[X(r) = X(r-1)] = \frac{1}{2} \text{ and } P[X(r) = X(r-1)+1] = \frac{1}{2}.$$

  It follows that

  $$E[X(r)] = \frac{1}{2}E[X(r-1)] + \frac{1}{2}E[X(r-1)+1] = E[X(r-1)] + \frac{1}{2}.$$

  Finally, the Principle of Mathematical Induction can be used to show that $E[X(r)] = \dfrac{r}{2}$.

  Also solved by Kee-Wai Lau, Hong Kong, China; Harry Sedinger, St. Bonvatenture, NY, and the proposers.

- 4981: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

  $$5^x + 3^x + 2^x - 28x + 18 = 0.$$

Solution by Paolo Perfetti, Dept. of Mathematics, University of Rome, Italy.

Let $f(x) = 5^x + 3^x + 2^x - 28x + 18$. The values for $x \le 0$ are excluded from being solutions because for these values $f(x) > 0$. It is immediately seen that $f(x) = 0$ for $x = 1, 2$. Moreover, the derivative $f'(x) = 5^x \ln 5 + 3^x \ln 3 + 2^x \ln 2 - 28$ is an increasing continuous function such that:

$$\begin{aligned} 1)\ f'(0) &= \ln 30 - 28 < 0, \quad \lim_{x+\infty} f'(x) = +\infty \\ 2)\ f'(1) &= 5\ln 5 + 3\ln 3 + 2\ln 2 - 28 < 10 + 6 + 2 - 28 = -10 \\ 3)\ f'(2) &= 25\ln 5 + 9\ln 3 + 4\ln 2 - 28 \ge 34 - 28 > 0. \end{aligned}$$

By continuity this implies that $f'(x) = 0$ for just one point x_o between 1 and 2, and that the graph of $f(x)$ has a minimum only at $x = x_o$. It follows that there are no values of x other than $x = 1, 2$ satisfying $f(x) = 0$.

Also solved by Brain D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M.N. Deshpande, Nagpur, India; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kee-Wai Lau, Hong Kong, China; Kenneth Korbin, NY, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins, Statesboro, GA, and the proposers.

- 4982: *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate

$$\lim_{n\to\infty} \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1} \frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1} \frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1} \frac{1}{i_1 i_2 \ldots i_n}\right).$$

Solution 1 by Paul M. Harms, North Newton, KS.

Let $S(n)$ be the addition of the summations inside the parentheses of the expression in the problem. When $n = 1$. The expression in the problem is

$$\frac{1}{2}\left(\left[\frac{1}{1} + \frac{1}{2}\right] + \left[\frac{1}{1(2)}\right]\right) = (\frac{1}{2})2 = 1, \text{ where } S(1) = 2.$$

When $n = 2$ the expression is

$$\begin{aligned} &= \frac{1}{3}\left(\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right] + \left[\frac{1}{1(2)} + \frac{1}{1(3)} + \frac{1}{(2)(3)}\right] + \left[\frac{1}{1(2)(3)}\right]\right) \\ &= \frac{1}{3}\left(S(1) + \frac{1}{3}\left[1 + S(1)\right]\right) \\ &= \frac{1}{3}\left(2 + \frac{1}{3}3\right) = 1, \text{ where } S(2) = 3. \end{aligned}$$

When $n = 3$ the expression is

$$\frac{1}{4}\left(S(2) + \frac{1}{4}\left[1 + S(2)\right]\right) = \frac{1}{4}\left(3 + \frac{1}{4}\left[1 + 3\right]\right) = 1, \text{ where } S(3) = 4.$$

When $n = k + 1$ the expression becomes

$$\frac{1}{k+2}\left(S(k) + \frac{1}{k+2}\left[1 + S(k)\right]\right) = 1, \text{ where } S(k) = k + 1.$$

The limit in the problem is one.

Solution 2 by David E. Manes,Oneonta, NY.

Let

$$a_n = \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}\right).$$

Then $a_1 = 3/4, a_2 = 17/18, a_3 = 95/96$, and $a_4 = 599/600$.

We will show that

$$a_n = 1 - \frac{1}{(n+1)(n+1)!}$$

Note that the equation is true for $n = 1$ and assume inductively that it is true for some integer $n \ge 1$. Then

$$\begin{aligned}
a_n &= \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}\right)\\
&= \frac{1}{n+2}\left[(n+1)a_n + \frac{1}{n+2} + \left(\frac{n+1}{n+2}\right)a_n + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[(n+1)\left(1-\frac{1}{(n+1)(n+1)!}\right) + \frac{1}{n+2} + \left(\frac{n+1}{n+2}\right)\left(1-\frac{1}{(n+1)(n+1)!}\right) + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[(n+1) - \frac{1}{(n+1)!} + 1 - \frac{1}{(n+2)(n+1)!} + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[n+2-\frac{1}{(n+2)!}\right] = 1 - \frac{1}{(n+2)(n+2)!}.
\end{aligned}$$

Therefore, the result is true for $n+1$. By induction $a_n = 1 - \dfrac{1}{(n+1)(n+1)!}$ is valid for all integers $n \ge 1$. Hence $\lim\limits_{n\to\infty} a_n = 1$.

Also solved by Carl Libis, Kingston, RI; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4983: *Proposed by Ovidiu Furdui, Kalamazoo, MI.*

Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\} dx,$$

where $\{a\}$ is the *fractional part* of a.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that

$$\int_0^1 \left\{\frac{k}{x}\right\} dx = k\left(\sum_{n=1}^{k}\frac{1}{n} - \ln k - \gamma\right),$$

where γ is Euler's constant. By substituting $x = ky$, we obtain

$\displaystyle\int_0^1 \left\{\frac{k}{x}\right\} dx = k\int_0^{1/k} \left\{\frac{1}{y}\right\} dy$. For any integer $M > k,$ we have

$$
\begin{aligned}
\int_{1/M}^{1/k} \left\{\frac{1}{y}\right\} dy &= \sum_{n=k}^{M-1} \int_{1/(n+1)}^{1/n} \left\{\frac{1}{y}\right\} dy \\
&= \sum_{n=k}^{M-1} \int_{1/(n+1)}^{1/n} \left\{\frac{1}{y} - n\right\} dy \\
&= \sum_{n=k}^{M-1} \left(\ln(\frac{n+1}{n}) - \frac{1}{n+1}\right) \\
&= \ln\left(\frac{M}{k}\right) - \sum_{n=k+1}^{M} \frac{1}{n} \\
&= \sum_{n=1}^{k} \frac{1}{n} - \ln k - \left(\sum_{n=1}^{M} \frac{1}{n} - \ln M\right).
\end{aligned}
$$

Since $\displaystyle\lim_{M\to\infty}\left(\sum_{n=1}^{M} \frac{1}{n} - \ln M\right) = \gamma$, we obtain our result.

Also solved by Brian D. Beasley, Clinton, SC; Jahangeer Kholdi, Portsmouth, VA; David E. Manes, Oneonta, NY; Paolo Perfetti, Dept. of Mathematics, University of Rome, Italy; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Acknowledgments

The name of **Dr. Peter E. Liley of Lafayette, IN** should have been listed as having solved problems 4966, 4973 and 4974. His name was inadvertently omitted from the listing; mea culpa.

Problem 4952 was posted in the January 07 issue of this column. It was proposed by **Michael Brozinsky of Central Islip, NY & Robert Holt of Scotch Plains, NJ.** I received one solution to this problem; it was from **Paul M. Harms of North Newton, KS.** His solution, which was different from the one presented by proposers, made a lot of sense to me and it was published in the October 07 issue of this column. Michael then wrote to me stating that he thinks Paul misinterpreted the problem. For the sake of completeness, here is the proposers' solution to their problem.

- 4952: An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

Solution by the proposers: a) Clearly these maximum numbers will be attained using the 3^k possible configurations for a house.

Note: For any two houses on a plot:
1) if they have the same number of any given feature then the third house will necessarily have this same number of that feature since the sum must be divisible by three, and
2) if they have a different number of a given feature then the third house will have a different number of that feature than the first two houses since the sum must be divisible by three.

It follows then that any fixed house can be adjoined with $\dfrac{3^k-1}{2}$ possible pairs of houses to be placed on a plot since the second house can be any of the remaining 3^k-1 house configurations but the third configuration is uniquely determined by the above note and the fact that no two houses on a plot can be identically configured. These 3^k-1 permutations of the second and third house thus must have arisen from the $\dfrac{3^k-1}{2}$ possible pairs claimed above. The answer is thus $\dfrac{3^k-1}{2}$.
b) The above note shows that for any two differently configured houses only one of the remaining 3^k-2 configurations will form a plot with these two. Hence, the probability that 3 configurations chosen randomly from the 3^k configurations are suitable for a plot is $\dfrac{1}{3^k-2}$. Since there are $\dbinom{3^k}{3}$ subsets of size three that can be formed from the 3^k configurations, it follows that the maximum number of distinct possible plots is
$$\frac{\dbinom{3^k}{3}}{3^k-2}=\frac{3^{k-1}(3^k-1)}{2}.$$

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
April 15, 2008

- 5002: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon with sides $3x, 3x, 3x, 5x, 5x$ and $5x$ is inscribed in a unit circle. Find the value of x.

- 5003: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\begin{aligned}\sqrt[3]{x+\sqrt{x^2-1}}+\sqrt[3]{x-\sqrt{x^2-1}} &= \frac{7}{2} \text{ and}\\ \sqrt[3]{y+\sqrt{y^2-1}}+\sqrt[3]{y-\sqrt{y^2-1}} &= \sqrt{10}\end{aligned}$$

- 5004: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{\sqrt{ab}}{1+a+b}+\frac{\sqrt{bc}}{1+b+c}+\frac{\sqrt{ca}}{1+c+a}$$

- 5005: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{\sqrt{3}}{2}\left(a+b+c\right)^{1/2} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

- 5006: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Find the sum
$$\sum_{k=2}^{\infty}(-1)^k \ln\left(1-\frac{1}{k^2}\right).$$

- 5007: *Richard L. Francis, Cape Girardeau, MO.*

  Is the centroid of a triangle the same as the centroid of its Morley triangle?

Solutions

- 4984: *Proposed by Kenneth Korbin, New York, NY.*

  Prove that
$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2009}+\sqrt{2011}} > \sqrt{120}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

The sum

$$\begin{aligned}
& \sum_{k=1}^{503}\frac{1}{\sqrt{4k-3}+\sqrt{4k-1}} \\
> \quad & \frac{1}{2}\sum_{k=1}^{503}\left(\frac{1}{\sqrt{4k-3}+\sqrt{4k-1}}+\frac{1}{\sqrt{4k-1}+\sqrt{4k+1}}\right) \\
= \quad & \frac{1}{2}\sum_{k=1}^{503}\left(\frac{\sqrt{4k-1}-\sqrt{4k-3}}{2}+\frac{\sqrt{4k+1}-\sqrt{4k-1}}{2}\right) \\
= \quad & \frac{1}{4}\sum_{k=1}^{503}\left(\sqrt{4k+1}-\sqrt{4k-3}\right) \\
= \quad & \frac{1}{4}\left(\sqrt{2013}-1\right) \\
= \quad & \frac{1}{4}\sqrt{2013-2\sqrt{2013}+1} \\
> \quad & \frac{1}{4}\left(\sqrt{2013-2(45)}+1\right) \\
> \quad & \frac{1}{4}\sqrt{1920} \\
= \quad & \sqrt{120}
\end{aligned}$$

as required.

Solution 2 by Kenneth Korbin, the proposer.

Let $K = \dfrac{1}{\sqrt{1}+\sqrt{3}} + \dfrac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \dfrac{1}{\sqrt{2009}+\sqrt{2011}}$.

Then, $K > \dfrac{1}{\sqrt{3}+\sqrt{5}} + \dfrac{1}{\sqrt{7}+\sqrt{9}} + \cdots + \dfrac{1}{\sqrt{2011}+\sqrt{2013}}$

and,

$$\begin{aligned}
2K &> \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{2011}+\sqrt{2013}} \\
&= \frac{\sqrt{3}-\sqrt{1}}{2} + \frac{\sqrt{5}-\sqrt{3}}{2} + \frac{\sqrt{7}-\sqrt{5}}{2} + \cdots + \frac{\sqrt{2013}-\sqrt{2011}}{2} \\
&= \frac{\sqrt{2013}-1}{2}. \text{ So,} \\
K &> \frac{\sqrt{2013}-1}{4} > \sqrt{120}.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Charles R. Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; Paolo Perfetti, Mathematics Department, U. of Rome, Italy, and David Stone & John Hawkins (jointly), Statesboro, GA.

- 4985: *Proposed by Kenneth Korbin, New York, NY.*

  A Heron triangle is one that has both integer length sides and integer area. Assume Heron triangle ABC is such that $\angle B = 2\angle A$ and with (a,b,c)=1.

  PartI : Find the dimensions of the triangle if side a = 25.
  PartII : Find the dimensions of the triangle if 100 < a < 200.

Solution by Brian D. Beasely, Clinton, SC.

Using the Law of Sines, we obtain
$$\frac{\sin A}{a} = \frac{\sin(2A)}{b} = \frac{\sin(180^\circ - 3A)}{c} = \frac{\sin(3A)}{c},$$
where $\angle B = 2\angle A$ forces $0^\circ < A < 60^\circ$. Since $\sin(2A) = 2\sin A\cos A$ and $\sin(3A) = 3\sin A - 4\sin^3 A$, we have $b = 2a\cos A$ and $c = a(3-4\sin^2 A)$. In particular, $a < b < 2a$, and using $A = \cos^{-1}\left(\dfrac{b}{2a}\right)$ implies
$$c = 3a - 4a\left(1 - \left(\frac{b}{2a}\right)^2\right) = -a + \frac{b^2}{a}.$$

Then a divides b^2, so we claim that a must be a perfect square: Otherwise, if a prime p divides a but p^2 does not, then p divides b^2; thus p divides b, yet p^2 does not divide a, which would imply that p divides b^2/a and hence p divides c, a contradiction of $(a,b,c)=1$.

Next, we note that the area of the triangle is $(1/2)bc\sin A$, which becomes
$$\frac{b(b+a)(b-a)}{2a}\sqrt{1-\left(\frac{b}{2a}\right)^2} = \frac{b(b+a)(b-a)}{4a^2}\sqrt{4a^2-b^2}.$$

I. Let $a = 25$. Then $25 < b < 50$ and $c = -25 + b^2/25$, so 5 divides b. Checking $b \in \{30, 35, 40, 45\}$ yields two solutions for which the area of the triangle is an integer:

$$(a, b, c) = (25, 30, 11) \text{ with area} = 132; \qquad (a, b, c) = (25, 40, 39) \text{ with area} = 468.$$

II. Let $100 < a < 200$. Then $a \in \{121, 144, 169, 196\}$.

If $a = 121$, then 11 divides b, so $b = 11d$ for $d \in \{12, 13, \ldots, 21\}$. Since the area formula requires $4a^2 - b^2 = 11^2(22^2 - d^2)$ to be a perfect square, we check that no such d produces a perfect square $22^2 - d^2$.

If $a = 144$, then 12 divides b, so $b = 12d$ for $d \in \{13, 14, \ldots, 23\}$. Since $4a^2 - b^2 = 12^2(24^2 - d^2)$ must be a perfect square, we check that no such d produces a perfect square $24^2 - d^2$.

If $a = 169$, then 13 divides b, so $b = 13d$ for $d \in \{14, 15, \ldots, 25\}$. Since $4a^2 - b^2 = 13^2(26^2 - d^2)$ must be a perfect square, we check that the only such d to produce a perfect square $26^2 - d^2$ is $d = 24$. This yields the triangle

$$(a, b, c) = (169, 312, 407) \text{ with area } 24{,}420.$$

If $a = 196$, then 14 divides b, so $b = 14d$ for $d \in \{15, 16, \ldots, 27\}$. Since $4a^2 - b^2 = 14^2(28^2 - d^2)$ must be a perfect square, we check that no such d produces a perfect square $28^2 - d^2$.

Comment: David Stone and John Hawkins of Statesboro, GA conjectured that in order to meet the conditions of the problem, a must equal p^2, where p is an odd prime congruent to 1 mod 4. With $p = m^2 + n^2$, there are one or two triangles, according to the ratio of m and n. If $\sqrt{3}n < m < (2 + \sqrt{3})n$, there are two solutions; if $m > (2 + \sqrt{3})n$, there is one solution; and if $n < m < \sqrt{3}n$, there is one solution.

Also solved by M.N. Deshpande, Nagpur, India; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; David Stone & John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4986: *Michael Brozinsky, Central Islip, NY.*

  Show that if $0 < a < b$ and $c > 0$, that

$$\sqrt{(a+c)^2 + d^2} + \sqrt{(b-c)^2 + d^2} \le \sqrt{(a-c)^2 + d^2} + \sqrt{(b+c)^2 + d^2}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

Squaring both sides and simplifying, we reduce the desired inequality to

$$2c(b-a) + \sqrt{(a-c)^2 + d^2}\sqrt{(b+c)^2 + d^2} \ge \sqrt{(a+c)^2 + d^2}\sqrt{b-c)^2 + d^2}.$$

Squaring the last inequality and simplifying we obtain

$$\sqrt{(a-c)^2 + d^2}\sqrt{(b+c)^2 + d^2} \ge ab + ac - bc - c^2 - d^2. \qquad (1)$$

If $ab + ac - bc - c^2 - d^2 \le 0$, (1) is certainly true. If $ab + ac - bc - c^2 - d^2 > 0$, we square both sides of (1) and the resulting inequality simplifies to the trivial inequality $(a+b)^2 d^2 \ge 0$. This completes the solution.

Solution 2 by Paolo Perfetti, Mathematics Department, U. of Rome, Italy.
The inequality is

$$\sqrt{(b-c)^2+d^2}-\sqrt{(a-c)^2+d^2}\le\sqrt{(b+c)^2+d^2}-\sqrt{(a+c)^2+d^2}.$$

Defining $f(x)=\sqrt{(b+x)^2+d^2}-\sqrt{(a+x)^2+d^2},\ -c\le x\le c$, the inequality becomes $f(-c)\le f(c)$ so we prove that

$$f'(x)=\frac{b+x}{\sqrt{(b+x)^2+d^2}}-\frac{a+x}{\sqrt{(a+x)^2+d^2}}>0.$$

There are three possibilities: 1) $b+x>a+x\ge 0$, 2) $a+x<b+x<0$, and 3) $b+x>0,\ a+x<0$. It is evident that 3) implies $f'(x)>0$. With the condition 1), after squaring, we obtain

$$\begin{aligned}(b+x)^2((a+x)^2+d^2) &> (a+x)^2((b+x)^2+d^2) \text{ or}\\ (b+x)^2 &> (a+x)^2\text{which is true.}\end{aligned}$$

As for 2) we have

$$\begin{aligned}\frac{|b+x|}{\sqrt{(b+x)^2+d^2}} &< \frac{|a+x|}{\sqrt{(a+x)^2+d^2}} \quad\text{or}\\ (b+x)^2 &< (a+x)^2\end{aligned}$$

and making the square root $-(b+x)<-(a+x)$ which is true as well.

Also solved by Angelo State University Problem Solving Group, San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY, and the proposer.

- 4987: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a,b,c be the sides of a triangle ABC with area S. Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\le 64S^3\csc 2A\csc 2B\csc 2C.$$

Solution by José Luis Díaz-Barrero, the proposer.

Let $A'\in BC$ be the foot of h_a. We have,

$$h_a=c\sin B\quad\text{and}\quad BA'=c\cos B \tag{1}$$

and

$$h_a=b\sin C\quad\text{and}\quad A'C=b\cos C \tag{2}$$

Multiplying up and adding the resulting expressions yields

$$h_a(BA'+A'C)=\frac{b^2\sin 2C}{2}+\frac{c^2\sin 2B}{2}$$

or

$$c^2\sin 2B+b^2\sin 2C=4S$$

Likewise, we have

$$a^2\sin 2C + c^2\sin 2A = 4S,$$
$$a^2\sin 2B + b^2\sin 2A = 4S.$$

Adding up the above expressions yields

$$(a^2+b^2)\sin 2C + (b^2+c^2)\sin 2A + (c^2+a^2)\sin 2B = 12S$$

Applying the AM-GM inequality yields

$$\sqrt[3]{(a^2+b^2)\sin 2C(b^2+c^2)\sin 2A(c^2+a^2)\sin 2B} \le 4S$$

from which the statement follows. Equality holds when $\triangle ABC$ is equilateral and we are done.

- 4988: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real solutions of the equation

$$3^{x^2-x-z} + 3^{y^2-y-x} + 3^{z^2-z-y} = 1.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak, and Paula Koca (jointly), San Angelo, TX.

By the Arithmetic - Geometric Mean Inequality,

$$\begin{aligned} 1 &= 3^{x^2-x-z} + 3^{y^2-y-x} + 3^{z^2-z-y} \\ &\ge 3\sqrt[3]{3^{x^2-2x+y^2-2y+z^2-2z}} \\ &= \sqrt[3]{3^{(x-1)^2+(y-1)^2+(z-1)^2}} \end{aligned}$$

and hence,

$$3^{(x-1)^2+(y-1)^2+(z-1)^2} \le 1.$$

It follows that

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 0,$$

i.e.,

$$x = y = z = 1.$$

Since it is easily checked that these values satisfy the original equation, the solution is complete.

Also solved by Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, U. of Rome, Italy; Boris Rays, Chesapeake, VA, and the proposer.

- 4989: *Proposed by Tom Leong, Scotrun, PA.*

The numbers $1, 2, 3, \cdots, 2n$ are randomly arranged onto $2n$ distinct points on a circle. For a chord joining two of these points, define its *value* to be the absolute value of the difference of the numbers on its endpoints. Show that we can connect the $2n$ points in disjoint pairs with n chords such that no two chords intersect inside the circle and the sum of the values of the chords is exactly n^2.

Solution 1 by Harry Sedinger, St. Bonaventure, NY.

First we show by induction that if there are n red points and n blue points (all distinct) on the circle, then there exist n nonintersecting chords, each connecting a read point an a blue point (with each point being used exactly once). This is obvious for $n = 1$.
Assume it is true for n and consider the case for $n + 1$. There obviously is a pair of adjacent points (no other points between them on one arc), one read and one blue. Clearly they can be connected by a chord which does not intersect any chord connecting two other points. Removing this chord and the two end points then reduces the problem to the case for n, which can be done according to the induction hypothesis. The desired result is then true for $n + 1$ and by induction true for all n.
Now for the given problem, color the points numbered $1, 2, \cdots, n$ red and color the ones numbered $n + 1, n + 2, \cdots, 2n$ blue. From above there exists n nonintersecting chords and the sum of their values is

$$\sum_{k=n+1}^{2n} k - \sum_{k=1}^{n} k = \sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2.$$

Solution 2 by Kenneth Korbin, New York, NY.

Arrange the numbers $1, 2, 3, \cdots, 2n$ randomly on points of a circle. Place a red checker on each point from 1 through n. Let

$$\sum R = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Place a black checker on each point numbered from $n + 1$ through $2n$. Let

$$\sum B = (n+1) + (n+2) + \cdots + (2n) = n^2 + \frac{n(n+1)}{2}.$$

Remove a pair of adjacent checkers that have different colors. Connect the two points with a chord. The value of this chord is $(B_1 - R_1)$.

Remove another pair of adjacent checkers with different colors. The chord between these two points will have value $(B_2 - R_2)$.

Continue this procedure until the last checkers are removed and the last chord will have value $(B_n - R_n)$.

The sum of the value of these n chords is

$$(B_1 - R_1) + (B_2 - R_2) + \cdots + (B_n - R_n) = \sum B - \sum R = n^2.$$

Also solved by N.J. Kuenzi, Oshkosh, WI; David Stone & John Hawkins (jointly), Statesboro,GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
May 15, 2008

- 5008: *Proposed by Kenneth Korbin, New York, NY.*

  Given isosceles trapezoid $ABCD$ with $\angle ABD = 60^o$, and with legs $\overline{BC} = \overline{AD} = 31$.

  Find the perimeter of the trapezoid if each of the bases has positive integer length with $\overline{AB} > \overline{CD}$.

- 5009: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with a cevian $\overline{CD}$ such that $\overline{AD}$ and $\overline{BD}$ have integer lengths. Find the side of the triangle $\overline{AB}$ if $\overline{CD} = 1729$ and if $(\overline{AB}, 1729) = 1$.

- 5010: *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be real numbers such that $0 < \alpha \le \beta \le \gamma < \pi/2$. Prove that

  $$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)(\cos\alpha + \cos\beta + \cos\gamma)} \le \frac{2}{3}.$$

- 5011: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\ge 0}$ be the sequence defined by $a_0 = a_1 = 2$ and for $n \ge 2,\ a_n = 2a_{n-1} - \dfrac{1}{2}a_{n-2}$. Prove that

  $$2^p a_{p+q} + a_{q-p} = 2^p a_p a_q$$

  where $p \le q$ are nonnegative integers.

- 5012 *Richard L. Francis, Cape Girardeau, MO.*

  Is the incenter of a triangle the same as the incenter of its Morley triangle?

- 5013: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Let $k \geq 2$ be a natural number. Find the sum

$$\sum_{n_1,n_2,\cdots,n_k \geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k}.$$

Solutions

- 4990: *Proposed by Kenneth Korbin, New York, NY.*

  Solve

$$40x + 42\sqrt{1-x^2} = 29\sqrt{1+x} + 29\sqrt{1-x}$$

with $0 < x < 1$.

Solution by Boris Rays, Chesapeake, VA.

Let $x = \cos\alpha$, where $\alpha \in (0, \pi/2)$. Then

$$\begin{aligned}
40\cos\alpha + 42\sqrt{1-\cos^2\alpha} &= 29\sqrt{1+\cos} + 29\sqrt{1-\cos\alpha} \\
&= 29\sqrt{2}\left(\sqrt{\frac{1+\cos\alpha}{2}} + \sqrt{\frac{1-\cos\alpha}{2}}\right) \\
&= 29\cdot\frac{2}{\sqrt{2}}\left(\sqrt{\frac{1+\cos\alpha}{2}} + \sqrt{\frac{1-\cos\alpha}{2}}\right) \\
&= 29\cdot 2\left(\frac{1}{\sqrt{2}}\cos\frac{\alpha}{2} + \frac{1}{\sqrt{2}}\sin\frac{\alpha}{2}\right) \\
&= 58\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2} + \sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right) = 58\cos(\frac{\pi}{4} - \frac{\alpha}{2}). \quad \text{Therefore,}
\end{aligned}$$

$$\begin{aligned}
40\cos\alpha + 42\sin\alpha &= 58\cos(\frac{\pi}{4} - \frac{\alpha}{2}). \\
\frac{40}{58}\cos\alpha + \frac{42}{58}\sin\alpha &= \cos(\frac{\pi}{4} - \frac{\alpha}{2}) \\
\frac{20}{29}\cos\alpha + \frac{21}{29}\sin\alpha &= \cos(\frac{\pi}{4} - \frac{\alpha}{2}).
\end{aligned}$$

Let $\cos\alpha_0 = \dfrac{20}{29}$. Then $\sin\alpha_0 = \sqrt{1 - \left(\dfrac{20}{29}\right)^2} = \dfrac{21}{29}$.

$$\cos\alpha_0\cos\alpha + \sin\alpha_0\sin\alpha = \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$$\cos(\alpha_0 - \alpha) = \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

Therefore we obtain from the above,

$$\begin{aligned}
1) \quad \alpha_0 - \alpha_1 &= \frac{\pi}{4} - \frac{\alpha_1}{2} \\
\alpha_1 &= 2\alpha_0 - \frac{\pi}{2}, \text{ where } \alpha_0 = \arccos\frac{20}{29}. \\
2) \quad \alpha_0 - \alpha_2 &= -\left(\frac{\pi}{4} - \frac{\alpha_2}{2}\right) = \frac{\alpha_2}{2} - \frac{\pi}{4} \\
\frac{3}{2}\alpha_2 &= \alpha_0 + \frac{\pi}{4} \\
\alpha_2 &= \frac{2}{3}\alpha_0 + \frac{\pi}{6}, \text{ where } \alpha_0 = \arccos\frac{20}{29}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
1) \quad x_1 &= \cos\left(2\alpha_0 - \frac{\pi}{2}\right) = \cos(2\alpha_0)\cos\frac{\pi}{2} + \sin(2\alpha_0)\sin\frac{\pi}{2} \\
&= 2\sin\alpha_0\cos\alpha_0 \cdot 1 = 2\cdot\frac{21}{29}\cdot\frac{20}{29} = \frac{840}{841}. \\
2) \quad x_2 &= \cos\left(\frac{2}{3}\alpha_0 + \frac{\pi}{6}\right) = \cos\left(\frac{2}{3}\arccos\left(\frac{20}{29}\right) + \frac{\pi}{6}\right).
\end{aligned}$$

The solution is:

$$x_1 = \frac{840}{841} \qquad x_2 = \cos\left(\frac{2}{3}\arccos\left(\frac{20}{29}\right) + \frac{\pi}{6}\right).$$

Remark: This solution is an adaptation of the solution to SSM problem 4966, which is an adaptation of the solution on pages 13-14 of *Mathematical Miniatures* by Savchev and Andreescu.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; José Hernández Santiago (student at UTM), Oaxaca, México; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Paolo Perfetti, Math Dept., U. of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4991: *Proposed by Kenneth Korbin, New York, NY.*

Find six triples of positive integers (a, b, c) such that

$$\frac{9}{a} + \frac{a}{b} + \frac{b}{9} = c.$$

Solution by David Stone and John Hawkins, Statesboro, GA,(with comments by editor).

David Stone and John Hawkins submitted a six page densely packed analysis of the problem, but it is too long to include here. Listed below is their solution and the gist of their analysis as to how they solved it. (Interested readers may obtain their full analysis by writing to David at <dstone@georgiasouthern.edu> or to me at <eisenbt@013.net>. Others who solved the problem programmed a computer.

David and John began by listing what they believe to be all ten solutions to the problem.

$$\begin{pmatrix} a & b & c \\ 2 & 12 & 6 \\ 9 & 9 & 3 \\ 14 & 588 & 66 \\ 18 & 36 & 5 \\ 54 & 12 & 6 \\ 162 & 4 & 41 \\ 378 & 588 & 66 \\ 405 & 25 & 19 \\ 11826 & 21316 & 2369 \\ 29565 & 133225 & 14803 \end{pmatrix}$$

The analysis in their words:
Rewriting the equation, we seek positive integer solutions to

$$(1) \qquad 81b + 9a^2 + ab^2 = 9abc.$$

<u>Theorem</u>. A solution must have the form $a = 3^i A, b = 3^j A^2$, where $(A, 3) = 1,\ i, j \geq 0$. At least one of i, j must be ≥ 1.

<u>Proof</u>. From equation (1), we see that 9 divides all terms but ab^2, so 9 divides ab^2, so 3 divides a or b so at least one of i, j must be ≥ 1.

Also from equation (1), it is clear that if p is a prime different from 3, then p divides a if and only if p divides b.

Suppose p is such a prime and $a = 3^i p^m C, b = 3^j p^n D$, where $m, n \geq 1$, and C and D are not divisible by 3 or p. Then equation (1) becomes

$$81\left(3^j p^n D\right) + 9\left(3^i p^m C\right)^2 + \left(3^i p^m C\right)\left(3^j p^n D\right)^2 = 9\left(3^i p^m C\right)\left(3^j p^n D\right)c,$$

or

$$(\#) \;\; 3^{j+4} p^n D + 3^{2i+2} p^{2m} C^2 + 3^{i+2j} p^{m+2n} C D^2 = 3^{i+j+2} p^{m+n} C D c.$$

If $n < 2m$, we can divide equation (#) by p^n to obtain

$$3^{j+4} D + 3^{2i+2} p^{2m-n} C^2 + 3^{i+2j} p^{m+n} C D^2 = 3^{i+j+2} p^m C D c.$$

But then p divides each term after the first, so p divides $3^{j+4}D$, which is impossible.

If $n > 2m$, we can divide through equation (#) by p^{2m} to obtain

$$\begin{aligned} 3^{j+4} p^{n-2m} D + 3^{2i+2} C^2 + 3^{i+2j} p^{2n-m} C D^2 &= 3^{i+j+2} p^{n-m} C D c \\ 81 p^{m-2n} D + 9C^2 + p^m C D^2 &= 9 p^{m-n} C D c. \end{aligned}$$

Noting that $2n > 4m > m$ and $n > 2m > m$, we see that p divides each term except $3^{2i+2}C^2$, so p divides $3^{2i+2}C^2$, which is impossible.

Therefore $n = 2m$.

That is, a and b have the same prime divisors, and in b, the power on each such prime is twice the corresponding power in a; therefore, in b, the product of all divisors other than 3 is the square of the analogous product in a. So the proof is complete.

They then substituted this result into equation (1) obtaining

$$81\Big(3^j A^2\Big) + 9\Big(3^i A^2\Big) + \Big(3^i A\Big)\Big(3^j A^2\Big)^2 = 9\Big(3^i A\Big)\Big(3^j A^2\Big)c,$$

or

$$(2) \quad \Big(2^{j+4} + 3^{2i+2}\Big) + 3^{i+2j}A^3 = 3^{i+j+2}Ac$$

and started looking for values of i, j, A and c satisfying this equation.

Analyzing the cases **(1)** where 3 divides b but not a; **(2)** where 3 divides a but not b; and **(3)** where 3 divides a and b led to the solutions listed above.

They ended their submission with comments about the patterns they observed in solving analogous equations of the form $\dfrac{N}{a} + b + \dfrac{c}{N} = c$ for various integral values of N.

Also solved by Charles Ashbacher, Marion, IA; Britton Stamper (student at Saint George's School), Spokane, WA, and the proposer.

- 4992: *Proposed by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX.*

A closed circular cone has integral values for its height and base radius. Find all possible values for its dimensions if its volume V and its total area (including its circular base) A satisfy $V = 2A$.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

$$\begin{aligned} \frac{1}{3}\pi r^2 h &= 2(\pi r^2 + \pi r\sqrt{r^2+h^2}) \text{ or} \\ rh &= 6r + 6\sqrt{r^2+h^2}. \end{aligned}$$

Squaring and simplifying gives $r^2 = 36\dfrac{h}{h-12}$. Therefore, $\dfrac{h}{h-12}$ is a square, and $\dfrac{h}{h-12} \in \{1, 4, 9, 16, \ldots\}$. Note that $f(h) = \dfrac{h}{h-12}$ is a decreasing function of h for $h > 12$ and that $h(16) = 4$. Note also that $f(13)$, $f(14)$ and $f(15)$ are not squares of integers. Therefore $(h, r) = (16, 24)$ is the only solution.

Also solved by Paul M.Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Britton Stamper (student at Saint George's School), Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4993: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real solutions of the equation

$$126x^7 - 127x^6 + 1 = 0.$$

Solution by N. J. Kuenzi, Oshkosh, WI.

Both 1 and 1/2 are easily seen to be positive rational roots of the given equation. So $(x-1)$ and $(2x-1)$ are both factors of the polynomial $126x^7 - 127x^6 + 1$. Factoring yields

$$126x^2 - 127x^6 + 1 = (x-1)(2x-1)(63x^5 + 31x^4 + 15c^3 + 7x^2 + 3x + 1).$$

The equation $(63x^5 + 31x^4 + 15c^3 + 7x^2 + 3x + 1)$ does not have any rational roots (Rational Roots Theorem) nor any positive real roots (Descartes' Rule of Signs).
Using numerical techniques one can find that -0.420834167 is the approximate value of a real root.
The four other roots are complex with approximate values:

$$0.1956354060 + 0.4093830251i \qquad 0.1956354060 - 0.4093830251i$$

$$-0.2312499936 + 0.3601917120i \qquad -0.2312499936 - 0.3601917120i$$

So the real solutions of the equation $126x^7 - 127x^6 + 1 = 0$ are 1, 1/2 and -0.420834167.

Also solved by Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro GA, and the proposer.

- 4994: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be three nonzero complex numbers lying on the circle $C = \{z \in \mathbf{C} : |\mathrm{z}| = \mathrm{r}\}$. Prove that the roots of the equation $az^2 + bz + c = 0$ lie in the ring shaped region $D = \left\{ z \in \mathbf{C} : \dfrac{1-\sqrt{5}}{2} \le |\mathrm{z}| \le \dfrac{1+\sqrt{5}}{2} \right\}$.

Solution by Kee-Wai Lau, Hong Kong, China.

By rewriting the equation as $az^2 = -bz - c$, we obtain

$$\begin{aligned} |a||z|^2 &= |az^2| = |bz + c| \le |b||z| + |c| \text{ or } |z|^2 - |z| - 1 \le 0 \\ &\text{or } \left(|z| + \frac{\sqrt{5}-1}{2}\right)\left(|z| - \frac{\sqrt{5}+1}{2}\right) \le 0 \text{ so that } |z| \le \frac{1+\sqrt{5}}{2}. \end{aligned}$$

By rewriting the equation as $c = -az^2 - bz$, we obtain

$$\begin{aligned} |c| &= |-az^2 - bz| \le |a||z|^2 + |b||z| \text{ or } |z|^2 + |z| - 1 \ge 0 \\ &\text{or } \left(|z| + \frac{\sqrt{5}+1}{2}\right)\left(|z| - \frac{\sqrt{5}-1}{2}\right) \ge 0 \text{ so that } |z| \ge \frac{\sqrt{5}-1}{2}. \end{aligned}$$

This finishes the solution.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Russell Euler and Jawad Sadek (jointly), Maryville, MO; Boris Rays, Chesapeake, VA; José Hernández Santiago (student at UTM) Oaxaca, México; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **4995:** *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India.*

Let A be a triangular array $a_{i,j}$ where $i = 1, 2, \cdots,$ and $j = 0, 1, 2, \cdots, i$. Let

$$a_{1,0} = 1,\ a_{1,1} = 2,\ \text{and } a_{i,0} = T(i+1) - 2 \text{ for } i = 2, 3, 4, \cdots,$$

where $T(i+1) = (i+1)(i+2)/2$, the usual triangular numbers. Furthermore, let $a_{i,j+1} - a_{i,j} = j + 1$ for all j. Thus, the array will look like this:

$$\begin{array}{c}
1 \quad 2 \\
4 \quad 5 \quad 7 \\
8 \quad 9 \quad 11 \quad 14 \\
13 \quad 14 \quad 16 \quad 19 \quad 23 \\
19 \quad 20 \quad 22 \quad 25 \quad 29 \quad 34
\end{array}$$

Show that for every pair (i, j), $4a_{i,j} + 9$ is the sum of two perfect squares.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

If we allow $T(0) = 0$, then for $i \geq 1$ and $j = 0, 1, \ldots, i$, it's clear from the definition of $a_{i,j}$ that

$$\begin{aligned}
a_{i,j} &= a_{i,0} + T(j) \\
&= T(i+1) - 2 + T(j) \\
&= \frac{i^2 + 3i - 2 + j^2 + j}{2}.
\end{aligned}$$

Therefore, for every pair (i, j),

$$\begin{aligned}
4a_{i,j} + 9 &= 2\left(i^2 + 3i - 2 + j^2 + j\right) + 9 \\
&= 2\left(i^2 + 3i + j^2 + j\right) + 5 \\
&= (i + j + 2)^2 + (i - j + 1)^2.
\end{aligned}$$

Solution 2 by Carl Libis, Kingston, RI.

For every pair $(i, j), 4a(i, j) + 9 = (i - j + 1)^2 + (i + j + 2)^2$ since

$$\begin{aligned}
4a(i,j) + 9 &= 4\left[a(i,0) + \frac{j(j+1)}{2}\right] + 9 = 4\left[\frac{(i+1)(i+2)}{2} - 2 + \frac{j(j+1)}{2}\right] + 9 \\
&= 2(i+1)(i+2) - 8 + 2j(j+1) + 9 \\
&= 2i^2 + 6i + 4 + 2j^2 + 2j + 1 \\
&= (i - j + 1)^2 + (i + j + 2)^2.
\end{aligned}$$

Also solved by Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro GA; José Hernándz Santiago (student at UTM), Oaxaca, México, and the proposers.

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Ted Eisenberg, Section Editor

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June 15, 2008

- 5014: *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with $a = 100, b = 105$, and with equal cevians $\overline{AD}$ and $\overline{BE}$. Find the perimeter of the triangle if $\overline{AE} \cdot \overline{BD} = \overline{CE} \cdot \overline{CD}$.

- 5015: *Proposed by Kenneth Korbin, New York, NY.*

  Part I: Find the value of
  $$\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$
  Part II: Find the value of
  $$\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

- 5016: *Proposed by John Nord, Spokane, WA.*

  Locate a point (p, q) in the Cartesian plane with integral values, such that for any line through (p, q) expressed in the general form $ax + by = c$, the coefficients a, b, c form an arithmetic progression.

- 5017: *Proposed by M.N. Deshpande, Nagpur, India.*

  Let ABC be a triangle such that each angle is less than 90^0. Show that
  $$\frac{a}{c \cdot \sin B} + \frac{1}{\tan A} = \frac{b}{a \cdot \sin C} + \frac{1}{\tan B} = \frac{c}{b \cdot \sin A} + \frac{1}{\tan C}$$

where $a = l(\overline{BC}), b = l(\overline{AC})$, and $c = l(\overline{AB})$.

- 5018: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Write the polynomial $x^{5020} + x^{1004} + 1$ as a product of two polynomials with integer coefficients.

- 5019: *Michael Brozinsky, Central Islip, NY.*

  In a horse race with 10 horses the horse with the number one on its saddle is referred to as the number one horse, and so on for the other numbers. The outcome of the race showed the number one horse did not finish first, the number two horse did not finish second, the number three horse did not finish third and the number four horse did not finish fourth. However, the number five horse did finish fifth. How many possible orders of finish are there for the ten horses assuming no ties?

Solutions

- 4996: *Proposed by Kenneth Korbin, New York, NY.*

  Simplify:

$$\sum_{i=1}^{N} \binom{N}{i} \left(2^{i-1}\right)\left(1+3^{N-i}\right).$$

Solution by José Hernández Santiago, (student, UTM, Oaxaca, México.)

$$\begin{aligned}
\sum_{i=1}^{N} \binom{N}{i} \left(2^{i-1}\right)\left(1+3^{N-i}\right) &= \sum_{i=1}^{N} \binom{N}{i} 2^{i-1} + \sum_{i=1}^{N} \binom{N}{i} 2^{i-1} \cdot 3^{N-i} \\
&= \frac{1}{2}\sum_{i=1}^{N} \binom{N}{i} 2^{i} + \frac{3^N}{2}\sum_{i=1}^{N} \binom{N}{i} \left(\frac{2}{3}\right)^{i} \\
&= \left(\frac{1}{2}\right)\left(\left(2+1\right)^{N} - 1\right) + \left(\frac{3^N}{2}\right)\left(\left(\frac{2}{3}+1\right)^{N} - 1\right) \\
&= \frac{3^N - 1}{2} + \frac{3^N}{2}\left(\frac{5^N - 3^N}{3^N}\right) \\
&= \frac{(3^N-1)3^N + 3^N(5^N - 3^N)}{2 \cdot 3^N} \\
&= \frac{15^N - 3^N}{2 \cdot 3^N} \\
&= \frac{5^N - 1}{2}
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Central

Islip, NY; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Carl Libis, Kingston, RI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4997: *Proposed by Kenneth Korbin, New York, NY.*

Three different triangles with integer-length sides all have the same perimeter P and all have the same area K.
Find the dimensions of these triangles if $K = 420$.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX.

Let a, b, c be the sides of the triangle and, for convenience, assume that $a \leq b \leq c$. By Heron's Formula,

$$(420)^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right) \tag{1}$$

Since a, b, c are positive integers, it is easily demonstrated that the quantities $(a+b-c)$, $(a-b+c)$, and $(-a+b+c)$ are all odd or all even. By (1), it is clear that in this case, they are all even. Therefore, there are positive integers x, y, z such that $a+b-c = 2x$, $a-b+c = 2y$, and $-a+b+c = 2z$. Then, $a = x+y$, $b = x+z$, $c = y+z$, $a+b+c = 2(x+y+z)$, and $a \leq b \leq c$ implies that $x \leq y \leq z$. With this substitution, (1) becomes

$$(420)^2 = xyz(x+y+z) \tag{2}$$

Since $x \leq y \leq z < x+y+z$, (2) implies that

$$x^4 < xyz(x+y+z) = (420)^2$$

and hence,

$$1 \leq x \leq \left\lfloor \sqrt{420} \right\rfloor = 20,$$

where $\lfloor m \rfloor$ denotes the greatest integer $\leq m$. Therefore, the possible values of x are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20$ (since x must also be a factor of $(420)^2$). Further, for each x, (2) implies that

$$y^3 < yz(x+y+z) = \frac{(420)^2}{x},$$

i.e.,

$$x \leq y \leq \left\lfloor \sqrt[3]{\frac{(420)^2}{x}} \right\rfloor,$$

(and y is a factor of $(420)^2 x$). Once we have assigned values to x and y, (2) becomes

$$z(x+y+z) = \frac{(420)^2}{xy}, \tag{3}$$

which is a quadratic equation in z. If (3) yields an integral solution $\geq y$, we have found a viable solution for x, y, z and hence, for a, b, c also. By finding all such solutions, we can find all Heronian triangles (triangles with integral sides and integral area) whose area is 420. Then, we must find three of these with the same perimeter to complete our solution.

The following two cases illustrate the typical steps encountered in this approach.

Case 1. If $x = 1$ and $y = 18$, (3) becomes

$$z^2 + 19z - 9800 = 0.$$

Since this has no integral solutions, this case does not lead to feasible values for a, b, c.

Case 2. If $x = 2$ and $y = 24$, (3) becomes

$$z^2 + 26z - 3675 = 0,$$

which has $z = 49$ as its only positive integral solution. These values of x, y, z yield $a = 26$, $b = 51$, $c = 73$, and $P = 150$.

The results of our approach are summarized in the following table.

x	y	z	a	b	c	P
1	6	168	7	169	174	350
1	14	105	15	106	119	240
1	20	84	21	85	104	210
1	25	72	26	73	97	196
1	40	49	41	50	89	180
2	24	49	26	51	73	150
2	35	35	37	37	70	144
4	21	35	25	39	56	120
5	9	56	14	61	65	140
5	21	30	26	35	51	112
6	15	35	21	41	50	112
8	21	21	29	29	42	100
9	20	20	29	29	40	98
10	15	24	25	34	39	98
12	12	25	24	37	37	98

Now, it is obvious that the last three entires constitute the solution of this problem.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4998: *Proposed by Jyoti P. Shiwalkar & M.N. Deshpande, Nagpur, India.*

  Let $A = [a_{i,j}]$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$ be a triangular array satisfying the following conditions:

$$\begin{aligned} 1)\ a_{i,1} &= L(i) \text{ for all } i \\ 2)\ a_{i,i} &= i \text{ for all } i \\ 3)\ a_{i,j} &= a_{i-1,j} + a_{i-2,j} + a_{i-1,j-1} - a_{i-2,j-1} \text{ for } 2 \le j \le (i-1). \end{aligned}$$

If $T(i) = \sum_{j=1}^{i} a_{i,j}$ for all $i \ge 2$, then find a closed form for $T(i)$, where $L(i)$ are the Lucas numbers, $L(1) = 1, L(2) = 3$, and $L(i) = L(i-1) + L(i-2)$ for $i \ge 3$.

Solution by Paul M. Harms, North Newton, KS.

Note that $a_{i-2,j}$ is not in the triangular array when $j=i-1$, so we set $a_{i-2,i-1}=0$. From Lucas numbers $a_{i,1}=a_{i-1,1}+a_{i-2,1}$ for $i>2$. For $i>2$,

$$\begin{aligned}
T(i) &= a_{i,1}+a_{i,2}+\cdots+a_{i,i-1}+i \\
&= (a_{i-1,1}+a_{i-2,1})+(a_{i-1,2}+a_{i-2,2}+a_{i-1,1}-a_{i-2,1})+\cdots \\
&\quad +(a_{i-1,i-1}+a_{i-2,i-1}+a_{i-1,i-2}-a_{i-2,i-2})+i.
\end{aligned}$$

Therefore we have

$$(a_{i-1,i-1}+a_{i-2,i-1}+a_{i-1,i-2}-a_{i-2,i-2})=(i-1)+0+a_{i-1,i-2}-(i-2).$$

Note that in $T(i)$ each term of row $(i-2)$ appears twice and subtracts out and each term of row $(i-1)$ except for the last term $(i-1)$, is added to itself. The term $(i-1)$ appears once. If we write the last term, i, of $T(i)$ as $i=(i-1)+1$, then $T(i)=2T(i-1)+1$. The values of the row sums are:

$$\begin{aligned}
T(1) &= 1 \\
T(2) &= 5 \\
T(3) &= 2(5)+1 \\
T(4) &= 2(2(5)+1)+1=2^2(5)+2+1 \\
T(5) &= 2\Big(2[2(5)+1]+1\Big)+1=2^3(5)+2^2+2+1, \text{ and in general} \\
T(i) &= 2^{i-2}(5)+(2^{i-3}+2^{i-4}+\cdots+1) \\
&= 2^{i-2}(5)+(2^{i-2}-1) \\
&= 2^{i-2}(6)-1 \\
&= 2^{i-1}(3) \text{ for } i\geq 2.
\end{aligned}$$

Also solved by Carl Libis, Kingston, RI; N. J. Kuenzi, Oshkosh, WI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4999: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real triplets (x,y,z) such that

$$\begin{aligned}
x+y+z &= 2 \\
2^{x+y^2}+2^{y+z^2}+2^{z+x^2} &= 6\sqrt[9]{2}
\end{aligned}$$

Solution by David E. Manes, Oneonta, NY.

The only real solution is $(x,y,z)=(\frac{2}{3},\frac{2}{3},\frac{2}{3})$. Note that these values do satisfy each of the equations.

By the Arithmetic-Geometric Mean Inequality,

$$6\sqrt[9]{2} = 2^{x+y^2}+2^{y+z^2}+2^{z+x^2}$$

$$\geq \quad 3\sqrt[3]{2^{x+y+z}\cdot 2^{x^2+y^2+z^2}} = 2\cdot 2^{2/3}\sqrt[3]{2^{x^2+y^2+z^2}}.$$

Therefore, $2^{x^2+y^2+z^2} \leq 2^{4/3}$ so that $x^2+y^2+z^2 \leq 4/3 \quad (\mathbf{1})$. Note that

$$\begin{aligned} 4 = (x+y+z)^2 &= x^2+y^2+z^2+2(xy+yz+zx), \text{ so that} \\ x^2+y^2+z^2 &= 4-2(xy+yz+zx). \end{aligned}$$

Substituting in ($\mathbf{1}$) yields the inequality $xy+yz+zx \geq \dfrac{4}{3}$. From $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$ with equality if and only if $x=y=z$, one now obtains the inequalities

$$\frac{4}{3} \geq x^2+y^2+z^2 \geq xy+yz+zx \geq \frac{4}{3}.$$

Hence

$$\begin{aligned} x^2+y^2+z^2 &= xy+yz+zx = \frac{4}{3} \\ (x-y)^2+(y-z)^2+(z-x)^2 &= 0, \text{ and } x=y=z=\frac{2}{3}. \end{aligned}$$

.

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie and Karl Havlak (jointly), San Angelo, TX; Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Math Dept. U. of Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 5000: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Of all the right triangles inscribed in the unit circle, which has the Morley triangle of greatest area?

Solution by Ken Korbin, New York, NY.

Given $\triangle ABC$ with circumradius $R=1$ and with $A+B=C=90^o$.
The side x of the Morley triangle is given by the formula

$$\begin{aligned} x &= 8\cdot R\cdot \sin(\frac{A}{3})\cdot\sin(\frac{B}{3})\cdot\sin(\frac{C}{3}) \\ &= 8\cdot 1\cdot\sin(\frac{A}{3})\cdot\sin(\frac{B}{3})\cdot\frac{1}{2} \\ &= 4\sin(\frac{A}{3})\sin(\frac{B}{3}). \end{aligned}$$

x will have a maximum value if

$$\frac{A}{3}=\frac{B}{3}=\frac{45^o}{3}=15^o.$$

Then,

$$x = 4\sin^2(15^o)$$

$$
\begin{aligned}
&= 4\Big(\frac{1-\cos 30^o}{2}\Big)\\
&= 2-2\cos 30^o\\
&= 2-\sqrt{3}.
\end{aligned}
$$

The area of this Morley triangle is

$$
\begin{aligned}
&\frac{1}{2}\cdot(2-\sqrt{3})^2\cdot\sin 60^o\\
= \ &\frac{1}{2}(7-4\sqrt{3})\cdot\frac{\sqrt{3}}{2}=\frac{7\sqrt{3}-12}{4}.
\end{aligned}
$$

Comment by David Stone and John Hawkins: "It may be the maximum, but it is pretty small!"

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; David Stone an John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5001:** *Proposed by Ovidiu Furdui, Toledo, OH.*

Evaluate:

$$\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx.$$

Solution by Kee-Wai Lau, Hong Kong, China.

We show that $\displaystyle\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx = 2\ln^2 2+\frac{11\pi^2}{6}$.

Denote the integral by I. Replacing x by $1/x$, we obtain

$$
\begin{aligned}
I &= \int_0^\infty \frac{\ln^2\Big((x+1)(2x+1)\Big)}{x^2}dx = \int_0^\infty \frac{\ln^2(x+1)}{x^2}dx+\int_0^\infty\frac{\ln^2(2x+1)}{x^2}dx\\
&\qquad +2\int_0^\infty\frac{\ln(x+1)\ln(2x+1)}{x}dx\\
&= I_1+I_2+2I_3,\ \text{say.}
\end{aligned}
$$

Integrating by parts, we obtain

$$I_1=\int_0^\infty \ln^2(x+1)d(\frac{-1}{x}) = 2\int_0^\infty\frac{\ln(x+1)}{x(x+1)}dx = 2\int_1^\infty\frac{\ln x}{x(x-1)}dx.$$

Replacing x by $/(1-x)$,we obtain $I_1=-2\displaystyle\int_0^1\frac{\ln(1-x)}{x}dx = 2\sum_{n=1}^\infty\frac{1}{n^2}=\frac{\pi^2}{3}$.

Replacing x by $x/2$ in I_2, we see that $I_2=2I_1=\dfrac{2\pi^2}{3}$. Next note that

$$I_3=\int_0^\infty\ln(x+1)\ln(2x+1)d(\frac{-1}{x}) = \int_0^\infty\frac{\ln(2x+1)}{x(x+1)}dx+2\int_0^\infty\frac{\ln(x+1)}{x(2x+1)}dx = J_1+2J_2,\ \text{say.}$$

Replacing x by $x/2$, then x by $x-1$ and then x by $1/x$, we have

$$\begin{aligned} J_1 = 2\int_0^\infty \frac{\ln(x+1)}{x(x+2)}dx &= 2\int_1^\infty \frac{\ln x}{(x-1)(x+1)}dx \\ &= -2\int_0^1 \frac{\ln x}{(1-x)(1+x)}dx = -\int_0^1 \ln x\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx. \end{aligned}$$

Integrating by parts, we have,

$$J_1 = \int_0^1 \frac{-\ln(1-x)+\ln(1+x)}{x}dx = \sum_{n=1}^\infty \frac{1+(-1)^{n-1}}{n^2} = \frac{\pi^2}{6}+\frac{\pi^2}{12} = \frac{\pi^2}{4}.$$

We now evaluate J_2. Replacing $x+1$ by x and then x by $1/x$, we have

$$J_2 = \int_1^\infty \frac{\ln x}{(x-1)(2x-1)}dx = -\int_0^1 \frac{\ln x}{(1-x)(2-x)}dx = -\int_0^1 \frac{\ln x}{1-x} + \int_0^1 \frac{\ln x}{2-x}dx = \frac{\pi^2}{6} + K, \text{ say.}$$

Replacing x by $1-x$

$$\begin{aligned} K = \int_0^1 \frac{\ln(1-x)}{1+x}dx &= \int_0^1 \frac{\ln(1+x)}{1+x}dx + \int_0^1 \frac{\ln(1-x)-\ln(1+x)}{1+x}dx \\ &= \frac{1}{2}\ln^2 2 + \int_0^1 \frac{\ln\left(\dfrac{1-x}{1+x}\right)}{1+x}dx. \end{aligned}$$

By putting $y = \dfrac{1-x}{1+x}$, we see that the last integral reduces to $\displaystyle\int_0^1 \frac{\ln y}{1+y}dy = -\frac{\pi^2}{12}$.

Hence, $K = \dfrac{1}{2}\ln^2 2 - \dfrac{\pi^2}{12}$, $J_2 = \dfrac{1}{2}\ln^2 2 + \dfrac{\pi^2}{12}$, $I_3 = \ln^2 2 + \dfrac{5\pi^2}{12}$ and finally

$$I = I_1 + I_2 + 2I_3 = \frac{\pi^2}{3} + \frac{2\pi^2}{3} + 2\left(\ln^2 2 + \frac{5\pi^2}{12}\right) = 2\ln^2 2 + \frac{11\pi^2}{6} \text{ as desired.}$$

Also solved by Paolo Perfetti, Math. Dept., U. of Rome, Italy; Worapol Rattanapan (student at Montfort College (high school)), Chiang Mai, Thailand, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before September 15, 2008

- 5020: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\begin{cases} x^7 - 13y = 21 \\ 13x - y^7 = 21 \end{cases}$$

- 5021: *Proposed by Kenneth Korbin, New York, NY.*

  Given

$$\frac{x+x^2}{1-34x+x^2} = x + 35x^2 + \cdots + a_n x^n + \cdots$$

  Find an explicit formula for a_n.

- 5022: *Proposed by Michael Brozinsky, Central Islip, NY .*

  Show that

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right)$$

  is proportional to $\sin(x)$.

- 5023: *Proposed by M.N. Deshpande, Nagpur, India.*

  Let $A_1A_2A_3\cdots A_n$ be a regular n-gon ($n \geq 4$) whose sides are of unit length. From A_k draw L_k parallel to $\mathrm{A}_{k+1}A_{k+2}$ and let L_k meet L_{k+1} at T_k. Then we have a "necklace" of congruent isosceles triangles bordering $A_1A_2A_3\cdots A_n$ on the inside boundary. Find the total area of this necklace of triangles.

- 5024: *Proposed by José Luis Díaz-Barrero and Josep Rubió-Masseqú, Barcelona, Spain.*

  Find all real solutions to the equation

$$\sqrt{1+\sqrt{1-x}} - 2\sqrt{1-\sqrt{1-x}} = \sqrt[4]{x}.$$

- 5025: *Ovidiu Furdui, Toledo, OH.*

  Calculate the double integral

$$\int_0^1 \int_0^1 \{x-y\} dxdy,$$

  where $\{a\} = a - [a]$ denotes the fractional part of a.

Solutions

- 5002: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon with sides $3x, 3x, 3x, 5x, 5x$ and $5x$ is inscribed in a unit circle. Find the value of x.

  Solution by David E. Manes, Oneonta, NY.

  The value of x is $\dfrac{\sqrt{3}}{7}$.

  Note that each inscribed side of the hexagon subtends an angle at the center of the circle that is independent of its position in the circle The sides are subject to the constraint that the sum of the angles subtended at the center equals 360^o. Therefore the sides of the hexagon can be permuted from $3x, 3x, 3x, 5x, 5x, 5x$ to $3x, 5x, 3x, 5x, 3x, 5x$. In problem **4974** : (December 2007, Korbin, Lau) it is shown that the circumradius r is then given by

$$r = \sqrt{\frac{(3x)^2 + (5x)^2 + (3x)(5x)}{3}}.$$

  With $r = 1$, one obtains $x = \dfrac{\sqrt{3}}{7}$.

  Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; John Boncek, Montgomery, AL; M.N. Deshpande, Nagpur, India; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student at St George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Minerva P. Harwell (student at Auburn University), Montgomery, AL; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; Amanda Miller (student at St. George's School), Spokane, WA; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5003: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\sqrt[3]{x+\sqrt{x^2-1}} + \sqrt[3]{x-\sqrt{x^2-1}} \quad = \quad \frac{7}{2} \text{ and}$$

$$\sqrt[3]{y+\sqrt{y^2-1}}+\sqrt[3]{y-\sqrt{y^2-1}} \;=\; \sqrt{10}$$

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX.

Let $A=\sqrt[3]{x+\sqrt{x^2-1}}$ and $B=\sqrt[3]{x-\sqrt{x^2-1}}$. Note that

$$\begin{aligned} A^3+B^3 &= 2x \quad \text{and} \\ AB &= 1. \end{aligned}$$

Since $A+B=\dfrac{7}{2}$,

$$\begin{aligned} \frac{343}{8} &= (A+B)^3 \\ &= A^3+3A^2B+3AB^2+B^3 \\ &= A^3+B^3+3AB(A+B) \\ &= 2x+\frac{21}{2}. \end{aligned}$$

Thus, $x=\dfrac{259}{16}$.

Similarly,

$$2y+3\sqrt{10}=10\sqrt{10}$$

and, thus, $y=\dfrac{7\sqrt{10}}{2}$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; M.N. Deshpande, Nagpur, India; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Amanda Miller (student at St. George's School), Spokane, WA; John Nord, Spokane, WA; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5004: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a,b,c be nonnegative real numbers. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\geq\frac{\sqrt{ab}}{1+a+b}+\frac{\sqrt{bc}}{1+b+c}+\frac{\sqrt{ac}}{1+c+a}$$

Solution by John Boncek, Montgomery, AL.

We use the arithmetic-geometric inequality: If $x,y\geq 0$, then $x+y\geq 2\sqrt{xy}$. Now

$$\begin{aligned} \frac{a}{1+a} &\geq \frac{a}{1+a+b}, \text{ and} \\ \frac{b}{1+b} &\geq \frac{b}{1+a+b}, \text{ so} \end{aligned}$$

$$\frac{a}{1+a}+\frac{b}{1+b} \quad \geq \quad \frac{a+b}{1+a+b} \geq \frac{2\sqrt{ab}}{1+a+b}.$$

Similarly,

$$\frac{a}{1+a}+\frac{c}{1+c} \geq \frac{2\sqrt{ac}}{1+a+c}, \text{ and}$$

$$\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{2\sqrt{bc}}{1+b+c}\ .$$

Summing up all three inequalities, we obtain

$$2\left(\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\right) \geq \frac{2\sqrt{ab}}{1+a+b}+\frac{2\sqrt{ac}}{1+a+c}+\frac{2\sqrt{bc}}{1+b+c}.$$

Divide both sides of the inequality by 2 to obtain the result.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M.N. Deshpande, Nagpur, India; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; Boris Rays, Chesapeake, VA, and the proposers.

- **5005:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{\sqrt{3}}{2}\Big(a+b+c\Big)^{1/2} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

Since $a+b \geq 2\sqrt{ab} = \dfrac{2}{\sqrt{c}}$ and so on, and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \\
\leq\ & \frac{\sqrt{c}+\sqrt{a}+\sqrt{b}}{2} \\
=\ & \frac{1}{2}\Big((1)\sqrt{a}+(1)\sqrt{b}+(1)\sqrt{c}\Big) \\
\leq\ & \frac{1}{2}\sqrt{1+1+1}\sqrt{a+b+c} \\
=\ & \frac{\sqrt{3}}{2}\Big(a+b+c)^{1/2}
\end{aligned}$$

as required.

Solution 2 by Charles McCracken, Dayton, OH.

Suppose a=b=c=1. Then the original inequality reduces to $\frac{3}{2} \geq \frac{3}{2}$ which is certainly true.

Let L represent the left side of the original inequality and let R represent the right side. Allow a, b, and c to vary and take partial derivatives.

$$\frac{\partial L}{\partial a} = \frac{\sqrt{3}}{2} \cdot \frac{1}{2}\left(a+b+c\right)^{-1/2} > 0. \text{ Similarly, } \frac{\partial L}{\partial b} > 0 \text{ and } \frac{\partial \mathrm{L}}{\partial c} > 0.$$

$$\frac{\partial R}{\partial a} = -(a+b)^{-2} - c(a+b)^{-2} < 0. \text{ Similarly, } \frac{\partial R}{\partial b} < 0 \text{ and } \frac{\partial R}{\partial c} < 0.$$

So any change in a, b or c results in an increase in L and a decrease in R so that L is always greater than R.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti (Department of Mathematics, University of Rome), Italy, and the proposer.

- 5006: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Find the sum

$$\sum_{k=2}^{\infty}(-1)^k \ln\left(1-\frac{1}{k^2}\right).$$

Solution 1 by Paul M. Harms, North Newton, KS.

Using $\ln\left(1-\frac{1}{k^2}\right) = \ln\left(\frac{k-1}{k}\right) + \ln\left(\frac{k+1}{k}\right)$, the summation is

$$\begin{aligned}
&\left(\ln\frac{1}{2}+\ln\frac{3}{2}\right) - \left(\ln\frac{2}{3}+\ln\frac{4}{3}\right) + \left(\ln\frac{3}{4}+\ln\frac{5}{4}\right) - \ln\left(\frac{4}{5}+\ln\frac{6}{5}\right) + \cdots \\
= \; & \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)^2 + \ln\left(\frac{3}{4}\right)^2 + \ln\left(\frac{5}{4}\right)^2 + \cdots \\
= \; & \ln\left(\frac{1}{2}\right) + 2\Big[\ln\left(\frac{3}{2}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{5}{6}\right) + \cdots\Big] \\
= \; & \ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\cdots.
\end{aligned}$$

Wallis' product for $\frac{\pi}{2}$ is

$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\left(\frac{6}{7}\right)\cdots.$$

Dividing both sides by 2 and taking the reciprocal yields

$$\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{7}{8}\right)\cdots.$$

The summation in the problem is then

$$\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{4}{\pi}\right) = \ln\left[\left(\frac{1}{2}\right)\left(\frac{16}{\pi^2}\right)\right] = \ln\left(\frac{8}{\pi^2}\right).$$

Solution 2 by Kee-Wai Lau, Hong Kong, China.

It can be proved readily by induction that for positive intergers n,

$$\sum_{k=2}^{2n}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 4\bigg(\ln((2n)!) - 2\ln(n!)\bigg) + \ln n + \ln(2n+1) - 2(4n-1)\ln 2.$$

By using the Stirling approximation $\ln(n!) = n\ln n - n + \dfrac{1}{2}\ln(2\pi n) + O\left(\dfrac{1}{n}\right)$ as $n \to \infty$, we obtain

$$\ln((2n)!) - 2\ln(n!) = 2n\ln 2 - \frac{\ln n}{2} - \frac{\ln \pi}{2} + O\left(\frac{1}{n}\right).$$

It follows that

$$\sum_{k=2}^{2n}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 3\ln 2 - 2\ln\pi + \ln\left(1+\frac{1}{2n}\right) + O\left(\frac{1}{n}\right) = 3\ln 2 - 2\ln\pi + O\left(\frac{1}{n}\right)$$

and that $\displaystyle\sum_{k=2}^{2n+1}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 3\ln 2 - 2\ln\pi + O\left(\frac{1}{n}\right)$ as well.

This shows that the sum of the problem equal $3\ln 2 - 2\ln\pi = \ln\left(\dfrac{8}{\pi^2}\right)$.

Also solved by Brian D. Beasley, Clinton, SC; Worapol Rattanapan (student at Montfort College (high school)), Chiang Mai, Thailand; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5007: *Richard L. Francis, Cape Girardeau, MO.*

Is the centroid of a triangle the same as the centroid of its Morley triangle?

Solution by Kenneth Korbin, New York, NY.

The centroids are not the same unless the triangle is equilateral.

For example, the isosceles right triangle with vertices at $(-6,0), (6,0)$ and $(0,6)$ has its centroid at $(0,2)$.

Its Morley triangle has verticies at $(0, 12-6\sqrt{3}), (-6+3\sqrt{3}, 3)$, and $(6-3\sqrt{3}, 3)$ and has its centroid at $(0, 6-2\sqrt{3})$.

Also solved by Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY, and the proposer.

Problems

Ted Eisenberg, Section Editor

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Solutions to the problems stated in this issue should be posted before
December 15, 2008

- 5026: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0)$, $B(12,0)$, $C(4,15)$, and $D(0,4)$. Point P has coordinates $(x,3)$. Find the value of x if

  $$\text{area }\triangle\text{PAD} + \text{area }\triangle\text{PBC} = \text{area }\triangle\text{PAB} + \text{area }\triangle\text{PCD}.$$

- 5027: *Proposed by Kenneth Korbin, New York, NY.*

  Find the x and y intercepts of

  $$y = x^7 + x^6 + x^4 + x^3 + 1.$$

- 5028: *Proposed by Michael Brozinsky, Central Islip, NY .*

  If the ratio of the area of the square inscribed in an isosceles triangle with one side on the base to the area of the triangle uniquely determine the base angles, find the base angles.

- 5029: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $x > 1$ be a non-integer number. Prove that

  $$\left(\frac{x+\{x\}}{[x]} - \frac{[x]}{x+\{x\}}\right) + \left(\frac{x+[x]}{\{x\}} - \frac{\{x\}}{x+[x]}\right) > \frac{9}{2},$$

  where $[x]$ and $\{x\}$ represents the entire and fractional part of x.

- 5030: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A_1, A_2, \cdots, A_n \in M_2(\mathbf{C}), (n \geq 2)$, be the solutions of the equation $X^n = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

  Prove that $\sum_{k=1}^{n} Tr(A_k) = 0$.

- 5031: *Ovidiu Furdui, Toledo, OH.*

  Let x be a real number. Find the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right).$$

Solutions

- 5008: *Proposed by Kenneth Korbin, New York, NY.*

  Given isosceles trapezoid $ABCD$ with $\angle ABD = 60^o$, and with legs $\overline{BC} = \overline{AD} = 31$.

  Find the perimeter of the trapezoid if each of the bases has positive integer length with $\overline{AB} > \overline{CD}$.

  Solution by David C. Wilson, Winston-Salem, N.C.

  Let the side lengths of $\overline{\text{AB}}= x$, $\overline{\text{BC}}$=31, $\overline{CD}= y$, $\overline{\text{DA}}$=31, and $\overline{\text{BD}}$=z.
  By the law of cosines

$$\begin{aligned}
31^2 &= x^2 + z^2 - 2xz\cos 60^o \text{ and} \\
31^2 &= y^2 + z^2 - 2yz\cos 60^o \Longrightarrow \\
961 &= z^2 + x^2 - xz \text{ and} \\
961 &= y^2 + z^2 - yz \Longrightarrow \\
0 &= (y^2 - x^2) - yz + xz \Longrightarrow \\
0 &= (y-x)(y+x) - z(y-x) = (y-x)(y+x-z) \Longrightarrow \\
y - x &= 0 \text{ or } y + x - z = 0.
\end{aligned}$$

  But $\overline{AB} > \overline{CD} \Longrightarrow x > y \Longrightarrow y - x \neq 0$. Thus, $y + x - z = 0 \Longrightarrow z = x + y$. Thus,

$$961 = (x+y)^2 + x^2 - x(x+y) = x^2 + 2xy + y^2 + x^2 - x^2 - xy = x^2 + xy + y^2.$$

  Consider $x = 30, 29, \cdots, 18$. After trial and error with a calculator, when $x = 24$ then $y = 11 \Longrightarrow z = 35$ and these check. Thus, the perimeter of $ABCD$ is $35 + 31 + 31 = 97$.

  Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Lauren Christenson, Taylor Brennan, Ross Hayden, and Meaghan Haynes (jointly; students at Taylor University), Upland, IN; Charles McCracken, Dayton, OH; Amanda Miller (student, St.George's School), Spokane, WA; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5009:** *Proposed by Kenneth Korbin, New York, NY.*

Given equilateral triangle ABC with a cevian $\overline{CD}$ such that $\overline{AD}$ and $\overline{BD}$ have integer lengths. Find the side of the triangle $\overline{AB}$ if $\overline{CD} = 1729$ and if $(\overline{AB}, 1729) = 1$.

Solution by David Stone and John Hawkins, Statesboro, GA.

The answer: $\overline{AB} = 1775, 1840, 1961, 1984$.

Let $x = \overline{AD}$ and $y = \overline{BD}$, with $s = x + y =$ the side length $\overline{AB}$. Applying the Law of cosines in each "subtriangle," we have

$$\begin{aligned} 1729^2 &= s^2 + x^2 - 2sx\cos\frac{\pi}{3} = s^2 + x^2 - sx \quad \text{and} \\ 1729^2 &= s^2 + y^2 - 2sy\cos\frac{\pi}{3} = s^2 + y^2 - sy. \end{aligned}$$

After adding equations and doing some algebra, we obtain the equation

$$y^2 + xy + x^2 = 1729^2.$$

Solving for y by the Quadratic Formula, we obtain

$$y = \frac{-x \pm \sqrt{4 \cdot 1729^2 - 3x^2}}{2} = \frac{-x \pm z}{2}$$

where $z = \sqrt{4 \cdot 1729^2 - 3x^2}$ must be an integer.

Because y must be positive, we have to choose $y = \dfrac{-x + z}{2}$.

Now we let Excel calculate, trying $x = 1, 2, \cdots, 1729$. We have 13 "solutions", but only four of them have $s = \overline{AB}$ relatively prime to 1729; hence only equilateral triangles of side length $AB = 1775, 1840, 1961$, and 1984 admit the cevian described in the problem.

x	$z = \sqrt{34586^2 - 3x^2}$	$y = (-x + z)/2$	$s = x + y$	$gcd(1729, s)$
96	3454	1679	1775	1
209	3439	1615	1824	19
249	3431	1591	1840	1
299	3419	1560	1859	13
361	3401	1520	1881	19
455	3367	1456	1911	91
504	3346	1421	1925	7
651	3269	1309	1960	7
656	3266	1305	1961	1
741	3211	1235	1976	247
799	3169	1185	1984	1
845	3133	1144	1989	13
931	3059	1064	1995	133

Note that we could let x run further, but the problem is symmetric in x and y, so we'd just recover these same solutions with x and y interchanged.

Comment by Kenneth Korbin, the proposer.

In the problem $\overline{CD} = (7)(13)(19)$ and there were exactly 4 possible answers. If $\overline{CD}$ would have been equal to $(7)(13)(19)(31)$then there would have been exactly 8 possible solutions.

Similarly, there are exactly 4 primitive Pythagorean triangles with hypotenuse (5)(13)(17) and there exactly 8 primitive Pythagorean triangles with hypotenuse (5)(13)(17)(29). And so on.

Also solved by Charles McCracken, Dayton, OH; David E. Manes, Oneonta, NY; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5010: *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona, Spain.*

Let α, β, and γ be real numbers such that $0 < \alpha \le \beta \le \gamma < \pi/2$. Prove that

$$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)(\cos\alpha + \cos\beta + \cos\gamma)} \le \frac{2}{3}.$$

Solution by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy.

Proof After some simple simplification the inequality is

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)$$

The concavity of $\sin(x)$ in the interval $[0, \pi]$ allows us to write $\sin(x+y) \ge (\sin(2x) + \sin(2y))/2$ thus

$$\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) \ge \sin 2\alpha + \sin 2\beta + \sin 2\gamma$$

concluding the proof.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeak, VA, and the proposers.

- 5011: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $\{a_n\}_{n\ge 0}$ be the sequence defined by $a_0 = a_1 = 2$ and for $n \ge 2$, $a_n = 2a_{n-1} - \dfrac{1}{2}a_{n-2}$. Prove that

$$2^p a_{p+q} + a_{q-p} = 2^p a_p a_q$$

where $p \le q$ are nonnegative integers.

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

Solving the characteristic equation

$$r^2 - 2r + \frac{1}{2} = 0$$

and using the intitial conditions, we obtain the solution

$$a_n = \left(\frac{2+\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}}{2}\right)^n.$$

Note that

$$2^p a_{p+q} \quad = \quad \frac{(2+\sqrt{2})^{p+q} + (2-\sqrt{2})^{p+q}}{2^q} \text{ and}$$

$$
\begin{aligned}
a_{q-p} &= \frac{(2+\sqrt{2})^{q-p} + (2-\sqrt{2})^{q-p}}{2^{q-p}} \text{ while} \\
2^p a_p a_q &= \frac{(2+\sqrt{2})^{p+q} + (2-\sqrt{2})^{p+q} + 2^p[(2+\sqrt{2})^{q-p} + (2-\sqrt{2})^{q-p}]}{2^q} \\
&= 2^p a_{p+q} + a_{q-p}.
\end{aligned}
$$

Solution 2 by Kee-Wai Lau, Hong Kong, China.

By induction, we obtain readily that for $n \geq 0$,

$$a_n = \left(\frac{2+\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}}{2}\right)^n.$$

Hence

$$
\begin{aligned}
a_p a_q &= \left(\left(\frac{2+\sqrt{2}}{2}\right)^p + \left(\frac{2-\sqrt{2}}{2}\right)^p\right)\left(\left(\frac{2+\sqrt{2}}{2}\right)^q + \left(\frac{2-\sqrt{2}}{2}\right)^q\right) \\
&= \left(\left(\frac{2+\sqrt{2}}{2}\right)^{p+q} + \left(\frac{2-\sqrt{2}}{2}\right)^{p+q}\right) + \left(\frac{2+\sqrt{2}}{2}\right)^p\left(\frac{2-\sqrt{2}}{2}\right)^q + \left(\frac{2-\sqrt{2}}{2}\right)^p\left(\frac{2+\sqrt{2}}{2}\right)^q \\
&= a_{p+q} + \left(\frac{2+\sqrt{2}}{2}\right)^p\left(\frac{2-\sqrt{2}}{2}\right)^q\left(\left(\frac{2-\sqrt{2}}{2}\right)^{q-p} + \left(\frac{2+\sqrt{2}}{2}\right)^q\right) \\
&= a_{p+q} + \frac{1}{2^p} a_{q-p},
\end{aligned}
$$

and the identity of the problem follows.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Jose Hernández Santiago (student, UTM), Oaxaca, México; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5012: *Richard L. Francis, Cape Girardeau, MO.*

Is the incenter of a triangle the same as the incenter of its Morley triangle?

Solution 1 by Kenneth Korbin, New York, NY.

The incenters are not the same unless the triangle is equilateral. For example, the isosceles right triangle with vertices at $(-6, 0), (6, 0)$ and $(0, 6)$ has its incenter at $(0, 6\sqrt{2} - 6)$.

Its Morely triangle has vertices at $(0, 12 - 6\sqrt{3}), (-6 + 3\sqrt{3}, 3)$, and $(6 - 3\sqrt{3}, 3)$ and has its incenter at $(0, 6 - 2\sqrt{3})$.

Solution 2 by Kee-Wai Lau, Hong-Kong, China.

We show that the incenter I of a triangle ABC is the same as the incenter I_M of its Morley triangle if and only if ABC is equilateral.

In homogeneous trilinear coordinates, I is $1:1:1$ and I_M is

$$\cos\left(\frac{A}{3}\right)+2\cos\left(\frac{B}{3}\right)\cos\left(\frac{C}{3}\right) : \cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) : \cos\left(\frac{C}{3}\right)+2\cos\left(\frac{A}{3}\right)\cos\left(\frac{B}{3}\right).$$

Clearly if ABC is equilateral, then $I = I_M$. Now suppose that $I = I_M$ so that

$$\cos\left(\frac{A}{3}\right) + 2\cos\left(\frac{B}{3}\right)\cos\left(\frac{C}{3}\right) = \cos\left(\frac{B}{3}\right) + 2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) \quad (1)$$

$$\cos\left(\frac{B}{3}\right) + 2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) = \cos\left(\frac{C}{3}\right) + 2\cos\left(\frac{A}{3}\right)\cos\left(\frac{B}{3}\right). \quad (2)$$

From (1) we obtain

$$\left(\cos\left(\frac{A}{3}\right) - \cos\left(\frac{B}{3}\right)\right)\left(1 - 2\cos\left(\frac{C}{3}\right)\right) = 0.$$

Since $0 < C < \pi$, so

$$1 - 2\cos\left(\frac{C}{3}\right) < 0.$$

Thus,

$$\cos\left(\frac{A}{3}\right) = \cos\left(\frac{B}{3}\right) \text{ or } A = B.$$

Similarly from (2) we obtain $B = C$. It follows that ABC is equilateral and this completes the solution.

Also solved by David E. Manes, Oneonta, NY, and the proposer.

- 5013: *Proposed by Ovidiu Furdui, Toledo, OH.*

Let $k \geq 2$ be a natural number. Find the sum

$$\sum_{n_1,n_2,\cdots,n_k\geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k}.$$

Solution by Kee-Wai Lau, Hong Kong, China.

For positive integers $M_1, M_2, \cdots, M_k$, we have

$$\sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2}\cdots\sum_{n_k=1}^{M_k} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2\cdots+n_k}$$

$$= \sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2}\cdots\sum_{n_k=1}^{M_k} (-1)^{n_1+n_2+\cdots+n_k}\int_0^1 x^{n_1+n_2+\cdots+n_k-1}dx$$

$$= \int_0^1 \left(\sum_{n_1=1}^{M_1}(-1)^{n_1}x^{n_1}\right)\left(\sum_{n_2=1}^{M_2}(-1)^{n_2}x^{n_2}\right)\cdots\left(\sum_{n_k=1}^{M_k}(-1)^{n_k}x^{n_k}\right)x^{-1}dx$$

$$= \int_0^1 \left(\frac{-x(1-(-x)^{M_1})}{1+x}\right)\left(\frac{-x(1-(-x)^{M_2})}{1+x}\right)\left(\frac{-x(1-(-x)^{M_k})}{1+x}\right)x^{-1}dx$$

$$
\begin{aligned}
&= (-1)^k \int_0^1 \frac{x^{k-1}(1-(-x)^{M_1})(1-(-x)^{M_2})\cdots(1-(-x)^{M_k}))}{(1+x)^k}dx \\
&= (-1)^k \int_0^1 \frac{x^{k-1}}{(1+x)^k}dx + O\Big(\int_0^1 x^{M_1}+x^{M_2}+\cdots+x^{M_k}\Big)dx \\
&= (-1)^k \int_0^1 \frac{x^{k-1}}{(1+x)^k}dx + O\Big(\frac{1}{M_1}+\frac{1}{M_2}+\cdots+\frac{1}{M_k}\Big)
\end{aligned}
$$

as $M_1, M_2, \cdots, M_k$ tend to infinity. Here the constants implied by the $O's$ depend at most on k.
It follows that the sum of the problem equals

$$
(-1)^k \int_0^1 \frac{x^{k-1}}{1+x)^k}dx = (-1)^k I_k, \quad \text{say.}
$$

Integrating by parts, we have for $k \geq 3$,

$$
\begin{aligned}
I_k &= \frac{1}{1-k}\int_0^1 x^{k-1} d((1+x)^{1-k}) \\
&= \frac{-1}{(k-1)2^{k-1}} + I_{k-1}.
\end{aligned}
$$

Since $I_2 = \ln 2 - \dfrac{1}{2}$, we obtain readily by induction that for $k \geq 2$.

$$
I_k = \ln 2 - \sum_{j=2}^{k} \frac{1}{(j-1)2^{j-1}}.
$$

we now conclude that for $k \geq 2$,

$$
\sum_{n_1,n_2,\cdots,n_k \geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k} = (-1)^k\Big(\ln 2 - \sum_{j=1}^{k-1}\frac{1}{j(2^j)}\Big).
$$

Also solved by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Paul M. Harms, North Newton, KS; Boris Rays, Chesapeake, VA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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Solutions to the problems stated in this issue should be posted before
January 15, 2009

- 5032: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive acute angles A, B, C such that
  $$\tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A = 1.$$
  Find the value of
  $$\frac{\sin A}{\cos B \cdot \cos C} + \frac{\sin B}{\cos A \cdot \cos C} + \frac{\sin C}{\cos A \cdot \cos B}.$$

- 5033: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15)$, and $D(0,4)$. Point P is on side $\overline{AB}$ and point Q is on side $\overline{CD}$. Find the coordinates of P and Q if area $\triangle PCD =$ area $\triangle QAB = \frac{1}{2}$area quadrilateral $ABCD$.

- 5034: *Proposed by Roger Izard, Dallas, TX.*

  In rectangle $MDCB$, $MB \perp MD$. F is the midpoint of BC, and points N, E and G lie on line segments DC, DM, and MB respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be A_1 and let the area of quadrilateral $MGFE$ be A_2. Determine the area of quadrilateral $EDNF$ in terms of A_1 and A_2.

- 5035: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that
  $$(a^a b^b c^c)^2 (a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)})^3 \geq 27.$$

- 5036: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all triples (x, y, z) of nonnegative numbers such that

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3^x + 3^y + 3^z = 5 \end{cases}$$

- 5037: *Ovidiu Furdui, Toledo, OH.*

  Let k, p be natural numbers. Prove that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$$

  for all $n \geq 1$ if and only if $k = p = 1$.

Solutions

- 5014: *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with $a = 100, b = 105$, and with equal cevians $\overline{AD}$ and $\overline{BE}$. Find the perimeter of the triangle if $\overline{AE} \cdot \overline{BD} = \overline{CE} \cdot \overline{CD}$.

  Solution by David Stone and John Hawkins, Statesboro, GA.

  The solution to this problem is more complex than expected. There are infinitely many triangles satisfying the given conditions, governed in a sense by two types of degeneracy. The nicest of these solutions is a right triangle with integer sides, dictated by the given data: $100 = 5(20)$ and $105 = 5(21)$ and $(20, 21, 29)$ is a Pythagorean triple.

  One type of degeneracy is the usual: if $AB = 5$ or $AB = 205$, we have a degenerate triangle which can be shown to satisfy the conditions of the problem.

  The other type of degeneracy is problem specific: when neither cevian intersects the interior of its targeted side, but lies along a side of the triangle. In these two situations, the problem's condition are also met.

  Let x = length of CE so $0 \leq x \leq 105$. The following table summarizes our results.

$x = CE$	$\cos(C)$	C	AB	BD	*Perimeter*	$AD = BE$	*note*
0	$\frac{21}{40}$	$\cos^{-1}\left(\frac{21}{40}\right)$	100	0	305	100	1
21	$\frac{194}{350}$	$\cos^{-1}\left(\frac{194}{350}\right)$	$\sqrt{9385}$	20	301.88	$\sqrt{8113}$	2
$\frac{1985}{41}$	1	0	5	$\frac{1900}{41}$	210	$\frac{2105}{41}$	3
excluded	values	*****	*****	*****	*****	*****	
$\frac{2205}{41}$	-1	180^o	205	$\frac{2100}{41}$	410	$\frac{6305}{41}$	3
$\frac{105}{41}(\sqrt{178081} - 400)$	0	90^o	145	≈ 53.65	350	≈ 114.775	4
105	$\frac{10}{21}$	$\cos^{-1}\left(\frac{10}{21}\right)$	105	100	310	105	5

Notes:

1. Cevian BE is side BC; Cevian AD is side AB. **2.** A "nice" value for x. **3.** Degenerate triangle. **4.** Right triangle. **5.** Cevian BE is side AB; Cevian AD is side AC.

In short, the perimeter assumes all values in $[210, 305] \cup [310, 410]$.

Now we support these assertions. Consider $\triangle ABC$ with cevians BE (from $\angle B$ to side AC) and AD (from $\angle A$ to side BC). Let $CE = x$, $AE = 105 - x$ and $CD = 100 - BD$. To find the perimeter, we only need to compute AB.

We have $AE = 105 - x$, so

$$\begin{aligned} AE \cdot BD &= CE \cdot CD \\ (105 - x)BD &= x(100 - BD) \\ 100x &= 105BD \\ \text{so } BD &= \frac{20}{21}x \text{ and } CD = \mathit{100} - \frac{\mathit{20}}{\mathit{21}}x. \end{aligned}$$

Applying the Law of Cosines three times, we have

$$\begin{array}{ll} (1) & BE^2 = x^2 + 100^2 - 2(100)x\cos C \\ (2) & AD^2 = CD^2 + 105^2 - 2(105)CD\cos(C) \text{ and} \\ (3) & AB^2 = 100^2 + 105^2 - 2 \cdot 100 \cdot 105\cos(C). \end{array}$$

Because we must have $AD = BE$, we combine (1) and (2) to get

$$\begin{aligned} CD^2 + 105^2 - 2(105)CD\cos(C) &= x^2 + 100^2 - 2(100)x\cos(C) \text{ or} \\ \left(100 - \frac{20}{21}x\right)^2 + 105^5 - 210\left(100 - \frac{20}{21}x\right)\cos(C) &= x^2 + 100^2 - 2(100)x\cos(C). \end{aligned}$$

Solving for $\cos(C)$, we obtain a rational expression in x:

$$(4) \quad \cos(C) = \frac{41x^2 + 84000x - 2205^2}{21^2 \cdot 200(2x - 105)}.$$

Substituting this value into (3) we have

$$\begin{aligned} AB^2 &= 21025 - 2100 \cdot \frac{41x^2 + 84000x - 2205^2}{21^2 \cdot 200(200x - 105)}, \text{ so} \\ (5) \ \ AB^2 &= \frac{5}{21}\frac{41x^2 + 92610x + 4410000}{105 - 2x}. \end{aligned}$$

Thus we can then calculate AB and the perimeter

$$P = 205 + \sqrt{\frac{5}{21}\frac{41x^2 + 92610x + 4410000}{105 - 2x}}.$$

The graphs of $\cos(C)$ and of AB have vertical asymptotes at $x = \dfrac{105}{2}$, in the center of our interval $[0, 105]$. Other than an interval bracketing this singularity, each value of x produces a solution to the problem.

We explore the endpoints and the "degenerate" solutions, obtaining the values exhibited in the table above.

I. $x = 0$: That is $CE = 0$, so $E = C$ and the cevian from vertex B is actually the side BC. Therefore, $BE = BC = 100$. Hence, the condition

$$\begin{aligned} AE \cdot BD &= CE \cdot CD \text{ becomes} \\ AC \cdot BD &= 0 \cdot CD \text{ or} \\ 100 \cdot BD &= 0. \end{aligned}$$

Thus $BD = 0$, so $D = B$ and the cevian from vertex A is actually the side AB; $AD = AB$.
Computing by (4) and (5): $\cos(C) = \dfrac{21}{40}$ and $AB = 100$. Thus $AD = AB = 100 = BC = BE$, so this triangle satisfies the required conditions. Its perimeter is 305.

II. $x = 105$**:** gives a similar result, a $(105, 105, 100)$ triangle with cevians lying along the sides and $P = 310$.

III. The degenerate case $C = 0$ occurs when $\cos(C) = 1$. By (4), this happens when $x = \dfrac{1985}{41}$. Also, $C = 0$ if and only if $AB = 5$, which is the smallest possible value (by the Triangle Inequality).

IV. The degenerate case $C = \pi$ occurs when $\cos(C) = -1$. By (4) this happens when $x = \dfrac{2205}{41}$. Also $C = \pi$ if and only if $AB = 205$, which is the largest possible value (by the Triangle Inequality).

The values of x appearing in III and IV are the endpoints of the interval of excluded values bracketing $\dfrac{105}{2}$.

V. The degenerate case $C = \pi/2$ occurs when $\cos(C) = 0$. By (4), this happens when x takes on the ugly irrational $\dfrac{105}{41}\left(\sqrt{178081} - 400\right)$. In this case, $AB = 145$ and our triangle is the $(20, 21, 29)$ Pythagorean triangle scaled up by a factor of 5. The common value of the cevians is $AD = BE = \dfrac{5}{41}\sqrt{49788121 - 352800\sqrt{178081}} \approx 114.775$.

VI. Because $BD = \dfrac{20}{21}x$, some nice results occur when x is a multiple of 21. The table shows the values for $x = 21$.

Excel has produced many values of these triangles, letting x range from 0 to 105, except for the excluded interval $\left(\dfrac{1985}{41}, \dfrac{2205}{41}\right)$, but in summary,

- the perimeter assumes all values in $[210, 305] \cup [310, 410]$.
- side AB assumes all values in $[5, 100] \cup [105, 205]$.
- $\angle C$ assumes all values in

$$\left[0, \cos^{-1}\left(\frac{21}{40}\right)\right] \cup \left[\cos^{-1}\left(\frac{10}{21}, 180^o\right)\right] = [0, 58.33^o] \cup [61.56^o, 180^o].$$

– The common cevians achieve the values
$$\left[\frac{2105}{41}, 100\right] \cup \left[105, \frac{6305}{41}\right] \approx [51.34, 100] \cup [105, 153.78].$$

Our final comment: AB assumes all integer values in $[5, 100] \cup [105, 205]$, so the right triangle described above is not the only solution with all sides integral. For any integer AB in $[5, 100] \cup [105, 205]$, we can use (5) to determine the appropriate value of x, C, etc. Of course, this raises another question: are any of these triangles Heronian?

Also solved by the proposer.

5015: *Proposed by Kenneth Korbin, New York, NY.*

Part I: Find the value of
$$\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

Part II: Find the value of
$$\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

Solution by David C. Wilson, Winston-Salem, N.C.

First, let's look for a pattern.

$\mathbf{x=1}:\quad \text{Arcsin}(\frac{4}{5}).$

$\mathbf{x=2}:\quad \text{Arcsin}(\frac{4}{5}) + \text{Arcsin}(\frac{16}{65}) = \text{Arcsin}(\frac{12}{13}).$

Let $\theta = \text{Arcsin}(\frac{4}{5})$ and $\phi = \text{Arcsin}(\frac{16}{65})$.

$$\sin\theta = \frac{4}{5} \qquad \sin\phi = \frac{16}{65}$$

$$\cos\theta = \frac{3}{5} \qquad \cos\phi = \frac{63}{65}$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = (\frac{4}{5})(\frac{63}{65}) + (\frac{3}{5})(\frac{16}{65}) = \frac{300}{325} = \frac{12}{13} = \text{Arcsin}(\frac{12}{13}).$$

$\mathbf{x=3}:\quad \text{Arcsin}(\frac{12}{13}) + \text{Arcsin}(\frac{36}{325}) = \text{Arcsin}(\frac{24}{25}).$

Let $\theta = \text{Arcsin}(\frac{12}{13})$ and $\phi = \text{Arcsin}(\frac{36}{325})$.

$$\sin\theta = \frac{12}{13} \qquad \sin\phi = \frac{36}{325}$$

$$\cos\theta = \frac{5}{13} \qquad \cos\phi = \frac{323}{325}$$

$$\sin(\theta+\phi) = (\frac{12}{13})(\frac{323}{325}) + (\frac{5}{13})(\frac{36}{325}) = \frac{4056}{4225} = \frac{24}{25} = \text{Arcsin}(\frac{24}{25}).$$

$\mathbf{x=4}:\quad \text{Arcsin}(\frac{24}{25}) + \text{Arcsin}(\frac{64}{1025}) = \text{Arcsin}(\frac{40}{41}).$

Let $\theta = \text{Arcsin}(\frac{24}{25})$ and $\phi = \text{Arcsin}(\frac{64}{1025})$.

$$\sin\theta = \frac{24}{25} \qquad \sin\phi = \frac{64}{1025}$$

$$\cos\theta = \frac{7}{25} \qquad \cos\phi = \frac{1023}{1025}$$

$$\sin(\theta+\phi) = (\frac{24}{25})(\frac{1023}{1025}) + (\frac{7}{25})(\frac{64}{1025}) = \frac{25000}{25625} = \frac{40}{41} = \text{Arcsin}(\frac{40}{41}).$$

Therefore, the conjecture is

$$\sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right).$$

Proof is by induction.

1) For $n = 1$, we obtain $\text{Arcsin}(\frac{4}{5}) = \text{Arcsin}(\frac{4}{5})$.

2) Assume true for n; i.e.,

$$\sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right).$$

3) For $n+1$, we have

$$\begin{aligned}\sum_{x=1}^{n+1} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) &= \sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) + \text{Arcsin}\left(\frac{4(n+1)^2}{4(n+1)^4+1}\right) \\ &= \sum_{x=1}^{n} \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right) + \text{Arcsin}\left(\frac{4(n+1)^2}{4(n+1)^4+1}\right).\end{aligned}$$

Let $\theta = \text{Arcsin}\left(\frac{2\text{n}^2+2\text{n}}{2\text{n}^2+2\text{n}+1}\right)$ and $\phi = \text{Arcsin}\left(\frac{4(\text{n}+1)^2}{4(\text{n}+1)^4+1}\right)$.

$$\sin\theta = \frac{2n^2+2n}{2n^2+2n+1} \qquad \sin\phi = \frac{4(n+1)^2}{4(n+1)^4+1}$$

$$\cos\theta = \frac{2n+1}{2n^2+2n+1} \qquad \cos\phi = \frac{4(n+1)^4-1}{4(n+1)^4+1}$$

$$\begin{aligned}\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi \\ &= \left(\frac{2n^2+2n}{2n^2+2n+1}\right)\left[\frac{4(n+1)^4-1}{4(n+1)^4+1}\right] + \left(\frac{2n+1}{2n^2+2n+1}\right)\left[\frac{4(n+1)^2}{4(n+1)^4+1}\right] \\ &= \frac{8n^6+40n^5+80n^4+88n^3+58n^2+22n+4}{(2n^2+2n+1)(2n^2+6n+5)(2n^2+2n+1)} \\ &= \frac{(2n^2+2n+1)^2(2n^2+6n+4)}{(2n^2+2n+1)^2(2n^2+6n+5)} = \frac{2n^2+6n+4}{2n^2+6n+5} = \frac{2(n+1)^2+2(n+1)}{2(n+1)^2+2(n+1)+1}.\end{aligned}$$

Thus $\sum_{x=1}^{n+1} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left[\frac{2(n+1)^2+2(n+1)}{2(n+1)^2+2(n+1)+1}\right]$ and this proves the conjecture.

Part I: $\displaystyle\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left[\frac{220}{221}\right].$

Part II:

$$\begin{aligned}
\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) &= \lim_{n\to\infty} \sum_{x=1}^{n} \text{Arcsin}\left[\frac{4x^2}{4x^4+1}\right] \\
&= \lim_{n\to\infty} \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right) \\
&= \text{Arcsin}\left[\lim_{n\to\infty} \frac{2n^2+2n}{2n^2+2n+1}\right] = \text{Arcsin}(1) = \frac{\pi}{2}.
\end{aligned}$$

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Roger Zarnowski (jointly), San Angelo, TX; Brian D. Beasley, Clinton, SC; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5016: *Proposed by John Nord, Spokane, WA.*

Locate a point (p, q) in the Cartesian plane with integral values, such that for any line through (p, q) expressed in the general form $ax + by = c$, the coefficients a, b, c form an arithmetic progression.

Solution 1 by Nate Wynn (student at Saint George's School), Spokane, WA.

As $\{a, b, c\}$ is an arithmetic progression, b can be written as $a + n$ and c can be written as $a + 2n$. Then using a series of two equations:

$$\begin{cases} ap + (a+n)q &= a+2n \\ tp + (t+u)q &= t+2u \end{cases}$$

Solving this system gives

$$(tn - au)q = 2tn - 2au, \text{thus } q = 2.$$

Placing this value into the first equation and solving gives

$$\begin{aligned}
ap + 2a + 2n &= a + 2n \\
a(p+1) &= 0 \\
p &= -1.
\end{aligned}$$

Therefore the point is $(-1, 2)$.

Solution 2 by Eric Malm (graduate student at Stanford University, and an alumnus of Saint George's School in Spokane), Stanford, CA.

The only such point is $(-1, 2)$.

Suppose that each line through (p, q) is of the form $ax + by = c$ with (a, b, c) an arithmetic progression. Then $c = 2b - a$. Taking $a = 0$ yields the line $by = 2b$ or $y = 2$, so $q = 2$. Taking $a \neq 0$, p= must satisfy $ap + 2b = 2b - a$, so $p = -1$.

Conversely, any line through $(p, q) = (-1, 2)$ must be of the form
$ax + by = ap + bq = 2b - a$, in which case the coefficients $(a, b, 2b - a)$ form an arithmetic progression.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Rachel Demeo, Matthew Hussey, Allison Reece, and Brian Tencher (jointly, students at Talyor University, Upland, IN); Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Raul A. Simon, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5017: *Proposed by M.N. Deshpande, Nagpur, India.*

Let ABC be a triangle such that each angle is less than 90^0. Show that

$$\frac{a}{c\cdot\sin B}+\frac{1}{\tan A}=\frac{b}{a\cdot\sin C}+\frac{1}{\tan B}=\frac{c}{b\cdot\sin A}+\frac{1}{\tan C}$$

where $a = l(\overline{BC}), b = l(\overline{AC})$, and $c = l(\overline{AB})$.

Solution by John Boncek, Montgomery, AL.

From the Law of Sines:

$$\begin{aligned}
a\sin B &= b\sin A \quad\rightarrow\quad \sin B=\frac{b\sin A}{a}\\
b\sin C &= c\sin B \quad\rightarrow\quad \sin C=\frac{c\sin B}{b}\\
c\sin A &= a\sin C \quad\rightarrow\quad \sin A=\frac{a\sin C}{c},
\end{aligned}$$

and from the Law of Cosines, we have

$$\begin{aligned}
bc\cos A &= \frac{1}{2}(b^2+c^2-a^2)\\
ac\cos B &= \frac{1}{2}(a^2+c^2-b^2)\\
ab\cos C &= \frac{1}{2}(a^2+b^2-c^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{a}{c\sin B}+\frac{1}{\tan A} &= \frac{a^2}{bc\sin A}+\frac{\cos A}{\sin A}\\
&= \frac{a^2+bc\cos A}{bc\sin A}\\
&= \frac{a^2+b^2+c^2}{2bc\sin A},
\end{aligned}$$

$$\frac{b}{a\sin C}+\frac{1}{\tan B} = \frac{b^2}{ac\sin B}+\frac{\cos B}{\sin B}$$

$$
\begin{aligned}
&= \frac{b^2 + ac\cos B}{ac\sin B} \\
&= \frac{a^2+b^2+c^2}{2ac\sin B} \\
&= \frac{a^2+b^2+c^2}{2c(a\sin B)} \\
&= \frac{a^2+b^2+c^2}{2bc\sin A},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{c}{b\sin A} + \frac{1}{\tan C} &= \frac{c^2}{ab\sin C} + \frac{\cos C}{\sin C} \\
&= \frac{c^2 + ab\cos C}{ab\sin C} \\
&= \frac{a^2+b^2+c^2}{2ab\sin C} \\
&= \frac{a^2+b^2+c^2}{2b(a\sin C)} \\
&= \frac{a^2+b^2+c^2}{2bc\sin A}.
\end{aligned}
$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5018: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Write the polynomial $x^{5020} + x^{1004} + 1$ as a product of two polynomials with integer coefficients.

Solution by Kee-Wai Lau, Hong Kong, China.

Clearly the polynomial $y^5 + y + 1$ has no linear factor with integer coefficients.
We suppose that for some integers a, b, c, d, e

$$
\begin{aligned}
y^5 + y + 1 &= (y^3 + ay^2 + by + c)(y^2 + dy + e) \\
&= y^5 + (a+d)y^4 + (b+e+ad)y^3 + (ae+bd+c)y^2 + (be+cd)y + ce.
\end{aligned}
$$

Hence

$$a + d = b + e + ad = ae + bd + c = 0, \quad be + cd = ce = 1.$$

It is easy to check that $a = -1, b = 0, c = 1, d = 1, e = 1$ so that

$$y^5 + y + 1 = (y^3 - y^2 + 1)(y^2 + y + 1)$$

and

$$x^{5020}+x^{1004}+1=\left(x^{3012}-x^{2008}+1\right)\left(x^{2008}+x^{1004}+1\right).$$

Comment by Kenneth Korbin, New York, NY. Note that (y^2+y+1) is a factor of (y^N+y+1) for all N congruent to 2(mod3) with $N>1$.

Also solved by Landon Anspach, Nicki Reishus, Jessi Byl, and Laura Schindler (jointly, students at Taylor University), Upland, IN; Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Matthew Hussey Rachel DeMeo, Brian Tencher, and Allison Reece (jointly, students at Taylor University), Upland IN; Kenneth Korbin, New York, NY; N. J. Kuenzi, Oshkosh, WI; Carl Libis, Kingston, RI; Eric Malm, Stanford, CA; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5019: *Michael Brozinsky, Central Islip, NY.*

In a horse race with 10 horses the horse with the number one on its saddle is referred to as the number one horse, and so on for the other numbers. The outcome of the race showed the number one horse did not finish first, the number two horse did not finish second, the number three horse did not finish third and the number four horse did not finish fourth. However, the number five horse did finish fifth. How many possible orders of finish are there for the ten horses assuming no ties?

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

There are 229,080 possible orders of finish.

For $k=0,1,2,3,4$ we perform the following calculations:

a) Choose the k horses numbered 1 through 4 which finish in places 1 through 4.

b) Arrange the k horses in places 1 through 4 and count the permutations with no "fixed points."

c) Arrange the remaining $(4-k)$ horses numbered 1 through 4 in places 6 through 10.

d) Arrange the 5 horses numbered 6 through 10 in the remaining 5 places.

Case 1: K=0.

${}_4C_0\cdot 5\cdot 4\cdot 3\cdot 2\cdot 5!=120\cdot 5!$

Case 2: K=1.

${}_4C_1\cdot 3\cdot 5\cdot 4\cdot 3\cdot 5!=720\cdot 5!$

Case 3: K=2.

${}_4C_2\cdot 7\cdot 5\cdot 4\cdot 5!=840\cdot 5!$

Case 4: K=3.

${}_4C_3\cdot 11\cdot 5\cdot 5!=220\cdot 5!$

Case 5: K=4.

${}_4C_4\cdot 9\cdot 5!=9\cdot 5!$

Solution 2 by Matt DeLong, Upland, IN.

We must count the total number of ways that 10 horses can be put in order subject to the given conditions. Since the number five horse always finishes fifth, we are essentially only counting the total number of way that 9 horses can be put in order subject to the other given conditions. Thus there are at most 9! possibilities.

However, this over counts, since it doesn't exclude the orderings with the number one horse finishing first, etc. By considering the number of ways to order the other eight horses, we can see that there are 8! ways in which the number one horse does finish first. Likewise, there are 8! ways in which each of the horses numbered two through four finish in the position corresponding to its saddle number. By eliminating these from consideration, we see that there are at least 9!-4(8!) possibilities.

However, this under counts, since we twice removed orderings in which both horse one finished first and horse two finished second etc. There are 6(7!) such orderings, since there are 6 ways to choose 2 horses from among 4, and once those are chosen the other 7 horses must be ordered. We can add these back in, but then we will again be over counting. We would need to subtract out those orderings in which three of the first four horses finish according to their saddle numbers. There are 4(6!) of these, since there are 4 ways to choose 3 horses from among 4, and once those are chosen the other 6 horses must be ordered. Finally, we would then need to add back in the number of orderings in which all four horses numbered one through four finish according to their saddle numbers. There are 5! such orderings.

In sum, we are applying the inclusion-exclusion principle, and the total that we are interested in is $9! - 4(8!) + 6(7!) - 4(6!) + 5! = 229,080$.

Also solved by Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Nate Kirsch and Isaac Bryan (students at Taylor University), Upland, IN; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2009

- 5038: *Proposed by Kenneth Korbin, New York, NY.*

  Given the equations

$$\begin{cases} \sqrt{1+\sqrt{1-x}} \;\; -5\cdot\sqrt{1-\sqrt{1-x}} = 4\cdot\sqrt[4]{x} \;\text{ and} \\ 4\cdot\sqrt{1+\sqrt{1-y}} \;\; -5\cdot\sqrt{1-\sqrt{1-y}} = \sqrt[4]{y}. \end{cases}$$

  Find the positive values of x and y.

- 5039: *Proposed by Kenneth Korbin, New York, NY.*

  Let d be equal to the product of the first N prime numbers which are congruent to 1(mod 4). That is

$$d = 5\cdot 13\cdot 17\cdot 29\cdots P_N.$$

  A convex polygon with integer length sides is inscribed in a circle with diameter d. Prove or disprove that the maximum possible number of sides of the polygon is the N^{th} term of the sequence $t=(4,8,20,32,80,\cdots,t_N,\cdots)$ where $t_N = 4t_{N-2}$ for $N>3$.

  Examples: If $N=1$, then $d=5$, and the maximum polygon has 4 sides $(3,3,4,4)$. If $N=2$, then $d=5\cdot 13=65$ and the maximum polygon has 8 sides $(16,16,25,25,25,25,33,33)$.

  Editor's comment: In correspondence with Ken about this problem he wrote that he has been unable to prove the formula for $N>5$; so it remains technically a conjecture.

- 5040: *Proposed by John Nord, Spokane, WA.*

  Two circles of equal radii overlap to form a lens. Find the distance between the centers if the area in circle A that is not covered by circle B is $\dfrac{1}{3}\left(2\pi+3\sqrt{3}\right)r^2$.

- 5041: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Quadrilateral $ABCD$ (with diagonals $AC=d_1$ and $BD=d_2$ and sides $AB=s_1, BC=s_2, CD=s_3$, and $DA=s_4$) is inscribed in a circle. Show that:

$$d_1^2+d_2^2+d_1d_2 > \frac{s_1^2+s_2^2+s_3^2+s_4^2}{2}.$$

- 5042: *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ $(a_k \neq 0)$ and $B(z) = z^{n+1} + \sum_{k=0}^{n} b_k z^k$ $(b_k \neq 0)$ be two prime polynomials with roots $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_{n+1}$ respectively. Prove that

  $$\frac{A(w_1)A(w_2)\ldots A(w_{n+1})}{B(z_1)B(z_2)\ldots B(z_n)}$$

  is an integer and determine its value.

- 5043: *Ovidiu Furdui, Toledo, OH.*

  Solve the following diophantine equation in positive integers k, m, and n

  $$k \cdot n! \cdot m! + m! + n! = (m+n)!.$$

Solutions

- 5020: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

  $$\begin{cases} x^7 - 13y = 21 \\ 13x - y^7 = 21 \end{cases}$$

Solution 1 by Brian D. Beasley, Clinton, SC.

Using the Fibonacci numbers $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for each integer $n \geq 3$, we generalize the given problem by finding numbers x and y such that

$$\begin{cases} x^n - F_n y & = & F_{n+1} \\ F_n x + (-1)^n y^n & = & F_{n+1} \end{cases}$$

for each positive integer n. (The given problem is the case $n = 7$.)

We let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ and apply the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ for each positive integer n to show that we may take $x = \alpha > 0$ and $y = -\beta > 0$:

$$\alpha^n - F_n(-\beta) = \frac{\alpha^n(\beta+\sqrt{5}) - \beta^{n+1}}{\sqrt{5}} = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} = F_{n+1};$$

$$F_n(\alpha) + (-1)^n(-\beta)^n = \frac{\alpha^{n+1} - \beta^n(\alpha-\sqrt{5})}{\sqrt{5}} = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} = F_{n+1}.$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

The solution anticipated by the poser is probably $(\alpha, -\beta)$, where $\alpha = \dfrac{1+\sqrt{5}}{2} \approx 1.618034$ is the Golden Ratio and $\beta = \dfrac{1-\sqrt{5}}{2} \approx -0.618034$ its companion (in the official terminology of *The Fibonacci Quarterly*).

Note that:

(#) if (x, y) is a solution to the system, then so is $(-y, x)$. Thus we may as well look for all solutions, not just positive solutions. We graph the system in the form

$$\begin{cases} y = \dfrac{x^7 - 21}{13} \\ \\ y = (13x - 21)^{1/7} \end{cases}.$$

There are five points of intersection. Graphically conditional (#) appears as symmetry of intersections across the line $y = -x$ (even though the curves themselves have no such symmetry).

Although we cannot determine all solutions analytically, we have their approximate numerical values:

$$\begin{array}{rcl} (x & , & y) \\ (1.6418599 & , & 0.85866981) \\ (1.61803399 & , & 0.61803399) \\ (1.249536927 & , & -1.249536927) \\ (-0.61803399 & , & -1.61803399) \\ (-0.85866981 & , & -1.6418599) \end{array}$$

(1) The second and fourth solutions seem to lie on the line $y = x - 1$, suggesting that x satisfies $x^7 - 13(x - 1) = 21$, so $x^7 - 13x - 8 = 0$. Factoring,

$$x^7 - 13x - 8 = (x^2 - x - 1)(x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8)$$

and the quadratic factor has (well-known) roots

$$\alpha = \frac{1 + \sqrt{5}}{2} \approx 1.618034 \text{ and } \beta = \frac{1 - \sqrt{5}}{2} \approx -0.618034.$$

Thus we actually know the second and fourth solutions are $(\alpha, -\beta)$ and $(\beta, -\alpha)$.

We verify that $(\alpha, -\beta)$ is indeed a solution to the given system. Note that by the first well-known relationship to the Fibonacci numbers, $\alpha^n = \alpha F_n + F_{n-1}$, we have $\alpha^7 = \alpha F_7 + F_6 = 13\alpha + 8$.

Now, substituting into the first equation:

$$\begin{array}{rcl} \alpha^7 - 13(-\beta) & = & \alpha^7 - 13(\alpha - 1) \\ & = & \alpha^7 - 13\alpha + 13 \\ & = & 13\alpha + 8 - 13\alpha + 13 = 21, \text{as desired.} \end{array}$$

It is also straight forward to verify the second equation: $13\alpha - (-\beta)^7 = 21$, using $\alpha\beta = -1$.

(2) The third solution lies on the line $y = -x$, so x is the sole real zero of $x^7 + 13x - 21 = 0$. This polynomial equation is not solvable in radicals – according to Maple, the Galois group of $x^7 + 13x - 21$ is S_7, which is not a solvable group. Hence, an approximation is probably the best we can do (barring some ingenious treatment employing transcendental functions.)

Unfortunately, we do not have any analytic characterization of the first and fifth solutions.

A final comment: the problem involves the exponent 7 and the Fibonacci numbers $F_7 = 13$ and $F_8 = 21$, so there is almost certainly a more general version with solution $(\alpha, -\beta)$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5021: *Proposed by Kenneth Korbin, New York, NY.*

Given

$$\frac{x+x^2}{1-34x+x^2} = x + 35x^2 + \cdots + a_n x^n + \cdots$$

Find an explicit formula for a_n.

Solution by David E. Manes, Oneonta, NY.

An explicit formula for a_n is given by

$$a_n = -\frac{1}{8}\Big[(4-3\sqrt{2})(17+12\sqrt{2})^n + (4+3\sqrt{2})(17-12\sqrt{2})^n\Big].$$

Let $F(x) = \dfrac{x+x^2}{1-34x+x^2}$ be the generating function for the sequence $\Big(a_n\Big)_{n\geq 1}$, where $a_1 = 1$ and $a_2 = 35$.

Then the characteristic equation is $\lambda^2 - 34\lambda + 1 = 0$, with roots $r_1 = 17 + 12\sqrt{2}$ and $r_2 = 17 - 12\sqrt{2}$.

Therefore,

$$a_n = \alpha\Big(17+12\sqrt{2}\Big)^n + \beta\Big(17-12\sqrt{2}\Big)^n$$

for some real numbers α and β. From the initial conditions one obtains

$$\begin{aligned} 1 &= \alpha\Big(17+12\sqrt{2}\Big) + \beta\Big(17-12\sqrt{2}\Big) \\ 35 &= \alpha\Big(17+12\sqrt{2}\Big)^2 + \beta\Big(17-12\sqrt{2}\Big)^2. \end{aligned}$$

The solution for this system of equations is

$$\begin{aligned} \alpha &= -\frac{1}{8}\Big(4-3\sqrt{2}\Big) \\ \beta &= -\frac{1}{8}\Big(4+3\sqrt{2}\Big). \end{aligned}$$

Hence, if $n \geq 1$, then

$$a_n = -\frac{1}{8}\Big[(4-3\sqrt{2})(17+12\sqrt{2})^n + (4+3\sqrt{2})(17-12\sqrt{2})^n\Big].$$

Also solved by Brian D. Beasley, Clinton, SC; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro

Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Chesapeake, VA; David Stone and John Hawkins, Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5022: *Proposed by Michael Brozinsky, Central Islip, NY.*

Show that

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right)$$

is proportional to $\sin(x)$.

Solution 1 by José Hernández Santiago, (student, UTM), Oaxaca, México.

From the well-known identity $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$, we derive that

$$\begin{aligned}
\sin 3\theta &= 4\sin\theta\Big(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\Big)\\
&= 4\sin\theta\Big(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\Big)\Big(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\Big)\\
&= 4\sin\theta\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right).
\end{aligned}$$

When we let $\theta = \dfrac{x}{3}$, the latter formula becomes:

$$\sin 3\Big(\frac{x}{3}\Big) = 4\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi-x}{3}\right)\sin\left(\frac{\pi+x}{3}\right) \qquad (1)$$

Now, the fact that

$$\begin{aligned}
\sin\left(\frac{x+2\pi}{3}\right) &= \sin\left(\frac{x-\pi}{3}+\pi\right)\\
&= \sin\left(\frac{x-\pi}{3}\right)\cos\pi\\
&= \sin\left(\frac{\pi-x}{3}\right)
\end{aligned}$$

allows us to put (1) in the form

$$\sin x = 4\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{x+2\pi}{3}\right);$$

and clearly this is equivalent to what the problem asked us to demonstrate.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

Since

$$\begin{aligned}
\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right) &= \frac{1}{2}\Big(\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2x}{3}\right)\Big)\\
&= \frac{1}{2}\Big(\frac{1}{2}+1-2\sin^2\left(\frac{x}{3}\right)\Big)\\
&= \frac{1}{4}\Big(3-4\sin^2\left(\frac{x}{3}\right)\Big),
\end{aligned}$$

so

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right)=\frac{1}{4}\left(3\sin\left(\frac{x}{3}\right)-4\sin^3\left(\frac{x}{3}\right)\right)=\frac{1}{4}\sin(x),$$

which is proportional to $\sin(x)$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kenneth Korbin, NY, NY: Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles, McCracken, Dayton, OH; John Nord, Spokane, WA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5023: *Proposed by M.N. Deshpande, Nagpur, India.*

Let $A_1A_2A_3\cdots A_n$ be a regular $n-$gon $(n\geq 4)$ whose sides are of unit length. From A_k draw L_k parallel to $\text{A}_{k+1}A_{k+2}$ and let L_k meet L_{k+1} at T_k. Then we have a "necklace" of congruent isosceles triangles bordering $A_1A_2A_3\cdots A_n$ on the inside boundary. Find the total area of this necklace of triangles.

Solution 1 by Paul M. Harms, North Newton, KS.

In order that the "necklace" of triangles have the $n-$gon as an inside boundary, it appears that line L_k (through A_k) should be parallel to $A_{k-1}A_{k+1}$ rather than $A_{k+1}A_{k+2}$. With this interpretation in mind, we now consider the n isosceles triangles with a vertex at the center of the $n-$gon and the opposite side being a side of unit length. The measure of the central angles are $360^o/n$. The angle inside the n-gon at the intersection of 2 unit sides is twice one of the equal angles of the isosceles triangles with a vertex at the center of the $n-$gon, so it has a degree measure of $180^{\text{o}}-(360^{\text{o}}/n)$.

The isosceles triangle $A_{k-1}A_kA_{k+1}$ has two equal angles (opposite the sides of unit length) with a measure of

$$\frac{1}{2}\left(180^o-(180^o-(360^o/n))\right)=\frac{180^o}{n}.$$

A side of length one intersects the two parallel lines $(A_{k-1}A_{k+1})$ and the line parallel to it through A_k. Using equal angles for a line intersecting parallel lines, we see that the equal angles in one necklace isosceles triangle has a measure of $180^o/n$.

Using the side of length one as a base, the area of one necklace triangle is

$$\frac{1}{2}(\text{base})\cdot(\text{height})=\frac{1}{2}(1)\left(\frac{1}{2}\tan(180^{\text{o}}/n)\right)=\frac{1}{4}\tan(180^{\text{o}}/n).$$

The total area of n necklace triangles is $\dfrac{n}{4}\tan(180^o/n)$. It is interesting to note that the total area approaches $\pi/4$ as n gets large.

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

David and John looked at the problem a bit differently than the other solvers. They wrote: "In order to get a clearer picture of what is going on, we introduce additional points that we will call B_k, where we define B_k to be the intersection of L_k and L_{k-2}, for $3 \le k \le n$ and the intersection of L_k and L_{k+n-2} for $k = 1$ or 2."

Doing this gave them a "necklace of isosceles triangles with bases along the interior boundary of the polygon: $\triangle A_1B_1A_2, \triangle A_2B_2A_3, \triangle A_3B_3A_4, \cdots, \triangle A_nB_nA_1$." *(Note that by doing this A_kT_k does pass through A_{k+3}.)*

They went on: "It is not clear that this was the intended necklace, because these triangles do not involve the points T_k. Let's call this the Perimeter Necklace."

There is a second necklace of isosceles triangle whose bases do involve the points $T_k : \triangle T_1B_3T_2,\ \triangle T_2B_4T_3,\ \triangle T_3B_5T_4,\ \cdots,\ \triangle T_{n-2}B_nTn_1,\ \triangle T_{n-1}B_1T_1,\ \triangle T_nB_2T_1$. Let's call this the Inner Necklace.

They then found the areas for both necklaces and summarized their results as follows:

$n = 4$: Area of Perimeter Necklace $= 0$. No Inner Necklace.
$n = 5$:

$$\begin{aligned}\text{Area of Perimeter Necklace} &= \frac{5}{4}\tan\left(\frac{\pi}{5}\right)\\ \text{Area of Inner Necklace} &= \frac{5}{4}\left(1-\tan^2\frac{\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)\end{aligned}$$

$n = 6$

$$\begin{aligned}\text{Area of Perimeter Necklace} &= \frac{3}{2}\tan\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}\\ \text{Area of Inner Necklace} &= 0.\end{aligned}$$

$n > 5$

$$\begin{aligned}\text{Area of Perimeter Necklace} &= \frac{n}{4}\tan\left(\frac{2\pi}{n}\right)\\ \text{Area of Inner Necklace} &= \frac{\pi}{4}\left(4\sin\frac{2\pi}{n}\cos\frac{2\pi}{n} - 4\sin\frac{2\pi}{n} + \tan\frac{2\pi}{n}\right)\end{aligned}$$

Note that these give the correct results for n=6.

Then they used Excel to compute the areas of the necklaces for various values of n, and proved that for large values of n, the ratio of the areas approaches one.

n	$PerimeterNecklace$	$InnerNecklace$
6	2.59876211	0
10	1.81635632	0.693786379
100	1.572747657	1.561067973
500	1.570879015	1.570382935

$$\lim_{n\to\infty} \frac{\frac{n}{4}\left(4\sin\frac{2\pi}{n}\sin\frac{2\pi}{n} - 4\sin\frac{2\pi}{n} + \tan\frac{2\pi}{n}\right)}{\frac{n}{4}\tan\frac{2\pi}{n}} = 1.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; Grant Evans (student, Saint George's School), Spokane, WA; Boris Rays, Chesapeake, VA, and the proposer.

- 5024: *Proposed by Luis Díaz-Barrero and Josep Rubió-Masseguú, Barcelona, Spain.*

  Find all real solutions to the equation

$$\sqrt{1+\sqrt{1-x}}-2\sqrt{1-\sqrt{1-x}}=\sqrt[4]{x}.$$

Solution by Jahangeer Kholdi, Portsmouth, VA.

Square both sides of the equation, simplify, and then factor to obtain

$$5(1-\sqrt{x})=3\sqrt{1-x}.$$

Squaring again gives $17x-25\sqrt{x}+8=0$, and now using the quadratic formula gives $x=1$ and $x=\dfrac{64}{289}$. But $x=1$ does not satisfy the original equation. The only real solution to the original equation is $x=\dfrac{64}{289}$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Grant Evans (student, Saint George's School), Spokane, WA; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kenneth Korbin, NY, NY; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Wattana Namkaew (student, Nakhon Ratchasima Rajabhat University), Thailand; John Nord, Spokane, WA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposers.

- 5025: *Ovidiu Furdui, Toledo, OH.*

  Calculate the double integral

$$\int_0^1\int_0^1\{x-y\}dxdy,$$

where $\{a\}=a-[a]$ denotes the fractional part of a.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

$$\begin{aligned}
\int_0^1\int_0^1\{x-y\}dxdy &= \int_0^1\int_0^x\{x-y\}dydx+\int_0^1\int_0^y\{x-y\}dxdy\\
&= \int_0^1\int_0^x(x-y)dydx+\int_0^1\int_0^y(x-y+1)dxdy\\
&= \int_0^1\left(xy-\frac{y^2}{2}\right)\Bigg|_0^x dx+\int_0^1\left(\frac{x^2}{2}-xy+x\right)\Bigg|_0^y dy\\
&= \int_0^1\frac{x^2}{2}dx+\int_0^1\left(y-\frac{y^2}{2}\right)dy
\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{x^3}{6}\right|_0^1 + \left.\left(\frac{y^2}{2} - \frac{y^3}{6}\right)\right|_0^1 \\
&= \frac{1}{2}.
\end{aligned}
$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Nate Kirsch and Isaac Bryan (jointly, students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; Matthew Hussey, Rachel DeMeo, Brian Tencher (jointly, students at Taylor University), Upland, IN; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Nicki Reishus, Laura Schindler, Landon Anspach and Jessi Byl (jointly, students at Taylor University), Upland, IN; José Hernández Santiago (student, UTM), Oaxaca, México, Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
March 15, 2009

- 5044: *Proposed by Kenneth Korbin, New York, NY.*

  Let N be a positive integer and let

  $$\begin{cases} x = 9N^2 + 24N + 14 \text{ and} \\ y = 9(N+1)^2 + 24(N+1) + 14. \end{cases}$$

  Express the value of y in terms of x, and express the value of x in terms of y.

- 5045: *Proposed by Kenneth Korbin, New York, NY.*

  Given convex cyclic hexagon ABCDEF with sides

  $$\begin{array}{rcl} \overline{AB} & = & \overline{BC} = 85 \\ \overline{CD} & = & \overline{DE} = 104, \text{ and} \\ \overline{EF} & = & \overline{FA} = 140. \end{array}$$

  Find the area of $\triangle BDF$ and the perimeter of $\triangle ACE$.

- 5046: *Proposed by R.M. Welukar of Nashik, India and K.S. Bhanu, and M.N. Deshpande of Nagpur, India.*

  Let $4n$ successive Lucas numbers $L_k, L_{k+1}, \cdots, L_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below:

  $$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 2n \end{array} \\ \begin{pmatrix} L_k & L_{k+3} & L_{k+4} & L_{k+7} & \cdots & L_{k+4n-1} \\ & & & & & \\ L_{k+1} & L_{k+2} & L_{k+5} & L_{k+6} & \cdots & L_{k+4n-2} \end{pmatrix} \end{array}$$

  Show that the sum of the elements of the first and second row denoted by R_1 and R_2 respectively can be expressed as

  $$R_1 = 2F_{2n}L_{2n+k}$$

  $$R_2 = F_{2n}L_{2n+k+1}$$

  where $\{L_n, n \geq 1\}$ denotes the Lucas sequence with $L_1 = 1, L_2 = 3$ and $L_{i+2} = L_i + L_{i+1}$ for $i \geq 1$ and $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence,
  $F_1 = 1, F_2 = 1, F_{n+2} = F_n + F_{n+1}$.

- 5047: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  Find a procedure for continuing the following pattern:

$$\begin{aligned}
S(n,0) &= \sum_{k=0}^{n}\binom{n}{k} = 2^n \\
S(n,1) &= \sum_{k=0}^{n}\binom{n}{k}k = 2^{n-1}n \\
S(n,2) &= \sum_{k=0}^{n}\binom{n}{k}k^2 = 2^{n-2}n(n+1) \\
S(n,3) &= \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}n^2(n+3) \\
&\vdots
\end{aligned}$$

- 5048: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*

  Let $a, b, c,$ be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \geq \frac{54}{(a+b+c)^2}\frac{(abc)^3}{\sqrt{(ab)^4+(bc)^4+(ca)^4}}.$$

- 5049: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find a function $f : \Re \to \Re$ such that

$$2f(x)+f(-x) = \begin{cases} -x^3-3, & x \leq 1, \\ 3-7x^3, & x > 1. \end{cases}$$

Solutions

- 5026: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0),\ B(12,0),\ C(4,15)$, and $D(0,4)$. Point P has coordinates $(x,3)$. Find the value of x if

$$\text{area } \triangle\text{PAD} + \text{area } \triangle\text{PBC} = \text{area } \triangle\text{PAB} + \text{area } \triangle\text{PCD}. \qquad (1)$$

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

$$(1) \quad \Leftrightarrow \quad \frac{1}{2}\left| det\begin{pmatrix} x & 3 & 1 \\ -3 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| + \frac{1}{2}\left| det\begin{pmatrix} x & 3 & 1 \\ 12 & 0 & 1 \\ 4 & 15 & 1 \end{pmatrix}\right|$$

$$+\frac{1}{2}\left|det\begin{pmatrix} x & 3 & 1\\ -3 & 0 & 1\\ 12 & 0 & 1\end{pmatrix}\right| + \frac{1}{2}\left|det\begin{pmatrix} x & 3 & 1\\ 4 & 15 & 1\\ 0 & 4 & 1\end{pmatrix}\right|$$

$$\Leftrightarrow \quad |-4x-3| \ + \ |156-15x| \ = \ 45 \ + \ |11x+4|. \qquad (2)$$

If $x \leq \dfrac{-3}{4}$, then (2) $\Leftrightarrow -4x-3-15x+156 = 45-11x-4 \Leftrightarrow x = 14$, impossible.
If $\dfrac{-3}{4} < x \leq \dfrac{-4}{11}$, then (2) $\Leftrightarrow 4x+3-15x+156 = 45-11x-4 \Leftrightarrow x = 159 = 41$, impossible.
If $\dfrac{-4}{11} < x \leq \dfrac{52}{5}$, then (2) $\Leftrightarrow 4x+3-15x+156 = 45+11x+4 \Leftrightarrow x = 5$.
If $x > \dfrac{52}{5}$, then (2) $\Leftrightarrow 4x+3+15x-156 = 45+11x+4 \Leftrightarrow x = \dfrac{101}{4}$.
Thus, there are two possible values of $x:\ x = 5$ and $\dfrac{101}{4}$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Mark Cassell (student, St. George's School), Spokane, WA; Grant Evans (student, St. George's School), Spokane, WA; John Hawkins and David Stone (jointly), Statesboro, GA; Peter E. Liley, Lafayette, IN; Paul M. Harms, North Newton, KS; Charles, McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Britton Stamper, (student, St. George's School), Spokane, WA; Vu Tran (student, Texas A&M University), College Station, TX, and the proposer.

- 5027: *Proposed by Kenneth Korbin, New York, NY.*

Find the x and y intercepts of

$$y = x^7 + x^6 + x^4 + x^3 + 1.$$

Solution by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.

The point $(0,1)$ is trivial. To find the x intercept we decompose $x^7+x^6+x^4+x^3+1 = (x^4+x^3+x^2+x+1)(x^3-x+1)$ and the value we are looking for is given by $x^3-x+1=0$ since

$$x^4+x^3+x^2+x+1 = (x^2 - x\frac{-1+\sqrt{5}}{2}+1)(x^2 - x\frac{-1-\sqrt{5}}{2}+1) \neq 0.$$

Applying the formula for solving cubic equations, the only real root of $x^3-x+1=0$ is

$$\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{27}}\right)^{1/3} + \left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{27}}\right)^{1/3} = \left(-\frac{1}{2}+\sqrt{\frac{69}{18}}\right)^{1/3} + \left(-\frac{1}{2}-\sqrt{\frac{69}{18}}\right)^{1/3}$$

whose approximate value is $-1.3247\ldots$

Also solved by Brian D. Beasley, Clinton, SC; Mark Cassell and Britton Stamper (jointly, students at St. George's School), Spokane, WA; Michael

Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA, and the propser.

- 5028: *Proposed by Michael Brozinsky, Central Islip, NY .*

If the ratio of the area of the square inscribed in an isosceles triangle with one side on the base to the area of the triangle uniquely determine the base angles, find the base angles.

Solution 1 by Brian D. Beasley, Clinton, SC.

Let θ be the measure of each base angle in the triangle, and let y be the length of each side opposite a base angle. Let x be the side length of the inscribed square. We first consider the right triangle formed with θ as an angle and x as a leg, denoting its hypotenuse by z. Then $x = z\sin\theta$. Next, we consider the isosceles triangle formed with the top of the inscribed square as its base; taking the right half of the top of the square as a leg, we form another right triangle with angle θ and hypotenuse $y-z$. Then $\frac{1}{2}x = (y-z)\cos\theta$, so $y = x(\csc\theta + \frac{1}{2}\sec\theta)$. Denoting the area of the square by S and the area of the original triangle by T, we have

$$\frac{T}{S} = \frac{\frac{1}{2}y^2\sin(\pi-2\theta)}{x^2} = \frac{1}{2}\sin(2\theta)\left(\csc\theta + \frac{1}{2}\sec\theta\right)^2 = \frac{1}{4}\tan\theta + \cot\theta + 1.$$

Let $f(\theta) = \frac{1}{4}\tan\theta + \cot\theta + 1$ for $0 < \theta < \pi/2$. Then it is straightforward to verify that

$$\lim_{\theta\to 0^+} f(\theta) = \lim_{\theta\to\frac{\pi}{2}^-} f(\theta) = \infty$$

and that f attains an absolute minimum value of 2 at $\theta = \arctan(2)$. Hence the ratio T/S (and thus S/T) is uniquely determined when $\theta = \arctan(2) \approx 63.435°$.

Solution 2 by J. W. Wilson, Athens, GA.

With no loss of generality, let the base of the isosceles triangle b be a fixed value and vary the height h of the triangle. Then if $f(h)$ is a function giving the ratio for the compared areas, in order for it to uniquely determine the base angles, there must be either a minimum or maximum value of the function. Let $f(h)$ represent the ratio of the area of the triangle to the area of the square.

It is generally known (and easy to show) that side s of an inscribed square on base b of a triangle is on-half of the harmonic mean of the base b and the altitude h to that base. Thus

$$\begin{aligned}
s &= \frac{hb}{h+b}. \text{ So,} \\
f(h) &= \frac{bh}{2s^2}. \text{ Substituting and simplifying this gives :} \\
f(h) &= \frac{h^2+2bh+b^2}{2bh}.
\end{aligned}$$

For $h > 0$ it can be shown, by using the arithmetic mean−geometric mean inequality, that this function has a minimum value of 2 when $h = b$.

$$f(h) = \frac{h^2+2bh+b^2}{2bh}$$

$$= \frac{h + 2b + \dfrac{b^2}{h}}{2b}.$$

Since b is fixed, and using the arithmetic mean$-$geometric mean inequaltiy, we may write:

$$h + \frac{b^2}{h} \quad \geq \quad 2\sqrt{h\frac{b^2}{h}} = 2b, \text{ with equality holding if, and only if,}$$

$$h \quad = \quad \frac{b^2}{h}.$$

Therefore $f(h)$ reaches a maximum if, and only if, $h = b$. This means the base angles can be uniquely determined when the altitude and the base are the same length. Thus, by considering the right triangle formed by the altitude and the base, the base angle would be arctan 2.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly; two solutions), Statesboro, GA; Peter E. Liley, Lafayette, IN; Kenneth Korbin, New York, NY; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA, and the proposer.

- 5029: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $x > 1$ be a non-integer number. Prove that

$$\left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right) > \frac{9}{2},$$

where $[x]$ and $\{x\}$ represents the entire and fractional part of x.

Solution by John Hawkins and David Stone, Statesboro, GA.

We improve the lower bound by verifying the more accurate inequality

$$\# \quad \left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right) > \frac{16}{3}.$$

In fact, $\dfrac{16}{3}$ is a sharp lower bound for $\left(\dfrac{x + \{x\}}{[x]} - \dfrac{[x]}{x + \{x\}}\right) + \left(\dfrac{x + [x]}{\{x\}} - \dfrac{\{x\}}{x + [x]}\right)$ for x in the interval $(1, 2)$, while this expression becomes much larger for larger x.

For convenience, we let

$$f(x) = \left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right).$$

The function f, defined for $x > 1$, x not an integer, has a"repetitive" behavior. Its graph has a vertical asymptote at each positive integer. On each interval $(n, n + 1)$, $f(x)$ decreases(strictly) from infinity to a specific limit, h_n (which we will specify), then repeats the behavior on the next interval, but does not drop down as far, because $h_n < h_{n+1}$ (so $f(x)$ never comes close to $h_1 = \frac{16}{3}$ again.)

We verify these statements by fixing n and examining the behavior on $f(x)$ on the interval $(n, n+1)$. In this case, we let $x = n+t$, where $0 < t < 1$; therefore, $[x] = n$ and $|x| = t$. Thus

$$\begin{aligned} f(x) &= \left(\frac{n+t+t}{n} - \frac{n}{n+t+t}\right) + \left(\frac{n+t+n}{t} - \frac{t}{n+t+n}\right) \\ &= \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t}. \end{aligned}$$

We handle the above claims in order:

(1) $\lim\limits_{t\to 0^+} f(x) = \lim\limits_{t\to 0^+} \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t} = +\infty$.

(2) Because $f(x)$ has been expressed in terms of t, say

$$g(t) = \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t},$$

we can show that $g(t)$ is decreasing by showing its derivative is negative.

We compute the derivative with respect to t:

$$g'(t) = \frac{2}{n} + \frac{2n}{(2t+n)^2} - \frac{2n}{t^2} - \frac{2n}{(t+2n)^2}.$$

Basically, this is negative because of the dominant term $\frac{-2n}{t^2}$, but we can make this more precise:

$$\begin{aligned} g'(t) &< 0 \\ \Leftrightarrow\ & \frac{2}{n} + \frac{2n}{(2t+n)^2} - \frac{2n}{t^2} - \frac{2n}{(t+2n)^2} < 0 \\ \Leftrightarrow\ & \frac{1}{n} + \frac{n}{(2t+n)^2} < \frac{n}{t^2} + \frac{n}{(t+2n)^2} \\ \Leftrightarrow\ & \frac{(2t+n)^2 + n^2}{n(2t+n)^2} < \frac{n(t+2n)^2 + nt^2}{t^2(t+2n)^2} \\ \Leftrightarrow\ & t^2(t+2n)^2\left[(2t+n)^2 + n^2\right] < n(2t+n)^2\left[n(t+2n)^2 + nt^2\right] \\ \Leftrightarrow\ & t^2(t+2n)^2\left[(2t^2 + 2tn + n^2\right] < n^2(2t+n)^2\left[t^2 + 2tn + 2n^2\right] \\ \Leftrightarrow\ & 2t^6 + 10t^5 n + 17t^4 n^2 + 12t^3 n^3 + 4t^2 n^4 < 2n^6 + 10n^5 t + 17n^4 t^2 + 12n^3 t^3 + 4n^2 t^4 \\ \Leftrightarrow\ & 0 < 2\left(n^6 - t^6\right) + 10nt\left(n^4 - t^4\right) + 17n^2t^2\left(n^2 - t^2\right) - 4n^2t^2\left(n^2 - t^2\right) \\ \Leftrightarrow\ & 0 < 2\left(n^6 - t^6\right) + 10nt\left(n^4 - t^4\right) + 13n^2t^2\left(n^2 - t^2\right), \end{aligned}$$

and this last inequality is true because $0 < t < 1 < n$.

(3) Finally, we compute the lower bound at the right-hand endpoint:

$$\lim_{t\to 1^-} f(x) = \lim_{t\to 1^-}\left[\frac{n+2t}{n} - \frac{n}{n+2} + \frac{2n+t}{t} - \frac{t}{2n+t}\right]$$

$$
\begin{aligned}
&= \frac{n+2}{n} - \frac{n}{n+2} + \frac{2n+1}{1} - \frac{1}{2n+1} \\
&= 2n+1 - \frac{1}{2n+1} + \frac{4(n+1)}{n(n+2)}.
\end{aligned}
$$

Thus, we see that $h_n = 2n+1 - \dfrac{1}{2n+1} + \dfrac{4(n+1)}{n(n+2)} \approx 2n+1$, so the intervals' lower bounds increase linearly with n.

Note that $h_1 = 3 + \dfrac{7}{3} = \dfrac{16}{3}$, so $f(x) > \dfrac{16}{3}$ for $1 < x < 2$. So inequality (#) has been verified.

As stated above, the lower bound on x then grows, for instance,

$h_2 = 5 + \dfrac{13}{10} = \dfrac{63}{10}$, so $f(x) > \dfrac{63}{10}$ for $2 < x < 3$,

and

$h_3 = 7 + \dfrac{97}{105} = \dfrac{832}{105}$, so $f(x) > \dfrac{832}{105}$ for $3 < x < 4$.

Comment: The inequality # is sharp in the sense that no value larger than $\dfrac{16}{3}$ can be used. That is, by (3) above, we know that values of x very close to 2 produce values of $f(x)$ just above and arbitrarily close to $\dfrac{16}{3}$. We can see this precisely:

$$
\begin{aligned}
f\left(2 - \frac{1}{m}\right) &= f\left(1 + \frac{m-1}{m}\right) \\
&= \left(\frac{\frac{2m-1}{m} + \frac{m-1}{m}}{1} - \frac{1}{\frac{2m-1}{m} + \frac{m-1}{m}} \right) + \left(\frac{\frac{2m-1}{m} + 1}{\frac{m-1}{m}} - \frac{\frac{m-1}{m}}{\frac{2m-1}{m} + 1} \right) \\
&= \frac{3m-2}{m} - \frac{m}{3m-2} + \frac{3m-1}{m-1} - \frac{m-1}{3m-1} \\
&= \frac{16}{3} + \frac{2}{3}\left[\frac{3}{m(m-1)} - \frac{1}{3(m-1)(3m-2)} \right].
\end{aligned}
$$

(John and David accompanied their above solution with a graph generated by Maple showing how the lower bounds increase from $\dfrac{16}{3}$ for various values of x.)

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; Vu Tran (student, Texas A&M University), College Station, TX; Boris Rays, Chesapeake, VA, and the proposer.

- 5030: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A_1, A_2, \cdots, A_n \in M_2(\mathbf{C}), (n \geq 2)$, be the solutions of the equation $X^n = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

Prove that $\sum_{k=1}^{n} Tr(A_k) = 0$.

Solution by John Hawkins and David Stone, Statesboro, GA.

The involvement of the Trace function is a red herring. Actually, for $A_1, A_2, A_3, \ldots, A_n$ as specified in the problem, we have $\sum_{k=1}^{n} A_k = 0$. Therefore, since Tr is linear, $\sum_{k=1}^{n} Tr(A_k) = Tr\left(\sum_{k=1}^{n} (A_k\right) = Tr(0) = 0$. In fact $\sum_{k=1}^{n} Tr(A_k) = 0$ for any linear transformation $T : M_2(C) \longrightarrow W$ to any complex vector space W.

Here is our argument. For convenience, let $B = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Note that $B^2 = 5B$. Thus $B^3 = BB^2 = B5B = 5B^2 = 5^2B$. Inductively, $B^k = 5^{k-1}B$ for $k \geq 1$.

Therefore, $B = \dfrac{1}{5^{n-1}}B^n = \left[\dfrac{1}{5^{(n-1)/n}}B\right]^n$, so $\mathrm{A}_1 = \dfrac{1}{5^{(n-1)/n}}B$ is an n^{th} root of B:

$$A_1^n = \left[\frac{1}{5^{(n-1)/n}}B\right]^n = \frac{1}{5^{n-1}}B^n = \frac{1}{5^{n-1}}5^{n-1}B = B.$$

Now let $\xi = e^{2\pi i/n}$ be the primitive n^{th} root of unity. Then

$$0 = \xi^n - 1 = (\xi - 1)(\xi^{n-1} + \xi^{n-2} + \xi^{n-3} + \cdots + \xi + 1),$$

so,

$$(\#) \qquad (\xi^{n-1} + \xi^{n-2} + \xi^{n-3} + \cdots + \xi + 1) = 0.$$

With $A_1 = \dfrac{1}{5^{(n-1)/n}}B$ as above, let $A_k = \xi^{k-1}A_1$ for $k = 2, 3, \ldots, n$. These n distinct matrices are the n^{th} roots of B, namely:

$$A_k^n = [\xi^{k-1}A_1]^n = \xi^{(k-1)n}A_1^n = (\xi^n)^{k-1}A_1^n = 1^{k-1}A_1^n = A_1^n = B.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} A_k &= \sum_{k=1}^{n} \xi^{k-1}A_1 = \left(\sum_{k=1}^{n} \xi^{k-1}\right)A_1 \\ &= 0 \cdot A_1 \text{ by } (\#) \\ &= 0. \end{aligned}$$

Comment 1: Implicit in the problem statement is that the given matrix equation has exactly n solutions. This is true for this particular matrix B. But it is not true in general. Gantmacher ("Matrix Theory", page 233) gives an example of a 3×3 matrix with infinitely many square roots: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Comment 2: The result would be true for B any 2×2 matrix having determinant zero but trace non-zero. In that case, we would have $B^2 = Tr(B)B$ and we use
$A_1 = \dfrac{1}{Tr(B)^{(n-1)/n}} B$.

Comment 3: More generally, let V be a vector space over C and $c_1, c_2, \ldots, c_n$ be complex scalars whose sum is zero. Also let A be any vector in V and let $A_k = c_k A$ for $k = 1, 2, \cdots, n$. Then

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} c_k A = \Big(\sum_{k=1}^{n} c_k\Big) A = 0 \cdot A = 0.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- 5031: *Ovidiu Furdui, Toledo, OH.*

Let x be a real number. Find the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \Big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\Big).$$

Solution 1 by Paul M. Harms, North Newton, KS.

We know that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$.

The expression
$(-1)^{n-1} n \Big(e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^n}{n!}\Big) = (-1)^{n-1} n \Big(\dfrac{x^{n+1}}{(n+1)!} + \dfrac{x^{n+2}}{(n+2)!} + \cdots\Big)$.

So the sum $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n \Big(\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots\Big)$ equals

$$\begin{aligned}
& \Big(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\Big) - 2\Big(\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\Big) + 3\Big(\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\Big) - 4\Big(\frac{x^5}{5!} + \cdots\Big) + \cdots \\
= & \frac{(1)x^2}{2!} + \frac{(1-2)x^3}{2!} + \frac{(1-2+3)x^4}{4!} + \frac{(1-2+3-4)x^5}{5!} + \cdots \\
= & \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{2x^4}{4!} - \frac{2x^5}{5!} + \frac{3x^6}{6!} - \frac{3x^7}{7!} + \frac{4x^8}{8!} - \frac{4x^9}{9!} \cdots
\end{aligned}$$

We need to find the sum of this alternating series..

We have

$$\begin{aligned}
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots \\
\frac{x}{2} \sinh x &= \frac{1}{2} x^2 + \frac{\frac{2}{4} x^4}{3!} + \frac{\frac{3}{6} x^6}{5!} + \frac{\frac{4}{8} x^8}{7!} + \cdots \\
&= \frac{x}{2!} + \frac{2x^4}{4!} + \frac{3x^6}{6!} + \frac{4x^8}{8!} + \cdots.
\end{aligned}$$

The positive terms of the alternating series sum to $\dfrac{x}{2} \sinh x$. Each negative term of the alternating series is an antiderivative of the previous term except for the minus sign. The general anitderivative of $\frac{x}{2}\sinh x$ is $\frac{1}{2}\Big[x\cosh x-\sinh x\Big]+C$. Using Taylor series we can show that $\frac{-1}{2}\Big[x\cosh x-\sinh x\Big]$ equals the sum of the negative terms of the alternating series. The sum in the problem is

$$\frac{x}{2}\sinh x-\frac{1}{2}\Big[x\cosh x-\sinh x\Big]=\frac{x+1}{2}\sinh x-\frac{x}{2}\cosh x.$$

Solution 2 by N. J. Kuenzi, Oshkosh, WI.

Let

$$F(x)=\sum_{n=1}^{\infty}(-1)^{n-1}n\Big(e^x-1-x-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\Big).$$

Differentiation yields

$$\begin{aligned}
F'(x) &= \sum_{n=1}^{\infty}\Big((-1)^{n-1}n(e^x-1-x-\cdots-\frac{x^{n-1}}{(n-1)!}\Big)\\
&= \sum_{n=1}^{\infty}\Big((-1)^{n-1}n(e^x-1-x-\cdots-\frac{x^{n-1}}{(n-1)!}-\frac{x^n}{n!}+\frac{x^n}{n!}\Big)\\
&= F(x)+\sum_{n=1}^{\infty}(-1)^{n-1}n\frac{x^n}{n!}\\
&= F(x)+x\Big(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+(-1)^m\frac{x^m}{m!}+\cdots\Big)\\
&= F(x)+xe^{-x}.
\end{aligned}$$

Solving the differential equation

$$\begin{aligned}
F'(x) &= F(x)+xe^{-x} \text{ with initial conditions F(0) = 0 yields}\\
F(x) &= \frac{e^x-(1+2x)e^{-x}}{4}.
\end{aligned}$$

Also solved by Charles Diminnie and Andrew Siefker (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
April 15, 2009

- 5050: *Proposed by Kenneth Korbin, New York, NY.*

  Given $\triangle ABC$ with integer-length sides, and with $\angle A = 120^o$, and with $(a,b,c) = 1$. Find the lengths of b and c if side $a = 19$, and if $a = 19^2$, and if $a = 19^4$.

- 5051: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (x, y) such that $\dfrac{(x-y)^2}{x+y} = 8$ with $x < y$.

  Find a formula for obtaining additional pairs of these integers.

- 5052: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

  If $a \geq 0$, evaluate:

$$\int_0^{+\infty} arctg \frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\,\frac{dx}{1+x^2}.$$

- 5053: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b and c be the sides, r the in-radius, and R the circum-radius of $\triangle ABC$. Prove or disprove that

$$\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c} \leq 2rR.$$

- 5054: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let x, y, z be positive numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

- 5055: *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

  Let α be a positive real number. Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^{\alpha}}.$$

Solutions

- 5032: *Proposed by Kenneth Korbin, New York, NY.*

Given positive acute angles A, B, C such that

$$\tan A\cdot\tan B+\tan B\cdot\tan C+\tan C\cdot\tan A=1.$$

Find the value of $$\frac{\sin A}{\cos B\cdot\cos C}+\frac{\sin B}{\cos A\cdot\cos C}+\frac{\sin C}{\cos A\cdot\cos B}.$$

Solution 1 by Brian D. Beasley, Clinton, SC.

Since A, B, and C are positive acute angles with

$$\begin{aligned}1 &= \frac{\sin A\sin B\cos C+\cos A\sin B\sin C+\sin A\cos B\sin C}{\cos A\cos B\cos C}\\ &= \frac{\cos A\cos B\cos C-\cos(A+B+C)}{\cos A\cos B\cos C},\end{aligned}$$

we have $\cos(A+B+C)=0$ and thus $A+B+C=90^\circ$. Then
$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}=\frac{\sin A\cos A+\sin B\cos B+\sin C\cos C}{\cos A\cos B\cos C}.$$
Letting N be the numerator of this latter fraction, we obtain

$$\begin{aligned}N &= \sin A\cos A+\sin B\cos B+\cos(A+B)\sin(A+B)\\ &= \sin A\cos A+\sin B\cos B+(\cos A\cos B-\sin A\sin B)(\sin A\cos B+\cos A\sin B)\\ &= \sin A\cos A(1+\cos^2 B-\sin^2 B)+\sin B\cos B(1+\cos^2 A-\sin^2 A)\\ &= \sin A\cos A(2\cos^2 B)+\sin B\cos B(2\cos^2 A)\\ &= 2\cos A\cos B(\sin A\cos B+\cos A\sin B)\\ &= 2\cos A\cos B\sin(A+B)\\ &= 2\cos A\cos B\cos C.\end{aligned}$$

Hence the desired value is 2.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

The condition $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$ is equivalent to $\cot A+\cot B+\cot C=\cot A\cot B\cot C$. Since it is well known that

$$\cos(A+B+C)=-\sin A\sin B\sin C\Big(\cot A+\cot B+\cot C-\cot A\cot B\cot C\Big),$$

so $\cos(A+B+C)=0$ and $A+B+C=\dfrac{\pi}{2}$. Hence,

$$\sin 2A+\sin 2B+\sin 2C \quad = \quad 2\sin(A+B)\cos(A-B)+2\sin C\cos C$$

$$
\begin{aligned}
&= 2\cos C\big(\cos(A-B)+\cos(A+B)\big)\\
&= 4\cos A\cos B\cos C.
\end{aligned}
$$

If follows that

$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}=\frac{\sin 2A+\sin 2B+\sin 2C}{2\cos A\cos B\cos C}=2.$$

Solution 3 by Boris Rays, Chesapeake, VA.

$\tan A\tan B+\tan B\tan C+\tan C\tan A=1$ implies,

$$
\begin{aligned}
\tan B(\tan A+\tan C) &= 1-\tan A\tan C\\
\frac{\tan A+\tan C}{1-\tan A\tan C} &= \frac{1}{\tan B}\\
\tan(A+C) &= \cot B=\tan(90^o-B).
\end{aligned}
$$

Similarly, we obtain:

$$
\begin{aligned}
\tan(B+C) &= \frac{1}{\tan A}=\cot A=\tan(90^o-A)\\
\tan(A+B) &= \frac{1}{\tan C}=\cot C=\tan(90^o-C),\text{which implies}\\
A &= 90^o-(B+C)\\
B &= 90^o-(A+C)\\
C &= 90^o-(A+B).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}\\
&= \frac{\sin(90^o-(B+C))}{\cos B\cos C}+\frac{\sin(90^o-(A+C))}{\cos A\cos C}+\frac{\sin(90^o-(A+B))}{\cos A\cos B}\\
&= \frac{\cos(B+C)}{\cos B\cos C}+\frac{\cos(A+C)}{\cos A\cos C}+\frac{\cos(A+B)}{\cos A\cos B}\\
&= \frac{\cos B\cos C-\sin B\sin C}{\cos B\cos C}+\frac{\cos A\cos C-\sin A\sin C}{\cos A\cos C}+\frac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}\\
&= \Big(1-\tan B\tan C\Big)\quad+\quad\Big(1-\tan A\tan C\Big)\quad+\quad\Big(1-\tan A\tan B\Big)\\
&= 1+1+1-(\tan A\tan B+\tan B\tan C+\tan A\tan C)\\
&= 3-1=2.
\end{aligned}
$$

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M.

Harms, North Newton, KS; John Hawkins, and David Stone (jointly), Statesboro, GA; Valmir Krasniqi, Prishtin, Kosovo; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5033: *Proposed by Kenneth Korbin, New York, NY.*

Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15)$, and $D(0,4)$. Point P is on side $\overline{AB}$ and point Q is on side $\overline{CD}$. Find the coordinates of P and Q if area $\triangle PCD =$ area $\triangle QAB = \frac{1}{2}$area quadrilateral $ABCD$. (1)

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

P is on side $\overline{AB} : y = 0 \Rightarrow P(p, 0)$.

Q is on side $\overline{CD} : y = \dfrac{11}{4}x + 4 \Rightarrow Q(4q,\ 11q + 4)$.

Area quadrilateral $ABCD$=area $\triangle ABD$+area $\triangle BCD$, so

$$(1) \quad \Leftrightarrow \quad \frac{1}{2}\left|\det\begin{pmatrix} p & 0 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix} 4q & 11q+4 & 1 \\ -3 & 0 & 1 \\ 12 & 0 & 1 \end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} -3 & 0 & 1 \\ 12 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| + \frac{1}{2}\left|\det\begin{pmatrix} 12 & 0 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right|$$

$$\Leftrightarrow \quad \left|11p + 16\right| = 30 + 74 = 15\left|11q + 4\right| \Leftrightarrow 11p + 16 = \pm 104 = 15(11q + 4)$$

$$\Leftrightarrow \quad P_1(8, 0) \text{ or } \mathrm{P}_2(-120/11, 0) \text{ and } \mathrm{Q}_1(16/15, 104/15) \text{ or } \mathrm{Q}_2(-656/165, -104/15).$$

Observations by Ken Korbin. The following four points are on a straight line: midpoint of $\overline{AC}$, midpoint of $\overline{BD}$, P_1, and Q_1. Moreover, the midpoint of $\overline{P_1P_2}$ = the midpoint of $\overline{Q_1, Q_2}$ = the intersection point of lines AB and CD.

Also solved by Brian D. Beasley, Clinton, SC; Michael N. Fried, Kibbutz Revivim, Israel; John Hawkins and David Stone (jointly), Statesboro, GA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5034: *Proposed by Roger Izard, Dallas, TX.*

In rectangle $MDCB$, $MB \perp MD$. F is the midpoint of BC, and points N, E and G lie on line segments DC, DM and MB respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be A_1 and let the area of quadrilateral $MGFE$ be A_2. Determine the area of quadrilateral $EDNF$ in terms of A_1 and A_2.

Solution by Paul M. Harms, North Newton, KS.

Put the rectangle $MDCB$ on a coordinate system. Assume all nonzero coordinates are positive with coordinates

$$M(0,0), B(0,b), C(c,b), D(c,0) \text{ and } E(e,0), F(c/2,b), G(0,g), N(c,g).$$

The coordinates satisfy $e < c$ and $g < b$. The area A_1 of the quadrilateral $MGFC =$ the area of $\triangle MGF+$ area of $\triangle MFC$. Then

$$A_1 = \frac{1}{2}g(c/2) + \frac{1}{2}(c/2)b = \frac{1}{2}(c/2)(b+g).$$

The area A_2 of the quadrilateral $MGFE =$ area of $\triangle MGF+$ area of $\triangle MEF$. Then

$$A_2 = \frac{1}{2}g(c/2) + \frac{1}{2}eb.$$

The area of the quadrilateral $EDNF =$ area of $\triangle EFD+$ area of $\triangle FDN$. The area of the quadrilateral $EDNF$ is then

$$\begin{aligned} &= \frac{1}{2}(c-e)b + \frac{1}{2}g(c/2) \\ &= 2(\frac{1}{2})(c/2)b - \frac{1}{2}eb + \frac{1}{2}g(c/2) \\ &= 2A_1 - A_2. \end{aligned}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly), Statesboro,GA; Kenneth Korbin, New York, NY; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

- 5035: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that

$$(a^a b^b c^c)^2 (a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)})^3 \ge 27.$$

Solution 1 by David E. Manes, Oneonta, NY.

Note that the inequality is equivalent to

$$\frac{3}{a^{\frac{1}{b+c}} + b^{\frac{1}{c+a}} + c^{\frac{1}{b+c}}} \le \sqrt[3]{a^{2a}b^{2b}c^{2c}}.$$

Since the problem is symmetrical in the variables a, b, and c, we can assume $a \ge b \ge c$. Therefore, $\ln a \ge \ln b \ge \ln c$. By the Rearrangement Inequality

$$\begin{aligned} a\ln a + b\ln b + c\ln c &\ge b\ln a + c\ln b + a\ln c \text{ and} \\ a\ln a + b\ln b + c\ln c &\ge c\ln a + a\ln b + b\ln c. \end{aligned}$$

Adding the two inequalities yields

$$2a\ln a + 2b\ln b + 2c\ln c \ge (b+c)\ln a + (c+a)\ln b + (a+b)\ln c.$$

Therefore,

$$\begin{aligned}
\ln\left(a^{2a}b^{2b}c^{2c}\right) &\geq \ln\left(a^{b+c}b^{c+a}c^{a+b}\right) \text{ or} \\
a^{2a}b^{2b}c^{2c} &\geq a^{b+c}b^{c+a}c^{a+b} \text{ and so} \\
\sqrt[3]{a^{2a}b^{2b}c^{2c}} &\geq \sqrt[3]{a^{b+c}b^{c+a}c^{a+b}}.
\end{aligned}$$

By the Harmonic-Geometric Mean Inequality

$$\frac{3}{a^{\frac{1}{b+c}}+b^{\frac{1}{c+a}}+c^{\frac{1}{b+c}}} \leq \sqrt[3]{a^{b+c}b^{c+a}c^{a+b}} \leq \sqrt[3]{a^{2a}b^{2b}c^{2c}}.$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.

Taking the logarithm we obtain,

$$2\sum_{\text{cyc}} \ln a + 3\ln\left(\sum_{\text{cyc}} a^{-(b+c)}\right) \geq 3\ln 3.$$

The concavity of the logarithm yields,

$$2\sum_{\text{cyc}} \ln a + 3\left(\ln 3 - \sum_{\text{cyc}} (b+c)\ln a\right) \geq 3\ln 3.$$

Defining $s = a+b+c$ gives,

$$\sum_{\text{cyc}} (3a-s)\ln a \geq 0.$$

Since the second derivative of the function $f(x) = (3x-s)\ln x$ is positive for any x and s, ($f''(x) = 3/x + s/x^2$) it follows that,

$$\sum_{\text{cyc}} (3a-s)\ln a \geq \sum_{\text{cyc}} (3a-s)\ln a\Big|_{a=s/3} = 0.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Boris Rays, Chesapeake, VA, and the proposer.

- 5036: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all triples (x, y, z) of nonnegative numbers such that

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3^x + 3^y + 3^z = 5 \end{cases}$$

Solution 1 by John Hawkins and David Stone, Statesboro, GA.

We are looking for all first octant points of intersection of the unit sphere with the surface $3^x + 3^y + 3^z = 5$. Clearly, the intercept points $(1,0,0), (0,1,0)$ and $(0,0,1)$ are solutions. We claim there no other solutions.

Consider the traces of our two surfaces in the xy-plane: the unit circle and the curve give by $3^x + 3^y = 4$. Our only concern is in the first quadrant, where we have a unit quarter circle and the curve $y = \dfrac{\ln(4-3^x)}{\ln 3}$. The two curves meet on the coordinate axes; otherwise graphing software shows that the logarithmic curve lies *inside* the quarter circle.

By the symmetry of the variables, we have the same behavior when we look at the traces in the xz- and yz-planes. That is, at our boundaries of concern, the exponential surface starts inside the sphere. By implicit differentiation of $3^x + 3^y + 3^z = 5$, we have the partial derivatives $\dfrac{\partial z}{\partial x} = -\dfrac{3^x}{3^z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{3^y}{3^z}$, which are both negative for nonnegative x, y and z. Therefore, the exponential surface *descends* from a trace inside the sphere to a trace which lies within the sphere. So the two surfaces have no points of intersection within the interior of the first octant.

Solution 2 by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Such triples are $(x, y, z) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. We note that the first equation implies that $x, y, z \in [0, 1]$. On the other hand, using Bernoulli's inequality we obtain that

$$\begin{cases} 3^x = (1+2)^x \le 1 + 2x \\ 3^y = (1+2)^y \le 1 + 2y \\ 3^z = (1+2)^z \le 1 + 2, \end{cases}$$

and hence, $5 = 3^x + 3^y + 3^z \le 3 + 2(x + y + z)$. It follows that $1 \le x + y + z$. This implies that $x^2 + y^2 + z^2 \le x + y + z$, and hence, $x(1-x) + y(1-y) + z(1-z) \le 0$. Since the left hand side of the preceding inequality is nonnegative we obtain that $x(1-x) = y(1-y) = z(1-z) = 0$ from which it follows that x, y, z are either 0 or 1. This combined with the first equation of the system shows that exactly one of x, y, and z is 1 and the other two are 0, and the problem is solved.

Solution 3 by the proposer.

By inspection we see that $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are solutions of the given system. We claim that they are the only solutions of the system. In fact, for all $t \in [0, 1]$ the function $f(t) = 3^t$ is greater than or equal to the function $g(t) = 2t^2 + 1$, as can be easily proven, for instance, by drawing their graphs when $0 \le t \le 1$.

Since $x^2 + y^2 + z^2 = 1$, then $x \in [0, 1], y \in [0, 1]$ and $z \in [0, 1]$. Therefore

$$\begin{aligned} 3^x &\ge 2x^2 + 1, \\ 3^y &\ge 2y^2 + 1, \\ 3^z &\ge 2z^2 + 1. \end{aligned}$$

Adding up the preceding expressions yields

$$3^x + 3^y + 3^z \ge 2(x^2 + y^2 + z^2) + 3 \ge 5$$

and we are done

Also solved by Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy, and Boris Rays, Chesapeake,VA.

- **5037:** *Ovidiu Furdui, Campia Turzii, Cluj, Romania*

Let k, p be natural numbers. Prove that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$$

for all $n \geq 1$ if and only if $k = p = 1$.

Solution 1 by Carl Libis, Kingston, RI.

Since $\Big(1 + 3 + \cdots + (2n+1)\Big)^p = \Big[(n+1)^2\Big]^p = (n+1)^{2p}$, it is clear that $\Big(1 + 3 + \cdots + (2n+1)\Big)^p$ is a monic polynomial of degree $2p$.

Let $S_k^{2n+1} = \sum_{i=1}^{2n+1} i^k$. Then

$$S_k^{2n+1} = \sum_{i=1}^{n+1}(2i-1)^k + \sum_{i=1}^{n}(2i)^k = \sum_{i=0}^{n}(2i+1)^k + 2^k \sum_{i=1}^{n} i^k = \sum_{i=0}^{n}(2i+1)^k + 2^k S_k^n.$$

Then $\sum_{i=0}^{n}(2i+1)^k = S_k^{2n+1} - 2^k S_k^n$. It is well known for sums of powers of integers S_k^n, that the leading term of S_k^n is $\dfrac{n^{k+1}}{k+1}$. Thus the leading term of $1^k + 3^k + 5^k + \cdots + (2n+1)^k$ is

$$\frac{(2n+1)^{k+1}}{k+1} - \frac{2^k n^{k+1}}{k+1} = \frac{2^{k+1}n^{k+1} - 2^k n^{k+1}}{k+1} = \frac{2^k n^{k+1}}{k+1}.$$

This is monic if, and only if, $k = 1$. When $k = 1$ we have that

$$\sum_{i=0}^{n}(2i+1) = S_1^{2n+1} - 2S_1^n = \frac{(2n+1)(2n+2)}{2} - 2\frac{n(n+1)}{2} = (n+1)^2.$$

For k, p natural numbers we have that $1^k + 3^k + 5^k + \cdots + (2n+1)^k = \Big(1 + 3 + \cdots + (2n+1)\Big)^p$ for all $n \geq 1$ if, and only if, $k = p = 1$.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

If $k = p = 1$, the equality $1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$ is trivial. Now suppose that the equality holds for all $n \geq 1$. By putting $n = 1, 2$, we obtain $1 + 3^k = 4^p$ and $1 + 3^k + 5^k = 9^p$. Hence

$$\begin{aligned} 3^k &= 4^p - 1 \text{ and} \\ 5^k &= 9^p - 4^p. \end{aligned}$$

Eliminating k from the last two equations, we obtain $9^p = 4^p + (4^p - 1)^{(\ln 5/\ln 3)}$. Hence,

$$\begin{aligned} 9^p &< 2\Big(4^{p(\ln 5/\ln 3)}\Big) \\ p\ln 9 &< \ln 2 + \frac{p(\ln 4)(\ln 5)}{\ln 3}, \text{and} \end{aligned}$$

$$p \quad < \quad \frac{(\ln 2)(\ln 3)}{(\ln 3)(\ln 9)-(\ln 4)(\ln 5)} = 4.16\cdots.$$

Thus $p = 1, 2, 3, 4$. But it is easy to check that only the case $p = 1$ and $k = 1$ admits solutions in the natural numbers for the equation $1 + 3^k = 4^p$, and this completes the solution.

Solution 3 by Paul M. Harms, North Newton, KS.

Clearly if $k = p = 1$, the equation holds for all appropriate integers n. For the *only if* part of the statement consider the contrapostive statement:

If $p \neq 1$ or $k \neq 1$, then for some $n \geq 1$ the equation does not hold.

Consider $n = 1$. Then the equation in the problem is $1^k + 3^k = (1 + 3)^p = 4^p$. If $k = 1$ with $p > 1$, then $4 < 4^p$ so the equation does not hold.

If $k > 1$ with $p = 1$, then $1^k + 3^k > 4$ so the equation does not hold.

Now consider both $p > 1$ and $k > 1$ using the equation in the form
$3^k = 4^p - 1^k = (2^p - 1)(2^p + 1)$.

If $p > 1$, then $2^p - 1 > 1$ and $2^p + 1 > 1$. Also, the expressions $2^p - 1$ and $2^p + 1$ are 2 units apart so that if 3 is a factor of one of these expressions then 3 is not a factor of the other expression. Since both expressions are greater than one, if 3 is a factor of one of the expressions, then the other expression has a prime number other than 3 as a factor. Thus $(2^p - 1)(2^p + 1)$ has a prime number other than 3 as a factor and cannot be equal to 3^k, a product of just the prime number 3. Thus the equation does not hold when both $p > 1$ and $k > 1$.

Solution 4 by John Hawkins and David Stone, Statesboro, GA.

Denote $1^k + 3^k + 5^k + \cdots + (2n + 1)^k = (1 + 3 + \cdots + (2n + 1))^p$ by (#). The condition requesting *all* $n \geq 1$ is overkill. Actually, we can prove the following are equivalent:

(a) condition (#) holds for all $n \geq 1$,

(b) condition (#) holds for all $n = 1$,

(c) $k = p = 1$.

Clearly, (a)$\Rightarrow$ (b).

Also (c) $\Rightarrow$ (a), for if $k = p = 1$, then (#) becomes the identity

$$1 + 3 + 5 + \cdots + (2n + 1) = (1 + 3 + \cdots + (2n + 1)).$$

Finally, we prove that (b) $\Rightarrow$ (c). Assuming the truth of (#) for $n = 1$ tells us that $3^k = 4^p - 1$.

If $k = 1$, we immediately conclude that $p = 1$ and we are finished.

Arguing by contradiction, suppose $k \geq 2$, so 3^k is actually a multiple of 9. Thus $4^p \equiv 1(\bmod 9)$. Now consider the powers of 4 modulo 9:

$$4^0 \equiv 1(\bmod 9)$$
$$4^1 \equiv 4(\bmod 9)$$

$$4^2 \equiv 7(\text{mod } 9)$$
$$4^3 \equiv 1(\text{mod } 9)$$

That is, 4 has order 3(mod 9), so $4^p \equiv 1(\text{mod } 9)$ if and only if p is a multiple of 3. Based upon some numerical testing, we consider 4^p modulo 7: $4^p = 4^{3t} \equiv 64^t \equiv 1^t \equiv 1(\text{mod } 7)$. That is, 7 divides $4^p - 1$, so $4^p - 1$ cannot be a power of 3. We have reached a contradiction.

Solution 5 by the proposer.

One implication is easy to prove. To prove the other implication we note that

$$1+3+\cdots+(2n+1) = \sum_{k=1}^{n+1}(2k-1) = 2\sum_{k=1}^{n+1} k-(n+1) = (n+1)(n+2)-(n+1) = (n+1)^2.$$

It follows that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (n+1)^{2p}.$$

We multiply the preceding relation by $2/(2n+1)^{k+1}$ and we get that

$$\frac{2}{2n+1}\left(\left(\frac{1}{2n+1}\right)^k + \left(\frac{3}{2n+1}\right)^k + \cdots + \left(\frac{2n+1}{2n+1}\right)^k\right) = 2\frac{(n+1)^{2p}}{(2n+1)^{k+1}}. \qquad (1)$$

Letting $n \to \infty$ in (1) we get that

$$\int_0^1 x^k dx = \frac{1}{k+1} = \lim_{n\to\infty} 2\frac{(n+1)^{2p}}{(2n+1)^{k+1}}.$$

It follows that $2p = k+1$ and that $\frac{1}{k+1} = \frac{1}{2^k}$. However, the equation $k+1 = 2^k$ has a unique positive solution namely $k = 1$. This can be proved by applying Bernouli's inequality as follows

$$2^k = (1+1)^k \geq 1 + k \cdot 1 = k+1,$$

with equality if and only if $k = 1$. Thus, $k = p = 1$ and the problem is solved.

Also solved by Boris Rays, Chesapeake, VA.

Late Solutions

Late solutions were received from **David C. Wilson of Winston-Salem, NC** to problems 5026, 5027, and 5028.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before June 15, 2009

- 5062: *Proposed by Kenneth Korbin, New York, NY.*

  Find the sides and the angles of convex cyclic quadrilateral ABCD if $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2$.

- 5063: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Euclid's inscribed polygon is a constructible polygon inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

  a) Does Euclid's inscribed n-gon exist for any prime n greater than 5?

  b) Does Euclid's n-gon exist for all composite numbers n greater than 2?

- 5064: *Proposed by Michael Brozinsky, Central Islip, NY.*

  The Lemoine point of a triangle is that point inside the triangle whose distances to the three sides are proportional to those sides. Find the maximum value that the constant of proportionality, say λ, can attain.

- 5065: *Mihály Bencze, Brasov, Romania.*

  Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$1) \qquad \sum_{i,j=1}^{n} |(i-j)(x_i - x_j)| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.$$

$$2) \qquad \sum_{i,j=1}^{n} (i-j)^2 = \frac{n^2(n^2-1)}{6}.$$

- 5066: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let $abc = 1$. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB respectively. Prove or disprove that

$$3(a+b)(b+c)(c+a) \geq 32(d_a+d_b+d_c)^2.$$

- 5067: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be complex numbers such that $a+b+c=0$. Prove that

  $$\max\{|a|, |b|, |c|\} \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.$$

Solutions

- 5044: *Proposed by Kenneth Korbin, New York, NY.*

  Let N be a positive integer and let

  $$\begin{cases} x = 9N^2+24N+14 \text{ and} \\ y = 9(N+1)^2+24(N+1)+14. \end{cases}$$

  Express the value of y in terms of x, and express the value of x in terms of y.

Solution by Armend Sh. Shabani, Republic of Kosova.

One easily verifies that

$$y - x = 18N + 33. \qquad (1)$$

From $9N^2+24N+14-x=0$ one obtains $N_{1,2} = \dfrac{-4 \pm \sqrt{2+x}}{3}$, and since N is a positive integer we have

$$N = \frac{-4+\sqrt{2+x}}{3}. \qquad (2)$$

Substituting (2) into (1) gives:

$$y = x + 9 + 6\sqrt{2+x}. \qquad (3)$$

From $9(N+1)^2+24(N+1)+14-y=0$ one obtains $N_{1,2} = \dfrac{-7 \pm \sqrt{2+y}}{3}$, and since N is a positive integer we have

$$N = \frac{-7+\sqrt{2+y}}{3}. \qquad (4)$$

Substituting (4) into (1) gives:

$$x = y + 9 - 6\sqrt{2+y}. \qquad (5)$$

Relations (3) and (5) are the solutions to the problem.

Comments: **1. Paul M. Harms** mentioned that the equations for x in terms of y, as well as for y in terms of x, are valid for integers satisfying the x, y and N equations in the problem. The minimum x and y values occur when $N=1$ and are $x=47$ and $y=98$. **2. David Stone and John Hawkins** observed that in addition to $(47, 98)$, other integer lattice points on the curve of $y = 9 + x + 6\sqrt{2+x}$ in the first quadrant are $(4, 98), (98, 167), (167, 254), (254, 359)$, and $(23, 62)$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; José Hernández Santiago (student UTM), Oaxaca, México; David Stone and John Hawkins (jointly), Statesboro, GA; David C.Wilson, Winston-Salem, NC, and the proposer.

- **5045:** *Proposed by Kenneth Korbin, New York, NY.*

Given convex cyclic hexagon ABCDEF with sides

$$\begin{aligned}\overline{AB} &= \overline{BC} = 85\\ \overline{CD} &= \overline{DE} = 104, \text{ and}\\ \overline{EF} &= \overline{FA} = 140.\end{aligned}$$

Find the area of $\triangle BDF$ and the perimeter of $\triangle ACE$.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that the area of $\triangle BDF$ iis 15390 and the perimeter of $\triangle ACE$ is $\dfrac{123120}{221}$.

Let $\angle AFE = 2\alpha$, $\angle EDC = 2\beta$, and $\angle$CBA=2γ so that

$$\angle ACE = \pi - 2\alpha,\ \angle CAE = \pi - 2\beta, \text{ and } \angle AEC = \pi - 2\gamma.$$

Since $\angle ACE + \angle CAE + \angle AEC = \pi$, so

$$\begin{aligned}\alpha + \beta + \gamma &= \pi\\ \cos\alpha + \cos\beta + \cos\gamma &= 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1 \text{ or}\\ (\cos\alpha + \cos\beta + \cos\gamma - 1)^2 &= 2(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma). \qquad (1)\end{aligned}$$

Denote the radius of the circumcircle by R. Applying the Sine Formula to $\triangle ACE$, we have

$$R = \frac{\overline{AE}}{2\sin 2\alpha} = \frac{\overline{EC}}{2\sin 2\beta} = \frac{\overline{CA}}{2\sin 2\gamma}.$$

By considering triangles AFE, EDC, and CBA respectively, we obtain

$$\overline{AE} = 280\sin\alpha,\ \overline{EC} = 208\sin\beta,\ \overline{CA} = 170\sin\gamma.$$

It follows that $\cos\alpha = \dfrac{70}{R}$, $\cos\beta = \dfrac{52}{R}$, and $\cos\gamma = \dfrac{85}{2R}$. Substituting into (1) and simplifying, we obtain

$$\begin{aligned}4R^3 - 37641R - 1237600 &= 0 \text{ or}\\ \Big(2R - 221\Big)\Big(2R^2 + 221R + 5600\Big) &= 0.\end{aligned}$$

Hence,

$$R=\frac{221}{2},\ \cos\alpha = \frac{140}{221},\ \sin\alpha=\frac{171}{221}$$
$$\cos\beta = \frac{104}{221},\ \sin\beta=\frac{195}{221}$$
$$\cos\gamma = \frac{85}{221},\ \sin\gamma=\frac{204}{221},$$

and our result for the perimeter of $\triangle ACE$.

It is easy to check that $\angle BFD=\alpha, \angle FDB=\beta, \angle DBF=\gamma$ so that $\angle BAF=\pi-\beta, \angle DEF=\pi-\gamma$.

Applying the cosine formula to $\triangle BAF$ and $\triangle DEF$ respectively, we obtain $BF=195$ and $DF=204$.

It follows, as claimed, that the area of

$$\triangle BDF=\frac{1}{2}\left(\overline{BF}\right)\left(\overline{DF}\right)\sin\angle BFD=\frac{1}{2}(195)(204)\frac{171}{221}=15390.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5046:** *Proposed by R.M. Welukar of Nashik, India and K.S. Bhanu, and M.N. Deshpande of Nagpur, India.*

  Let $4n$ successive Lucas numbers $L_k, L_{k+1}, \cdots, L_{k+4n-1}$ be arranged in a $2\times 2n$ matrix as shown below:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 2n \end{array}$$
$$\begin{pmatrix} L_k & L_{k+3} & L_{k+4} & L_{k+7} & \cdots & L_{k+4n-1} \\ L_{k+1} & L_{k+2} & L_{k+5} & L_{k+6} & \cdots & L_{k+4n-2} \end{pmatrix}$$

Show that the sum of the elements of the first and second row denoted by R_1 and R_2 respectively can be expressed as

$$R_1=2F_{2n}L_{2n+k}$$

$$R_2=F_{2n}L_{2n+k+1}$$

where $\{L_n, n\geq 1\}$ denotes the Lucas sequence with $L_1=1, L_2=3$ and $L_{i+2}=L_i+L_{i+1}$ for $i\geq 1$ and $\{F_n, n\geq 1\}$ denotes the Fibonacci sequence, $F_1=1, F_2=1, F_{n+2}=F_n+F_{n+1}$.

Solution by Angel Plaza and Sergio Falcon, Las Palmas, Gran Canaria, Spain.

$R_1=L_k+L_{k+3}+L_{k+4}+L_{k+7}+\cdots+L_{k+4n-2}+L_{k+4n-1}$, and since $L_i=F_{i-1}+F_{i+1}$, we have:

$$
\begin{aligned}
R_1 &= F_{k-1}+F_{k+1}+F_{k+2}+F_{k+4}+F_{k+3}+F_{k+5}+\cdots+F_{k+4n-2}+F_{k+4n}\\
&= F_{k-1}+\sum_{j=1}^{4n}F_{k+j}-F_{k+4n-1}\\
&= F_{k-1}-F_{k+4n-1}+\sum_{j=0}^{4n+k}F_j-\sum_{j=0}^{k}F_j
\end{aligned}
$$

And since $\sum_{j=0}^{m}F_j=F_{m+2}-1$ we have:

$$
R_1 = F_{k-1}-F_{k+4n-1}+F_{k+4n+2}-1-F_{k+2}+1=2F_{k+4n}-2F_k
$$

where in the last equation it has been used that $F_{i+2}-F_i=F_{i+1}+F_i-F_{i-1}=2F_i$. Now, using the relation $L_nF_m=F_{n+m}-(-1)^mF_{n-m}$ (S. Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover Press (2008)) in the form $L_{2n+k}F_{2n}=F_{4n+k}-(-1)^{2n}F_{2n+k-2n}$ it is deduced $R_1=2F_{2n}L_{2n+k}$.
In order to prove the fomula for R_2 note that

$$
R_1+R_2=\sum_{j=0}^{4n-1}L_{k+j}=\sum_{j=0}^{4n+k-1}L_j-\sum_{j=0}^{k-1}L_j
$$

As before, $\sum_{j=0}^{4n+k-1}L_j=F_{k+4n}+F_{k+4n+2}$, while $\sum_{j=0}^{k-1}L_j=F_k+F_{k+2}$, so

$$
\begin{aligned}
R_1+R_2 &= F_{k+4n}-F_k+F_{k+4n+2}-F_{k+2}\\
&= L_{2n+k}F_{2n}+L_{2n+k+2}F_{2n}
\end{aligned}
$$

And therefore,

$$
R_2 = F_{2n}\left(L_{2n+k+2}-L_{2n+k}\right)=F_{2n}L_{2n+k+1}
$$

Also solved by Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA, and the proposers.)

- 5047: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  Find a procedure for continuing the following pattern:

$$
\begin{aligned}
S(n,0) &= \sum_{k=0}^{n}\binom{n}{k}=2^n\\
S(n,1) &= \sum_{k=0}^{n}\binom{n}{k}k=2^{n-1}n\\
S(n,2) &= \sum_{k=0}^{n}\binom{n}{k}k^2=2^{n-2}n(n+1)
\end{aligned}
$$

$$S(n,3) = \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}n^2(n+3)$$

$$\vdots$$

Solution by David E. Manes, Oneonta, NY.

Let $f(x) = (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. For $m \geq 0$,

$S(n,m) = \left(x\dfrac{d}{dx}\right)^m (f(x))\Big|_{x=1}$, where $\left(x\dfrac{d}{dx}\right)^m$ is the procedure $x\dfrac{d}{dx}$ iterated m times and then evaluating the resulting function at $x=1$. For example,

$$\begin{aligned}
S(n,0) = f(1) &= \sum_{k=0}^{n}\binom{n}{k} = 2^n. \text{ Then}\\
x\frac{d}{dx}(f(x)) &= x\frac{d}{dx}(1+x)^n = x\frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right) \text{ implies}\\
nx(1+x)^{n-1} &= \sum_{k=0}^{n}\binom{n}{k}k\cdot x^k. \text{ If } x=1, \text{then}\\
\sum_{k=0}^{n}\binom{n}{k}k &= n\cdot 2^{n-1} = S(n,1).
\end{aligned}$$

For the value of $S(n,2)$ note that if

$$\begin{aligned}
x\frac{d}{dx}\left[nx(1+x)^{n-1}\right] &= x\frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}kx^k\right], \text{ then}\\
nx(nx+1)(1+x)^{n-2} &= \sum_{k=0}^{n}\binom{n}{k}k^2x^k. \text{ If } x=1, \text{ then}\\
n(n+1)2^{n-2} &= \sum_{k=0}^{n}\binom{n}{k}k^2 = S(n,2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
S(n,3) &= \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}\cdot n^2(n+3) \text{ and}\\
S(n,4) &= \sum_{k=0}^{n}\binom{n}{k}k^4 = 2^{n-4}\cdot n(n+1)(n^2+5n-2.)
\end{aligned}$$

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro GA, and the proposer.

- 5048: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*

Let $a, b, c,$ be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \ \geq \ \frac{54}{(a+b+c)^2}\frac{(abc)^3}{\sqrt{(ab)^4+(bc)^4+(ca)^4}}.$$

Solution1 by Boris Rays, Chesapeake, VA.

Rewrite the inequality into the form:

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\cdot\Big(a+b+c\Big)^2\cdot\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq 54(abc)^3 \qquad (1)$$

We will use the Arithmetic-Geometric Mean Inequality (e.g., $x+y+z \geq 3\sqrt[3]{xyz}$ and $u+v \geq 2\sqrt{uv}$) for each of the three factors on the left side of (1).

$$\begin{aligned}
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \ &\geq \ \sqrt{3\sqrt[3]{c^2(a^2+b^2)^2\cdot b^2(c^2+a^2)^2\cdot a^2(b^2+c^2)^2}}\\
&\geq \ \sqrt{3\sqrt[3]{(abc)^2(a^2+b^2)^2(c^2+a^2)^2(b^2+c^2)^2}}\\
&\geq \ \sqrt{3\sqrt[3]{(abc)^2(4a^2b^2)(4c^2a^2)(4b^2c^2)}}\\
&= \ \sqrt{3(abc)^{2/3}\sqrt[3]{4^3a^4b^4c^4}}\\
&= \ \sqrt{3(abc)^{2/3}4(abc)^{4/3}}\\
&= \ \sqrt{3\cdot 2^2(abc)^2}\\
&= \ 2\sqrt{3}(abc) \qquad (2)
\end{aligned}$$

Also, since $(a+b+c) \geq 3\sqrt[3]{abc}$, we have

$$(a+b+c)^2 \geq 3^2\left(\sqrt[3]{abc}\right)^2 = 3^2(abc)^{2/3} \qquad (3)$$

$$\begin{aligned}
\sqrt{(ab)^4+(bc)^4+(ca)^4} \ &\geq \ \sqrt{3\sqrt[3]{(ab)^4(bc)^4(ca)^4}}\\
&= \ \sqrt{3\sqrt[3]{a^8b^8c^8}}\\
&= \ \sqrt{3(abc)^{8/3}}\\
&= \ \sqrt{3}(abc)^{4/3} \qquad (4)
\end{aligned}$$

Combining (2), (3), and (4) we obtain:

$$
\begin{aligned}
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \quad \cdot \quad & \Big(a+b+c\Big)^2 \cdot \sqrt{(ab)^4+(bc)^4+(ca)^4} \\
\geq \quad & 2\sqrt{3}(abc)\cdot 3^2(abc)^{2/3}\sqrt{3}(abc)^{4/3} \\
= \quad & 2\cdot 3^3(abc)^{1+2/3+4/3} \\
= \quad & 54(abc)^3.
\end{aligned}
$$

Hence, we have shown that (1) is true, with equality holding if $a=b=c$.

Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain.

The inequality claimed is equivalent to

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq \frac{54(abc)^3}{(a+b+c)^2}$$

Applying Cauchy's inequality to the vectors $\vec{u}=(c(a^2+b^2), b(c^2+a^2), a(b^2+c^2))$ and $\vec{v}=(a^2b^2, c^2a^2, b^2c^2)$ yields

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\sqrt{(ab)^4+(bc)^4+(ca)^4}$$

$$\geq abc(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))$$

So, it will be suffice to prove that

$$(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))(a+b+c)^2 \geq 54a^2b^2c^2 \tag{1}$$

Taking into account GM-AM-QM inequalities, we have

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \geq 2(a^2b^2+b^2c^2+c^2a^2) \geq 6abc\sqrt[3]{abc}$$

and

$$(a+b+c)^2 \geq 9\sqrt[3]{a^2b^2c^2}$$

Multiplying up the preceding inequalities (1) follows and the proof is complete

Solution 3 by Kee-Wai Lau, Hong Kong, China.

By homogeneity, we may assume without loss of generality that $abc=1$. We have

$$
\begin{aligned}
& \sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \\
= \quad & \sqrt{\left(\frac{a^2+b^2}{ab}\right)^2+\left(\frac{c^2+a^2}{ca}\right)^2+\left(\frac{b^2+c^2}{bc}\right)^2} \\
= \quad & \sqrt{\left(\frac{a^2-b^2}{ab}\right)^2+\left(\frac{c^2-a^2}{ca}\right)^2+\left(\frac{b^2-c^2}{bc}\right)^2+12}
\end{aligned}
$$

$$\geq \quad 2\sqrt{3}.$$

By the arithmetic-geometric mean inequality, we have $(a+b=c)^2 \geq 9(abc)^{2/3} = 9$ and $\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq \sqrt{3}(abc)^{4/3} = \sqrt{3}$. The inequality of the problem now follows immediately.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ovidiu Furdui, Campia Turzii, Cluj, Romania; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Armend Sh. Shabani, Republic of Kosova, and the proposer.

5049: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find a function $f : \Re \to \Re$ such that

$$2f(x) + f(-x) = \begin{cases} -x^3 - 3, & x \leq 1, \\ 3 - 7x^3, & x > 1. \end{cases}$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX .

If $x > 1$, then

$$2f(x) + f(-x) = 3 - 7x^3. \qquad (1)$$

Also, since $-x < -1$, we have

$$2f(-x) + f(x) = -(-x)^3 - 3 = x^3 - 3. \qquad (2)$$

By (1) and (2), $f(x) = 3 - 5x^3$ and $f(-x) = -3 + 3x^3$ when $x > 1$. Further, $f(-x) = -3 + 3x^3$ when $x > 1$ implies that $f(x) = -3 + 3(-x)^3 = -3 - 3x^3$ when $x < -1$.

Finally, when $-1 \leq x \leq 1$, we get $-1 \leq -x \leq 1$ also, and hence,

$$\begin{aligned} 2f(x) + f(-x) &= -x^3 - 3, & (3) \\ 2f(-x) + f(x) &= -(-x)^3 - 3 = x^3 - 3. & (4) \end{aligned}$$

As above, (3) and (4) imply that $f(x) = -x^3 - 1$ when $-1 \leq x \leq 1$.

Therefore, $f(x)$ must be of the form

$$f(x) = \begin{cases} -3 - 3x^3 & \text{if } \quad x < -1, \\ -1 - x^3 & \text{if } \ -1 \leq x \leq 1, \\ 3 - 5x^3 & \text{if } \ x > 1. \end{cases} \qquad (5)$$

With some perseverance, this can be condensed to

$$f(x) = \left|x^3 + 1\right| - 2\left|x^3 - 1\right| - 4x^3$$

for all $x \in \Re$. Since it is straightforward to check that this function satisfies the given conditions of the problem, this completes the solution.

Also solved by Brian D. Beasely, Clinton, SC; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

Late Solutions

Late solutions were received from **Pat Costello of Richmond, KY** to problem 5027; **Patrick Farrell of Andover, MA** to 5022 and 5024, and from **David C. Wilson of Winston-Salem, NC** to 5038.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent by e-mail to eisenbt@013.net. Solutions to previously stated problems can be seen at http://ssmj.tamu.edu

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Solutions to the problems stated in this issue should be posted before
October 15, 2009

- 5068: *Proposed by Kenneth Korbin, New York, NY*

  Find the value of
  $$\sqrt{1+2009\sqrt{1+2010\sqrt{1+2011\sqrt{1+\cdots}}}}.$$

- 5069: *Proposed by Kenneth Korbin, New York, NY*

  Four circles having radii $\dfrac{1}{14}$, $\dfrac{1}{15}$, $\dfrac{1}{x}$ and $\dfrac{1}{y}$ respectively, are placed so that each of the circles is tangent to the other three circles. Find positive integers x and y with $15 < x < y < 300$.

- 5070: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Find all real solutions to the system
  $$\left.\begin{array}{c} 9(x_1^2+x_2^2-x_3^2)=6x_3-1, \\ \\ 9(x_2^2+x_3^2-x_4^2)=6x_4-1, \\ \cdots\cdots\cdots \\ 9(x_n^2+x_1^2-x_2^2)=6x_2-1. \end{array}\right\}$$

- 5071: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let h_a, h_b, h_c be the altitudes of $\triangle ABC$ with semi-perimeter s, in-radius r and circum-radius R, respectively. Prove that
  $$\frac{1}{4}\left(\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c}\right) \le \frac{R^2}{r}\left(\sin^2 A+\sin^2 B+\sin^2 C\right).$$

- 5072: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  Let a, b and c be the sides, l_a, l_b, l_c the bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the heights of $\triangle ABC$. Prove or disprove that

  a) $$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c} \ge \frac{4}{3}\left(m_a\cdot l_a+l_b\cdot h_b+h_c\cdot m_c\right)$$

b) $\quad 3\sum_{cyc}\dfrac{(-a+b+c)^3}{a} \geq 2\sum_{cyc}[m_a(l_a+h_a)].$

- 5073: *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania*

  Let $m > -1$ be a real number. Evaluate

  $$\int_0^1 \{\ln x\} x^m dx,$$

  where $\{a\} = a - [a]$ denotes the fractional part of a.

Solutions

- 5050: *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with integer-length sides, and with $\angle A = 120^o$, and with $(a,b,c) = 1$.
  Find the lengths of b and c if side $a = 19$, and if $a = 19^2$, and if $a = 19^4$.

Solution 1 by Paul M. Harms, North Newton, KS

Using the law of cosines we have $a^2 = b^2 + c^2 - 2bc\cos 120^o = b^2 + c^2 + bc.$

When $a = 19$ we have $19^2 = 361 = b^2 + c^2 + bc$. The result $b = 5, c = 16$ with $a = 19$ satisfies the problem.

Some books indicate that the Diophantine equation $a^2 = b^2 + c^2 + bc$ has solutions of the form

$$b = u^2 - v^2,\ c = 2uv + v^2,\ \text{and}\ a = u^2 + v^2 + uv.$$

For the above $u = 3, v = 2$ and $a = 19 = 3^2 + 2^2 + 2(3)$.

Let $a_1^2 = b_1^2 + c_1^2 + b_1c_1$ be another Diophantine equation which has solutions of the form $b_1 = u_1^2 - v_1^2,\ c_1 = 2u_1v_1 + v^2$, and $a_1 = u_1 + v_1^2 + u_1v_1$. Let u$_1$ be the largest and v_1 be the smallest of the numbers $\{b,c\}$. If $b = c$, the Diophantine equation becomes $a_1^2 = 3b_1^2$ which has no integer solutions. Suppose $c > b$. (If $b > c$, a procedure similar to that below can be used).

Let $u_1 = c$ and $v_1 = b$. Then $b_1 = c^2 - b^2$ and $c_1 = 2cb + b^2$. The expression $b_1^2 + c_1^2 + b_1c_1 = (c^2-b^2)^2 + (2cb+b^2)^2 + (c^2-b^2)(2cb+b^2) = (c^2+b^2+bc)^2 = (a^2)^2 = a^4 = a_1^2.$ In this case $a_1 = a^2$.

Now start with the above solution where $a = 19, u = 3, v = 2, b = 5$, and $c = 16$. For $a = 19^2$, let $u = 16$ and $v = 5$. Then we have the solution $b = 231^2, c = 185$ where a$^2 = 19^4 = 231 + 185^2 + 231(185)$.

For $a = 19^4$, let $u = 231$ and $v = 185$. Then $b = 19136, c = 119695$ and $a^2 = 19^8 = 19136^2 + 119695^2 + 19136(119695)$. Since 19 is not a factor of the b and c solutions above, $(a,b,c) = 1$.

The solutions I have found are $(19, 5, 16)$, $(19^2, 231, 185)$, and $(19^4, 19136, 119695)$.

Solution 2 by Bruno Salguerio Fanego, Viveiro, Spain

If $\triangle ABC$ is such a triangle, by the cosine theorem $a^2 = b^2 + c^2 - 2bc\cos A$, that is

$$c^2 + bc + b^2 - a^2 = 0,\ c = \frac{-b \pm \sqrt{4a^2 - 3b^2}}{2} \text{ and } 4a^2 - 3b^2$$

must be positive integers and the latter a perfect square, with $(a,b,c)=1$.

When $a=19,\ 0<b\le 2\cdot 19/\sqrt{3}\ \Rightarrow\ 0<b\le 21;\ \ 4\cdot 19^2-3b^2$ is a positive perfect square for $b\in\{2^4,5\}$ so $c\in\{5,2^4\}$, and $(a,b,c)=1$.

When $a=19^2,\ 0<b\le 2\cdot 19^2/\sqrt{3}\ \Rightarrow\ 0<b\le 416;\ 4\cdot 19^4-3b^2$ is a positive perfect square that is not a multiple of 19 for $b\in\{3\cdot 7\cdot 11,\ 5\cdot 37\}$, so $c\in\{5\cdot 37,\ 3\cdot 7\cdot 11\}$, and $(a.b.c)=1$.

When $a=19^4,\ 0<b\le 2\cdot 19^4/\sqrt{3}\ \Rightarrow\ 0<b\le 150481;\ 4\cdot 19^8-3b^2$ is a positive perfect square that is not a multiple of 19 for $b\in\{5\cdot 37\cdot 647,\ 2^6\cdot 13\cdot 23\}$. So $c\in\{2^6\cdot 13\cdot 23,\ 5\cdot 37\cdot 647\}$, and $(a,b,c)=1$.

And reciprocally, the triangular inequalities are verified by $a=19,16,5$, by $a=19^2,231,185$, and by $a=19^4,\ 119695,19136$, so there is a $\triangle ABC$ with sides $a,\ b$ and c with these integer lengths, and with $\angle A=120^o$, and $(a,b,c)=1$.

Thus, if $a=19$, then $\{b,c\}=\{5,16\}$; if $a=19^2$, then $\{b,c\}=\{185,231\}$, and if $a=19^4$, then $\{b,c\}=\{19136,119695\}$.

Note: When $a=19^2, 4\cdot 19^4-3b^2$ is a perfect square for $b\in\{2^4\cdot 19,\ \ 3\cdot 7\cdot 11,\ \ 5\cdot 37,\ \ 5\cdot 19\}$.

When $a=19^4, 4\cdot 19^8-3b^2$ is a perfect square for $b\in\{5\cdot 37\cdot 647,\ \ 2^4\cdot 19^3,\ \ 2^4\cdot 3^2\cdot 5\cdot 7\cdot 19,\ \ 3\cdot 7\cdot 11\cdot 19^2,\ \ 5\cdot 19^2\cdot 37,\ \ 17\cdot 19\cdot 163,\ \ 5\cdot 19^3,\ \ 2^6\cdot 13\cdot 23\}$.

Also solved by John Hawkins and David Stone (jointly), Statesboro, GA; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5051: *Proposed by Kenneth Korbin, New York, NY*

  Find four pairs of positive integers (x,y) such that $\dfrac{(x-y)^2}{x+y}=8$ with $x<y$.

  Find a formula for obtaining additional pairs of these integers.

Solution 1 by Charles McCracken, Dayton, OH

The given equation can be solved for y in term of x by expanding the numerator and multiplying by the denominator to get

$$x^2-2xy+y^2=8((x+y)\Longrightarrow y^2-(2x+8)y+(x^2-8x)=0.$$

Solving this by the quadratic formula yields $y=x+4+4\sqrt{x+1}$.

Since the problem calls for integers we choose values of x that will make $x+1$ a square. Specifically

$$\begin{aligned} x &= 3,8,15,24,35,\cdots \text{or} \\ x &= k^2+2k,\ k\ge 1 \end{aligned}$$

The first four pairs are $(3,15),\ (8,24),\ (15,35),\ (24,48)$.

In general, $x=k^2+2k$ and $y=k^2+6k+8,\ k\ge 1$.

Solution 2 by Armend Sh. Shabani, Republic of Kosova

The pairs are $(3,15),(8,24),(15,35),(24,48)$. In order to find a formula for additional pairs we write the given relation $(y-x)^2=8(x+y)$ in its equivalent form $y-x=2\sqrt{2(x+y)}$.

From this it is clear that $x+y$ should be of the form $2s^2$, and this gives the system of equations:

$$\begin{cases} x+y=2s^2 \\ y-x=4s \end{cases}$$

The solutions to this system are $x=s^2-2s,\ y=s^2+2s$, and since the solutions should be positive, we choose $s\geq 3$.

Solution 3 by Boris Rays, Brooklyn, NY

Let

$$\begin{cases} x+y=a \\ y-x=b \end{cases}$$

Since $x<y$ and a and b are positive integers, it follows that $b^2=8a$ and that b=$2\sqrt{2a}$. Since b is a positive integer we may choose values of a so that $2a$ is a perfect square. Specifically, let $a=2^{2n-1}$, where $n=1,2,3,\cdots$. Therefore, $2a=2\cdot 2^{2n-1}=2^{2n}=(2^n)^2$, where $n=1,2,3,\cdots$. Similarly, $b=2^{n+1}\ n=1,2,3,\cdots$.
Substituting these values of a and of b into the original system gives:

$$\begin{aligned} x &= \frac{2^{2n-1}-2^{n+1}}{2}=2^n(2^{n-2}-1) \\ y &= \frac{2^{2n-1}+2^{n+1}}{2}=2^n(2^{n-2}+1) \end{aligned}$$

and since we want $x,y>0$ we choose $n=3,4,5,\cdots$. The ordered triplets

$$(n,x,y):(3,8,24),(4,48,80),(5,224,288),(6,960,1088).$$

satisfy the problem. It can also be easily shown that our general values of x and y also satisfy the original equation.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Jahangeer Kholdi (with John Viands and Tyler Winn (students),Western Branch High School, Chesapeake, VA), Portsmouth, VA; Tuan Le (student, Fairmont, High School), Anaheim, CA; David E. Manes, Oneonta, NY; Melfried Olson, Honolulu, HI; Jaquan Outlaw (student, Heritage High School) Newport News, VA and Robert H. Anderson (jointly), Chesapeake, VA; Boris Rays, Brooklyn, NY; Vicki Schell, Pensacola, FL; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5052: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain*

  If $a\geq 0$, evaluate:

$$\int_0^{+\infty} arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\ \frac{dx}{1+x^2}.$$

Solution by Kee-Wai Lau, Hong Kong, China

Denote the integral by I. We show that

$$I = \begin{cases} \frac{\pi}{4} arctg \frac{2a}{1-a^2}, & 0 \le a < 1; \\ \\ \frac{\pi^2}{8}, & a = 1; \\ \\ \frac{\pi}{4}\Big(\pi - arctg\frac{2a}{a^2-1} - 4arctg\frac{\sqrt{a^4+a^2-1}-a}{1+a^2}\Big), & a > 1. \end{cases} \tag{1}$$

Let $J = \int_0^{+\infty} \frac{2a(ax^2+2x+a)arctg(x)}{(1+x^2)\Big((a^2+1)x^2+4ax+a^2+1\Big)}dx$. Integrating by parts, we see that for $0 \le a < 1$,

$$\begin{aligned} I &= \int_0^{+\infty} arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} d(arctg(x)) \\ &= \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_0^{+\infty} \\ &- \int_0^{+\infty} arctg(x) d\Big(arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\Big) \\ &= J. \end{aligned}$$

For $a \ge 1$, let $r_a = \frac{\sqrt{a^4+a^2-1}-a}{1+a^2}$ be the non-negative root of the quadratic equation $(1+a^2)x^2+2ax+1-a^2=0$ so that

$$\begin{aligned} I &= \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_0^{r_a} \\ &+ \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_{r_a}^{+\infty} + J \\ &= -\pi arctg(r_a) + J. \end{aligned}$$

By substituting $x = \frac{1}{y}$ and making use of the fact that $arctg(1/y) = \frac{\pi}{2} - arctg(y)$ we obtain

$$J = 2a \int_0^{+\infty} \frac{(ay^2+2y+a)arctg(1/y)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy$$

$$= \quad 2a\Big(\frac{\pi}{2}\int_0^{+\infty}\frac{(ay^2+2y+a)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy\Big)-J$$

so that $J=\dfrac{\pi a}{2}\displaystyle\int_0^{+\infty}\frac{(ay^2+2y+a)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy$. Resolving into partial fractions we obtain

$$J=\frac{\pi}{4}\Big(\int_0^{+\infty}\frac{dy}{1+y^2}+(a^2-1)\int_0^{+\infty}\frac{dy}{(1+a^2)y^2+4ay+1+a^2}\Big).$$

Clearly, $J=\dfrac{\pi^2}{8}$ for $a=1$. For $p>0,\ pr>q^2$, we have the well know result

$$\int_0^{+\infty}\frac{dy}{py^2+2qy+r}=\frac{1}{\sqrt{pr-q^2}}arctg\frac{q}{\sqrt{pr-q^2}},$$

so that for $a\geq 0,\ a\neq 1$

$$J=\frac{\pi}{4}\Big(\frac{\pi}{2}+\frac{a^2-1}{|a^2-1|}arctg\frac{2a}{|a^2-1|}\Big).$$

Hence (1) follows and this completes the solution.

Also solved by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy, and the proposer.

- 5053: *Proposed by Panagiote Ligouras, Alberobello, Italy*

Let a,b and c be the sides, r the in-radius, and R the circum-radius of $\triangle ABC$. Prove or disprove that

$$\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c}\leq 2rR.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Roger Zarnowski (jointly), San Angelo, TX

The given inequality is essentially the same as Padoa's Inequality which states that

$$abc\geq(a+b-c)\,(b+c-a)\,(c+a-b)\,,$$

with equality if and only if $a=b=c$. We will prove this using the approach presented in [1]. Let $x=\dfrac{a+b-c}{2}$, $y=\dfrac{b+c-a}{2}$, and $z=\dfrac{c+a-b}{2}$. Then, $x,y,z>0$ by the Triangle Inequality and $a=x+z$, $b=x+y$, $c=y+z$. By the Arithmetic - Geometric Mean Inequality,

$$\begin{aligned}abc &= (x+z)(x+y)(y+z)\\ &\geq (2\sqrt{xz})(2\sqrt{xy})(2\sqrt{yz})\\ &= (2x)(2y)(2z)\\ &= (a+b-c)(b+c-a)(c+a-b),\end{aligned}$$

with equality if and only if $x=y=z$, i.e., if and only if $a=b=c$.

If $A = Area(\triangle ABC)$ and $s = \dfrac{a+b+c}{2}$, then

$$R = \frac{abc}{4A} \quad \text{and} \quad A = rs = r\left(\frac{a+b+c}{2}\right),$$

which imply that $2rR = \dfrac{abc}{a+b+c}$. Hence, the problem reduces to Padoa's Inequality.

Reference:

[1] R. B. Nelsen, Proof Without Words: Padoa's Inequality, **Mathematics Magazine** 79 (2006) 53.

Also solved by Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Tuan Le (student, Fairmont High School), Anaheim, CA; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students), HUS, Vietnam; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.)

- 5054: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let x, y, z be positive numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

Solution 1 by Ovidiu Furdui, Campia Turzii, Cluj, Romania

First we note that if a and b are two positive numbers then the following inequality holds

$$\frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{1}{3} \qquad (1).$$

Let

$$S = \frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2}.$$

We have,

$$\begin{aligned}
S &= \frac{x^3-y^3+y^3}{x^2+xy+y^2} + \frac{y^3-z^3+z^3}{y^2+yz+z^2} + \frac{z^3-x^3+x^3}{z^2+zx+x^2} \\
&= (x-y) + \frac{y^3}{x^2+xy+y^2} + (y-z) + \frac{z^3}{y^2+yz+z^2} + (z-x) + \frac{x^3}{z^2+zx+x^2} \\
&= \frac{y^3}{x^2+xy+y^2} + \frac{z^3}{y^2+yz+z^2} + \frac{x^3}{z^2+zx+x^2}.
\end{aligned}$$

It follows, based on (1), that

$$\begin{aligned}
S &= \frac{1}{2}(S+S) \\
&= \frac{1}{2}\left(\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2}\right) \\
&= \frac{1}{2}\left((x+y)\frac{x^2-xy+y^2}{x^2+xy+y^2} + (y+z)\frac{y^2-yz+z^2}{y^2+yz+z^2} + (z+x)\frac{z^2-xz+x^2}{z^2+zx+x^2}\right)
\end{aligned}$$

$$\geq \frac{1}{2}\left(\frac{x+y}{3}+\frac{y+z}{3}+\frac{z+x}{3}\right)$$
$$= \frac{x+y+z}{3} \geq \sqrt[3]{xyz}=1, \text{ and the problem is solved.}$$

Solution 2 by Manh Dung Nguyen (student, Special High School for Gifted Students) HUS, Vietnam

Firstly, we have,

$$\sum \frac{x^3-y^3}{(x^2+xy+y^2)} = \sum \frac{(x-y)(x^2+xy+y^2)}{(x^2+xy+y^2)} = \sum (x-y) = 0.$$

Hence,

$$\sum \frac{x^3}{x^2+xy+y^2} = \sum \frac{y^3}{x^2+xy+y^2}.$$

So it suffices to show that,

$$\sum \frac{x^3+y^3}{x^2+xy+y^2} \geq 2.$$

On the other hand,

$$3(x^2-xy+y^2)-(x^2+xy+y^2) = 2(x-y)^2 \geq 0.$$

Thus,

$$\sum \frac{x^3+y^3}{x^2+xy+y^2} = \sum \frac{(x+y)(x^2-xy+y^2)}{x^2+xy+y^2} = \sum \frac{x+y}{3} = \frac{2(x+y+z)}{3}.$$

By the AM-GM Inequality, we have,

$$x+y+z \geq 3\sqrt[3]{xyz} = 3,$$

so we are done.
Equality hold if and only if $x=y=z=1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that,

$$\frac{x^3}{x^2+xy+y^2} = \frac{(2x-y)}{3} + \frac{(x+y)(x-y)^2}{3(x^2+xy+y^2)} \geq \frac{(2x-y)}{3}.$$

Similarly, $\dfrac{y^3}{y^2+yz+z^2} \geq \dfrac{(2y-z)}{3}$, $\dfrac{z^3}{z^2+zx+x^2} \geq \dfrac{(2z-x)}{3}$.

Hence by the arithmetic mean-geometric mean inequality, we have:

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2}$$
$$\geq \frac{x+y+z}{3}$$

$$\geq \sqrt[3]{xyz}$$

$$= 1.$$

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Tuan Le (student, Fairmont High School), Anaheim, CA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, Republic of Kosova, and the proposer.

- 5055: *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania*

Let α be a positive real number. Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^\alpha}.$$

Solution 1 by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy

Answer:

$$\text{The limit is}\begin{cases}0, & \text{if } \alpha>1;\\ 1, & \text{if } 0<\alpha<1;\\ \ln 2, & \text{if } \alpha=1.\end{cases}$$

Proof: Let $\alpha > 1$.

Writing $k^\alpha = \sum_{i=1}^{N}\frac{k^\alpha}{N}$, by the AGM we have

$$\frac{1}{n+k^\alpha} = \frac{1}{\frac{n}{2}+\frac{n}{2}+\frac{k^\alpha}{N}+\ldots+\frac{k^\alpha}{N}} \leq \frac{1}{\frac{n}{2}+\left(\frac{n}{2}\frac{k^{\alpha N}}{N^N}\right)^{\frac{1}{N+1}}}$$

$$= \frac{1}{\frac{n}{2}+\frac{n^{\frac{1}{N+1}}k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}} \leq \frac{1}{n^{\frac{1}{N+1}}\left(\frac{1}{2}+\frac{k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}\right)}$$

and we observe that $\alpha N/(N+1) > 1$ if $N > 1/(\alpha-1)$. Thus we write

$$0 < \sum_{k=1}^{n}\frac{1}{n+k^\alpha} \leq n^{-1/(N+1)}\sum_{k=1}^{\infty}\frac{1}{\left(\frac{1}{2}+\frac{k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}\right)}$$

The series converges and the limit is zero.

Let $\alpha < 1$. Trivially we have $\sum_{k=1}^{n}\frac{1}{n+k^\alpha} \leq \sum_{k=1}^{n}\frac{1}{n} = 1$.

Moreover,

$$\sum_{k=1}^{n}\frac{1}{n+k^\alpha} \geq \sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+\frac{k^\alpha}{n}} \geq \sum_{k=1}^{n}\frac{1}{n}(1-\frac{k^\alpha}{n}) = 1-\sum_{k=1}^{n}\frac{k^\alpha}{n^2} \geq 1-\frac{n^{1+\alpha}}{n^2},$$

$1 \geq (1-x^2)$ has been used. By comparison the limit equals one since

$$1 \leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1 - \frac{n^{1+\alpha}}{n^2}$$

The last step is $\alpha = 1$. We need the well known equality $H_n \approx \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$ and then

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=n+1}^{2n} (H_{2n} - H_n) = \ln(2n) - \ln n + o(1) \to \ln 2$$

The proof is complete.

Solution 2 by David Stone and John Hawkins, Statesboro, GA

Below we show that for $0 < \alpha < 1$, the limit is 1; for $\alpha = 1$, the limit is $\ln 2$; and for $\alpha > 1$, the limit is 0.

For $\alpha = 1$ we get

$$\int_0^1 \frac{1}{1+u} du \geq \sum_{k=1}^{n} \frac{1}{n+k} \geq \int_{1/n}^{(n+1)/n} \frac{1}{1+u} du.$$

Since $\frac{1}{2} \leq \frac{1}{1+u} \leq 1$, we know that the limit exists as n approaches infinity and is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = \int_0^1 \frac{1}{1+u} du = \ln(1+u)\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Next suppose $\alpha < 1$. Then

$$0 < k^\alpha \leq n^\alpha \text{ for } 1 \leq k \leq n, \text{ so}$$

$$\begin{aligned}
n < n + k^\alpha &\leq n + n^\alpha \quad \text{and} \\
\frac{1}{n+n^n} &\leq \frac{1}{n+k^\alpha} < \frac{1}{n}. \quad \text{Thus,} \\
\sum_{k=1}^{n} \frac{1}{n+n^\alpha} &\leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} < \sum_{k=1}^{n} \frac{1}{n} = 1, \quad \text{or} \\
\frac{n}{n+n^\alpha} &\leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} < 1. \quad \text{Hence,} \\
\lim_{n\to\infty} \frac{n}{n+n^\alpha} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1 \\
\lim_{n\to\infty} \frac{1}{1+\alpha n^{\alpha-1}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1. \quad \text{But,} \\
\lim_{n\to\infty} \frac{1}{1+\alpha n^{\alpha-1}} &= 1, \ \text{since } \alpha - 1 < 0. \text{ Therefore,} \\
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} &= 1.
\end{aligned}$$

Finally, suppose $\alpha > 1$.

We note that $\dfrac{1}{n+k^\alpha}$ is a decreasing function of k and as a result we can write

$$0 \le \sum_{k=1}^{\infty} \frac{1}{n+k^\alpha} \le \int_0^n \frac{1}{n+k^\alpha}dk = \frac{1}{n}\int_0^1 \frac{1}{1+\dfrac{k^n}{n^{\alpha/\alpha}}}dk.$$

Using the substitution $u = \dfrac{k}{u^{1/\alpha}}$ with $du = \dfrac{1}{n^{1/\alpha}}dk$, the above becomes,

$$\begin{aligned}
0 \le \sum_{k=1}^{n} \frac{1}{n+k^\alpha} &\le \frac{n^{1/\alpha}}{n}\int_0^{n^{(n-1)/n}} \frac{1}{1+u^\alpha}du = \frac{1}{n^{(\alpha-1)/\alpha}}\int_0^{n^{(n-1)/\alpha}} \frac{1}{1+u^\alpha}du \\
&\le \frac{1}{n^{(\alpha-1)/\alpha}}\int_0^{n} \frac{1}{1+u^\alpha}du \\
&\le \frac{1}{n^{(\alpha-1)}\alpha}\int_0^{1} \frac{1}{1+u^\alpha}du + \frac{1}{n^{(\alpha-1)/\alpha}}\int_1^{n} \frac{1}{1+u^\alpha}du \\
&\le \frac{1}{n^{(\alpha-1)/\alpha}}(1) + \frac{1}{n^{(\alpha-1)/\alpha}}\int_1^{n} \frac{1}{1+u}du \\
&= \frac{1}{n^{(\alpha-1)/\alpha}}(1) + \frac{1}{n^{(\alpha-1)/\alpha}}(1)\Big[\ln(1+n)-\ln 2\Big].
\end{aligned}$$

That is,

$$0 \le \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^\alpha} \le \lim_{n\to\infty}\frac{1}{n^{(\alpha-1)/\alpha}} + \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{2}\right)}{n^{(\alpha-1)/\alpha}}.$$

Using L'Hospital's rule repeatedly we get,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^{(\alpha-1)/\alpha}} + \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{2}\right)}{n^{(\alpha-1)/\alpha}} &= 0 + \lim_{n\to\infty}\frac{\frac{2}{n+1}}{\left(\frac{\alpha-1}{\alpha}\right)n^{-1/\alpha}} \\
&= \lim_{n\to\infty}\frac{2\alpha n^{1/\alpha}}{(\alpha-1)(n+1)} \\
&= \lim_{n\to\infty}\frac{2}{(\alpha-1)(n)^{1-1/\alpha}} \\
&= 0.
\end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^\alpha} = 0$ for $\alpha > 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^\alpha} = \begin{cases} 1, & 0<\alpha<1; \\ \ln 2, & \alpha = 1; \\ 0, & \alpha > 1. \end{cases}$

For $0 < \alpha < 1$, we have

$$\frac{1}{1+n^{\alpha-1}} = \sum_{k=1}^{n} \frac{1}{n+n^{\alpha}} \le \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} < \sum_{k=1}^{n} \frac{1}{n} = 1 \text{ and so } \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = 1.$$

For $\alpha = 1$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{(1+k/n)} = \int_0^1 \frac{dx}{1+x} = \ln 2.$$

For $\alpha > 1$, let t be any real number satisfying $\dfrac{1}{\alpha} < t < 1$ and let $m = \lfloor n^t \rfloor$.

We have

$$0 < \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = \sum_{k=1}^{m} \frac{1}{n+k^{\alpha}} + \sum_{k=m+1}^{n} \frac{1}{n+k^{\alpha}} < \frac{m}{n} + \frac{n-m}{(m+1)^{\alpha}} \le \frac{1}{n^{1-t}} + \frac{1}{n^{\alpha t-1}},$$

which tends to 0 as n tends to infinity. It follows that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = 0.$

This completes the solution.

Also solved by Valmir Krasniqi, Prishtina, Kosova, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
December 15, 2009

- 5074: *Proposed by Kenneth Korbin, New York, NY*

  Solve in the reals:

$$\sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}}=\frac{3}{x\sqrt{x}}.$$

- 5075: *Proposed by Kenneth Korbin, New York, NY*

  An isosceles trapezoid is such that the length of its diagonal is equal to the sum of the lengths of the bases. The length of each side of this trapezoid is of the form $a+b\sqrt{3}$ where a and b are positive integers.

  Find the dimensions of this trapezoid if its perimeter is $31+16\sqrt{3}$.

- 5076: *Proposed by M.N. Deshpande, Nagpur, India*

  Let a, b, and m be positive integers and let F_n satisfy the recursive relationship

$$F_{n+2}=mF_{n+1}+F_n, \text{ with } F_0=a, F_1=b, n\geq 0.$$

  Furthermore, let $a_n=F_n^2+F_{n+1}^2, n\geq 0$. Show that for every a, b, m, and n,

$$a_{n+2}=(m^2+2)a_{n+1}-a_n.$$

- 5077: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Find all triplets (x, y, z) of real numbers such that

$$\left.\begin{array}{l} xy(x+y-z)=3, \\ yz(y+z-x)=1, \\ zx(z+x-y)=1. \end{array}\right\}$$

- 5078: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy*

  Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)}\geq\frac{3}{2}\frac{1}{\sqrt{ab+ac+cb}}.$$

- 5079: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $x \in (0,1)$ be a real number. Study the convergence of the series

$$\sum_{n=1}^{\infty} x^{\sin\frac{1}{1} + \sin\frac{1}{2} + \cdots + \sin\frac{1}{n}}.$$

Solutions

- 5056: *Proposed by Kenneth Korbin, New York, NY*

  A convex pentagon with integer length sides is inscribed in a circle with diameter $d = 1105$. Find the area of the pentagon if its longest side is 561.

  Solution by proposer

  The answer is 25284.

  The sides are $561, 169, 264, 105$, and 47 (in any order).

  Check: $\arcsin\left(\frac{561}{d}\right) = \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right) + \arcsin\left(\frac{105}{d}\right) + \arcsin\left(\frac{47}{d}\right)$.

  Let $\overline{AB} = 561, \overline{BC} = 105, \overline{CD} = 47, \overline{DE} = 169, \overline{EA} = 264$. Then Diag $\overline{AC} = 468$.

  Check: $\arcsin\left(\frac{468}{d}\right) = \arcsin\left(\frac{47}{d}\right) + \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right)$.

  Area $\triangle ABC = \sqrt{567 \cdot 99 \cdot 462 \cdot 6} = 12474$.

  Diag $\overline{AD} = 425$.

  Check: $\arcsin\left(\frac{425}{d}\right) = \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right)$.

  Area $\triangle ACD = \sqrt{470 \cdot 45 \cdot 423 \cdot 2} = 4230$, and
  Area $\triangle ADE = \sqrt{429 \cdot 260 \cdot 165 \cdot 4} = 8580$.

  Area pentagon $= 12474 + 4230 + 8580 = 25284$.

  Editor's comments: Several solutions to this problem were received each claiming, at least initially, that the problem was impossible. I sent these individuals Ken's proof and some responded with an analysis of their errors. **Brian Beasley of Clinton, SC** responded as follows:

  "My assumption was that the inscribed pentagon was large enough to contain the center of the circle, so that I could subdivide the pentagon into five isosceles triangles, each with two radii as sides along with one side of the pentagon. But this pentagon is very small compared to the circle; it does not contain the center of the circle, and the ratio of its area to the area of the circle is only bout 2.64%. Attached is a rough diagram with two attempts to draw such an inscribed pentagon."

  "This has been a fascinating exercise! I found a Wolfram site and a Monthly paper with results about cyclic pentagons: <http://mathworld.wolfram.com/CyclicPentagon.html> and Areas of Polygons Inscribed in a Circle, by D. Robbins, American Mathematical Monthly, 102(6), 1995, 523-530."

"I salute Ken for creating this problem and for finding the arcsine identities to make it work."

David Stone and John Hawkins of Statesboro GA wrote: "Using MATLAB, we found the following four cyclic pentagons which have a side of length 561 and can be inscribed in a circle of diameter 1105. The first one has longest side 561, as required by the problem."

561	264	169	105	47	Area = 25284
817	663	663	561	520	Area = 705276
817	744	576	561	520	Area = 699984
817	744	663	561	425	Area = 692340

- 5057: *Proposed by David C. Wilson, Winston-Salem, N.C.*

We know that $1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ where $-1 < x < 1$.

Find formulas for $\sum_{k=1}^{\infty} kx^k$, $\sum_{k=0}^{\infty} k^2x^k$, $\sum_{k=0}^{\infty} k^3x^k$, $\sum_{k=0}^{\infty} k^4x^k$, and $\sum_{k=0}^{\infty} k^5x^k$.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

By differentiating the geometric series when $|x| < 1$,

$$\begin{aligned}\sum_{k=1}^{\infty} x^k &= \frac{1}{1-x} \\ &\Rightarrow \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \\ &\Rightarrow \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \qquad (1)\end{aligned}$$

Similarly, by differentiating (1),

$$\begin{aligned}\sum_{k=1}^{\infty} k^2x^{k-1} &= \frac{1+x}{(1-x)^3} \\ &\Rightarrow \sum_{k=1}^{\infty} k^2x^k = \frac{x(1+x)}{(1-x)^3}.\end{aligned}$$

Continuing this technique, it can be shown that

$$\begin{aligned}\sum_{k=1}^{\infty} k^3x^k &= \frac{x(x^2+4x+1)}{(1-x)^4} \\ \sum_{k=1}^{\infty} k^4x^k &= \frac{x(x^3+11x^2+11x+1)}{(1-x)^5} \\ \sum_{k=1}^{\infty} k^5x^k &= \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}\end{aligned}$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy

The sums are respectively:

$$\frac{x}{(1-x)^2},\quad \frac{x(x+1)}{(1-x)^3},\quad \frac{x(x^2+4x+1)}{(1-x)^4},$$

$$\frac{x(x^3+11x^2+11x+1)}{(1-x)^5},\quad \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}$$

One might invoke standard theorems about the differentiability of convergent power series, but we propose the following proof which we believe is attributed to Euler.
We define
$S_p(x) \doteq \sum_{k=1}^{\infty} k^p x^k$, $p=1,\ldots,5$ and employ $\sum_{k=1}^{\infty} x^k = \left(\sum_{k=0}^{\infty} x^k\right) - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$.

To compute $\sum_{k=0}^{\infty} x^k - 1 = \frac{1}{1-x}$ we proceed as follows:

$$P \doteq \sum_{k=0}^{\infty} x^k = 1 + x(1+x+x^2+\ldots) = 1 + xP \quad\Longrightarrow\quad P = \frac{1}{1-x}.$$

$\mathbf{S_1(x)}$:

$$\sum_{k=1}^{\infty} kx^k = \sum_{k=2}^{\infty}(k-1)x^k + \sum_{k=0}^{\infty} x^k - 1 = x\sum_{n=1}^{\infty} nx^n + \frac{1}{1-x} - 1 \text{ or}$$

$$(1-x)\sum_{k=1}^{\infty} kx^k = \frac{x}{1-x} \quad\Longrightarrow\quad \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

$\mathbf{S_2(x)}$:

$$\sum_{k=1}^{\infty} k^2x^k = \sum_{k=2}^{\infty}(k-1)^2x^k + 2\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \text{ or}$$

$$\begin{aligned}\sum_{k=1}^{\infty} k^2x^k - x\sum_{n=1}^{\infty} n^2x^n &= 2\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \\ &= \frac{2x}{(1-x)^2} - \frac{x}{(1-x)} \Longrightarrow S_2(x) = \frac{x(x+1)}{(1-x)^3}.\end{aligned}$$

$\mathbf{S_3(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^3x^k &= \sum_{k=2}^{\infty}(k-1)^3x^k + 3\sum_{k=1}^{\infty} k^2x^k - 3\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \\ &= x\sum_{k=1}^{\infty} k^3x^k + 3\sum_{k=1}^{\infty} k^2x^k - 3\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$(1-x)\sum_{k=1}^{\infty} k^3x^k = 3S_2(x) - 3S_1(x) + \frac{x}{1-x} \Longrightarrow S_3(x) = \frac{x(x^2+4x+1)}{(1-x)^4}.$$

$\mathbf{S_4(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^4x^k &= \sum_{k=2}^{\infty}(k-1)^4x^k + 4\sum_{k=1}^{\infty} k^3x^k - 6\sum_{k=1}^{\infty} k^2x^k + 4\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \\ &= x\sum_{k=1}^{\infty} k^4x^k + 4\sum_{k=1}^{\infty} k^3x^k - 6\sum_{k=1}^{\infty} k^2x^k + 4\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$(1-x)\sum_{k=1}^{\infty} k^4x^k = 4S_3(x) - 6S_2(x) + 4S_1(x) - \frac{x}{1-x} \Longrightarrow S_4(x) = \frac{x(x^3+11x^2+11x+1)}{(1-x)^5.}$$

$\mathbf{S_5(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^5x^k &= \sum_{k=2}^{\infty}(k-1)^5x^k + 5\sum_{k=1}^{\infty} k^4x^k - 10\sum_{k=1}^{\infty} k^3x^k + 10\sum_{k=1}^{\infty} k^2x^k - 5\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \\ &= x\sum_{k=1}^{\infty} k^5x^k + 5S_4(x) - 10\sum_{k=1}^{\infty} k^3x^k + 10\sum_{k=1}^{\infty} k^2x^k - 5\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$\begin{aligned}(1-x)\sum_{k=1}^{\infty} k^5x^k &= 5S_4(x) - 10S_3(x) + 10S_2(x) - 5S_1(x) + \frac{x}{1-x} \\ &\Longrightarrow S_5(x) = \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}.\end{aligned}$$

Also solved by Matei Alexianu (student, St. George's School), Spokane,WA; Brian D. Beasley, Clinton, SC; Sully Blake (student, St. George's School), Spokane,WA; Michael Brozinsky, Central Islip, NY; Mark Cassell (student, St. George's School), Spokane,WA; Richard Caulkins (student, St. George's School), Spokane,WA; Pat Costello, Richmond, KY; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Nguyen Pham and Quynh Anh (jointly; students, Belarusian State University), Belarus; Boris Rays, Brooklyn, NY, and the proposer.

- 5058: *Proposed by Juan-Bosco Romero Márquez, Avila, Spain.*

If p, r, a, A are the semi-perimeter, inradius, side, and angle respectively of an acute triangle, show that

$$r + a \le p \le \frac{p}{\sin A} \le \frac{p}{\tan\dfrac{A}{2}},$$

with equality holding if, and only if, $A = 90^o$.

Solution by Manh Dung Nguyen,(student, Special High School for Gifted Students) HUS, Vietnam

1) $\mathbf{r+a\le p}$:

$\tan\dfrac{A}{2}\le 1$ for all $A\in(0,\pi/2]$, so by the well known formula $\tan\dfrac{A}{2}=\dfrac{(p-b)(p-c)}{p(p-a)}$ we have $(p-b)(p-c)\le p(p-a)$. Letting S be the area of $\triangle ABC$ and using Heron's formula,

$$\begin{aligned} S^2=p^2r^2 &= p(p-a)(p-b)(p-c)\le p^2(p-a)^2. \text{ Thus}\\ r &\le p-a \text{ or } r+a\le p. \end{aligned}$$

2) $\mathbf{p\le \dfrac{p}{\sin A}}$:

We have $\sin A\le 1$ for all $A\in(0,\pi)$, so $p\le\dfrac{p}{\sin A}$.

3) $\mathbf{\dfrac{p}{\sin A}\le\dfrac{p}{\tan\frac{A}{2}}}$:

For $A\in(0,\pi/2]$ we have

$$\sin A-\tan\frac{A}{2}=\sin\frac{A}{2}\left(2\cos\frac{A}{2}-\frac{1}{\cos\frac{A}{2}}\right)=\frac{\sin\frac{A}{2}\cos A}{\cos\frac{A}{2}}\ge 0. \text{ Hence}$$

$$\frac{p}{\sin A}\le\frac{p}{\tan\frac{A}{2}}.$$

Equality holds if and only if $A=90^0$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Scott H. Brown, Montgomery, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY, and the proposer.

- 5059: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Prove that for all triangles ABC

$$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\le\frac{6\sqrt{3}+1}{8}.$$

Editor's comment: Many readers noted that the inequality as stated in the problem is incorrect. It should have been $\dfrac{3(2\sqrt{3}+1)}{2}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We need the following inequalities

$$\begin{aligned} \sin(A)+\sin(B)+\sin(C) &\ge \sin(2A)+\sin(2B)+\sin(2C) &\quad (1)\\ \sin(A)+\sin(B)+\sin(A) &\le \frac{3\sqrt{3}}{2} &\quad (2) \end{aligned}$$

$$\sin(\frac{A}{2}) + \sin(\frac{B}{2}) + \sin(\frac{C}{2}) \quad \leq \quad \frac{3}{2} \qquad (3)$$

Inequalities (1), (2), (3) appear respectively as inequalities 2.4, 2.2(1),and 2.9 in Geometric Inequalities by O. Bottema, R.Z. Dordevic, R.R. Janic, D.S. Mitrinovic, and P.M. Vasic, (Groningen), 1969.

It follows from (1),(2),(3) that

$$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \leq \frac{3(2\sqrt{3}+1)}{2}.$$

Solution 2 by John Hawkins and David Stone, Statesboro, GA

We treat this as a Lagrange Multiplier Problem: let

$$f(A,B,C) = \sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right).$$

We wish to find the maximum value of this function of three variables, subject to the constraint $g(A,B,C) : A + B + C = \pi$. That is, (A,B,C) lies in the closed, bounded, triangular region in the first octant with vertices on the coordinate axes: $(\pi,0,0),(0,\pi,0),(0,0,\pi)$.

By taking partial derivatives with respect to the variables A, B, and C and setting $\nabla f(A,B,C) = \lambda \nabla g(A,B,C)$ or $\left\langle f_A, f_B, f_C \right\rangle = \lambda \left\langle g_A, g_B, g_C \right\rangle = \lambda\langle 1,1,1\rangle$, we are lead to the system

$$\begin{cases} 2\cos(2A) + \cos(A) + \frac{1}{2}\cos\left(\frac{A}{2}\right) = \lambda \\ 2\cos(2B) + \cos(B) + \frac{1}{2}\cos\left(\frac{B}{2}\right) = \lambda \\ 2\cos(2C) + \cos(C) + \frac{1}{2}\cos\left(\frac{C}{2}\right) = \lambda \end{cases}$$

It is clear that one solution is to let $A = B = C$. We claim there are no others in our domain.

To show this, we investigate the fuction $h(\theta) = 2\cos(2\theta) + \cos(\theta) + \dfrac{1}{2}\cos\left(\dfrac{\theta}{2}\right)$ on the interval $0 \leq \theta \leq \pi$. Finding a solution to our system is equivalent to finding values A, B and C such that $h(A) = h(B) = h(C) = \lambda$.

We determine that $h(0) = 3.5$; then the function h decreases, passing through height 1 at (0.802,1), reaching a minimum at $(1.72, -1.73)$, then rising to height 1 at π. No horizontal line crosses the graph three times, so we cannot find distinct A, B and C with $h(A) = h(B) = h(C)$. In fact, because the function is decreasing from 0 to 1.72, and increasing from 1.72 to π, any horizontal line crossing the graph more than once must do so after $\theta = 0.802$. That is all of A, B and C would have to be greater than 0.802, and at least one of them greater than 1.72. Because $0.802 + 0.802 + 1.72 = 3.324 > \pi$, this violates the condition that $A + B + C = \pi$.

Thus the maximum value occurs when $A = B = C = \dfrac{\pi}{3}$:

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = 3\sin\left(\frac{2\pi}{3}\right) + 3\sin\left(\frac{\pi}{3}\right) + 3\sin\left(\frac{\pi}{6}\right) = 6\frac{\sqrt{3}}{2} + \frac{3}{2} = \frac{6\sqrt{3}+3}{2}.$$

This method tells us that the only point on the plane $A + B + C = \pi$ (in the first octant) where the function f achieves a maximum value is the point we just found. We must check the boundaries for a minimum.

Note that $f(\pi, 0, 0) = 1 = f(0, \pi, 0) = f(0, 0, \pi)$. That is f achieves the lower bound 1 at the vertices of our triangular region.

We also consider the behavior of the function f along the edges of this region. For instance, in the AB-plane where $C = 0$, we have $A + B = \pi$. Then $f(A, \pi - A, 0) = 2\sin A + \sin\left(\dfrac{A}{2}\right) + \cos\left(\dfrac{A}{2}\right)$, which has value 1 (of course) at the endpoints $A = 0$ and $A = \pi$, and climbs to a local maximum value of $2 + \sqrt{2}$ when $A = \dfrac{\pi}{2}$. This value is less than $f\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

There is identical behavior along the other two edges.

In summary, the function f achieves an absolute maximum of $\dfrac{6\sqrt{3}+3}{2}$ at the interior point $A = B = C = \dfrac{\pi}{3}$, and f achieves its absolute minimum of 1 at the vertices.

However, for a non-degenerate triangle ABC

$$1 < \sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \le \frac{6\sqrt{3}+3}{2},$$

and the lower bound is never actually achieved.

Solution 3 by Tom Leong, Scranton, PA

This inequality follows from summing the three known inequalities labeled (1), (2), and (3) below. Both $\sin x$ and $\sin\dfrac{x}{2}$ are concave down on $(0, \pi)$. Applying the AM-GM inequality followed by Jensen's inequality gives

$$\sin A \sin B \sin C \le \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3 \le \sin^3\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{8} \qquad (1)$$

and

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \left(\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3}\right)^3 \le \sin^3\left(\frac{A+B+C}{6}\right) = \frac{1}{8}. \qquad (2)$$

For the third inequality, we use the AM-GM inequality along with the identity

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

and (1):

$$\sin 2A \sin 2B \sin 2C \quad \le \quad \left(\frac{\sin 2A + \sin 2B + \sin 2C}{3}\right)^3 = \left(\frac{4\sin A \sin B \sin C}{3}\right)^3$$

$$\le \left(\frac{4}{3}\cdot\frac{3\sqrt{3}}{8}\right)^3 = \frac{3\sqrt{3}}{8}. \tag{3}$$

Equality occurs if and only if $A = B = C = \pi/3$ as it does in every inequality used above.

Also solved by Brian D. Beasley, Clinton, SC; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Elsie Campbell, Dionne Bailey, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students) HUS, Vietnam; Boris Rays, Brooklyn, NY, and the proposer.

- 5060: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Show that there exists $c \in (0, \pi/2)$ such that

$$\int_0^c \sqrt{\sin x}\,dx + c\sqrt{\cos c} = \int_c^{\pi/2} \sqrt{\cos x}\,dx + (\pi/2 - c)\sqrt{\sin c}$$

Solution 1 by Paul M. Harms, North Newton, KS

Let

$$f(x) = \int_0^x \sqrt{\sin t}\,dt + x\sqrt{\cos x} - \int_x^{\pi/2} \sqrt{\cos t}\,dt - (\frac{\pi}{2} - x)\sqrt{\sin x} \text{ where } x \in [0, \pi/2].$$

For $x \in [0, \pi/2]$, each term of $f(x)$ is continuous including the integrals of continuous functions. Then $f(x)$ is continuous for $x \in [0, \pi/2]$. For any $x \in [0, \pi/2]$, the two integrals of nonnegative functions are positive except when the lower limit equals the upper limit. We have

$$f(0) = -\int_0^{\pi/2} \sqrt{\cos t}\,dt < 0 \text{ and } f(\pi/2) = \int_0^{\pi/2} \sqrt{\sin t}\,dt > 0.$$

Since $f(x)$ is continuous for $x \in [0, \pi/2]$, $f(0) < 0$ and $f(\pi/2) > 0$, there is at least one $c \in (0, \pi/2)$ such that

$$f(c) = 0 = \int_0^c \sqrt{\sin t}\,dt + c\sqrt{\cos c} - \int_c^{\pi/2} \sqrt{\cos t} - (\pi/2 - c)\sqrt{\sin c}.$$

This last equation is equivalent to the equation in the problem.

Solution 2 by Michael C. Faleski, University Center, MI

The given equation will hold if the integrals and their constants of integration are the same on each side of the equality.
For the integral $\int_0^c \sqrt{\sin x}dx$ we substitute $x = \frac{\pi}{2} - y$ to obtain

$$\int_0^c \sqrt{\sin x}dx = \int_{\pi/2}^{\pi/2-c} \sqrt{\sin\left(\frac{\pi}{2} - y\right)}(-dy) = \int_{\pi/2-c}^{\pi/2} \sqrt{\cos y}dy.$$

We substitute this into the original statement of the problem and equate the integrals on each side of the equation.

$$\int_{\pi/2-c}^{\pi/2} \sqrt{\cos y}dy = \int_{c}^{\pi/2} \sqrt{\cos y}dy$$

For equality to hold the lower limits of integration must be the same; that is,
$\dfrac{\pi}{2} - c = c \Longrightarrow c = \dfrac{\pi}{4}$

We now check the constants of integration on each side of the equality when $c = \dfrac{\pi}{4}$, and we see that they are equal.

$$\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^{1/2} = \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^{1/2}$$

Hence, the value of $c = \dfrac{\pi}{4}$ satisfies the original equation.

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker (jointly), San Angelo, TX; Brian D. Beasley, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ovidiu Furdui, Cluj, Romania; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Nguyen Pham and Quynh Anh (jointly; students, Belarusian State University), Belarus; Angel Plaza, Las Palmas, Spain; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly) Statesboro, GA , and the proposer.

- 5061: *Michael P. Abramson, NSA, Ft. Meade, MD.*

Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers. Prove that

$$\sum_{i_m=1}^{n} \sum_{i_{m-1}=1}^{i_m} \cdots \sum_{i_1=1}^{i_2} a_{i_1} = \sum_{i=1}^{n} \binom{n-i+m-1}{m-1} a_i.$$

Solution by Tom Leong, Scranton, PA

We treat the a's as variables; they don't necessarily have to be integers. Fix an i, $1 \le i \le n$, and imagine completely expanding all the sums on the lefthand side. We wish to show that, in this expansion, the number of times that the term a_i appears is $\dbinom{n-i+m-1}{m-1}$. Now each term in this expansion corresponds to some m-tuple of indices in the set

$$I = \{(i_1, i_2, \ldots, i_m) : 1 \le i_1 \le i_2 \le \cdots \le i_m \le n\}.$$

We want to count the number of elements of I of the form $(i, i_2, \ldots, i_m)$. Equivalently, using the one-to-one correspondence between I and

$$J = \{(j_1, j_2, \ldots, j_m) : 1 \le j_1 < j_2 < \cdots < j_m \le n+m-1\}$$

given by

$$(i_1, i_2, \ldots, i_m) \leftrightarrow (j_1, j_2, \ldots, j_m) = (i_1, i_2+1, i_3+2 \ldots, i_m+m-1),$$

we wish to count the number elements of J of the form $(i, j_2, \ldots, j_m)$. This number is simply the number of $(m-1)$-element subsets of $\{i+1, i+2, \ldots, n+m-1\}$ which is just $\dbinom{n-i+m-1}{m-1}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
January 15, 2010

- 5080: *Proposed by Kenneth Korbin, New York, NY*

  If p is a prime number congruent to 1 (mod4), then there are positive integers a, b, c, such that
  $$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) + \arcsin\left(\frac{c}{p^3}\right) = 90^o.$$
  Find a, b, and c if $p = 37$ and if $p = 41$, with $a < b < c$.

- 5081: *Proposed by Kenneth Korbin, New York, NY*

  Find the dimensions of equilateral triangle ABC if it has an interior point P such that $\overline{PA} = 5, \overline{PB} = 12$, and $\overline{PC} = 13$.

- 5082: *Proposed by David C. Wilson, Winston-Salem, NC*

  Generalize and prove:
  $$\begin{aligned}
  \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= 1 - \frac{1}{n+1}\\
  \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} &= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}\\
  \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)} &= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}\\
  \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)(n+4)} &= \frac{1}{96} - \frac{1}{4(n+1)(n+2)(n+3)(n+4)}
  \end{aligned}$$

- 5083: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \Re$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by
  $$x_n = \begin{cases} \displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \dfrac{1}{\alpha};\\ 0, & n \leq \dfrac{1}{\alpha} \end{cases}$$
  is convergent and determine its limit.

- 5084: *Charles McCracken, Dayton, OH*

  A natural number is called a "repdigit" if all of its digits are alike.
  Prove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit.

- 5085: *Proposed by Valmir Krasniqi, (student, Mathematics Department,) University of Prishtinë, Kosova*

  Suppose that a_k, $(1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

  Prove that
  $$\prod_{j=1}^{n} \sum_{k=1}^{n} e_{j,k} a_k^2 \ge \prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^2.$$

Solutions

- 5062: *Proposed by Kenneth Korbin, New York, NY.*

  Find the sides and the angles of convex cyclic quadrilateral ABCD if
  $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2$.

  Solution 1 by David E. Manes, Oneonta, NY

  Let $x = \overline{AB} = \overline{BC} = \overline{CD}$ and let $y = \overline{BD}$. Then $\overline{AD} = \overline{AC} = x + 2$.

  Let $\alpha = \angle CAB$, $\beta = \angle ABD$, and $\gamma = \angle DBC$. Finally, in quadrilateral $ABCD$, we denote the angle at vertex A by $\angle A$ and similarly for the other three vertices. Then $\overline{AB} = \overline{BC}$ implies $\alpha = \angle BCA$. Since angles inscribed in the same arc are congruent, it follows that

  $$\begin{aligned} \alpha &= \angle CAB = \angle CDA, \\ \alpha &= \angle BCA = \angle BDA, \\ \beta &= \angle ABD = \angle ACD, \text{and} \\ \gamma &= \angle DBC = \angle DAC \end{aligned}$$

  Therefore,

  $$\angle A = \alpha + \gamma,\ \angle B = \beta + \gamma,\ \angle C = \alpha + \beta \text{ and } \angle D = 2\alpha = \beta \text{ since } \overline{AC} = \overline{AD}.$$

  From Ptolemy's Theorem, one obtains

  $$\begin{aligned} \overline{AC} \cdot \overline{BD} &= \overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} \text{ or} \\ (x+2)y &= x^2 + x(x+2) \\ y &= \frac{2x(x+1)}{x+2}. \end{aligned}$$

  In triangles ACD and BCD, the law of cosines implies $\cos\gamma = \dfrac{2(x+2)^2 - x^2}{2(x+2)^2}$ and $\cos\gamma = \dfrac{y}{2x} = \dfrac{x+1}{x+2}$ respectively. Setting the two values equal yields the quadratic equation $x^2 - 2x - 4 = 0$ with positive solution $x = 1 + \sqrt{5}$. Hence,

  $$\overline{AB} = \overline{BC} = \overline{CD} = 1 + \sqrt{5} \text{ and } \overline{AD} = 3 + \sqrt{5}.$$

Moreover, note that

$$\begin{aligned}\cos\gamma &= \frac{x+1}{x+2}=\frac{2+\sqrt{5}}{3+\sqrt{5}}=\frac{1+\sqrt{5}}{4} \text{ implies that}\\ \gamma &= \arccos\left(\frac{1+\sqrt{5}}{4}\right)=36^0\end{aligned}$$

In $\triangle ACD, \gamma+\beta+2\alpha=180^o$ or $\gamma+2\beta=180^0$ so that $\beta=\dfrac{180^0-36^0}{2}=72^0$ and $\alpha=\beta/2=36^0$.

Therefore,

$$\angle A=\alpha+\gamma=72^0=2\alpha=\angle D \text{ and}$$

$$\angle B=\beta+\gamma=108^0=\alpha+\beta=\angle C.$$

Solution 2 by Brian D. Beasley, Clinton, SC

We let $a=\overline{AB}$, $b=\overline{BC}$, $c=\overline{CD}$, $d=\overline{AD}$, $p=\overline{BD}$ and $q=\overline{AC}$. Then $a=b=c=d-2=q-2$. According to the Wolfram MathWorld web site [1], for a cyclic quadrilateral, we have

$$pq=ac+bd \text{ (Ptolemy'sTheorem)} \quad \text{and} \quad q=\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

Thus $a+2=\sqrt{2a^2+2a}$, so the only positive value of a is $a=1+\sqrt{5}$. Hence $a=b=c=1+\sqrt{5}$ and $d=p=q=3+\sqrt{5}$. Using the Law of Cosines, it is straightforward to verify that $\angle ABC=\angle BCD=108°$ and $\angle CDA=\angle DAB=72°$.

[1] Weisstein, Eric W. "Cyclic Quadrilateral." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CyclicQuadrilateral.html

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

We show that the sides are $1+\sqrt{5}, 1+\sqrt{5}, 1+\sqrt{5}, 3+\sqrt{5}$ and the angles are $108^0,\ 72^0,\ 72^0,\ 108^0$.

Let $\alpha=\overline{AB}=\overline{BC}=\overline{CD}=\overline{AD}-2=\overline{AC}-2, \beta=\angle CBA$ and R the circumradius of $ABCD$.

By solution 1 of SSM problem 4961,

$$R=\frac{1}{4}\sqrt{\frac{[aa+a(a+2)][a(a+2)+aa][aa+a(a+2)]}{(2a+1-a)(2a+1-a)(2a+1-a)[2a+1-(a+2)]}} = \frac{a}{2}\sqrt{\frac{2a}{a-1}}.$$

From this and the generalized sine theorem in $\triangle ABC$,

$$\frac{a}{2R}=\sin\left(\frac{180^0-\beta}{2}\right)\Longrightarrow\cos\left(\frac{\beta}{2}\right)=\sqrt{\frac{a-1}{2a}}.$$

By the law of cosines in $\triangle ABC$,

$$\cos\beta=\frac{a^2+a^2-(a+2)^2}{2a^2}\Longrightarrow\cos\left(\frac{\beta}{2}\right)=\sqrt{\frac{1+\cos\beta}{2}}=\frac{\sqrt{3a^2-4a-4}}{2a}.$$

Hence,

$$\sqrt{\frac{a-1}{2a}} = \frac{\sqrt{3a^2-4a-4}}{2a} \Longrightarrow a^2-2a-4=0 \Longrightarrow a = 1+\sqrt{5} = 2\phi,$$

so the sides are

$$\overline{AB} = \overline{BC} = \overline{CD} = 1+\sqrt{5} \text{ and } \overline{AD} = a+2 = 3+\sqrt{5}.$$

Then $\beta = 2\arccos\sqrt{\dfrac{\sqrt{5}}{2(1+\sqrt{5})}} = 108^0$, so the angles are
$\angle CBA = 108^0,\ \angle DCB = \angle CBA = 108^0,\ \angle ADC = 180^0 - 108^0 = 72^0$ and $\angle BAD = 72^0$.

Also solved by Michael Brozinsky, Central Islip, NY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5063: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Euclid's inscribed polygon is a constructible polygon inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

a) Does Euclid's inscribed n-gon exist for any prime n greater than 5?

b) Does Euclid's n-gon exist for all composite numbers n greater than 2?

Solution by Joseph Lupton, Jacob Erb, David Ebert, and Daniel Kasper, students at Taylor University, Upland, IN

a) For an inscribed polygon to fit this description, there has to be an arithmetic sequence of positive integers where the number of terms in the sequence is equal to the number of sides of the polygon and the terms sum to 360. So if the first term is f and the constant difference between the terms is d, the sum of the terms is

$$f \cdot n + \frac{n(n-1)}{2}d = 360.$$

Thus, $f \cdot n + \dfrac{n(n-1)}{2}d = 360 \Longrightarrow n \Big| 360$. That is, n is a prime number greater than five and $n \Big| 2^3 \cdot 3^3 \cdot 5$. But there is no prime number greater than five that divides 360. So there is no Euclidean polygon that can be inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

b) Euclid's inscribed n-gon does not exist for all composite numbers greater than two. Obviously, if n gets too large, then the terms $\frac{n(n-1)}{2}d$ will be greater than 360 even if $d=1$ which is the minimal d allowed. There is no Eculidean inscribed n-gon for $n=21$. If there were, the the sum of central angles would be $f \cdot n + n \cdot d \cdot \dfrac{n-1}{2}$ implies that 21 divides 360. Similarly, there is no 14-gon for if there were, it would imply that 7 divides 360.

- **Comments and elaborations by David Stone and John Hawkins, Statesboro GA**

We note that this problem previously appeared as part of Problem 4708 in this journal in March, 1999; however the solution was not published. Also, a Google search on the internet turned up a paper by the proposer in the Bulletin of the Malaysian Mathematical Sciences Society in which the answer to both questions is presented as being "no". {See "The Euclidean Inscribed Polygon" (Bull. Malaysian Math Sc. Soc (Second series) 27 (2004), 45-52).}

David and John solved the problem and then elaborated on it by considering the possibility that the inscribed polygon many not enclose the center of the circle. And it is here that things start to get interesting.

(In the case where the inscribed polygon does not include the center of the circle, and letting a be the first term in the arithmetic sequence and d the common difference, they noted that the largest central angle must be the sum of the previous $n-1$ central angles, and they proceeded as follows:)

$$\begin{aligned}
a+(n-1)d=S_{n-1} &= \frac{n-1}{2}\Big(2a+(n-2)d\Big) \text{ or} \\
2a+2(n-1)d &= 2a(n-1)+(n-1)(n-2)d \text{ or} \\
2a(n-2) &= -(n-1)(n-4)d.
\end{aligned}$$

For $n=3$, this happens exactly when $a=d$; although $n=3$ is of no concern for the stated problem, we shall return to this case later.

For $n \geq 4$, this condition is never satisfied because the left-hand side is positive and the right-hand side ≤ 0.

David and John then determined all Euclidean inscribed n-gons as follows:

The cited paper by the poser points out that 3^0 is the smallest constructible angle of positive integral degree. In fact, it is well known that an angle is constructible if, and only if, its degree measure is an integral multiple of 3^0. This implies that a and d must both be multiples of 3. We wish to find all solutions of the Diophantine equation
(1) $\quad n(2a+(n-1)d)=2^4\cdot 3^2\cdot 5$, where a and d are multiples of 3.

Letting $a=3A$ and $d=3D$, the above equation becomes
(2) $\quad n\Big(2A+(n-1)D\Big)=2^4\cdot 3\cdot 5=240$, so n must be a divisor of 240.

Moreover, the cofactor $2A+(n-1)D$ is bounded below. That is

$$\begin{aligned}
2A+(n-1)D &\geq 2+(n-1)=n+1. \text{ So} \\
\frac{240}{n} &= 2A+(n-1)D \geq 1, \text{ and} \\
n(n+1) &\leq 240.
\end{aligned}$$

These conditions allow only $n=3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,$ and 15.
First we show that $n=12$ fails. For in this case (2) becomes

$$\begin{aligned}
12(2A+11D)=240, &\text{ or} \\
2A+11D=20&,
\end{aligned}$$

and this linear Diophantine equation has no positive solutions.

All other possible values of n do produce corresponding Euclidean $n-$gons.

The case $n = 3$ is perhaps the most interesting. There are twenty triangles inscribed in semi-circle: $(3A,\ 6A,\ 9A)$ for $A = 1, 2, \dots 20$, each having $a = d$, and nineteen more triangles which properly enclose the center of the circle: $(3t, 120, 240 - 3t)$, for $t = 21,\ 22, \dots,\ 39$, each with $d = 120 - a$.

We consider in detail the case $n = 4$, in which case Equation (2) becomes $4(2A + 3D) = 2^4 \cdot 3 \cdot 5$, or $2A + 3D = 60$. The solution of this Diophantine equation is given by

$$\begin{cases} A = 3t \\ D = 20 - 2t \end{cases}$$

where the integer parameter t satisfies $0 < t < 10$.

We exhibit the results in tabular form, with all angles in degrees:

t	A	$a = 3A$	D	$d = 3D$	*Central angles of inscribed quarilateral*
1	3	9	18	54	9, 63, 117, 171
2	6	18	16	48	18, 66, 114, 162
3	9	27	14	42	27, 69, 111 153
4	12	36	12	36	36, 72, 108, 144
5	15	45	10	30	45, 75, 105, 135
6	18	54	8	24	54, 78, 102, 126
7	21	63	6	18	63, 81, 99, 117
8	24	72	4	12	72, 84, 96, 108
9	27	81	2	6	81, 87, 93, 99

That is, the central angles are $(9t,\ 60 + 3t,\ 120 - 3t,\ 180 - 9t)$ for $t = 1, 2, \dots, 9$. Thus we have nine Euclidean inscribed quadrilaterals.

Similarly for $n = 5$, we have eleven Euclidean inscribed pentagons, with central angles $(6t,\ 36 + 3t,\ 72,\ 108 - 3t,\ 144 - 6t)$ for $t = 1,\ 2, \dots,\ 11$.

Similarly for $n = 6$, we have three Euclidean inscribed hexagons, with central angles $(45,\ 51,\ 57,\ 63,\ 75), (30,\ 42,\ 54,\ 66,\ 78,\ 90)$ and $(15,\ 33,\ 52,\ 69,\ 105)$.

For $n = 8$, we have two Euclidean inscribed octagons with central angles $(24,\ 30,\ 36,\ 42,\ 48,\ 54,\ 60,\ 66)$ and $(3,\ 15,\ 27,\ 39,\ 51,\ 63,\ 75,\ 87)$.

For $n = 10$, we have one Euclidean inscribed decagon, with central angles $(9,\ 15,\ 21,\ 27,\ 33,\ 39,\ 45,\ 51,\ 57,\ 63)$.

For $n = 15$, we have one Euclidean inscribed 15-gon with central angles $(3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24\ 27,\ 30,\ 33,\ 36,\ 39,\ 42,\ 45)$.

There is a grand total of 66 Euclidean inscribed n-gons!

A final note: If $n(n + 1)$ divides 240, then $a = d = 3\dfrac{240}{n(n+1)} = \dfrac{720}{n(n+1)}$ produces a Euclidean inscribed $n-$gon.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Boris Rays, Brooklyn, NY, and the proposer.

- 5064: *Proposed by Michael Brozinsky, Central Islip, NY.*

The Lemoine point of a triangle is that point inside the triangle whose distances to the three sides are proportional to those sides. Find the maximum value that the constant of proportionality, say λ, can attain.

Solution 1 by David E. Manes, Oneonta, NY

The maximum value of λ is $\sqrt{3}/6$ and is attained when the triangle is equilateral.

Given the triangle ABC let $[ABC]$ denote its area. The distance from the Lemoine point to the three sides are in the ratio $\lambda a,\ \lambda b,\ \lambda c$ where $\lambda = \dfrac{2[ABC]}{a^2+b^2+c^2}$ and a,b,c denote the length of the sides $BC,\ CA$ and AB respectively. Let $\alpha = \angle BAC, \beta = \angle CBA$, and $\gamma = \angle ACB$. Then

$$[ABC] = \frac{1}{2}bc \cdot \sin\alpha = \frac{1}{2}ac \cdot \sin\beta = \frac{1}{2}ab \cdot \sin\gamma.$$

Therefore,

$$a^2+b^2+c^2 \geq ab+bc+ca = [ABC]\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma}\right).$$

The function $f(x) = \dfrac{1}{\sin x}$ is convex on the interval $(0,\pi)$. Jensen's inequality then implies

$$f(\alpha)+f(\beta)+f(\gamma) \geq 3f\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3f\left(\frac{\pi}{3}\right) = \frac{3}{\sin\left(\frac{\pi}{3}\right)} = 2\sqrt{3}$$

with equality if and only if $\alpha=\beta=\gamma=\pi/3$. Therefore, $a^2+b^2+c^2 \geq 4\sqrt{3}\cdot[ABC]$ so that

$$\lambda = \frac{2[ABC]}{a^2+b^2+c^2} \leq \frac{2[ABC]}{4\sqrt{3}\cdot[ABC]} = \frac{\sqrt{3}}{6}$$

with equality if and only if the triangle ABC is equilateral.

Solution 2 by John Nord, Spokane, WA

Without loss of generality we can denote the coordinates of $\triangle ABC$ as $A(0,0), B(1,0), C(b,c)$, the coordinates of the Lemoine point L as (x_1,y_1), the constant of proportionality from L to the sides as λ, the coordinates on AB of the foot of the perpendicular from L to AB as $D(x_1,0)$, the coordinates on BC of the foot of the perpendicular from L to BC as $E(x_2,y_2)$ and the coordinates on AC of the foot of the perpendicular from L to AC as $F(x_3,y_3)$.

The distance from L to AB equals $LD = \lambda\cdot 1$.
The distance from L to BC equals $LE = \lambda\cdot\sqrt{(1-b)^2+c^2}$ and
The distance from L to AC equals $LF = \lambda\cdot\sqrt{b^2+c^2}$.

The coordinates of E can be found by finding the intersection of LE and BC. That is, by solving:

$$\begin{cases} y = \dfrac{c}{b-1}x + \dfrac{c}{1-b}, \text{ and} \\ y = \dfrac{1-b}{c}x + y_1 + \dfrac{b-1}{c}x_1. \end{cases}$$

And the coordinates of F can be found by finding the intersection of LF and AC. That is, by solving,

$$\begin{cases} y = \dfrac{c}{b}x \text{ and} \\ \\ y = \dfrac{-b}{c}x + y_1 + \dfrac{b}{c}x_1. \end{cases}$$

Once we have computed (x_2, y_2) and (x_3, y_3) in terms of b, c, x_1 and λ, we apply the distance relationships above. This results in:

$$x_1 = \frac{b + b^2 + c^2}{2(1 - b + b^2 + c^2)} \qquad y_1 = \lambda = \frac{c}{2(1 - b + b^2 + c^2)}.$$

The maximum value of λ is obtained by solving the system of partial derivatives

$$\begin{cases} \dfrac{\partial \lambda}{\partial b} = 0 \\ \dfrac{\partial \lambda}{\partial c} = 0. \end{cases}$$

This yields: $c = \dfrac{\sqrt{3}}{2}$ and $b = \dfrac{1}{2}$. Substituting these values into y_1 above gives $\lambda = \dfrac{\sqrt{3}}{6}$ as the maximum value of the constant of proportionality.

Solution 3 by Charles Mc Cracken, Dayton, OH

The Lemoine point is also the intersection of the symmedians.

The medians of a triangle divide the triangle in two equal areas.

The medians intersect at the centroid, G.

Any point other than G is closer than G to one side of the triangle.

In $\triangle ABC$ let a denote the side (and its length) opposite $\angle A$, b the side opposite $\angle B$, and c the side opposite $\angle C$. Let L denote the Lemoine point.

If the distance from L to side a is λa, then λa less the distance from G to a we call γa.

Similarly for sides b and c.

For $\lambda = \gamma$, L must coincide with G.

This will happen when the medians and symmedians coincide.

This occurs when the triangle is equiangular $(60^0 - 60^0 - 60^0)$ and hence equilateral $(a = b = c)$.

In that case, $\lambda = \dfrac{\sqrt{3}}{6} \equiv 0.289$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, John Hawkins and David Stone (jointly), Statesboro, GA; Kee-Wai Lau, Hong Kong, China; Tom Leong, Scranton, PA, and the proposer.

- 5065: *Mihály Bencze, Brasov, Romania.*

  Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$1) \qquad \sum_{i,j=1}^{n} |(i-j)(x_i - x_j)| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.$$

$$2)\qquad \sum_{i,j=1}^{n}(i-j)^2=\frac{n^2(n^2-1)}{6}.$$

Solution 1 by Paul M. Harms, North Newton, KS

1) Both summations in part 1) have the same terms for $i>j$ that they have for $i<j$ and have 0 for $i=j$. Equality will be shown for $i>j$.

Each row below is the left summation of part 1) of the problem for $i>j$ and for a fixed j starting with $j=1$.

$$\begin{array}{l}1(x_2-x_1)+2(x_3-x_1)+\ldots+(n-1)(x_n-x_1)\\ 1(x_3-x_2)+2(x_4-x_2)+\ldots+(n-2)(x_n-x_2)\\ \qquad\qquad\qquad\vdots\\ 1(x_{n-1}-x_{n-2})+2(x_n-x_{n-2})\\ 1(x_n-x_{n-1})\end{array}$$

The coefficient of x_1 is $(-1)[1+2+\ldots+(n-1)]=\dfrac{-(n-1)n}{2}$. Note that the coefficient of x_n (looking at the diagonal from lower left to upper right is $1+2+\ldots+(n-1)=\dfrac{(n-1)n}{2}$.

The coefficient of x_2 is $(-1)[1+2+\ldots+(n-2)]+1=\dfrac{-(n-2)(n-1)}{2}+1$, where the one is the coefficient of x_2 in row 1.

The coefficient of x_{n-1} is the negative of the coefficient of x_2.

The coefficient of x_r where r is a positive integer less than $\dfrac{n+1}{2}$ is

$$\begin{aligned}(-1)[1+2+\ldots(n-r)]+0+1+\ldots(r-1)&=\frac{(-1)(n-r)(n-r+1)}{2}+\frac{(r-1)r}{2}\\ &=\frac{(-1)n(n-2r+1)}{2}\\ &=(-1)\frac{n}{2}[(n-r)+(1-r)].\end{aligned}$$

The coefficients of x_r andx_{n+1-r} are the negatives of each other.

If we write out the right summation of part 1) for $i>j$, we can obtain a triangular form like that above except that each coefficient of the difference of the $x's$ is 1. Using the form just explained, the coefficient of x_1 is $(-1)(n-1)$ and the coefficient of x_n along the diagonal is $(n-1)$.

The coefficient of x_2 is $(-1)(n-2)+1$ where the $(+1)$ is the coefficient of x_2 in row 1.

For x_r, where r is a positive integer less than $\dfrac{n+1}{2}$, the coefficient is $(-1)(n-r)+(r-1)$ where $(r-1)$ comes from the x_r having coefficients of one in each of the first $(r-1)$ rows. The coefficient of x_r on the right side of the inequaity of part 1) is then $\dfrac{n}{2}(-1)[(n-r)+(1-r)]$ which is the same as the left side of the inequality.

Also, the coefficients of x_r and x_{n+1-r} are negative of each other.

2) To show part 2), first consider the summation of each of the three terms $i^2, j^2, -2ij$.

For each j, the summation of i^2 from $i=1$ to n is $1^2+2^2+\ldots n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Then the summation of i^2 where both i and j go from 1 to n is $\dfrac{n(n+1)(2n+1)}{6}$. The summation of j^2 is the same value.

The summation of ij is

$$\begin{aligned} 1(1+2+\ldots+n)+2(1+2+\ldots+n)+\ldots+n(1+2+\ldots+n) &= (1+2+\ldots+n)^2 \\ &= \frac{n^2(n+1)^2}{2^2} \end{aligned}$$

The total summation of the left side of part 2) is

$$\begin{aligned} \frac{2n^2(n+1)(2n+1)}{6} - \frac{2n^2(n+1)^2}{2^2} &= n^2(n+1)\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] \\ &= \frac{n^2(n+1)(n-1)}{6}. \end{aligned}$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy

We begin with 1). The result is achieved by a double induction. For $n=1$ there is nothing to say. Let's suppose that 1) holds for any $1 \le n \le m$. For $n=m+1$ the equality reads as

$$\begin{aligned} &\sum_{i,j=1}^{m+1} |i-j|\,|x_i-x_j| = \\ &\sum_{i,j=1}^{m} |i-j|\,|x_i-x_j| + \sum_{i=1}^{m+1} |i-m-1|\,|x_i-x_{m+1}| + \sum_{j=1}^{m+1} |m+1-j|\,|x_{m+1}-x_j| = \\ &\frac{m}{2}\sum_{i,j=1}^{m} |x_i-x_j| + 2\sum_{i=1}^{m+1} |i-m-1|\,(x_{m+1}-x_i). \end{aligned}$$

(in the second passage the induction hypotheses has been used) and we need it equal to

$$\frac{m+1}{2}\sum_{i,j=1}^{m+1} |x_i-x_j| = \frac{m}{2}\sum_{i,j=1}^{m} |x_i-x_j| + \frac{1}{2}\sum_{i,j=1}^{m} |x_i-x_j| + (m+1)\sum_{i=1}^{m} |x_i-x_{m+1}|.$$

Comparing the two quantities we have to prove

$$2\sum_{i=1}^{m+1} (m+1-i)(x_{m+1}-x_i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i-x_j| + (m+1)\sum_{i=1}^{m} |x_i-x_{m+1}|$$

or

$$\sum_{i=1}^{m} (x_{m+1}-x_i)(m+1-2i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i-x_j|$$

or

$$-\sum_{i=1}^{m} x_i(m+1-2i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i - x_j| \qquad \text{since} \qquad \sum_{i=1}^{m}(m+1-2i) = 0.$$

Here starts the second induction. For $m = 1$ there is nothing to do as well. Let's suppose that the equality holds true for any $1 \le m \le r$. For $m = r+1$ we have to prove that

$$-\sum_{i=1}^{r+1} x_i(r+2-2i) = \frac{1}{2}\sum_{i,j=1}^{r} |x_i - x_j| + \frac{1}{2}\sum_{i=1}^{r+1}(x_{r+1} - x_i) + \frac{1}{2}\sum_{i=1}^{r+1}(x_{r+1} - x_i).$$

which, by using the induction hypotheses is

$$-\sum_{i=1}^{r} x_i(r+1-2i) - \sum_{i=1}^{r} x_i + r x_{r+1} = -\sum_{i=1}^{r} x_i(r+1-2i) + \sum_{i=1}^{r+1}(x_{r+1} - x_i).$$

or

$$-\sum_{i=1}^{r} x_i + r x_{r+1} = (r+1)x_{r+1} - x_{r+1} - \sum_{i=1}^{r} x_i.$$

namely the expected result.

To prove 2) we employ 1) by calculating $\frac{n}{2}\sum_{i,j=1}^{n} |i-j|$. The symmetry of the absolute value yields

$$\frac{n}{2}\sum_{i,j=1}^{n} |i-j| = n\sum_{1\le i<j\le n}(j-i) = n\sum_{i=1}^{n}\sum_{j=i+1}^{n}(j-i) = n\sum_{i=1}^{n}\sum_{k=1}^{n-i} k = \frac{n}{2}\sum_{i=1}^{n}(n-i)(n-i+1).$$

The last sum is equal to $\frac{n}{2}\sum_{k=1}^{n-1} k(k+1)$.

In the last step we show that $\sum_{k=1}^{n-1} k(k+1) = \frac{n^3-n}{3}$.
For $n = 1$ both sides are 0. Let's suppose it is true for $1 \le n \le m-1$.
For $n = m$ we have

$$\sum_{k=1}^{m-1} k(k+1) + m(m+1) = \frac{m^3-m}{3} + m(m+1) = m(m+1)\frac{m+2}{3} = \frac{(m+1)^3-(m+1)}{3}.$$

Finally,

$$\frac{n}{2}\frac{n^3-n}{3} = n^2\frac{n^2-1}{6}$$

The proof is complete.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5066: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let $abc = 1$. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB respectively. Prove or disprove that

$$3(a+b)(b+c)(c+a) \geq 32(d_a + d_b + d_c)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the inequality. First we have $(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 8$. Hence it suffices to prove that $d_a + d_b + d_c \leq \dfrac{\sqrt{3}}{2}$. Let s, r, R be respectively the semi-perimeter, in-radius and circumradius of triangle ABC. Let the foot of the perpendicular from A to BC be D and the foot of the perpendicular from B to AC be E so that $\triangle BCE \sim \triangle BHD$. Hence,

$$\begin{aligned} d_a &= \overline{DH} = \frac{(\overline{BD})(\overline{CE})}{\overline{BE}} \\ &= \frac{(c\cos B)(a\cos C)}{c\sin A} = 2R\cos B\cos C, \text{ and similarly,} \\ d_b &= 2R\cos C\cos A \text{ and } d_c = 2R\cos A\cos B. \end{aligned}$$

Therefore, by the well known equality

$$\begin{aligned} \cos A\cos B + \cos B\cos C + \cos C\cos A &= \frac{r^2+s^2-4R^2}{4R^2}, \text{ we have} \\ d_a + d_b + d_c &= \frac{r^2+s^2-rR^2}{2R}. \end{aligned}$$

And by a result of J. C. Gerretsen: Ongelijkheden in de Driehoek Nieyw Tijdschr.Wisk. 41(1953), 1-7, we have $s^2 \leq 4R^2 + 4Rr + 3r^2$. Thus

$$d_a + d_b + d_c = \frac{2r(R+r)}{R} \leq 3r,$$

which follows from L. Euler's result that $R \geq 2r$.

It remains to show that $r \leq \dfrac{1}{2\sqrt{3}}$. But this follows from the well known result that $s \geq 3\sqrt{3}r$ and the fact that $1 = abc = 4rsR \geq 4r(3\sqrt{3})r(2r) = 24\sqrt{3}r^3$.

This completes the solution.

Also solved by the proposer.

- 5067: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be complex numbers such that $a + b + c = 0$. Prove that

$$\max\{|a|, |b|, |c|\} \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.$$

Solution by Tom Leong, Scranton, PA

Since $a+b+c=0$, $|a|$, $|b|$, and $|c|$ form the sides of a (possibly degenerate) triangle. It follows from the triangle inequality that the longest side, $\max\{|a|,|b|,|c|\}$, cannot exceed half of the perimeter, $\frac{1}{2}(|a|+|b|+|c|)$, of the triangle. Using this fact along with the Cauchy-Schwarz inequality gives the desired result:

$$\begin{aligned}
\max\{|a|,|b|,|c|\} &\leq \frac{1}{2}(|a|+|b|+|c|) \\
&= \frac{1}{2}(1\cdot|a|+1\cdot|b|+1\cdot|c|) \\
&\leq \frac{1}{2}\sqrt{1^2+1^2+1^2}\sqrt{|a|^2+|b|^2+|c|^2} \\
&= \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Centeral Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students), HUS, Vietnam; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2010

- 5086: *Proposed by Kenneth Korbin, New York, NY*

  Find the value of the sum
  $$\frac{2}{3}+\frac{8}{9}+\cdots+\frac{2N^2}{3^N}.$$

- 5087: *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers a, b, c, and d such that $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ with $a < b < c < d$. Rationalize and simplify
  $$\frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} \quad \text{if} \quad \begin{cases} x = bc+bd+cd, & \text{and} \\ y = ab+ac+ad. \end{cases}$$

- 5088: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b be positive integers. Prove that
  $$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2)+\varphi^2(b^2)}} \le \frac{\sqrt{2}}{2},$$
  where $\varphi(n)$ is Euler's totient function.

- 5089: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  In $\triangle ABC$ let $AB = c, BC = a, CA = b, r =$ the in-radius and r_a, r_b, and $r_c=$ the ex-radii, respectively.

  Prove or disprove that
  $$\frac{(r_a-r)(r_b+r_c)}{r_ar_c+rr_b}+\frac{(r_c-r)(r_a+r_b)}{r_cr_b+rr_a}+\frac{(r_b-r)(r_c+r_a)}{r_br_a+rr_c} \ge 2\left(\frac{ab}{b^2+ca}+\frac{bc}{c^2+ab}+\frac{ca}{a^2+bc}\right).$$

- 5090: *Proposed by Mohsen Soltanifar (student), University of Saskatchewan, Canada*

  Given a prime number p and a natural number n. Calculate the number of elementary matrices $E_{n\times n}$ over the field Z_p.

- 5091: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k, p \geq 0$ be nonnegative integers. Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}}dx.$$

Solutions

- 5068: *Proposed by Kenneth Korbin, New York, NY.*

Find the value of

$$\sqrt{1+2009\sqrt{1+2010\sqrt{1+2011\sqrt{1+\cdots}}}}.$$

Solution by Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA

To solve this, we apply Ramanujan's nested radical. Consider the identity $(x+n)^2 = x^2+2nx+n^2$, which can be rewritten as

$$x+n = \sqrt{n^2+x((x+n)+n)}.$$

Now, the $(x+n)+n$ term has the same form as the left-hand side, so we can write it in terms of a radical:

$$x+n = \sqrt{n^2+x\sqrt{n^2+(x+n)((x+2n)+n)}}$$

Repeating this process, ad infinitum, yields Ramanujan's nested radical:

$$x+n = \sqrt{n^2+x\sqrt{n^2+(x+n)\sqrt{n^2+\cdots}}}$$

With $n=1$ and $x=2009$, the right-hand side becomes the expression in the problem. It follows that the value is 2010.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Pat Costello, Richmond, KY; Michael N. Fried, Kibbutz Revivim, Israel; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Nguyen Van Vinh (student, Belarusian State University), Minsk, Belarus, and the proposer.

- 5069: *Proposed by Kenneth Korbin, New York, NY.*

Four circles having radii $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{x}$ and $\frac{1}{y}$ respectively, are placed so that each of the circles is tangent to the other three circles. Find positive integers x and y with $15 < x < y < 300$.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain

If all the circles are tangent in a point, the problem is not interesting because x and y can take on any value for which $15 < x < y < 300$. So we assume that the circles are not mutually tangent at a point.
By Descarte's circle theorem with ϵ_1, ϵ_2 and ϵ_3 being the curvature of the first three circles, the curvature ϵ_4 of the fourth circle can be obtained with Soddy's formula:

$$\begin{aligned}
\epsilon_4 &= \epsilon_1 + \epsilon_2 + \epsilon_3 \pm 2\sqrt{\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1}, \text{ that is,} \\
y &= 14 + 15 + x \pm 2\sqrt{14 \cdot 15 + 15 \cdot x + x \cdot 14} \\
y &= 29 + x + \pm 2\sqrt{210 + 29x}
\end{aligned}$$

Then, $210 + 29x$ must be a perfect square, say a^2. Since, $15 < x < 300$,

$$25^2 < 210 + 29x < 95^2, \text{ so}$$
$$26 \leq a \leq 94.$$

Thus,

$$29 \Big| (a^2 - 210).$$

The only integers a, $26 \leq a \leq 94$, which satisfy this condition are $35, 52, 64, 81$, and 93. Taking into account that $15 < x < y < 300$, we have:

$$\begin{aligned}
\text{For } a &= 35, \ x = 35 \text{ and so } y = 29 + x \pm 2a = 134 \\
\text{For } a &= 52, \ x = 86 \text{ and } y = 219; \\
\text{For } a &= 64, \ x = 134 \text{ and } y = 291;
\end{aligned}$$

and for $a \in \{81, 93\}$, none of the obtained values of y is valid.

Thus the only pairs of integers x and y with $15 < x < y < 300$ are

$$(x, y) \in \Big\{(35, 134), (86, 219), (134, 291)\Big\}.$$

Also solved by Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Antonio Ledesma Vila, Requena-Valencia, Spain, and the proposer.

- **5070:** *Proposed by Isabel Iriberri Díaz and José Luis Díaz- Barrero, Barcelona, Spain.*

Find all real solutions to the system

$$\left.\begin{aligned}
9(x_1^2 + x_2^2 - x_3^2) &= 6x_3 - 1, \\
9(x_2^2 + x_3^2 - x_4^2) &= 6x_4 - 1, \\
&\cdots\cdots\cdots \\
9(x_n^2 + x_1^2 - x_2^2) &= 6x_2 - 1.
\end{aligned}\right\}$$

Solution by Antonio Ledesma Vila, Requena -Valencia, Spain

Add all

$$\begin{aligned}
9(x_1^2+x_2^2-x_3^2) &= 6x_3-1\\
9(x_2^2+x_3^2-x_4^2) &= 6x_4-1\\
&\cdots\\
9(x_n^2+x_1^2-x_2^2) &= 6x_2-1
\end{aligned}$$

$$9\left(\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_i^2\right)=6\sum_{i=1}^{n}x_i-n$$

$$9\sum_{i=1}^{n}x_i^2=6\sum_{i=1}^{n}x_i-n$$

$$\sum_{i=1}^{n}(3x_i)^2=2\sum_{i=1}^{n}(3x_i)-n$$

$$\sum_{i=1}^{n}(3x_i)^2-2\sum_{i=1}^{n}(3x_i)+n=0$$

$$\sum_{i=1}^{n}(3x_i-1)^2=0,$$

$$x_i=\frac{1}{3}\text{ for all }i$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong; China; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY; Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA, and the proposer.

- 5071: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let h_a, h_b, h_c be the altitudes of $\triangle ABC$ with semi-perimeter s, in-radius r and circum-radius R, respectively. Prove that

$$\frac{1}{4}\left(\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c}\right)\leq\frac{R^2}{r}\left(\sin^2 A+\sin^2 B+\sin^2 C\right).$$

Solution by Charles McCracken, Dayton, OH

Multiply both sides of the inequality by 4 to obtain

$$\begin{aligned}
\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c} &\leq \frac{(2R)^2}{r}\left[\sin^2 A+\sin^2 B+\sin^2 C\right]\\
\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c} &\leq \frac{1}{r}\left[(2R)^2\sin^2 A+(2R)^2\sin^2 B+(2R)^2\sin^2 C\right].
\end{aligned}$$

Now $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ so the inequality becomes

$$\frac{s(2s-a)}{h_a} + \frac{s(2s-b)}{h_b} + \frac{s(2s-c)}{h_c} \leq \frac{1}{r}\left(a^2+b^2+c^2\right).$$

From Johnson (Roger A. Johnson, Advanced Euclidean Geometry, Dover, 2007, p. 11) we have

$$h_a = \frac{2\Delta}{a},\ h_b = \frac{2\Delta}{b},\ h_c = \frac{2\Delta}{c},\ \text{where } \Delta \text{ represents the area of the triangle.}$$

The inequality now takes the form

$$\frac{as(2s-a)}{2\Delta} + \frac{bs(2s-b)}{2\Delta} + \frac{cs(2s-c)}{2\Delta} \leq \frac{1}{r}\left(a^2+b^2+c^2\right).$$

Since $\Delta = rs$, we now have our inequality in the form

$$\begin{aligned}
\frac{as(2s-a)}{2rs} + \frac{bs(2s-b)}{2rs} + \frac{cs(2s-c)}{2rs} &\leq \frac{1}{r}\left(a^2+b^2+c^2\right) \\
\frac{a(2s-a)}{2} + \frac{b(2s-b)}{2} + \frac{c(2s-c)}{2} &\leq \left(a^2+b^2+c^2\right)
\end{aligned}$$

Substituting $a+b+c$ for $2s$ we have

$$\begin{aligned}
a(b+c)+b(c+a)+c(a+b) &\leq 2a^2+2b^2+2c^2 \\
ab+ac+bc+ba+ca+cb &\leq 2a^2+2b^2+2c^2 \\
ab+bc+ca &\leq a^2+b^2+c^2
\end{aligned}$$

This last inequality, $ab+bc+ca \leq a^2+b^2+c^2$, can be readily proved true for any triple of positive numbers a, b, c by letting $b = a+\delta$ and $c = a+\epsilon$ with $0 < \delta < \epsilon$. Hence the original inequality holds.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5072: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b and c be the sides, l_a, l_b, l_c the bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the heights of $\triangle ABC$. Prove or disprove that

  a) $\dfrac{(-a+b+c)^3}{a} + \dfrac{(a-b+c)^3}{b} + \dfrac{(a+b-c)^3}{c} \geq \dfrac{4}{3}\left(m_a \cdot l_a + l_b \cdot h_b + h_c \cdot m_c\right)$

  b) $3\sum_{cyc} \dfrac{(-a+b+c)^3}{a} \geq 2\sum_{cyc}[m_a(l_a+h_a)]$.

Solution by proposer

We have

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\geq a^2+b^2+c^2. \qquad (1)$$

In fact, the equality is homogeneous and putting $a+b=c=1$ gives

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\geq a^2+b^2+c^2 \Leftrightarrow \sum_{cyc}\frac{(1-2a)^3}{a}\geq\sum_{cyc}a^2. \qquad (2)$$

Applying Chebyshev's Inequality gives

$$\sum_{cyc}\frac{(1-2a)^3}{a}=\sum_{cyc}\frac{1}{a}(1-2a)^3\geq\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)\cdot\left[\sum_{cyc}(1-2a)^3\right]. \qquad (3)$$

Using the well known equalities

$$\sum x^3=\left(\sum x\right)^3-3(x+y)(y+z)(z+x). \quad (4)$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 3^2=9 \qquad (5)$$

and applying (4), (3), and (5) we have

$$\begin{aligned}
&\sum_{cyc}\frac{(1-2a)^3}{a}\geq\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)\cdot\left[\sum_{cyc}(1-2a)^3\right]\\
=\ &\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)\cdot\left[(1-2a+1-2b+1-2c)^3-3(1-2a+1-2b)(1-2b+1-2c)(1-2c+1-2a)\right]\\
=\ &\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)\cdot[1-24abc]\\
=\ &\frac{1}{3}\left(\sum_{cyc}\frac{1}{a}\right)\cdot\left(\sum a\right)-\frac{24}{3}\left(\sum_{cyc}ab\right)\\
\geq\ &\frac{1}{3}\cdot 9-8\left(\sum_{cyc}ab\right)\\
\Leftrightarrow\ &\sum_{cyc}\frac{(1-2a)^3}{a}\geq 3-8\left(\sum_{cyc}ab\right). \qquad (6)
\end{aligned}$$

We have

$$3-8\left(\sum_{cyc}ab\right)\geq\sum_{cyc}a^2. \qquad (7)$$

In fact,

$$3-8\left(\sum_{cyc}ab\right)\geq\sum_{cyc}a^2\Leftrightarrow 3-6\left(\sum_{cyc}ab\right)\geq\sum_{cyc}a^2+2\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow \quad 3-6\Big(\sum_{cyc} ab\Big) \geq \Big(\sum_{cyc} a\Big)^2 = 1 \Leftrightarrow 3-6\Big(\sum_{cyc} ab\Big) \geq 1-3$$

$$\Leftrightarrow \quad \sum_{cyc} ab \leq \frac{1}{3} = \frac{\Big(\sum a\Big)^2}{3}$$

$$\Leftrightarrow \quad \sum_{cyc}(a-b)^2 \geq 0, \text{ and this last statement is true.}$$

Using (6) and (7) we have

$$\sum_{cyc} \frac{(1-2a)^3}{a} \geq 3-8\Big(\sum_{cyc} ab\Big) \geq \sum_{cyc} a^2$$
$$\Leftrightarrow \sum_{cyc} \frac{(1-2a)^3}{a} \geq \sum_{cyc} a^2, \text{ and (1) is true.}$$

Is well known that

$$a^2+b^2 = 2m_c^2+\frac{1}{2}c^2 \qquad \text{(A)}$$
$$c^2+b^2 = 2m_a^2+\frac{1}{2}a^2 \qquad \text{(B)}$$
$$c^2+a^2 = 2m_b^2+\frac{1}{2}b^2 \qquad \text{(C)}$$

For (A),(B), and (C)

$$m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2) \text{ and}$$
$$a^2+b^2+c^2 = \frac{4}{3}(m_a^2+m_b^2+m_c^2) \quad (8)$$

It is also well known that

$$m_a \geq l_a \geq h_a, \quad m_b \geq l_b \geq h_b, \quad m_c \geq l_c \geq h_c. \quad (9)$$

Using (9) we have

$$m_a^2 \geq m_a \cdot l_a \geq m_a \cdot h_a, \quad m_b^2 \geq m_b \cdot l_b \geq m_b \cdot h_b, \quad m_c^2 \geq m_c \cdot l_c \geq m_c \cdot h_c \quad \text{(D)}$$

$$m_a^2 \geq l_a \cdot h_a, \quad m_b^2 \geq l_b \cdot h_b, \quad m_c^2 \geq l_c \cdot h_c, \quad \text{(E)}$$

Using (8) and (D) we have

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a l_a + m_b l_b + m_c l_c). \quad (10)$$

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a h_a + m_b h_b + m_c h_c). \quad (11)$$

And using (8), (D), and (E) we have

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a l_a + l_b h_b + h_c m_c). \quad (12)$$

For part a of the problem, using (1) and (12) we have

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c} \geq \frac{4}{3}\Big(m_a\cdot l_a + l_b\cdot h_b + h_c\cdot m_c\Big)$$

For part b of the problem, using (1), (10) and (11) we have

$$2\left[\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\right] \geq$$
$$\frac{4}{3}\Big(m_a\cdot l_a + m_b\cdot l_b + m_c\cdot l_c\Big)+\frac{4}{3}\Big(m_a\cdot h_a + m_b\cdot h_b + m_c\cdot h_c\Big)$$

$$\Leftrightarrow \quad \frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}$$

$$\geq \quad \frac{2}{3}\Big(m_a\cdot l_a + m_b\cdot l_b + m_c\cdot l_c + m_a\cdot h_a + m_b\cdot h_b + m_c\cdot h_c\Big)$$

$$\Leftrightarrow \quad \frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}$$

$$\geq \quad \frac{2}{3}\Big[m_a\cdot(l_a+h_a)+m_b\cdot(l_b+h_b)+m_c\cdot(l_c+h_c)\Big]$$

- 5073: *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania.*

  Let $m > -1$ be a real number. Evaluate

$$\int_0^1 \{\ln x\} x^m dx,$$

where $\{a\} = a - [a]$ denotes the fractional part of a.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

$$I_m = \int_0^1 \{\ln x\} x^m dx = \int_0^1 (\ln x - [\ln x]) x^m dx = \int_0^1 (\ln x) x^m dx - \int_0^1 [\ln x] x^m dx = A - B$$

where $A = \int_0^1 (\ln x) x^m dx$ and $B = \int_0^1 [\ln x] x^m dx$. Integrating by parts $\Big(\int udv = uv - \int vdu$ with $u = \ln x$ and $dv = x^m dx\Big)$, and by using Barrow's and L'Hospital's rule we obtain,

$$\int (\ln x) x^m dx \quad = \quad \frac{(\ln x)x^{m+1}}{m+1} - \int \frac{x^m}{m+1}dx = \frac{(\ln x)x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2}$$

$$\begin{aligned}
\Longrightarrow A &= \left.\frac{(\ln x)x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2}\right|_0^1 \\
&= \frac{(\ln 1)1^{m+1}}{m+1} - \frac{1^{m+1}}{(m+1)^2} - \left(\lim_{x\to 0^+}\frac{(\ln x)x^{m+1}}{m+1} - \frac{0^{m+1}}{(m+1)^2}\right) \\
&= \frac{-1}{(m+1)^2} - \lim_{x\to 0^+}\frac{(\ln x)}{(m+1)x^{-(m+1)}} \\
&= \frac{-1}{(m+1)^2} - \lim_{x\to 0^+}\frac{x^{-1}}{-(m+1)^2x^{-(m+2)}} \\
&= \frac{-1}{(m+1)^2} + \lim_{x\to 0^+}\frac{x^{m+1}}{(m+1)^2} \\
&= \frac{-1}{(m+1)^2}
\end{aligned}$$

With the partition $\left\{\ldots, e^{-n},\ e^{-n+1},\ e^{-n+2}, \ldots,\ e^{-2},\ e^{-1},\ e^0 = 1\right\}$ of $(0,1]$, being $[\ln x] = -n$ for $e^{-n} \le x < e^{-n+1}$, and $\left|e^{-m-1}\right| < 1$,

$$\begin{aligned}
B &= \int_0^1 [\ln x]x^m dx = \sum_{n=1}^{\infty}\int_{e^{-n}}^{e^{-n+1}} [\ln x]x^m dx \\
&= \sum_{n=1}^{\infty}\int_{e^{-n}}^{e^{-n+1}} (-n)x^m dx = \sum_{n=1}^{\infty}\left.\frac{-nx^{m+1}}{m+1}\right|_{e^{-n}}^{e^{-n+1}} \\
&= \sum_{n=1}^{\infty}\frac{-n\left(e^{(-n+1)(m+1)} - e^{-n(m+1)}\right)}{m+1} \\
&= \sum_{n=1}^{\infty}\frac{-n\left(e^{m+1}e^{(-n)(m+1)} - e^{-n(m+1)}\right)}{m+1} \\
&= \sum_{n=1}^{\infty}\frac{-n\left(e^{(m+1)} - 1\right)e^{-n(m+1)}}{m+1} \\
&= \sum_{n=1}^{\infty}\frac{\left(1 - e^{(m+1)}\right)n\left(e^{-m-1}\right)^n}{m+1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(1-e^{m+1}\right)e^{-m-1}}{m+1}\sum_{n=1}^{\infty}\left(e^{-m-1}\right)^{n-1} \\
&= \frac{e^{-m-1}-1}{m+1}\sum_{n=1}^{\infty}\frac{d}{dx}x^{n}\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{m+1}\frac{d}{dx}\sum_{n=1}^{\infty}x^{n}\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{m+1}\frac{d}{dx}\frac{x}{1-x}\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{(m+1)(1-x)^2}\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{(m+1)(e^{-m-1}-1)^2} = \frac{1}{(m+1)(e^{-m-1}-1)}, \text{ so} \\
I_m &= A-B = -\frac{1}{(m+1)^2} - \frac{1}{(m+1)(e^{-m-1}-1)} \\
&= \frac{me^{m+1}+1}{(m+1)^2(e^{m+1}-1)}.
\end{aligned}
$$

Solution 2 by the proposer

The integral equals

$$\frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2}.$$

We have, by making the substitution $\ln x = y$, that

$$
\begin{aligned}
\int_0^1 \{\ln x\}\, x^m dx &= \int_{-\infty}^{0} \{y\}\, e^{(m+1)y} dy \\
&= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} \{y\}\, e^{(m+1)y} dy \\
&= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} \left(y-(-k-1)\right) e^{(m+1)y} dy \\
&= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} \left(y+k+1\right) e^{(m+1)y} dy
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\left(\frac{y+k+1}{m+1}e^{(m+1)y}\Big|_{-k-1}^{-k} - \frac{e^{(m+1)y}}{(m+1)^2}\Big|_{-k-1}^{-k}\right)\\
&= \sum_{k=0}^{\infty}\frac{e^{-(m+1)k}}{m+1} - \frac{1}{(m+1)^2}\sum_{k=0}^{\infty}\left(e^{-(m+1)k} - e^{-(m+1)(k+1)}\right)\\
&= \frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2},
\end{aligned}
$$

and the problem is solved.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; and David Stone and John Hawkins (jointly), Statesboro, GA.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before March 15, 2010

- 5092: *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with altitude h and with cevian $\overline{CD}$. A circle with radius x is inscribed in $\triangle ACD$, and a circle with radius y is inscribed in $\triangle BCD$ with $x < y$. Find the length of the cevian $\overline{CD}$ if x, y and h are positive integers with $(x, y, h) = 1$.

- 5093: *Proposed by Worapol Ratanapan (student), Montfort College, Chiang Mai, Thailand*

  $6 = 1 + 2 + 3$ is one way to partition 6, and the product of $1, 2, 3$ is 6. In this case, we call each of $1, 2, 3$ a **part** of 6.

  We denote the maximum of the product of all **parts** of natural number n as $N(n)$.

  As a result, $N(6) = 3 \times 3 = 9,\ N(10) = 2 \times 2 \times 3 \times 3 = 36$, and $N(15) = 3^5 = 243$.

  More generally, $\forall n \in N,\ N(3n) = 3^n,\ N(3n + 1) = 4 \times 3^{n-1}$, and $N(3n + 2) = 2 \times 3^n$.

  Now let's define $R(r)$ in the same way as $N(n)$, but each **part** of r is positive real. For instance $R(5) = 6.25$ and occurs when we write $5 = 2.5 + 2.5$

  Evaluate the following:

  $$\begin{array}{ll} i) & R(2e) \\ ii) & R(5\pi) \end{array}$$

- 5094: *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let a, b, c be real positive numbers such that $a + b + c + 2 = abc$. Prove that

  $$2(a^2 + b^2 + c^2) + 2(a + b + c) \geq (a + b + c)^2.$$

- 5095: *Proposed by Zdravko F. Starc, Vršac, Serbia*

  Let F_n be the Fibonacci numbers defined by

  $$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n = 1, 2, \cdots).$$

Prove that

$$\sqrt{F_{n-2}F_{n-1}}+1 \le F_n \le \sqrt{(n-2)F_{n-2}F_{n-1}}+1 \ (n=3,4,\cdots).$$

- 5096: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a,b,c be positive real numbers. Prove that

$$\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \ge \frac{3}{2}.$$

- 5097: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $p \ge 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\lfloor \sqrt[p]{n}\rfloor},$$

where $\lfloor a\rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4\rfloor = 2$).

Solutions

- 5074: *Proposed by Kenneth Korbin, New York, NY*

  Solve in the reals:

$$\sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}}=\frac{3}{x\sqrt{x}}.$$

Solution by Antonio Ledesma Vila, Requena-Valencia, Spain

Note that the domain of definition is $x \ge 1$, and that the two radicands are perfect squares:

$$\begin{aligned} 25+9x+30\sqrt{x} &= \left(3\sqrt{x}+5\right)^2 \\ 16+9x+30\sqrt{x-1} &= \left(3\sqrt{x-1}+5\right)^2 \end{aligned}$$

So

$$\begin{aligned} \sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}} &= \frac{3}{x\sqrt{x}} \\ \sqrt{\left(3\sqrt{x}+5\right)^2}-\sqrt{\left(3\sqrt{x-1}+5\right)^2} &= \frac{3}{x\sqrt{x}} \\ \left|3\sqrt{x}+5\right|-\left|3\sqrt{x-1}+5\right| &= \frac{3}{x\sqrt{x}} \\ (3\sqrt{x}+5)-(3\sqrt{x-1}+5) &= \frac{3}{x\sqrt{x}} \end{aligned}$$

$$
\begin{aligned}
\sqrt{x}-\sqrt{x-1} &= \frac{1}{x\sqrt{x}}\\
\frac{1}{\sqrt{x}-\sqrt{x-1}} &= x\sqrt{x}\\
\sqrt{x}+\sqrt{x-1} &= x\sqrt{x}\\
\sqrt{x-1} &= (x-1)\sqrt{x}\\
(x-1) &= (x-1)^2x\\
(x-1)\Big(1-(x-1)x\Big) &= 0
\end{aligned}
$$

Therefore, $x=1$ or $x^2-x-1=0 \Longrightarrow x=\dfrac{1\pm\sqrt{5}}{2}$. But $\dfrac{1-\sqrt{5}}{2}$ is an extraneous root.

Hence, the only two real solutions are $x=1$ and $x=\dfrac{1+\sqrt{5}}{2}=\phi$, the golden ratio.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; José Luis Díaz-Barrero, Barcelona, Spain; Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Katherine Janell Eyre (student, Angelo State University), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy; David C. Wilson, Winston-Salem, NC, and the proposer.

5075: *Proposed by Kenneth Korbin, New York, NY*

An isosceles trapezoid is such that the length of its diagonal is equal to the sum of the lengths of the bases. The length of each side of this trapezoid is of the form $a+b\sqrt{3}$ where a and b are positive integers.

Find the dimensions of this trapezoid if its perimeter is $31+16\sqrt{3}$.

Solution by Michael N. Fried, Kibbutz Revivim, Israel

Let the equal sides be $s=a+b\sqrt{3}$ and the bases be $b_1=p+q\sqrt{3}$ and $b_2=u+v\sqrt{3}$. Since each of its diagonals d is the sum of the bases, we have:

$$d=b_1+b_2=(p+u)+(q+v)\sqrt{3}=y+x\sqrt{3},$$

where a,b,p,q,u,v, and accordingly, y and x are all positive integers.

We begin by making some observations.

I. Since the diagonal $d = b_1 + b_2$, we have $P = 2s + d = 31 + 16\sqrt{3} \quad (1)$

II. From (1), we have,

$$\begin{aligned} s = a + b\sqrt{3} &= \left(\frac{31-y}{2}\right) + \left(\frac{16-x}{2}\right)\sqrt{3} \text{ or} \\ a &= \frac{31-y}{2} \quad (2) \\ b &= \frac{16-x}{2} \quad (3) \end{aligned}$$

And since a and b are positive integers, (2) and (3) imply that *y is odd and x even.*

III. Since any isosceles trapezoid can be inscribed in a circle, we can apply Ptolemy's theorem here to obtain the equation: $d^2 - s^2 = b_1 b_2$ (4). This, together with the fact that $d = b_1 + b_2$, implies that the bases b_1 and b_2 are the solutions of the equation $\mathrm{b}^2 - db + (d^2 - s^2) = 0$. Thus:

$$\begin{aligned} b_1 &= \frac{1}{2}\left(d + \sqrt{4s^2 - 3d^2}\right) \quad (5) \\ b_2 &= \frac{1}{2}\left(d - \sqrt{4s^2 - 3d^2}\right) \quad (6) \end{aligned}$$

IV. Since $b_1 = p + q\sqrt{3}$ and $b_2 = u + v\sqrt{3}$ where $p, q, u,$ and v are integers, it follows from (5) and (6) that

$$4s^2 - 3d^2 = \left(K + L\sqrt{3}\right)^2 = K^2 + 3L^2 + 2KL\sqrt{3} \quad (7)$$

where K and L are integers.

Now, let us find bounds for d and, from those, bounds for y and x. But to start, let us find bounds for $\frac{s}{d}$.

From equation (4), we have:

$$\begin{aligned} \frac{s^2}{d^2} = 1 - \frac{b_1 b_2}{d^2} &= 1 - \frac{b_1 b_2}{(b_1+b_2)^2} \\ &= 1 - \frac{1}{4}\left(\frac{(b_1+b_2)^2 - (b_1-b_2)^2}{(b_1+b_2)^2}\right) \\ &= \frac{3}{4} + \frac{1}{4}\left(\frac{b_1-b_2}{b_1+b_2}\right)^2 \end{aligned}$$

Thus,

$$\frac{3}{4} < \frac{s^2}{d^2} < 1$$

or

$$\frac{\sqrt{3}}{2} < \frac{s}{d} < 1.$$

From this, we can write,

$$1 + \sqrt{3} < \frac{2s+d}{d} < 3.$$

By (1), we can substitue $31+16\sqrt{3}$ for $2s+d$, thus eliminating s. With that, we obtain:

$$\frac{31+16\sqrt{3}}{3} < d < \frac{31+16\sqrt{3}}{1+\sqrt{3}} \quad (8)$$

Replacing d by $y+x\sqrt{3}$, we can rewrite (8) as bounds for y in terms of x:

$$\frac{31+(16-3x)\sqrt{3}}{3} < y < \frac{(31-3x)+(16-x)\sqrt{3}}{1+\sqrt{3}} \quad (9)$$

Since y must be a positive integer, x cannot exceed 11, otherwise y will be either negative or less than 1. Also, recalling observation II, x must be even and y must be odd. Replacing x successively by $2, 4, 6, 8$, and 10, then, we find by (9) that the corresponding values of y will be $17, 13, 11, 7$, and 3. From these values, in turn, we can then find a and b by equations (2) and (3). The five possibilities we have to check are summarized in the following table.

$$\left\{\begin{array}{cc} d = y + x\sqrt{3} & s = a + b\sqrt{3} \\ x=2 \;\; y=17 & a=7 \;\; b=7 \\ x=4 \;\; y=13 & a=9 \;\; b=6 \\ x=6 \;\; y=11 & a=10 \;\; b=5 \\ x=8 \;\; y=7 & a=12 \;\; b=4 \\ x=10 \;\; y=3 & a=14 \;\; b=3 \end{array}\right\}$$

Now, in observation IV, we found $4s^2-3d^2=\left(K+L\sqrt{3}\right)^2=K^2+3L^2+2KL\sqrt{3}$ which of course must be a positive number. This immediately eliminates the first and last possibilities, $d=17+2\sqrt{3}, s=7+7\sqrt{2}$, and $d=3+10\sqrt{3}, s=14+3\sqrt{2}$ since the rational part of $4s^2-3d^2$ (that is, the part not multiplying $\sqrt{3}$) is negative for these pairs.

This leaves only the second, third, and fourth possibilities. The rational parts of $4s^2-3d^2$ for these are, respectively, $105, 13$, and 45. It is then easy to check that only $13=1^2+3\times 2^2$ corresponding to $d=11+6\sqrt{3}, s=10+5\sqrt{3}$ can be written in the form K^2+3L^2, and the irrational part is also $4=2KL$.

Hence, these together with equations (5) and (6), give us our solution:

$$\begin{array}{rcl} s & = & 10+5\sqrt{3} \\ b_1 & = & 6+4\sqrt{3} \\ b_2 & = & 5+2\sqrt{3} \end{array}$$

Also solved by Mayer Goldberg, Beer-Sheva, Israel; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5076: *Proposed by M.N. Deshpande, Nagpur, India*

Let a, b, and m be positive integers and let F_n satisfy the recursive relationship

$$F_{n+2}=mF_{n+1}+F_n, \text{ with } F_0=a, F_1=b, n\geq 0.$$

Furthermore, let $a_n = F_n^2 + F_{n+1}^2, n \geq 0$. Show that for every a, b, m, and n,

$$a_{n+2} = (m^2+2)a_{n+1} - a_n.$$

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX

From the given,

$$\begin{aligned}
a_{n+2} &= F_{n+2}^2 + F_{n+3}^2 \\
&= F_{n+2}^2 + (mF_{n+2} + F_{n+1})^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}F_{n+2} + mF_{n+1}F_{n+2} + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}F_{n+2} + mF_{n+1}(F_n + mF_{n+1}) + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}(F_{n+2} + F_n) + m^2F_{n+1}^2 + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + (F_{n+2} - F_n)(F_{n+2} + F_n) + m^2F_{n+1}^2 + F_{n+1}^2 \\
&= F_{n+2}^2(m^2+2) + F_{n+1}^2(m^2+1) - F_n^2 \\
&= (F_{n+2}^2 + F_{n+1}^2)(m^2+2) - (F_n^2 + F_{n+1}^2) \\
&= (m^2+2)a_{n+1} - a_n.
\end{aligned}$$

Solution 2 by G. C. Greubel, Newport News, VA

Changing the terms slightly we shall use the more familiar Fibonacci polynomial terminology. The fibonacci polynomials are given by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

The Binet form of the Fibonacci polynomials is given by

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\begin{aligned}
\alpha &= \alpha(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right) \\
\beta &= \beta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right).
\end{aligned}$$

Also, the Lucas polynomials are given by

$$L_n(x) = \alpha^n + \beta^n$$

and satisfies the recurrence relation

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x).$$

The term to be considered is

$$a_n = F_{n+1}^2(x) + F_n^2(x).$$

It can be seen that

$$F_n^2(x) = \frac{1}{x^2+4}\left(L_{2n}(x) - 2(-1)^n\right).$$

This leads to the relation

$$a_n = \frac{1}{x^2+4}\left(L_{2n+1}(x) + L_{2n}(x)\right).$$

The relation being asked to show is given by

$$a_{n+2} = \left(x^2+2\right)a_{n+1} - a_n.$$

Let $\phi_n = (x^2+2)\,a_{n+1} - a_n$ for the purpose of demonstration. With the use of the above equations we can see the following:

$$\begin{aligned}
\left(x^2+4\right)\phi_n &= \left(x^2+4\right)\left[\left(x^2+2\right)a_{n+1} - a_n\right] \\
&= \left(x^2+2\right)\left(L_{2n+3} + L_{2n+2}\right) - \left(L_{2n+1} + L_{2n}\right) \\
&= \left(x^2+2\right)\left(\left(x^2+x+1\right)L_{2n+1} + (x+1)L_{2n}\right) - \left(L_{2n+1} + L_{2n}\right) \\
&= \left(x^4+x^3+3x^2+2x+1\right)L_{2n+1} + \left(x^3+x^2+2x+1\right)L_{2n} \\
&= \left(x^3+x^2+2x+1\right)L_{2n+2} + \left(x^2+x+1\right)L_{2n+1} \\
&= \left(x^2+x+1\right)L_{2n+3} + (x+1)L_{2n+2} \\
&= xL_{2n+4} + L_{2n+4} + L_{2n+3} \\
&= L_{2n+5} + L_{2n+4}. \quad \mathbf{(1)}
\end{aligned}$$

From the equation $a_n = \dfrac{1}{x^2+4}\left(L_{2n+1}(x) + L_{2n}(x)\right)$ we have

$$\left(x^2+4\right)a_{n+2} = L_{2n+5} + L_{2n+4}. \quad \mathbf{(2)}$$

Comparing the result of (1) to that of (2) leads to $\phi_n = a_{n+2}$. Thus we have the relation

$$a_{n+2} = \left(x^2+2\right)a_{n+1} - a_n$$

and this provides the relation being sought.

Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University) Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

5077: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

Find all triplets (x, y, z) of real numbers such that

$$\left.\begin{array}{r} xy(x+y-z)=3, \\ yz(y+z-x)=1, \\ zx(z+x-y)=1. \end{array}\right\}$$

Solution by Ercole Suppa, Teramo, Italy

From the second and third equation it follows that

$$yz(y+z) = zx(z+x) \qquad \Longleftrightarrow \qquad (x-y)(x+y+z) = 0.$$

If $x+y+z=0$ the first two equations yield $-2xyz=3$ and $-xyz=1$ which is impossible.

If $x=y$ then the system can rewritten as

$$\begin{aligned} x^2(2x-z) &= 3 \\ z^2y &= 1 \\ z^2x &= 1 \end{aligned}$$

Thus $x = \dfrac{1}{z^2}$ and

$$\begin{aligned} \frac{1}{z^4}\left(\frac{2}{z^2}-z\right) &= 3 \\ 3z^6+z^3-2 &= 0 \\ (3z^3-2)(z^3+1) &= 0 \end{aligned}$$

The equation $(3z^3-2)(z^3+1)=0$ factors into

$$\left(3^{1/3}z-2^{1/3}\right)\left(3^{2/3}z^2+(3^{1/3}\cdot 2^{1/3})z+2^{2/3}\right)(z+1)(z^2-z+1)=0.$$

Setting each factor equal to zero we see that only the first and third factors give real roots for the unknown z. So, the real roots are $z=\sqrt[3]{\dfrac{2}{3}}$ and $z=-1$. And since $x=y=\dfrac{1}{z^2}$ we see that $(1,1,-1)$ and $\left(\sqrt[3]{\dfrac{9}{4}}, \sqrt[3]{\dfrac{9}{4}}, \sqrt[3]{\dfrac{2}{3}}\right)$ are the only real triplets (x,y,z) that satisfy the given system.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M. N. Deshpande, Nagpur, India; Bruno Salgueiro Fanego, Viveiro, Spain; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY;

Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Antonio Ledesma Vila, Requena-Valencia, Spain, and the proposers.

5078: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)}\geq \frac{3}{2}\frac{1}{\sqrt{ab+ac+cb}}.$$

Solution by Kee-Wai Lau, Hong Kong, China

For $x>0$, let $f(x)$ be the convex function x^{-1} so that we have

$$\begin{aligned}
&\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)}\\
=\ & af\Big(\sqrt{b}(b+c)\Big)+bf\Big(\sqrt{c}(a+c)\Big)+cf\Big(\sqrt{a}(a+b)\Big)\\
\geq\ & f\Big(a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)\Big)\\
=\ & \frac{1}{a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)}. \qquad (1)
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)\\
=\ & \Big(\sqrt{ab(b+c)}\Big)\Big(\sqrt{a(b+c)}\Big)+\Big(\sqrt{bc(a+c)}\Big)\Big(\sqrt{b(a+c)}\Big)+\Big(\sqrt{ca(a+b)}\Big)\Big(\sqrt{c(a+b)}\Big)\\
\leq\ & \Big(\sqrt{ab(b+c)+bc(a+c)+ca(a+b)}\Big)\Big(\sqrt{a(b+c)+b(a+c)+c(a+b)}\Big)\\
=\ & \Big(\sqrt{ab^2+bc^2+ca^2+3abc}\Big)\Big(\sqrt{2(ab+bc+ca)}\Big). \qquad (2)
\end{aligned}$$

By (1) and (2), it suffices for us to show that $ab^2+bc^2+ca^2+3abc\leq \dfrac{2}{9}$. In fact,

$$ab^2+bc^2+ca^2+3abc$$

$$
\begin{aligned}
&= \left(a+b+c-\frac{2}{3}\right)(ab+bc+ca)+\frac{a+b+c}{9}-b\left(a-\frac{1}{3}\right)^2-c\left(b-\frac{1}{3}\right)^2-a\left(c-\frac{1}{3}\right)^2 \\
&\le \frac{ab+bc+ca}{3}+\frac{1}{9} \\
&= \frac{(a+b+c)^2}{9}-\frac{(a-b)^2+(b-c)^2+(c-a)^2}{18}+\frac{1}{9} \\
&\le \frac{2}{9}.
\end{aligned}
$$

This completes the solution.

Also solved by Boris Rays, Brooklyn, NY, and the proposer.

5079: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $x \in (0,1)$ be a real number. Study the convergence of the series

$$\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For positive integers n and $x \in (0,1)$, let $a_n = a_n(x) = x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}$.

Since $\sin\dfrac{1}{n+1} = \dfrac{1}{n} + O\left(\dfrac{1}{n^2}\right)$ as n tends to infinity, so

$$
\begin{aligned}
\left|\frac{a_n}{a_{n+1}}\right| &= \exp\left(\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)\right) \\
&= 1+\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)+\sum_{m=2}^{\infty}\frac{\left(\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)\right)^m}{m!} \\
&= 1+\frac{1}{n}\ln\left(\frac{1}{x}\right)+O\left(\frac{1}{n^2}\right),
\end{aligned}
$$

where the constant implied by the last O depends at most on x. Hence, by Gauss' test, the series of the problem is convergent if $0 < x < \dfrac{1}{e}$ and is divergent if $\dfrac{1}{e} \le x < 1$.

Solution 2 by David Stone and John Hawkins (jointly), Statesboro, GA

Our answer: we have convergence if $0 < x < \dfrac{1}{e}$ and divergence if $\dfrac{1}{e} \le x < 1$.

We start by looking at the sum $\sum_{i=1}^{n} \sin\dfrac{1}{k}$. Each term of the sum, $\sin\dfrac{1}{k}$, can be expanded in an alternating series $\sin\dfrac{1}{k} = \dfrac{1}{k} - \dfrac{1}{3!}\left(\dfrac{1}{k}\right)^3 + \cdots$. The error from terminating the series after the first term does not exceed the second term. Thus we have

$$\begin{aligned}
\left|\sin\frac{1}{k} - \frac{1}{k}\right| &< \frac{1}{3!}\left(\frac{1}{k}\right)^3, \text{ so} \\
-\frac{1}{6k^3} &< \sin\frac{1}{k} - \frac{1}{k} < \frac{1}{6k^3} \\
\frac{1}{k} - \frac{1}{6k^3} &< \sin\frac{1}{k} < \frac{1}{k} + \frac{1}{6k^3}. \text{ Therefore,} \\
\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} &< \sum_{k=1}^{n}\sin\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3}.
\end{aligned}$$

The series $\sum_{k=1}^{\infty}\dfrac{1}{k^3}$ is known to be convergent, say to L, which is grater than any of its partial sums.

Moreover, by looking at the graph of $y = 1/x$ we see that

$$\begin{aligned}
\frac{1}{k} &< \int_{k-1}^{k}\frac{1}{u}du = \ln k - \ln(k-1), \text{ and} \\
\frac{1}{k} &> \int_{k}^{k+1}\frac{1}{u}du = \ln(k+1) - \ln(k).
\end{aligned}$$

Using these for our bound on the partial sum of $\sin\dfrac{1}{k}$, we obtain

$$\sum_{k=1}^{n}\Big(\ln(k+1) - \ln k\Big) - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\sin\frac{1}{k}, \text{ so}$$

$$\ln(n+1) - \frac{1}{6}L < \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\sin\frac{1}{k}.$$

On the other hand,

$$\sum_{k=1}^{n}\sin\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < 1 + \ln n + \frac{1}{6}L.$$

Thus we have bounds on the sine sum:

$$\ln(n+1) - \frac{1}{6}L < \sum_{i=1}^{n}\sin\frac{1}{k} < 1 + \ln n + \frac{1}{6}L.$$

We use this to investigate the convergence so the series $\displaystyle\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}$.

Since $0 < x < 1$, we know that x^u is a decreasing function of u. Thus

$$x^{-\frac{1}{6}L+\ln(n+1)} > x^{\sum\limits_{k=1}^{n}\sin\frac{1}{k}} > x^{\frac{1}{6}L+\ln n}$$

and we have

$$x^{\frac{1}{6}L+1}\sum_{n=1}^{t} x^{\ln n} < \sum_{n=1}^{t} x^{\sum\limits_{k=1}^{n}\sin\frac{1}{k}} < x^{-\frac{1}{6}L}\sum_{n=1}^{t} x^{\ln(n+1)}.$$

Noting that

$$x^{\ln n} = e^{\ln(x^{\ln n})} = e^{(\ln n)(\ln x)} = e^{\ln n^{\ln x}} = n^{\ln x}$$

we can rewrite the outside sums to obtain

$$x^{\frac{1}{6}L+1}\sum_{n=1}^{t} n^{\ln x} < \sum_{n=1}^{t} x^{\sum\limits_{k=1}^{n}\sin\frac{1}{k}} < x^{-\frac{1}{6}L}\sum_{n=1}^{t}(n+1)^{\ln x}.$$

It is well known that the series $\displaystyle\sum_{n=1}^{\infty} n^{\alpha}$ diverges if $\alpha \geq -1$. Hence, if $\ln x \geq -1$, the series $\displaystyle\sum_{n=1}^{\infty} x^{\sum\limits_{k=1}^{n}\sin\frac{1}{k}}$ dominates the divergent series $\displaystyle\sum_{n=1}^{\infty} x^{\ln x}$ and thus diverges. That is, we have divergence if $1 > x \geq \dfrac{1}{e}$.

Likewise, it is well known that $\displaystyle\sum_{n=1}^{\infty}(n+1)^{\alpha}$ converges if $\alpha < -1$. So if $\ln x < -1$, the series $\displaystyle\sum_{n=1}^{\infty} x^{\sum\limits_{k=1}^{n}\sin\frac{1}{k}}$ is dominated by the convergent series $\displaystyle\sum_{n=1}^{\infty}(n+1)^{\ln x}$ and thus converges.

That is, we have convergence if $0 < x < \dfrac{1}{e}$.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before April 15, 2010

- 5098: *Proposed by Kenneth Korbin, New York, NY*

  Given integer-sided triangle ABC with $\angle B = 60^\circ$ and with $a < b < c$. The perimeter of the triangle is $3N^2 + 9N + 6$, where N is a positive integer. Find the sides of a triangle satisfying the above conditions.

- 5099: *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $d - 1$ bisects two of its sides.

- 5100: *Proposed by Mihály Bencze, Brasov, Romania*

  Prove that
  $$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \le \sqrt{\frac{n(2^{n+1}-n)2^{n-1}}{n+1}}$$

- 5101: *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India*

  An unbiased coin is tossed repeatedly until r heads are obtained. The outcomes of the tosses are written sequentially. Let R denote the total number of runs (of heads and tails) in the above experiment. Find the distribution of R.

  Illustration: if we decide to toss a coin until we get 4 heads, then one of the possibilities could be the sequence $T\ T\ H\ H\ T\ H\ T\ H$ resulting in 6 runs.

- 5102: *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a positive integer and let $a_1, a_2, \cdots, a_n$ be any real numbers. Prove that
  $$\frac{1}{1+a_1^2+\ldots+a_n^2} + \frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n} \frac{a_kF_k}{1+a_1^2+\ldots+a_k^2}\right)^2 \le 1,$$
  where F_k represents the k^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \ge 3, F_n = F_{n-1} + F_{n-2}$.

- 5103: *Proposed by Roger Izard, Dallas, TX*

  A number of circles of equal radius surround and are tangent to another circle. Each of the outer circles is tangent to two of the other outer circles. No two outer circles intersect in two points. The radius of the inner circle is a and the radius of each outer circle is b. If

$$a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0,$$

determine the number of outer circles.

Solutions

- 5080: *Proposed by Kenneth Korbin, New York, NY*

If p is a prime number congruent to 1 (mod 4), then there are positive integers a, b, c, such that

$$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) + \arcsin\left(\frac{c}{p^3}\right) = 90^\circ.$$

Find a, b, and c if $p = 37$ and if $p = 41$, with $a < b < c$.

Solution 1 by Paul M. Harms, North Newton, KS

The equation in the problem is equivalent to

$$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) = 90^\circ - \arcsin\left(\frac{c}{p^3}\right).$$

Taking the cosine of both sides yields

$$\begin{aligned} \frac{(p^6 - a^2)^{1/2}(p^6 - b^2)^{1/2}}{p^6} - \frac{ab}{p^6} &= \frac{c}{p^3}. \\ (p^6 - a^2)^{1/2}(p^6 - b^2)^{1/2} - ab &= cp^3. \end{aligned}$$

Since p^3 is a factor on the right side I made some assumptions on a and b so that the left side also had p^3 as a factor.
Assume $a = p^2a_1$ and $b = pb_1$ where all numbers are positive integers. Then we have

$$c = (p^2 - a_1)^{1/2}(p^4 - b_1^2)^{1/2} - a_1b_1.$$

I then looked for perfect squares for $(p^2 - a_1^2)$ and $(p^4 - b_1^2)$.

When $p = 37, (37^2 - a_1^2) = (37 - a_1)(37 + a_1)$ and $a_1 = 12$ yields a product of the squares 25 and 49.

When $p = 37, (37^4 - b_1^2) = (37^2 - b_1)(37^2 + b_1)$.

I checked for a number b_1 where both $(37^2 - b_1)$ and$(37^2 + b_1)$ were perfect squares. The numbers b_1 which make $(37^2 - b_1)$ a square are

$$0,\ 37 + 36 = 73,\ 73 + (36 + 35) = 144,\ 144 + (35 + 34) = 213,\ \cdots.$$

When $b_1 = 840$, both factors involving b_1 are perfect squares.

When $p = 37$ a result is $a = (12)37^2 = 16428, b = 840(37) = 31080$ and $c = 27755$.

Since the problem conditions state that $a < b < c$, I will switch notation. One answer is

$$a = 16428, b = 27755, \text{and } = 31080$$

with approximate angles $18.925°,\ 33.226°$ and $37.849°$.

When $p = 41$, $(41 - a_1)(41 + a_1)$ is a perfect square when $a_1 = 9$ or 40. The product $(41^2 - b_1)(41^2 + b_1)$ is a perfect square when $b_1 = 720$. One answer is

$$a = 9(41^2) = 15129,\ b = 720(41) = 29520 \text{ and } c = 54280$$

with approximate angles $12.757°,\ 25,361°$, and $51.959°$.

When $a_1 = 40$ and $b_1 = 720$, c was less than zero so this did not satisfy the problem.

Solution 2 by Tom Leong, Scotrun, PA

Fermat's Two-Square Theorem implies that every prime congruent to 1 mod 4 can be represented as the sum of two distinct squares. We give a solution to the following modest generalization. Suppose the positive integer n is the sum of two distinct squares, say, $n = x^2 + y^2$ with $0 < x < y$. Then a solution to

$$\arcsin\frac{A}{n} + \arcsin\frac{B}{n^2} + \arcsin\frac{C}{n^3} = 90°$$

in positive integers A, B, C is

$$(A,B,C) = \begin{cases} (s, 2st, 2(xs+yt)(xt-ys)) & \text{if } 1 < \dfrac{y}{x} < \sqrt{3} \\ (t, t^2 - s^2, 2(xs+yt)(ys-xt)) & \text{if } \sqrt{3} < \dfrac{y}{x} < 1+\sqrt{2} \\ (s, s^2 - t^2, (xs+yt)^2 - (ys-xt)^2) & \text{if } 1+\sqrt{2} < \dfrac{y}{x} < 2+\sqrt{3} \\ (t, 2st, (ys-xt)^2 - (xs+yt)^2) & \text{if } \dfrac{y}{x} > 2+\sqrt{3} \end{cases}$$

where $s = y^2 - x^2$ and $t = 2xy$.
We can verify this as follows. Since $\arcsin(A/n) + \arcsin(B/n^2)$ and $\arcsin(C/n^3)$ are complementary,

$$\tan\left(\arcsin\frac{A}{n} + \arcsin\frac{B}{n^2}\right) = \cot\left(\arcsin\frac{C}{n^3}\right).$$

Using the angle sum formula for tangent and $\tan(\arcsin z) = z/\sqrt{1-z^2}$, this reduces to

$$\frac{A\sqrt{n^4 - B^2} + B\sqrt{n^2 - A^2}}{\sqrt{n^2 - A^2}\sqrt{n^4 - B^2} - AB} = \frac{\sqrt{n^6 - C^2}}{C}.$$

Now verifying the solutions is straightforward using the following identities

$$n = x^2 + y^2, \qquad n^2 = s^2 + t^2, \qquad n^3 = (xs+yt)^2 + (ys-xt)^2$$

and the following inequalities

$$\frac{y}{x} < \sqrt{3} \iff ys < xt, \qquad \frac{y}{x} < 1+\sqrt{2} \iff s < t, \qquad \frac{y}{x} < 2+\sqrt{3} \iff ys - xt < xs + yt.$$

As for the original problem, for $n = 37$, since $37 = 1^2 + 6^2$, we have
$x = 1, y = 6, s = 35, t = 12$ which gives

$$\arcsin\frac{12}{37} + \arcsin\frac{840}{37^2} + \arcsin\frac{27755}{37^3} = \arcsin\frac{16428}{37^3} + \arcsin\frac{31080}{37^3} + \arcsin\frac{27755}{37^3} = 90°.$$

For $n = 41$, since $41 = 4^2 + 5^2$, we have $x = 4, y = 5, s = 9, t = 40$ which gives

$$\arcsin\frac{9}{41} + \arcsin\frac{720}{41^2} + \arcsin\frac{54280}{41^3} = \arcsin\frac{15129}{41^3} + \arcsin\frac{29520}{41^3} + \arcsin\frac{54280}{41^3} = 90^\circ.$$

Comment by editor: David Stone and John Hawkins of Statesboro, GA developed equations:

$$\begin{aligned} b &= \sqrt{\frac{p^3(p^3 - c)}{2}} \\ a &= \frac{-bc + \sqrt{b^2c^2 + p^6\left(p^6 - b^2 - c^2\right)}}{p^3}. \end{aligned}$$

Using Matlab they found four solutions for $p = 37$,

$$\begin{array}{lll} a = 16428 & b = 27755 & c = 31080 \\ a = 3293 & b = 32157 & c = 36963 \\ a = 7363 & b = 27188 & c = 38332 \\ a = 352 & b = 25123 & c = 43808 \end{array}$$

and two solutions for $p = 41$,

$$\begin{array}{lll} a = 15129 & b = 29520 & c = 54280 \\ a = 5005 & b = 31529 & c = 58835. \end{array}$$

Also solved by Brian D. Beasley, Clinton, SC, and the proposer.

- **5081:** *Proposed by Kenneth Korbin, New York, NY*

Find the dimensions of equilateral triangle ABC if it has an interior point P such that $\overline{PA} = 5, \overline{PB} = 12$, and $\overline{PC} = 13$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let the length of the sides of the equilateral triangle be x. We show that $x = \sqrt{169 + 60\sqrt{3}}$.

Applying the cosine formula to triangles APB, BPC, and CPA respectively, we obtain

$$\cos\angle APB = \frac{169 - x^2}{120},\ \cos\angle BPC = \frac{313 - x^2}{312},\ \cos\angle CPA = \frac{194 - x^2}{130}.$$

Since

$$\begin{aligned} \angle APB + \angle BPC + \angle CPA &= 360^\circ \text{ so} \\ \cos\angle CPA &= \cos(\angle APB + \angle BPC) \text{ and} \\ \sin\angle APB \sin\angle BPC &= \cos\angle APB\cos\angle BPC - \cos\angle CPA. \end{aligned}$$

Hence,

$$\left(\frac{\sqrt{338x^2 - x^4 - 14161}}{120}\right)\left(\frac{\sqrt{626x^2 - x^4 - 625}}{312}\right) = \left(\frac{169 - x^2}{120}\right)\left(\frac{313 - x^2}{312}\right) - \frac{194 - x^2}{130} \text{ or}$$

$$\sqrt{338x^2 - x^4 - 14161}\sqrt{626x^2 - x^4 - 625} \;=\; (169 - x^2)(313 - x^2) - 288(194 - x^2).$$

Squaring both sides and simplifying, we obtain

$$\begin{aligned} 576x^6 - 194668x^4 + 10230336x^2 &= 0 \text{ or} \\ 576x^2(x^4 - 338x^2 + 17761) &= 0. \end{aligned}$$

It follows that $x = \sqrt{169 - 60\sqrt{3}},\ \sqrt{160 + 60\sqrt{3}}$. Since $\angle APB,\ \angle BPC, \angle CPA$ are not all acute, the value of $\sqrt{169 - 60\sqrt{3}}$ must be rejected.

This completes the solution.

Comments and Solutions 2 & 3 by Tom Leong, Scotrun, PA

Comments: This problem is not new and has appeared in, e.g., the 1998 Irish Mathematical Olympiad and T. Andreescu & R. Gelca, *Mathematical Olympiad Challenges*, Birkhäuser, 2000, p5. A nice elementary solution to this problem uses a rotation argument (Solution 2 below). A quick solution to a more general problem can be found using a somewhat obscure result of Euler on tripolar coordinates (Solution 3 below).

Solution 2

Rotate the figure about the point C by $60°$ so that B maps onto A. Let P' denote the image of P under this rotation. Note that triangle PCP' is equilateral since $PC = P'C$ and $\angle PCP' = 60°$. So $\angle P'PC = 60°$. Furthermore, since $PP' = 13$, triangle APP' is a 5-12-13 right triangle. Consequently,

$$\cos\angle APC = \cos(\angle APP' + 60°) = \frac{5}{13}\cdot\frac{1}{2} - \frac{12}{13}\cdot\frac{\sqrt{3}}{2} = \frac{5 - 12\sqrt{3}}{26}.$$

So by the Law of Cosines,

$$AC = \sqrt{5^2 + 13^2 - 2\cdot 5\cdot 13\cdot\frac{5 - 12\sqrt{3}}{26}} = \sqrt{169 + 60\sqrt{3}}$$

Solution 3

A generalization follows from a result of Euler on *tripolar coordinates* (see, e.g., van Lamoen, Floor and Weisstein, Eric W. “Tripolar Coordinates” From MathWorld–A Wolfram Web Resource.
`http://mathworld.wolfram.com/TripolarCoordinates.html`.) Suppose triangle ABC is equilateral with side length s, and P is a point in the plane of ABC. The triple $(x, y, z) = (PA, PB, PC)$ is the tripolar coordinates of P in reference to triangle ABC. A result of Euler implies these tripolar coordinates satisfy

$$s^4 - (x^2 + y^2 + z^2)s^2 + x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2 = 0$$

which gives the positive solutions

$$s = \sqrt{\frac{x^2 + y^2 + z^2 \pm \sqrt{(x^2 + y^2 + z^2)^2 - 2(x - y)^2 - 2(y - z)^2 - 2(z - x)^2}}{2}}.$$

The larger solution refers to the case where P is interior to the triangle, while the smaller solution refers to the case where P is exterior to the triangle. In the case where (x, y, z) is a Pythatgorean triple with $x^2 + y^2 = z^2$, this simplifies to the surprisingly terse

$$s = \sqrt{z^2 \pm xy\sqrt{3}}.$$

In the original problem, with $(x, y, z) = (5, 12, 13)$, we find

$$s = \sqrt{169 \pm 60\sqrt{3}}$$

with the larger solution $s = \sqrt{169 + 60\sqrt{3}}$ being the desired answer.

A conjecture by David Stone and John Hawkins, Statesboro, GA

If a, b, c form a *right* triangle with $a^2 + b^2 = c^2$, then

1. the side length of the unique equilateral triangle ABC having an *interior* point P such that $\overline{PA} = a, \overline{PB} = b$, and $\overline{PC} = c$ is $s\sqrt{c^2 + ab\sqrt{3}}$, and
2. the side length of the unique equilateral triangle with an *exterior* point P satisfying $\overline{PA} = a, \overline{PB} = b$, and $\overline{PC} = c$ is $s\sqrt{c^2 - ab\sqrt{3}}$.

Also solved by Scott H. Brown, Montgomery, AL; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Pat Costello, Richmon, KY; Paul M. Harms, North Newton, KS; Antonio Ledesma López, Requena-Valencia, Spain; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, Republic of Kosova; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 5082: *Proposed by David C. Wilson, Winston-Salem, NC*

  Generalize and prove:

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= 1 - \frac{1}{n+1} \\
\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} &= \frac{1}{4} - \frac{1}{2(n+1)(n+2)} \\
\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)} &= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)} \\
\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)(n+4)} &= \frac{1}{96} - \frac{1}{4(n+1)(n+2)(n+3)(n+4)}
\end{aligned}$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We will give two different proofs, each relies on the telescoping property.

First proof:

Our quantity may be written as $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)}$ where m is a positive integer.

Next we observe

$$\frac{1}{k(k+1)\cdots(k+m-1)}-\frac{1}{(k+1)\cdots(k+m)}=\frac{m}{k(k+1)\cdots(k+m)}$$

yielding, also by telescoping,

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)\cdots(k+m)} &= \frac{1}{m}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)\cdots(k+m-1)}-\frac{1}{(k+1)\cdots(k+m)}\right)\\ &= \frac{1}{m}\left(\frac{1}{m!}-\frac{1}{(n+1)\cdots(n+m)}\right)\end{aligned}$$

Second proof:

If $a_k=\dfrac{1}{k(k+1)\cdots(k+m)}$, then $\dfrac{a_{k+1}}{a_k}=\dfrac{k\cdot(k+1)\cdots(k+m)}{(k+1)(k+2)\cdots(k+m)}=\dfrac{k}{k+1+m}$ and then $ma_k=ka_k-(k+1)a_{k+1}$ and therefore

$$\begin{aligned}m\sum_{k=1}^{n}a_k=m\sum_{k=0}^{n-1}a_{k+1} &= m\sum_{k=0}^{n-1}(ka_k-(k+1)a_{k+1})\\ &= \frac{1}{m!}-\frac{1}{(n+1)(n+2)\cdots(n+m)}\end{aligned}$$

and the result is immediate.

Solution 2 by G. C. Greubel, Newport News, VA

It can be seen that all the series in question are of the form

$$S_n^m=\sum_{k=1}^{n}\frac{(k-1)!}{(k+m)!}.$$

Making a slight change we have

$$S_n^m=\frac{1}{m!}\sum_{k=1}^{n}\frac{(k-1)!m!}{(k+m)!}=\frac{1}{m!}\sum_{k=1}^{n}B(k,m+1),$$

where $B(x,y)$ is the Beta function. By using an integral form of the Beta function, namely,

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

the series becomes

$$\begin{aligned}S_n^m &= \frac{1}{m!}\sum_{k=1}^{n}\int_0^1 t^m(1-t)^{k-1}dt\\ &= \frac{1}{m!}\int_0^1 t^m(1-t)^{-1}\cdot\frac{(1-t)(1-(1-t)^n)}{t}dt\\ &= \frac{1}{m!}\int_0^1 t^{m-1}(1-(1-t)^n)dt\end{aligned}$$

$$= \frac{1}{m!}\left(\int_0^1 t^{m-1}dt - B(n+1,m)\right)$$

$$= \frac{1}{m!}\left(\frac{1}{m} - B(n+1,m)\right)$$

$$= \frac{1}{m}\left[\frac{1}{m!} - \frac{n!}{(n+m)!}\right].$$

The general result is given by

$$\sum_{k=1}^{n} \frac{(k-1)!}{(k+m)!} = \frac{1}{m}\left[\frac{1}{m!} - \frac{n!}{(n+m)!}\right].$$

As examples let $m = 1$ to obtain

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

and when $m = 2$ the series becomes

$$\sum_{k=1}^{n} \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

The other series follow with higher values of m.

Comments by Tom Leong, Scotrun, PA

This series is well-known and has appeared in the literature in several places. Some references include

1. Problem 241, *College Mathematics Journal* (Nov 1984, p448–450)
2. Problem 819, *College Mathematics Journal* (Jan 2007, p65–66)
3. K. Knopp, *Theory and Application of Infinite Series*, 2nd ed., Blackie & Son, 1951, p233
4. D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman and Company, 1962, p30

In the first reference above, four different perspectives on this series are given.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Antonio Ledesma López, Requena-Valencia, Spain; Tom Leong, Scotrun, PA; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; Armend Sh. Shabani, Republic of Kosova; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5083:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \Re$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n = \begin{cases} \sum\limits_{k=1}^{n} f\left(\dfrac{k}{n^2}\right), & n > \dfrac{1}{\alpha}; \\ 0, & n \leq \dfrac{1}{\alpha} \end{cases}$$

is convergent and determine its limit.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Clearly, for n large enough, we will have $n > \dfrac{1}{\alpha}$. Therefore, we only need to show that $\sum\limits_{k=1}^{n} f\left(\dfrac{k}{n^2}\right)$ converges and to find *its* limit as $n \to \infty$.

Since $f(0) = 0$ and $f'(x)$ exist in $[0, k/n^2] \subset [0, 1/n] \subset [-\alpha, \alpha]$, there is some $\xi_k \in [0, k/n^2]$ such that $f\left(\dfrac{k}{n^2}\right) = f'\left(\xi_k\right)\dfrac{k}{n^2}$ by the mean value theorem.

Let $f'(M_n) = \max_k f'(\xi_k)$ and $f'(m_n) = \min_k f'(\xi_k)$.

Then, since $\sum\limits_{k=1}^{n} f\left(\dfrac{k}{n^2}\right) = \sum\limits_{k=0}^{n} f'(\xi_k)\dfrac{k}{n^2}$, we have:

$$f'(m_n)\sum_{k=1}^{n}\frac{k}{n^2} \leq \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \leq f'(M_n)\sum_{k=1}^{n}\frac{k}{n^2}, \text{ or}$$

$$f'(m_n)\left(\frac{1}{2}+\frac{1}{2n}\right) \leq \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \leq f'(M_n)\left(\frac{1}{2}+\frac{1}{2n}\right).$$

But f' is bounded in $[-\alpha, \alpha]$ and, thus, in every subinterval of $[-\alpha, \alpha]$. Therefore, f' is continuous in every subinterval of $[-\alpha, \alpha]$. Hence,

$$\lim_{n\to\infty} f'(m_n) = \lim_{n\to\infty} f'(M_n) = f'(0), \text{ so that}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \frac{f'(0)}{2}$$

Heuristically, we can approach the problem in a slightly different way. Keeping in mind that $f(0) = 0$, write:

$$\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = n^2\sum_{k=0}^{n}\left(\frac{k}{n}\times\frac{1}{n}\right)\frac{1}{n^2} \approx n^2\int_0^{\frac{1}{n}} f\left(\xi\right)d\xi.$$

The approximation become exact as $n \to \infty$ (this is the heuristic part!)
Since f' is bounded in $(0, \alpha)$ (being bounded in $(-\alpha, \alpha)$), and since $f(0) = 0$ we can write, for some $s \in (0, 1/n)$:

$$n^2\int_0^{\frac{1}{n}} f(\xi)d\xi = n^2\int_0^{\frac{1}{n}}\left(f'(0)\xi + \frac{f''(s)}{2}\xi^2\right)d\xi$$

$$= n^2\left(\frac{f'(0)}{2}\frac{1}{n^2}+\frac{f''(s)}{6}\frac{1}{n^3}\right)$$

$$= \frac{f'(0)}{2}+\frac{f''(s)}{6}\frac{1}{n}$$

$$= \frac{f'(0)}{2} \text{ as } n \to \infty.$$

Solution 2 by Ovidiu Furdui, Cluj, Romania

The limit equals

$$\frac{f'(0)}{2}.$$

We have, since $f(0)=0$, that for all $n > \dfrac{1}{\alpha}$ one has

$$\begin{aligned} x_n = \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) &= \sum_{k=1}^{n}\left(f\left(\frac{k}{n^2}\right)-f(0)\right) \\ &= \sum_{k=1}^{n}\frac{k}{n^2}f'\left(\theta_{k,n}\right) \\ &= \sum_{k=1}^{n}\frac{k}{n^2}\left(f'(\theta_{k,n})-f'(0)\right)+\sum_{k=1}^{n}\frac{k}{n^2}f'(0) \\ &= \sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}f''(\beta_{k,n})+\frac{f'(0)(n+1)}{2n}. \qquad (1) \end{aligned}$$

We used, in the preceding calculations, the **Mean Value Theorem** twice where $0 < \beta_{k,n} < \theta_{k,n} < \dfrac{k}{n^2}$. Now,

$$\left|\sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}f''(\beta_{k,n})\right| \le M\sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n} \le M\sum_{k=1}^{n}\frac{k^2}{n^4} = M\frac{(n+1)(2n+1)}{6n^3},$$

where $M = \sup\limits_{x\in(-\alpha,\alpha)} |f''(x)|$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}f''(\beta_{k,n}) = 0. \qquad (2)$$

Combining (1) and (2) we get that the desired limit holds and the problem is solved.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Tom Leong, Scotrun, PA; Paolo Perfetti, Department of Mathematics, Tor Vergata Universtiy, Rome, Italy, and the proposer.

- 5084: *Charles McCracken, Dayton, OH*

A natural number is called a "repdigit" if all of its digits are alike.
Prove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit.

Comments by David E. Manes, Oneonta, NY; Michael N. Fried, Kibbutz Revivim, Israel, the proposer, and the editor.

Manes: The website <http://www.research.att.com/ njas/sequences/A158235> appears to have many counterexamples to problem 5084.
Editor: Following are some examples and comments from the above site.

$$11, 20, 39, 40, 49, 78, 133, 247, 494, 543, 1086, 1218,$$

$$1651, 1729, 2172, 2289, 2715, 3097, 3258, 3458, 3801,$$

$$171, 4344, 4503, 4578, 4887, 5187, 5430, 6194, 6231.$$

(And indeed, each number listed above can be written as repdigit *in some base.* For example:)

$$\begin{array}{rcl} 11^2 & = & 11111 \text{ in base } 3 \\ 20^2 & = & 1111 \text{ in base } 7 \\ 39^2 & = & 333 \text{ in base } 22 \\ 40^2 & = & 4444 \text{ in base } 7 \\ 49^2 & = & 777 \text{ in base } 18 \\ 78^2 & = & (12)(12)(12) \text{ in base } 22 \\ 1218^2 & = & (21)(21)(21)(21) \text{ in base } 41 \end{array}$$

McCracken: When I wrote the problem I intended that the number and it's power be written *in the same base.*

Editor: Charles McCracken sent in a proof that was convincing to me that the statement, as he had intended it to be, was indeed correct. No natural number with two or more digits (written in base b), when raised to a positive integral power, will produce a repdigit (in base b). I showed the problem, its solution, and Manes' comment, to my colleague Michael Fried, and he finally convinced me that although the intended statement might be true, the proof was in error.

Fried: The Sloan Integer Sequence site (mentioned above) also cites a paper which among other things, refers to Catalan's conjecture, now proven, stating that the only solution to $x^k - y^n = 1$ is $3^2 - 2^3 - 9 - 8 = 1$. This is the fact one needs to show that Charles' claim is true for base 2 repdigits. For in base 2 only numbers of the form $11111\ldots1$ are repdigits. These numbers are equal to $2^n - 1$. So if one of these numbers were equal to x^k, we would have $2^n - 1 = x^k$ or $2^n - x^k = 1$. But by the proven Catalan conjecture, the latter can never be satisfied.

Editor: So, dear readers, let's rephrase the problem: Prove or disprove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit in base b.

- 5085: *Proposed by Valmir Krasniqi, (student, Mathematics Department,) University of Prishtinë, Kosova*

Suppose that $a_k,\ (1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

Prove that

$$\prod_{j=1}^{n}\sum_{k=1}^{n} e_{j,k}a_k^2 \ge \prod_{j=1}^{n}\left(\sum_{k=1}^{n} d_{j,k}a_k\right)^2.$$

Solution by proposer

On expanding each side and reducing, the inequality becomes

$$\prod_{k=1}^{n}\left[(n-2)S + a_k^2\right] \quad \ge \quad \prod_{k=1}^{n}(T - a_k)\,,\ \text{where}$$

$$S = \sum_{k=1}^{n} a_k^2 \quad \text{and} \quad T = \sum_{k=1}^{n} a_k.$$

Since $(T - a_1)^2 \le (n-1)(S - a_1^2)$, etc., it suffices to prove that

$$\prod_{k=1}^{n}\left[(n-2)S + a_k^2\right] \ge (n-1)^n \prod_{k=1}^{n}(S - a_k)\,. \qquad (1)$$

If we now let $x_k = S - a_k^2$ where $k = 1, 2, 3, \ldots, n$ so that $S = \dfrac{x_1 + x_2 + \ldots + x_n}{n-1}$ and $a_k^2 = S - x_k$, then (1) becomes

$$\prod_{k=1}^{n}\left(S^{'} - x_k\right) \ge (n-1)^n \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n,\ \text{where } S^{'} = \sum_{k=1}^{n} x_k.$$

The result now follows by applying the AM-GM inequality to each of the factors $(S^{'} - x_k)$ on the left hand side. There is equality if, and only if, all the a_k's are equal.

Also solved by Tom Leong, Scotrun, PA

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
May 15, 2010

- 5104: *Proposed by Kenneth Korbin, New York, NY*

  There are infinitely many primitive Pythagorean triangles with hypotenuse of the form $4x^4+1$ where x is a positive integer. Find the dimensions of all such triangles in which at least one of the sides has prime integer length.

- 5105: *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation
  $$x+y-\sqrt{x^2+xy+y^2}=2+\sqrt{5}$$
  if x and y are of the form $a+b\sqrt{5}$ where a and b are positive integers.

- 5106: *Proposed by Michael Brozinsky, Central Islip, NY*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC,CA, and AB respectively.

  Show that
  $$d_a+d_b+d_c \le \frac{3}{4}D$$
  where D is the diameter of the circumcircle.

- 5107: *Proposed by Tuan Le (student, Fairmont, H.S.), Anaheim, CA*

  Let a, b, c be positive real numbers. Prove that
  $$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2} \ge \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}$$

- 5108: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute
  $$\lim_{n\to\infty}\frac{1}{n}\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right].$$

- 5109 *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the value of

$$\lim_{n\to\infty} \frac{(k\sqrt[n]{n}-k+1)^n}{n^k}.$$

Solutions

- 5086: *Proposed by Kenneth Korbin, New York, NY*

Find the value of the sum

$$\frac{2}{3}+\frac{8}{9}+\cdots+\frac{2N^2}{3^N}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

If $x \neq 1$, the formula for a geometric sum yields

$$\sum_{k=0}^{N} x^k = \frac{x^{N+1}-1}{x-1}.$$

If we differentiate and simplify, we obtain

$$\sum_{k=1}^{N} kx^{k-1} = \frac{Nx^{N+1}-(N+1)\,x^N+1}{(x-1)^2}.$$

Next, multiply by x and differentiate again to get

$$\sum_{k=1}^{N} kx^{k} = \frac{Nx^{N+2}-(N+1)\,x^{N+1}+x}{(x-1)^2}$$

and

$$\sum_{k=1}^{N} k^2x^{k-1} = \frac{N^2x^{N+2}-\left(2N^2+2N-1\right)x^{N+1}+(N+1)^2\,x^N-x-1}{(x-1)^3}.$$

Finally, multiply by x once more to yield

$$\sum_{k=1}^{N} k^2x^{k} = \frac{N^2x^{N+3}-\left(2N^2+2N-1\right)x^{N+2}+(N+1)^2\,x^{N+1}-x^2-x}{(x-1)^3}.$$

In particular, when we substitute $x=\dfrac{1}{3}$ and simplify, the result is

$$\sum_{k=1}^{N} \frac{k^2}{3^k} = \frac{3^{N+1}-\left(N^2+3N+3\right)}{2\cdot 3^N}.$$

Therefore, the desired sum is

$$\sum_{k=1}^{N} \frac{2k^2}{3^k} = \frac{3^{N+1}-\left(N^2+3N+3\right)}{3^N}.$$

Solution 2 by Ercole Suppa, Teramo, Italy

The required sum can be written as $S_N = \dfrac{2}{3^N} \cdot x_n$, where x_n denotes the sequence

$$x_n = 1^2 \cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} + \cdots + n^2 \cdot 3^0.$$

Since

$$x_{n+1} = 1^2 \cdot 3^n + 2^2 \cdot 3^{n-1} + 3^2 \cdot 3^{n-2} + \cdots + n^2 \cdot 3^1 + (n+1)^2 \cdot 3^0,$$

such a sequence satisfies the linear recurrence

$$x_{n+1} - 3x_n = (n+1)^2. \qquad (*)$$

Solving the characteristic equation $\lambda - 3 = 0$, we obtain the homogeneous solutions $x_n = A \cdot 3^n$, where A is a real parameter. To determine a particular solution, we look for a solution of the form $x_n^{(p)} = Bn^2 + Cn + D$. Substituting this into the difference equation, we have

$$\begin{aligned} B\,(n+1)^2 + C\,(n+1) + D - 3\left[Bn^2 + Cn + D\right] &= (n+1)^2 \Leftrightarrow \\ -2Bn^2 + 2\,(B-C)\,n + B + C - 2D &= n^2 + 2n + 1. \end{aligned}$$

Comparing the coefficients of n and the constant terms on the two sides of this equation, we obtain

$$B = -\frac{1}{2}, \qquad C = -\frac{3}{2}, \qquad D = -\frac{3}{2}$$

and thus

$$x_n^{(p)} = -\frac{1}{2}n^2 + -\frac{3}{2}n - \frac{3}{2}$$

The general solution of $(*)$ is simply the sum of the homogeneous and particular solutions, i.e.,

$$x_n = A \cdot 3^n - \frac{1}{2}n^2 + -\frac{3}{2}n - \frac{3}{2}$$

From the boundary condition $x_1 = 1$, the constant is determined as $\dfrac{3}{2}$.
Finally, the desired sum is

$$S_N = \frac{3^{N+1} - N^2 - 3N - 3}{3^N}$$

and we are done.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Pat Costello, Richmond, KY; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Paolo Perfetti, Department of Mathematics, Tor Vergata Universtiy, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Taylor University Problem Solving Group, Upland, IN, and the proposer.

- **5087:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers $a, b, c,$ and d such that $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ with $a < b < c < d$. Rationalize and simplify

$$\frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} \quad \text{if} \quad \begin{cases} x = bc+bd+cd, & \text{and} \\ y = ab+ac+ad. \end{cases}$$

Solution by Paul M. Harms, North Newton, KS

From the equation given in the problem we have

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd = 2\left(a^2+b^2+c^2+d^2\right).$$

From the last equation we have

$$2\left(ab+ac+ad+bc+bd+cd\right) = a^2+b^2+c^2+d^2.$$

We note that,

$$\begin{aligned} x+y &= ab+ac+ad+bc+bd+cd, \text{ then} \\ 2\left(x+y\right) &= a^2+b^2+c^2+d^2 \end{aligned}$$

From the identity in the problem,

$$\begin{aligned} 2\left(x+y\right) &= \frac{(a+b+c+d)^2}{2} \text{ or} \\ (x+y) &= \frac{(a+b+c+d)^2}{2^2} \end{aligned}$$

Also note that,

$$\begin{aligned} y &= a(b+c+d) \text{ or} \\ \frac{y}{a} &= b+c+d. \text{ Then} \\ x+y &= \frac{(a+(y/a))^2}{2^2} = \frac{(a^2+y)^2}{(2a)^2}. \end{aligned}$$

We have,

$$\begin{aligned} x &= (x+y)-y \\ &= \frac{(a^2+y)^2}{(2a)^2} - y \\ &= \frac{a^4+2a^2y+y^2-4a^2y}{4a^2} \\ &= \frac{(a^2-y)^2}{(2a)^2}. \end{aligned}$$

From $a < b < c < d$, we see that

$$\begin{aligned} a^2 - y &= a^2 - a(b+c+d) < 0. \text{ Thus} \\ \sqrt{\left(a^2 - y\right)^2} &= y - a^2. \end{aligned}$$

Working with the expression to be simplified, we have

$$\begin{aligned} \frac{\sqrt{x+y} - \sqrt{x}}{\sqrt{x+y} + \sqrt{x}} &= \frac{\left(\sqrt{x+y} - \sqrt{x}\right)^2}{y} \\ &= \frac{\left[(a^2+y)/(2a) - (y-a^2)/(2a)\right]^2}{y} \\ &= \frac{\left(2a^2/2a\right)^2}{y} \\ &= \frac{a^2}{y} \\ &= \frac{a}{b+c+d}. \end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; G. C., Greubel, Newport News, VA; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5088: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b be positive integers. Prove that

$$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2) + \varphi^2(b^2)}} \le \frac{\sqrt{2}}{2},$$

where $\varphi(n)$ is Euler's totient function.

Solution by Tom Leong, Scotrun, PA

We show

$$\varphi(ab) \le \sqrt{\varphi(a^2)\varphi(b^2)} \le \sqrt{\frac{\varphi^2 a^2) + \varphi^2(b^2)}{2}}$$

which implies the desired result. The second inequality used here is simply the AM-GM Inequality. To prove the first inequality, let p_i denote the prime factors of both a and b, and let q_j denote the prime factors of a only and r_k the primes factors of b only. Then

$$\begin{aligned} \varphi(ab) &= ab \prod_i \left(1 - \frac{1}{p_i}\right) \prod_j \left(1 - \frac{1}{q_j}\right) \prod_k \left(1 - \frac{1}{r_k}\right) \\ \varphi(a^2)\varphi(b^2) &= \left[a^2 \prod_i \left(1 - \frac{1}{p_i}\right) \prod_j \left(1 - \frac{1}{q_j}\right)\right] \left[b^2 \prod_i \left(1 - \frac{1}{p_i}\right) \prod_k \left(1 - \frac{1}{r_k}\right)\right] \end{aligned}$$

where we understand the empty product to be 1. Then $\varphi(ab) \le \sqrt{\varphi(a^2)\varphi(b^2)}$ reduces to

$$\prod_j \left(1 - \frac{1}{q_j}\right) \prod_k \left(1 - \frac{1}{r_k}\right) \le 1$$

which is obviously true.

Editor's comment: **Kee-Wai Lau of Hong Kong, China** mentioned in his solution to this problem that in the *Handbook of Number Theory I* (Section 1.2 of Chapter I by J. Sándor, D.S. Mitrinovi, and B. Crstic, Springer, 1995), the proof of $(\varphi(mn))^2 \le \varphi(m^2)\varphi(n^2)$, for positive integers m and n is attributed to a 1940 paper by T. Popoviciu. Kee-Wai then wrote $\sqrt{\varphi^2(a^2) + \varphi^2(b^2)} \ge \sqrt{2\varphi(a^2)\varphi(b^2)} \ge \sqrt{2}\varphi(ab)$, proving the inequality.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy; and the proposers.

- 5089: *Proposed by Panagiote Ligouras, Alberobello, Italy*

In $\triangle ABC$ let $AB = c, BC = a, CA = b, r =$ the in-radius and r_a, r_b, and r_c= the ex-radii, respectively.

Prove or disprove that

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} + \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} + \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \ge 2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right).$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the inequality.

Let s and S be respectively the semi-perimeter and area of $\triangle ABC$. It is well known that

$$r = \frac{S}{s},\ r_a = \frac{S}{s-a},\ r_b = \frac{S}{s-b},\ r_c = \frac{S}{s-c}.$$

Using these relations, we readily simplify

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} \text{ to } \frac{a}{c},\ \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} \text{ to } \frac{c}{b},\ \text{and } \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \text{ to } \frac{b}{a}.$$

Since $b^2 + ca \ge 2b\sqrt{ca}$, $c^2 + ab \ge 2c\sqrt{ab}$, and $a^2 + bc \ge 2a\sqrt{bc}$, so

$$2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right) \le \sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}}.$$

By the Cauchy-Schwarz inequality, we have

$$\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \le \sqrt{3\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)},$$

and by the arithmeic mean-geometric mean inequality we have

$$3 = 3\left(\sqrt[3]{\left(\frac{a}{c}\right)\left(\frac{b}{a}\right)\left(\frac{c}{b}\right)}\right) \leq \frac{a}{c} + \frac{b}{a} + \frac{c}{b}.$$

It follows that $\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \leq \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$ and this completes the solution.

Also solved by Tom Leong, Scotrun, PA; Ercole Suppa, Teramo, Italy, and the proposer.

- 5090: *Proposed by Mohsen Soltanifar (student), University of Saskatchewan, Canada*

Given a prime number p and a natural number n. Calculate the number of elementary matrices $E_{n\times n}$ over the field Z_p.

Solution by Paul M. Harms, North Newton, KS

The notation 0 and 1 will be used for the additive and multiplicative identities, respectively.

There are three types of matrices which make up the set of elementary matrices. One type is a matrix where two rows of the identity matrix are interchanged. Since there are n rows and we interchange two at a time, the number of elementary matrices of this type is $\frac{n(n-1)}{2}$, the combination of n things taken two at a time.

Another type of elementary matrix is a matrix where one of the elements along the main diagonal is replaced by an element which is not 0 or 1. There are $(p-2)$ elements which can replace a 1 on the main diagonal. The number of elementary matrices of this type is $(p-2)n$.

The third type of elementary matrix is the identity matrix where at most one position, not on the main diagonal, is replaced by a non-zero element. There are (n^2-n) positions off the main diagonal and $(p-1)$ non-zero elements. Then there are $(n^2-n)(p-1)$ different elementary matrices where a non-zero element replaces one zero element in the identity matrix. If the identity matrix is included here, the number of elementary matrices of this type is $(n^2-n)(p-1)+1$.

The total number of elementary matrices is

$$\frac{n(n-1)}{2} + (p-2)n + (n^2-n)(p-1) + 1 = n^2\left(p-\frac{1}{2}\right) - \frac{3n}{2} + 1.$$

Comment by David Stone and John Hawkins of Statesboro, GA. There doesn't seem to be any need to require that p be prime as we form and count these elementary matrices. However, if m were not prime then Z_m would not be a field and the algebraic properties would be affected. For instance, it's preferable that any elementary matrix be invertible and the appearance of non-invertible scalars would produce non-invertible elementary matrices such as $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ over Z_4.

Also solved by David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5091: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k, p \geq 0$ be nonnegative integers. Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the integral equals $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$, independent of k.

Here $(-1)!! = 0!! = 1,\ n!! = n(n-2)\ldots(3)(1)$ if n is a positive odd integer and $n!! = n(n-2)\ldots(4)(2)$ if n is a positive even integer.

By substituting $x = -y$, we have

$$\int_{-\pi/2}^{0} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx = \int_{0}^{\pi/2} {}_0 \frac{\sin^{2p} y}{1-\sin^{2k+1} y+\sqrt{1+\sin^{4k+2} y}} \text{ so that}$$

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx \\
= &\int_{0}^{\pi/2} \sin^{2p} x \left(\frac{1}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} + \frac{1}{1-\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} \right) dx \\
= &\ 2\int_{0}^{\pi/2} \sin^{2p} x \left(\frac{1+\sqrt{1+\sin^{4k+2} x}}{\left(1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}\right)\left(1-\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}\right)} \right) dx \\
= &\int_{0}^{\pi/2} \sin^{2p} x dx.
\end{aligned}$$

The last integral is standard and its value is well known to be $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy

The answer is: $\dfrac{(2p)!}{2^{2p}(p!)^2}\dfrac{\pi}{2}$ for any k.

Proof Let's substitute $\sin x = t$

$$\int_{-1}^{1} \frac{t^{2p}}{1+t^{2k+1}+\sqrt{1+t^{4k+2}}} \frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{t^{2p}(1+t^{2k+1}-\sqrt{1+t^{4k+2}})}{2t^{2k+1}} \frac{dt}{\sqrt{1-t^2}}$$

Now

$$\int_{-1}^{1} \frac{t^{2p}}{2t^{2k+1}} \frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{t^{2p}\sqrt{1+t^{4k+2}}}{2t^{2k+1}} \frac{dt}{\sqrt{1-t^2}} = 0$$

since the integrands are odd functions. It remains

$$\frac{1}{2}\int_{-1}^{1}\frac{t^{2p}}{\sqrt{1-t^2}}dt=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx$$

after changing variable $t=\sin x$. Integrating by parts we obtain

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx &= \int_{-\pi/2}^{\pi/2}(-\cos x)'(\sin x)^{2p-1}dx\\ &= -\cos x(\sin x)^{2p-1}\Big|_{-\pi/2}^{\pi/2}+(2p-1)\int_{-\pi/2}^{\pi/2}\cos^2 x(\sin x)^{2p-2}dx\\ &= (2p-1)\int_{-\pi/2}^{\pi/2}(\sin x)^{2p-2}dx-(2p-1)\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx\end{aligned}$$

and if we call $I_{2p}=\displaystyle\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx$, then we have $I_{2p}=\dfrac{2p-1}{2p}I_{2p-2}$. It results that $I_{2p}=\dfrac{(2p-1)!!}{(2p)!!}\pi=\dfrac{(2p)!}{2^{2p}(p!)^2}\pi$ and then $\dfrac{1}{2}\displaystyle\int_{-1}^{1}\frac{t^{2p}}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\frac{(2p-1)!!}{(2p)!!}=\frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$

Editor's comment: The two solutions presented, $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$ and $\dfrac{(2p)!}{2^{2p}(p!)^2}\dfrac{\pi}{2}$, are equivalent to one another.

Also solved by Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before June 15, 2010

- **5110:** *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with an interior point P and with coordinates $A(0,0), B(6,8)$, and $C(21,0)$. The distance from point P to side $\overline{AB}$ is a, to side $\overline{BC}$ is b, and to side $\overline{CA}$ is c where $a:b:c = \overline{AB}:\overline{BC}:\overline{CA}$.

  Find the coordinates of point P.

- **5111:** *Proposed by Michael Brozinsky, Central Islip, NY.*

  In Cartesianland where immortal ants live, it is mandated that any anthill must be surrounded by a triangular fence circumscribed in a circle of unit radius. Furthermore, if the vertices of any such triangle are denoted by A, B, and C, in counterclockwise order, the anthill's center must be located at the interior point P such that $\angle PAB = \angle PBC = \angle PCA$.

  Show $\overline{PA} \cdot \overline{PB} \cdot \overline{PC} \leq 1$.

- **5112:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

  Let $0 < a < b$ be real numbers with a fixed and b variable. Prove that

$$\lim_{b \to a} \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} = \lim_{b \to a} \int_a^b \frac{dx}{\ln \dfrac{b(a+x)}{a(b+x)}}.$$

- **5113:** *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \leq \sqrt{xy} + \frac{x+y}{2} + \frac{\left(\dfrac{x+y}{6} - \dfrac{\sqrt{xy}}{3}\right)^2}{\dfrac{2xy}{x+y}}.$$

- **5114:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let M be a point in the plane of triangle ABC. Prove that

$$\frac{\overline{MA}^2+\overline{MB}^2+\overline{MC}^2}{\overline{AB}^2+\overline{BC}^2+\overline{CA}^2}\geq\frac{1}{3}.$$

  When does equality hold?

- **5115:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

  Let G be a finite cyclic group. Compute the number of distinct composition series of G.

Solutions

- **5092:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with altitude h and with cevian $\overline{CD}$. A circle with radius x is inscribed in $\triangle ACD$, and a circle with radius y is inscribed in $\triangle BCD$ with $x<y$. Find the length of the cevian $\overline{CD}$ if x,y and h are positive integers with $(x,y,h)=1$.

  Solution by David Stone and John Hawkins (jointly), Statesboro, GA;

  We let the length of cevian $=d$. Since the altitude of the equilateral triangle is h, the length of the side $\overline{AC}$ is $\dfrac{2h}{\sqrt{3}}$. Let F be the center of the circle inscribed in $\triangle ACD$. Let $\alpha=\angle ACF=\angle FCD$. Therefore $\angle ACD=2\alpha$.

  Let E be the point where the inscribed circle in $\triangle ACD$ is tangent to side $\overline{AC}$. Since $\overline{AF}$ bisects the base angle of 60°, we know that $\triangle AEF$ is a $30^\circ-60^\circ-90^\circ$ triangle, implying that $\overline{AE}=\sqrt{3}x$. Thus the length of $\overline{CE}$ is $\overline{AC}-\overline{AE}=\dfrac{2h}{\sqrt{3}}-\sqrt{3}x=\dfrac{2h-3x}{\sqrt{3}}$.

  Applying the Law of Sines in triangle $\triangle ADC$, we have

$$\frac{\sin 2\alpha}{\overline{AD}}=\frac{\sin 60^\circ}{d}=\frac{\sin(\angle ADC)}{\overline{AC}}. \qquad (1)$$

  Because $\angle ADC=180^\circ-60^\circ-2\alpha=120^\circ-2\alpha$, we have

$$\begin{aligned}\sin(\angle ADC)&=\sin(120^\circ-2\alpha)\\ &=\sin 120^\circ\cos 2\alpha-\cos 120^\circ\sin 2\alpha\\ &=\frac{\sqrt{3}}{2}\cos 2\alpha+\frac{1}{2}\sin 2\alpha\\ &=\frac{\sqrt{3}}{2}\left(\cos^2\alpha-\sin^2\alpha\right)+\frac{1}{2}(2\sin\alpha\sin\alpha).\end{aligned}$$

Thus from (1) we have

$$\frac{\sqrt{3}}{2d} = \frac{\left[\sqrt{3}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(2\sin\alpha\sin\alpha\right)\right]\sqrt{3}}{4h}.$$

Therefore, we can solve for d in terms of h and α:

$$d = \frac{2h}{\left[\sqrt{3}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(2\sin\alpha\sin\alpha\right)\right]}.$$

In the right triangle $\triangle EFC$, we have

$$\overline{FC} = \sqrt{x^2 + \left(\frac{2h-3x}{\sqrt{3}}\right)^2} = \sqrt{\frac{3x^2 + 4h^2 - 12hx + 9x^2}{3}} = \frac{2}{\sqrt{3}}\sqrt{3x^2 + h^2 - 3hx}.$$

Thus, $\sin\alpha = \dfrac{\sqrt{3}x}{2\sqrt{3x^2+h^2-3hx}}$ and $\cos\alpha = \dfrac{2h-3x}{2\sqrt{3x^2+h^2-3hx}}$. Therefore,

$$\begin{aligned} cos^2\alpha - \sin^2\alpha &= \frac{(2h-3x)^2}{4\left(3x^2+h^2-3hx\right)} - \frac{3x^2}{4\left(3x^2+h^2-3hx\right)} \\ &= \frac{4h^2-12hx+6x^2}{4\left(3x^2+h^2-3hx\right)} = \frac{2h^2-6hx+3x^2}{2\left(3x^2+h^2-3hx\right)}. \end{aligned}$$

and $2\sin\alpha\cos\alpha = \dfrac{\sqrt{3}x\left(2h-3x\right)}{2\left(3x^2+h^2-3hx\right)}$.

Therefore the denominator in the expression for d becomes

$$\frac{\sqrt{3}(2h^2-6hx+3x^2)}{2\left(3x^2+h^2-3hx\right)} + \frac{\sqrt{3}x\left(2h-3x\right)}{2\left(3x^2+h^2-3hx\right)} = \sqrt{3}\frac{2h^2-4hx}{2\left(3x^2+h^2-3hx\right)}.$$

Thus, $d = \dfrac{2h}{\dfrac{\sqrt{3}(2h^2-4hx)}{2(3x^2+h^2-3hx)}} = \dfrac{2(3x^2+h^2-3hx)}{\sqrt{3}(h-2x)}$.

Similarly, working in $\triangle BCD$, we can show that $d = \dfrac{2(3y^2+h^2-3hy)}{\sqrt{3}(h-2y)}$.

We note that x and y both satisfy the same equation when set equal to d. Thus for a given value of d, the equation should have two solutions. The smaller one can be used for x and the larger for y.

We also note that if x, h and y are integers, then d has the form $d = \dfrac{r}{\sqrt{3}}$, for r a rational number. We substitute this into the equation x:

$$d \quad = \quad \frac{2\left(3x^2+h^2-3hx\right)}{\sqrt{3}(h-2x)} = \frac{r}{\sqrt{3}}, \text{ so}$$

$$r = \frac{2(3x^2+h^2-3hx)}{h-2x}.$$

Now we solve this for x:

$$rh - 2xr = 6x^2 + 2h^2 - 6hx$$

$$6x^2 - (6h-2r)\,x + 2h^2 - rh = 0$$

$$x = \frac{6h - 2r \pm \sqrt{36h^2 - 24hr + 4r^2 - 48h^2 + 24hr}}{12} = \frac{3h - r \pm \sqrt{r^2 - 3h^2}}{6}.$$

Of course we would have the exact same expression for y.

Thus we take $x = \dfrac{3h - r - \sqrt{r^2-3h^2}}{6}$ and $y = \dfrac{3h - r - \sqrt{r^2-3h^2}}{6}$ and find h and r so that x and y turn out to be positive integers.

Subtracting x from y gives $y - x = \dfrac{\sqrt{r^2-3h^2}}{3}$. Thus we need r and h such that $\dfrac{\sqrt{r^2-3h^2}}{3}$ is an integer.

It must be the case that $r^2 - 3h^2 \geq 0$, which requires $0 < \sqrt{3}h \leq r$. In addition it must be true that

$$\begin{aligned}
3h - r - \sqrt{r^2-3h^2} &> 0 \\
9h^2 - 6hr + r^2 &> r^2 - 3h^2 \\
12h^2 - 6hr &> 0 \\
6h(2h - r) &> 0 \\
0 < \quad r &< 2h. \quad \text{Thus,} \\
\sqrt{3}h &\leq r < 2h.
\end{aligned}$$

If we restrict our attention to integer values of r, then both h and r must be divisible by 3.

For $h = 3, 6$ and 9, no integer values of r divisible by 3 satisfy $\sqrt{3}h \leq r < 2h$. So the first allowable value of h is 12. Then the condition $12\sqrt{3} \leq r < 24$ forces $r = 21$. From this we find that $x = 2$ and $y = 3$ and $d = 7\sqrt{3}$. (Note that $(2, 3, 12) = 1$.)

This is only the first solution. We programmed these constraints and let MatLab check for integer values of h and appropriate integer values of r which make x and y integers satisfying $(x, y, h) = 1$. There are many solutions:

$$\begin{array}{ccccc} r & y & x & y & \text{cevian} \\ \end{array}$$
$$\begin{pmatrix} 21 & 12 & 2 & 3 & 7\sqrt{3} \\ 78 & 45 & 9 & 10 & 26\sqrt{3} \\ 111 & 60 & 5 & 18 & 37\sqrt{3} \\ 114 & 63 & 7 & 18 & 38\sqrt{3} \\ 129 & 72 & 9 & 20 & 43\sqrt{3} \end{pmatrix}$$

Editor's note: David and John then listed another 47 solutions. They capped their search at $h = 1000$, but stated that solutions exist for values of $h > 1000$. They ended the write-up of their solution with a formula for expressing the cevian in terms of x, y and h.

$$\begin{aligned} y - x &= \frac{\sqrt{r^2 - 3h^2}}{3} \\ 9(y-x)^2 &= r^2 - 3h^2 \\ r^2 &= 3h^2 + 9(y-x)^2 \\ r &= \sqrt{3h^2 + 9(y-x)^2} \\ \text{Length of cevian } \frac{r}{\sqrt{3}} &= \sqrt{h^2 + 3(y-x)^2}. \end{aligned}$$

Ken Korbin, the proposer of this problem, gave some insights into how such a problem can be consructed. He wrote:

Begin with any prime number P congruent to 1(mod 6). Find positive integers $[a, b]$ such that $a^2 + ab + b^2 = P^2$. Construct an equilateral triangle with side $a + b$ and with Cevian P. The Cevian will divide the base of the triangle into segments with lengths a and b. Find the altitude of the triangle and the inradii of the 2 smaller triangles. Multiply the altitude, the inradii and the Cevian P by $\sqrt{3}$ and then by their LCD. This should do it. Examples: $P = 7, [a, b] = [3, 5]$. $P = 13, [a, b] = [7, 8]$. And so on.

- **5093:** *Proposed by Worapol Ratanapan (student), Montfort College, Chiang Mai, Thailand*

$6 = 1 + 2 + 3$ is one way to partition 6, and the product of $1, 2, 3$, is 6. In this case, we call each of $1, 2, 3$ a **part** of 6.

We denote the maximum of the product of all **parts** of natural number n as $N(n)$.

As a result, $N(6) = 3 \times 3 = 9, N(10) = 2 \times 2 \times 3 \times 3 = 36$, and $N(15) = 3^5 = 243$.

More generally, $\forall n \in N, N(3n) = 3^n, N(3n+1) = 4 \times 3^{n-1}$, and $N = (3n+2) = 2 \times 3^n$.

Now let's define $R(r)$ in the same way as $N(n)$, but each **part** of r is positive real. For instance $R(5) = 6.25$ and occurs when we write $5 = 2.5 + 2.5$

Evaluate the following:

$$i)\ \ R(2e)$$
$$ii)\ \ R(5\pi)$$

Solution by Michael N. Fried, Kibbutz Revivim, Israel

Let $R(r) = \prod_i x_i$, where $\sum_i x_i = r$ and $x_i > 0$ for all i. For any given r, find the maximum of $R(r)$.

Since for any given r and n the arithmetic mean of every set $\{x_i\}$ $i = 1, 2, 3 \ldots n$ is $\dfrac{r}{n}$ by assumption, the geometric-arithemetic mean inequality implies that

$$R(r) = \prod_{i=1}^{n} x_i \leq \left(\frac{r}{n}\right)^n.$$

Hence the maximum of $R(r)$ is a function of n. Let us then find the maximum of the function $R(x) = \left(\dfrac{r}{x}\right)^x$, which is the same as the maximum of

$$L(x) = \ln\left(R(x)\right) = x\ln r - x\ln x.$$

$L(x)$ indeed has a *single* maximum at $x = \dfrac{r}{e}$.

Let $m = \lfloor \dfrac{r}{e} \rfloor$ and $M = \lceil \dfrac{r}{e} \rceil$. Then the maximum value of $R(r)$ is

$$\max\left(\left(\frac{r}{m}\right)^m, \left(\frac{r}{M}\right)^M\right).$$

To make this concrete consider $r = 5,\ 2e$, and 5π.

For $r = 5, r/e = 1.8393\ldots$, so $\max R(5) = \max\left(5, (5/2)^2\right) = \max(5, 6.25) = 6.25$

For $r = 2e, r/e = 2$, so $\max R(2e) = e^2$.

For $r = 5\pi, r/e = 5.7786\ldots$, so $\max R(5\pi) = \max\left(\left(\dfrac{5\pi}{5}\right)^5, \left(\dfrac{5\pi}{6}\right)^6\right) = \left(\dfrac{5\pi}{6}\right)^6$.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA; The Taylor University Problem Solving Group, Upland, IN, and the proposer.

- **5094:** *Proposed by Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy*

Let a, b, c be real positive numbers such that $a + b + c + 2 = abc$. Prove that

$$2\left(a^2 + b^2 + c^2\right) + 2\left(a + b + c\right) \geq \left(a + b + c\right)^2.$$

Solution 1 by Ercole Suppa, Teramo, Italy

We will use the "magical" substitution given in "Problems from The Book" by Titu Andreescu and Gabriel Dospinescu, which is explained in the following lemma:

If a, b, c are positive real numbers such that $a + b + c + 2 = abc$, then there exists three real numbers $x, y, z > 0$ such that

$$a = \frac{y+z}{x}, \ b = \frac{z+x}{y}, \text{ and } c = \frac{x+y}{z}. \qquad (*)$$

Proof: By means of a simple computation the condition $a+b+c+2 = abc$ can be written in the following equivalent form

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1.$$

Now if we take

$$x = \frac{1}{1+a}, \quad y = \frac{1}{1+b}, \text{ and } \ z = \frac{1}{1+c},$$

then $x + y + z = 1$ and $a = \frac{1-x}{x} = \frac{y+z}{x}$. Of course, in the same way we find $b = \frac{z+x}{y}$ and $c = \frac{x+y}{z}$.

By using the substitution $(*)$, after some calculations, the given inequality rewrites as

$$\frac{z^4(x-y)^2 + x^4(y-z)^2 + y^4(x-z)^2 + 2(x^3y^3 + x^3z^3 + y^3z^3 - 3x^2y^2z^2)}{x^2y^2z^2} \geq 0,$$

which is true since

$$x^3y^3 + x^3z^3 + y^3z^3 \geq 3x^2y^2z^2$$

by virtue of the AM-GM inequality.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

First let $x = a + b$ and $y = ab$. Hence $x \geq 2\sqrt{y}$.

From $a + b + c + 2 = abc$, we have $c = \frac{x+2}{y-1}$. Hence, $y > 1$.

Noting that $x^2 - 2y = a^2 + b^2$, it follows readily that the original inequality can be rewritten as

$$(y-2)^2 x^2 + 2\left(y^2 - 3y + 4\right) x - 4y^3 + 8y^2 \geq 0, \qquad (1)$$

where $y > 1$ and $x \geq 2\sqrt{y}$. For $y > 1$ arbitrary but fixed, we denote by $f_y(x)$, for $x \geq 2\sqrt{y}$, the function on the left-hand side of (1).

Trivially, $f_y(x) \geq 0$ for $y = 2$. For $y \neq 2$ (which we henceforth assume), $f_y(\cdot)$, when extended to $\Re$, is a quadratic function (parabola) attaining its minimum at $x_0 = \frac{-\left(y^2 - 3y + 4\right)}{(y-2)^2}$.

Noting that $x_0 < 0$, it follows that

$$\min_{\{x:x\geq 2\sqrt{y}\}} f_y(x) \quad = \quad f_y(2\sqrt{y})$$

$$= \quad 4\sqrt{y}\left(y^2 - 3y + 4 - 2y^{3/2} + 4\sqrt{y}\right).$$

Thus the inequality (1) will be proved if we show that

$$\varphi(y) := y^2 - 3y + 4 - 2y^{3/2} + 4\sqrt{y} \geq 0. \qquad (2)$$

This is trivial for $1 < y < 2$ since in this case both $y^2 - 3y + 4$ and $-2y^{3/2} + 4\sqrt{y}$ are greater than 0.

For $y > 2$, it is readily seen that $\varphi''(y) > 0$. Hence, $\varphi'(y)$ is increasing for $y > 2$. Noting that $\varphi'(4) = 0$, it thus follows that $\min_{\{y>2\}} \varphi(y) = \varphi(4)$. Since $\varphi(4) = 0$, inequality (2) is proved.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Firstly, we have

$$2(a^2 + b^2 + c^2) + 2(a + b + c) - (a + b + c)^2 = (a + b + c)(a + b + c + 2) - 4(ab + bc + ca)$$

Let $p = a + b + c,\ q = ab + bc + ca,\ r = abc$, so that $r = p + 2$.

We need to show that $q \leq \dfrac{p(p+2)}{4}$ (1)

It is well known that $a,\ b$, and c are the positive real roots of the cubic equation

$$x^3 - px^2 + qx - r \quad = \quad 0 \ \text{ if, and only if,}$$

$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 \geq 0.$$

By substituting $r = p + 2$ and simplifying, we reduce the last inequality to $f(q) \leq 0$, where

$$\begin{aligned} f(q) &= 4q^3 - p^2q^2 - \left(36p + 18p^2\right)q + 4p^4 + 8p^3 + 27p^2 + 108p + 108 \\ &= (q + 2p + 3)\left(4q^2 - (p^2 + 8p + 12)q + 2p^3 + p^2 + 12p + 36\right). \text{ Thus} \end{aligned}$$

$$4q^2 - (p^2 + 8p + 12)q + 2p^3 + p^2 + 12p + 36 \leq 0. \qquad (2)$$

By the arithmetic mean-geometric inequality we have
$abc = a + b + c + 2 \geq 4(2abc)^{1/4}$ so that $abc \geq 8$ and $p = a + b + c \geq 6$.

From (2) we obtain $q \leq \dfrac{1}{8}\left(p^2 + 8p + 12 + \sqrt{(p+2)(p-6)^3}\right)$ and it remains to show that

$$\frac{1}{8}\left(p^2 + 8p + 12 + \sqrt{(p+2)(p-6)^3}\right) \leq \frac{p(p+2)}{4}. \qquad (3)$$

Now (3) is equivalent to $\sqrt{(p+2)(p-6^3)} \leq (p-6)(p+2)$ or, on squaring both sides and simplifying, $-8(p+2)(p-6)^2 \leq 0$.

Since the last inequality is clearly true, we see that (1) is true, and this completes the solution.

Also solved by Tom Leong, Scotrun, PA; Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5095:** *Proposed by Zdravko F. Starc, Vršac, Serbia*

  Let F_n be the Fibonacci numbers defined by

$$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \ (n = 1, 2, \cdots).$$

  Prove that

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq F_n \leq \sqrt{(n-2)F_{n-2}F_{n-1}} + 1 \quad (n = 3, 4, \cdots).$$

Solution 1 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

First, using mathematical induction, we show that

$$F_n^2 = F_{n-1}F_{n+1} + (-1)^{n+1}, \text{for } n = 2, 3, \ldots \qquad (2).$$

For $n = 2$ we have:

$$F_2^2 = 1 = 1 \cdot 2 - 1 = F_1F_3 + (-1)^3.$$

Assume that (2) holds for n. We show that it is true also for $n+1$.

$$\begin{aligned}
F_nF_{n+2} + (-1)^{n+2} &= F_n\left(F_n + F_{n+1}\right) + (-1)^{n+2} \\
&= F_n^2 + F_nF_{n+1} + (-1)^{n+2} \\
&= F_{n-1}F_{n+1} + F_nF_{n+1} + (-1)^{n+1} + (-1)^{n+2} \\
&= F_{n+1}\left(F_{n-1} + F_n\right) = F_{n+1}^2.
\end{aligned}$$

So (2) hold for any $n \geq 2$.

Next we show that,

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq F_n, \text{ holds.}$$

By applying (2) several times we obtain:

$$\begin{aligned}
F_n^2 &= F_{n-1}F_{n+1} + (-1)^{n+1} \\
&= F_{n-1}\left(F_n + F_{n-1}\right) + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-1}^2 + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-2}F_n + (-1)^n + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-2}F_{n-1} + F_{n-2}^2 \\
&= 2F_{n-1}F_{n-2} + F_{n-2}F_n + F_{n-2}^2 + (-1)^{n+1} \\
&= 3F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \\
&= F_{n-1}F_{n-2} + 2F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1}
\end{aligned}$$

$$\geq F_{n-1}F_{n-2} + 2\sqrt{F_{n-1}F_{n-2}} + 1$$
$$= \left(\sqrt{F_{n-1}F_{n-2}} + 1\right)^2$$

Taking the square root of both sides we obtain:

$$F_n \geq \sqrt{F_{n-1}F_{n-2}} + 1,$$

which is the first part of (1).

To prove the second part of (1), we proceed similarly. That is:

$$\begin{aligned}
F_n^2 &= F_{n-1}F_{n+1} + (-1)^{n+1} \\
&= F_{n-1}(F_n + F_{n-1}) + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-1}^2 + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-2}F_n + (-1)^n + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-2}F_{n-1} + F_{n-2}^2 \\
&= 2F_{n-1}F_{n-2} + F_{n-2}F_n + F_{n-2}^2 + (-1)^{n+1} \\
&= 3F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \\
&\leq 3F_{n-1}F_{n-2} + 2F_{n-1}F_{n-2} + 1 \\
&= 5F_{n-1}F_{n-2} + 1 \\
&\leq (n-2)F_{n-1}F_{n-2} + 1 \text{ for } n \geq 7.
\end{aligned}$$

Taking the square root of both sides we obtain:

$$F_n \leq \sqrt{(n-2)F_{n-1}F_{n-2} + 1} \leq \sqrt{(n-2)F_{n-1}F_{n-2}} + 1, \qquad (4)$$

which proves the second part of (1) for $n \geq 7$.

On can easily show that (4) also holds for $n = 3, 4, 5$, and 6 by checking each of these cases separately. So combining (3) and (4) we have proved that:

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq F_n \leq \sqrt{(n-2)F_{n-2}F_{n-1}} + 1 \quad (n = 3, 4, \cdots).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Given $n = 3, 4, \cdots$, we can use (because all the F_n are positive) the Geometric Mean-Arithmetic Mean Inequality applied to $F_i, i = n-1, n-2$, the facts that $F_n = F_{n-1} + F_{n-2}$ and $F_n \geq 2$ with equality if, and only if, $n = 3$, to obtain:

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq \frac{F_{n-2} + F_{n-1}}{2} + 1 = \frac{F_n}{2} + 1 \leq F_n,$$

which is the first inequality to prove, with equality if, and only if, $n = 3$.

The second inequality, if $n = 3, 4, \cdots$ can be proved using that $F_n = \sum_{i=1}^{n-2} F_i + 1$, the Quadratic Mean-Arithmetic Mean inequality applied to the positive numbers F_i, $i = 1, 2, \cdots, n-2$, and that $F_{n-2}F_{n-1} = \sum_{i=1}^{n-2} F_i^2$, because

$$F_n = \sum_{i=1}^{n-2} F_i + 1 \leq \sqrt{(n-2)\sum_{i=1}^{n-2} F_i^2} + 1 = \sqrt{(n-2)F_{n-2}F_{n-1}} + 1,$$

with equality if, and only if, $n = 3$ or $n = 4$.

Solution 3 by Shai Covo, Kiryat-Ono, Israel

The left inequality is trivial. Indeed, for any $n \geq 3$,

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq \sqrt{F_{n-1}F_{n-1}} + F_{n-2} = F_n.$$

As for the right inequality, the result is readily seen to hold for $n = 3, 4, 5, 6$. Hence, it suffices to show that for any $n \geq 7$ the following inequality holds:

$$F_n = F_{n-2} + F_{n-1} < \sqrt{5F_{n-2}F_{n-1}}.$$

With x and y playing the role of F_{n-2} and F_{n-1} ($n \geq 7$), respectively, it thus suffices to show that $x + y < \sqrt{5xy}$, subject to $x < y < 2x$ ($x \geq F_5 = 5$).
It is readily checked that, for any fixed $x > 0$ (real), the function $\phi_x(y) = \sqrt{5xy} - (x+y)$, defined for $y \in [x, 2x]$, has a global minimum at $y = 2x$, where $\phi_x(y) = (\sqrt{10} - 3)x > 0$. The result is now established.

Solution 4 by Brian D. Beasley, Clinton, SC

Let $L_n = \alpha\sqrt{\alpha F_{n-2}F_{n-1}} - 1$ and $U_n = \alpha\sqrt{\alpha F_{n-2}F_{n-1}} + 1$, where $\alpha = (1+\sqrt{5})/2$. We prove the stronger inequalities $L_n \leq F_n \leq U_n$ for $n \geq 3$, with improved lower bound for $n \geq 5$ and improved upper bound for $n \geq 7$.

First, we note that the inequalities given in the original problem hold for $3 \leq n \leq 6$. Next, we apply induction on n, verifying that $L_3 \leq F_3 \leq U_3$ and assuming that $L_n \leq F_n \leq U_n$ for some $n \geq 3$. Then $(F_n - 1)^2 \leq \alpha^3 F_{n-2}F_{n-1} \leq (F_n + 1)^2$, which implies

$$(F_{n+1} - 1)^2 = (F_n - 1)^2 + 2F_{n-1}(F_n - 1) + F_{n-1}^2 \leq \alpha^3 F_{n-2}F_{n-1} + 2F_{n-1}(F_n - 1) + F_{n-1}^2$$

and

$$(F_{n+1} + 1)^2 = (F_n + 1)^2 + 2F_{n-1}(F_n + 1) + F_{n-1}^2 \geq \alpha^3 F_{n-2}F_{n-1} + 2F_{n-1}(F_n + 1) + F_{n-1}^2.$$

Since $\alpha^3 F_{n-1}F_n = \alpha^3 F_{n-2}F_{n-1} + \alpha^3 F_{n-1}^2$, it suffices to show that

$$2F_{n-1}(F_n - 1) + F_{n-1}^2 \leq \alpha^3 F_{n-1}^2 \leq 2F_{n-1}(F_n + 1) + F_{n-1}^2,$$

that is, $2(F_n - 1) + F_{n-1} \leq \alpha^3 F_{n-1} \leq 2(F_n + 1) + F_{n-1}$. Using the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\beta = (1 - \sqrt{5})/2$, these latter inequalities are equivalent to $2\beta^{n-1} - 2 \leq 0 \leq 2\beta^{n-1} + 2$, both of which hold since $-1 < \beta < 0$. (We also used the identities $2\alpha + 1 - \alpha^3 = 0$ and $\alpha^3 - 1 - 2\beta = 2\sqrt{5}$.)

Finally, we note that U_n is smaller than the original upper bound for $n \geq 7$, since $\alpha^3 + 2 < 7$. Also, a quick check verifies that L_n is larger than the original lower bound for $n \geq 5$; this requires

$$(\alpha^3 - 1)^2 (F_{n-2}F_{n-1})^2 - 8(\alpha^3 + 1)F_{n-2}F_{n-1} + 16 \geq 0,$$

which holds if $F_{n-2}F_{n-1} \geq 4$.

Also solved by Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA; Boris Rays, Brooklyn NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5096:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b + \sqrt[4]{ab^3}} + \frac{b}{c + \sqrt[4]{bc^3}} + \frac{c}{a + \sqrt[4]{ca^3}} \geq \frac{3}{2}.$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

We have, since $\sqrt[4]{xy^3} \leq \dfrac{x+3y}{4}$, that

$$\sum_{cyclic} \frac{a}{b + \sqrt[4]{ab^3}} \geq 4 \sum_{cyclic} \frac{a}{7b + a} = 4 \sum_{cyclic} \frac{a^2}{7ba + a^2} \geq 4 \frac{(a+b+c)^2}{\sum a^2 + 7 \sum ab},$$

and hence it suffices to prove that

$$8(a+b+c)^2 \geq 3(a^2 + b^2 + c^2) + 21(ab + bc + ca).$$

However, the last inequality reduces to proving that

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

and the problem is solved since the preceding inequality holds for all real a, b, and c.

Solution 2 by Ercole Suppa, Teramo, Italy

By the weighted AM-GM inequality we have

$$\begin{aligned}
&\frac{a}{b + \sqrt[4]{ab^3}} + \frac{b}{c + \sqrt[4]{bc^3}} + \frac{c}{a + \sqrt[4]{ca^3}} \\
\geq\ & \frac{a}{b + \frac{1}{4}a + \frac{3}{4}b} + \frac{b}{c + \frac{1}{4}b + \frac{3}{4}c} + \frac{c}{a + \frac{1}{4}c + \frac{3}{4}a} \\
=\ & \frac{4a}{a+7b} + \frac{4b}{b+7c} + \frac{4c}{c+7a}.
\end{aligned}$$

So it suffices to prove that

$$\frac{a}{a+7b} + \frac{b}{b+7c} + \frac{c}{c+7a} \geq \frac{3}{8}.$$

This inequality is equivalent to

$$\frac{7(13a^2b+13b^2c+13ac^2+35ab^2+35a^2c+35bc^2-144abc)}{8(a+7b)(b+7c)(c+7a)} \geq 0$$

which is true. Indeed according to the AM-GM inequality we obtain

$$13a^2b+13b^2c+13ac^2 \geq 13\cdot 3\cdot \sqrt[3]{a^3b^3c^3} \quad = \quad 39abc$$

$$35ab^2+35a^2c+35bc^2 \geq 35\cdot 3\cdot \sqrt[3]{a^3b^3c^3} \quad = \quad 105abc$$

and, summing these inequalities we obtain:

$$13a^2b+35ab^2+35a^2c+13b^2c+13ac^2+35bc^2 \geq 144abc.$$

This ends the proof. Clearly, equality occurs for $a=b=c$.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

We start by considering the function

$$f(t)=\frac{1}{e^t+e^{\frac{3}{4}t}}$$

on $\Re$. Then for all $t\in\Re$,

$$f''(t)=\frac{16e^{2t}+23e^{\frac{7}{4}t}+9e^{\frac{3}{2}t}}{16\left(e^t+e^{\frac{3}{4}t}\right)^3}>0,$$

and hence, $f(t)$ is strictly convex on $\Re$.

If $x=\ln\left(\dfrac{b}{a}\right)$, $y=\ln\left(\dfrac{b}{a}\right)$, and $z=\ln\left(\dfrac{b}{a}\right)$, then

$$x+y+z=\ln\left(\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{a}{c}\right)=\ln 1=0.$$

By Jensen's Theorem,

$$\begin{aligned}
\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \quad &= \quad \frac{1}{\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^{3/4}}+\frac{1}{\left(\frac{c}{b}\right)+\left(\frac{c}{b}\right)^{3/4}}+\frac{1}{\left(\frac{a}{c}\right)+\left(\frac{a}{c}\right)^{3/4}}\\
&= \quad f(x)+f(y)+f(z)\\
&\geq \quad 3f\left(\frac{x+y+z}{3}\right)
\end{aligned}$$

$$= \quad 3f(0)$$
$$= \quad \frac{3}{2}.$$

Further, equality is attained if, and only if, $x = y = z = 0$, i.e., if, and only if, $a = b = c$.

Solution 4 by Shai Covo, Kiryat-Ono, Israel

Let us first represent b and c as $b = xa$ and $c = yxa$, where x and y are arbitrary positive real numbers. By doing so, the original inequality becomes

$$\frac{1}{x + x^{3/4}} + \frac{1}{y + y^{3/4}} + \frac{yx}{1 + (yx)^{1/4}} \geq \frac{3}{2}. \qquad (1)$$

Let us denote by $f(x, y)$ the expression on the left-hand side of this inequality. Clearly, $f(x, y)$ has a global minimum at some point $(\alpha, \beta) \in (0, \infty) \times (0, \infty)$, a priori not necessarily unique. This point is, in particular, a critical point of f; that is, $f_x(\alpha, \beta) = f_y(\alpha, \beta) = 0$, where f_x and f_y denote the partial derivatives of f with respect to x and y. Calculating derivatives, the conditions $f_x(\alpha, \beta) = 0$ and $f_y(\alpha, \beta) = 0$ imply that

$$\begin{cases} \dfrac{1 + \frac{3}{4}\alpha^{-1/4}}{\left(\alpha + \alpha^{3/4}\right)^2} = \dfrac{\beta\left[1 + \frac{3}{4}(\beta\alpha)^{1/4}\right]}{\left[1 + (\beta\alpha)^{1/4}\right]^2} \text{ and} \\ \\ \dfrac{1 + \frac{3}{4}\beta^{-1/4}}{\left(\beta + \beta^{3/4}\right)^2} = \dfrac{\alpha\left[1 + \frac{3}{4}(\beta\alpha)^{1/4}\right]}{\left[1 + (\beta\alpha)^{1/4}\right]^2} \end{cases}, \qquad (2)$$

respectively. From this it follows straight forwardly, that

$$\frac{1 + \frac{3}{4}\alpha^{-1/4}}{\alpha\left(1 + \alpha^{-1/4}\right)^2} = \frac{1 + \frac{3}{4}\beta^{-1/4}}{\beta\left(1 + \beta^{-1/4}\right)^2}.$$

Writing this equality as $\varphi(\alpha) = \varphi(\beta)$ and noting that φ is strictly decreasing, we conclude (by virtue of φ being one-to-one) that $\alpha = \beta$. Substituting this into (2) gives

$$\frac{1 + \frac{3}{4}\alpha^{-1/4}}{\left(\alpha + \alpha^{3/4}\right)^2} = \frac{\alpha\left(1 + \frac{3}{4}\alpha^{1/2}\right)}{\left(1 + \alpha^{1/2}\right)^2}.$$

Comparing the numerators and denominators of this equation shows that the right-hand side is greater than the left-hand side for $\alpha > 1$, while the opposite is true for $\alpha < 1$. We conclude that $\alpha = \beta = 1$. Thus f has a unique global minimum at $(x, y) = (1, 1)$, where $f(x, y) = 3/2$. The inequality (1), and hence the one stated in the problem, is thus proved.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Tom Leong, Scotrun, PA; Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy, and the proposer.

- **5097:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $p \geq 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor},$$

where $\lfloor a \rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4 \rfloor = 2$).

Solution 1 by Paul M. Harms, North Newton, KS

Since the series is an alternating series it is important to check whether the number of terms with the same denominator is even or odd. It is shown below that the number of terms with the same denominator is an odd number.

Consider p=2. The series starts:

$$\frac{(-1)^1}{1} + \frac{(-1)^2}{1} + \frac{(-1)^3}{1} + \frac{(-1)^4}{1} + \ldots + \frac{(-1)^8}{2} + \frac{(-1)^9}{3} + \ldots$$

$$\begin{aligned} &= \frac{(-1)^3}{1} + \frac{(-1)^8}{2} \frac{(-1)^{15}}{3} + \ldots \\ &= -1 + \frac{1}{2} - \frac{1}{3} + \ldots . \end{aligned}$$

The terms with 1 in the denominator are from $n = 1^2$ up to (not including) $n = 2^2$, and the terms with 2 in the denominator come from $n = 2^2$ up to $n = 3^2$. The number of terms with 1 in the denominator is $2^2 - 1^2 = 3$ terms.

For $p = 2$ the number of terms with a positive integer m in the denominator is $(m+1)^2 - m^2 = 2m + 1$ terms which is an odd number of terms.

For a general positive integer p, the number of terms with a positive integer m in the denominator is $(m+1)^p - m^p$ terms. Either $(m+1)$ is even and m is odd or vice versa. An odd integer raised to a positive power is an odd integer, and an even integer raised to a positive power is an even integer. Then $(m+1)^p - m^p$ is the difference of an even integer and an odd integer which is an odd integer. Since, for every positive integer p the series starts with $\dfrac{(-1)^1}{1} = -1$ and we have an odd number of terms with denominator 1, the last term with 1 in the denominator is $\dfrac{-1}{1}$ and the other terms cancel out.

The terms with denominator 2 start and end with positive terms. They all cancel out except the last term of $\dfrac{1}{2}$.

Terms with denominator 3 start and end with negative terms. For every p we have the series

$$\frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots = -\ln 2.$$

Solution 2 by The Taylor University Problem Solving Group, Upland, IN

First note that the denominators of the terms of this series will be increasing natural numbers, because $\sqrt[p]{n}$ will always be a real number greater than or equal to 1 for $n \geq 1$, meaning that its floor will be a natural number. Furthermore, for a natural number a, a^p is the smallest n for which a is the denominator, because $\lfloor \sqrt[p]{a^p} \rfloor = \lfloor a \rfloor = a$. In other words, the denominator increases by 1 each time n is a perfect *pth* power. Thus, a natural number k occurs as the denominator $(k+1)^p - k^p$ times in the series. Because multiplying a number by itself preserves parity and $k+1$ and k always have opposite parity, $(k+1)^p$ and k^p also have opposite parity, hence their difference is odd. So each denominator occurs an odd number of times. Because the numberator alternates between 1 and -1, all but the last of the terms with the same denominator will cancel each other out. This leaves an alternating harmonic series with a negative first term, which converges to $-\ln 2$.

This can be demonstrated by the fact that the alternating harmonic series with a positive first term is the Mercator series evaluated at $x = 1$, and this series is simply the opposite of that.

Incidentally, this property holds for $p = 1$ as well.

Also solved by Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before September 15, 2010

- **5116:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on side AB, and with point Q on side BC such that

  $$\frac{AP}{PB}=\frac{BQ}{QC}>5.$$

  The cevians DP and DQ divide diagonal AC into three segments with each having integer length. Find those three lengths, if $AC=84$.

- **5117:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive acute angles A and B such that

  $$\sin A+\sin B=2\sin A\cdot\cos B.$$

- **5118:** *Proposed by David E. Manes, Oneonta, NY*

  Find the value of

  $$\sqrt{2011+2007\sqrt{2012+2008\sqrt{2013+2009\sqrt{2014+\cdots}}}}$$

- **5119:** *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a non-negative integer. Prove that

  $$2+\frac{1}{2^{n+1}}\prod_{k=0}^{n}\csc\left(\frac{1}{F_k}\right)<F_{n+1}$$

  where F_n is the n^{th} Fermat number defined by $F_n=2^{2^n}+1$ for all $n\geq 0$.

- **5120:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Calculate

  $$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left(\frac{2n-k}{2n+k}\right).$$

- **5121:** *Proposed by Tom Leong, Scotrun, PA*

  Let n, k and r be positive integers. It is easy to show that

$$\sum_{n_1+n_2+\cdots+n_r=n} \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k} = \binom{n+r-1}{kr+r-1}, \quad n_1, n_2, \cdots n_r \in N$$

  using generating functions. Give a combinatorial argument that proves this identity.

Solutions

- **5098:** *Proposed by Kenneth Korbin, New York, NY*

  Given integer-sided triangle ABC with $\angle B = 60^\circ$ and with $a < b < c$. The perimeter of the triangle is $3N^2 + 9N + 6$, where N is a positive integer. Find the sides of a triangle satisfying the above conditions.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Since $3n^2 + 9n + 6 = 3(n+1)(n+2) = 3K$, we can rephrase the problem as follows: Find an integer-sided triangle ABC with angle $B = 60^\circ$ and $a < b < c$ whose perimeter is the same as an equilateral triangle PBQ whose side is K.

Let us then consider a triangle derived from PBQ by lengthening PB by an integer x and shortening BQ by an integer y such that the resulting triangle still has perimeter $3K$.

Thus, we can write the following expression:

$$\text{Perimeter } (\triangle ABC) = (K+x)+(K-y)+\left[(K+x)^2 + (K-y)^2 - (K+x)(K-y)\right]^{1/2} = 3K \qquad (1)$$

Also we must make sure that,

$$(K+x)^2 + (K-y)^2 - (K+x)(K-y) = M^2, \text{ for some interger } M \qquad (2)$$

Note also that $BC < CA < AB$ since $\angle BAC < 60^\circ < \angle BCA$.

Equation (1) can be transformed into the much simpler equation,

$$xy = K(y-x) = (n+1)(n+2)(y-x) \qquad (3)$$

The most obvious solution of (3) is $x = n+1$ and $y = n+2$.

Substituting these expressions into the left hand side of (2) and simplifying, we get

$$(K+x)^2+(K-y)^2-(K+x)(K-y) = (n+1)^4+2(n+1)^3+3(n+1)^2+2(n+1)+1 \qquad (4)$$

But the right hand side of (4) is just $\left[(n+1)^2 + (n+1) + 1\right]^2$, so that (2) is satisfied when $x = n+1$ and $y = n+2$.

Hence, we have at least one solution:

$$\begin{array}{rcl} AB & = & K + x = (n+1)(n+2) + (n+1) = (n+1)(n+3) \\ BC & = & K - y = (n+1)(n+2) - (n+2) = n(n+2) \\ CA & = & (n+1)^2 + (n+1) + 1 \end{array}$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We show that the following triangle satisfies the conditions posed in the problem:

$$\begin{aligned} a &= N2 + 2N = N(N+2) \\ b &= N^2 + 3N + 3 = (N+1)(N+2) + 1 \\ c &= N^2 + 4N + 3 = (N+1)(N+3). \end{aligned}$$

But by no means does this give all acceptable triangles and we exhibit some others (and methods to produce them).

The given sum for the perimeter does have a connection to triangles: $3N^2 + 9N + 6$ is $6T_{N+1}$, the $N+1st$ triangular number!

Since $a < b < c$ are all integers, we let m and n be positive integers such that $b = a + m$ and $c = a + m + n$.

By the Law of Cosines, $b^2 = a^2 + c^2 - 2ac\cos 60^\circ = a^2 - ac + c^2$. Replacing $b = a + m$ and $c = a + m + n$ we get

$$\begin{aligned} &(a+m)^2 = a^2 - a(a+m+n) + (a+m+n)^2 \text{ or} \\ &-am + an + n^2 + 2mn = 0. \qquad (1) \end{aligned}$$

Likewise, substituting $b = a + m$ and $c = a + m + n$ into the proscribed perimeter conditions produces

$$3a + 2m + n = 3(N+1)(N+2). \qquad (2)$$

From equation (1), we have $am = n(a + 2m + n)$; and from this we see that n must be a factor of am. There are many ways for this to happen, but the simplest possible is that $n|s$ or $n|m$.

Case I: $a = nA$.

Then

$$\begin{aligned} nAm &= n(nA + 2m + n), \\ Am &= nA + 2m + n, \text{ or} \\ (A-2)m &= n(A+1). \qquad (1a) \end{aligned}$$

The simplest possible solution to Equation (1a) is

$$\begin{aligned} n &= A - 2 \\ m &= A + 1 \end{aligned}$$

In this case, equation (2) becomes

$$\begin{aligned} 3nA + 2m + n &= 3\,(N+1)\,(N+2)\,, \\ 3(A-2)A + 2(A+1) + (A-2) &= 3(N+1)(N+2), \\ 3A^2 - 3A &= 3(N+1)(N+2), \text{ or} \\ (A-1)A &= (N+1)(N+2). \end{aligned}$$

Because $A - 1$ and A are consecutive integers, as are $N + 1$ and $N + 2$, we must have $A = N + 2$ (so $n = N$ and $m = N + 3$). It then follows that

$$a = nA = N(N+2)$$

$$
\begin{aligned}
b &= a+m = N(N+2)+(N+3) = N^2+3N+3 \\
c &= a+m+n = N(N+2)+(N+3)+N = N^2+4N+3.
\end{aligned}
$$

It is straightforward to check that such a, b, c satisfy equations (1) and (2). Here are the first few solutions:

N	a	b	c
1	3	7	8
2	8	13	15
3	15	21	24
4	24	31	35
5	35	43	48
6	48	57	63
7	63	73	80
8	80	91	99
9	99	111	120
10	120	133	143
11	143	157	168
12	168	183	195
13	195	211	224
14	224	241	255
15	225	273	288

There are more solutions to the equation $(1a) : (A-2)m = n(A+1)$. For instance, we could look for solutions with $m = d(A+1)$ and $n = d(A-2)$, with $d > 1$. In this case, equation (2) becomes $dA(A-1) = (N+1)(N+2)$ which is quadratic in A. By varying d (and using Excel) we find more solutions:

d	N	a	b	c
2	2	6	14	16
2	19	390	422	448
3	8	72	93	105
3	34	1197	1263	1320
5	4	15	35	40
5	13	175	215	240
5	98	9675	9905	10120

Note that these solutions are scalar multiples of the (fundamental?) solutions found above. Many more solutions are possible.

Case II: $n|m$, or $m = nC$.

In this case, $am = n(a+2m+n)$ becomes

$$
\begin{aligned}
anC &= n(a+2nC+n) \text{ or} \\
aC &= a+2nC+n \text{ or} \\
(C-1)a &= n(2C+1) \qquad (1b)
\end{aligned}
$$

Once again, there are many ways to find solutions to this, but no general solution valid for all values of N. We stop by giving one more: with $N = 54$ we find $a = 231, b = 4449, c = 4560$.

Also solved by Brian D. Beasley, Clinton, SC; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; David C. Wilson, Winston-Salem, NC, and the proposer.

- **5099:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $d-1$ bisects two of its sides.

Solution 1 by Boris Rays, Brooklyn, NY

Let O be the center of the inscribed equilateral triangle ABC. Let the intersection of the altitude from vertex A with side BC be F; from vertex B with side AC be H, and from vertex C with side AB be E . Since $\triangle ABC$ is equilateral, AF, BH, and CE are also the respective angle bisectors, perpendicular bisectors, and medians of the equilateral triangle, and $AH = HC = CE = FB = BE = EA$.

Let line segment EF be extended in each direction, intersecting BH at K, and the circumscribing circle of $\triangle ABC$ at points D and G, where D is on the minor arc$\widehat{AB}$ and G is on the minor arc $\widehat{BC}$. Note that points D, E, K, F, and G lie on line segment $\overline{DG}$ and that $AO = OG$. Also note, by the givens of the problem, that

$$\begin{aligned} DG &= d-1 \text{ and} \\ AO &= BO = CO = r = \frac{d}{2}, \qquad (1) \end{aligned}$$

where r and d are correspondingly the radius and diameter of the circumscribed circle.

$$\begin{aligned} BH &\perp AC, AH = HC, \angle BAO = \angle OAH = 30^\circ. \\ OH &= \frac{1}{2}AO = \frac{d}{4}. \\ AH &= \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{d}{4}\right)^2} = \frac{d}{4}\sqrt{3}. \\ AC &= 2AH = \frac{d}{2}\sqrt{3} \qquad (2) \end{aligned}$$

The perimeter P of triangle $\triangle ABC$ will be

$$\begin{aligned} P &= 3 \cdot AC = \frac{3}{2}\sqrt{3}d. \qquad (3) \\ BK &= \frac{1}{2}BH = \frac{1}{2} \cdot 3 \cdot OH = \frac{3}{8}d. \\ KO &= BO - BK = \frac{d}{2} - \frac{3}{8}d = \frac{d}{8}. \\ GK &= \frac{1}{2}DG = \frac{d-1}{2}. \end{aligned}$$

Triangle $\triangle GKO$ is a right triangle with $DG \perp BH$ and $GK \perp BO$. Therefore,

$$GO^2 = GK^2 + KO^2 \qquad (4)$$

Substituting the values of the component parts of $\triangle GKO$ into (4),

$$GO = r = \frac{d}{2},\ GK = \frac{d-1}{2},\ KO = \frac{d}{8},$$

we obtain

$$\left(\frac{d-1}{2}\right)^2 - \left(\frac{d}{8}\right)^2 = \left(\frac{d}{2}\right)^2. \qquad (5)$$

Simplifying the last equation (5) we find that $d = 4\cdot\left(4+\sqrt{15}\right)$. Therefore,

$$\begin{aligned} AC &= \frac{4(4+\sqrt{15})}{2}\sqrt{3} = 2\left(4\sqrt{3}+3\sqrt{5}\right), \text{and} \\ P &= 3\cdot 2\left(4\sqrt{3}+3\sqrt{5}\right) = 24\sqrt{3}+18\sqrt{5}. \end{aligned}$$

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Solution 2 by Brian D. Beasley, Clinton, NC

We model the circle using $x^2+y^2=r^2$, where $r=d/2$, and place the triangle with one vertex at $(0,r)$, leaving the other two vertices in the third and fourth quadrants. Labeling the fourth quadrant vertex as (a,b), we have $b=r-\sqrt{3}a$ and thus $a=\sqrt{3}r/2$, $b=-r/2$. Then two of the midpoints of the triangle's sides are $\left(\frac{\sqrt{3}}{4}r, \frac{1}{4}r\right)$ and $\left(0, -\frac{1}{2}r\right)$. We find the endpoints of the chord through these two midpoints by substituting its equation, $y=\sqrt{3}x-r/2$, into the equation of the circle; the two x-coordinates of these endpoints are $x=sr$ and $x=tr$, where

$$s=\frac{\sqrt{3}+\sqrt{15}}{8} \quad \text{and} \quad t=\frac{\sqrt{3}-\sqrt{15}}{8}.$$

Hence the length of the chord is

$$\sqrt{(s-t)^2r^2+(\sqrt{3}(s-t))^2r^2} = d(s-t).$$

If the chord length is $d-k$, where $0<k<d$, then $d=k/(1-s+t)=4k(4+\sqrt{15})$. Thus the perimeter of the triangle is $P=3\sqrt{3}r=k(24\sqrt{3}+18\sqrt{5})$. For the given problem, since $k=1$, we obtain $P=24\sqrt{3}+18\sqrt{5}$.

Also solved by Michael Brozinsky, Central Islip, NY; Paul M. Harms, North Newton, KS; John Nord, Spokane, WA; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5100:** *Proposed by Mihály Bencze, Brasov, Romania*

  Prove that

$$\sum_{k=1}^{n}\sqrt{\frac{k}{k+1}}\binom{n}{k} \le \sqrt{\frac{n(2^{n+1}-n)2^{n-1}}{n+1}}$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We need the identities

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}x^k &= (1+x)^n \qquad (1) \\ \sum_{k=0}^{n}k\binom{n}{k}x^{k-1} &= n\,(1+x)^{n-1} \qquad (2) \end{aligned}$$

$$\text{and} \quad \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1} \quad = \quad \frac{(1+x)^{n+1}-1}{n+1} \qquad (3)$$

Identity (1) is the well known binomial expansion, whilst identities (2) and (3) follow respectively by differentiating and integrating (1). By the Cauchy-Schwarz inequality and putting $x = 1$ in (2) and (3) we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} &= \sum_{k=1}^{n} \left(\sqrt{k\binom{n}{k}}\right)\left(\sqrt{\binom{n}{k}\frac{1}{k+1}}\right) \\
&\leq \sqrt{\left(\sum_{k=1}^{n} k\binom{n}{k}\right)\left(\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k+1}\right)} \\
&= \sqrt{(n2^{n-1})\left(\frac{2^{n+1}-1}{n+1}-1\right)} \\
&= \sqrt{\frac{n\left(2^{n+1}-n-2\right)2^{n-1}}{n+1}},
\end{aligned}$$

and the inequality of the problem follows.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We shall prove a substantially better upper bound than the one stated in the problem. Namely, we show that

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} < \frac{n}{n+1}\left(2^{n}-\frac{1}{2}\right).$$

It is readily checked that our bound is less than the bound of $\sqrt{\dfrac{n(2^{n+1}-n)2^{n-1}}{n+1}}$ that the problem suggests; moreover, we have verified numerically that it is much tighter.

Now to the proof. The key observation is that

$$\sqrt{\frac{k}{k+1}} < 1 - \frac{1}{2(k+1)}$$

for all $k \in N$ (actually, for any real $k > 0$; its origin lies in the *mean value theorem* applied to the function $f(x) = \sqrt{x}$ and points $a = k/(k+1), b = 1$.

Thus, using the elementary identity $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ (twice), we get

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} \quad < \quad \sum_{k=1}^{n}\binom{n}{k} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+1}\binom{n}{k}$$

$$= \quad 2^n - 1 - \frac{1}{2(n+1)} \sum_{k=1}^{n} \binom{n+1}{k+1}$$

$$= \quad 2^n - 1 - \frac{2^{n+1} - (n+1) - 1}{2(n+1)}$$

$$= \quad \frac{n}{n+1}\left(2^n - \frac{1}{2}\right).$$

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

We prove the slightly more general statement

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \leq \sqrt{\frac{(2^n - 1)\left[(n-1)\,2^n + 1\right]}{n+1}}. \qquad (1)$$

To show that this implies the desired inequality, we begin by letting $P(n)$ be the statement: $2^{n+1} > (n-1)^2 + 3$. $P(1)$ is obvious and if we assume $P(n)$ is true for some $n \geq 1$, then

$$\begin{aligned} 2^{n+2} &= 2 \cdot 2^{n+1} > 2\left[(n-1)^2 + 3\right] = \left(n^2 + 3\right) + \left(n^2 - 4n + 5\right) \\ &= \left(n^2 + 3\right) + \left[(n-2)^2 + 1\right] > n^2 + 3, \end{aligned}$$

and $P(n+1)$ is also true. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

Then, for $n \geq 1$,

$$\begin{aligned} & n\left(2^{n+1} - n\right) 2^{n-1} - (2^n - 1)\left[(n-1)\,2^n + 1\right] \\ = \; & 2^{n-1}\left[2^{n+1} - (n-1)^2 - 3\right] + 1 \\ > \; & 0 \end{aligned}$$

and we have

$$(2^n - 1)\left[(n-1)\,2^n + 1\right] < n\left(2^{n+1} - n\right) 2^{n-1}. \qquad (2)$$

It follows that statement (1) implies the given inequality.

To prove statement (1), we note that since $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we get $\displaystyle\sum_{k=1}^{n} \frac{\binom{n}{k}}{2^n - 1} = 1$.

Because $f(x) = \sqrt{x}$ is concave down on $[0, \infty)$, Jensen's Theorem implies that

$$\sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n - 1} \sqrt{\frac{k}{k+1}} \leq \sqrt{\sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n - 1} \frac{k}{k+1}} = \sqrt{\frac{1}{2^n - 1} \sum_{k=1}^{n} \binom{n}{k} \frac{k}{k+1}},$$

and hence,

$$\sum_{k=1}^{n} \binom{n}{k} \sqrt{\frac{k}{k+1}} \leq \sqrt{(2^n - 1) \sum_{k=1}^{n} \binom{n}{k} \frac{k}{k+1}}. \qquad (3)$$

For $k = 1, 2, \ldots, n$,

$$\binom{n}{k}\frac{k}{k+1} = \frac{k}{n+1}\binom{n+1}{k+1}$$

and we get

$$\sum_{k=1}^{n}\binom{n}{k}\frac{k}{k+1} = \frac{1}{n+1}\sum_{k=1}^{n}k\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}. \qquad (4)$$

Finally, the Binomial Theorem yields

$$\sum_{k=0}^{n+1}\binom{n+1}{k}x^k = (1+x)^{n+1}.$$

It follows that when $x \neq 0$,

$$\sum_{k=1}^{n+1}\binom{n+1}{k}x^{k-1} = \frac{(1+x)^{n+1}-1}{x}$$

and, by differentiating,

$$\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}x^{k-2} = \frac{x(n+1)(1+x)^n - \left[(1+x)^{n+1}-1\right]}{x^2}.$$

In particular, when $x = 1$,

$$\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k} = (n+1)2^n - 2^{n+1} + 1 = (n-1)2^n + 1. \qquad (5)$$

Then, statements (3), (4), and (5) imply statement (1), which (by statement (2)) yields the desired inequality.

Also solved by G. C. Greubel, Newport News, VA, and the proposer

- **5101:** *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India*

An unbiased coin is tossed repeatedly until r heads are obtained. The outcomes of the tosses are written sequentially. Let R denote the total number of runs (of heads and tails) in the above experiment. Find the distribution of R.

Illustration: if we decide to toss a coin until we get 4 heads, then one of the possibilities could be the sequence $T\ T\ H\ H\ T\ H\ T\ H$ resulting in 6 runs.

Solution by Shai Covo, Kiryat-Ono, Israel

It is readily seen that R can be represented as

$$R = 1 + Y_1 + 2\sum_{i=2}^{r} Y_i, \qquad (1)$$

where $Y_i,\ i = 1, \ldots, r$, is a random variable equal to 1 if the i-th head follows a tail and equal to 0 otherwise. The $Y_{i's}$ are thus *independent* Bernoulli (1/2) variables, that is

$P(Y_i = 1) = P(Y_i = 0) = 1/2$. Noting that R is odd if and only if $Y_1 = 0$, and even if and only if $Y_1 = 1$, it follows straightforwardly from (1) that

$$\begin{aligned} P(R=n) &= \frac{1}{2}P\left(\sum_{i=2}^{r} Y_i = \frac{n-1}{2}\right) \text{ for } n = 1,3,\ldots,(2r-1) \text{ and} \\ P(R=n) &= \frac{1}{2}P\left(\sum_{i=2}^{r} Y_i = \frac{n-2}{2}\right) \text{ for } n = 2,4,\ldots,2r. \end{aligned} \tag{2}$$

Finally, since $\displaystyle\sum_{i=2}^{r} Y_i$ has a binomial distribution with parameters $r-1$ and $\dfrac{1}{2}$ (defined as 0 if $r = 1$), we conclude that

$$P(R=n) = \binom{r-1}{(n-1)/2}\frac{1}{2^r} \text{ for } n = 1,3,\ldots,(2r-1)$$

and

$$P(R=n) = \binom{r-1}{(n-2)/2}\frac{1}{2^r} \text{ for } n = 2,4,\ldots,2r.$$

Remark 1. More generally, if the probability of getting a head on each throw is $p \in (0,1)$, then $P(R=n)$ is given, in a shorter form, by

$$P(R=n) = \binom{r-1}{\lfloor\frac{n-1}{2}\rfloor}(1-p)^{\lfloor n/2\rfloor}p^{r-\lfloor n/2\rfloor},\ n = 1,2,\ldots,2r,$$

where $\lfloor\cdot\rfloor$ denotes the floor function. This is proved in the same way as in the unbiased case, only that now the Y_i are Bernoulli $(1-p)$ variables.

Remark 2. From (1) and the fact that $E(Y_i) = 1/2$ and $Var(Y_i) = 1/4$, we find that the expectation and variance of R are given by
$E(R) = 1 + \dfrac{1}{2} + 2(r-1)\dfrac{1}{2} = r + \dfrac{1}{2}$ and $Var(R) = \dfrac{1}{4} + 4(r-1)\dfrac{1}{4} = r - \dfrac{3}{4}$.
In the more general case of Remark 1, where $E(Y_i) = 1-p$ and $Var(Y_i) = (1-p)p$, the expectation and variance of R are given by

$$E(R) = 2(1-p)r + p \quad \text{and} \quad Var(R) = 4(1-p)pr - 3(1-p)p.$$

Also solved by David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5102:** *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain*

Let n be a positive integer and let $a_1, a_2, \cdots, a_n$ be any real numbers. Prove that

$$\frac{1}{1+a_1^2+\ldots+a_n^2} + \frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n}\frac{a_kF_k}{1+a_1^2+\ldots+a_k^2}\right)^2 \le 1,$$

where F_k represents the k^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution by Kee-Wai Lau, Hong Kong, China

By Cauchy-Schwarz's inequality and the well known identity $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ we have

$$\begin{aligned}
&\frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k F_k}{1 + a_1^2 + \ldots + a_k^2} \right)^2 \\
= &\frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \left(\frac{a_k}{1 + a_1^2 + \ldots + a_k^2} \right) F_k \right)^2 \\
\leq &\frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k^2}{(1 + a_1^2 + \ldots + a_k^2)^2} \right) \left(\sum_{k=1}^{n} F_k^2 \right) \\
= &\sum_{k=1}^{n} \frac{a_k^2}{(1 + a_1^2 + \ldots + a_k^2)^2}
\end{aligned}$$

Hence it remains for us to show that

$$\frac{1}{1 = a_1^2 + \ldots + a_n^2} + \sum_{k=1}^{n} \frac{a_k^2}{(1 + a_1^2 + \ldots + a_k^2)^2} \leq 1. \qquad (1)$$

Denote the left hand side of (1) by $f(n)$. Since $f(1) = \dfrac{1 + 2a_1^2}{1 + 2a_1^2 + a_1^4}$, so $f(1) \leq 1$.

Now

$$\begin{aligned}
&f(m+1) - f(m) \\
= &\frac{1}{1 + a_1^2 + \ldots + a_{m+1}^2} + \frac{a_{m+1}^2}{(1 + a_1^2 + \ldots + a_{m+1}^2)^2} - \frac{1}{1 + a_1^2 + \ldots + a_m^2} \\
= &\frac{(1 + a_1^2 + \ldots + a_m^2)(1 + a_1^2 + \ldots + a_m^2 + 2a_{m+1}^2) - (1 + a_1^2 + \ldots + a_{m+1}^2)^2}{(1 + a_1^2 + \ldots + a_{m+1}^2)^2 (1 + a_1^2 + \ldots + a_m^2)} \\
= &- \frac{a_{m+1}^4}{(1 + a_1^2 + \ldots + a_{m+1}^2)^2 (1 + a_1^2 + \ldots + a_m^2)} \\
\leq &\ 0,
\end{aligned}$$

so in fact $f(n) \leq 1$ for all positive integers n. Thus (1) holds and this completes the solution.

Also solved by the proposers.

- **5103:** *Proposed by Roger Izard, Dallas, TX*

A number of circles of equal radius surround and are tangent to another circle. Each of the outer circles is tangent to two of the other outer circles. No two outer circles intersect in two points. The radius of the inner circle is a and the radius of each outer circle is b. If

$$a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0,$$

determine the number of outer circles.

Solution by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo, TX

Let $\mathcal{C}_{\mathcal{A}}$ be the inner circle centered at point A with radius a. Similarly, let $\mathcal{C}_{\mathcal{B}}$ be a fixed outer circle centered at point B with radius b. Circle $\mathcal{C}_{\mathcal{B}}$ is tangent to two other outer circles; let T_1 and T_2 be these points of tangency. Then,

$$\overline{BT_1} \perp \overline{AT_1} \quad \text{and} \quad \overline{BT_2} \perp \overline{AT_2}.$$

If θ is the measure of $\angle T_1AT_2$, then $0^\circ < \theta < 180^\circ$. Further, triangle T_1AB is a right triangle where

$$m\angle T_1AB = \frac{\theta}{2}, \quad T_1B = b, \quad \text{and} \quad AB = a + b$$

which yields

$$\sin\left(\frac{\theta}{2}\right) = \frac{b}{a+b}. \qquad (1)$$

The given condition $a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0$ implies that

$$\begin{aligned}
a^4 + 4a^3b + b^4 &= 10a^2b^2 + 28ab^3 \\
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 &= 16a^2b^2 + 32ab^3 \\
(a+b)^4 &= 16b^2(a^2 + 2ab) \\
(a+b)^4 &= 16b^2(a^2 + 2ab + b^2) - 16b^4 \\
(a+b)^4 &= 16b^2(a+b)^2 - 16b^4 \\
1 &= \frac{16b^2(a+b)^2 - 16b^4}{(a+b)^4} \\
1 &= 16\left(\frac{b}{a+b}\right)^2 - 16\left(\frac{b}{a+b}\right)^4.
\end{aligned}$$

By equation (1) and the half-angle formula, $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$, it follows that:

$$\begin{aligned}
1 &= 16\left(\frac{1-\cos\theta}{2}\right) - 16\left(\frac{1-\cos\theta}{2}\right)^2 \\
1 &= 8(1-\cos\theta) - 4(1-\cos\theta)^2 \\
1 &= 4 - 4\cos^2\theta
\end{aligned}$$

$$\begin{aligned}\cos^2\theta &= \frac{3}{4}\\ \cos\theta &= \pm\frac{\sqrt{3}}{2}\\ \theta &= 30^\circ \text{ or } 150^\circ.\end{aligned}$$

Since the number of outer circles is $\dfrac{360^\circ}{\theta}$, then $\theta = 30^\circ$ and there must be 12 outer circles.

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** observed in their solution that "the circle passing through the centers of the outer bracelet of circles has circumference almost equal, but slightly larger than, the perimeter of the regular polygon determined by these centers: $2\pi(a+b) \approx n(2b)$. Thus $n \approx \dfrac{a+b}{b}\pi$ (in fact, n must be slightly smaller than $\dfrac{a+b}{b}\pi$)."

They went on to say that since

$$\begin{aligned}a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 &= 0,\\ \frac{a^4}{b^4} + \frac{4a^3b}{b^4} - \frac{10a^2b^2}{b^4} - \frac{28ab^3}{b^4} + \frac{b^4}{b^4} &= 0, \text{ implies}\\ x^4 + 4x^3 - 10x^2 - 28x + 1 &= 0, \text{ where } x = \frac{a}{b}.\end{aligned}$$

Therefore, $\dfrac{a}{b} = \sqrt{6} \pm \sqrt{2} - 1$, and since $n \approx \dfrac{a+b}{b}\pi$, $n = 12$. But then they went further. The equation $\sin\left(\dfrac{\pi}{n}\right) = \dfrac{b}{a+b} = \dfrac{1}{1+\frac{a}{b}}$, provides the link between n and the ratio $\dfrac{a}{b}$; we can solve for either:

$$n = \frac{\pi}{\sin^{-1}\left(\dfrac{1}{1+a/b}\right)} \quad\text{and}\quad \frac{a}{b} = \frac{1}{\sin(\pi/n)} - 1.$$

The problem poser cleverly embedded a nice ratio for $\dfrac{a}{b}$ in the fourth degree polynomial; nice in the sense that the n turned out to be an integer. In fact, the graph of the increasing function $y = \dfrac{\pi}{\sin^{-1}\left(\dfrac{1}{1+r}\right)}$ is continuous and increasing for the positive ratio r. Thus **any** lager value of n is uniquely attainable (given the correct choice of $r = \dfrac{a}{b}$).

Or we can reverse the process: fix the number of surrounding circles and calculate $r = \dfrac{a}{b}$.

A nice example (by letting $b = 1$): if we want to surround a circle with a bracelet of 100 unit circles, how large should it be? Answer:

$$\text{radius } = a = \frac{a}{1} = \frac{1}{\sin\frac{\pi}{100}} - 1 = 30.836225.$$

Also solved by Michael Brozinsky, Central Islip, NY; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; Kenneth

Korbin, New York, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; The Taylor University Problem Solving Group, Upland, IN, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before December 15, 2010

- **5122:** *Proposed by Kenneth Korbin, New York, NY*

  Partition the first 32 non-negative integers from 0 to 31 into two sets A and B so that the sum of any two distinct integers from set A is equal to the sum of two distinct integers from set B and vice versa.

- **5123:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles triangle ABC with $\overline{AB} = \overline{BC} = 2011$ and with cevian $\overline{BD}$. Each of the line segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ have positive integer length with $\overline{AD} < \overline{CD}$.

  Find the lengths of those three segments when the area of the triangle is minimum.

- **5124:** *Proposed by Michael Brozinsky, Central Islip, NY*

  If $n > 2$ show that $\displaystyle\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2}$.

- **5125:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

  $$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{11}{32}.$$

- **5126:** *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be positive real numbers and $f : [a, b] \to [c, d]$ be a function such that $|f(x) - f(y)| \geq |g(x) - g(y)|$, for all $x, y \in [a, b]$, where $g : R \to R$ is a given injective function, with $g(a), g(b) \in \{c, d\}$.

  Prove

  (i) $f(a) = c$ and $f(b) = d$, or $f(a) = d$ and $f(b) = c$.

  (ii) If $f(a) = g(a)$ and $f(b) = g(b)$, then $f(x) = g(x)$ for $a \leq x \leq b$.

- **5127:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be an integer and let $T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \dfrac{x^{2k-1}}{(2k-1)!}$, denote the $(2n-1)$th Taylor polynomial of the sine function at 0. Calculate

$$\int_0^{\infty} \frac{T_n(x) - \sin x}{x^{2n+1}} dx.$$

Solutions

- **5104:** *Proposed by Kenneth Korbin, New York, NY*

  There are infinitely many primitive Pythagorean triangles with hypotenuse of the form $4x^4 + 1$ where x is a positive integer. Find the dimensions of all such triangles in which at least one of the sides has prime integer length.

  Solution by Brian D. Beasley, Clinton, SC

  It is well-known that a primitive Pythagorean triangle (a, b, c) satisfies $a = 2st$, $b = s^2 - t^2$, and $c = s^2 + t^2$ for integers $s > t > 0$ of opposite parity with $\gcd(s, t) = 1$. Then a is never prime. Letting x be a positive integer and taking

$$c = 4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1),$$

  we see that c can only be prime if $2x^2 - 2x + 1 = 1$, meaning $x = 1$. Thus $s = 2$ and $t = 1$, which produces the triangle $(4, 3, 5)$. Similarly, $b = (s + t)(s - t)$ can only be prime if $s - t = 1$, which would yield

$$4x^4 + 1 = 2t^2 + 2t + 1 \quad \text{and hence} \quad 2x^4 = t(t + 1).$$

  But this would force one of the consecutive positive integers t or $t + 1$ to be a fourth power and the other to be twice a fourth power, meaning $t = 1$. Once again, our only solution is the triangle $(4, 3, 5)$.

  Also solved by Paul M. Harms, North Newton, KS; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer

- **5105:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation

$$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5}$$

  if x and y are of the form $a + b\sqrt{5}$ where a and b are positive integers.

  Solution by Shai Covo, Kiryat-Ono, Israel

  We let $x = a + b\sqrt{5}$ and $y = c + d\sqrt{5}$, with $a, b, c, d \in N$. Since the solution of

$$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5} \qquad (1)$$

  is symmetric in x and y, it suffices to consider the case $x \leq y$. Hence, we let $y = x\alpha$ with $\alpha \geq 1$. Substituting into (1) gives

$$(a + b\sqrt{5})\left[(1 + \alpha) - \sqrt{1 + \alpha + \alpha^2}\right] = 2 + \sqrt{5} \qquad (2)$$

It is immediately verified by taking the derivative that the function $\varphi(\alpha) = (1+\alpha) - \sqrt{1+\alpha+\alpha^2}$ is increasing . From $\varphi(\alpha)\left[(1+\alpha)+\sqrt{1+\alpha+\alpha^2}\right] = \alpha$ it is readily seen that $\varphi(\alpha) \to \dfrac{1}{2}$ as $\alpha \to \infty$. On the other hand, $\varphi(1) = 2 - \sqrt{3}$. We thus conclude from (2) that

$$4 + 2\sqrt{5} < a + b\sqrt{5} \le \frac{2+\sqrt{5}}{2-\sqrt{3}}.$$

We verify numerically that this leaves us with the following set of pairs (a,b):

$$\begin{array}{ccccccc}
\{(1,4), & (1,5), & (1,6), & (2,3), & (2,4), & (2,5), & (2,6), \\
(3,3), & (3,4), & (3,5), & (4,3), & (4,4), & (4,5), & (5,2), \\
(5,3), & (5,4), & (6,2), & (6,3), & (6,4), & (7,1), & (7,2), \\
(7,3), & (8,1), & (8,2), & (8,3), & (9,1), & (9,2), & (9,3), \\
(10,1), & (10,2), & (11,1), & (11,2), & (12,1), & (13,1)\}. &
\end{array}$$

It follows straightforwardly from (1) that

$$y = \frac{\left(4+2\sqrt{5}\right)x - 9 - 4\sqrt{5}}{x - 4 - 2\sqrt{5}}.$$

Substituting $x = a + b\sqrt{5}$ and multiplying the numerator and denominator on the right hand side by $(a-4)-(b-2)\sqrt{5}$ gives, after some algebra,

$$y = \frac{4a^2 - 5a + 20b - 20b^2 - 4}{(a-4)^2 - 5(b-2)^2} + \frac{2a^2 - 4a + 13b - 10b^2 - 2}{(a-4)^2 - 5(b-2)^2}\sqrt{5}. \qquad (3)$$

This determines the constants c and d forming y in an obvious manner, since $a, b, c, d \in N$. In particular, we see that

$$c - 2d = \frac{3a - 6b}{(a-4)^2 - 5(b-2)^2}. \qquad (4)$$

From this, noting that $c - 2d$ is an integer, it follows readily that a and b cannot be both odd; furthermore if a and b are both even, then a must be divisible by 4. This restricts the set of all possible pairs (a, b)given above to

$$\begin{array}{ccccccc}
\{(1,4), & (1,6), & (2,3), & (2,5), & (3,4), & (4,3), & (4,4), \\
(4,5), & (5,2), & (5,4), & (6,3), & (7,2), & (8,1), & (8,2), \\
(8,3), & (9,2), & (10,1), & (11,2), & (12,1)\}. & &
\end{array}$$

The requirement that the right-handside of (4) be an integer further restricts the set to

$$\{(2,3),\ (5,2),\ (6,3),\ (7,2)\}.$$

With these values of a and b, calculating c and d according to (3) give the following x, y pairs:

$$x = 2 + 3\sqrt{5}, \qquad y = 118 + 53\sqrt{5}$$

$$x = 5 + 2\sqrt{5}, \qquad y = 31 + 14\sqrt{5}$$

$$x = 6 + 3\sqrt{5}, \qquad y = 10 + 5\sqrt{5}$$

$$x = 7 + 2\sqrt{5}, \qquad y = 13 + 6\sqrt{5}.$$

Substituting into (1) show that these x, y pairs constitute the solution of (1) for $x \leq y$. The complete solution then follows by symmetry in x and y.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro,GA, and the proposer.

- **5106:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let a, b, and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC, CA, and AB respectively.

Show that
$$d_a + d_b + d_c \leq \frac{3}{4}D$$
where D is the diameter of the circumcircle.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

From the published solution of SSM problem 5066 and Gerretsen and Euler's inequality, we have that

$$d_a + d_b + d_c = \frac{r^2 + s^2 - 4R^2}{2R} \leq \frac{r^2 + 4Rr + 3r^2}{2R} = \frac{2}{R}(r + R) \leq 1\left(\frac{R}{2} + R\right) = \frac{3}{4}D,$$

with equality if and only if $\triangle ABC$ is equilateral.

Solution 2 by Ercole Suppa, Teramo, Italy

Let H_a, H_b, H_c be the feet of A, B, C onto the sides BC, CA, AB respectively and let R be the circumradius of $\triangle ABC$. We have

$$d_a = BH_a \cdot \tan(90^\circ - C) = c \cos B \cot C.$$

Hence, taking into account the extended sine law, we get

$$d_a = 2R \sin C \cos B \cot C = 2R \cos B \cos C. \qquad (1)$$

Now, by using (1) and its cyclic permutations, the given inequality rewrites as

$$\begin{aligned} 2R\cos B\cos C + 2R\cos C\cos A + 2R\cos A\cos B &\leq \frac{3}{4}\cdot 2R \\ \cos B\cos C + \cos C\cos A + \cos A\cos B &\leq \frac{3}{4} \qquad (2) \end{aligned}$$

which is true. In fact, from the well known formulas

$$\sum \cos^2 A = 1 - 2\cos A\cos B\cos C$$

and
$$0 \leq \cos A\cos B\cos C \leq \frac{1}{8},$$

each of which is valid for an acute-angled triangle, we immediately obtain

$$\sum \cos^2 A \geq \frac{3}{4}. \qquad (3)$$

Hence, by applying the known inequality

$$1 < \cos A + \cos B + \cos C \leq \frac{3}{2},$$

we obtain

$$\begin{aligned}
(\cos A + \cos B + \cos C)^2 &\leq \frac{9}{4} \Rightarrow \\
\sum \cos^2 A + 2\sum \cos B \cos C &\leq \frac{9}{4} \Rightarrow \\
2\sum \cos B \cos C \leq \frac{9}{4} - \sum \cos^2 A &\leq \frac{9}{4} - \frac{3}{4} = \frac{3}{2} \Rightarrow \\
\sum \cos B \cos C &\leq \frac{3}{4},
\end{aligned}$$

and the conclusion follows. Equality holds for $a = b = c$.

Also solved by Scott H. Brown, Montgomery, AL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China;

- **5107:** *Proposed by Tuan Le (student, Fairmont, H.S.), Anaheim, CA*

Let a, b, c be positive real numbers. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} + \frac{\sqrt{b^3+c^3}}{b^2+c^2} + \frac{\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}$$

Solution by Kee-Wai Lau, Hong Kong, China

By the Cauchy-Schwarz inequality, we have

$$a^2+b^2 \leq \sqrt{(a+b)\left(a^3+b^3\right)},\ b^2+c^2 \leq \sqrt{(b+c)\left(b^3+c^3\right)},\ c^2+a^2 \leq \sqrt{(c+a)\left(c^3+a^3\right)}.$$

Hence it suffices to show that

$$\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}} \text{ or}$$

$$\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} \geq \frac{6(ab+bc+ac)}{(a+b+c)}.$$

By the arithmetic mean-geometric mean-harmonic inequalities, we have

$$\begin{aligned}
&\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} \\
\geq\ & 3\sqrt[3]{(a+b)(b+c)(c+a)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{9}{\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}} \\
&= \frac{9(a+b)((b+c)(c+a)}{a^2+b^2+c^2+3(ab+bc+ca)}.
\end{aligned}
$$

It remains to show that

$$3(a+b+c)(a+b)(b+c)(c+a) \geq 2(ab+bc+ca)\left(a^2+b^2+c^2+3\,(ab+bc+ca)\right).$$

But this follows from the fact that

$$
\begin{aligned}
& 3(a+b+c)(a+b)(b+c)(c+a)-2(ab+bc+ca)\left(a^2+b^2+c^2+3(ab+bc+ca)\right) \\
= \;& a^3b+ab^3+a^3c+ac^3+b^3c+bc^3-2a^2bc-2ab^2c-2abc^2 \\
= \;& a(b+c)(b-c)^2+b(c+a)(c-a)^2+c(a+b)(a-b)^2 \\
\geq \;& 0,
\end{aligned}
$$

and this completes the solution.

Also solved by Pedro H.O. Pantoja (student, UFRN), Natal, Brazil; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

- **5108:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute
  $$\lim_{n\to\infty}\frac{1}{n}\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right].$$

  Solution 1 by Ovidiu Furdui, Cluj, Romania

  The limit equals 4. A calculation shows that

$$
\begin{aligned}
\arctan\left(1+\frac{2}{k(k+1)}\right) &= \arctan(1)+\arctan\frac{1}{k^2+k+1}, \\
&= \frac{\pi}{4}+\arctan\frac{1}{k}-\arctan\frac{1}{k+1}.
\end{aligned}
$$

And it follows that

$$
\begin{aligned}
\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right) &= (4n+1)\frac{\pi}{4}+\arctan 1-\arctan\frac{1}{4n+2} \\
&= (4n+1)\frac{\pi}{4}+\arctan\frac{4n+1}{4n+3}.
\end{aligned}
$$

Thus,

$$\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right] = \tan\left((4n+1)\frac{\pi}{4}+\arctan\frac{4n+1}{4n+3}\right)$$

$$= \frac{\tan((4n+1)\frac{\pi}{4}) + \frac{4n+1}{4n+3}}{1 - \tan((4n+1)\frac{\pi}{4})\frac{4n+1}{4n+3}}$$

$$= \frac{1 + \frac{4n+1}{4n+3}}{1 - \frac{4n+1}{4n+3}}$$

$$= 4n+2.$$

So the limit equals 4, and the problem is solved.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We will show that

$$\lim_{n\to\infty} \frac{1}{n} \tan\left[\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right)\right] = 4. \qquad (1)$$

From the identity $\tan(x + m\pi) = \tan(x)$, m integer, it follows that the equality (1) will be proved if we show that

$$\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right) = \arctan(4n+2) + m\pi, \qquad (2)$$

for some integer m. In fact, as we will see at the end, the m in (2) is equal to n. We first prove the following lemma.

Lemma. Define a sequence $(a_k)_{k\geq 1}$ recursively by $a_1 = 2$ and, for $k \geq 2$,

$$a_k = \frac{a_{k-1} + \left(1 + \frac{2}{k(k+1)}\right)}{1 - a_{k-1}\left(1 + \frac{2}{k(k+1)}\right)}. \qquad (3)$$

If $a_{k-1} = k$, for some $k \geq 2$, then

$$a_k = -\frac{k+2}{k},\ a_{k+1} = -\frac{1}{k+2},\ a_{k+2} = \frac{k+2}{k+4},\ a_{k+3} = k+4.$$

Hence, in particular, $a_{4n+1} = 4n+2$ for all $n \geq 0$.

Proof. Suppose that $a_{k-1} = k, k \geq 2$. Substituting this into (3) gives

$$a_k = -\frac{(k^2+1)(k+2)}{(k^2+1)k} = -\frac{k+2}{k}. \qquad (4)$$

From (3) and (4) we find

$$a_{k+1} = -\frac{k^2+2k+2}{(k^2+2k+2)(k+2)} = -\frac{1}{k+2}. \qquad (5)$$

From (3) and (5) we find

$$a_{k+2} = \frac{(k^2+4k+5)(k+2)}{(k^2+4k+5)(k+4)} = \frac{k+2}{k+4}. \qquad (6)$$

Finally, from (3) and (6) we find

$$a_{k+3} = \frac{(k^2+6k+10)(k+4)}{k^2+6k+10} = k+4.$$

The lemma is thus established.

We make use of the addition formula for arctan:

$$\arctan(x)+\arctan(y) = \begin{cases} \arctan\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy<1, \\ \arctan\left(\dfrac{x+y}{1-xy}\right)+\pi\text{sign}(x), & \text{if } xy>1, \end{cases} \qquad (7)$$

the case where $xy=1$ being irrelevant here. Now, let $(a_k)_{k\ge 1}$ be the sequence defined in the above lemma. If follows readily from (7) and the lemma that, for all $k\ge 2$,

$$\arctan(a_{k-1}) + \arctan\left(1+\frac{2}{k(k+1)}\right) = \arctan(a_k)+\pi\sigma_k,$$

where $\sigma_k = 1$ or 0 accordingly, as k is or is not of the form $k=4j+2, j\ge 0$ integer. From this it follows that

$$\sum_{k=1}^{l}\arctan\left(1+\frac{2}{k(k+1)}\right) = \arctan(a_l)+\pi\sum_{k=1}^{l}\sigma_k.$$

Recalling the conclusion in the lemma, it thus follows that (2) holds with $m=n$, and so we are done.

Remark: From (2), where $m=n$, and the fact that

$$\begin{aligned}\int \arctan\left(1+\frac{2}{x(x+1)}\right)dx &= \frac{1}{2}\log(x^2+1)-\frac{1}{2}\log(x^2+2x+2) \\ &+ \arctan\left(1+\frac{2}{x(x+1)}\right)+\arctan(x+1)+C,\end{aligned}$$

it follows readily the following interesting result:

$$\int_0^{4n+1}\arctan\left(1+\frac{2}{x(x+1)}\right)dx - \sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right) \longrightarrow \frac{1}{2}\log 2, \text{as } n\to\infty.$$

Also solved by Kee-Wai, Hong Kong, China, and the proposer.

- **5109:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k\ge 1$ be a natural number. Find the value of

$$\lim_{n\to\infty}\frac{(k\sqrt[n]{n}-k+1)^n}{n^k}.$$

Solution 1 by Angel Plaza and Sergio Falcon, Las Palmas de Gran Canaria, Spain

Let

$$
\begin{aligned}
x_n &= \frac{(k\sqrt[n]{n}-k+1)^n}{n^k}. \quad \text{Then,}\\
\ln x_n &= \ln(k\sqrt[n]{n}-k+1)^n - \ln n^k = n\ln(k\sqrt[n]{n}-k+1) - k\ln n\\
&= n\left(\ln(k\sqrt[n]{n}-k+1) - k\ln\sqrt[n]{n}\right)\\
&= \frac{\ln\dfrac{k\sqrt[n]{n}-k+1}{(\sqrt[n]{n})^k}}{\dfrac{1}{n}} \approx \frac{\dfrac{k\sqrt[n]{n}-k+1}{(\sqrt[n]{n})^k}-1}{\dfrac{1}{n}} = \frac{k\sqrt[n]{n}-k+1-(\sqrt[n]{n})^k}{(\sqrt[n]{n})^k\,\dfrac{1}{n}}.
\end{aligned}
$$

Now, taking into account that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ and the equivalence of the infinitesimals $k(x-1)+1-x^k \approx \dfrac{k(k-1)}{2}(x-1)^2$ when $x\to 1$, we have

$$
\begin{aligned}
\lim_{n\to\infty}\ln x_n &= \lim_{n\to\infty}\frac{\dfrac{k(k-1)}{2}(\sqrt[n]{n}-1)^2}{\dfrac{1}{n}} = \frac{k(k-1)}{2}\lim_{n\to\infty}\frac{(\ln n)^2}{\dfrac{1}{n}}\\
&= \frac{k(k-1)}{2}\lim_{n\to\infty}\frac{(\ln n)^2}{n} = 0. \quad \text{Therefore,}\\
\lim_{n\to\infty} x_n &= 1.
\end{aligned}
$$

Solution 2 by Kee-Wai Lau of Hong Kong, China

As $n\to\infty$, we have $\sqrt[n]{n} = e^{\ln n/n} = 1 + \dfrac{\ln n}{n} + O\left(\dfrac{\ln^2 n}{n^2}\right)$. Since $\ln(1+x) = x + O(x^2)$ as $x\to 0$, so

$$
n\ln\left(1+k\left(\sqrt[n]{n}-1\right)\right) - k\ln n = n\left(\frac{k\ln n}{n} + O\left(\frac{\ln^2 n}{n^2}\right)\right) - k\ln n = O\left(\frac{\ln^2 n}{n}\right),
$$

where the constant implied by the last O depends at most on k. It follows that the limit of the problem equal 1, independent of k.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

Late Solution

A late solution to 5099 was received from **Charles McCracken of Dayton, OH.**

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before January 15, 2011

- **5128:** *Proposed by Kenneth Korbin, New York, NY*

  Find all positive integers less than 1000 such that the sum of the divisors of each integer is a power of two.

  For example, the sum of the divisors of 3 is 2^2, and the sum of the divisors of 7 is 2^3.

- **5129:** *Proposed by Kenneth Korbin, New York, NY*

  Given prime number c and positive integers a and b such that $a^2+b^2=c^2$, express in terms of a and b the lengths of the legs of the primitive Pythagorean Triangles with hypotenuses c^3 and c^5, respectively.

- **5130:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, calculus has not been discovered. A bride and groom start out from $A(-a,0)$ and $B(b,0)$ respectively where $a \neq b$ and $a>0$ and $b>0$ and walk at the rate of one unit per second to an altar located at the point P on line $L: y=mx$ such that the time that the first to arrive at P has to wait for the other to arrive is a maximum. Find, without calculus, the locus of P as m varies through all nonzero real numbers.

- **5131:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a,b,c be positive real numbers. Prove that

  $$\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c}\geq\frac{15}{8}.$$

- **5132:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Find all all functions $f: C \to C$ such that $f(f(z))=z^2$ for all $z \in C$.

- **5133:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be a natural number. Calculate

  $$I_n=\int_0^1\int_0^1 (x-y)^n dxdy.$$

- **5110:** *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with an interior point P and with coordinates $A(0,0), B(6,8)$, and $C(21,0)$. The distance from point P to side $\overline{AB}$ is a, to side $\overline{BC}$ is b, and to side $\overline{CA}$ is c where $a:b:c = \overline{AB}:\overline{BC}:\overline{CA}$.

  Find the coordinates of point P

  Solution 1 by Boris Rays, Brooklyn, NY

  From the given triangle we have $\overline{AB} = 10, \overline{BC} = 17$ and $\overline{CA} = 21$. Also $a:b:c = 10:17:21$.

  Let $a = 10t,\ b = 17t$, and $c = 21t$, where t is real number, $t > 0$. (1)

$$\text{Area } \triangle\text{ABC} = \text{Area } \triangle\text{APB} + \text{Area } \triangle\text{BPC} + \text{Area } \triangle\text{CPA}. \qquad (2)$$

Express all of the terms in (2) by using formulas in (1).

$$\begin{aligned} \frac{1}{2}\cdot 21\cdot 8 &= \frac{1}{2}\cdot 10\cdot 10t + \frac{1}{2}\cdot 17\cdot 17t + \frac{1}{2}\cdot 21\cdot 21t \\ &= \frac{1}{2}t\left(10^2+17^2+21^2\right) = \frac{1}{2}830t \end{aligned}$$

From the above we find that $t = \dfrac{84}{415} = \dfrac{2^2\cdot 3\cdot 7}{5\cdot 83}$.

The y-coordinate of point P is c, the distance to side $\overline{CA}$.

$$y_P = c = 21t = 21\cdot\frac{84}{415} = \frac{1764}{415}.$$

Let points E and F lie on side $\overline{CA}$, where $\overline{PE}\perp\overline{CA}$ and $\overline{BF}\perp\overline{CA}$.

Hence we have $\overline{PE} = C = \dfrac{42^2}{415}$, $\overline{BF} = 8$, and $\overline{AF} = 6$.

$$\text{Area } \triangle\text{APB} + \text{Area } \triangle\text{APE} + \text{Area BPEF} = \text{Area } \triangle\text{ABF}.$$

Letting $\overline{AE} = x$ we have $\overline{EF} = 6 - x$. Therefore,

$$\begin{aligned} \frac{1}{2}\cdot 10\cdot a + \frac{1}{2}\cdot x\cdot c + \frac{1}{2}\left(\overline{PE}+\overline{BF}\right)\cdot\overline{EF} &= \frac{1}{2}\overline{AF}\cdot\overline{BF} \\ \frac{1}{2}\cdot 100\cdot\frac{84}{415} + \frac{1}{2}\cdot x\cdot\frac{42^2}{415} + \frac{1}{2}\left(\frac{42^2}{415}+8\right)(6-x) &= \frac{1}{2}6\cdot 8. \end{aligned}$$

From the above equation we find x.

$$x = \frac{1}{8}\left(\frac{8400+6(42)^2}{415}\right) = \frac{2373}{415}.$$

Hence, the coordinates of point P are $\left(\dfrac{2373}{415}, \dfrac{1764}{415}\right)$.

Solution 2 by Charles McCracken, Dayton, OH

$$\overline{AB} = 10 \qquad \overline{BC} = 17 \qquad \overline{CA} = 21$$

The equations of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ are respectively,

$$4x - 3y = 0 \qquad 8x + 15y - 168 = 0 \qquad y = c$$

Then,

$$a = \frac{4x-3y}{5} \qquad b = \frac{8x+15y-168}{17} \qquad c = y$$

$$\frac{\left(\dfrac{4x-3y}{5}\right)}{y} = \frac{10}{21} \qquad \frac{\left(\dfrac{8x+15y-168}{-17}\right)}{y} = \frac{17}{21}$$

$$21\,(4x-3y) = 50y \qquad 21\,(8x+15y-168) = -289y$$

$$84x - 113y = 0 \qquad 168x + 604y = 3528$$

These last two equations give:

$$(x, y) = \left(\frac{2373}{415}, \frac{1764}{415}\right)$$

Note that P is the Lemoine point of $\triangle ABC$, that is, the intersection of the symmedians. (*Editor:* A symmedian is the reflection of a median about its corresponding angle bisector.)

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Nord, Spokane, WA; Raúl A. Simón, Santiago, Chile; Danielle Urbanowicz, Jennie Clinton, and Bill Solyst (jointly; students at Taylor University), Upland, IN; David Stone and John Hawkins (jointly), Satetesboro, GA, and the proposer.

- **5111:** *Proposed by Michael Brozinsky, Central Islip, NY.*

In Cartesianland where immortal ants live, it is mandated that any anthill must be surrounded by a triangular fence circumscribed in a circle of unit radius. Furthermore, if the vertices of any such triangle are denoted by A, B, and C, in counterclockwise order, the anthill's center must be located at the interior point P such that $\angle PAB = \angle PBC = \angle PCA$.

Show $\overline{PA} \cdot \overline{PB} \cdot \overline{PC} \leq 1$.

Solution by Kee-Wai Lau, Hong Kong, China

It is easy to check that $\angle APB = 180^\circ - B$, $\angle BPC = 180^\circ - C$, and $\angle CPA = 180^\circ - A$.

It is well known that the area of $\triangle ABC = 2R^2 \sin A \sin B \sin C$, where R is the circumradius of the triangle. Here we have $R = 1$. Since the area of $\triangle ABC$ equals the sum of the areas of triangles APB, BPC and CPA, we have

$$\text{Area } \triangle\text{ABC} = \text{Area } \triangle\text{APB} + \text{Area } \triangle\text{BPC} + \text{Area } \triangle\text{CPA}$$

$$2\sin A \sin B \sin C = \frac{1}{2}\left(\overline{PA}\cdot\overline{PB}\sin B + \overline{PB}\cdot\overline{PC}\sin C + \overline{PC}\cdot\overline{PA}\sin A\right).$$

By the arithmetic mean-geometric mean inequality, we have

$$\overline{PA}\cdot\overline{PB}\sin B + \overline{PB}\cdot\overline{PC}\sin C + \overline{PC}\cdot\overline{PA}\sin A \geq 3\left(\overline{PA}\cdot\overline{PB}\cdot\overline{PC}\right)^{2/3}(\sin A \sin B \sin C)^{1/3}.$$

It follows that

$$\left(\overline{PA}\cdot\overline{PB}\cdot\overline{PC}\right)^{2/3} \leq \frac{4}{3}(\sin A \sin B \sin C)^{2/3}. \qquad (1)$$

By the concavity of the function $\ln(\sin x)$ for $0 < x < \pi$, we obtain

$$\ln(\sin A) + \ln(\sin B) + \ln(\sin C) \leq 3\left(\sin\left(\frac{A+B+C}{3}\right)\right) = 3\ln\left(\frac{\sqrt{3}}{2}\right).$$

Therefore,

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}. \qquad (2)$$

The result $\overline{PA}\cdot\overline{PB}\cdot\overline{PC} \leq 1$ now follows easily from (1) and (2) immediately above.

Comments: The proposer, **Michael Brozinsky,** mentioned in his solution that point P is precisely the Brocard point of the triangle, and **David Stone and John Hawkins** noted in their solution that given an inscribed triangle and letting $\theta = \angle PAB = \angle PBC = \angle PCA$, then the identity

$$\sin\theta = \frac{abc}{2\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}$$

allows one to find the unique angle θ and thus sides $\overline{PA}, \overline{PB}$, and $\overline{PC}$.

Also solved by David Stone and John Hawkins (jointly), Satetesboro, GA, and the proposer.

- **5112:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

Let $0 < a < b$ be real numbers with a fixed and b variable. Prove that

$$\lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b+x}{a+x}} = \lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b(a+x)}{a(b+x)}}.$$

Solution by Shai Covo, Kiryat-Ono, Israel

We begin with the left-hand side limit. Writing $\ln\dfrac{b+x}{a+x}$ as $\ln(b+x) - \ln(a+x)$, we have by the mean value theorem that this expression is equal to $\dfrac{1}{\xi}(b-a)$ where $\xi = \xi(a,b,x)$ is some point between $(a+x)$ and $(b+x)$. Since x varies from a to b, it thus follows that

$$\frac{b-a}{2b} \leq \ln\frac{b+x}{a+x} \leq \frac{b-a}{2a}.$$

Hence,

$$2a = \int_a^b \frac{2a}{b-a}dx \le \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} \le \int_a^b \frac{2b}{b-a}dx = 2b,$$

and so

$$\lim_{b\to a} \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} = 2a.$$

Applying this technique to the computation of the right-hand side limit gives

$$\frac{a(b-a)}{ab+b^2} \le \ln \frac{b(a+x)}{a(b+x)} \le \frac{b(b-a)}{ab+a^2},$$

from which it follows immediately that also

$$\lim_{b\to a} \int_a^b \frac{dx}{\ln \dfrac{b(a+x)}{a(b+x)}} = 2a.$$

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5113:** *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \le \sqrt{xy} + \frac{x+y}{2} + \frac{\left(\dfrac{x+y}{6} - \dfrac{\sqrt{xy}}{3}\right)^2}{\dfrac{2xy}{x+y}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By homogeneity, we may assume without loss of generality that $xy = 1$. Let $t = x + y \ge 2\sqrt{xy} = 2$. Then the inequality of the problem is equivalent to

$$\begin{aligned}
& \frac{2}{t} + \sqrt{\frac{t^2-2}{2}} \le 1 + \frac{t}{2} + \frac{t(t-2)^2}{72} \\
\Leftrightarrow\quad & 36t\sqrt{2\left(t^2-2\right)} \le t^4 - 4t^3 + 40t^2 + 72t - 144 \\
\Leftrightarrow\quad & \left(t^4 - 4t^3 + 40t^2 + 72t - 144\right) - 2592t^2\left(t^2-2\right) \ge 0 \\
\Leftrightarrow\quad & t^8 - 8t^7 + 96t^6 - 176t^5 - 1856t^4 + 6912t^3 - 1152t^2 - 20376t + 20376 \ge 0 \\
\Leftrightarrow\quad & (t-2)^2\left(t^6 - 4t^5 + 76t^4 + 144t^3 - 1584t^2 + 5184\right) \ge 0.
\end{aligned}$$

Since

$$\begin{aligned} & t^6 - 4t^5 + 76t^4 + 144t^3 - 1584t^2 + 5184 \\ = \; & t^4(t-2)^2 + 72(t-2)^4 + \frac{16(3t-8)^2(15t+11) + 832}{3} > 0, \end{aligned}$$

the inequality of the problem holds.

Solution 2 by Paul M. Harms, North Newton, KS

Le $w = \dfrac{x+y}{2\sqrt{xy}}$ and $z = \sqrt{xy}$. For x and y positive

$$\left(\sqrt{x} - \sqrt{y}\right)^2 = x + y - 2\sqrt{xy} \geq 0 \Longrightarrow w = \frac{x+y}{2\sqrt{xy}} \geq 1. \text{ Also } z > 0.$$

From the substitutions we have the following expressions :

$$\begin{aligned} 2xy &= 2z^2 \\ x + y &= 2zw \\ x^2 + y^2 = (x+y)^2 - 2xy &= 4z^2w^2 - 2z^2 = 2z^2(2w^2 - 1) \end{aligned}$$

The inequality becomes

$$\frac{2z^2}{2zw} + \sqrt{\frac{2z^2\left(2w^2-1\right)}{2}} \leq z + \frac{2zw}{2} + \frac{\left(\dfrac{2zw - 2z}{6}\right)^2}{\dfrac{2z^2}{2zw}}$$

Simplifying and dividing both sides of the inequality by z yields the inequality

$$\frac{1}{w} + \sqrt{2w^2 - 1} \leq 1 + w + \frac{1}{9}\left(w-1\right)^2 w.$$

After multiplying both sides by $9w$ and isolating the square root term we get

$$9w\sqrt{2w^2-1} \leq -9 + 9w + 9w^2 + (w-1)^2w^2 = w^4 - 2w^3 + 10w^2 + 9w - 9.$$

Now let $w = L + 1$. Since $w \geq 1$, we check the resulting inequality for $L \geq 0$. Replacing w by $L+1$ and squaring both sides of the inequality we obtain

$$\begin{aligned} 81\left(L+1\right)^2\left[2L^2 + 4L + 1\right] &= 81\left(2L^4 + 8L^3 + 11L^2 + 6L + 1\right) \\ &\leq \left(L^4 + 2L^3 + 10L^2 + 27L + 9\right)^2 \\ &= L^8 + 4L^7 + 24L^6 + 94L^5 + 226L^4 + 576L^3 + 909L^2 + 486L + 81 \end{aligned}$$

Moving all terms to the right side, we need to show for $L \geq 0$, that

$$0 \leq L^2\left[L^6 + 4L^5 + 24L^4 + 94L^3 + 64L^2 - 72L + 18\right].$$

Let

$$g(L) = 94L^3 + 64L^2 - 72L + 18.$$

If $g(L) \geq 0$ for $L \geq 0$, then the inequality holds since all other terms and factors of the inequality not involved with $g(L)$ are non-negative.

The derivative $g'(L) = 2\left[141L^2 + 64L - 36\right]$. The zeroes of $g'(L)$ are $L = -0.7810$ and $L = 0.3297$ with a negative derivative between these two L values. It is easy to check that $g(0.3297) > 0$ is the only relative minimum and that $g(L) > 0$ for all $L \geq 0$. Thus the inequality holds.

A comment by the editor: **David Stone and John Hawkins of Statesboro, GA** sent in a solution path that was dependent on a computer, and this bothered them. They let $y = ax$ in the statement of the problem and then showed that the original inequality was equivalent to showing that

$$\frac{2a}{1+a} + \sqrt{\frac{1+a^2}{2}} \leq \frac{(\sqrt{a}+1)^2}{2} + \frac{(a+1)(\sqrt{a}-1)^4}{72a}.$$

They then had Maple graph the left and right hand sides of the inequality respectively; they analyzed the graphs and concluded that the inequality held (with equality holding for $a = 1$.) But this approach bothered them and so they let $a = z^2$ in the above inequality and they eventually obtained the following:

$$\begin{aligned}&(z-1)^4\Big(z^{12} - 4z^{11} + 82z^{10} + 124z^9 - 1265z^8\\ &+392z^7 + 2492z^6 + 392z^5 - 1265z^4 + 124z^3 + 82z^2 - 4z + 1\Big) \leq 0.\end{aligned}$$

Again they called on Maple to factor the above polynomial, and it did into linear and irreducible quadratic factors. They then showed that there were no positive real zeros and so the inequality must be true. They also noted that equality holds if and only if $z = 1$; that is, equality holds for the original statement if and only if $x = y$. They ended their submission with the statement:

"The bottom line: with the use of a machine's assistance, we believe the original inequality to be true."

In their letter submitting the above to me David wrote:

"Last week I mentioned that our solution to Problem 5113 was dependent upon machine help. We are still in that position, so I send this to you as a comment, not as a solution. There is a nice reduction to an inequality in a single variable, but we never found an analytic verification for the inequality."

All of this reminded me of the comments in 1976 surrounding Appel and Haken's proof of the four color problem which was done with the aid of a computer. The concerns raised then, still exist today.

Also solved by Shai Covo, Kiryat-Ono, Israel; Boris Rays, Brooklyn, NY, and the proposer.

- **5114:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let M be a point in the plane of triangle ABC. Prove that

$$\frac{\overline{MA}^2 + \overline{MB}^2 + \overline{MC}^2}{\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2} \geq \frac{1}{3}.$$

When does equality hold?

Solution by Michael Brozinsky, Central Islip, NY

Without loss of generality let the vertices of the triangle be $A(0,0), B(a,0)$, and $C(b,c)$ and let M be (x,y). Now completing the square shows

$$\overline{AM}^2+\overline{BM}^2+\overline{CM}^2-\frac{1}{3}\left(\overline{AB}^2+\overline{BC}^2+\overline{AC}^2\right)$$

$$= \left(x^2+y^2+(x-a)^2+y^2+(x-b)^2+(y-c)^2-\frac{1}{3}\left(a^2+\left(b-a\right)^2+c^2+b^2+c^2\right)\right)$$

$$= 3\cdot\left(\left(x-\frac{a+b}{3}\right)+\left(y-\frac{c}{3}\right)^2\right)$$

and thus the given inequality follows at once and equality holds iff M is $\dfrac{2}{3}$ of the way from vertex C to side $\overline{AB}$. Relabeling thus implies that M is the centroid of the triangle.

Comments in the solutions of others: 1) From Kee-Wai Lau, Hong Kong, China. The inequality of the problem can be found at the top of p. 283, Chapter XI in *Recent Advances in Geometric Inequalities* by Mitrinovic, Pecaric, and Volenec, (Kluwer Academic Press), 1989.

The inequality was obtained using the Leibniz identity

$$\overline{MA}^2+\overline{MB}^2+\overline{MC}^2=3\overline{MG}^2+\frac{1}{3}\left(\overline{AB}^2+\overline{BC}^2+\overline{CA}^2\right)$$

where G is the centroid of triangle ABC. Equality holds if and only if $M=G$.

2) From Bruno Salgueiro Fanego, Viveiro Spain. This problem was solved for any point M in space using vectors. (See page 303 in *Problem Solving Strategies* by Arthur Engel, (Springer-Verlag), 1998.) Equality holds if, and only if, M is the centroid of ABC.

Another solution and a discussion of where the problem mostly likely originated can be found on pages 41 and 42 of

$$http://www.cpohoata.com/wp-conent/uploads/2008/10/inf081019.pdf.$$

Also, a local version of the Spanish Mathematical Olympiad of 1999 includes a version of this problem and it can be seen at
$http://platea.pntic.mec.es/\sim csanchez/local99.htm.$

3) From David Stone and John Hawkins (jointly), Statesboro, GA. Because the given problem has the sum of the squares of the triangle's sides as the denominator, one might conjecture the natural generalization

$$\frac{\sum_{i=1}^{n}\overline{MA_i}^2}{\sum_{i=1}^{n}\overline{A_iA_{i+1}}^2}\geq\frac{1}{n},$$

but this is not true. Instead, we must also allow all squares of diagonals to appear in the sum in the denominator. Of course, a triangle has no diagonals.

Also solved by Shai Covo, Kiryat-Ono, Israel; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; Michael N. Fried, Kibbutz Revivim, Israel; Raúl A. Simón, Santiago, Chile, and the proposer.

- **5115:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

Let G be a finite cyclic group. Compute the number of distinct composition series of G.

Solution 1 by Kee-Wai Lau,Hong Kong, China

Denote the order of a group S by $|S|$. Let $E = G_0, < G_1 < G_2 < \ldots < G_m = G$ be a composition series for G, where E is the subgroup of G consisting of the identity element only. A composition series is possible if and only if the factor groups $G_1/G_0, G_2/G_1, \ldots, G_m/G_{m-1}$ are simple. For cyclic group G, where all these factor groups are also cyclic, this is equivalent to saying that

$$|G_1/G_0| = p_1,\ |G_2/G_1| = p_2,\ \ldots,\ |G_m/G_{m-1}| = p_m,$$

where $p_1, p_2, \ldots, p_m$ are primes, not necessarily distinct. By the Jordan-Hölder theorem, m is uniquely determined and the prime divisors, $p_1, p_2, \cdots, p_m$ themselves are unique. Any other composition series therefore correspond with a permutation of the primes $p_1, p_2, \ldots, p_m$. Note that

$$|G| = |G_m| = \frac{|G_m|}{|G_{m-1}|}\frac{|G_{m-1}|}{|G_{m-2}|}\cdots\frac{|G_2|}{|G_1|}\frac{|G_1|}{1} = p_m p_{m-1}\ldots p_2 p_1.$$

We rewrite $|G|$ in standard form $|G| = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k}$, where $a_1, a_2, \ldots, a_k$ are positive integers and $q_1 < q_2 < \ldots q_k$ are primes. The number of distinct composition series of G then equals

$$\frac{(a_1 + a_2 + \cdots + a_k)!}{a_1! a_2! \ldots a_k!}$$

Solution 2 by David Stone and John Hawkins (jointly), Statesboro, GA

Let G have order n, where n has prime factorization $n = \prod_{i=1}^{m} p_i^{e_i}$. Then the number of distinct composition series of G is the multinomial coefficient $\binom{e_1+e_2+e_3+\ldots+e_m}{e_1,e_2,e_3,\ldots,e_m}$. Letting e=e$_1 + e_2 + e_3 + \ldots + e_m$, this can be computed as

$$\binom{e}{e_1}\binom{e-e_1}{e_2}\binom{e-e_1-e_2}{e_3}\cdots\binom{e_{m-1}+e_m}{e_{m-1}}\binom{e_m}{e_m} = \frac{e!}{(e_1!)(e_2!)(e_3!)\cdots(e_m!)}.$$

Our rationale follows.

We'll simply let G be Z_n, written additively and denote the cyclic subgroup generated by a as $< a >= \{ka| \in Z\}$.

Note that $< a >$ is a subgroup of $< b >$ if and only if $a = bc$ for some c in G. We'll denote this by $< a > \leq < b >$. That is, to enlarge the subgroup $< a >$ to $< b >$, we divide a by some group element c to obtain b. In particular, if we divide a by a prime p to obtain b, then the factor group $< b > / < a >$ is isomorphic to the simple group Z_p.

In the lattice of subgroups of G, any two subgroups have a greatest lower bound, given by intersection , and a least upper bound, given by summation.The maximal length (ascending) chains are the distinct composition series. All such chain have the same length (by the Jordan-Hölder Theorem).

For a specific example, let $n = 12 = 2^2 \cdot 3^1$. In Z_{12}, the distinct subgroups are:

$$\begin{aligned}
0 &= \{0\},\\
<2> &= \{0,2,4,6,8,10\},\\
<4> &= \{0,4,8\},\\
<3> &= \{0,3,6,9\},\\
<6> &= \{0,6\},\\
<1> &= \{0,1,2,3,4,5,6,7,8,9,10,11\} = Z_{12},
\end{aligned}$$

and the maximal length ascending chains (composition series) are

$$0 \leq <4> \ \leq\ <2> \ \leq\ <1>,$$

$$0 \leq <6> \ \leq\ <2> \ \leq\ <1>,$$

$$0 \leq <6> \ \leq\ <3> \ \leq\ <1>.$$

Note that the composition factors (the simple factor groups) of the first chain are

$$\begin{aligned}
<4>/0 &\cong Z_3\\
<2>/<4> &\cong Z_2, \text{ and}
\end{aligned}$$

$$< 1 > / < 2 > \quad \cong \quad Z_2.$$

Thus, the sequence of composition factors is Z_3, Z_2, Z_2.

Similarly for the second chain, the sequence of composition factors is Z_2, Z_3, Z_2, and for the third chain the sequence of composition factors is Z_2, Z_2, Z_3. The three elements of each chain are Z_2, Z_2, and Z_3, forced by the factorization of 12. The number of possible chains is simply the number of ways to arrange these three simple groups: 3. Note that

$$\binom{2+1}{2,1} = \binom{3}{2,1} = \binom{3}{2} \cdot \binom{1}{1} = 3.$$

Method: For arbitrary $n = \prod_{i=1}^{m} p_i^{e_i}$, this example demonstrates a constructive method for generating (and counting) all such maximal chains:

(i) Start with $0 =< n >$.

(ii) Divide (in the usual sense, not mod n) by one of $n's$ prime divisors, p, to obtain $k = \frac{n}{p}$, so that $0 =< n > \leq < k >$ and the factor group $< k > / < n > \cong Z_p$.

(iii) Next, divide k by any unused prime divisor, say q of n to obtain $h = \frac{k}{q}$, so that $< k > \leq < h >$ and the factor group $< h > / < k > \cong Z_q$.

(In this process, each prime factor p will be used e_i times, so there will be $e = e_1 + e_2 + e_3 = \ldots + e_m$ steps.)

We now have the beginning of a composition series: $0 \ \leq < k > \leq < h >$. Continue with the division steps until the supply of prime divisors of n is exhausted, so the final division will produce the final element of the chain: $< 1 > = \ Z_n$. We will have thus constructed a composition series. In the procedure there will be e_1 divisions by p_1, e_2 divisions by p_2, etc.

Therefore, the number of ways to carry out this procedure is the number of ways to carry out these dvisions: choose e_1 places from e possible spots to divide by p, choose e_2 places from the remaining $e - e_1$ possible spots to divide by p_2 etc.
So we can count the total number of ways to carry out the process:

$$\binom{e}{e_1}\binom{e-e_1}{e_2}\binom{e-e_1-e_2}{e_3}\cdots\binom{e_{m-1}+e_m}{e_{m-1}}\binom{e_m}{e_m}.$$

Moreover, if we let S be the sequence of simple groups consisting of e_1 copies of Z_{p_1}, e_2 copies of Z_{p_2}, etc., then S will have $e = e_1 + e_2 + e_3 + \cdots + e_m$ elements and each of our composition series will have some rearrangement of S as its sequence of compositions factors.

Example: Let $n = 360 = 2^3 \cdot 3^2 \cdot 5^1$.
Then the sequence of divisors $3, 5, 2, 2, 3, 2$ will produce he composition series

$$0 =< 360 > \leq < 120 > \leq < 24 > \leq < 12 > \leq < 6 > \leq < 2 > \leq < 1 > = Z_{360},$$

with composition factors $Z_3, \ Z_5, \ Z_2, \ Z_2, \ Z_3, \ Z_2$.

There are $\binom{3+2+1}{3,2,1} = \binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = 60$ different ways to construct a divisors sequence from $2, 2, 2, 3, 3, 5$, so Z_{360} has 60 distinct composition series.

Also solved by the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2011

- **5134:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles $\triangle ABC$ with cevian CD such that $\triangle CDA$ and $\triangle CDB$ are also isosceles, find the value of
  $$\frac{AB}{CD} - \frac{CD}{AB}.$$

- **5135:** *Proposed by Kenneth Korbin, New York, NY*

  Find a, b, and c such that
  $$\begin{cases} ab + bc + ca = -3 \\ a^2b^2 + b^2c^2 + c^2a^2 = 9 \\ a^3b^3 + b^3c^3 + c^3a^3 = -24 \end{cases}$$
  with $a < b < c$.

- **5136:** *Proposed by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico*

  Prove that for every positive integer n, the real number
  $$\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} + \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$$
  is irrational.

- **5137:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive numbers such that $abc \geq 1$. Prove that
  $$\prod_{cyclic} \frac{1}{a^5 + b^5 + c^2} \leq \frac{1}{27}.$$

- **5138:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $n \geq 2$ be a positive integer. Prove that
  $$\frac{n}{F_nF_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_2^2} + \cdots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n}\sum_{k=1}^{n} \frac{1}{F_k^2},$$
  where F_n is the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2,\ F_n = F_{n-1} + F_{n-2}$.

- **5139:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\zeta(n+m)-1}{n+m},$$
  where ζ denotes the Riemann Zeta function.

Solutions

- **5116:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on side AB, and with point Q on side BC such that
$$\frac{AP}{PB}=\frac{BQ}{QC}>5.$$
  The cevians DP and DQ divide diagonal AC into three segments with each having integer length. Find those three lengths, if $AC=84$.

Solution by David E. Manes, Oneonta, NY

Let E and F be the points of intersection of AC with DP and DQ respectively. Then $AE=40,\ EF=37$ and $FC=7$.

Since $ABCD$ is a square with diagonal of length 84, it follows that the sides of the square have length $42\sqrt{2}$. Let $\dfrac{AP}{PB}=\dfrac{BQ}{QC}=t>5$. Then $AP=t\cdot PB$ and $AP+PB=AB=42\sqrt{2}$. Therefore,

$$\begin{aligned}PB(t+1)&=42\sqrt{2}\\ PB&=\frac{42\sqrt{2}}{1+t},\quad\text{and}\\ AP&=\frac{42\sqrt{2}\cdot t}{1+t}.\end{aligned}$$

Similarly, $QC=\dfrac{42\sqrt{2}}{1+t}$ and $BQ=\dfrac{42\sqrt{2}\cdot t}{1+t}$.

Coordinatize the problem so that

$$\begin{aligned}A&=(0,0),\quad B=(42\sqrt{2},0),\quad C=(42\sqrt{2},42\sqrt{2}),\quad D=(0,42\sqrt{2}),\\ P&=\left(\frac{42\sqrt{2}\cdot t}{1+t},0\right),\ \text{and}\quad Q=\left(42\sqrt{2},\frac{42\sqrt{2}\cdot t}{1+t}\right).\end{aligned}$$

Let L_1 be the line through the points D and P. Then the equation of L_1 is $y-42\sqrt{2}=-\left(\dfrac{1+t}{t}\right)x$. The point of intersection of L_1 and the line $y=x$ is the point E. Therefore,

$$x-42\sqrt{2}=-\left(\frac{1+t}{t}\right)x,\ \text{and so}$$

$$
\begin{aligned}
x &= \frac{42\sqrt{2}\cdot t}{2t+1}. \text{ Thus,} \\
E &= \left(\frac{42\sqrt{2}\cdot t}{2t+1}, \frac{42\sqrt{2}\cdot t}{2t+1}\right) \text{ so that} \\
AE &= \sqrt{2\left(\frac{42\sqrt{2}\cdot t}{2t+1}\right)^2} = \frac{84\cdot t}{2t+1}.
\end{aligned}
$$

Let L_2 be the line through D and Q. Then the equation of L_2 is $y - 42\sqrt{2} = -\left(\dfrac{1}{1+t}\right)x$. Since F is the point of intersection of L_2 and $y = x$, we obtain $x = \dfrac{42\sqrt{2}(t+1)}{t+2}$. Thus,

$$
\begin{aligned}
F &= \left(\frac{42\sqrt{2}(t+1)}{t+2}, \frac{42\sqrt{2}(t+1)}{t+2}\right) \text{ so that} \\
AF &= \frac{84(t+1)}{t+2}.
\end{aligned}
$$

Using the distance formula, one obtains

$$CF = \sqrt{2\left(42\sqrt{2} - \frac{42\sqrt{2}(t+1)}{t+2}\right)^2} = \frac{84}{t+2}.$$

As a result,

$$AE = \frac{84\cdot t}{2t+1},\ AF = \frac{84(t+1)}{t+2},\ \text{and } CF = \frac{84}{t+2}$$

If $t = 10$, then $AE = 40, AF = 77$, and $CF = 7$. Therefore $EF = AF - AE = 37$, yielding the claimed values. Finally, one checks that for these values all triangles in the figure are defined.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paul M. Harms, North Newton, KS; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5117:** *Proposed by Kenneth Korbin, New York, NY*

Find positive acute angles A and B such that

$$\sin A + \sin B = 2\sin A\cdot\cos B.$$

Solution by David Stone and John Hawkins (jointly), Statesboro, GA

There are infinitely many solutions, given by

$$A = \sin^{-1}\left(\frac{\sqrt{1-t^2}}{2t-1}\right),\ \mathrm{B} = \cos^{-1} t, \quad \text{where } \frac{4}{5} < t < 1.$$

Here's why.

The given condition is equivalent to

$$2\sin A(2\cos B - 1) = \sin B$$

so we see that $2\cos B - 1 > 0$, that is, $0 < B < \frac{\pi}{3}$.

Solving for $\sin A$, we must have $\sin A = \frac{\sin B}{2\cos B - 1}$, which requires $0 \le \frac{\sin B}{2\cos B - 1} \le 1$.

Upon squaring, this is equivalent to

$$\begin{aligned} \sin^2 B &\le 4\cos^2 B - 4\cos B + 1 \\ 1 - \cos^2 B &\le 4\cos^2 B - 4\cos B + 1 \\ \cos B &\ge \frac{4}{5}. \end{aligned}$$

So if we choose angle B to make $\cos B \ge \frac{4}{5}$, then we can choose angle A to make $\sin A = \frac{\sin B}{2\cos B - 1}$.

Since cosine is decreasing in the first quadrant, the size condition on $\cos B$ forces $B \le \cos^{-1}\left(\frac{4}{5}\right) \approx 36.87°$.

In fact, for any t, with $\frac{4}{5} \le t \le 1$, we can let $B = \cos^{-1} t$, in which case $\sin B = \sqrt{1-t^2}$, and let $A = \sin^{-1}\left(\frac{\sqrt{1-t^2}}{2t-1}\right)$.

Note that the endpoint "solution" given by $t = 1$ is $A = 0, B = 0$, which we disregard.

Also, the endpoint solution given by $t = \frac{4}{5}$ is $A = \frac{\pi}{2}, B = \cos^{-1}\frac{4}{5}$.

It is worth noting that we thus have a right triangle solution, but it doesn't quite meet the problem's criteria, so we'll disregard this one. Thus, there are infinitely many solutions, given in terms of the parameter t for $\frac{4}{5} < t < 1$.

We also note that one could also say that all solutions are given by $\sin A = \frac{\sin B}{2\cos B - 1}$, where angle B is chosen so that $\cos B > \frac{4}{5}$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Michael Brozinsky, Central Islip, NY; Shai Covo, Kiryat-Ono, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Raúl A. Simón, Santiago, Chile; Taylor University Problem Solving Group; Upland, IN, and the proposer.

- **5118:** *Proposed by David E. Manes, Oneonta, NY*

Find the value of

$$\sqrt{2011+2007\sqrt{2012+2008\sqrt{2013+2009\sqrt{2014+\cdots}}}}$$

Solution 1 by Shai Covo, Kiryat-Ono, Israel

The value is 2009. More generally, for any integer $n \geq 3$ we have

$$n=\sqrt{(n+2)+(n-2)\sqrt{(n+3)+(n-1)\sqrt{(n+4)+n\sqrt{(n+5)+\cdots}}}}$$

($n=2009$ corresponds to the original problem.) The claim follows from an iterative application of the identity $n=\sqrt{(n+2)+(n-2)(n+1)}$, as follows:

$$\begin{aligned}
n &= \sqrt{(n+2)+(n-2)(n+1)} \\
&= \sqrt{(n+2)+(n-2)\sqrt{(n+3)+(n-1)(n+2)}} \\
&= \sqrt{(n+2)+(n-2)\sqrt{(n+3)+(n-1)\sqrt{(n+4)+n(n+3)}}} \\
&= \cdots.
\end{aligned}$$

Solution 2 by Taylor University Problem Solving Group, Upland, IN

We use Ramanujan's nested radical approach. Beginning with

$$(x+n+a)^2=x^2+n^2+a^2+2ax+2nx+2an,$$

we see that

$$\begin{aligned}
x+n+a &= \sqrt{x^2+n^2+a^2+2ax+2nx+2an} \\
&= \sqrt{ax+n^2+a^2+2an+x\,(x+2n+a)} \\
&= \sqrt{ax+(n+a)^2+x\,(x+2n+a)}.
\end{aligned}$$

However, the $(x+2n+a)$ term on the right is basically of the same form as the left (with n replaced by $2n$). We can make the corresponding substitution, and continue this process indefinitely, until we are left with $x+n+a=$

$$\sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{a\,(x+2n)+(n+a)^2+(x+2n)\sqrt{\cdots}}}}$$

Substituting in $x=2007$, $n=a=1$ produces

$$\begin{aligned}
2009 &= \sqrt{2007+4+2007\sqrt{2008+4+2008\sqrt{2009+4+2009\sqrt{\cdots}}}} \\
&= \sqrt{2011+2007\sqrt{2012+2008\sqrt{2013+2009\sqrt{\cdots}}}}.
\end{aligned}$$

Hence, the value is 2009.

Also solved by Scott H. Brown, Auburn University, Montgomery, AL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS: Kenneth Korbin, NY, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Stateboro GA, and the proposer.

- **5119:** *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain*

Let n be a non-negative integer. Prove that

$$2 + \frac{1}{2^{n+1}} \prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) < F_{n+1}$$

where F_n is the n^{th} Fermat number defined by $F_n = 2^{2^n} + 1$ for all $n \geq 0$.

Solution by Charles R. Diminnie, San Angelo, TX

To begin, we note that for $x \in \left(0, \dfrac{\pi}{3}\right)$, $\cos x$ is decreasing and the Mean Value Theorem for Derivatives implies that there is a point $c_x \in (0, x)$ such that

$$\begin{aligned} \sin x &= \sin x - \sin 0 \\ &= \cos c_x \,(x - 0) \\ &> \cos\frac{\pi}{3} \cdot x \\ &= \frac{x}{2}. \end{aligned}$$

As a result, when $x \in \left(0, \dfrac{\pi}{3}\right)$,

$$x \csc x < 2.$$

Since $F_n \geq F_0 = 3$ for all $n \geq 0$, it follows that $0 < \dfrac{1}{F_n} \leq \dfrac{1}{3} < \dfrac{\pi}{3}$ and hence,

$$\begin{aligned} \frac{1}{F_n} \csc\left(\frac{1}{F_n}\right) &< 2, \text{ or} \\ \csc\left(\frac{1}{F_n}\right) &< 2F_n \qquad (1) \end{aligned}$$

Let $P(n)$ be the statement

$$\prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) < 2^{n+1}\left(F_{n+1} - 2\right) \qquad (2)$$

By (1),

$$\csc\left(\frac{1}{F_0}\right) < 2F_0 = 2 \cdot 3 = 2\left(F_1 - 2\right)$$

and $P(0)$ is true. If $P(n)$ is true for some $n \geq 0$, then by (1),

$$\begin{aligned}
\prod_{k=0}^{n+1} \csc\left(\frac{1}{F_k}\right) &= \csc\left(\frac{1}{F_{n+1}}\right) \prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) \\
&< \csc\left(\frac{1}{F_{n+1}}\right) \cdot 2^{n+1}\left(F_{n+1}-2\right) \\
&< 2F_{n+1} \cdot 2^{n+1}\left(F_{n+1}-2\right) \\
&= 2^{n+2}\left(2^{2^{n+1}}+1\right)\left(2^{2^{n+1}}-1\right) \\
&= 2^{n+2}\left(2^{2^{n+2}}-1\right) \\
&= 2^{n+2}\left(F_{n+2}-2\right)
\end{aligned}$$

and $P(n+1)$ follows. By Mathematical Induction, $P(n)$ is true for all $n \geq 0$.

Since (2) is equivalent to the given inequality, the proof is complete.

Also solved by Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5120:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Calculate

$$\lim_{n\to\infty} \frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left(\frac{2n-k}{2n+k}\right).$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

The limit equals 0. More generally, we prove that if $f:[0,1]\to\Re$ is a continuous function then

$$\lim_{n\to\infty} \frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k} f\left(\frac{k}{n}\right) = 0.$$

Before we give the solution of the problem we collect the following equality from **[1]** (Formula 0.154(3), p.4): If $p \geq 0$ is a nonnegative integer, then the following equality holds

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^p = 0. \qquad (1)$$

Now we are ready to solve the problem. First we note that for a polynomial $P(x)=\sum_{j=0}^{m} a_j x^j$ we have, based on (1), that

$$\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}P\left(\frac{k}{n}\right) = \sum_{j=0}^{m}\frac{a_j}{n^j}\cdot\frac{1}{2^n}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j\right) = 0. \qquad (2)$$

Let $\epsilon > 0$ and let P_ϵ be the polynomial that uniformly approximates f, i.e. $|f(x)-P_\epsilon(x)| < \epsilon$ for all $x \in [0,1]$. We have, based on (2), that

$\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_\epsilon\left(\frac{k}{n}\right)=0$. Thus,

$$
\begin{aligned}
\left|\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f\left(\frac{k}{n}\right)\right| &= \left|\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(f\left(\frac{k}{n}\right)-P_\epsilon\left(\frac{k}{n}\right)\right)\right| \\
&\le \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-P_\epsilon\left(\frac{k}{n}\right)\right| \\
&\le \frac{\epsilon}{2^n}\sum_{k=0}^{n}\binom{n}{k} \\
&= \epsilon.
\end{aligned}
$$

Thus, the limit is 0 and the problem is solved.

[1] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, Sixth Edition, Alan Jeffrey, Editor, Daniel Zwillinger, Associate Editor, 2000.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We will show that

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left(\frac{2n-k}{2n+k}\right)=0. \qquad (1)$$

(The log function in (1) has no significant role in the analysis below, we could replace it by any other continuous function.)

The lemma below follows straightforwadly from the Central Limit Theorem (CLT). We recall that, according to the CLT, if $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d) random variables with expectation μ and variance σ^2, then

$$P\left(a<\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}}\le b\right)\to\Phi(b)-\Phi(a) \qquad (2)$$

as $n\to\infty$, for any $a,b\in\Re$ with $a<b$ where Φ is the distribution function of the Normal $(0,1)$ distribution (i.e., $\Phi(x)=(2\pi)^{-1/2}\int_{-\infty}^{x}e^{-\mu^2/2}du$).

Lemma: For any $\epsilon>0$, there exists an $r>0$ such that

$$\frac{1}{2^n}\sum_{\substack{0\le k\le n/2-r\sqrt{n}\\ n/2+r\sqrt{n}<k\le n}}\binom{n}{k}<\epsilon \qquad (3)$$

for all n sufficiently large.

Proof: Fix $\epsilon>0$. Choose $r>0$ sufficiently large so that $\Phi(2r)-\Phi(-2r)>1-\epsilon$. Let $X_1, X_2\ldots$ be a sequence of i.i.d. variables with $P(X_i=0)=P(X_i=1)=1/2$. Put $Y_n=\sum_{i=1}^{n}X_i$. Thus Y_n has a binomial $(n,1/2)$ distribution. The X_i's have expectation $\mu=1/2$ and variance $\sigma^2=1/4$. Hence by (2) (with $a=-2r$ and $b=2r$),

$$P(n/2-r\sqrt{n}<Y_n\le n/2+r\sqrt{n})>1-\epsilon$$

for all n sufficiently large. In turn, by taking complements, we conclude (3), since the distribution of Y_n is given by $P(Y_n = k) = \dfrac{1}{2^n}\dbinom{n}{k}, k = 0, \ldots, n.$

It follows from the lemma and the fact that $\left|(-1)^k \log\left(\dfrac{2n-k}{2n+k}\right)\right|$ is bounded uniformly in k (say, by 2) that (1) will be proved if we show that

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{n/2-r\sqrt{n}<k<n/2+r\sqrt{n}} (-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right) = 0 \qquad (4)$$

for any fixed $r > 0$. This is shown as follows. We first write

$$\begin{aligned} &\left|(-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right) + (-1)^k \binom{n}{k+1} \log\left(\frac{2n-(k+1)}{2n+(k+1)}\right)\right| \\ = &\binom{n}{k}\left|\log\left(\frac{2n-k}{2n+k}\right) - \frac{n-k}{k+1}\log\left(\frac{2n-(k+1)}{2n+(k+1)}\right)\right|. \qquad (5) \end{aligned}$$

Clearly, the expression multipying $\dbinom{n}{k}$ on the right of the equality in (5) can be made arbitrarily small uniformly in $k \in [n/2 - r\sqrt{n},\ n/2 + r\sqrt{n}]$, where $r > 0$ is fixed, by choosing n sufficiently large. Then, in view of the triangle inequality, (4) follows from $\dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}\epsilon = \epsilon$ (where $\epsilon > 0$ is arbitrarily small) and $\dbinom{n}{k}\Big/2^n \xrightarrow{\text{unif.}} 0$ (to be used if the sum in (4) consists of an odd number of terms). The desired result (1) is thus proved.

Also proved by Boris Rays, Brooklyn, NY and the proposer.

5121: *Proposed by Tom Leong, Scotrun, PA*

Let n, k and r be positive integers. It is easy to show that

$$\sum_{n_1+n_2+\cdots+n_r=n} \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k} = \binom{n+r-1}{kr+r-1}, \quad n_1, n_2, \cdots n_r \in N$$

using generating functions. Give a combinatorial argument that proves this identity.

Solution 1 by Shai Covo, Kiryat-Ono, Israel

Suppose we have n identical boxes and kr $(\leq n)$ identical balls. The stated equality is trivial if $r = 1$, hence we can assume $r > 1$.

We begin with the left-hand side of the stated equality. Assuming $n_1, \ldots, n_r \geq k$, it gives the number of ways to divide the n boxes into r groups–the ith group having $n_i \geq k$ elements–and put exactly k balls in each group.

As for the right-hand side, suppose that in addition to the n boxes and the kr balls we have $r-1$ separators. This gives rise to an $(n+r-1)$-tuple of boxes and separators. We denote this tuple by M. We identify a sequence $(i_1, i_2, \ldots, i_{kr+r-1})$ such that $1 \le i_1 < i_2 < \cdots < i_{kr+r-1} \le n+r-1$ with the following arrangement: the i_jth $(j = 1, \ldots, kr+r-1)$ element of M is a separator if j is a multiple of $k+1$ and a box containing a ball otherwise. (The remaining $n-kr$ elements are empty boxes.) We thus conclude that $\binom{n+r-1}{kr+r-1}$ gives the number of ways to place $r-1$ separators between the n boxes and kr balls into the boxes, such that each of the resulting r groups contains exactly k balls. This establishes the equality of the left-and right-hand sides.

Solution 2 by the proposer

Both sides count the number of possible ways to arrange $kr+r-1$ green balls and $n-kr$ red balls in a row. This is clearly true for the right side. In the left side, note that any term in the sum with $n_i < k$ for some i is equal to zero; so we may assume $n_i \ge k$ for all i. For each *composition* $n_1 + \cdots + n_r = n$ of n, consider the row of n red and $r-1$ green balls arranged as

$$\underbrace{RR\cdots R}_{n_1 \text{ balls}} G \underbrace{RR\cdots R}_{n_2 \text{ balls}} G \underbrace{RR\cdots R}_{n_3 \text{ balls}} G \cdots G \underbrace{RR\cdots R}_{n_{r-1} \text{ balls}} G \underbrace{RR\cdots R}_{n_r \text{ balls}}$$

From each block of red balls, choose k of them and paint them green. The number of ways to do this is $\binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k}$. This results in a row consisting of $kr+r-1$ green balls and $n-kr$ red balls. Conversely, in any row consisting of $kr+r-1$ green balls and $n-kr$ red balls, we can determine a unique composition $n_1+n_2+\cdots+n_r = n$ of n by reversing the process.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
March 15, 2011

- **5140:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with an interior point P such that

$$\begin{aligned}\overline{AP} &= 22+16\sqrt{2}\\ \overline{BP} &= 13+9\sqrt{2}\\ \overline{CP} &= 23+16\sqrt{2}.\end{aligned}$$

  Find $\overline{AB}$.

- **5141:** *Proposed by Kenneth Korbin, New York, NY*

  A quadrilateral with sides $259, 765, 285, 925$ is constructed so that its area is maximum. Find the size of the angles formed by the intersection of the diagonals.

- **5142:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let CD be an arbitrary diameter of a circle with center O. Show that for each point A distinct from O, C, and D on the line containing CD, there is a point B such that the line from D to any point P on the circle distinct from C and D bisects angle APB.

- **5143:** *Proposed by Valmir Krasniqi (student), Republic of Kosova*

  Show that

$$\sum_{n=1}^{\infty} Cos^{-1}\,\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} \quad = \quad \frac{\pi}{2}.$$

$$(Cos^{-1} = Arccos)$$

- **5144:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right].$$

- **5145:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the sum of

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-1-x-x^2-\cdots-x^n\right)^k, \quad \text{for } |x|<1.$$

Solutions

- **5122:** *Proposed by Kenneth Korbin, New York, NY*

Partition the first 32 non-negative integers from 0 to 31 into two sets A and B so that the sum of any two distinct integers from set A is equal to the sum of two distinct integers from set B and vice versa.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Suppose A contains 0. This means that any other number in A must be the sum of two numbers in B. The next number in A, therefore, must be at least 3 since 3 is the smallest number that is the sum of two positive integers. On the other hand, the next number in A cannot be greater than 3, for 1 and 2 must still be in B. This group of four numbers forms a kind of unit, which we can represent graphically as follows:

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \qquad \text{or} \qquad \begin{array}{ccc} 1 & & 2 \\ & \sqcap & \\ 0 & & 3 \end{array}$$

The symmetry of the unit reflects the fact that $a+b=c+d$ if and only if $b-d=a-c$, that is if and only if there is some number k such that $c=a+k$ and $d=b-k$. Thus any four consecutive integers forming such a figure will have the property that the sum of the top pair of numbers equals the sum of the bottom pair.

(This makes the problem almost a geometrical one, for arranging the numbers in set A and B in parallel lines as in the figure above, the condition of the problem becomes that every pair of numbers in the first line corresponds to a pair of numbers in the second line.)

So our strategy for the problem will be to assemble units such as those above to produce larger units satisfying in each case the condition of the problem.

Let us then start with two. The first, as before is:

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array}$$

And as we have already argued, the first two numbers of A and B *must* be arranged in this way. The second unit, then, will be either

$$\begin{array}{ccc} 4 & & 7 \\ & \sqcup & \\ 5 & & 6 \end{array} \quad \text{or} \quad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}$$

The symmetrical combination,

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \qquad \begin{array}{ccc} 4 & & 7 \\ & \sqcup & \\ 5 & & 6 \end{array}$$

fails, because the pair $(0,4)$ in the upper row has no matching pair in the second row.

However, the non-symmetrical combination works:

$$\begin{array}{ccccccc} 0 & & 3 & \quad & 5 & & 6 \\ & \sqcup & & & & \sqcap & \\ 1 & & 2 & & 4 & & 7 \end{array}$$

Again, these two form a new kind of unit, and, as before, any eight consecutive integers forming a unit such as the above, will have the property that any pair of numbers in the top row will have the same sum as some pair in the bottom row.

So, let us try and fit together two units of this type, and let us call them R and S. As before, there are two possibilities, one symmetric and one anti-symmetric.
Since the anti-symmetric option worked before, let us try it again and call the top row **A** and the bottom row **B**.

$$\overbrace{\begin{array}{ccccccc} 0 & & 3 & \quad & 5 & & 6 \\ & \sqcup & & & & \sqcap & \\ 1 & & 2 & & 4 & & 7 \end{array}}^{R} \qquad\qquad \overbrace{\begin{array}{ccccccc} 8 & & 11 & \quad & 13 & & 14 \\ & \sqcap & & & & \sqcup & \\ 9 & & 10 & & 12 & & 15 \end{array}}^{S}$$

$$\begin{aligned} \mathbf{A} &= \{0,3,5,6,8,11,13,14\} \\ \mathbf{B} &= \{1,2,4,7,9,10,12,15\} \end{aligned}$$

Now, to check whether this combination works we do not have to check $\binom{8}{2} = 28$ pairs of numbers.

All of the subunits will satisfy the condition of the problem. Indeed, we do not have to check pairs contained in the first and second, second and third and third and fourth terms, because they represent eight consecutive integers as discussed above. And we do not have to check pairs from the first and fourth terms because these also behave like a single unit R (where for example the pair (0,13) corresponds to (1,12) just as (0,5) corresponded to (1,4). So we only have to check pairs of numbers coming from the first and third elements and the second and fourth. But here we find a problem, for (2,10) in **B** cannot have a corresponding pair in **A.**

Let us then check the symmetrical arrangement:

$$\overbrace{\begin{array}{ccccccc} 0 & & 3 & \quad & 5 & & 6 \\ & \sqcup & & & & \sqcap & \\ 1 & & 2 & & 4 & & 7 \end{array}}^{R} \qquad\qquad \overbrace{\begin{array}{ccccccc} 9 & & 10 & \quad & 12 & & 15 \\ & \sqcup & & & & \sqcap & \\ 8 & & 11 & & 13 & & 14 \end{array}}^{S}$$

$$\begin{aligned} \mathbf{A} &= \{0,3,5,6,9,10,12,15\} \\ \mathbf{B} &= \{1,2,4,7,8,11,13,14\} \end{aligned}$$

As in the anti-symmetrical arrangement, we need not check pairs of numbers in R or S, or, in this case, pairs if the first and third elements or second and fourth, which behave exactly as R and S individually. We need only check non-symmetrical pairs in the first and fourth elements and in the second and third. For the former this means (3,15) and (0,12) in **A** and (1,13) and (2,14) in **B.** For these we have corresponding pairs (3,15) to (7,8), (0,12) to (4,8), (1,13) to (5,9) and (2,14) to (6,10). Similarly, corresponding pairs exist for each non-smmetric pair in **A** and **B** in the second and third elements.

The above arrangement is then a new unit of 16 consecutive numbers satisfying the condition that every pair in the upper row **A,** has a correspnding pair of numbers in the second row **B,** with the same sum.

Finally, then, we want to join together two units, each of 16 consecutive integers as above, to partition the set of 32 consecutive integers $\{0, 1, 2, \ldots, 31\}$.
Reasoning as above, and checking only the critical elements in the unit for corresponding sums, we see that the symmetric case works.

$$\text{The symmetric case :}$$
$$\overbrace{\begin{matrix}0\\1\end{matrix}\sqcup\begin{matrix}3\\2\end{matrix}\quad\begin{matrix}5\\4\end{matrix}\sqcap\begin{matrix}6\\7\end{matrix}\quad\begin{matrix}9\\8\end{matrix}\sqcap\begin{matrix}10\\11\end{matrix}\quad\begin{matrix}12\\13\end{matrix}\sqcup\begin{matrix}15\\14\end{matrix}}\quad\text{and}\quad\overbrace{\begin{matrix}16\\17\end{matrix}\sqcup\begin{matrix}19\\18\end{matrix}\quad\begin{matrix}21\\20\end{matrix}\sqcap\begin{matrix}22\\23\end{matrix}\quad\begin{matrix}25\\24\end{matrix}\sqcap\begin{matrix}26\\27\end{matrix}\quad\begin{matrix}28\\29\end{matrix}\sqcup\begin{matrix}31\\30\end{matrix}}$$

Thus,

$$\begin{aligned}\mathbf{A} &= \{0,3,5,6,9,10,12,15,16,19,21,22,25,26,28,31\}\\ \mathbf{B} &= \{1,2,4,7,8,11,13,14,17,18,20,23,24,27,29,30\}\end{aligned}$$

Editor's comment: In Michael's solution each element in the set of four consecutive integers was written as being the vetex of an isosceles trapezoid. (The trapezoids were oriented with the bases being parallel to the top and bottom edges of page; Michael then manitpulated the trapezoids by flipping their bases.)

Adoración Martínez Ruiz of the Mathematics Club of the Institute of Secondary Education (No. 1) in Requena-Valencia, Spain also approached the problem geometrically in an almost identical manner as Michael. I adopted Adoración Martínez' notation of "cups" $\sqcup$ and "caps" $\sqcap$ instead of Michael's isosceles trapezoids in writing-up Michael's solution. (If the shorter base of the trapezoid was closer to the bottom edge of the page than the longer base, then that trapezoid became a cup, $\sqcup$; whereas if the shorter base of the trapezoid was closer to the top edge of the page than the longer base, then that trapezoid became a cap, $\sqcap$.

Michael's solution and Adoración Martínez' solution were identical to one another up until the last step. At that point Michael took the symmetric extension in moving from the first 16 non-negative integers to the first 32 non-negative integers, whereas Adoración Martínez took the anti-symmetric extension, and surprisingly (at least to me), each solution worked.

$$\text{Adoración Martínez}' \text{ anti} - \text{symmetric case :}$$
$$\overbrace{\begin{matrix}0\\1\end{matrix}\sqcup\begin{matrix}3\\2\end{matrix}\quad\begin{matrix}5\\4\end{matrix}\sqcap\begin{matrix}6\\7\end{matrix}\quad\begin{matrix}9\\8\end{matrix}\sqcap\begin{matrix}10\\11\end{matrix}\quad\begin{matrix}12\\13\end{matrix}\sqcup\begin{matrix}15\\14\end{matrix}}\quad\text{and}\quad\overbrace{\begin{matrix}17\\16\end{matrix}\sqcap\begin{matrix}18\\19\end{matrix}\quad\begin{matrix}20\\21\end{matrix}\sqcup\begin{matrix}23\\22\end{matrix}\quad\begin{matrix}24\\25\end{matrix}\sqcup\begin{matrix}27\\26\end{matrix}\quad\begin{matrix}29\\28\end{matrix}\sqcap\begin{matrix}30\\31\end{matrix}}$$

$$\begin{aligned}\mathbf{A} &= \{0,3,5,6,9,10,12,15,17,18,20,23,24,27,29,30\}\\ \mathbf{B} &= \{1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31\}\end{aligned}$$

So now we have two solutions to the problem, each motivated by geometry, and it was assumed (at least by me) that their were no other solutions. Michael challenged **Mayer Goldberg,** a colleague in CS here at BGU, to find other solutions, and he did; many of them! Following is his approach.

Solution 2 by Mayer Goldberg, Beer-Sheva, Israel

Notation: For any set S of integers, the set $aS + b$ is the set $\{ak + b : k \in S\}$.

Construction: We start with the set $A_0 = \{0, 4\}, b_0 = \{1, 2\}$. We define A_n, B_n inductively as follows:

$$
\begin{aligned}
A_{n+1} &= (2A_n + 1) \cup (2B_n) \\
B_{n+1} &= (2A_n) \cup (2B_n + 1)
\end{aligned}
$$

Claim: The sets A_n, B_n partition the set $\{0, \ldots, 2^{n+2}\}$ according to the requirements of the problem.

Proof: By Induction. The sets $A_0 B_0$ satisfy the requirement trivially, since they each contain one pair, and by inspection, we see that the sums are the same. Assume that A_n, B_n satisfy the requirement. Pick $x_1, x_2 \in A_{n+1}$.

- **Case I:** $x_1 = 2x_3 + 1, x_2 = 2x_4 + 1$, for $x_3, x_4 \in A_n$. Then by the induction hypothesis (IH), there exists $y_3, y_4 \in B_n$, such that $x_3 + x_4 = y_3 + y_4$. Consequently,

$$x_1 + x_2 = 2(x_3 + x_4) + 2 = 2(y_3 + y_4) + 2 = (2y_3 + 1) + (2y_4 + 1).$$

  So let $y_1 = 2y_3 + 1, y_2 = 2y_4 + 1 \in B_{n+1}$.

- **Cases II & III:** $x_1 = 2x_3 + 1, x_2 = 2y_4$, for $x_3 \in A_n, y_4 \in B_n$.

$$x_1 + x_2 = 2(x_3 + 1) + 2y_4 = 2x_3 + (2y_4 + 1).$$

  So let $y_1 = 2x_3, y_2 = 2y_4 + 1 \in B_{n+1}$.

- **Case IV:** $x_1 = 2y_3, x_2 = 2y_4$, for $y_3, y_4 \in B_n$.Then by the IH, there exists $x_3, x_4 \in A_n$, suc that $y_3 + y_4 = x_3 + x_4$. Consequently,

$$x_1 + x_2 = 2y_3 + 2y_4 = 2(y_3 + y_4) = 2(x_3 + x_4) = 2x_3 + 2x_4.$$

  So let $y_1 = 2x_3, y_2 = 2y_3 \in B_{n+1}$

Editor: This leads to potentially thousands of such pairs of sets that satisfy the criteria of the problem. Mayer listed about one hundred such examples, a few of which are reproduced below:

$$
\begin{aligned}
A &= \{0, 3, 5, 6, 9, 10, 13, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\
B &= \{1, 2, 4, 7, 8, 11, 12, 14, 16, 20, 21, 22, 25, 26, 28, 31\}
\end{aligned}
$$

$$
\begin{aligned}
A &= \{0, 3, 5, 6, 9, 10, 11, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\
B &= \{1, 2, 4, 7, 8, 12, 13, 14, 16, 20, 21, 22, 25, 26, 28, 31\}
\end{aligned}
$$

$$
\begin{aligned}
A &= \{0, 3, 5, 6, 9, 10, 14, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\
B &= \{1, 2, 4, 7, 8, 11, 12, 13, 16, 20, 21, 22, 25, 26, 28, 31\}
\end{aligned}
$$

$$
\begin{aligned}
A &= \{0, 3, 5, 6, 9, 10, 13, 15, 17, 18, 22, 23, 24, 27, 29, 30\}, \\
B &= \{1, 2, 4, 7, 8, 11, 12, 14, 16, 19, 20, 21, 25, 26, 28, 31\}
\end{aligned}
$$

$$
\begin{aligned}
A &= \{0, 3, 5, 6, 9, 10, 14, 15, 17, 18, 22, 23, 24, 27, 29, 30\}, \\
B &= \{1, 2, 4, 7, 8, 11, 12, 13, 16, 19, 20, 21, 25, 26, 28, 31\}
\end{aligned}
$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,15,16,17,18,19,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,13,14,20,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,15,16,17,18,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,13,14,19,20,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,13,15,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,11,12,14,16,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,11,15,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,12,13,14,16,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,15,16,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,11,12,13,14,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,13,15,17,19,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,14,16,18,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,11,15,17,19,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,12,13,14,16,18,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,13,15,17,20,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,14,16,18,19,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,11,15,17,20,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,12,13,14,16,18,19,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,14,15,17,20,22,23,24,27,29,30\}, \\ B &= \phantom{\{}1,2,4,7,8,11,12,13,16,18,19,21,25,26,28,31\} \end{aligned}$$

Editor (again): **Edwin Gray of Highland Beach, FL** working together with **John Kiltinen of Marquette, MI** claimed and proved by induction the following more general theorem:

Let $S = \{0,1,2,3,\ldots,2^n-1\},\ n > 1.$ Then there is a partition of S, say A, B such that

1) $A \cup B = S,\ A \cap B = \emptyset,$ and

2) For all $x, y \in A$, there is an $r, s \in B,$ such that $x + y = r + s,$ and vice versa.

That is, the sum of any two elements in B has two elements in A equal to their sum.

David Stone and John Hawkins both of Statesboro, GA also claimed and proved a more general statement: They showed that: for $n \geq 2$, the set $S_n = \{0,1,2,\ldots,2^n-1\}$ consists of the non-negaitve integers which can be written with n or fewer binary digits. E.g.,

$$S_2 = \{0,1,2,3\} = \{00,01,10,11\} \text{ and}$$

$$S_3 \;=\; \{0,1,2,3,4,5,6,7\} = \{000,001,010,011,100,101,110,111\}$$

Their proof consisted of partitioning S_n into two subsets: E_n: those elements of S_n whose binary representation uses an even number of ones, and O_n: those numbers in S_n whose binary reprsentation uses an odd number of ones. Hence, for any $x \neq y$ in E_n, $x + y$ can be written as $x + y = w + z$ for some $w \neq z$ in O_n, and vice versa. This lead them to Adoración Martínez' solution, and they speculated on its uniqueness.

All of this seemed to be getting out-of-hand for me; at first I thought the solution is unique; then I thought that there are only two solutions, and then I thought that there are many solutions to the problem. **Shai Covo's** solutio/Users/admin/Desktop/SSM/For Jan 11/For Jan 11; Jerry.texn however, shows that the answer can be unique if one uses a notion of *sum multiplicity.*

Solution 3 by Shai Covo, Kiryat-Ono, Israel

We give two solutions, the first simple and original, the second sophisticated and more interesting, thanks to the Online Encyclopedia of Integer sequences(OEIS).

Assuming that $0 \in A$, one checks that we must have either

$$\begin{array}{ccc} \{0,3,5,6\} \cup \{25,26,28,31\} \subset A & \text{and} & \{1,2,4,7\} \cup \{24,27,29,30\} \subset B \\ & \text{or} & \\ \{0,3,5,6\} \cup \{24,27,29,30\} \subset A & \text{and} & \{1,2,4,7\} \cup \{25,26,28,31\} \subset B. \end{array}$$

In view of the first possibility, it is natural to examine the following sets:

$$\begin{aligned} A &= \{0,3,5,6,9,10,13,14,17,18,21,22,25,26,28,31\} \\ B &= \{1,2,4,7,8,11,12,15,16,19,20,23,24,27,29,30\}. \end{aligned}$$

To see why this is natural, connect the numbers with arrows, in increasing order, starting with a vertical arrow pointing down to 1. Now, define

$$\begin{aligned} C &= \{a_1 + a_2 \mid a_1, a_2 \in A, a_1 \neq a_2\} \subset \{3,4,5,\ldots,59\} \text{ and} \\ D &= \{b_1 + b_2 \mid b_1, b_2 \in B, b_1 \neq b_2\} \subset \{3,4,5,\ldots,59\}. \end{aligned}$$

We want to show that $C = D$, or equivalently, for every $x \in \{3,4,5,\ldots,59\}$ either $x \in C \cap D$ or $x \notin C \cup D$. Checking each x value, we find that

$$C \cap D = \{3,4,5,\ldots,59\} \setminus \{4,7,55,58\} \text{ and } \{4,7,55,58\} \cap (\mathrm{C} \cup \mathrm{D}) = \emptyset.$$

Thus, $C = D$, and so the problem is solved with A and B as above.

We now turn to the second solution. OEIS sequences $A001969$ (numbers with an even number of 1's in their binary expansion) and $A000069$ (numbers with an odd number of 1's in their binary expansion) "give the unique solution to the problem of splitting the nonnegative integers into two classes in such a way that sums of pairs of distinct elements from either class occur *with the same multiplicities.* [Lambek and Moser]." We have verified (by computer) that, in the case at hand, the sets

$$A \;=\; \{A001969(n) : A001969(n) \leq 30\}$$

$$
\begin{aligned}
&= \{0,3,5,6,9,10,12,15,17,18,20,23,24,27,29,30\} \text{ and}\\
B &= \{A000069(n): \ A000069(n) \le 31\}\\
&= \{1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31\}
\end{aligned}
$$

split the first 32 nonnegative integers from 0 to 31 in the manner stated for splitting the nonnegative integers. (The number 32 plays an important role here.) However, this is not the case for the sets A and B from the previous solution (consider, for example,12=3+9 versus 12=1+11, 12=4+8; there are seven more such examples.)

Editor (still again): I did not understand the notion about sums having the *same multiplicity,* but this is the key for having *a unique solution* to the problem, as it states in the OEIS. So I asked Shai to elaborate on this notion. Here is what he wrote:

The point is that "given the unique solution to the problem of splitting the nonnegative integers..." refers to the infinite set $\{0,1,2,3,...\}$ and not the finite set $\{0,1,2,...,31\}$. I should have stressed this point in my solution. As far as I can recall, I considered doing so, but decided not to, based on the following: "... the manner stated for splitting the nonnegative integers" only refers to "splitting the nonnegative integers into two classes in such a way that sums of pairs of distinct elements from either class occur with the same multiplicities," and not to "give the unique solution to the problem of splitting the nonnegative integers...".

In explaining the notion of itself, Shai wrote:

Consider Michael Fried's sets:

$$
\begin{aligned}
A &= \{0,3,5,6,9,10,12,15,16,19,21,22,25,26,28,31\}\\
B &= \{1,2,4,7,8,11,13,14,17,18,20,23,24,27,29,30\}.
\end{aligned}
$$

For set A, the number 16 can be decomposed as 0+16 and 6+10; hence the multiplicity is 2. For set B, on the other hand, 16 can only be decomposed as 2+14 (8+8 does not count, since we consider distinct elements only); hence the multiplicity is 1.

Also solved by Brian D. Beasley, Clinton, SC; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Kiltinen, Marquette, MI; Charles McCracken, Dayton, OH; Adoración Martínez Ruiz, Requena-Valencia, Spain; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5123:** *Proposed by Kenneth Korbin, New York, NY*

Given isosceles triangle ABC with $\overline{AB} = \overline{BC} = 2011$ and with cevian $\overline{BD}$. Each of the line segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ have positive integer length with $\overline{AD} < \overline{CD}$.

Find the lengths of those three segments when the area of the triangle is minimum.

Solution by Shai Covo, Kiryat-Ono, Israel

We begin by observing that $\overline{AC} \in \{3, 4, \ldots, 4021\}$. This follows from $\overline{AC} < \overline{AB} + \overline{BC} = 4022$ and the assumption that $\overline{AC} = \overline{AD} + \overline{CD}$ is the sum of the distinct positive integers. The area S of triangle ABC can be expressed in terms of $\overline{AC}$ as

$$S = S\left(\overline{AC}\right) = \frac{\overline{AC}}{2}\sqrt{2011^2 - \left(\frac{\overline{AC}}{2}\right)^2}.$$

Define $f(x) = x^2(2011^2 - x^2), x \in [0, 2011]$. Then $S\left(\overline{AC}\right) = \sqrt{f\left(\overline{AC}/2\right)}$. It is readily verified that the function f (and hence $\sqrt{f}$) is unimodal with mode $m = 2011/\sqrt{2}$; that is, it is increasing for $x \leq m$ and decreasing for $x \geq m$. If thus follows from $f(4021/2) < f(127/2)$ that $S(4021) < S(k)$ for any integer $127 \leq k \leq 4020$. Next by the law of cosines, we find that

$$\overline{BD}^2 = 2011^2 + \overline{AD}^2 - 2 \cdot 2011 \cdot \overline{AD} \cdot \frac{\overline{AC}/2}{2011}.$$

Hence,

$$\overline{AD}^2 - \overline{AC} \cdot \overline{AD} + \left(2011^2 - \overline{BD}^2\right) = 0.$$

The roots of this quadratic equation are given by the standard formula as

$$\overline{AD}_{1,2} = \frac{\overline{AC} \pm \sqrt{\overline{AC}^2 - 4\left(2011^2 - \overline{BD}^2\right)}}{2}.$$

However, we are given that $\overline{AD} < \overline{CD}$; hence $\overline{AD} = \overline{AD_2}$ and $\overline{CD} = \overline{AD_1}$, and we must have $\overline{AC}^2 > 4\left(2011^2 - \overline{BD}^2\right)$. Since, obviously, $\overline{BD} \leq 2010$, we must have $\overline{AC^2} > 4\left(2011^2 - 2010^2\right) = 4 \cdot 4021$; hence, $127 \leq \overline{AC} \leq 4021$.

Thus, under the condition that S is minimum, we wish to find an integer value of $\overline{BD}(\leq 2010)$ that makes $\overline{AD_{1,2}}$ (that is, $\overline{CD}$ and $\overline{AD}$) distinct integers when $\overline{AC}$ is set to 4021.

We thus look for $\overline{BD} \in \{1, 2, \ldots, 2010\}$ for which the discriminant $\Delta = 4021^2 - 4(2011^2 - BD^2)$ is a positive perfect square, say $\Delta = j^2$ with $j \in N$ (actually, $j = \overline{CD} - \overline{AD}$). This leads straightforwardly to the following equation:

$$\left(2\overline{BD} + j\right)\left(2\overline{BD} - j\right) = 3 \cdot 7 \cdot 383.$$

Since 3,7, and 383 are primes, we have to consider the following four cases:

- $\left(2\overline{BD} - j\right) = 1$ and $\left(2\overline{BD} + j\right) = 3 \cdot 7 \cdot 383$. This leads to $\overline{BD} = 2011$; however, $\overline{BD}$ must be less than 2011.
- $\left(2\overline{BD} - j\right) = 3$ and $\left(2\overline{BD} + j\right) = 7 \cdot 383$. This leads to $\overline{BD} = 671$ and $j = 1339$, and hence to our first solution:

$$\overline{AD} = 1341,\ \overline{BD} = 671,\ \overline{CD} = 2680.$$

- $\left(2\overline{BD} - j\right) = 7$ and $\left(2\overline{BD} + j\right) = 3 \cdot 383$. This leads to $\overline{BD} = 289$ and $j = 571$, and hence to our second solution:

$$\overline{AD} = 1725,\ \overline{BD} = 289,\ \overline{CD} = 2296.$$

- $\left(2\overline{BD} - j\right) = 3 \cdot 7$ and $\left(2\overline{BD} + j\right) = 383$. This leads to $\overline{BD} = 101$ and $j = 181$, and hence to our third solution:

$$\overline{AD} = 1920,\ \overline{BD} = 101,\ \overline{CD} = 2101.$$

Editor: **David Stone and John Hawkins** made two comments in their solution. They started off their solution by letting $r = \overline{AC}$, the length of the triangle's base. By Heron's formula, they obtained the triangle's area: $K = \frac{r}{4}\sqrt{4022^2 - r^2}$ and then they made the following observations.

- **a)** $\overline{BD} = 1$ and $\overline{CD} = 2011$ gives us a triangle ABC with area $\left(\frac{1}{2} - \frac{1}{4\,(2011)^2}\right)\sqrt{4\,(2011^2) - 1} \approx 2010.999689$ which is the smallest value that can be obtained **not** requiring $\overline{AD}$ to be an integer.
- **b)** Letting $m = \overline{AD}, n = \overline{CD}, k = \overline{BD}$, (where $1 \leq m < n$ and $\overline{AC} = m + n \leq 4021$), and letting α be the base angle at vertex A (and at C), and dropping an altitude from B to side AC, we obtain a right triangle and see that

$$\cos\alpha = \frac{\overline{AC}/2}{2011} = \frac{m+n}{2 \cdot 2011}.$$

Using the Law of Cosines in triangle BDC, we have

$$k^2 = n^2 + 2011^2 - 2 \cdot 2011 \cdot n\cos\alpha = 2011^2 + n^2 - n(m+n),$$

so we have a condition which the integers m, n and k must satisfy

$$k^2 = 2011^2 - mn \qquad (\mathbf{1})$$

There are many triangles satisfying condition (**1**), some with interesting characteristics. There are no permissible triangles with base 4020, five with base 4019 and six with base 4018. All have larger areas than the champions listed above.

The altitude of each triangle in our winners group is 44.8 so the "shape ratio", altitude/base, is very small: 0.011. A wide flat triangle indeed!

One triangle with base 187 has a relatively small area: 187,825.16. This is as close as we can come to a tall, skinny triangle with small area. Its altitude/base ratio is 10.7.

In general, the largest isosceles triangle is an isosceles right triangle. With side lengths 2011, this would require a hypotenuse (our base) of $2011\sqrt{2} \approx 2843.98$. There are no permissible triangles with $r = 2844$. Letting $r = 2843$, we find the two largest permissible triangles:

$$m \;=\; 291,\ n = 2552, \text{cevian} = 1817 \text{ and area } 2,022,060.02$$

$$m \;=\; 883,\ n = 1960, \text{cevian} = 1521 \text{ and area } 2,022,060.02$$

The triangle with $m = 3, n = 2680$ (hence base =2683) has a large area: 2,009,788,52. The cevian has length 2009; it is very close to the side AB.

The triangle with $m = 1524, n = 1560$ and cevian=1291, comes closer than any other we found to having the cevian bisect the base. Its area is 1,990,528.49

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5124:** *Proposed by Michael Brozinsky, Central Islip, NY*

If $n > 2$ show that $\displaystyle\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2}$.

Solution 1 by Piriyathumwong P. (student, Patumwan Demonstration School), Bangkok, Thailand

Since $\cos 2\theta = 1 - 2\sin^2\theta$, we have

$$\begin{aligned}\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) &= \frac{1}{2}\sum_{i=1}^{n}\left(1 - \cos\left(\frac{4\pi i}{n}\right)\right)\\ &= \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n}\cos\left(\frac{4\pi i}{n}\right)\end{aligned}$$

We now have to show that $S = \displaystyle\sum_{i=1}^{n}\cos\left(\frac{4\pi i}{n}\right) = 0.$

Multiplying both sides of S by $2\sin\left(\dfrac{2\pi}{n}\right)$, gives

$$\begin{aligned}2\sin\left(\frac{2\pi}{n}\right)\cdot S &= 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right) + 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right) + \ldots + 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{4n\pi}{n}\right)\\ &= \left(\sin\left(\frac{6\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\right) + \left(\sin\left(\frac{10\pi}{n}\right) - \sin\left(\frac{6\pi}{n}\right)\right) + \ldots\\ &\quad + \left(\sin\left(\frac{(4n+2)\pi}{n}\right) - \sin\left(\frac{(4n-2)\pi}{n}\right)\right)\\ &= \sin\left(\frac{(4n+2)\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\\ &= 0\end{aligned}$$

Hence, $S = 0$, and we are done.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

To avoid confusion with the complex number $i = \sqrt{-1}$, we will consider

$$\sum_{k=1}^{n} \sin^2\left(\frac{2\pi k}{n}\right).$$

If $R = e^{(4\pi i/n)}$, with $n > 2$, then $R \neq 1$ and $R^n = e^{4\pi i} = 1$. Then, using the formula for a geometric sum, we get

$$\sum_{k=1}^{n} R^k = R\frac{R^n - 1}{R - 1} = 0,$$

and hence,

$$\sum_{k=1}^{n} \cos\left(\frac{4\pi k}{n}\right) = \sum_{k=1}^{n} Re\left(R^k\right) = Re\left(\sum_{k=1}^{n} R^k\right) = 0.$$

Therefore, by the half-angle formula,

$$\sum_{k=1}^{n} \sin^2\left(\frac{2\pi k}{n}\right) = \frac{1}{2}\sum_{k=1}^{n}\left[1 - \cos\left(\frac{4\pi k}{n}\right)\right] = \frac{n}{2}.$$

Also solved by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Pedro H. O. Pantoja, Natal-RN, Brazil; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5125:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{11}{32}.$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the sharp inequality

$$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{1}{3}. \qquad (1)$$

Let $x = \dfrac{a}{a+b+c}$, $y = \dfrac{b}{a+b+c}$, $z = \dfrac{c}{a+b+c}$ so that (1) can be written as

$$(a+b+c)\left(\frac{xy}{3y+2} + \frac{yz}{3z+2} + \frac{zx}{3x+2}\right) \le \frac{1}{3}. \qquad (2)$$

Since

$$a+b+c = \sqrt{3\left(a^2+b^2+c^2\right) - (a-b)^2 - (b-c)^2 - (c-a)^2} \le \sqrt{3\left(a^2+b^2+c^2\right)} = 3$$

so to prove (2), we need only prove that

$$\frac{xy}{3y+2}+\frac{yz}{3z+2}+\frac{zx}{3x+2}\le\frac{1}{9}. \qquad (3)$$

whenever x, y, z are positive and $x+y+z=1$. It is easy to check that (3) is equivalent to

$$\frac{x}{3y+2}+\frac{y}{3z+2}+\frac{z}{3x+2}\ge\frac{1}{3}. \qquad (4)$$

By the convexity of the function $\dfrac{1}{t}$, for $t>0$ and Jensen's inequality, we have

$$\frac{x}{3y+2}+\frac{y}{3z+2}+\frac{z}{3x+2}\ge\frac{1}{x(3y+2)+y(3z+2)+z(3x+2)}=\frac{1}{3\left(xy+yz+zx\right)+2}.$$

Now

$$xy+yz+zx=\frac{2\left(x+y+z\right)^2-\left(x-y\right)^2-\left(y-z\right)^2-\left(z-x\right)^2}{6}\le\frac{1}{3}$$

and so (4) holds. This proves (1) and equality holds when $a=b=c=1$.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

- **5126:** *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be positive real numbers and $f:[a,b]\to[c,d]$ be a function such that $|f(x)-f(y)|\ge|g(x)-g(y)|$, for all $x, y\in[a,b]$, where $g:R\to R$ is a given injective function, with $g(a),\ g(b)\in\{c,d\}$.

Prove

(i) $f(a)=c$ and $f(b)=d,$ or $f(a)=d$ and $f(b)=c.$

(ii) If $f(a)=g(a)$ and $f(b)=g(b),$ then $f(x)=g(x)$ for $a\le x\le b.$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

To avoid trivial situations, we will assume that $a<b$. Then, since $g\left(x\right)$ is injective and $g\left(a\right), g\left(b\right)\in\{c,d\}$, it follows that $c<d$ also.

First of all, the fact that $f\left(x\right)\in[c,d]$ for all $x\in[a,b]$ implies that

$$\left|f\left(x\right)-f\left(y\right)\right|\le d-c$$

for all $x, y\in[a,b]$.

(i) In particular, since $g\left(a\right), g\left(b\right)\in\{c,d\}$, we have

$$d-c\ge\left|f\left(a\right)-f\left(b\right)\right|\ge\left|g\left(a\right)-g\left(b\right)\right|=d-c.$$

Hence, $\left|f\left(a\right)-f\left(b\right)\right|=d-c$ with $c\le f\left(a\right), f\left(b\right)\le d$, and we get $f\left(a\right)=c$ and $f\left(b\right)=d$, or $f\left(a\right)=d$ and $f\left(b\right)=c$.

(ii) Suppose $f(a) = g(a) = c$ and $f(b) = g(b) = d$. The proof in the other case is similar. Then, since $c \le f(x) \le d$ for all $x \in [a,b]$, we obtain

$$\begin{aligned} d - c &= (d - f(x)) + (f(x) - c) \\ &= |d - f(x)| + |f(x) - c| \\ &= |f(b) - f(x)| + |f(x) - f(a)| \\ &\ge |g(b) - g(x)| + |g(x) - g(a)| \\ &= |d - g(x)| + |g(x) - c| \\ &\ge |d - c| \\ &= d - c. \end{aligned}$$

Thus, for all $x \in [a,b]$,

$$|d - f(x)| = |d - g(x)| \text{ and } |f(x) - c| = |g(x) - c|.$$

If there is an $x_0 \in [a,b]$ such that $f(x_0) \ne g(x_0)$, then

$$d - f(x_0) = g(x_0) - d \text{ and } f(x_0) - c = c - g(x_0)$$

and hence,

$$2d = f(x_0) + g(x_0) = 2c.$$

This is impossible since $c \ne d$. Therefore, $f(x) = g(x)$ for all $x \in [a,b]$.

Remark. The condition that $a, b, c, d > 0$ seems unnecessary for the solution of this problem.

Editor: **Shai Covo** suggested that the problem can be made more interesting by adding a third condition. Namely:

iii) If $f(a) \ne g(a)$ (or equivalently, $f(b) \ne g(b)$), then $f(x) + g(x) = c + d$ for all $x \in [a,b]$ and, hence, $f(x) - f(y) = g(y) - g(x)$ for all $x, y \in [a,b]$.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5127:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \ge 1$ be an integer and let $T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \dfrac{x^{2k-1}}{(2k-1)!}$, denote the $(2n-1)$th Taylor polynomial of the sine function at 0. Calculate

$$\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx.$$

Solution by Paolo Perfetti, Department of Mathematics, University of Rome, Italy

Answer: $\dfrac{\pi(-1)^{n-1}}{2(2n)!}$

Proof: Integrating by parts:

$$\begin{aligned}
\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx &= -\frac{1}{2n}\int_0^\infty (T_n(x) - \sin x)(x^{-2n})' dx \\
&= \frac{T_n(x) - \sin x}{-2nx^{2n}}\Big|_0^\infty + \frac{1}{2n}\int_0^\infty \frac{T_n'(x) - \cos x}{x^{2n}} dx \\
&= \frac{1}{2n}\int_0^\infty \frac{T_n'(x) - \cos x}{x^{2n}} dx
\end{aligned}$$

using $T_n(x) - \sin x = -\sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$ in the last equality.

After writing $T_n'(x) - \cos x = -\sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{x^{2k-2}}{(2k-2)!}$, we do the second step.

$$\begin{aligned}
\int_0^\infty \frac{T_n'(x) - \cos x}{(2n)x^{2n}} dx &= \frac{-1}{2n(2n-1)}\int_0^\infty (T_n'(x) - \cos x)(x^{-2n+1})' dx \\
&= \frac{T_n'(x) - \cos x}{-2n(2n-1)x^{2n-1}}\Big|_0^\infty + \frac{1}{2n(2n-1)}\int_0^\infty \frac{T_n''(x) + \sin x}{x^{2n-1}} dx \\
&= \frac{1}{2n(2n-1)}\int_0^\infty \frac{T_n''(x) + \sin x}{x^{2n-1}} dx.
\end{aligned}$$

After $2n$ steps we obtain

$$\frac{(-1)^{n-1}}{(2n)!}\int_0^\infty \frac{\sin}{x} dx = \frac{\pi(-1)^{n-1}}{2(2n)!}$$

Also solved by Shai Covo, Kiryat-Ono, Israel; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before April 15, 2011

- **5146:** *Proposed by Kenneth Korbin, New York, NY*

  Find the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$.

- **5147:** *Proposed by Kenneth Korbin, New York, NY*

  Let
  $$\begin{cases} x = 5N^2 + 14N + 23 \text{ and} \\ y = 5(N+1)^2 + 14(N+1) + 23 \end{cases}$$
  where N is a positive integer. Find integers a_i such that
  $$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0.$$

- **5148:** *Proposed by Pedro Pantoja (student, UFRN), Natal, Brazil*

  Let a, b, c be positive real numbers such that $ab + bc + ac = 1$. Prove that
  $$\frac{a^2}{\sqrt[3]{b(b+2c)}} + \frac{b^2}{\sqrt[3]{c(c+2a)}} + \frac{c^2}{\sqrt[3]{a(a+2b)}} \geq 1.$$

- **5149:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  A regular n-gon $A_1, A_2 \cdots, A_n$ $(n \geq 3)$ has center F, the focus of the parabola $y^2 = 2px$, and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \cdots, FA_n$ cut the parabola at points $B_1, B_2, \cdots, B_n$.

  Prove that
  $$\frac{1}{n}\sum_{k=1}^{n} FB_k^2 > p^2.$$

- **5150:** *Proposed by Mohsen Soltanifar(student, University of Saskatchewan), Saskatoon, Canada*

  Let $\{A_n\}_{n=1}^{\infty}, (A_n \in M_{n\times n}(C))$ be a sequence of matrices such that $\det(A_n) \neq 0, 1$ for all $n \in N$. Calculate:
  $$\lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|\det\left(adj^{\circ n}\left(A_n\right)\right)|\right)},$$
  where $adj^{\circ n}$ refers to $adj \circ adj \circ \cdots \circ adj, n$ times, the n^{th} iterate of the classical adjoint.

- **5151:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Find the value of
$$\prod_{n=1}^{\infty}\left(\sqrt{\frac{\pi}{2}}\cdot\frac{(2n-1)!!\sqrt{2n+1}}{2^n n!}\right)^{(-1)^n}.$$

  More generally, if $x \neq n\pi$ is a real number, find the value of
$$\prod_{n=1}^{\infty}\left(\frac{x}{\sin x}\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{(n\pi)^2}\right)\right)^{(-1)^n}.$$

Solutions

- **5128:** *Proposed by Kenneth Korbin, New York, NY*

  Find all positive integers less than 1000 such that the sum of the divisors of each integer is a power of two.

  For example, the sum of the divisors of 3 is 2^2, and the sum of the divisors of 7 is 2^3.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

For $n \geq 1$, let $\sigma(n)$ denote the sum of the positive divisors of n. The problem is to find all positive integers $n < 1000$ such that $\sigma(n) = 2^k$ for some integer $k \geq 0$. We note first that $n = 1$ is a solution since $\sigma(1) = 1 = 2^0$. For the remainder, we will assume that $n \geq 2$. Our key result is the following:

Lemma. If p is prime and k and e are positive integers such that $\sigma(p^e) = 2^k$, then $e = 1$ and $p = 2^k - 1$ (i.e., p is a Mersenne prime).

Proof. First of all, $p \neq 2$ since $\sigma(2^e) = 1 + 2 + \ldots + 2^e$, which is odd. Further, since p must be odd,
$$2^k = \sigma(p^e) = 1 + p + \ldots + p^e$$
implies that e is also odd. It follows that
$$\begin{aligned} 2^k &= (1+p) + \left(p^2+p^3\right) + \left(p^4+p^5\right) + \ldots + \left(p^{e-1}+p^e\right) \\ &= (1+p)\left(1+p^2+p^4+\ldots+p^{e-1}\right). \qquad (*) \end{aligned}$$
Then, $1+p$ divides 2^k and $1+p>1$, which leads us to conclude that $1+p=2^m$, with $1 \leq m \leq k$. Statement (*) reduces to
$$2^{k-m} = 1 + p^2 + p^4 + \ldots + p^{e-1}.$$
If $e \geq 3$, then $m < k$ and using the same reasoning as above, we get
$$\begin{aligned} 2^{k-m} &= \left(1+p^2\right) + \left(p^4+p^6\right) + \ldots + \left(p^{e-3}+p^{e-1}\right) \\ &= \left(1+p^2\right)\left(1+p^4+\ldots+p^{e-3}\right), \end{aligned}$$
which implies that $1+p^2 = 2^i$, for some positive integer $i \leq k-m$. Thus,
$$2^i = 1 + p^2 = 1 + (2^m-1)^2 = 2^{2m} - 2^{m+1} + 2,$$

or

$$2^{i-1} = 2^{2m-1} - 2^m + 1 = 2^m \left(2^{m-1} - 1\right) + 1.$$

This requires $i = m = 1$, which is impossible since this would entail $p = 2^m - 1 = 2 - 1 = 1$. Therefore, $e = 1$ and $2^k = \sigma(p) = p + 1$, i.e., $p = 2^k - 1$.

To return to our problem, we may write

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$$

for distinct primes $p_1, \ldots, p_m$ and positive integers $e_1, \ldots, e_m$. Since σ is multiplicative and $p_1^{e_1}, \ldots, p_m^{e_m}$ are pairwise relatively prime,

$$2^k = \sigma(n) = \sigma\left(p_1^{e_1}\right) \sigma\left(p_2^{e_2}\right) \cdots \sigma\left(p_m^{e_m}\right).$$

Further, for $i = 1, \ldots, m$, $\sigma\left(p_i^{e_i}\right) \geq p_i + 1 > 1$. Hence, there are positive integers $k_1, \ldots, k_m$ such that

$$\sigma\left(p_i^{e_i}\right) = 2^{k_i}$$

for $i = 1, \ldots, m$. By the Lemma, $e_1 = e_2 = \ldots = e_m = 1$ and

$$p_i = 2^{k_i} - 1$$

for $i = 1, \ldots, m$. Therefore, $n = p_1 p_2 \cdots p_m$, where each p_i is a distinct Mersenne prime.

To solve our problem, we need to find all Mersenne primes < 1000 and all products of distinct Mersenne primes for which the product < 1000. The Mersenne primes < 1000 are $3, 7, 31$, and 127. All solutions of $\sigma(n) = 2^k$, with $n < 1000$, are listed below.

n	$\sigma(n)$
1	2^0
3	2^2
7	2^3
$21 = 3 \cdot 7$	2^5
31	2^5
$93 = 3 \cdot 31$	2^7
127	2^7
$217 = 7 \cdot 31$	2^8
$381 = 3 \cdot 127$	2^9
$651 = 3 \cdot 7 \cdot 31$	2^{10}
$889 = 7 \cdot 127$	2^{10}

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; Harry Sedinger, St. Bonaventure, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; Tran Trong Hoang Tuan (student, Bac Lieu High School for the Gifted), Bac Lieu City, Vietnam, and the proposer.

- **5129:** *Proposed by Kenneth Korbin, New York, NY*

Given prime number c and positive integers a and b such that $a^2 + b^2 = c^2$, express in terms of a and b the lengths of the legs of the primitive Pythagorean Triangles with hypotenuses c^3 and c^5, respectively.

Solution 1 by Howard Sporn, Great Neck, NY

A Pythagorean Triple (a, b, c) can be represented by the complex number $a + bi$, with modulus c. By multiplying two Pythagorean Triples in this form, one can generate another Pythagorean Triple. For instance, the complex representation of the 3-4-5 triangle is $3 + 4i$. By multiplying the complex number by itself, (and taking the absolute value of the real and imaginary parts), one obtains the 7-24-25 triangle:

$$\begin{aligned}(3+4i)(3+4i) &= -7+24i \\ 7^2+24^2 &= 25^2\end{aligned}$$

By cubing $a + bi$, one can obtain a Pythagorean Triple whose hypotenuse is c^3.

$$\begin{aligned}(a+bi)^3 &= (a+bi)^2(a+bi) \\ &= (a^2-b^2+2abi)(a+bi) \\ &= a^3-3ab^2+i\left(3a^2-b^3\right)\end{aligned}$$

One can verify that the modulus of this complex number is $\left(a^2+b^2\right)^3 = c^3$. Thus we obtain the Pythagorean Triple $(\left|a^3-3ab^2\right|, \left|3a^2b-b^3\right|, c^3)$.

That this Pythagorean Triangle is primitive can be seen by factoring the lengths of the legs:

$$\begin{aligned}a^3-3ab^2 &= a\left(a^2-3b^2\right), \text{ and} \\ 3a^2b-b^3 &= b\left(3a^2-b^2\right),\end{aligned}$$

generally have no factors in common.

Example: If we let $(a, b, c) = (3, 4, 5)$, we obtain the Pythagorean Triple $(117, 44, 125)$.

By a similar procedure , one can obtain a Pythagorean Triple whose hypotenuse is c^5.

$$\begin{aligned}(a+bi)^5 &= (a+bi)^3(a+bi)(a+bi) \\ &= \left[a^3-3ab^2+i\left(3a^2b-b^3\right)\right](a+bi)(a+bi) \\ &= \left[a^4-6a^2b^2+b^4+i\left(4a^3b-4ab^3\right)\right](a+bi) \\ &= a^5-10a^3b^2+5ab^4+i\left(5a^4b-10a^2b^3+b^5\right).\end{aligned}$$

Thus we obtain the Pythagorean Triple

$$\left(\left|a^5-10a^3b^2+5ab^4\right|,\ \left|5a^4b-10a^2b^3+b^5\right|,\ c^5\right).$$

Example: If we let $(a, b, c) = (3, 4, 5)$, we obtain the Pythagorean Triple $(237, 3116, 3125)$.

Solution 2 by Brian D. Beasley, Clinton, SC

Given positive integers a, b, and c with c prime and $c^2 = a^2 + b^2$, we may assume without loss of generality that $a < b < c$. Also, we note that c must be odd and that c divides neither a nor b. Using the classic identity

$$(w^2 + x^2)(y^2 + z^2) = (wy + xz)^2 + (wz - xy)^2,$$

we proceed from $c^2 = a^2 + b^2$ to obtain $c^4 = (-a^2 + b^2)^2 + (2ab)^2$. Similarly, we have

$$c^6 = (-a^3 + 3ab^2)^2 + (3a^2b - b^3)^2$$

and

$$c^{10} = (a^5 - 10a^3b^2 + 5ab^4)^2 + (-5a^4b + 10a^2b^3 - b^5)^2.$$

Thus the leg lengths for the Primitive Pythagorean Triangle (PPT) with hypotenuse c^3 are

$$m = |-a^3 + 3ab^2| \quad \text{and} \quad n = |3a^2b - b^3|,$$

while the leg lengths for the PPT with hypotenuse c^5 are

$$q = |a^5 - 10a^3b^2 + 5ab^4| \quad \text{and} \quad r = |-5a^4b + 10a^2b^3 - b^5|.$$

To show that these triangles are primitive, we first note that $(-a^2 + b^2, 2ab, c^2)$ is a PPT, since c cannot divide $2ab$. Next, we prove that (m, n, c^3) is also a PPT: If not, then c divides both $a(-a^2 + 3b^2)$ and $b(3a^2 - b^2)$, so c divides $-a^2 + 3b^2$ and $3a^2 - b^2$; thus c divides the linear combination $(-a^2 + 3b^2) + 3(3a^2 - b^2) = 8a^2$, a contradiction. Similarly, we prove that (q, r, c^5) is a PPT: If not, then c divides both $a(a^4 - 10a^2b^2 + 5b^4)$ and $b(-5a^4 + 10a^2b^2 - b^4)$, so c divides $a^4 - 10a^2b^2 + 5b^4$ and $-5a^4 + 10a^2b^2 - b^4$; thus c divides the linear combinations

$$(a^4 - 10a^2b^2 + 5b^4) + 5(-5a^4 + 10a^2b^2 - b^4) = 8a^2(-3a^2 + 5b^2)$$

and

$$5(a^4 - 10a^2b^2 + 5b^4) + (-5a^4 + 10a^2b^2 - b^4) = 8b^2(-5a^2 + 3b^2).$$

But this means that c divides the linear combination $3(-3a^2 + 5b^2) - 5(-5a^2 + 3b^2) = 16a^2$, a contradiction.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; David E. Manes, Oneonta, NY, and the proposer.

- **5130:** *Proposed by Michael Brozinsky, Central Islip, NY*

In Cartesianland, where immortal ants live, calculus has not been discovered. A bride and groom start out from $A(-a, 0)$ and $B(b, 0)$ respectively where $a \neq b$ and $a > 0$ and $b > 0$ and walk at the rate of one unit per second to an altar located at the point P on line $L : y = mx$ such that the time that the first to arrive at P has to wait for the other to arrive is a maximum. Find, without calculus, the locus of P as m varies through all nonzero real numbers.

Solution 1 by Michael N. Fried, Kibbtuz Revivim, Israel

Let OQ be the line $y = mx$. Since it is the total time which must be a minimum, we might as well consider the minimum time from A to a point P on OQ and then from P to B. But since the speed is equal and constant for both the bride and groom the minimum time will be achieved for the path having the minimum distance. This, as is well-known, occurs when $\angle APO = \angle BPQ$. Accordingly, OP is the *external* angle bisector of angle APB, and, thus, $\dfrac{BP}{AP} = \dfrac{BO}{OA}$ = a constant ratio. So, P lies on a circle (an Apollonius circle) whose diameter is OAC, where OC is the harmonic mean between OA and OB.

Solution 2 by the proposer

Since the bride and groom go at the same rate, then for a given m, P is the point such that the maximum of $||AQ| - |BQ||$ for points Q on L occurs when Q is P. Let A' denote the reflection of A about this line.

Now since $||AQ| - |BQ|| = ||A'Q| - |BQ|| \geq |A'B|$ (from the triangle inequality) we have this maximum must be $|A'B|$ since it is attained when P is the point of intersection of the line through B and A', with L. (Note that the line through A' and B is not parallel to L because that would imply that the origin is the midpoint of AB because the line through the midpoint of AA' and the midpoint of AB is parallel to the line through A' and B.)

Let M be the midpoint of segment AA'. Now, since triangles $A'PM$ and APM are congruent, L is the angle bisector at P in triangle ABP, and since an angle bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides we have $\dfrac{AP}{BP} = \dfrac{a}{b}$ (1).

Denoting P by $P(X, Y)$ we thus have $Y \neq 0$ and thus $X \neq 0$ and so from (1)

$$\frac{\sqrt{(X+a)^2 + (mX)^2}}{\sqrt{(X-b)^2 + (mX)^2}} = \frac{a}{b},$$

and since $X \neq 0$, we have by squaring both sides and solving for X, that

$$\begin{aligned} X &= \frac{2ab}{(a-b)(m^2+1)}, \text{ and thus} \\ Y &= \frac{2mab}{(a-b)(m^2+1)} \end{aligned}$$

are parametric equations of the locus. Now replacing m by $\dfrac{Y}{X}$ and simplifying, we obtain

$$X = \frac{2abX^2}{(X^2+Y^2)(a-b)}$$

which is just the circle

$$(X^2 + Y^2)(a - b) = 2abX$$

with the endpoints of the diameter deleted. The endpoints of the diameter occur when $Y = 0$; that is, at $(0, 0)$, and at $\left(\dfrac{2ab}{a-b}, 0\right)$.

Note that if the line $x = 0$ were a permissible altar line, then we would add $(0, 0)$ to the locus, while if the $x-$axis were a permissible altar line, then the union of the rays $(-\infty, -a] \cup [b, \infty)$ would be part of the locus, and in particular, this includes $\left(\dfrac{2ab}{a-b}, 0\right)$.

- **5131:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c}\geq\frac{15}{8}.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

The inequality is homogeneous, so we can assume without loss of generality that $a+b+c=1$, being equivalent to

$$\frac{1+2c}{3-c}+\frac{1+2b}{3-b}+\frac{1+2a}{3-a}\geq\frac{15}{8},$$

which is Jensen's inequality $f(c)+f(b)+f(a)\geq 3f\left(\dfrac{c+b+a}{3}\right)$ applied to the convex function $f\left(x\right)=\dfrac{1+2x}{3-x}$ and the numbers $c,\ b,\ a$ on the interval $(0,1)$; equality occurs if and only if $a=b=c$.

Solution 2 by Javier García Cavero (student, Mathematics Club of the Instituto de Educación Secundaria- N^o 1), Requena-Valencia, Spain

Changing the variables, that is to say, calling

$$\begin{aligned} x &= 2a+3b+3c, \\ y &= 3a+2b+3c, \text{ and} \\ z &= 3a+3b+2c \end{aligned}$$

it is easy to see, solving the corresponding system of equations, that

$$\begin{aligned} a+b+c &= \frac{x+y+z}{8} \text{ and that} \\ a &= \frac{-5x+3y+3z}{8} \\ b &= \frac{3x-5y+3z}{8}, \text{ and} \\ c &= \frac{3x+3y-5z}{8}. \end{aligned}$$

The numerators of the fractions will thus be:

$$a+b+3c=\frac{7x+7y-9z}{8}, \quad a+3b+c=\frac{7x-9y+7z}{8}, \quad 3a+b+c=\frac{-9x+7y+7z}{8}$$

Replacing everything in the initial expression:

$$\begin{aligned} &\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c} \\ = \quad &\frac{7x+7y-9z}{8z}+\frac{7x-9y+7z}{8y}+\frac{-9x+7y+7z}{8x} \end{aligned}$$

$$= \left(\frac{7x}{8z} + \frac{7y}{8z} + \frac{-9}{8}\right) + \left(\frac{7x}{8y} + \frac{-9}{8} + \frac{7z}{8y}\right) + \left(\frac{-9}{8} + \frac{7y}{8x} + \frac{7z}{8x}\right)$$

$$= 3 \cdot \left(\frac{-9}{8}\right) + \frac{7}{8}\left(\frac{x}{z} + \frac{y}{z} + \frac{x}{y} + \frac{z}{y} + \frac{y}{x} + \frac{z}{x}\right)$$

$$\frac{-27}{8} + \frac{7}{8}\left(\left(\frac{x}{z} + \frac{z}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{x}{y} + \frac{y}{x}\right)\right)$$

$$\geq \frac{-27}{8} + \frac{42}{8}$$

$$= \frac{15}{8},$$

since $r + \frac{1}{r} \geq 2$. Equality occurs for $x = y = z$ and, therefore, for $a = b = c$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Since

$$\frac{a+b+3c}{3a+3b+2c} + \frac{b+c+3a}{3b+3c+2a} + \frac{c+a+3b}{3c+3a+2b} - \frac{15}{8}$$

$$= \frac{7\left(6a^3 + 6b^3 + 6c^3 - a^2b - ab^2 - b^2c - bc^2 - c^2a - ca^2 - 12abc\right)}{8\left(3a+3b+2c\right)\left(3b+3c+2a\right)\left(3c+3a+2b\right)}$$

$$= \frac{7\Big((3a+3b+2c)(a-b)^2 + (3b+3c+2a)(b-c)^2 + (3c+3a+2b)(c-a)^2\Big)}{8(3a+3b+2c)(3b+3c+2a)(3c+3a+2b)}$$

$$\geq 0,$$

the inequality of the problem follows.

Solution 4 by P. Piriyathumwong (student, Patumwan Demonstration School), Bangkok, Thailand

The given inequality is equivalent to the following:

$$\sum_{cyc}\left(\frac{a+b+3c}{3a+3b+2c} - \frac{5}{8}\right) \geq 0 \quad\Leftrightarrow\quad \sum_{cyc}\left(\frac{-a-b+2c}{3a+3b+2c}\right) \geq 0$$

$$\Leftrightarrow \quad \sum_{cyc}\left(\frac{(c-a)+(c-b)}{3a+3b+2c}\right) \geq 0$$

$$\Leftrightarrow \quad \sum_{cyc}(a-b)\left(\frac{1}{2a+3b+3c} - \frac{1}{3a+2b+3c}\right) \geq 0$$

$$\Leftrightarrow \quad \sum_{cyc}\frac{(a-b)^2}{(2a+3b+3c)(3a+2b+3c)} \geq 0,$$

which is obviously true.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata", Rome, Italy; Boris Rays, Brooklyn, NY; Tran Trong Hoang Tuan (student, Bac Lieu High School for the Gifted), Bac Lieu City, Vietnam, and the proposer.

- **5132:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Find all all functions $f : C \to C$ such that $f(f(z)) = z^2$ for all $z \in C$.

Solution by Kee-Wai Lau, Hong Kong, China

We show that no such functions $f(z)$ exist by considering the values of $f(1), f(-1), f(i), f(-i)$, where $i = \sqrt{-1}$.

From the given relation

$$f(f(z)) = z^2 \qquad (1)$$

we obtain $f(f(f(z))) = f(z^2)$ so that

$$(f(z))^2 = f\left(z^2\right). \qquad (2)$$

Replacing z by z^2 in (2), we get

$$f(z^4) = (f(z))^4. \qquad (3)$$

By putting $z = 1$ into (2), we obtain $f(1) = 0$ or 1. If $f(1) = 0$, then by putting $z = i$ into (3), we get $0 = f(i^4) = (f(i))^4$, so that $f(i) = 0$. Putting $z = i$ into (1) we get $f(0) = -1$ and putting $z = 0$ into (2) we obtain $(-1)^2 = -1$ which is false. It follows that

$$f(1) = 1. \qquad (4)$$

By putting $z = -1$ into (2) we get $(f(-1))^2$ so that $f(-1) = -1$ or 1.

If $f(-1) = -1$ then by (1), $-1 = f(f(-1)) = (-1)^2 = 1$, which is false.

Hence,

$$f(-1) = 1. \qquad (5)$$

By putting $z = i$ into (3), we are $(f(i))^4 = 1$, so that $f(i) = -1, 1, i, -i$.

If $f(i) = \pm 1$, then by (1), (4) and (5), $1 = f(f(i)) = i^2 = -1$, which is false.

If $f(i) = i$, then by (1), $i = f\left(f((i)\right) = -1$, which is also false. Hence,

$$f(i) = -i \qquad (6)$$

By putting $z = -i$ into (3), we have $(f(-i))^4 = 1$, so that $f(-i) = -1, 1, i, -i$.

If $f(-i) = \pm 1$, then by (1), (4), and (5) $1 = f(f(-i)) = (-i)^2 = -1$, which is false.

If $f(-i) = \pm i$, then by (1) and (6) $-i = f(f(-i)) = (-i)^2 = -1$, which is also false.

Thus $f(-i)$ can take no value, showing that no such $f(z)$ exists.

Also solved by Howard Sporn and Michael Brozinsky (jointly), of Great Neck and Central Islip, NY (respectively), and the proposer.

- **5133:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \geq 1$ be a natural number. Calculate

$$I_n = \int_0^1 \int_0^1 (x-y)^n dx dy.$$

Solutions 1 and 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

Solution 1) We first calculate $\int_0^1 (x-y)^n dx$.

Letting $u = x - y$ we get

$$\begin{aligned} \int_0^1 (x-y)^n &= \int_{-y}^{1-y} u^n du \\ &= \frac{1}{n+1}\left[(1-y)^{n+1} + (-1)^n y^{n+1}\right]. \end{aligned}$$

Now,

$$\begin{aligned} I_n &= \int_0^1 \int_0^1 (x-y)^n dx dy \\ &= \frac{1}{n+1}\int_0^1 \left[(1-y)^{n+1} + (-1)^n y^{n+1}\right] dy \\ &= \begin{cases} \dfrac{2}{(n+1)(n+2)} & : n \text{ even} \\ 0 & : n \ \text{ odd} \end{cases} \end{aligned}$$

Solution 2) Using the fact that

$$(x-y)^n = \sum_{k=0}^{n} C_n^k \Big(-1\Big)^k x^{n-k} y^k,$$

we get

$$\begin{aligned} I_n &= \int_0^1 \int_0^1 (x-y)^n dx dy \\ &= \int_0^1 \int_0^1 \sum_{k=0}^{n} C_n^k (-1)^k x^{n-k} y^k dx dy \end{aligned}$$

$$= \sum_{k=0}^{n} C_n^k (-1)^k \frac{1}{(n-k+1)(k+1)}.$$

Comment: Comparing Solution 1 with Solution 2, we obtain an interesting *side-result:* namely the identity

$$\sum_{k=0}^{n} C_n^k (-1)^k \frac{1}{(n-k+1)(k+1)} = \begin{cases} \dfrac{2}{(n+1)(n+2)} & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases},$$

which one can verify directly, as well.

Solution 3 by Paul M. Harms, North Newton, KS

Let $f(x,y) = (x-y)^n$. The integration region is the square in the x,y plane with vertices at $(0,0),(1,0),(1,1)$, and $(0,1)$. The line $y=x$ divides this region into two congruent triangles. I will use the terms *lower triangle* and *upper triangle,* for these two congruent triangles.

The points (x,y) and (y,x) are symmetric with respect to the line $y=x$. Let n be an odd integer. For each point (x,y) in the lower (upper) triangle we have a point (y,x) in the upper (lower) triangle such that $f(y,x) = -f(x,y)$. Thus the value of $I_n = 0$ when n is an odd integer.

When n is an even integer, $f(y,x) = f(x,y)$ and the value of the original double integral should equal $2\int_0^1 \int_y^1 (x-y)^n dxdy$ where the region of the integration is the lower triangle. The first integration of the last double integral yields

$$\left.\frac{(x-y)^{n+1}}{n+1}\right|_y^1 = \frac{(1-y)^{n+1}}{n+1}.$$

The second integration of the double integral then yields the expression

$$\left.\frac{-2(1-y)^{n+2}}{(n+1)(n+2)}\right|_0^1 = \frac{2}{(n+1)(n+2)} = I_n$$

when n is an even integer.

Also solved by Brian D. Beasley, Clinton, SC; Michael C. Faleski, University Center, MI; G. C. Greubel, Newport News, VA; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; James Reid (student, Angelo State University), San Angelo, TX; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
May 15, 2011

- **5152:** *Proposed by Kenneth Korbin, New York, NY*

  Given prime numbers x and y with $x > y$. Find the dimensions of a primitive Pythagorean Triangle which has hypotenuse equal to $x^4 + y^4 - x^2y^2$.

- **5153:** *Proposed by Kenneth Korbin, New York, NY*

  A trapezoid with sides $(1, 1, 1, x)$ and a trapezoid with sides $(1, x, x, x)$ are both inscribed in the same circle. Find the diameter of the circle.

- **5154:** *Proposed by Andrei Răzvan Băleanu (student, George Cosbuc National College) Motru, Romania*

  Let a, b, c be the sides, m_a, m_b, m_c the lengths of the medians, r the in-radius, and R the circum-radius of the triangle ABC. Prove that:

  $$\frac{m_a^2}{1+\cos A} + \frac{m_b^2}{1+\cos B} + \frac{m_c^2}{1+\cos C} \ge 6Rr\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

- **5155:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be the roots of the equation $x^4 + 6x^3 + 7x^2 + 6x + 1 = 0$. Find the value of

  $$\frac{3-2a}{1+a} + \frac{3-2b}{1+b} + \frac{3-2c}{1+c} + \frac{3-2d}{1+d}.$$

- **5156:** *Proposed by Yakub N. Aliyev, Khyrdalan, Azerbaijan*

  Given two concentric circles with center O and let A be a point different from O in the interior of the circles. A ray through A intersects the circles at the points B and C. The ray OA intersects the circles at the points B_1 and C_1, and the ray through A perpendicular to line OA intersects the circles at the points B_2 and C_2. Prove that

  $$B_1C_1 \le BC \le B_2C_2.$$

- **5157:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

  Let $p \ge 2, \lambda \ge 1$ be real numbers and let $e_k(x)$ for $1 \le k \le n$ be the symmetric elementary functions in the variables $x = (x_1, \ldots, x_n)$ and $x^p = (x_1^p, \ldots x_n^p)$, with $n \ge 2$ and $x_i > 0$ for all $i = 1, 2, \ldots, n$.

Prove that

$$e_n^{(pk/n)}(x) \leq \frac{e_k(x^p) + \lambda\left(e_k^p(x) - e_k(x^p)\right)}{\binom{n}{k} + \lambda\left(\binom{n}{k}^p - \binom{n}{k}\right)} \leq \left(\frac{e_1(x)}{n}\right)^{pk}, \quad 1 \leq k \leq n.$$

Solutions

- **5134:** *Proposed by Kenneth Korbin, New York, NY*

Given isosceles $\triangle ABC$ with cevian CD such that $\triangle CDA$ and $\triangle CDB$ are also isosceles, find the value of

$$\frac{AB}{CD} - \frac{CD}{AB}.$$

Solution 1 by David Stone and John Hawkins, Statesboro, GA,

Because the cevian origiantes at the vertex C, angle C plays a special role. We consider two cases. Then, ignoring degenerate triangles, we have the following solutions. Note that solving simple algebraic equations involving the angles (and a little trig) are all that is needed.

Case 1 – Angle C is one of the base angles of our isosceles triangle

$\angle A$	$\angle B$	$\angle C$	$\angle ACD$	$\angle BCD$	$\angle ADC$	$\angle BDC$	$\dfrac{AB}{CD} - \dfrac{CD}{AB}$
$\frac{2\pi}{5}$	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{\pi}{5}$	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	1
$\frac{3\pi}{7}$	$\frac{\pi}{7}$	$\frac{3\pi}{7}$	$\frac{2\pi}{7}$	$\frac{\pi}{7}$	$\frac{2\pi}{7}$	$\frac{5\pi}{7}$	$2\cos\left(\frac{\pi}{7}\right) - \dfrac{1}{2\cos\left(\frac{\pi}{7}\right)}$ ≈ 1.24698

Case 2 – Angles A and B are the base angles

$\angle A$	$\angle B$	$\angle C$	$\angle ACD$	$\angle BCD$	$\angle ADC$	$\angle BDC$	$\dfrac{AB}{CD} - \dfrac{CD}{AB}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3}{2}$
$\frac{\pi}{5}$	$\frac{\pi}{5}$	$\frac{3\pi}{5}$	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	$\frac{2\pi}{5}$	$\sqrt{5}$

The derivation:

Case 1: $\angle C$ is one of the base angles

Without loss of generality we may assume $\angle A = \angle C$. For convenience, let $\angle A = \alpha$ and $\angle$ADC=β.

$\triangle CDA$ must be isosceles, so we have three possibilities:

$$\begin{array}{ll} (a) & \alpha = \angle ACD \\ (b) & \quad \alpha = \beta \\ (c) & \angle ACD = \beta \end{array}$$

Subcase (a) is impossible because $\angle ACD\ <$ the base angle $\angle C = \alpha$,

Subcase (b):

Because $\alpha = \beta$, we have $\angle$ACD$=\pi - 2\alpha$, so

$$\angle BCD = \angle C - \angle ACD = \alpha - (\pi - 2\alpha) = 3\alpha - \pi.$$

We also see that $\alpha = \beta$ must be acute angles forcing $\angle BDC$ to be obtuse. Thus the isosceles triangle $\triangle BDC$ must have $\angle B = \angle BCD$; i.e. $\pi - 2\alpha = 3\alpha - \pi$.

Hence, $\alpha = \dfrac{2\pi}{5} = \beta$. The values of all angles then follow.

Applying the Law of Sines, we learn that $\dfrac{AB}{CD} = \dfrac{1}{2\cos\alpha}$. Because $\alpha = \dfrac{2\pi}{5}$ and $\cos\left(\dfrac{2\pi}{5}\right) = \dfrac{-1+\sqrt{5}}{2}$, we calculate $\dfrac{AB}{CD} - \dfrac{CD}{AB} = \dfrac{1}{2\cos\alpha} - 2\cos\alpha = 1$.

Subcase (c):

Because $\angle ACD = \beta$, we have $\alpha = \pi - 2\beta$, and because $\angle A = \angle C = \alpha$, we have

$$\angle B = \pi - 2\alpha = \pi - 2(\pi - 2\beta) = 4\beta - \pi.$$

Since β is the size of two angles in the triangle it must be acute. Thus the supplement $\angle BDC$ is obtuse. Therefore the equal angles in the isosceles triangle $\triangle CDB$ must be $\angle B$ and $\angle DCB$ which equals $4\beta - \pi$.

Hence, in $\triangle BDC$, we have

$$\pi = \angle B + \angle DCB + \angle BDC = (4\beta - \pi) + (4\beta - \pi) + (\pi - \beta),$$

forcing $\beta = \dfrac{2\pi}{7}$.

Thus $\alpha = \pi - 2\beta = \pi - 2\left(\dfrac{2\pi}{7}\right) = \dfrac{3\pi}{7}$. The other angles follow and by the Law of Sines in $\triangle CDB$, we see that $\dfrac{CD}{AB} = \dfrac{1}{2\cos\left(\dfrac{\pi}{7}\right)}$.

Thus

$$\frac{AB}{CD} - \frac{CD}{AB} = 2\cos\left(\frac{\pi}{7}\right) - \frac{1}{2\cos\left(\frac{\pi}{7}\right)} \approx 1.24698$$

Case 2: Angles A and B are the base angles.

We let $\angle A = \angle B = \alpha$ and $\angle ADC = \beta$. Again, $\triangle CDA$ must be isosceles, so we have three possibilities:

$$\begin{array}{ll} (a) & \alpha = \angle ACD \\ (b) & \quad \alpha = \beta \\ (c) & \beta = \angle ACD \end{array}$$

In an analysis similar to that above, subcases (b) and (c) lead to degenerate triangles. Subcase (a) must be spit again, depending upon the isosceles nature of $\triangle CDB$, but the two possibilities lead to the two triangles presented in the table.

Editor's comment: Most of the solutions received to this problem essentially followed the above track, but the solvers often missed one of the possible answers; Michael Fried's solution and Boris Rays' solution were exceptions, they too were complete. But **Paul M. Harms** took a different approach. He placed the isosceles triangle on a coordinate system and then considered cevians from a given vertex to various points on the opposite side. The conditions of the problem led him to solving a system of equations which then enabled him to find the lengths of the sides. His method also missed one of the solutions and I spent hours in vain trying to understand why, but for the sake of seeing an alternative analysis, I present his solution.

Solution 2 by Paul M. Harms, North Newton, KS

Since some similar triangles would give the same ratios required in the problem, I will fix one side of the large triangle and check for cevians which make the small triangles isosceles.

Case 1: Let A be at $(-1,0)$, B be at $(1,0)$, and let C be at $(0,c)$ where $c>0$. If D is at $(0,0)$ and C is at $(0,1)$, then the conditions of the problem are satisfied with

$$\frac{AB}{CD}-\frac{CD}{AB}=\frac{2}{1}-\frac{1}{2}=\frac{3}{2}.$$

Case 2: Keeping the coordinates A,B as above and letting C having the coordinates $(0,c)$, we let D be at $(d,0)$ where $0<d<1$. To get the two smaller triangles to be isosceles we need $AD=AC$ and $CD=DB$.

The distance equations that need to be satisfied are

$$\begin{cases}\sqrt{c^2+d^2} &= 1-d\\ 1+d &= \sqrt{c^2+1}.\end{cases}$$

Solving this system:

$$d=-2+\sqrt{5},\quad c=\sqrt{5-2\sqrt{5}}.$$

Hence, $CD=\sqrt{14-6\sqrt{5}}$ and $\dfrac{AB}{CD}-\dfrac{CD}{AB}=\sqrt{5}$.

Case 3: Now consider C at a vertex other than the intersection of the equal sides.

Let C be at $(2,0)$, A be at $(0,0)$ and B be at $(1,b)$.

a) If $AC>AB=BC$, then $CD>BC>DB$ and $\triangle CDB$ cannot be isosceles.

b) If $AB=BC>AC$, then the smaller triangles would be isosceles when $CD=DB$ and $AC=AD$. Let D be at (d,bd). From the distance equations we have

$$\begin{cases}(d-1)^2+(bd-b)^2 &= b^2d^2+(d-2)^2\\ b^2d^2+d^2 &= 4.\end{cases}$$

From the second equation $(b^2+1)=\dfrac{4}{d^2}$. The first equation is

$(d-1)^2\left(b^2+1\right)=(b^2+1)d^2-4d+4$. Substituting for b^2+1 we obtain

$$d^3-d^2-2d+1=0.$$

Using approximations, the value of d between 0 and 1 is approximately $d=0.4451$. Then $b=4.3807, AB=\sqrt{b^2+1}=4.4934$ and $CD=\sqrt{(2-d)^2+b^2d^2}=2.4939$. Then $\dfrac{AB}{CD}-\dfrac{CD}{AB}=1.2467$.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Michael N. Fried, Kibbutz Revivim, Israel; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany (jointly); Antonio Ledesma López, Mathematical Club of the Instituto de Educación Secundaria-N^o 1, Requena-Valencia, Spain; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; Ercole Suppa, Teramo, Italy, and the proposer.

- **5135:** *Proposed by Kenneth Korbin, New York, NY*

Find a, b, and c such that

$$\begin{cases} ab+bc+ca=-3 \\ a^2b^2+b^2c^2+c^2a^2=9 \\ a^3b^3+b^3c^3+c^3a^3=-24 \end{cases}$$

with $a<b<c$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $x=ab, y=bc, z=ca$, so that $x+y+z=-3, x^2+y^2+z^2=9$ and $x^3+y^3+z^3=-24$. We have

$$\begin{aligned} xy+yz+zx &= \frac{(x+y+z)^2-x^2-y^2-z^2}{2}=0 \text{ and} \\ xyz &= \frac{x^3+y^3+z^3-(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}{3}=1. \end{aligned}$$

Thus $abc(a+b+c)=0$ and $(abc)^2=1$. Hence, either

$$\begin{cases} a+b+c=0 \\ ab+bc+ca=-3 \\ abc=1 \end{cases} \text{ or } \begin{cases} a+b+c=0 \\ ab+bc+ca=-3 \\ abc=-1. \end{cases}$$

In the former case, a, b, c are the roots of the equation $t^3-3t-1=0$ and in the latter case, the roots of the equation $t^3-3t+1=0$. By standard formula, we obtain respectively

$$a=-2\cos\left(\frac{2\pi}{9}\right),\ b=-2\cos\left(\frac{4\pi}{9}\right),\ c=2\cos\left(\frac{\pi}{9}\right)$$

and

$$a=-2\cos\left(\frac{\pi}{9}\right),\ b=2\cos\left(\frac{4\pi}{9}\right),\ c=2\cos\left(\frac{\pi}{9}\right)$$

Solution 2 by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo, TX

If $a = 0$, then the given system of equations becomes

$$bc = -3, \quad b^2c^2 = 9, \quad b^3c^3 = -24,$$

which is impossible. Thus, $a \neq 0$, and similarly $b \neq 0$ and $c \neq 0$.
Let $x = ab$, $y = bc$, and $z = ca$, then

$$\begin{aligned}
(ab+bc+ca)^2 &= (x+y+z)^2 \\
&= x^2+y^2+z^2+2(xy+yz+zx) \\
&= a^2b^2+b^2c^2+c^2a^2+2(ab^2c+bc^2a+ca^2b) \\
&= a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c) \qquad \mathbf{(1)}
\end{aligned}$$

and $(ab+bc+ca)^3 = (x+y+z)^3$

$$\begin{aligned}
&= x^3+y^3+z^3+3(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)+6xyz \\
&= 3(x^3+xy^2+xz^2+yx^2+y^3+yz^2+zx^2+zy^2+z^3)-2(x^3+y^3+z^3)+6xyz \\
&= 3x(x^2+y^2+z^2)+3y(x^2+y^2+z^2)+3z(x^2+y^2+z^2)-2(x^3+y^3+z^3)+6xyz \\
&= 3(x+y+z)(x^2+y^2+z^2)-2(x^3+y^3+z^3)+6xyz \\
&= 3(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2)-2(a^3b^3+b^3c^3+c^3a^3)+6a^2b^2c^2. \qquad \mathbf{(2)}
\end{aligned}$$

Using **(1)** and **(2),** it can easily be shown that the given system of equations is equivalent to

$$\begin{cases} a+b+c=0 \\ ab+bc+ca=-3 \\ a^2b^2c^2=1 \end{cases} \qquad \mathbf{(3)}$$

If $abc = 1$, then the solutions of **(3)** are roots of the cubic equation

$$\begin{aligned}
0 &= (t-a)(t-b)(t-c) \\
&= t^3-(a+b+c)t^2+(ab+bc+ca)t-abc \\
&= t^3-3t-1, \qquad \mathbf{(4)}
\end{aligned}$$

and must be strictly between -2 and 2 by the Upper and Lower Bound theorem.

Let $t = 2\cos\theta$, with $0 < \theta < \pi$, by **(4),**

$$\begin{aligned}
0 &= 8\cos^3\theta - 6\cos\theta - 1 \\
\frac{1}{2} &= 4\cos^3\theta - 3\cos\theta \\
\frac{1}{2} &= \cos(3\theta) \\
\theta &= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}.
\end{aligned}$$

Thus,

$$a = 2\cos\left(\frac{7\pi}{9}\right) = -2\cos\left(\frac{2\pi}{9}\right), \quad b = 2\cos\left(\frac{5\pi}{9}\right) = -2\cos\left(\frac{4\pi}{9}\right), \quad c = 2\cos\left(\frac{\pi}{9}\right).$$

Similarly, when $abc = -1$,

$$a = -2\cos\left(\frac{\pi}{9}\right), \quad b = 2\cos\left(\frac{4\pi}{9}\right), \quad c = 2\cos\left(\frac{2\pi}{9}\right).$$

Also solved by Brian D. Beasley, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany (jointly); Antonio Ledesma López, Mathematical Club of the Instituto de Educación Secundaria-N^{o} 1, Requena-Valencia, Spain; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5136:** *Proposed by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico*

Prove that for every natural n, the real number

$$\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} + \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$$

is irrational.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $x = \left(\sqrt{19} - 3\sqrt{2}\right)^{1/n}$, $y = \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$ and $z = x + y$.

It is easy to check that $y > x$, $xy = 1$ and that x and y are the two roots of the equation, $t + \dfrac{1}{t} = z$. Solving for t, we obtain $x = \dfrac{z - \sqrt{z^2 - 4}}{2}$ and $y = \dfrac{z + \sqrt{z^2 - 4}}{2}$.

Applying the binomial theorem, we have

$$\begin{aligned}
& 2^{n+1}\sqrt{19} \\
= \ & 2^n\left(x^n + y^n\right) \\
= \ & \left(z - \sqrt{z^2 - 4}\right)^n + \left(z - \sqrt{z^2 + 4}\right)^n \\
= \ & \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^{n-k}\left(z^2 - 4\right)^{k/2} + \sum_{k=0}^{n} \binom{n}{k} z^{n-k}\left(z^2 - 4\right)^{k/2} \\
= \ & 2\sum_{j=0}^{m} \binom{n}{2j} z^{n-2j}\left(z^2 - 4\right)^{j},
\end{aligned}$$

where m is the greatest integer not exceeding $\dfrac{n}{2}$. Hence, if z is rational, then $\sqrt{19}$ is also rational, which is false. Thus z is in fact irrational and this completes the solution.

Solution 2 by Valmir Bucaj (student, Texas, Lutheran University), Seguin, TX

To contradiction, suppose that

$$\left(\sqrt{19}-3\sqrt{2}\right)^{1/n}+\left(\sqrt{19}+3\sqrt{2}\right)^{1/n}$$

is rational.

Then it is easy to see that both $\left(\sqrt{19}-3\sqrt{2}\right)^{1/n}$ and $\left(\sqrt{19}+3\sqrt{2}\right)^{1/n}$ have to be rational. Let

$$\left(\sqrt{19}-3\sqrt{2}\right)^{1/n}=\frac{a}{b} \quad \text{and} \quad \left(\sqrt{19}+3\sqrt{2}\right)^{1/\mathrm{n}}=\frac{c}{d} \qquad (\mathbf{1})$$

where $a, b, c, d \in Z$.

Raising both expressions in (**1**) to the power of n, we get

$$\left(\sqrt{19}-3\sqrt{2}\right)=\left(\frac{a}{b}\right)^{n} \quad \text{and} \quad \left(\sqrt{19}+3\sqrt{2}\right)=\left(\frac{c}{d}\right)^{\mathrm{n}} \qquad (\mathbf{2})$$

Adding the left sides and the right sides of the expressions in **(2)** and dividing by 2 we get

$$\sqrt{19}=\frac{1}{2}\left[\left(\frac{a}{b}\right)^{n}+\left(\frac{c}{d}\right)^{n}\right]. \qquad (\mathbf{3})$$

However, since $\frac{a}{b} \in Q$ and $\frac{c}{d} \in Q$, it follows that $\frac{1}{2}\left[\left(\frac{a}{b}\right)^{n}+\left(\frac{c}{d}\right)^{n}\right] \in Q$; that is, the right-hand side of the expression in (**3**) is a rational number, while the left-hand side, namely $\sqrt{19}$, is an irrational number.

Therefore, the contradiction that we arrived at shows that our initial assumption that$\left(\sqrt{19}-3\sqrt{2}\right)^{1/n}+\left(\sqrt{19}+3\sqrt{2}\right)^{1/n}$ is a rational number is not correct, hence the statement of the problem holds.

Solution 3 by Pedro H.O. Pantoja (student, UFRN), Natal, Brazil;

Using the identity

$$x^{n+1}+\frac{1}{x^{n+1}}=\left(x+\frac{1}{x}\right)\left(x^{n}+\frac{1}{x^{n}}\right)-\left(x^{n-1}+\frac{1}{x^{n-1}}\right)$$

if $x+\frac{1}{x}$ is rational, then so would be $x^{n}+\frac{1}{x^{n}}$, where $x=\left(\sqrt{19}+\sqrt{18}\right)^{\frac{1}{n}}$

$$\Rightarrow \frac{1}{x}=\left(\sqrt{19}-\sqrt{18}\right)^{\frac{1}{n}}.$$

Hence,

$$x^{n}+\frac{1}{x^{n}}=\sqrt{19}+\sqrt{18}+\sqrt{19}-\sqrt{18}=2\sqrt{19}$$

which is irrational.

It follows that $x+\frac{1}{x}$ must be irrational too.

Editor's comment: **David E. Manes** cited the paper "On A Substitution Made In Solving Reciprocal Equations" by Arnold Singer that appeared in the **Mathematics Magazine** [38(1965), p. 212] as being helpful in solving such equations. This paper starts off as follows: "The standard procedure employed in solving reciprocal equations requires the substitution $y = x + 1/x$. One is then required to write $x^2 + 1/x^2, x^3 + 1/x^3 \ldots$, as polynomials in y. Many texts give the relationships up to $x^4 + 1/x^4$ or so, and some give the recurrence relation....... This note derives the expression for $x^n + 1/x^n$ as a polynomial in y for general n."

Also solved by Michael N. Fried, Kibbutz Revivim, Israel: David E. Manes, Oneonta, NY, and the proposer.

- **5137:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive numbers such that $abc \geq 1$. Prove that

$$\prod_{cyclic} \frac{1}{a^5 + b^5 + c^2} \leq \frac{1}{27}.$$

Solution 1 by Ercole Suppa, Teramo, Italy

By AM-GM inequality we have:

$$\begin{aligned} a^5 + b^5 + c^2 &\geq 3\sqrt[3]{a^5 b^5 c^2} \\ a^2 + b^5 + c^5 &\geq 3\sqrt[3]{a^2 b^5 c^5} \\ a^5 + b^2 + c^5 &\geq 3\sqrt[3]{a^5 b^2 c^5} \end{aligned}$$

Therefore

$$\prod_{cyclic} \left(a^5 + b^5 + c^2\right) \geq 27\sqrt[3]{a^{12} b^{12} c^{12}} = 27(abc)^4 \geq 27 \qquad \Rightarrow$$

$$\prod_{cyclic} \frac{1}{a^5 + b^5 + c^2} \leq \frac{1}{27}$$

and the conclusion follows.

Solution 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin TX

Editor's comment: All solutions received were similar to the above. But Valmir Bucaj submitted two solutions to the problem. One solution was similar to the above, the other is listed below.

The inequality to be proved is equivalent to

$$(a^5 + b^5 + c^2)(a^5 + c^5 + b^2)(b^5 + c^5 + a^2) \geq 27.$$

Multiplying out the left-hand side and using the fact that $abc \geq 1$, after rearranging we get

$$(a^5 + b^5 + c^2)(a^5 + c^5 + b^2)(b^5 + c^5 + a^2) \geq \left(\frac{a^5}{b^5} + \frac{b^5}{c^5} + \frac{c^5}{a^5}\right) + \left(\frac{a^5}{c^5} + \frac{c^5}{b^5} + \frac{b^5}{a^5}\right)$$

$$
\begin{aligned}
&+ \quad \left(\frac{a^2}{b^5}+\frac{b^5}{a^2}\right)+\left(\frac{a^5}{c^2}+\frac{c^2}{a^5}\right)+\left(\frac{a^2}{c^5}+\frac{c^5}{a^2}\right)\\
&+ \quad \left(\frac{b^2}{c^5}+\frac{c^5}{b^2}\right)+\left(\frac{c^2}{b^5}+\frac{b^5}{c^2}\right)+\left(\frac{a^5}{b^2}+\frac{b^2}{a^5}\right)\\
&+ \quad a^{12}+b^{12}+c^{12}+3+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\\
&\geq \quad 21+a^{12}+b^{12}+c^{12}+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}.
\end{aligned}
$$

The last inequality follows from the fact that $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}\geq 3$ and from $\dfrac{x}{y}+\dfrac{y}{x}\geq 2$, where x,y,z are positive real numbers.

It remains to show that

$$a^{12}+b^{12}+c^{12}+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 6.$$

But this follows immediately from the AM-GM inequality and from the fact that $abc\geq 1$. That is

$$a^{12}+b^{12}+c^{12}+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 6\cdot\sqrt[6]{(a\cdot b\cdot c)^{12}\cdot\frac{1}{(a\cdot b\cdot c)^3}}=6\cdot\underbrace{\sqrt[6]{(abc)^9}}_{\geq 1}\geq 6.$$

This proves $(a^5+b^5+c^2)(a^5+c^5+b^2)(b^5+c^5+a^2)\geq 27$, and thus the statement of the problem.

Also solved by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany (jointly); Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Pedro H.O. Pantoja (student, UFRN), Natal, Brazil; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY, and the proposer.

- **5138:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $n\geq 2$ be a positive integer. Prove that

$$\frac{n}{F_nF_{n+1}}\leq\frac{1}{(n-1)F_1^2+F_2^2}+\cdots+\frac{1}{(n-1)F_n^2+F_1^2}\leq\frac{1}{n}\sum_{k=1}^{n}\frac{1}{F_k^2},$$

where F_n is the n^{th} Fibonacci number defined by $F_0=0, F_1=1$ and for all $n\geq 2,\ F_n=F_{n-1}+F_{n-2}$.

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

First we prove the LHS inequality. The function $f(x) = \frac{1}{x}$ is a convex function since the second derivative is positive, and so according to Jensen's inequality we have $\sum_{k=1}^{n} f(x) \geq nf\left(\frac{\sum_{k=1}^{n} x}{n}\right)$ and implying this together with the known fact that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ we have

$$
\begin{aligned}
\frac{1}{(n-1)F_1^2 + F_2^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} &\geq n\frac{1}{\frac{(n-1)F_1^2+F_1^2+\dots+(n-1)F_n^2+F_1^2}{n}} \\
&= \frac{n}{\sum_{k=1}^{n} F_k^2} \\
&= \frac{n}{F_n F_{n+1}}
\end{aligned}
$$

Now we will prove the RHS inequality. First using the AM-GM inequality we have

$$
\begin{aligned}
\frac{1}{(n-1)F_1^2 + F_2^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} &\leq \frac{1}{n\sqrt[n]{F_1^{2(n-1)}F_2^2}} + \dots + \frac{1}{n\sqrt[n]{F_n^{2(n-1)}F_1^2}} \\
&= \frac{\sum_{cyclic} F_1^{\frac{2}{n}} F_2^{\frac{2(n-1)}{n}} \dots F_n^2}{n\prod_{k=1}^{n} F_k^2}
\end{aligned}
$$

Now since the sequence $[2, 0, 2, ..., 2]$ majorizes the sequence $[\frac{2}{n}, \frac{2(n-1)}{n}, 2, ..., 2]$, using Muirhead's Inequality we have

$$
\begin{aligned}
\frac{1}{(n-1)F_1^2 + F_2^2} + \dots + \frac{1}{(n-1)F_n^2 + F_1^2} &\leq \frac{\sum_{cyclic} F_1^{\frac{2}{n}} F_2^{\frac{2(n-1)}{n}} \dots F_n^2}{n\prod_{k=1}^{n} F_k^2} \\
&\leq \frac{\sum_{cyclic} F_1^2 F_2^0 \dots F_n^2}{n\prod_{k=1}^{n} F_k^2} \\
&= \sum_{k=1}^{n} \frac{1}{F_k^2}
\end{aligned}
$$

and this is the end of the proof.

Solution 2 by the Proposer

Applying AM-HM inequality to the positive numbers $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$, we have

$$\frac{n}{(n-1)x_k + x_{k+1}} = \frac{n}{\underbrace{x_k + \ldots + x_k}_{n-1} + x_{k+1}} \leq \frac{1}{n}\left(\underbrace{\frac{1}{x_k} + \frac{1}{x_k} + \ldots + \frac{1}{x_k}}_{n-1} + \frac{1}{x_{k+1}}\right)$$

or

$$\frac{n}{(n-1)x_k + x_{k+1}} \leq \frac{1}{n}\left(\frac{n-1}{x_k} + \frac{1}{x_{k+1}}\right)$$

Adding the preceding inequalities for $1 \leq k \leq n$, and putting $x_{n+1} = x_1$, yields

$$\sum_{k=1}^{n} \frac{n}{(n-1)x_k + x_{k+1}} \leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{n-1}{x_k} + \frac{1}{x_{k+1}}\right) = \sum_{k=1}^{n}\frac{1}{x_k}$$

On the other hand, applying again AM-HM inequality to the positive numbers $\frac{1}{(n-1)x_k + x_{k+1}}$, $1 \leq k \leq n, (x_{n+1} = x_1)$ we have

$$\sum_{k=1}^{n} \frac{1}{(n-1)x_k + x_{k+1}} \geq \frac{n^2}{\sum_{k=1}^{n}(n-1)x_k + x_{k+1}} = \frac{n}{\sum_{k=1}^{n} x_k}$$

Combining the preceding results, we obtain

$$\left(\frac{1}{n^2}\sum_{k=1}^{n} x_k\right)^{-1} \leq \sum_{k=1}^{n} \frac{n}{(n-1)x_k + x_{k+1}} \leq \sum_{k=1}^{n}\frac{1}{x_k}$$

Setting $x_k = F_k^2$, $1 \leq k \leq n$, in the preceding inequalities, we get

$$\left(\frac{1}{n^2}\sum_{k=1}^{n} F_k^2\right)^{-1} \leq \sum_{k=1}^{n} \frac{n}{(n-1)F_k^2 + F_{k+1}^2} \leq \sum_{k=1}^{n}\frac{1}{F_k^2}$$

or

$$\left(\frac{F_nF_{n+1}}{n^2}\right)^{-1} \leq \sum_{k=1}^{n} \frac{n}{(n-1)F_k^2 + F_{k+1}^2} \leq \sum_{k=1}^{n}\frac{1}{F_k^2}$$

on account of the well-known formulae $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$. From the above the statement follows. Equality holds when $n = 2$, and we are done.

Editor's comment: **Valmir Bucaj (student, Texas Lutheran University), Seguin TX,** solved a slight variation of the given inequality. He showed that

$$\frac{n}{F_nF_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_n^2} + \cdots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{F_k^2}.$$

- **5139:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Calculate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\zeta(n+m)-1}{n+m},$$

where ζ denotes the Riemann Zeta function.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the value of the required sum is $\gamma = 0.577\ldots$, the Euler constant.

Since $\dfrac{1}{(n+m)k^{n+m}} > 0$ for positive integers, m, n and k, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\zeta(n+m)-1}{n+m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n+m}\sum_{k=2}^{\infty}\frac{1}{k^{n+m}} = \sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)k^{n+m}}.$$

For each integer $t > 1$, the number of solutions of the equation $n+m=t$ in positive integers n and m is $t-1$. Hence, $\displaystyle\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)k^{n+m}} = \sum_{k=2}^{\infty}\sum_{t=2}^{\infty}\frac{t-1}{tk^t}$.

For real $x > 1$, we have the well known series $\displaystyle\sum_{t=1}^{\infty}\frac{1}{x^t} = \frac{1}{x-1}$ and $\displaystyle\sum_{t=1}^{\infty}\frac{1}{tx^t} = -\ln\left(1-\frac{1}{x}\right)$ so that $\displaystyle\sum_{t=1}^{\infty}\frac{t-1}{tk^t} = \frac{1}{k-1} + \ln\left(\frac{k-1}{k}\right)$.

It follows that for any integer $M > 1$, we have

$$\sum_{k=2}^{M}\sum_{t=2}^{\infty}\frac{t-1}{tk^t} = \sum_{k=1}^{M-1}\frac{1}{k} - \ln(M),$$

which tends to γ as M tends to infinity. This proves our claim.

Solution 2 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

Answer: γ

Proof: We will need the well known $\displaystyle\sum_{k=1}^{n}\frac{1}{k} = \ln n + \gamma + o(1)$ and γ is of course the Euler–Mascheroni constant.

Setting $n+m=k$ the series is

$$\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\frac{\zeta(k)-1}{k} = \sum_{k=2}^{\infty}(k-1)\frac{\zeta(k)-1}{k} = \sum_{k=2}^{\infty}(\zeta(k)-1) - \sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}$$

$$\sum_{k=2}^{\infty}(\zeta(k)-1) = \sum_{k=2}^{\infty}\sum_{p=2}^{\infty}\frac{1}{p^k} = \sum_{p=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{p^k} = \sum_{p=2}^{\infty}\frac{1}{p^2}\frac{p^2}{p^2-1} = \sum_{p=2}^{\infty}\left(\frac{1}{p-1}-\frac{1}{p+1}\right) = 1$$

$$-\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k} = -\sum_{k=2}^{\infty}\sum_{p=2}^{\infty}\frac{1}{k}\frac{1}{p^k} = \sum_{p=2}^{\infty}\left(\ln(1-\frac{1}{p})+\frac{1}{p}\right) = \lim_{N\to\infty}\sum_{p=2}^{N}\left(\ln(1-\frac{1}{p})+\frac{1}{p}\right)$$

$$= \lim_{N\to\infty} \sum_{p=2}^{N} \left(\ln(p-1) - \ln p) + \frac{1}{p} \right) = \lim_{N\to\infty} ((-\ln N) + \ln N + \gamma + o(1)) = \gamma$$

Thus the result easily follows

Also solved by the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before June 15, 2011

- **5158:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with integer length sides $\overline{AB} = \overline{BC} = x$, and $\overline{CD} = \overline{DA} = x + 1$.

  Find the distance between the incenter and the circumcenter.

- **5159:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on diagonal $\overline{AC}$ and with point Q at the midpoint of side $\overline{AB}$.

  Find the perimeter of cyclic quadrilateral $ADPQ$ if its area is one unit less than the area of square $ABCD$.

- **5160:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, there are n (where $n \geq 2$) roads $\{l_i\}$ whose equations are

  $$l_i: \ x \cos\left(\frac{2\pi i}{n}\right) + y \sin\left(\frac{2\pi i}{n}\right) = i, \text{where } i = 1, 2, 3, \ldots, n.$$

  Any anthill must be located so that the sum of the squares of its distances to these n lines is $\dfrac{n(n+1)(2n+1)}{6}$. Two queen ants are (im)mortal enemies and have their anthills as far apart as possible. If the distance between these queens' anthills is 4 units, find n.

- **5161:** *Proposed by Paolo Perfetti, Department of Mathematics, University "Tor Vergata, Rome, Italy*

  It is well known that for any function $f: \Re \rightarrow \Re$, continuous or not, the set of points on the y-axis where it attains a maximum or a minimum can be at most denumerable. Prove that any function can have at most a denumerable set of inflection points, or give a counterexample.

- **5162:** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Barcelona, Spain*

Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}}+\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}}+\sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} \le \sqrt{3}.$$

- **5163:** *Proposed by Pedro H. O. Pantoja, Lisbon, Portugal*

Prove that for all $n \in N$

$$\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2}-1\right)dx = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\frac{1}{k_1\cdots k_n\,(k_1+\cdots+k_n)}.$$

Solutions

- **5140:** *Proposed by Kenneth Korbin, New York, NY*

Given equilateral triangle ABC with an interior point P such that

$$\begin{aligned}\overline{AP} &= 22+16\sqrt{2}\\ \overline{BP} &= 13+9\sqrt{2}\\ \overline{CP} &= 23+16\sqrt{2}.\end{aligned}$$

Find $\overline{AB}$.

Solution 1 Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

Let α be the length of each side of triangle ABC with vertices $A\left(0, \frac{\sqrt{3}}{2}\alpha\right)$, $B\left(-\frac{\alpha}{2}, 0\right)$, and $C\left(\frac{\alpha}{2}, 0\right)$, and let $P(x, y)$ be an interior point of the triangle. Then,

$$\begin{aligned}\overline{AP}^2 &= (22+16\sqrt{2})^2 &&= x^2+(y-\frac{\sqrt{3}}{2}\alpha)^2, &(1)\\ \overline{BP}^2 &= (13+9\sqrt{2})^2 &&= (x+\frac{\alpha}{2})^2+y^2, &(2)\\ \overline{CP}^2 &= (23+16\sqrt{2})^2 &&= (x-\frac{\alpha}{2})^2+y^2. &(3)\end{aligned}$$

Using (2)and (3), it follows that

$$(13+9\sqrt{2})^2-(23+16\sqrt{2})^2 = 2\alpha x,$$

and

$$x = \frac{(13+9\sqrt{2})^2-(23+16\sqrt{2})^2}{2\alpha}$$

$$\begin{aligned}
&= \frac{-710-502\sqrt{2}}{2\alpha} \\
&= -\frac{355+251\sqrt{2}}{\alpha}. \qquad (4)
\end{aligned}$$

Therefore, using (1), (2), and (3),

$$(13+9\sqrt{2})^2 + (23+16\sqrt{2})^2 - 2(22+16\sqrt{2})^2 = 2\sqrt{3}\alpha y - \alpha^2, \text{ and}$$

$$\begin{aligned}
y &= \frac{\alpha^2 + (13+9\sqrt{2})^2 + (23+16\sqrt{2})^2 - 2(22+16\sqrt{2})^2}{2\sqrt{3}\alpha} \\
&= \frac{\alpha^2 - 620 - 438\sqrt{2}}{2\sqrt{3}\alpha} \qquad (5)
\end{aligned}$$

Hence, using (2), (4), and (5),

$$\begin{aligned}
(13+9\sqrt{2})^2 &= \left(-\frac{355+251\sqrt{2}}{\alpha} + \frac{\alpha}{2}\right)^2 + \left(\frac{\alpha^2-620-438\sqrt{2}}{2\sqrt{3}\alpha}\right)^2 \\
331+234\sqrt{2} &= \frac{(\alpha^2-710-502\sqrt{2})^2}{4\alpha^2} + \frac{(\alpha^2-620-438\sqrt{2})^2}{12\alpha^2} \\
12(331+234\sqrt{2})\alpha^2 &= 3(\alpha^2-710-502\sqrt{2})^2 + (\alpha^2-620-438\sqrt{2})^2 \\
0 &= 4\alpha^4 - (9472+6696\sqrt{2})\alpha^2 + (3,792,412+2,681,640\sqrt{2}) \\
0 &= \alpha^4 - (2368+1674\sqrt{2})\alpha^2 + (948,103+670,410\sqrt{2}).
\end{aligned}$$

Thus, using the quadratic formula and a computer algebra system, the solutions are

$$\begin{aligned}
\alpha^2 = 2147+1518\sqrt{2} \qquad &\text{or} \qquad \alpha^2 = 221+156\sqrt{2} \\
\Rightarrow \alpha = 33+23\sqrt{2} \qquad &\text{or} \qquad \alpha = \sqrt{13}(3+2\sqrt{2}).
\end{aligned}$$

The solution $\alpha = \sqrt{13}(3+2\sqrt{2})$ is extraneous since this will make $y < 0$. Thus,

$$\overline{AB} = \alpha = 33+23\sqrt{2}.$$

Remark: If we substitute $\alpha = 33+23\sqrt{2}$ into (4) and (5), we obtain

$$x = -\frac{169+118\sqrt{2}}{31} \quad \text{and} \quad y = \frac{\sqrt{3}}{62}(237+173\sqrt{2}).$$

Solution 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

This problem is of the same nature as Problem 5081. It can be solved using *Tripolar Coordinates* and a result from Euler.

Here, we us another method and instead solve the more general problem: Let ABC be an equilateral triangle and P an interior point such that $\overline{AB} = a, \overline{BO} = b, \overline{CP} = c$. We will give a general formula for the dimensions of $\triangle ABC$

Let $(x, y), (s, 0), (0, 0), \left(\frac{s}{2}, \frac{\sqrt{3}}{2}s\right)$, be the coordinates of the points P, A, B, C respectively. Then,

$$\begin{aligned} d(P,B) &= \sqrt{x^2+y^2} = b, && (1) \\ d(P,A) &= \sqrt{(x-s)^2+y^2} = a, && (2) \\ d(C,P) &= \sqrt{\left(\frac{s}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}s - y\right)^2} = c. && (3) \end{aligned}$$

Solving (1) for y and substituting into (2) we obtain an expression for x:

$$x = \frac{s^2+b^2-a^2}{2s}.$$

Now, substituting in for $y = \sqrt{b^2 - x^2}$ and x in (3) we eventually obtain the following biquadratic equation on s:

$$s^4 - (a^2+b^2+c^2)s^2 + (a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2) = 0. \qquad (4)$$

$$s = \pm\sqrt{\frac{a^2+b^2+c^2}{2} \pm \sqrt{b^2c^2 - \left(\frac{b^2+c^2-a^2}{2}\right)^2} \cdot \sqrt{3}}.$$

Finally, the length of the sides of the given equilateral triangle are calculated by

$$s = \sqrt{\frac{a^2+b^2+c^2}{2} + \sqrt{b^2c^2 - \left(\frac{b^2+c^2-a^2}{2}\right)^2} \cdot \sqrt{3}}.$$

For the given problem we have

$$s = \sqrt{1518\sqrt{2}+2147} = \sqrt{\left(33+23\sqrt{2}\right)^2} = 33+23\sqrt{2}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\overline{AB} = 33 + 23\sqrt{2}$.

We first note that the given conditions determine $\overline{AB}$ uniquely. Suppose that $\overline{AB} = x$ and $\overline{AB} = y$ are two distinct solutions, say $y > x$. Then all the angles $\angle APB, \angle BPC, \angle CPA$ for the solution $\overline{AB} = y$ will be greater than the corresponding angles $\angle APB, \angle BPC, \angle CPA$ for the solution $\overline{AB} = x$. Ths is impossible because $\angle APB + \angle BPC + \angle CPA = 2\pi$ for both solutions.

We now need only show that $\overline{AB} = 33 + 23\sqrt{2}$ is a solution. We let

$$\cos\angle APB = \frac{-5}{7},\ \cos\angle BPC = \frac{-\left(5+3\sqrt{2}\right)}{14},\ \cos\angle CPA = \frac{-\left(5-3\sqrt{2}\right)}{14},\ \text{and}$$

$$\sin\angle APB = \frac{\sqrt{24}}{7},\ \sin\angle BPC = \frac{\sqrt{150}-\sqrt{3}}{14},\ \sin\angle CPA = \frac{\sqrt{150}+\sqrt{3}}{14}.$$

By the standard compound angle formula, we readily check that

$$\begin{aligned} \cos\left(\angle APB + \angle BPC + \angle CPA\right) &= 1, \text{ so in fact} \\ \angle APB + \angle BPC + \angle CPA &= 2\pi. \end{aligned}$$

By the cosine formula, we obtain, $\overline{AB} = \overline{BC} = \overline{CA} = 33+23\sqrt{2}$ as well and this completes the solution.

Editor's comment: Several other solution paths were used in solving this problem. One used a theorem that states that in any equilateral triangle ABC with side length a and with P being any point in the plane whose distances to the vertices A, B, C are respectively p, q and t, then $3(p^4+q^4+t^4+a^4) = (p^2+q^2+t^2+a^2)^2$. **Bruno Salgueiro Fanego** stated that a reference for this theorem can be found in the article *Curious properties of the circumcircle and incircle of an equilateral triangle,* by Prithwijit De, *Mathematical Spectrum* 41(1), 2008-2009, 32-35. This solution path works but one ends up solving the following equation:

$$13108182 + 9268884\sqrt{2} + 3\overline{AB}^4 = \left(2368 + 1674\sqrt{2}\right)^2 + 2\left(2368 + 1674\sqrt{2}\right)\overline{AB}^2 + \overline{AB}^4.$$

Another solution path dealt with using Heron's formula for the area of a triangle and using the fact that the area of ABC equals the sum of the areas of the three interior triangles APB, BPC, and CPA.

Also solved by Brian D. Beasley, Clinton, SC; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; Edwin Gray, Highland Beach, FL; Bruno Salgueiro, Fanego (two solutions), Viveiro, Spain; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.

- **5141:** *Proposed by Kenneth Korbin, New York, NY*

A quadrilateral with sides $259, 765, 285, 925$ is constructed so that its area is maximum. Find the size of the angles formed by the intersection of the diagonals.

Solution by David E. Manes, Oneonta, NY

Given the four sides of a quadrilateral, the one with maximum area is the convex, cyclic quadrilateral. However, the four sides can be permuted to give different quadrilateral shapes. Furthermore, there are only three different shapes that yield maximized convex quadrilaterals; all others are simply rotations and reflections of these three.

For the given problem these three quadrilaterals can be denoted by

$$\begin{aligned} I:\quad a &= 259,\ b = 765,\ c = 285,\ d = 925 \\ II:\quad a &= 259,\ b = 285,\ c = 765,\ d = 925 \end{aligned}$$

$$III: \; a \;=\; 259, \; b = 765, \; c = 925, \; d = 285$$

All three quadrilaterals yield the same maximum area A given by Brahmagupta's formula; that is , if

$$s = \frac{1}{2}(a+b+c+d) = \frac{1}{2}(259+765+285+925) = 1117 \text{ is the semiperimeter, then}$$

$$\begin{aligned} A &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ &= \sqrt{(1117-259)(1117-765)(1117-285)(1117-9250)} \\ &= 219648 \end{aligned}$$

Let θ denote the intersection angle of the diagonals. If $\theta \neq 90^\circ$, then

$$A = \frac{|\tan\theta|}{4}\left|a^2+b^2-b^2-d^2\right|.$$

For the quadrilateral in case I,

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{4A}{|a^2+c^2-b^2-d^2|}\right) \\ &= \tan^{-1}\left(\frac{4\cdot 219648}{|259^2+285^2-765^2-925^2|}\right) = 34.21^\circ. \end{aligned}$$

Similarly, for case II, $\theta = 72.05^\circ$ and in case III, $\theta = 73.74^\circ$.

Also solved by Scott H. Brown, Montgomery, AL; Bruno Salgueiro, Fanego, Viveiro, Spain; Edwin Gray, Highland Beach, FL; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5142:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let CD be an arbitrary diameter of a circle with center O. Show that for each point A distinct from O, C, and D on the line containing CD, there is a point B such that the line from D to any point P on the circle distinct from C and D bisects angle APB.

Solution by Paul M. Harms, North Newton, KS

Consider the unit circle with its center at the origin C at $(-1,0)$ and D at $(1,0)$. Let A be at $(a,0)$ were $0<a<1$ and let B be at $(b,0)$, where $b>1$.

To find the point B which satisfies the problem first consider P at $(a,\sqrt{1-a^2})$ just above point A. From the right triangle $APD, \tan\angle APD = \dfrac{1-a}{\sqrt{1-a^2}}$. The right triangle APB

has $\angle APB = 2\angle APD$. We have

$$\begin{aligned}
\tan\angle APB = \frac{b-a}{\sqrt{1-a^2}} &= \tan 2\angle APD \\
&= 2\left(\frac{\dfrac{1-a}{\sqrt{1-a^2}}}{1-\dfrac{(1-a)^2}{1-a^2}}\right) \\
&= \frac{\sqrt{1-a^2}}{a}.
\end{aligned}$$

Then, $b-a = \dfrac{1-a^2}{a}$, and $b = \dfrac{1}{a}$.

Now we check whether $B(1/a, 0)$ satisfies the problem for all points P on the upper half of the circle.

Let P be at $\left(t, \sqrt{1-t^2}\right)$ and let T be at $(t,0)$ where $-1 < t < a$. Then

$$\begin{aligned}
\tan\angle APD = \tan\left(\angle TPD - \angle TPA\right) &= \frac{\dfrac{1-t}{\sqrt{1-t^2}} - \dfrac{a-t}{\sqrt{1-t^2}}}{1+\dfrac{(1-t)(a-t)}{1-t^2}} \\
&= \frac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}.
\end{aligned}$$

Also

$$\begin{aligned}
\tan\angle DPB = \tan\left(\angle TPB - \angle TPD\right) &= \frac{\dfrac{\frac{1}{a}-t}{\sqrt{1-t^2}} - \dfrac{1-t}{\sqrt{1-t^2}}}{1+\dfrac{\left(\frac{1}{a}-t\right)(1-t)}{\sqrt{1-t^2}}} \\
&= \frac{\left(\frac{1}{a}-1\right)\sqrt{1-t^2}}{1+\frac{1}{a}-t-\frac{t}{a}} \\
&= \frac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}
\end{aligned}$$

Since the tangents of the angles are equal we conclude that for the points P given above, $\angle APD$ bisects $\angle$APB.

Now consider points $P(t, \sqrt{1-t^2})$ where $a < t < 1$. The tangent of $\angle DPB$ is the same as above,

it is still $\dfrac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}$. Now

$$\begin{aligned}
\tan\angle APD = \tan\left(\angle TPD + \angle APT\right) &= \frac{\dfrac{1-t}{\sqrt{1-t^2}} + \dfrac{t-a}{\sqrt{1-t^2}}}{1 - \dfrac{(1-t)(t-a)}{1-t^2}} \\
&= \frac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}.
\end{aligned}$$

From this we see that problem is also satisfied for these points. When P is on the bottom half of the circle at $\left(t, -\sqrt{1-t^2}\right)$, the triangles and angles used with P at $\left(t, \sqrt{1-t^2}\right)$, on the top half of the circle are congruent with those on the bottom half of the circle and should satisfy the problem.

If A is at $(a, 0)$, were $-1 < a < 0$, then using symmetry with respect the y-axis and point B at $\left(\dfrac{1}{a}, 0\right)$ satisfies the problem since all angles and triangles are congruent to those with A at $(|a|, 0)$.

If the circle has a radius of R, then we expect the same conclusion looking at similar triangles to those we used above. Consider C at $(-R, 0)$, D at $(R, 0)$, A at $(aR, 0)$, P at $(tR, R\sqrt{1-t^2})$, T at $(tR, 0)$, and B at $(R/a, 0)$ with the same restrictions on a and t as above. Using the new points, the tangents of the given lettered angles would have the same value as those given earlier since R would cancel for all the ratios involved with these tangents.

Also solved by Bruno Salgueiro, Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5143:** *Proposed by Valmir Krasniqi (student), Republic of Kosova*

Show that

$$\sum_{n=1}^{\infty} Cos^{-1}\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = \frac{\pi}{2}.$$

$$\left(Cos^{-1} = Arccos\right)$$

Solution 1 by Bruno Sagueiro Fanego, Viveiro, Spain

For each $n = 1, 2, \ldots, \dfrac{1}{n+1} \in (0,1]$ and $\dfrac{1}{n} \in (0,1]$, there exists $\alpha_n, \beta_n \in \left[0, \dfrac{\pi}{2}\right)$ such that

$cos\ \alpha_n = \dfrac{1}{n+1}$, and $cos\ \beta_n = \dfrac{1}{n}$. That is, $\alpha_n = Cos^{-1}\dfrac{1}{n+1}$, and $\beta_n = Cos^{-1}\dfrac{1}{n}$. Hence,

$$sin\ \alpha_n = \sqrt{1 - cos^2\ \alpha_n} = \sqrt{1 - \frac{1}{(n+1)^2}} = \frac{\sqrt{n^2+2n}}{n+1} \text{ and}$$

$$sin\ \beta_n = \sqrt{1 - cos^2\ \beta_n} = \sqrt{1 - \frac{1}{(n)^2}} = \frac{\sqrt{n^2-1}}{n}.$$

Therefore,

$$\begin{aligned}
\frac{1 + \sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} &= \frac{1}{n\cdot(n+1)} + \frac{\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\cdot(n+1)} \\
&= \frac{1}{n+1}\cdot\frac{1}{n} + \frac{\sqrt{n^2-1}}{n}\cdot\frac{\sqrt{n^2+2n}}{n+1} \\
&= cos\ \alpha_n \cdot cos\ \beta_n + sin\ \alpha_n \cdot \sin\ \beta_n \\
&= cos\,(\alpha_n - \beta_n)\,.
\end{aligned}$$

So,

$Cos^{-1}\dfrac{1 + \sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = Cos^{-1}cos(\alpha_n - \beta_n) = \alpha_n - \beta_n = Cos^{-1}\dfrac{1}{n+1} - Cos^{-1}\dfrac{1}{n}$, from where

$$\sum_{n=1}^{\infty} Cos^{-1}\frac{1 + \sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = \sum_{n=1}^{\infty} Cos^{-1}\frac{1}{n+1} - Cos^{-1}\frac{1}{n} = \lim_{m\to\infty}\sum_{n=1}^{m} Cos^{-1}\frac{1}{n+1} - Cos^{-1}\frac{1}{n}$$

$$= \lim_{m\to\infty} Cos^{-1}\frac{1}{2} - Cos^{-1}\frac{1}{1} + Cos^{-1}\frac{1}{3} - Cos^{-1}\frac{1}{2} + \cdots + Cos^{-1}\frac{1}{m} - Cos^{-1}\frac{1}{m-1} + Cos^{-1}\frac{1}{m+1} - Cos^{-1}\frac{1}{m}$$

$$= \lim_{m\to\infty} Cos^{-1}\frac{1}{m+1} - Cos^{-1}\frac{1}{1} = Cos^{-1}0 - Cos^{-1}1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Solution 2 by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

Let

$$\theta_n = \cos^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{\sqrt{n^2-1}}{n}\right)$$

so that

$$\begin{aligned}
\sum_{n=1}^{N} \cos^{-1}\left(\frac{1 + \sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n(n+1)}\right) &= \sum_{n=1}^{N} \cos^{-1}\left(\cos\theta_{n+1}\cos\theta_n + \sin\theta_{n+1}\sin\theta_n\right) \\
&= \sum_{n=1}^{N} \cos^{-1}\left(\cos(\theta_{n+1} - \theta_n)\right) = \theta_{n+1} - \theta_1.
\end{aligned}$$

which converges to $\pi/2 - 0 = \pi/2$ as $N \to \infty$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Since $Cos^{-1}\dfrac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = Cos^{-1}\left(\dfrac{1}{n+1}\right) - Cos^{-1}\left(\dfrac{1}{n}\right)$, so for any positive integers N, $\displaystyle\sum_{n=1}^{\infty} Cos^{-1}\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = Cos^{-1}\left(\frac{1}{N+1}\right)$.

The result of the problem follows as we allow N to tend to infinity.

Solution 4 by G. C. Greubel, Newport News, VA

Let $\cos x = \dfrac{1}{n+1}$ and $\cos y = \dfrac{1}{n}$. Using $\sin\theta = \sqrt{1-\cos^2\theta}$ with the corresponding cosine values leads to

$$\frac{1+\sqrt{n^2+2n}\sqrt{n^2-1}}{n(n+1)} = \cos x\cos y + \sin x \sin y = \cos(x-y).$$

From these reductions, now consider a finite version of the series in question.

$$\begin{aligned} S_m &= \sum_{n=1}^{m} \cos^{-1}\left(\frac{1+\sqrt{n^2+2n}\sqrt{n^2-1}}{n(n+1)}\right) \\ &= \sum_{n=1}^{m} \cos^{-1}(\cos(x-y)) \\ &= \sum_{n=1}^{m} (x-y). \end{aligned}$$

Now using the values for x and y the finite series becomes

$$\begin{aligned} S_m &= \sum_{n=1}^{m}\left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n}\right)\right) \\ &= \cos^{-1}\left(\frac{1}{m+1}\right) - \cos^{-1}(1). \end{aligned}$$

The series stated in the problem can be obtained by taking the limit m goes to infinity. Considering this leads to

$$\begin{aligned} S_\infty &= \lim_{m\to\infty} S_m \\ &= \lim_{m\to\infty}\left[\cos^{-1}\left(\frac{1}{m+1}\right) - \cos^{-1}(1)\right] \\ &= \cos^{-1}(0) - \cos^{-1}(1) = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \end{aligned}$$

That is, $\displaystyle\sum_{n=1}^{\infty} \cos^{-1}\left(\frac{1+\sqrt{n^2+2n}\sqrt{n^2-1}}{n(n+1)}\right) = \frac{\pi}{2}.$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Valmir Bucaj (student, Texas Luthern University), Seguin, TX; Edwin Gray, Highland

Beach, FL; David E. Manes, Oneonta, NY; Pedro H. O. Pantoja, Natal-RN, Brazil; Paolo Perfetti, Department of Mathematics, University "Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY, and the proposer.

- **5144:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Compute
$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right].$$

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Let $A=\int_0^1 \ln(x+\sqrt{1+x^2})$ and doing easy calculations we have

$$\begin{aligned}
A &= \int_0^1 \ln(x+\sqrt{1+x^2})\\
&= \int_0^1 \sinh^{-1}(x)\\
&= \left[-\sqrt{1+x^2}+x\sinh^{-1}(x)\right]_0^1\\
&= 1-\sqrt{2}+\sinh^{-1}(1)\\
&\simeq 0.46716
\end{aligned}$$

Using the Riemann sums we have

$$\begin{aligned}
\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{k^2+n^2}}{n}\right)^{1/n}\right] &= \lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{k^2+n^2}}{n}\right)\right]\\
&= \lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\frac{k^2}{n^2}}\right)\right]\\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{A^k}{k!}\\
&= \sum_{k=0}^{\infty}\frac{A^k}{k!}\\
&= e^A\\
&= e^{1-\sqrt{2}+\sinh^{-1}(1)}\\
&\simeq 1.5956
\end{aligned}$$

Solution 2 by Ovidiu Furdui, Cluj, Romania

More generally, we prove that if $f : [0,1] \to \Re$ is an integrable function then

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right) = e^{\int_0^1 f(x)dx}.$$

Let

$$x_n = \prod_{k=1}^{n} \left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right).$$

Then,

$$\ln x_n = \sum_{k=1}^{n} \ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right) = \sum_{k=1}^{n} \frac{\ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right)}{\frac{1}{n} f\left(\frac{k}{n}\right)} \cdot \ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right),$$

and it follows that

$$(1) \min_{1\le k\le n} \frac{\ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right)}{\frac{1}{n} f\left(\frac{k}{n}\right)} \cdot \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \le \ln x_n \le \max_{1\le k\le n} \frac{\ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right)}{\frac{1}{n} f\left(\frac{k}{n}\right)} \cdot \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Since $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = 0$, one has that

$$(2) \lim_{n\to\infty} \min_{1\le k\le n} \frac{\ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right)}{\frac{1}{n} f\left(\frac{k}{n}\right)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \max_{1\le k\le n} \frac{\ln\left(1 + \frac{1}{n} f\left(\frac{\mathrm{k}}{\mathrm{n}}\right)\right)}{\frac{1}{n} f\left(\frac{\mathrm{k}}{\mathrm{n}}\right)} = 1.$$

Letting $n \to \infty$ in (1) and using (2) one has that

$$\lim_{n\to\infty} \ln x_n = \int_0^1 f(x)dx,$$

and the problem is solved.

In particular, when $f(x) = \ln(x + \sqrt{1+x^2})$ one has that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \ln\left(\frac{k + \sqrt{k^2+n^2}}{n}\right)^{1/n}\right) = e^{\int_0^1 \ln(x+\sqrt{1+x^2})} = e^{\ln(1+\sqrt{2})+1-\sqrt{2}} = (1+\sqrt{2})e^{1-\sqrt{2}}.$$

Solution 3 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

Answer: $(1+\sqrt{2})e^{1-\sqrt{2}}$

Proof: By taking the logarithm we obtain

$$\ln\left\{\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right]\right\} = \sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)\right]$$

We observe that

$$\frac{k+\sqrt{n^2+k^2}}{n} \le \frac{n+\sqrt{2}n}{n} = 1+\sqrt{2}$$

thus the increasing monotonicity implies

$$\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) \le \frac{\ln(1+\sqrt{2})}{n}$$

By employing

$$\ln(1+x) = x + O(x^2), \quad x \to 0$$

we may write

$$\ln\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)\right] = \frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) + O(n^{-2})$$

and this in turn implies

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)\right] &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) + O(\frac{1}{n^2})\right] \\
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\frac{k^2}{n^2}}\right)
\end{aligned}$$

Of course we have

$$\sum_{k=1}^{n}\frac{1}{n}O(\frac{1}{n^2}) = O(\frac{1}{n^2})$$

Now it is easy to recognize that the last limit is actually the Riemann sum of the following integral

$$\int_0^1 \ln(x+\sqrt{1+x^2})dx$$

and the integral is easily calculated by the standard methods. Integrating by parts

$$\begin{aligned}
\int_0^1 \ln(x+\sqrt{1+x^2})dx &= x\ln(x+\sqrt{1+x^2})|_0^1 - \int_0^1 \frac{x}{\sqrt{1+x^2}}dx \\
&= \ln(1+\sqrt{2}) - (1+x^2)^{1/2}|_0^1 = \ln(1+\sqrt{2}) - \sqrt{2} + 1 = \ln\left(1+\sqrt{2}\right)e^{1-\sqrt{2}}
\end{aligned}$$

By exponentiating we obtain the desired result.

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Let us denote $P = \lim\limits_{n\to} P_n$ and $P_n = \prod\limits_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right]$.

Then

$$\begin{aligned}\ln P = \lim_{n\to\infty}\ln P_n &= \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right]\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\right].\end{aligned}$$

Taking, for each $k=1,2,\ldots,n,\ x=\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)$ in the

inequalities $\frac{x}{1+x}\le\ln(1+x)\le x$ (for any $x>0$), and being, for any $k=1,2,\ldots,n$,

$\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\le 1+\sqrt{1+1^2}=1+\sqrt{2}$, we obtain:

$$\begin{aligned}\frac{\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}{1+\frac{1}{n}\ln\left(1+\sqrt{2}\right)} &\le \frac{\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}{1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}\\ &\le \ln\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\right]\\ &\le \frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right). \qquad \mathbf{(1)}\end{aligned}$$

From **(1)** we obtain:

$$\begin{aligned}\sum_{k=1}^{n}\frac{\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}{1+\frac{1}{n}\ln\left(1+\sqrt{2}\right)} &\le \sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\right]\\ &\le \sum_{k=1}^{n}\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right) \qquad \mathbf{(2)}\end{aligned}$$

And from (**2**) we obtain:

$$\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}\ln\left(1+\sqrt{2}\right)}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\le \ln P\le \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right).$$

That is:

$$\frac{1}{1+0}\int_0^1 \ln\left(x+\sqrt{1+x^2}\right)dx\le \ln P\le \int_0^1 \ln\left(x+\sqrt{1+x^2}\right)dx$$

So,

$$\begin{aligned}\ln P=\int_0^1 \ln\left(x+\sqrt{1+x^2}\right)dx &= \left[x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}\right]_0^1\\ &= \ln(1+\sqrt{2})+1-\sqrt{2},\end{aligned}$$

and therefore,

$$P=e^{\ln(1+\sqrt{2})+1-\sqrt{2}}=\left(1+\sqrt{2}\right)e^{1-\sqrt{2}.}$$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5145:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k\ge 1$ be a natural number. Find the sum of

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-1-x-x^2-\cdots-x^n\right)^k,\quad \text{for } |x|<1.$$

Solution by Michael C. Faleski, University Center, MI

The summation can be rewritten as

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-\left(1+x+x^2+\cdots+x^n\right)\right)^k$$

where we can write the second term in the parentheses using the geometric series expression. That is, $(1+x+x^2+\cdots+x^n)=\dfrac{1-x^{n+1}}{1-x}$. Substitution of this result yields the original sum to

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-\left(1+x+x^2+\cdots+x^n\right)\right)^k=\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-\frac{1-x^{n+1}}{1-x}\right)^k=\sum_{n=1}^{\infty}\left(\frac{x^{n+1}}{1-x}\right)^k.$$

Since the denominator has no "n" dependence, we now have

$$\sum_{n=1}^{\infty}\left(\frac{x^{n+1}}{1-x}\right)^k=\left(\frac{1}{(1-x)^k}\right)\sum_{n=1}^{\infty}x^{(n+1)k}.$$

Once again, we use the geometric series relation with multiplicative term x^k yielding a result of

$$\sum_{n=1}^{p} x^{(n+1)k} = \frac{x^{2k} - x^{(p+2)k}}{1 - x^k}.$$

Now, as the upper limit of the sum is $p \to \infty$, then $x^{(p+2)k} \to 0$ since $|x| < 1$ and $k \geq 1$. Hence

$$\sum_{n=1}^{\infty} x^{(n+1)k} = \frac{x^{2k}}{1 - x^k}.$$

So, finally, we have our result of the original sum as

$$\sum_{n=1}^{\infty} \left(\frac{1}{1-x} - 1 - x - x^2 - \cdots - x^n \right)^k = \frac{x^{2k}}{(1-x)^k (1 - x^k)}.$$

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie and Andrew Siefker (jointly), San Angelo, TX; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro, Fanego, Viveiro,Spain; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Angel Plaza, Gran Canaria, Spain; Paolo Perfetti, Department of Mathematics, University "Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

From the Editor; Mea Culpa

In the March 2011 issue of the column I inadvertently forgot to mention that **Enkel Hysnelaj, of the University of Technology, in Sydney, Australia and Elton Bojaxhiu of Kriftel, Germany** had also solved problems 5136 and 5139. Sorry.

Mistakes happen to all of us; they are embarrassing, but they are part of life. **Albert Stadler of Herrliberg, Switzerland** pointed out an error in the first solution to 5138 that appeared in last month's column. The problem challenged us to prove for all natural numbers $n \geq 2$ that

$$\frac{n}{F_n F_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_2^2} + \cdots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{F_k^2},$$

where F_n is the n^{th} Fibonacci number defined in the usual way with $F_0 = 0, F_1 = 1$. At a crucial step in the proof Muirhead's inequality was employed and this is what triggered Albert's suspicions; the conditions to use the inequality were not met. (Muirhead's inequality generalizes the arithmetic-geometric means inequality. See: http://en.wikipedia.org/wiki/Muirhead) In further correspondence with Albert he pointed out a paper by Yufei Zhao (yufeiz@mit.edu) on Inequalities that contains two practical notes with respect to the Muirhead inequality. Zhao wrote: "Don't try to apply Muirhead when there are more than 3 variables, since mostly likely you won't succeed (and never, ever try to use Muirhead when the inequality is only cyclic but not symmetric, since it is incorrect to use Muirhead there) (2) when writing up your solution, it is probably safer to just deduce the inequality using weighted AM-GM by finding the appropriate weights, as this can always be done. The reason is that it is not always clear that Muirhead will be accepted as a quoted theorem." The second solution listed for 5138 is correct.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
October 15, 2011

- **5164:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle has integer length sides (a, b, c) such that $a - b = b - c$. Find the dimensions of the triangle if the inradius $r = \sqrt{13}$.

- **5165:** *Proposed by Thomas Moore, Bridgewater, MA*

  "Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast."

  Let $\sigma(n)$ denote the sum of all the different divisors of the positive integer n. Then n is perfect, deficient, or abundant according as $\sigma(n) = 2n, \sigma(n) < 2n$, or $\sigma(n) > 2n$. For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

- **5166:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be lengths of the sides of a triangle ABC. Prove that

  $$\left(3^{a+b} + \frac{c}{b}3^{-b}\right)\left(3^{b+c} + \frac{a}{c}3^{-c}\right)\left(3^{c+a} + \frac{b}{a}3^{-a}\right) \geq 8.$$

- **5167:** *Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy*

  Find the maximum of the real valued function

  $$f(x, y) = x^4 - 2x^3 - 6x^2y^2 + 6xy^2 + y^4$$

  defined on the set $D = \{(x, y) \in \Re^2 : \ x^2 + 3y^2 \leq 1\}$.

- **5168:** *Proposed by G. C. Greubel, Newport News, VA*

  Find the value of a_n in the series

  $$\frac{7t + 2t^2}{1 - 36t + 4t^2} = a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n} + \cdots.$$

- **5169:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be an integer and let i be such that $1 \leq i \leq n$. Calculate:

  $$\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n.$$

- **5146:** *Proposed by Kenneth Korbin, New York, NY*

  Find the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$.

  Solution 1 by Kee-Wai Lau, Hong Kong, China

  Let the lengths of the sides of the triangle be a, b, and c with $c \le b \le a$.

  Let $x = b + c - a,\ y = c + a - b,\ z = a + b - c$ so that x, y, z are integers and $0 < x \le y \le z$.

  It is well known that $\dfrac{1}{2}\sqrt{\dfrac{xyz}{x+y+z}}$ or $\dfrac{xyz}{x+y+z} = 52$.

  From $xyz < xy(x+y+z)$, we see that $xy > 52$ and from $xy < \dfrac{3xyz}{x+y+z}$, we have $xy \le 156$. Since $a = \dfrac{y+z}{2},\ b = \dfrac{z+x}{2},\ c = \dfrac{x+y}{2}$, so we have to find positive integers x, y satisfying

  $$\begin{cases} x & \le y \\ 1 & \le x \le 12 \\ 52 < xy \le 156 \end{cases}$$

  such that $z = \dfrac{52(x+y)}{xy-52}$ is a positive integer greater than or equal to y and that x, y, z are of the same parity. With the help of a computer we find that

  $(x,y,z) = (2,28,390),\ (2,30,208),\ (2,40,78),\ (2,52,54),\ (4,14,234),\ (4,26,30),\ (6,10,104),\ (6,16,26)$

  are the only solutions. Since $a + b + c = x + y + z$, so the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$ is 420.

  Solution 2 by Brian D. Beasley, Clinton, SC

  We designate the integer side lengths of the triangle by a, b, and c. We also let $x = a + b - c$, $y = c + a - b$, and $z = b + c - a$ and note that $x + y + z = a + b + c$. Then the formula for the in-radius r of a triangle becomes

  $$r = \frac{1}{2}\sqrt{\frac{(a+b-c)(c+a-b)(b+c-a)}{a+b+c}} = \frac{1}{2}\sqrt{\frac{xyz}{x+y+z}}.$$

  For the given triangle, we thus have $52(x+y+z) = xyz$. Then xyz is even; combined with the fact that x, y, and z have the same parity, this implies that all three are even. Writing $x = 2u$, $y = 2v$, and $z = 2w$, we obtain $13(u+v+w) = uvw$. Then 13 divides uvw, so without loss of generality, we assume $w = 13k$ for some natural number k. This produces $v = (u+13k)/(uk-1)$. Using this equation, a computer search reveals eight solutions for (u,v,w) (with $u \le v$) and hence for (a,b,c):

  $$\begin{array}{lllll}
(u,v,w) = (2,15,13) & \Longrightarrow & (a,b,c) = (17,15,28) & \Longrightarrow & \text{perimeter } = 60 \\
(u,v,w) = (3,8,13) & \Longrightarrow & (a,b,c) = (11,16,21) & \Longrightarrow & \text{perimeter } = 48 \\
(u,v,w) = (1,27,26) & \Longrightarrow & (a,b,c) = (28,27,53) & \Longrightarrow & \text{perimeter } = 108 \\
(u,v,w) = (1,20,39) & \Longrightarrow & (a,b,c) = (21,40,59) & \Longrightarrow & \text{perimeter } = 120 \\
(u,v,w) = (3,5,52) & \Longrightarrow & (a,b,c) = (8,55,57) & \Longrightarrow & \text{perimeter } = 120 \\
(u,v,w) = (1,15,104) & \Longrightarrow & (a,b,c) = (16,105,119) & \Longrightarrow & \text{perimeter } = 240 \\
(u,v,w) = (2,7,117) & \Longrightarrow & (a,b,c) = (9,119,124) & \Longrightarrow & \text{perimeter } = 252 \\
(u,v,w) = (1,14,195) & \Longrightarrow & (a,b,c) = (15,196,209) & \Longrightarrow & \text{perimeter } = 420
\end{array}$$

Thus the maximum value of the perimeter is 420.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5147:** *Proposed by Kenneth Korbin, New York, NY*

Let

$$\begin{cases} x = 5N^2 + 14N + 23 \text{ and} \\ y = 5(N+1)^2 + 14(N+1) + 23 \end{cases}$$

where N is a positive integer. Find integers a_i such that

$$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0.$$

Solution 1 by G. C. Greubel, Newport News, VA

The equations for x and y are given by $x = 5n^2 + 14n + 23$ and $y = 5n^2 + 24n + 42$. We are asked to find the values of a_i such that the equation

$$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0$$

is valid. In order to do so we need to calculate the values of x^2, y^2, and xy. For this we have

$$\begin{aligned} x^2 &= 25n^4 + 140n^3 + 426n^2 + 644n + 529 \\ y^2 &= 25n^4 + 240n^3 + 996n^2 + 2016n + 1764 \\ xy &= 25n^4 + 190n^3 + 661n^2 + 1140n + 966. \end{aligned}$$

Using the above results we then have the equation

$$\begin{aligned} 0 = \; & 25(a_1 + a_2 + a_3)n^4 + 10(14a_1 + 24a_2 + 19a_3)n^3 \\ & +(426a_1 + 996a_2 + 661a_3 + 5a_4 + 5a_5)n^2 \\ & +2(322a_1 + 1008a_2 + 570a_3 + 7a_4 + 12a_5)n \\ & +(529a_1 + 1764a_2 + 966a_3 + 23a_4 + 42a_5 + a_6). \end{aligned}$$

From this we have five equations for the coefficients a_i given by

$$\begin{aligned} 0 &= a_1 + a_2 + a_3 \\ 0 &= 14a_1 + 24a_2 + 19a_3 \\ 0 &= 426a_1 + 996a_2 + 661a_3 + 5a_4 + 5a_5 \\ 0 &= 322a_1 + 1008a_2 + 570a_3 + 7a_4 + 12a_5 \\ 0 &= 529a_1 + 1764a_2 + 966a_3 + 23a_4 + 42a_5 + a_6. \end{aligned}$$

From $0 = 14a_1 + 24a_2 + 19a_3$ we have $0 = 14(a_1 + a_2 + a_3) + 10a_2 + 5a_3 = 5(2a_2 + a_3)$, where we used the fact that $0 = a_1 + a_2 + a_3$. This yields $a_3 = -2a_2$. Using this result in $0 = a_1 + a_2 + a_3$ yields $a_2 = a_1$. The three remaining equations can be reduced to

$$\begin{aligned} 0 &= 20a_1 + a_4 + a_5 \\ 0 &= 190a_1 + 7a_4 + 12a_5 \\ 0 &= 361a_1 + 23a_4 + 42a_5 + a_6. \end{aligned}$$

Solving this system we see that

$$a_1 = a_1,\ a_2 = a_1,\ a_3 = -2a_1,\ a_4 = -10a_1,\ a_5 = -10a_1,\ a_6 = 289a_1.$$

We now verify that the above coefficients work.

$$\begin{aligned}
a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 &= 0, \text{ becomes} \\
a_1\left(x^2 + y^2 - 2xy - 10x - 10y + 289\right) &= 0, \text{ and since } a_1 \neq 0 \\
x^2 + y^2 - 2xy - 10x - 10y + 289 &= 0, \text{ and} \\
(x-y)^2 - 10(x+y) + 289 &= 0.
\end{aligned}$$

From the values of x and y presented to us in terms of n at the start of the problem, we see that $x - y = -(10n + 19)$ and $x + y = 10n^2 + 38n + 65$.
Substituting these values into the above equations we obtain:

$$\begin{aligned}
0 &= (x-y)^2 - 10(x+y) + 289 \\
&= (10n+19)^2 - 10(10n^2 + 38n + 65) + 289 \\
&= (100n^2 + 380n + 361) - (100n^2 + 380n + 650) + 289 \\
&= 361 - 650 + 289 \\
&= 0.
\end{aligned}$$

We have thus verified that for the coefficients we have obtained, and for the vaules of x and y that are given, $a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

By putting $N = 1, 2, 3, 4, 5$, we obtain the system of equations

$$\begin{cases}
1764a_1 + 5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 = 0 \\
5041a_1 + 12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 = 0 \\
12100a_1 + 25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 = 0 \\
25281a_1 + 47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 = 0 \\
47524a_1 + 82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 = 0.
\end{cases} \qquad (1)$$

If $a_1 = 0$, then (1) reduces to

$$\begin{cases}
5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 = 0 \\
12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 = 0 \\
25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 = 0 \\
47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 = 0 \\
82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 = 0.
\end{cases} \qquad (2)$$

Since the determinant $\begin{vmatrix} 5041 & 2982 & 42 & 71 & 1 \\ 12100 & 7810 & 71 & 110 & 1 \\ 25281 & 17490 & 110 & 159 & 1 \\ 47524 & 34662 & 159 & 218 & 1 \\ 82369 & 62566 & 218 & 287 & 1 \end{vmatrix} = -18000000$, so (2) has the unique solution $a_2 = a_3 = a_4 = a_5 = a_6 = 0$.

If $a_1 \neq 0$, we write $a_2 = a_1 b_2,\ a_3 = a_1 b_3, a_4 = a_1 b_4,\ a_5 = a_1 b_5,\ a_6 = a_1 b_6$, so that (1) becomes

$$\begin{cases} 1764 + 5041b_2 + 2982b_3 + 42b_4 + 71b_5 + b_6 = 0 \\ 5041 + 12100b_2 + 7810b_3 + 71b_4 + 110b_5 + b_6 = 0 \\ 12100 + 25281b_2 + 17490b_3 + 110b_4 + 159b_5 + b_6 = 0 \\ 25281 + 47524b_2 + 34662b_3 + 159b_4 + 218b_5 + b_6 = 0 \\ 47524 + 82369b_2 + 62566b_3 + 218b_4 + 287b_5 + b_6 = 0. \end{cases} \qquad (3)$$

By Cramer's rule, we find the unique solution of (3) to be

$$b_2 = 1,\ b_3 = -2,\ b_4 = -10,\ b_5 = -10,\ b_6 = 289.$$

It follows that the general solution to (1) is

$$a_1 = k,\ a_2 = k,\ a_3 = -2k,\ a_4 = -10k,\ a_5 = -10k,\ a_6 = 289k, \qquad (4)$$

where k is any integer. It can be checked readily by direct expansion that $kx^2 + ky^2 - 2kxy - 10kx - 10ky + 289k = 0$ for any positive integer N, and so the general solution to the equation of the problem is given by (4).

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

As in the published solutions to SSMJ problem 5144, we first compute

$$y - x = 5\left[(N+1)^2 - N^2\right] + 14\,(N+1-N) + 23 - 23 = 5\,(2N+1) + 14 = 10N + 19 \qquad (1)$$

From $x = 5N^2 + 14N + 23$ that is $5\mathrm{N}^2 + 14N + 23 - x = 0$, one obtains

$$N_{1,2,} = \frac{-14 \pm \sqrt{14^2 - 20(23-x)}}{10} = \frac{-7 \pm \sqrt{5x - 66}}{5}$$

and since N is a positive integer, we choose $N = \dfrac{-7 + \sqrt{5x-66}}{5}$ (2).

Substituting (2) into (1) gives

$$y - x = 2\left(-7 + \sqrt{5x-66}\right) + 19 = 5 + 2\sqrt{5x-66}. \qquad (3)$$

From (3) one obtains

$$(y - x - 5)^2 = \left(2\sqrt{5x-66}\right)^2, \text{that is}$$

$$x^2 + y^2 - 2xy - 10x - 10y + 289 = 0 \qquad (4)$$

Relation (4) shows that it suffices to take the following integers for a_i

$$a_1 = a_2 = 1; \quad a_3 = -2; \quad a_4 = a_5 = -10; \quad a_6 = 289$$

Comment: Relation (4) shows that for any positive integer N, all of the points with coordinates (x, y)=(u_N, u_{N+1}) for $u_N = 5N^2 + 14N + 23$, are points situated on the parabola (*) with equation

$$(1\ X\ Y)\begin{pmatrix} 289 & -5 & -5 \\ -5 & 1 & -1 \\ -5 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} = 0.$$

$(*)$ Because $\det\begin{pmatrix} 289 & -5 & -5 \\ -5 & 1 & -1 \\ -5 & -1 & 1 \end{pmatrix} = 100 \neq 0$ and $\det\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0$.

Solution 4 by David Stone and John Hawkins, Statesboro, GA

We will show that the proscribed points (x, y) lie on the conic

$$x^2 + y^2 - 2xy - 10xy - 10y + 289 = 0.$$

This is a parabola. In fact, it is the parabola $x = \dfrac{1}{2\sqrt{2}}y^2$ roatated counterclockwise $\dfrac{\pi}{4}$ and translated "up the diagonal $y = x$" by a distance $\dfrac{289}{20}\sqrt{2}$, having its vertex at $\left(\dfrac{289}{20}, \dfrac{289}{20}\right)$.

We will actually consider the more general problem

$$\begin{cases} x = aN^2 + bN + c \\ y = a\,(N+1)^2 + b(N+1) + c \end{cases}$$

with the restrictions on N removed.
Treating these as parametric equations, we can eliminate the parameter N (without getting bogged down in the quadratic formula).

Expanding the expression for y gives

$$\begin{aligned} y &= aN^2 + 2aN + a + bN + b + c \\ &= \left(aN^2 + bN + c\right) + 2aN + a + b \\ &= x + 2aN + a + b. \end{aligned}$$

Solving for N gives $N = \dfrac{y - x - (a+b)}{2a}$.
Substituting back into the expression for x:

$$x = a\left(\frac{y - x - a - b}{2a}\right)^2 + b\left(\frac{y - x - a - b}{2a}\right) + c,$$

which simplifies to

$$(1) \quad x^2 + y^2 - 2xy - 2ax - 2ay + \left(a^2 - b^2 + 4ac\right) = 0.$$

This is our solution for the general problem. So we do indeed have a quadratic equation for our figure; the discriminate equals zero.

From calculus,we know that a 45° rotation will remove the xy term. The transformation equations are

$$x = \frac{1}{\sqrt{2}}\left(x' - y'\right) \quad \text{and} \quad y = \frac{1}{\sqrt{2}}\left(x' + y'\right)$$

Substituting these into Equation (1), we get

$$\frac{(x'-y')^2}{2} + \frac{(x'+y')^2}{2} - 2\frac{(x'-y')(x'+y')}{2} - \frac{2a}{\sqrt{2}}\left(x' - y'\right) - \frac{2a}{\sqrt{2}}\left(x' + y'\right) + \left(a^2 - b^2 + 4ac\right) = 0.$$

This simplifies to

$$2\left(y'\right)^2 - \frac{4a}{\sqrt{2}}x' + \left(a^2 - b^2 + 4ac\right) = 0.$$

This becomes more familiar as

$$x' - \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}} = \frac{1}{a\sqrt{2}}\left(y'\right)^2.$$

We recognize a nice parabola in the x', y' plane. In fact, if we translate to the new origin, $\left(\dfrac{a^2 - b^2 + 4ac}{2a\sqrt{2}}, 0\right)$ (in the x', y' plane) and let

$$x'' = x' - \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}} \quad \text{and} \quad y'' = y'$$

our equation becomes

$$x'' = \frac{1}{a\sqrt{2}}\left(y''\right)^2.$$

Substituting the values $a = 5, b = 14, c = 23$ produces the solution to the given problem.

Comment 1: We see that x and y are interchangeable in Equation (1), reflecting the fact that the line $y = x$ is the axis of symmetry of our parabola. Therefore, more lattice points than originally mandated fall on the parabola.

For convenience, let $u_n = aN^2 + bN + c$. By the given condition, for any integer N, the point (u_N, u_{N+1}) lies on the parabola. By symmetry, (u_{N+1}, u_N) also lies on the parabola.

Comment 2: We see that this sequence satisfies the first order non-linear recurrence: $u_{N+1} = u_N + (2N+1)\,a + b$. We have shown that the points $(u_N, u_{N+1}),\ N \in Z$, lie on the parabola given by Equation (1) (as do the points (u_{N+1}, u_N)). This is reminiscent of the result that pairs of Fibonacci numbers (F_N, F_{N+1}) lie on the hyperbolas $y^2 - xy - x^2 = \pm 1$. In fact, such pairs are the *only* lattice points on these hyperbolas.

So we wonder if the points (u_N, u_{N+1}) and (u_{N+1}, u_N) are the only lattice points on the parabola given by Equation (1).

Also solved by Brian D. Beasley, Clinton, SC; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile, and the proposer.

- **5148:** *Proposed by Pedro Pantoja (student, UFRN), Natal, Brazil*

Let a, b, c be positive real numbers such that $ab + bc + ac = 1$. Prove that

$$\frac{a^2}{\sqrt[3]{b(b+2c)}} + \frac{b^2}{\sqrt[3]{c(c+2a)}} + \frac{c^2}{\sqrt[3]{a(a+2b)}} \geq 1.$$

Solution 1 by David E. Manes, Oneonta, NY

Let $L = \dfrac{a^2}{\sqrt[3]{b(b+2c)}} + \dfrac{b^2}{\sqrt[3]{c(c+2a)}} + \dfrac{c^2}{\sqrt[3]{a(a+2b)}}$. To prove that $L \geq 1$, we will use

Jensen's inequality that states if $\lambda_1, \lambda_2 \ldots, \lambda_n$ are positive numbers satisfying $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$, and $x_1, x_2, \ldots, x_n$ are any n points in an interval where f is continuous and convex, then

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f\Big(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\Big).$$

The function $f(x) = \dfrac{1}{\sqrt[3]{x}}$ is continuous and convex on the interval $(0, \infty)$. Let

$$\alpha = a^2 + b^2 + c^2 \qquad \lambda_1 = \frac{a^2}{\alpha} \qquad \lambda_2 = \frac{b^2}{\alpha} \qquad \lambda_3 = \frac{c^2}{\alpha}$$

$$x_1 = b^2 + 2bc \qquad x_2 = c^2 + 2ac \qquad x_3 = a^2 + 2ab$$

Then $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and Jensen's inequality implies

$$\begin{aligned}
\frac{1}{\alpha} L &= \frac{a^2}{\alpha} f\left(b^2 + 2bc\right) + \frac{b^2}{\alpha} f\left(c^2 + 2ac\right) + \frac{c^2}{\alpha} f\left(a^2 + 2ab\right) \\
&\geq f\left(\frac{a^2\left(b^2 + 2bc\right) + b^2\left(c^2 + 2ac\right) + c^2\left(a^2 + 2ab\right)}{\alpha}\right) \\
&= \sqrt[3]{\frac{\alpha}{a^2b^2 + b^2c^2 + c^2a^2 + 2a^2bc + 2ab^2c + 2abc^2}} \\
&= \sqrt[3]{\frac{\alpha}{(ab + bc + ac)^2}} = \sqrt[3]{\alpha}.
\end{aligned}$$

Hence, $L \geq \alpha^{4/3} = \left(a^2 + b^2 + c^2\right)^{4/3} \geq 1$ since the inequality $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$ with the constraint $ab + bc + ac = 1$ implies $a^2 + b^2 + c^2 \geq 1$. Note that equality occurs if and only if $a = b = c = \dfrac{1}{\sqrt{3}}$.

Solution 2 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Using Cauchy-Schwarz inequality we have,

$$\left(\sqrt[3]{b(b + 2c)} + \sqrt[3]{c(c + 2a)} + \sqrt[3]{a(a + 2b)}\right)\left(\frac{a^2}{\sqrt[3]{b(b + 2c)}} + \frac{b^2}{\sqrt[3]{c(c + 2a)}} + \frac{c^2}{\sqrt[3]{a(a + 2b)}}\right) \geq (a+b+c)^2,$$

which implies that,

$$\frac{a^2}{\sqrt[3]{b(b + 2c)}} + \frac{b^2}{\sqrt[3]{c(c + 2a)}} + \frac{c^2}{\sqrt[3]{a(a + 2b)}} \geq \frac{(a + b + c)^2}{\sqrt[3]{b(b + 2c)} + \sqrt[3]{c(c + 2a)} + \sqrt[3]{a(a + 2b)}}.$$

Using the fact that the function $f(x) = \sqrt[3]{x}$ is a concave function, since the second derivative is negative, we have that any three numbers x, y, z, according to Jensen's inequality, satisfy the inequality $f(x)+f(y)+f(z) \le 3f\left(\dfrac{x+y+z}{3}\right)$ and applying this we have

$$
\begin{aligned}
\frac{a^2}{\sqrt[3]{b(b+2c)}}+\frac{b^2}{\sqrt[3]{c(c+2a)}}+\frac{c^2}{\sqrt[3]{a(a+2b)}} &\ge \frac{(a+b+c)^2}{\sqrt[3]{b(b+2c)}+\sqrt[3]{c(c+2a)}+\sqrt[3]{a(a+2b)}}\\
&\ge \frac{(a+b+c)^2}{3\sqrt[3]{\left(\dfrac{b(b+2c)+c(c+2a)+a(a+2b)}{3}\right)}}\\
&= \frac{(a+b+c)^2}{\dfrac{3}{\sqrt[3]{3}}\sqrt[3]{(a+b+c)^2}}
\end{aligned}
$$

So it is enough to prove that

$$
\begin{aligned}
\frac{(a+b+c)^2}{\dfrac{3}{\sqrt[3]{3}}\sqrt[3]{(a+b+c)^2}} &\ge 1, \quad \text{which implies}\\
\frac{(a+b+c)^2}{\sqrt[3]{(a+b+c)^2}} &\ge \frac{3}{\sqrt[3]{3}}\\
(a+b+c)^2 &\ge 3
\end{aligned}
$$

Using the given condition and the AM-GM inequality we have

$$
\begin{aligned}
(a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ac\\
&\ge 3ab+3bc+3ac\\
&= 3(ab+bc+ac)\\
&= 3
\end{aligned}
$$

and this is the end of the proof.

Solution 3 by Andrea Fanchini, Cantú, Italy

Recall Holder's inequality that states that if $a_{ij}, 1 \le i \le m, 1 \le j \le n$ are positive real numbers, then:

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right) \ge \left(\sum_{j=1}^{n}\sqrt[m]{\prod_{i=1}^{m} a_{ij}}\right)^m .$$

Setting $n=3$ and $m=4$ and using this inequality we have

$$\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}\frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc}a^2\left(b^2+2bc\right)\right) \ge \left(a^2+b^2+c^2\right)^4,$$

and being that $a^2+b^2+c^2 \geq ab+bc+ca$,

$$\left(\sum_{cyc} \frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc} \frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc} \frac{a^2}{\sqrt[3]{b^2+2bc}}\right)\left(\sum_{cyc} a^2\left(b^2+2bc\right)\right) \geq (ab+bc+ca)^4 = 1$$

because

$$\left(\sum_{cyc} a^2\left(b^2+2bc\right)\right) = (ab+bc+ca)^2 = 1,$$

and so the proposed inequality holds.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy, and the proposer.

- **5149:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

A regular n-gon $A_1, A_2 \cdots, A_n$ $(n \geq 3)$ has center F, the focus of the parabola $y^2 = 2px$, and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \cdots, FA_n$ cut the parabola at points $B_1, B_2, \cdots, B_n$.

Prove that

$$\frac{1}{n}\sum_{k=1}^{n} FB_k^2 > p^2.$$

Solution by Ángel Plaza (University of Las Palmas de Gran Canaria) and Javier Sánchez-Reyes (University of Castilla-La Mancha), Spain

In polar coordinates (r,θ) centered at the focus the parabola is given by $r = p/(1+\cos\theta)$. Defining the arguments $\theta_k = \theta_n + 2k\pi/n$ for $k = 1, 2, \ldots, n$ corresponding to the vertices A_k of the polygon, we have to prove that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} \frac{p^2}{(1+\cos\theta_k)^2} &> p^2,\\ \frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+\cos\theta_k)^2} &> 1,\end{aligned}$$

where $\theta_k \neq 0$ and $\theta_k \neq \pi$. Since the function $f(x) = 1/x^2$ is strictly convex and $\sum_{k=1}^{n} \cos\theta_k = 0$, for example because the sum of all the nth complex roots of unity is zero, it follows that

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+\cos\theta_k)^2} > \left(1 + \frac{\sum_{k=1}^{n} \cos\theta_k}{n}\right)^{-2} = 1.$$

Also solved by Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA and the proposer.

- **5150:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

  Let $\{A_n\}_{n=1}^{\infty}, (A_n \in M_{n\times n}(C))$be a sequence of matrices such that $\det(A_n) \neq 0, 1$ for all $n \in N$. Calculate:
$$\lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|\det\left(adj^{\circ n}\left(A_n\right)\right)|\right)},$$
  where $adj^{\circ n}$ refers to $adj \circ adj \circ \cdots \circ adj, n$ times, the n^{th} iterate of the classical adjoint.

  Solved 1 by the proposer

  A simple calculation of $adj^{\circ n}(A),\ m = 1, 2, \cdots, 5$ using equalities:

$$(i) \qquad adj(A)\cdot A = A \cdot adj(A) = \det(A)\cdot I_{n\times n}.$$

$$(ii) \qquad \det(A^{-1}) = (det(A))^{-1}$$

$$(iii) \qquad \det(kA) = k^n \det(A)$$

  suggests the following conjecture:

$$adj^{om}(A) = \det(A)^{P_m(n)} A^{(-1)^m}; \quad P_m(n) = \frac{(n-1)^m + (-1)^{m-1}}{n},\ m, n \in N \qquad (**)$$

  We prove the conjecture by induction on the positive integer m. The assertion trivially holds for the case $m = 1$. Let it hold for some positive integer $m > 1$. Then

$$\begin{aligned}
adj^{om+1}(A) &= adj\left(adj^{om}(A)\right) \\
&= \det\left(adj^{om}(A)\right)\left(adj^{om}(A)\right)^{-1} \\
&= \det\left(\det(A)^{P_m(n)} A^{(-1)^m}\right)\left(\det(A)^{P_m(n)} A^{(-1)^m}\right)^{-1} \\
&= \det(A)^{(n-1)P_m(n)+(-1)^m}(A)^{(-1)^{m+1}}.
\end{aligned}$$

  Besides,
$$P_{m+1}(n) = (n-1)P_m(n) + (-1)^m = \frac{(n-1)^{m+1} + (-1)^m}{n},$$
  proving the assertion for positive integer $m+1$. Accordingly, using $(**)$ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln(|\det(adj^{on}(A_n))|)} &= \lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|\det(\det(A_n)^{P_n(n)} A_n^{(-1)^n})|\right)} \\
&= \lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|(\det(A_n)^{nP_n(n)} \det(A_n{}^{(-1)^n})|\right)} \\
&= \lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|(\det(A_n)^{nP_n(n)+(-1)^n}|\right)}
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{n^n}{nP_n(n) + (-1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n$$

$$= e.$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We shall find a formula for $adj^{\circ n}(nA)$ and then show the limit is e.

First recall some properties of the inverse and the classical adjoint, where A is $n \times n$ and invertible and c a non-zero scalar.

$$(1) \qquad adj(A) = \det(A)A^{-1}$$

$$(2) \qquad adj\,(A)^{-1} = \frac{1}{\det(A)}A = adj\left(A^{-1}\right)$$

$$(3) \qquad \det\left[adj(A)\right] = \left[\det A\right)]^{n-1}$$

$$(4) \qquad \det(cA) = c^n \det(A)$$

$$(5) \qquad (cA)^{-1} = \frac{1}{c}A^{-1}$$

$$(6) \qquad adj(cA) = c^{n-1}adj(A)$$

Then we see

$$\begin{aligned}
(7)\quad adj^{\circ 2}(A) &= adj\left[adj(A)\right] \\
&= \det\left[adj(A)\right]\left[adj(A)\right]^{-1} \text{ by (1)} \\
&= \left[\det(A)\right]^{n-1}\frac{1}{\det(A)}A \text{ by (3) and (2)} \\
&= \left[\det(A)\right]^{n-2}A.
\end{aligned}$$

Continuing with our calculations, we have

$$\begin{aligned}
(8)\quad adj^{\circ 3}(A) &= adj\left[adj^{\circ 2}(A)\right] \\
&= adj\left[\left[\det(A)\right]^{n-2}A\right] \text{ by (7)} \\
&= \left\{\left[\det(A)\right]^{n-2}\right\}^{n-1} adj(A) \text{ by (6)} \\
&= \left[\det(A)\right]^{(n-1)(n-2)}\det(A)A^{-1}\text{by (1)} \\
&= \left[\det(A)\right]^{n^2-3n+3}A^{-1}
\end{aligned}$$

We observe that repeated applications of adj will produce terms of the form $[\det(A)]^{[p_k(n)} A^{(-1)k}$, where $p_i(n)$ is a polynomial of degree $k-1$ in n.

Specifically, for $k = 1, 2, 3, \ldots, n-1$, we have

$$
\begin{aligned}
(9) \quad adj^{\circ(k+1)}(A) &= adj\left[adj^{\circ k}(A)\right] \\
&= adj\left[[\det(A)]^{p_k(n)} A^{(-1)^k}\right] \text{ by induction} \\
&= \left\{[\det(A)]^{p_k(n)}\right\}^{n-1} adj\left(A^{(-1)^k}\right) \text{ by (6)} \\
&= [\det(A)]^{(n-1)p_k(n)} \det\left(A^{(-1)^k}\right)\left[A^{(-1)^k}\right]^{-1} \text{ by (1)} \\
&= [\det(A)]^{(n-1)p_k(n)+(-1)^k} A^{(-1)^{k+1}}
\end{aligned}
$$

Therefore, we can recursively compute the polynomials which give the exponent on $\det(A)$ and obtain a concrete formula for $adj(A) : p_{k+1}(n) = (n-1)p_k(n) + (-1)^k$.

By (1) $adj(A) = \det(A)A^{-1}$, so $p_1(n) = 1$.

By (7) $adj^{\circ 2}(A) = [\det(A)]^{n-2} A$, so $p_2(n) = n-2$.

Then $p_3(n) = (n-1)p_2(n) + (-1)^2 = (n-1)(n-2) + 1 = n^2 - 3n + 3$, agreeing with (8).

Continuing, we find that

$p_4(n) = n^3 - 4n^2 + 6n - 4$ and

$p_5(n) = n^4 - 5n^3 + 10n^2 - 10n + 5$.

The appearance of the binomial coefficients is unmistakable. We deduce that, for $k = 1, 2, 3, \ldots, n$,

$p_k(n) = \dfrac{(n-1)^k + (-1)^{k-1}}{n}$, a polynomial of degree $k-1$.

The capstone of this sequence of polynomials: $p_n(n) = \dfrac{(n-1)^n + (-1)^{n-1}}{n}$, allows us to calculate $adj^{\circ n}(A)$ as:

$$(10) \quad adj^{\circ n}(A) = [\det(A)]^{\frac{(n-1)^n + (-1)^{n-1}}{n}} A^{(-1)^n}.$$

Therefore, $A_n \in M_{n\times n}(\mathbf{C})$,

$$
\begin{aligned}
\det\left(adj^{\circ n}(A_n)\right) &= \det\left\{[\det(A)]^{\frac{(n-1)^n + (-1)^{n-1}}{n}} A^{(-1)^n}\right\} \text{ by(10)} \\
&= \left([\det(A)]^{\frac{(n-1)^n + (-1)^{n-1}}{n}}\right)^n \det\left[A^{(-1)^n}\right] \text{ by (4)} \\
&= [\det(A)]^{(n-1)^n + (-1)^{n-1}} \det\left[A^{(-1)^n}\right]
\end{aligned}
$$

$$= \quad [\det(A)]^{(n-1)^n + (-1)^{n-1} + (-1)^n}$$

$$= \quad [\det(A)]^{(n-1)^n}.$$

Thus,

$$\ln\left(\left|\det\left(adj^{\circ n}\left(A_n\right)\right)\right|\right) = \ln\left|[\det(A_n)]^{(n-1)^n}\right| = (n-1)^n \ln|\det(A_n)|,$$

so, for $n \geq 2$,

$$\frac{n^n \ln\left(|\det(A_n)|\right)}{\ln\left(\left|\det\left(adj^{\circ n}(A_n)\right)\right|\right)} = \frac{n^n \ln\left(|\det(A_n)|\right)}{(n-1)^n \ln\left(|\det(A_n)\right)|} = \frac{n^n}{(n-1)^n} = \left(\frac{n}{n-1}\right)^n.$$

That is, the individual A_n has disappeared and our complex fraction has become very simple.

Now it is easy to show by calculus that the limit is e.

- **5151:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Find the value of

$$\prod_{n=1}^{\infty}\left(\sqrt{\frac{\pi}{2}}\cdot\frac{(2n-1)!!\sqrt{2n+1}}{2^n n!}\right)^{(-1)^n}.$$

More generally, if $x \neq n\pi$ is a real number, find the value of

$$\prod_{n=1}^{\infty}\left(\frac{x}{\sin x}\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{(n\pi)^2}\right)\right)^{(-1)^n}.$$

Solution by the proposer

The first product equals $\sqrt{\dfrac{2\sqrt{2}}{\pi}}$ and the second one equals $\dfrac{2\sin\frac{x}{2}}{x}$. Recall the infinite product representation for the sine function

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

Since the first product can be obtained from the second one, when $x = \pi/2$, we concentrate on the calculation of the second product. Let

$$\begin{aligned}
S_{2n} &= \sum_{k=1}^{2n}(-1)^k\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\cdots+\ln\left(1-\frac{x^2}{k^2\pi^2}\right)+\ln\frac{x}{\sin x}\right)\\
&= -\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\frac{x}{\sin x}\right)+\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\left(1-\frac{x^2}{2^2\pi^2}\right)+\ln\frac{x}{\sin x}\right)\\
&\ldots
\end{aligned}$$

$$
\begin{aligned}
& -\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\left(1-\frac{x^2}{2^2\pi^2}\right)+\cdots+\ln\left(1-\frac{x^2}{(2n-1)^2\pi^2}\right)+\ln\frac{x}{\sin x}\right) \\
& +\left(\ln\left(1-\frac{x^2}{\pi^2}\right)+\ln\left(1-\frac{x^2}{2^2\pi^2}\right)+\cdots+\ln\left(1-\frac{x^2}{(2n-1)^2\pi^2}\right)+\ln\left(1-\frac{x^2}{(2n)^2\pi^2}\right)+\ln\frac{x}{\sin x}\right) \\
= \; & \ln\left(\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(4\pi)^2}\right)\cdots\left(1-\frac{x^2}{(2n\pi)^2}\right)\right) \\
= \; & \ln\left(\left(1-\frac{(x/2)^2}{\pi^2}\right)\left(1-\frac{(x/2)^2}{(2\pi)^2}\right)\cdots\left(1-\frac{(x/2)^2}{(n\pi)^2}\right)\right).
\end{aligned}
$$

Letting n tend to infinity in the preceding equality we get that $\lim_{n\to\infty} S_{2n} = \ln\frac{2\sin(x/2)}{x}$, and the problem is solved

Also solved by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2011

- **5170:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $DEFG$ has coordinates $D(-6,-3)$ and $E(2,12)$. The midpoints of the diagonals are on line l.

  Find the area of the quadrilateral if line l intersects line FG at point $P\left(\dfrac{672}{33}, \dfrac{-9}{11}\right)$.

- **5171:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle has integer length sides $x, x+y$, and $x+2y$.

  Part I: Find x and y if the inradius $r = 2011$.

  Part II: Find x and y if $r = \sqrt{2011}$.

- **5172:** *Proposed by Neculai Stanciu, Buzău, Romania*

  If a, b and c are positive real numbers, then prove that,

$$\frac{a\,(b-c)}{c\,(a+b)} + \frac{b\,(c-a)}{a\,(b+c)} + \frac{c\,(a-b)}{b\,(c+a)} \geq 0.$$

- **5173:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

  Find all triples x, y, z of non-negative real numbers that satisfy the system of equations,

$$\begin{cases} x^2(2x^2+x+2) = xy(3x+3y-z) \\ y^2(2y^2+y+2) = yz(3y+3z-x) \\ z^2(2z^2+z+2) = xz(3z+3x-y) \end{cases}$$

- **5174:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a positive integer. Compute:

$$\lim_{n\to\infty} \frac{n^2}{2^n} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}.$$

- **5175:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Find the value of,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{i^2+j^2}.$$

Solutions

- **5152:** *Proposed by Kenneth Korbin, New York, NY*

  Given prime numbers x and y with $x > y$. Find the dimensions of a primitive Pythagorean Triangle which has hypotenuse equal to $x^4 + y^4 - x^2y^2$.

  Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

  We will use the Euclid's formula for generating Pythagorean triples given an arbitrary pair of positive integers m and n with $m > n$, which states that the integers $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if m and n are coprime and one of them is even. Obviously c will be the hypotenuse, since it is the largest side.

  Now using the fact that the hypotenuse is equal to $x^4 + y^4 - x^2y^2$ we have

  $$c = x^4 + y^4 - x^2y^2 = \left(x^2 - y^2\right)^2 + x^2y^2$$

  and this implies that $(m, n) = (x^2 - y^2, xy)$ since $x > y$. Using the fact that the numbers x and y are primes with $x > y$, we have $\gcd(m, n) = \gcd(x^2 - y^2, xy) = 1$, so the numbers m and n are coprime.

  Now if one of the numbers x or y is even, then xy is even, and if both are odd then $x^2 - y^2$ is even; the case when both x and y are even is not possible since $\gcd(x^2 - y^2, xy) = 1$. So at least one of the numbers m or n is even.

  By Euclid's formula we produce primitive Pythagorean Triangles which will have side lengths:

  $$\begin{aligned} a &= m^2 - n^2 = \left(x^2 - y^2\right)^2 - x^2y^2 = x^4 + y^4 - 3x^2y^2 \\ b &= 2mn = 2\left(x^2 - y^2\right)xy = 2x^3y - 2xy^3 \\ c &= m^2 + n^2 = \left(x^2 - y^2\right)^2 + x^2y^2 = x^4 + y^4 - x^2y^2 \end{aligned}$$

  Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

  Recall that (a, b, c) is a primitive Pythagorean triple (with a even) if and only if there are positive integers m, n with $m > n$, $\gcd(m, n) = 1$, and $m \not\equiv n \pmod 2$ such that $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$.

  Let $m = \max\left\{x^2 - y^2, xy\right\}$ and $n = \min\left\{x^2 - y^2, xy\right\}$. Since x and y are distinct primes, it is easily shown that $\gcd(m, n) = 1$ and $m > n$. If $y = 2$, then since x is prime and $x > y = 2$, x must be odd and consequently, xy is even and $x^2 - y^2$ is odd. If $y > 2$, then since x and y are primes and $x > y > 2$, x and y must be odd. In this case, xy is odd and $x^2 - y^2$ is even. It follows that $m \not\equiv n \pmod 2$ in all cases.

As a result, $a = 2mn = 2xy\left(x^2 - y^2\right)$, $b = m^2 - n^2 = \left|\left(x^2 - y^2\right)^2 - x^2y^2\right|$, $c = m^2 + n^2 = \left(x^2 - y^2\right)^2 + x^2y^2 = x^4 + y^4 - x^2y^2$ yields a primitive Pythagorean triple (a, b, c). Some examples are listed in the following table:

x	y	a	b	c
3	2	60	11	61
5	2	420	341	541
7	2	1260	1829	2221
5	3	480	31	481
7	3	1680	1159	2041
7	5	1680	649	1801

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5153:** *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with sides $(1, 1, 1, x)$ and a trapezoid with sides $(1, x, x, x)$ are both inscribed in the same circle. Find the diameter of the circle.

Solution 1 by David E. Manes, Oneonta, NY

Let D be the diameter of the circle. If $x = 1$, then $D = \sqrt{2}$. If $x = \dfrac{3+\sqrt{5}}{2}$ or $x = \dfrac{3-\sqrt{5}}{2}$, then $D = \sqrt{6 + 2\sqrt{5}}$ or $\sqrt{6 - 2\sqrt{5}}$ respectively.

Given a cyclic quadrilaterals with successive sides a, b, c, d and semiperimeter s, then the diameter D of the circumscribed circle is given by

$$D = \frac{1}{2}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

For the trapezoid with sides $(1, 1, 1, x)$,

$$D = \frac{1}{2}\sqrt{\frac{(1+x)(1+x)(1+x)}{\left(\frac{1+x}{2}\right)\left(\frac{1+x}{2}\right)\left(\frac{1+x}{2}\right)\left(\frac{3-x}{2}\right)}} = \frac{2}{\sqrt{3x-1}}.$$

For the trapezoid with sides $(1, x, x, x)$,

$$D = \frac{1}{2}\sqrt{\frac{(x+x^2)(x+x^2)(x+x^2)}{\left(\frac{3x-1}{2}\right)\left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right)}} = \frac{2x\sqrt{x}}{\sqrt{3-x}}.$$

Setting the two expressions for D equal and simplifying,one obtains the quartic equation

$$x^4 - 3x^3 + 3x - 1 = \left(x^2 - 1\right)\left(x^2 - 3x + 1\right) = 0$$

whose positive roots are $x = 1$, or $x = \dfrac{3 \pm \sqrt{5}}{2}$.

If $x = 1$, then $D = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

If $x = \dfrac{3+\sqrt{5}}{2}$, then $D = \dfrac{2}{\sqrt{3 - \left(\dfrac{3+\sqrt{5}}{2}\right)}} = \sqrt{6 + 2\sqrt{5}}$.

If $x = \dfrac{3-\sqrt{5}}{2}$, then $D = \sqrt{6 - 2\sqrt{5}}$.

Finally, note that if $x = 1$, then the two trapezoids are the same unit square.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Call the first trapezoid $ABCD$ such that $AB = BC = CD = 1$ and $DA = x$.

Let $\angle ABC = \theta$ so that $\angle ADC = \pi - \theta$. Applying the cosine formula to triangles ABC and ADC, we obtain respectively

$$AC^2 = 2(1 - \cos\theta) \text{ and } AC^2 = x^2 + 2x\cos\theta + 1.$$

Eliminating AC from these two equations, we obtain $\cos\theta = \dfrac{1-x}{2}$ and hence $AC = \sqrt{x+1}$.

Let the diameter of the circle be d. By the sine formula, we have

$$d = \frac{AC}{\sin\theta} = \frac{2}{\sqrt{3-x}}. \qquad (1)$$

Call the second trapezoid $PQRS$ such $PQ = QR = RS = x$ and $SP = 1$.

Let $\angle$PQR=ϕ so that $\angle PSR = \pi - \phi$. By the procedure similar to that for trapezoid $ABCD$, we obtain

$$d = \frac{PR}{\sin\phi} = \frac{2x\sqrt{x}}{\sqrt{3x-1}}. \qquad (2)$$

From (1) and (2), we obtain $x^4 - 3x^3 + 3x - 1 = 0$, whose positive roots are $1,\ \dfrac{3-\sqrt{5}}{2},\ \dfrac{3+\sqrt{5}}{2}$. The corresponding values of d are $\sqrt{2}$, $\sqrt{5} - 1$, and $\sqrt{5} + 1$.

Also solved by Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia (jointly with) Elton Bojaxhiu, Kriftel, Germany; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; Raul A. Simon, Santiago, Chile; Trey Smith, San Angelo, TX; Jim Wilson, Athens, GA, and the proposer.

- **5154:** *Proposed by Andrei Răzvan Băleanu (student, George Cosbuc National College) Motru, Romania*

Let a, b, c be the sides, m_a, m_b, m_c the lengths of the medians, r the in-radius, and R the circum-radius of the triangle ABC. Prove that:

$$\frac{m_a^2}{1+\cos A} + \frac{m_b^2}{1+\cos B} + \frac{m_c^2}{1+\cos C} \geq 6Rr\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Solution by Arkady Alt, San Jose, California, USA

Since

$$
\begin{aligned}
\frac{m_a^2}{1+\cos A} &= \frac{m_a^2}{2\cos^2\frac{A}{2}} = \frac{m_a^2}{2}\left(1+\tan^2\frac{A}{2}\right) = \frac{m_a^2}{2}+\frac{m_a^2}{2}\tan^2\frac{A}{2} \\
&= \frac{m_a^2}{2}+\frac{m_a^2}{2}\cdot\frac{r^2}{(s-a)^2} = \frac{2\left(b^2+c^2\right)-a^2}{8}+\frac{(b+c)^2-a^2+(b-c)^2}{8}\cdot\frac{r^2}{(s-a)^2} \\
&\geq \frac{2\left(b^2+c^2\right)-a^2}{8}+\frac{s(s-a)r^2}{2(s-a)^2} = \frac{2\left(b^2+c^2\right)-a^2}{8}+\frac{sr^2}{2(s-a)},
\end{aligned}
$$

then

$$\sum_{cyc}\frac{m_a^2}{1+\cos A} \geq \frac{3\left(a^2+b^2+c^2\right)}{8}+\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a}.$$

Noting that

$$
\begin{aligned}
\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a} &= \frac{(s-a)(s-b)(s-c)}{2}\sum_{cyc}\frac{1}{s-a} = \frac{1}{2}\sum_{cyc}(s-b)(s-c) \\
&= \frac{2ab+2bc+2ca-a^2-b^2-c^2}{8},
\end{aligned}
$$

we obtain

$$\frac{3\left(a^2+b^2+c^2\right)}{8}+\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a} = \frac{ab+bc+ca+a^2+b^2+c^2}{4}.$$

Hence,

$$\sum_{cyc}\frac{m_a^2}{1+\cos A} \geq A,\ \text{where } \mathrm{A} = \frac{ab+bc+ca+a^2+b^2+c^2}{4}.$$

Also since,

$$6Rr = \frac{12Rrs}{2s} = \frac{3abc}{2s} \quad\text{and}\quad \frac{a}{\mathrm{b}+\mathrm{c}} \leq \frac{a^2(b+c)}{4abc},$$

we have,

$$B \geq 6Rr\sum_{cyc}\frac{a}{b+c},\ \text{where } \mathrm{B} = \frac{3abc}{2s}\sum_{\mathrm{cyc}}\frac{a^2(b+c)}{4abc} = \frac{3}{4(a+b+c)}\sum_{\mathrm{cyc}}a^2(b+c).$$

Thus, it suffices to prove inequality $A \geq B$.

Since $\sum_{cyc} a(a-b)(a-c) \geq 0$ (by the Schur Inequality), we have

$$4(a+b+c)(A-B) = (a+b+c)\left(ab+bc+ca+a^2+b^2+c^2\right)-3\sum_{cyc}a^2(b+c)$$

$$
\begin{aligned}
&= && (a+b+c)\left((a+b+c)^2-ab-bc-ca\right) \\
&- && 3(a+b+c)(ab+bc+ca)+9abc \\
&\Longleftrightarrow && 9abc+(a+b+c)^3 \geq 4(a+b+c)(ab+bc+ca) \\
&\Longleftrightarrow && \sum_{cyc} a(a-b)(a-c) \geq 0.
\end{aligned}
$$

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5155:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be the roots of the equation $x^4+6x^3+7x^2+6x+1=0$. Find the value of

$$
\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d}.
$$

Solution 1 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

Since a, b, c, d are the roots of the equation $x^4+6x^3+7x^2+6x+1=0$, by Viéte's formulas we have

$$
\begin{aligned}
a+b+c+d &= -6 \\
ab+ac+ad+bc+bd+cd &= 7 \\
abc+abd+acd+bcd &= -6 \\
abcd &= 1.
\end{aligned}
$$

For convenience we adopt the following notation:

$$
\begin{aligned}
x &= a+b+c+d \\
y &= ab+ac+ad+bc+bd+cd \\
z &= abc+abd+acd+bcd \\
w &= abcd.
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d} &= \frac{-(8w+3z-2y-7x-12)}{w+z+y+x+1} \\
&= \frac{-(8\times 1+3\times(-6)-2\times 7-7\times(-6)-12)}{1-6+7-6+1} \\
&= 2.
\end{aligned}
$$

Solution 2 by Brian D. Beasley, Clinton, SC

Since $x^4+6x^3+7x^2+6x+1=(x^2+x+1)(x^2+5x+1)=0$, we calculate the four roots and assign the values $a=(-1+i\sqrt{3})/2$, $b=(-1-i\sqrt{3})/2$, $c=(-5+\sqrt{21})/2$, and $d=(-5-\sqrt{21})/2$. This yields:

$$\frac{3-2a}{1+a}=\frac{1-5i\sqrt{3}}{2};\qquad \frac{3-2b}{1+b}=\frac{1+5i\sqrt{3}}{2};$$
$$\frac{3-2c}{1+c}=\frac{3+5\sqrt{21}}{6};\qquad \frac{3-2d}{1+d}=\frac{3-5\sqrt{21}}{6}.$$

Hence the desired sum is 2.

Solution 3 by David E. Manes, Oneonta, NY

The value of the expression is 2.

Note that if r is a root of the equation, then $r \neq 0$ and moreover $\frac{1}{r}$ also satisfies the equation since

$$\frac{1}{r^4}+\frac{6}{r^3}+\frac{7}{r^2}+\frac{6}{r}+1=\frac{1+6r+7r^2+6r^3+r^4}{r^4}=0.$$

Therefore, the roots of the equation can be labeled $a, b=\frac{1}{a}, c$, and $d=\frac{1}{c}$. Then

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}=\frac{3-2a}{1+a}+\frac{3-\frac{2}{a}}{1+\frac{1}{a}}=\frac{3-2a}{1+a}+\frac{3a-2}{1+a}=1.$$

Similarly, $\frac{3-2c}{1+c}+\frac{3-2d}{1+d}=1$. Hence,

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d}=2.$$

Solution 4 by Michael Brozinsky, Central Islip, New York

We first note that if $f(x)=\frac{3-2x}{1+x}$, then $f^{-1}(x)=\frac{-x+3}{x+2}$.

If we denote the given polynomial by $P(x)$, then the given expression is just the sum of the roots of the equation $-3x^4+6x^3-17x^2+14x+817=0$, obtained by clearing fractions in the equation $P\left(f^{-1}(x)\right)=0$. So the answer is $-6/(-3)=2$.

Solution 5 by Pedro H. O. Pantoja, UFRN, Brazil

Let a, b, c, d be the roots of equation $x^4+6x^3+7x^2+6x+1=0$, then $1+a, 1+b, 1+c, 1+d$ will be the roots of the equation $x^4+2x^3-5x^2+6x-3=0$. So,

$$\frac{1}{1+a}, \frac{1}{1+b}, \frac{1}{1+c}, \frac{1}{1+d},$$

will be the roots of the equation $3x^4-6x^3+5x^2-2x-1=0$.

Then,

$$\frac{1}{1+a}, \frac{1}{1+b}, \frac{1}{1+c}, \frac{1}{1+d} \quad = \quad \frac{-(-6)}{3}=2, \quad \text{implies}$$

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d} \quad = \quad 5\cdot 2-4\cdot 2=2.$$

Comments by David Stone and John Hawkins of Statesboro, GA

(1) If we fix h and $j=k$ and define $G(x)=\dfrac{h+kx}{1+x}$, then

$$G(r)+G\left(\frac{1}{r}\right)=\frac{h+kr}{1+r}+\frac{h+k\dfrac{1}{r}}{1+\dfrac{1}{r}}=\frac{h+kr}{1+r}+\frac{rh+k}{r+1}=\frac{h+rh+k+kr}{1+r}=h+k.$$

Thus, in the setting of the posed problem,

$$G(a)+G(b)+G(c)+G(d)=G(a)+G\left(\frac{1}{a}\right)+G(b)+G\left(\frac{1}{b}\right)=2\left(h+k\right).$$

In fact, if $p(x)=\sum\limits_{i=0}^{2n} a_i x^k$ is any palindromic polynomial (i.e., $a_k=a_{2n-k}$) of even degree with distinct zeros unequal to ± 1 and $G(x)=\dfrac{h+kx}{1+x}$, then $\sum G(r)=n(h+k)$, where the sum is taken over all zeros of $p(x)$.

(2) We had to be careful with the pairing of the zeros because it is conceivable that some r is being paired with multiple reciprocals$-$say all the zeros were $2,2,2$, and $\dfrac{1}{2}$. Actually, the polynomial given in the problem has a pair of real zeros (both negative and reciprocals of each other) and a pair or complex zeros (which must lie on the unit circle since the reciprocal equals the conjugate).

(3) We wonder if the expression $\dfrac{3-2x}{1+x}$ and the total sum have some deep connection to the polynomial. Perhaps there is some algebraic relationship, since algebraists carefully analyze properties of the zeros of polynomials.

Also solved by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico; Brian D. Beasley (two solutions), Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), Angelo TX; Bruno Salgueiro Fanego, Viveiro, Spain; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia with Elton Bojaxhiu, Kriftel, Germany; Talbot Knighton, Stephen Chou and Tom Peller (jointly, students at Taylor University). Upland, IN; Bradley Luderman, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Sugie Lee, Jon Patton, and Matthew Fox (jointly, students at Taylor University), Upland, IN; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Aaron Milauskas, Daniel Perrine, and Kari Webster (jointly, students at Taylor University), Upland, IN; John Nord, Spokane WA; Boris, Rays, Brooklyn, NY, and the proposer.

- **5156:** *Proposed by Yakub N. Aliyev, Khyrdalan, Azerbaijan*

Given two concentric circles with center O and let A be a point different from O in the interior of the circles. A ray through A intersects the circles at the points B and C. The ray OA intersects the circles at the points B_1 and C_1, and the ray through A

perpendicular to line OA intersects the circles at the points B_2 and C_2. Prove that

$$B_1C_1 \le BC \le B_2C_2.$$

Solution 1 by Charles McCracken, Dayton, OH

$$B_1C_1 < BC$$

(because the shortest path between two concentric circle is along a ray from the center.)

Rotate $\triangle OBC$ until C is at C_2, and let D be the new position of B.

$$\begin{array}{rcl}
\triangle ODC_2 & \cong & \triangle OBC \\
OD = OB, \quad OC_2 & = & OC, \quad \angle DOC = \angle BOC \\
\triangle ODC & \textit{lies inside} & \triangle OB_2C_2 \\
DC_2 & < & B_2C_2 \\
DC_2 & = & BC \\
BC & < & B_2C_2 \\
B_1C_1 < & BC & < B_2C_2.
\end{array}$$

This is almost a proof without words!

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We will employ the *line* through the point A, rather than a ray. This is satisfactory because the two line segments formed by the line intersecting the given annulus have the same length. So if we impose a coordinate system with origin at the circles' center and assume that the point A is on the non-negative $y-$axis, we can restrict our attention to the right half plane.

Suppose the two concentric circles have radii $r_1 < r_2$. By a rotation we can position A at $(0, b)$, where $b \ge 0$ in the case where the line is vertical, the distance BC is B$_1C_1 = r_2 - r_1$.
A non-vertical line though A has equation $y = mx + b$ where m may vary from $-\infty$ to $+\infty$.

It is straight forward to determine that the line that intersects the right half circle of radius r at the point $(x, mx + b)$, where $x = \dfrac{-mb + \sqrt{(1 + m^2)r^2 - b^2}}{1 + m^2}$. Now we have $B = (x_1, y_1)$ the point in the right half plane where the line $y = mx + b$ intersects the inner circle of radius r_1 and $C = (x_2, y_2)$ the point in the right half pane where the line $y = mx + b$ intersects the outer circle of radius r_2.

Thus, $x_1 = \dfrac{-mb + \sqrt{(1 + m^2)r_1^2 - b^2}}{1 + m^2}$ and $x_2 = \dfrac{-mb + \sqrt{(1 + m^2)r_2^2 - b^2}}{1 + m^2}$. Note that $x_1 < x_2$.

We compute the distance BC, which depends only upon m by the distance formula:

$$\begin{aligned} d(m) &= \sqrt{(x_2-x_1)^2+(mx_2+b-mx_1-b)^2} \\ &= \sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2} \\ &= \sqrt{(1+m^2)(x_2-x_1)^2} \\ &= \sqrt{(1+m^2)}\,(x_2-x_1) \end{aligned}$$

Because $x_2 - x_1 = \dfrac{\sqrt{(1+m^2)\,r_2^2-b^2}-\sqrt{(1+m^2)\,r_1^2-b^2}}{1+m^2}$, we see that

$$d(m) = \frac{\sqrt{(1+m^2)\,r_2^2-b^2}-\sqrt{(1+m^2)\,r_1^2-b^2}}{\sqrt{1+m^2}} = \sqrt{r_2^2-\frac{b^2}{1+m^2}}-\sqrt{r_1^2-\frac{b^2}{1+m^2}}.$$

Our goal is to show that the length of the line segment BC is a maximum when $m=0$ and is a minimum when the line is vertical, using $BC = d(m)$ for all non-vertical lines.

Note the behavior of the function d:

(1) d is even, so its graph is symmetric about the $y-$axis.

(2) As m grows to positive or negative infinity, $d(m)$ approaches $r_2 - r_1$.

(3) $d(0) = \sqrt{r_2^2-b^2}-\sqrt{r_1^2-b^2} = B_2C_2$.

(4) $d'(m) = -\dfrac{mb^2}{(1+m^2)^2\sqrt{r_1^2-\dfrac{b^2}{1+m^2}}\sqrt{r_2^2-\dfrac{b^2}{1+m^2}}}d(m)$.

By the expression for the derivative, we see:

$$\begin{aligned} &\text{for } m<0,\ d'(m)>0, \text{ so } d \text{ is increasing;} \\ &\text{for } m>0,\ d'(m)<0, \text{ so } d \text{ is decreasing.} \end{aligned}$$

Therefore, d achieves its maximum, given in (3), when $m=0$; that is, in the direction perpendicular to the line along OA. The minimum value of BC is $B_1C_1 = r_2 - r_1$. All other values of BC lie between these extremes.

Also solved by Michael Brozinsky, Central Islip, NY; Michael N. Freid, Kibbutz Revivim, Israel; Raul A. Simon, Santiago, Chile, and the proposer.

- **5157:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

Let $p \geq 2, \lambda \geq 1$ be real numbers and let $e_k(x)$ for $1 \leq k \leq n$ be the symmetric elementary functions in the variables $x = (x_1, \ldots, x_n)$ and $x^p = (x_1^p, \ldots, x_n^p)$, with $n \geq 2$ and $x_i > 0$ for all $i = 1, 2, \ldots, n$.

Prove that

$$e_n^{(pk/n)}(x) \leq \frac{e_k(x^p)+\lambda\left(e_k^p(x)-e_k(x^p)\right)}{\binom{n}{k}+\lambda\left(\binom{n}{k}^p-\binom{n}{k}\right)} \leq \left(\frac{e_1(x)}{n}\right)^{pk}, \quad 1 \leq k \leq n.$$

Solution by the proposer

The elementary symmetric functions in the real variables $x_1, x_2, \ldots, x_n$ for $1 \le k \le n$ are defined as follows:

$$\begin{aligned}
e_1(x) &= e_1(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i = \sum_{1 \le i \le n} x_i = \sum_{1 \le i \le \binom{n}{1}} x_i \\
e_2(x) &= \sum_{1 \le i < j \le n} x_i x_j = \sum_{1 \le i \le \binom{n}{2}} x_i^*, \\
\textit{where } x_i^* &= x_{i_1} x_{i_2}, 1 \le i_1 < i_2 \le n; \textit{ and similarly,} \\
e_k(x) &= \sum_{1 \le i \le \binom{n}{k}} x_i^* = \sum_{1 \le i_1 < i_2 \ldots < k_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k} \\
\textit{where } x_i^* &= x_{i_1} x_{i_2} \ldots x_{i_k}, \quad 1 \le i_1 < i_2 < \ldots < i_k \le n, \\
&\ldots \\
e_n(x) &= x_1 x_2 \ldots x_n.
\end{aligned}$$

We present some results, that we will need:

Theorem 1 (Mac Laurin's Inequalities)

$$If \ E_k(x) = \frac{e_k(x)}{\binom{n}{k}}, 1 \le k \le n, \ is \ the \ k^{th} \ symmetric \ function \ mean, \ then$$

$$(E_n(x))^{1/n} \le \ldots \le (E_k(x))^{1/k} \le \ldots \le (E_2(x))^{1/2} \le E_1(x).$$

Theorem 2 (Power Means Inequality)

$$If x_i > 0, i = 1, 2, \ldots, n, \ p > 1 \ are \ real \ numbers \ then,$$

$$\sum_{i=1}^{n} x_i^p < \left(\sum_{i=1}^{n} x_i \right)^p \le n^{p-1} \sum_{i=1}^{n} x_i^p.$$

See, reference [1].

And by reference [2], we have:

Corollary (Shanse Wu)

$$If \ x_i > 0, i = 1, 2, \ldots, n; \ n \ge 2, \ p \ge 2, \ 1 \le k \le n \ are \ real \ numbers, \ then$$

$$(e_k(x))^p - e_k(x^p) \ge \left(\binom{n}{k}^p - \binom{n}{k} \right) (e_n(x))^{pk/n}.$$

We denote the homographic function of the real variable $\lambda \ge 1, \ with \ x \in \Re_+^n \ and \ p \ge 2$ fixed as follows:

$$f(\lambda) = \frac{a + \lambda(b - a)}{\binom{n}{k} + \lambda\left(\binom{n}{k}^p - \binom{n}{k}\right)}$$

where $a = e_k(x^p)$ and $b = e_k^p(x), \ 1 \le k \le n.$

Properties of the function f

1) f is a positive function for $\lambda \geq 1$ since by Corollary 1, we have

$$b - a = e_k^p(x) - e_k(x^p) \geq \left(\binom{n}{k}^p - \binom{n}{k} \right) e_n^{pk/n}(x) > 0, \ for \ 1 \leq k \leq n,$$

and so by the definition of f we obtain, $f(\lambda) \geq 0$, for $\lambda \geq 1$.

2) f is an infinitely differentiable continuous function for $\lambda \geq 1$.

Monotonicity of f.

We have:

$$f(1) = \frac{a + b - a}{\binom{n}{k}^p} = \frac{b}{\binom{n}{k}^p} = \frac{e_k^p(x)}{\binom{n}{k}^p} = \left(\frac{e_k(x)}{\binom{n}{k}} \right)^p = E_k^p(x) \leq (E_1^k(x)^p = [E_1(x)]^{pk}$$

by application of the Theorem 1 (MacLaurin's Inequalities).

Now by computing and evaluating the first derivative of the function f we obtain:

$$\begin{aligned} f'(\lambda) &= \frac{\left[\binom{n}{k} + \lambda \left(\binom{n}{k}^p - \binom{n}{k}\right)(b-a)\right] - \left[a + \lambda(b-a)\right]\left[\binom{n}{k}^p - \binom{n}{k}\right]}{\left[\binom{n}{k} + \lambda\left[\binom{n}{k}^p - \binom{n}{k}\right]\right]^2} = \frac{\binom{n}{k} b - a \binom{n}{k}^p}{D} \\ &= \frac{\binom{n}{k}}{D} \left[b - a \binom{n}{k}^{p-1} \right] \\ &= \frac{\binom{n}{k}}{D} \left[\Big(\sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} x_{j_1} x_{j_2} \ldots x_{j_k} \Big)^p - \binom{n}{k}^{p-1} \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} (x_{j_1} x_{j_2} \ldots x_{j_k})^p \right] \\ &= \frac{\binom{n}{k}}{D} \left[\left(\sum_{1 \leq j \leq \binom{n}{k}} x_j^* \right)^p - \binom{n}{k}^{p-1} \sum_{1 \leq j \leq \binom{n}{k}} x^{p_j} \right] \leq 0. \end{aligned}$$

by application of Theorem 2 and where $D = \left[\binom{n}{k} + \lambda\left[\binom{n}{k}^p - \binom{n}{k}\right]\right]^2 > 0$, and where the products are defined as: $x_j^* = x_{j_1} x_{j_2} \ldots x_{j_k}$ where j is the number of elements in (or cardinality of) each set $J = \{j_1, j_2, \ldots, j_k\}$ with k elements. The number of all the products is also equal to all subsets of k elements of the set $\{1, 2, \ldots, n\}$ which is in total $\binom{n}{k}$.
Using Theorem 2 (the power means inequality), function f is decreasing for $\lambda \geq 1$. And so,

$$f(+\infty) = \lim_{\lambda \to +\infty} f(\lambda) \leq f(\lambda) \leq f(1), \ for \ \lambda \geq 1.$$

From the above corollary we have:

$$f(+\infty) = \lim_{\lambda \to +\infty} f(\lambda) = \frac{b - a}{\binom{n}{k}^p - \binom{n}{k}} = \frac{e_k^p(x) - e_k(x^p)}{\binom{n}{k}^p - \binom{n}{k}} \geq \left(\prod_{i=1}^n x_i^{p/n} \right)^k = e_n^{pk/n}(x)).$$

And so, for the conditions of the problem, we have shown that the original inequality holds.

$$e_n^{pk/n}(x) \leq \frac{e_k(x^p) + \lambda \left(e_k^p(x) - e_k(x^p)\right)}{\binom{n}{k} + \left(\binom{n}{k}^p - \binom{n}{k}\right)} \leq \left(\frac{e_1(x)}{n} \right)^{pk}.$$

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[1] P.S. Bullen, D.S. Mitrinovic, P.M. Vasic. **The Handbook of Means and their Inequalities,** Reidel, Dordrecht, 1988.

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[4] Mihaly Bencze. "New means and new refinements of Cesaro's inequality and of the AM-GM-HM inequalities," **Libertas Mathematica,** Vol 28 (2008; pp.123-124).

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2012

- **5176:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
$$\begin{cases} x^2+xy+y^2=3^2 \\ y^2+yz+z^2=4^2 \\ z^2+xz+x^2=5^2. \end{cases}$$

- **5177:** *Proposed by Kenneth Korbin, New York, NY*

  A regular nonagon $ABCDEFGHI$ has side 1.
  Find the area of $\triangle ACF$.

- **5178:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Prove: If x, y and z are positive real numbers such that $xyz \geq 7+5\sqrt{2}$, then
$$x^2+y^2+z^2-2(x+y+z) \geq 3.$$

- **5179:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Find all positive real solutions $(x_1, x_2, \ldots, x_n)$ of the system
$$\begin{cases} x_1+\sqrt{x_2+11}=\sqrt{x_2+76}, \\ x_2+\sqrt{x_3+11}=\sqrt{x_3+76}, \\ \cdots\cdots\cdots \\ x_{n-1}+\sqrt{x_n+11}=\sqrt{x_n+76}, \\ x_n+\sqrt{x_1+11}=\sqrt{x_1+76}. \end{cases}$$

- **5180:** *Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy*

  Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that
$$\frac{1+a}{bc}+\frac{1+b}{ac}+\frac{1+c}{ab} \geq \frac{4}{\sqrt{a^2+b^2-ab}}+\frac{4}{\sqrt{b^2+c^2-bc}}+\frac{4}{\sqrt{a^2+c^2-ac}}.$$

- **5181:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate:
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\cdot m}{(n+m)!}.$$

Solutions

- **5158:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with integer length sides $\overline{AB}=\overline{BC}=x$, and $\overline{CD}=\overline{DA}=x+1$.

  Find the distance between the incenter and the circumcenter.

  Solution by Michael Brozinsky, Central Islip, NY

  Since the perpendicular bisector of the base of an isosceles triangle passes through the vertex angle and the circumcenter of that triangle, it follows (by considering isosceles triangles CBA and CDA) that the line segment joining B and D is a diameter of the circumcircle, and thus the inscribed angles A and C are right angles.

  The circumcenter **E** is also the circumcenter of right triangle BAC, and thus it is the midpoint of the hypotenuse BD, and so it is $\dfrac{\sqrt{x^2+(x+1)^2}}{2}$ from B. The incenter **F** (being equidistant from BA and BD) is on the angle bisector of angle A and also on BD (by symmetry as triangles ABD and CBD are congruent), and so $\dfrac{BF}{FD}=\dfrac{x}{x+1}$ (since an angle bisector of a triangle divides the opposite side into segments proportional to the adjacent sides).

  Since $BF+FD=\sqrt{x^2+(x+1)^2}$ we have $BF=\dfrac{x}{2x+1}\cdot\sqrt{x^2+(x+1)^2}$ and hence the distance between **E** and **F** is
$$\sqrt{x^2+(x+1)^2}\cdot\left(\frac{1}{2}-\frac{x}{2x+1}\right)=\frac{\sqrt{2x^2+2x+1}}{4x+2}.$$

  Comments: Most of the solvers realized that there is no need to restrict x to being an integer; x can be any positive real number. **David Stone and John Hawkins** also mentioned in their solution that even though the inradius (ρ) and the circumradius (r) grow large with x, as does the difference $r-\rho$, the distance d between the centers has the limiting value of $\dfrac{\sqrt{2}}{4}\approx 0.35355339$. So for large x, the incircle and the circumcircle are relatively concentric.

  Also solved by Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays of Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5159:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on diagonal $\overline{AC}$ and with point Q at the midpoint of side $\overline{AB}$.

  Find the perimeter of cyclic quadrilateral $ADPQ$ if its area is one unit less than the area of square $ABCD$.

  Solution by Trey Smith, San Angelo, TX

  Fix a point E on $\overline{AB}$ such that $\overline{PE}$ is perpendicular to $\overline{AB}$. Similarly, fix a point F on $\overline{AD}$ such that $\overline{PF}$ is perpendicular to $\overline{AD}$. Let k =length($\overline{AB}$). $AQPD$ is a cyclic quadrilateral, so it must be the case that $\angle QPD$ is a right angle, since it and $\angle DAQ$ are supplementary. Now $\angle FPE$ is also a right angle which forces $\angle QPE \cong \angle DPF$. And since $\triangle EPQ$ and $\triangle FPD$ are both right triangles with $\overline{PE} \cong \overline{PF}$, it is the case that $\triangle FPD \cong \triangle EPQ$. Finally, observing that $\overline{EQ} \cong \overline{FD} \cong \overline{EB}$ we have that E is the midpoint of $\overline{QB}$ and so the length of $\overline{AE}$ is $\dfrac{3k}{4}$.

  Since $\triangle FPD \cong \triangle EPQ$, it is easy to see that the area of $AQPD$ is the same as the area of square $AEPF$. Thus, the area of $AQPD$ is $\dfrac{9k^2}{16}$ and so the difference in the area of $ABCD$ and $AQPD$ is $\dfrac{7k^2}{16}$. Setting this equal to 1 and solving, we obtain $k = \dfrac{4}{\sqrt{7}}$. Now

$$\begin{aligned}
\text{length}(\overline{AQ}) = \frac{k}{2} &= \frac{2}{\sqrt{7}}, \\
\text{length}(\overline{QP}) = \text{length}(\overline{PD}) &= \sqrt{\left(\frac{k}{4}\right)^2 + \left(\frac{3k}{4}\right)^2} = \frac{k\sqrt{10}}{4} = \frac{\sqrt{10}}{\sqrt{7}}, \text{and} \\
\text{length}(\overline{DA}) &= k = \frac{4}{\sqrt{7}}.
\end{aligned}$$

  Summing these, we obtain the perimeter $\dfrac{6+2\sqrt{10}}{\sqrt{7}} = \dfrac{6\sqrt{7}+2\sqrt{70}}{7}$.

  Comment by David Stone and John Hawkins. It is easy to show that the point P must be located $\dfrac{3}{4}$ of the way from A to C along the diagonal AC in order to make $ADPQ$ a cyclic quadrilateral.

  Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Tania Moreno García, UHO, Cuba jointly with Jose P. Suárez, ULPGC, Spain; Paul M. Harms, North Newton, KS; Caleb Hemmick, Kaleb Davis, Logan Belgrave and Brianna Leever (jointly, students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; Sugie Lee, Jon Patton, and Matthew Fox (jointly, students at Taylor University), Upland, IN; David E. Manes, Oneonta, NY; Aaron Milauksas, Daniel Perrine, Kari Webster (jointly, students at Taylor University), Upland, IN; Tom Peller, Stephen Chou and Tal Knighton (jointly, students at Taylor University), Upland, IN; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5160:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, there are n (where $n \geq 2$) roads $\{l_i\}$ whose equations are

$$l_i: \; x\cos\left(\frac{2\pi i}{n}\right) + y\sin\left(\frac{2\pi i}{n}\right) = i, \text{where } i = 1, 2, 3, \ldots, n.$$

  Any anthill must be located so that the sum of the squares of its distances to these n lines is $\dfrac{n(n+1)(2n+1)}{6}$. Two queen ants are (im)mortal enemies and have their anthills as far apart as possible. If the distance between these queens' anthills is 4 units, find n.

Solution by Kee-Wai Lau, Hong Kong, China

We show that the anthills are $2\csc\left(\dfrac{\pi}{n}\right)$ units apart for $n \geq 3$. In the present case that they are 4 units apart, we see that $n = 6$. If $n = 2$, then the anthill can be located anywhere on the $y-$axis, so that the distance between them can be as large as possible.

For simplicity, we denote π/n by m. Let the coordinates of an anthill be $(r\cos\theta, r\sin\theta)$, where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. Its distance to l_i is given by

$$|r\cos\theta\cos(2mi) + r\sin\theta\sin(2mi) - i| = |r\cos(2mi - \theta) - i|.$$

Since $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so according to the rule for location , we have $\displaystyle\sum_{i=1}^{n}(r\cos(2mi-\theta) - i)^2 = \sum_{i=1}^{n} i^2$. Clearly the origin satisfies the rule. If $r \neq 0$, then

$$\sum_{i=1}^{n}\Big(r\cos^2(2mi-\theta) - 2i\cos(2mi-\theta)\Big) = 0. \qquad (1)$$

As $\cos^2(2mi-\theta) = \dfrac{1}{2}\left(1 + \cos(4mi - 2\theta)\right)$, so (1) is equivalent to

$$\begin{aligned} rn + r\cos 2\theta\sum_{i=1}^{n}\cos(4mi) \quad &+ \quad r\sin 2\theta\sum_{i=1}^{n}(4mi) \\ -4\cos\theta\sum_{i=1}^{n} i\cos(2mi) \quad &- \quad 4\sin\theta\sum_{i=1}^{n} i\sin(2mi) = 0. \qquad (2) \end{aligned}$$

For $\sin(x/2) \neq 0$ and positive integers k, we have the following known results,

$$\sum_{i=1}^{k}\cos(ix) = \frac{\sin(kx/2)\cos(k+1)x/2}{\sin(x/2)}, \qquad \sum_{i=1}^{k}\sin(ix) = \frac{\sin(kx/2)\sin(k+1)x/2}{\sin(x/2)},$$

$$\sum_{i=1}^{k} i\cos(ix) = \frac{(k+1)\cos(kx)k\cos(k+1)x - 1}{4\sin^2(x/2)}, \qquad \sum_{i=1}^{k} i\sin(ix) = \frac{(k+1)\sin(kx) - k\sin(k+1)x}{4\sin^2(x/2)},$$

which can be proved readily by induction on k. Thus for $n \geq 3$, we have

$$\sum_{i=1}^{n} \cos(4mi) = \sum_{i=1}^{n} \sin(4mi) = 0, \quad \sum_{i=1}^{n} i\cos(2mi) = \frac{n}{2}, \quad \sum_{i=1}^{n} \sin(2mi) = \frac{-n\cot(m)}{2},$$

and from (2) we deduce that for $m - \pi < \theta < m$,

$$r = \frac{2\sin(m-\theta)}{\sin m}. \qquad (3)$$

Together with the origin, (3) represents the locus of a circle. In rectangular coordinates the equation of the circle is $(x-1)^2\,(y+\cot m)^2 = \csc^2 m$. Thus the distance between the anthills equals the diameter $2\csc m$ of the circle and this completes the solution.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5161:** *Proposed by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy*

It is well known that for any function $f : \Re \to \Re$, continuous or not, the set of points on the y-axis where it attains a maximum or a minimum can be at most denumerable. Prove that any function can have at most a denumerable set of inflection points, or give a counterexample.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let C be the Cantor ternary set, defined by

$$C = [0,1] - \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

(see [1] as a reference). It is well known that C is uncountable and has Lebesgue measure zero. (Therefore C does not contain any interval).

For any point $x \in [0,1]$ define the distance from x to C by $d(x,C) = \inf_{y \in C} |x-y|$. If $x \in [0,1]$ there is (at least) one point $y(x) \in C$ such that $d(x,C) = |x - y(x)|$, since C is closed. Furthermore $d(x,C) = 0$ if and only if $x \in C$.

Define a real function $f : [0,1] \longrightarrow R$ by $f(x) = (x - y(x))^3$, if the point $y(x) \in C$ that is closest to x is unique, and put $f(x) = 0$ if there is not a unique closest point to x. Extend f to a 1-periodic function to the whole of the real line. f is a piecewise cubic polynomial (and therefore f is piecewise differentiable). Any point of the form $n + y$ where n is an integer and $y \in C$ is an inflection point.

We produce a second counterexample and show that there is even a continuously differentiable function with uncountably many inflection points by "lifting" the previous example to a continuously differentiable function. We put $C^* = \{n + y | n$ an integer, $y \in C\}$ and define

$$f(x) = \int_0^x d^2\,(t, C^*)\,dt.$$

f is differentiable and $f'(x) = d^2(x, C^*)$ where $f'(x$ is continuous, since $d(x, C*)$ is. The derivative is zero if and only if $x \in C^*$. The points of zero derivatives are uncountable, since C^* is uncountable, and every point of C^* is an inflection point.

Reference: [1] <http://en.wikipedia.org/wiki/Cantor set>

Solution 2 by proposer

We propose the counter example.

Let $f(x) = \displaystyle\int_0^x \rho(t, C)dt$ where $0 \le x \le 1$, C is a Cantor set (ternary for example) and let $\rho(t, C) = \inf_{t' \in C} |t - t'| = \min_{t' \in C} |t - t'|$ (the equality due to the closeness of $\overline{C} = C$).

We know that C is non-denumerable and nowhere dense. The nowhere density means that for any $t \in C$, $t \in C$, there exists an open interval $I = (a, b)$ such that $t < a < b < t'$ and $I \cap C = \emptyset$.

Now we observe that:

1) $f(x)$ is differentiable since $\rho(t, C)$ is continuous, and

2) $f'(x) = 0$ if $x \in C$ and $f'(x) > 0$ if $x \notin C$, (this is due to the closeness of C).

The non-denumerability of C implies the non-denumerability of set of points x where $f'(x) = 0$ and moreover they are inflection points because $f'(x) > 0$ if $x \notin C$.

The nowhere density of C together with $\rho(t, C) > 0$ imply that the ordinates of two different points are necessarily different so getting the non-denumerability of the ordinates of these inflection points.

Editor's comment: Several readers stated that at most there can be a denumerable number of inflection points. **Michael Fried of Kibbutz Revivim in Israel** was one them, but upon seeing Paolo's proof he wrote:

Yes, Paolo is right. The mistake in my objection was to assume implicitly that the inflection points corresponded to *distinct* maximum/minimum values of the derivative function. This would indeed imply that the distinct ordinates of the inflection points were as numerous as the those of the maximum/minimum, and, therefore, at most denumerable.

But think about what this function $\rho(t, C) = \inf\limits_{t' \in C} |t - t'|$ looks like.

The set of all points at which the function ρ has a minimum is precisely the Cantor set, as Paolo claimed, so that set is non-denumerable. All its minimum *values,* which occur at every point of the Cantor set, however, are all equal to zero. As for its maximum values, there is one for each step in the process producing the ternary Cantor set (i.e. one maximum value for each "removal of the middle third"), so that the ordinates of the maximum values of ρ are denumerable. There is no contradiction, then, of the fact that the maximum/minimum values of the functions can be at most denumerable.

Hence, we have the following situation:

1) ρ is the derivative function of $f(x) = \displaystyle\int_0^x \rho(t, C)dt$ where $0 \le x \le 1$. Therefore, f has a non-denumerable set of infection points.

2) Since f is defined as an integral, it is an increasing function. Therefore, the value of f at each inflection point is unique.

Very interesting!

Each of the other solvers came up with the opposite conclusion, namely that the number of inflection points must be at most denumerable. Their reasoning is reflected in Michael Brozinsky's argument. He stated: "If a function $f(x)$ has an inflection point at $x = x_0$ then there is an open interval $a_{x_0} < x < b_{x_0}$ containing x_0 such that the concavity on (a_{x_0}, x_0) and (x_0, b_{x_0}) is different and thus (a_{x_0}, b_{x_0}) cannot contain another inflection point of $f(x)$. Thus the inflection points of $f(x)$ are isolated points and hence at most denumerable. (We can, without loss of generality, take a_{x_0} and b_{x_0} to be rational since the rational numbers are dense and then associate to x_0 the midpoint of the aforementioned interval, i.e., the rational number $\dfrac{a_{x_0} + b_{x_0}}{2}$. Since the rationals are denumerable, the inflection points of $f(x)$ are at most denumerable."

David Stone and John Hawkins were in correspondence with me about this problem because I took issue with their solution, which was in the spirit of Michael Brozinsky's. I sent them Paolo's proof and Michael Fried's comment about it, and they responded as follows:

John and I looked at Paolo's counterexample and Michael's comment and now the reason for the confusion is clear. We're using the standard calculus notion – an inflection point is a place where the concavity changes. Moreover, concavity is defined over an interval, not at a point. You can see our meaning in the proof we sent you.... But Paolo and Michael seem to be using a different definition, more like "an inflection point is a place where the derivative achieves a max or min". Their work never mentions "concavity" – not in their mind at all. Wikipedia mostly agrees with this notion. (It's not an issue here, but what if the derivative didn't exist? What would be meant by "inflection point"?) I think we are all correct, subject to the differing definitions (and the problem statement proscribed no particular meaning of the term "inflection point").
– –

As editor of this column, I agree with them, that both solutions are correct, depending upon which definition of inflection point is used. But using the change in concavity definition of an inflection point makes this problem much less challenging than using the extremities of the first derivative definition. Here is what Albert (proof #1) wrote about his initial thoughts on the problem.

I have given problem 5161 a few thoughts. It is clear that the number of inflection points is countable if the function $f(x)$ is sufficiently smooth, let's say two times continuously differentiable. The inflection points are then the extrema of the function $f'(x)$, and the set of (local) extrema of a function is countable. So if we want to find a counterexample we should concentrate on more "exotic" functions. I have in mind to construct a counterexample that is based on the Cantor set. We start from the function $f(x) = x$ defined on the interval $[0, 1]$ and replace linear segments by cubics in the following sense: if $0 < a < b < 1$ then we replace the function $f(x) = x$ by $g(x) = a + 3\dfrac{(x-a)^2}{(b-a)} - 2\dfrac{(x-a)^3}{(b-a)^2}$. Then $g(a) = a, g(b) = b, g'(a) = g'(b) = 0$. In the first iteration we take $a = 1/3, b = 2/3$ and do the replacement for the third in the middle. We continue the Cantor construction and do similar replacements for the first and third third. Continuing this way we get a continuous function that is piecewise differentiable (multiple times). We now have to analyze in more detail what happens at the points of the Cantor set, and see whether all these points are inflection points. Kind regards - Albert

Also solved by: Michael Brozionsky, Central Islip, NY; Michael N. Fried, Kibbtuz Revivim, Israel, and David Stone and John Hawkins (jointly), Statesboro, GA,

- **5162:** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Barcelona, Spain*

Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}}+\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}}+\sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} \leq \sqrt{3}.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Using the Law of Cosines and the Arithmetic - Geometric Mean Inequality, we get

$$b^2+c^2-a^2 = 2bc\cos A$$

and

$$a^2+2bc = b^2+c^2-2bc\cos A+2bc \geq 4bc-2bc\cos A = 2bc\,(2-\cos A)\,.$$

Since $0 < A < \dfrac{\pi}{2}$, we have

$$\frac{b^2+c^2-a^2}{a^2+2bc} \leq \frac{2bc\cos A}{2bc\,(2-\cos A)} = \frac{\cos A}{2-\cos A} = \frac{1}{2\sec A-1}$$

and hence,

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}} \leq \frac{1}{\sqrt{2\sec A-1}}.$$

Further, equality is attained if and only if $b = c$.

Similar steps show that

$$\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}} \leq \frac{1}{\sqrt{2\sec B-1}} \quad \text{and} \quad \sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} \leq \frac{1}{\sqrt{2\sec C-1}},$$

with equality if and only if $a = b = c$.

Consider the function $f\,(x) = \dfrac{1}{\sqrt{2\sec x-1}}$ on $\left(0, \dfrac{\pi}{2}\right)$. Since

$$\begin{aligned} f''\,(x) &= \frac{-\sec x\,(\sec^3 x-2\sec^2 x+\sec x+1)}{(2\sec x-1)^{\frac{5}{2}}} \\ &= \frac{-\sec x\left[\sec x\,(\sec x-1)^2+1\right]}{(2\sec x-1)^{\frac{5}{2}}} \\ &< 0 \end{aligned}$$

on $\left(0, \frac{\pi}{2}\right)$, it follows that $f(x)$ is concave down on $\left(0, \frac{\pi}{2}\right)$. Then, by Jensen's Theorem and our comments above,

$$\begin{aligned}
\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}}+\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}}+\sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} &\leq f(A)+f(B)+f(C) \\
&\leq 3f\left(\frac{A+B+C}{3}\right) \\
&= 3f\left(\frac{\pi}{3}\right) \\
&= 3\cdot\frac{1}{\sqrt{3}} \\
&= \sqrt{3},
\end{aligned}$$

with equality if and only if $a=b=c$. That is, if and only if $\triangle ABC$ is equilateral.

Also solved by Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH, and the proposers.

- **5163:** *Proposed by Pedro H. O. Pantoja, Lisbon, Portugal*

  Prove that for all $n \in N$

$$\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2}-1\right)dx = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\frac{1}{k_1\cdots k_n\,(k_1+\cdots+k_n)}.$$

Solution 1 by G. C. Greubel, Newport News, VA

It can be seen that
$$\coth\frac{x}{2}-1=\frac{2}{e^x-1}.$$
With this the integral in question becomes

$$\begin{aligned}
I &= \int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2}-1\right)dx \\
&= \int_0^\infty \frac{x^n}{e^x-1}dx \\
I &= \Gamma(n+1)\zeta(n+1).
\end{aligned}$$

Now we have to show that the $n-$ sums are equal to the same value. This can be done by considering the integral
$$\int_0^\infty e^{-ax}dx=\frac{1}{a}.$$
Using this we then have

$$S = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\frac{1}{k_1\cdots k_n\,(k_1+\cdots+k_n)}$$

$$\begin{aligned}
&= \int_0^\infty \sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \frac{1}{k_1\cdots k_n} e^{-(k_1+\cdots+k_n)x} dx \\
&= \int_0^\infty \left(\sum_{k_1=1}^\infty \frac{e^{-k_1 x}}{k_1}\right)\cdots\left(\sum_{k_n=1}^\infty \frac{e^{-k_n x}}{k_n}\right) dx \\
&= \int_0^\infty \left(\sum_{k_1=1}^\infty \frac{e^{-kx}}{k_1}\right)^n dx \\
S &= \int_0^\infty \left(-\ln\left(1-e^{-x}\right)\right)^n dx.
\end{aligned}$$

By making the substitution $t = -\ln(1-e^{-x})$ we then have

$$S = \int_0^\infty \frac{t^n}{e^t - 1} dt = \Gamma(n+1)\zeta(n+1).$$

We have shown that $\displaystyle\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2} - 1\right) dx$ and $\displaystyle\sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \frac{1}{k_1\cdots k_n\,(k_1+\cdots+k_n)}$ is each equal to $\Gamma(n+1)\zeta(n+1)$, thus they are equal to each other.

Solution 2 by Paolo Perfetto, Department of Mathematics, "Tor Vergatta" University, Rome, Italy

Proof: We write

$$\frac{1}{k_1+\cdots+k_n} = \int_0^1 t^{k_1+\cdots+k_n-1} dt$$

and then

$$\begin{aligned}
\sum_{k_1,\ldots,k_n=1}^\infty \frac{1}{k_1k_2\cdots k_n}\int_0^1 t^{k_1+\cdots+k_n-1}dt &= \int_0^1 t^{-1} \sum_{k_1,\ldots,k_n=1}^\infty \frac{t^{k_1+\cdots+k_n}}{k_1k_2\cdots k_n} dt \\
&= \int_0^1 t^{-1}(-1)^n(\ln(1-t))^n dt \\
&= (-1)^n \int_0^1 (1-t)^{-1}(\ln t)^n dt.
\end{aligned}$$

Now we change variables letting $\ln t = -x$. Therefore,

$$(-1)^n \int_0^\infty \frac{(-x)^n}{1-e^{-x}} dx = \int_0^\infty \frac{x^n}{1-e^{-x}} dx.$$

The proof concludes by observing that

$$\coth\frac{x}{2} - 1 = \frac{e^{\frac{x}{2}} + e^{\frac{-x}{2}}}{e^{\frac{x}{2}} - e^{\frac{-x}{2}}} - 1 = \frac{2e^{\frac{-x}{2}}}{e^{\frac{x}{2}} - e^{\frac{-x}{2}}} = \frac{2}{1-e^{-x}}.$$

Comment by Paolo: Apart from $p = 0$ the series in the statement is the same as in problem #174 in the **Missouri Journal of Mathematical Sciences,** 22(1); downloadable at $<$ http : //www.math − cs.ucmo.edu/mjms/2010.1/Prob7.pdf $>$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We find that:

$$\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2}-1\right)dx = \int_0^\infty \frac{x^n}{2}\left(\frac{e^{\frac{x}{2}}+e^{\frac{-x}{2}}}{e^{\frac{x}{2}}-e^{\frac{-x}{2}}}-1\right)dx = \int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx. \tag{1}$$

We perform a change of variables: $y = 1 - e^{-x},\ dy = e^{-x}dx$. So

$$\begin{aligned}
\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx = \int_0^1 \frac{(-\log(1-y))^n}{y}dy \quad &= \quad \int_0^1 \frac{\left(\displaystyle\sum_{k=1}^\infty \frac{y^k}{k}\right)^n}{y}dy \\
&= \int_0^1 \sum_{k_1\ge 1, k_2\ge 1\ldots, k_n\ge 1} \frac{y^{k_1+k_2+\ldots+k_n-1}}{k_1\cdot k_2\cdots k_n}dy \\
&= \sum_{k_1\ge 1, k_2\ge 1\ldots, k_n\ge 1} \frac{y^{k_1+k_2+\ldots+k_n-1}}{k_1\cdot k_2\cdots k_n}\int_0^1 y^{k_1+k_2+\ldots+k_n-1}dy \\
&= \sum_{k_1\ge 1, k_2\ge 1\ldots, k_n\ge 1} \frac{1}{k_1\cdot k_2\cdots k_n\,(k_1+k_2+\ldots+k_n)}.
\end{aligned}$$

The interchange of summation and integration is allowed becasue of absolute convergence (all involved terms are positive).

It is noteworthy that the integral (1) can be explicitly evaluated in terms of the Riemann zeta function:

$$\int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}dx = \sum_{k=1}^\infty \int_0^\infty x^n e^{-kx}dx = \sum_{k=1}^\infty \frac{1}{k^{n+1}}\int_0^\infty x^n e^{-x}dx = n!\sum_{k=1}^\infty \frac{1}{k^{n+1}} = n!\zeta(n+1).$$

It is well known that $\zeta(n+1)$ is a rational multiple of π^{n+1}, if n is odd.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
February 15, 2012

- **5182:** *Proposed by Kenneth Korbin, New York, NY*

  Part I: An isosceles right triangle has perimeter P and its Morley triangle has perimeter x. Find these perimeters if $P = x + 1$.

  Part II: An isosceles right triangle has area K and its Morley triangle has area y. Find these areas if $K = y + 1$

- **5183:** *Proposed by Kenneth Korbin, New York, NY*

  A convex pentagon ABCDE, with integer length sides, is inscribed in a circle with diameter $\overline{AE}$.

  Find the minimum possible perimeter of this pentagon.

- **5184:** *Proposed by Neculai Stanciu, Buzău, Romania*

  If x, y and z are positive real numbers, then prove that

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} \geq \frac{8}{9}.$$

- **5185:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Calculate, without using a computer, the value of

$$\sin\left[\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{7}\right) + \arctan\left(\frac{1}{11}\right) + \arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{111}{121}\right)\right].$$

- **5186:** *Proposed by John Nord, Spokane, WA*

  Find k so that $\displaystyle\int_0^k \left(-\frac{b}{a}x + b\right)^n dx = \frac{1}{2}\int_0^a \left(-\frac{b}{a}x + b\right)^n dx.$

- **5187:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $f:[0,1]\to(0,\infty)$ be a continuous function. Find the value of

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f(\frac{1}{n})}+\sqrt[n]{f(\frac{2}{n})}+\cdots+\sqrt[n]{f(\frac{n}{n})}}{n}\right)^n.$$

Solutions

- **5164:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle has integer length sides (a,b,c) such that $a-b=b-c$. Find the dimensions of the triangle if the inradius $r=\sqrt{13}$.

  Solution 1 by Albert Stadler, Herrliberg, Switzerland

  If a,b and c are the side lengths of the triangle then the inradius r is given by the formula

  $r=\dfrac{1}{2}\sqrt{\dfrac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}}$. (see, e.g., http//mathworld.wolfram.com/Inradius.html).

  By assumption, $c=2b-a$. So

  $$\sqrt{13}=\frac{1}{2}\sqrt{\frac{(3b-2a)(2a-b)}{3}},\text{ or equivalently}$$

  $$(3b-2a)(2a-b)=156.$$

  Obviously b is even. (If b were odd, then both $3b-2a$ and $2a-b$ are odd, and therefore their product would be odd, which is not true.) So $b=2b'$ and this gives the equation

  $$(3b'-a)(a-b')=39.$$

  Note that $39=xy$ is the product of two integers. So,

  $$(x,y)\in\{(1,39),(3,13),(13,3),(39,1),(-1,-39),(-3,-13),(-13,-3),(-39,-1)\}.$$

  If $3b'-a=x$ and $a-b'=y$, then

  $$\begin{aligned} b' &= \frac{x+y}{2}, \text{ and}\\ a &= \frac{x+3y}{2}. \end{aligned}$$

  We find $(a,b,c)\in\{(59,40,21),(21,16,11),(11,16,21),(21,40,59)\}$, and we easily verify that each triplet satisfies the triangle inequality.

  Solution 2 by Arkady Alt, San Jose, CA

  Let F and s be the area and semiperimeter. Since $a+c=2b$ then $s=\dfrac{a+b+c}{2}=\dfrac{3b}{2}$, and using $F=\sqrt{s(s-a)(s-b)(s-c)}=sr$ we obtain

$$(s-a)(s-b)(s-c)=sr^2 \iff \left(\frac{3b}{2}-a\right)\left(\frac{3b}{2}-b\right)\left(\frac{3b}{2}-c\right)=13\cdot\frac{3b}{2}$$

$$\iff \quad \left(\frac{3b}{2}-a\right)\left(\frac{3b}{2}-c\right)=39$$

$$\iff \quad \left(\frac{9b^2}{4}-(a+c)\frac{3b}{2}+ac\right)=39 \iff \left(\frac{9b^2}{4}-2b\cdot\frac{3b}{2}+ac\right)=39$$

$$\iff \quad 4ac-3b^2=12\cdot 13.$$

Thus we have

$$\begin{cases} a+c=2b \\ 4ac-3b^2=156 \end{cases} \implies \begin{cases} 4a(2b-a)-3b^2=156 \\ c=2b-a \end{cases} \quad \text{if, and only if,}$$

$$\begin{cases} 4a(2b-a)-3b^2=156 \\ c=2b-a \end{cases} \iff \begin{cases} 8ab-a^2-3b^2=156 \\ c=2b-a. \end{cases}$$

Since $8ab-a^2-3b^2=(3b-2a)(2a-b)$ and

$$\begin{cases} a<s \\ b<s \\ c<s \end{cases} \iff \begin{cases} 2a<3b \\ c<s \end{cases} \iff \begin{cases} 2a<3b \\ 2(2b-a)<3b \end{cases} \iff b<2a<3b$$

then the problem is equivalent to the system

$$(1) \qquad \begin{cases} (3b-2a)(2a-b)=156 \\ b<2a<3b. \end{cases}$$

Since $3b-2a\equiv 2a-b \pmod 2$ and $156=2^2\cdot 3\cdot 13=2\cdot 78=6\cdot 26$ then (1) in positive integers is equivalent to

$$\begin{cases} 3b-2a=k \\ 2a-b=m \end{cases} \iff \begin{cases} 2b=k+m \\ 4a=k+3m \end{cases} \iff \begin{cases} a=\dfrac{k+3m}{4} \\ b=\dfrac{k+m}{2} \end{cases},$$

where $(k,m)\in\{(2,78),(78,2),(6,26),(26,6)\}$.

Noting that the inequality $b<2a<3b \iff \dfrac{k+m}{2}<\dfrac{k+3m}{2}<\dfrac{3(k+m)}{2}$ holds for any positive k,m we finally obtain

$$(a,b)\in\{(59,40),(21,40),(21,16),(11,16)\}.$$

Thus, $(a,b,c) \in \{(59,40,21), (21,40,59), (21,16,11), (11,16,21)\}$ are all solutions of the problem.

Comment by David Stone and John Hawkins, Statesboro, GA. In their featured solutions to SSM 5146 (May 2011 issue) both Kee-Wai Lau and Brian Beasley found all integral triangles with in-radius $\sqrt{13}$. Note that the condition $a-b=b-c$ is equivalent to $b=(a+c)/2$. That is, irrespective of how one might label or order the sides, the side b must be the "middle-length" side , the average of the other two sides.

Also solved by Brain D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo TX; Bruno Salgueiro Fanego, Viveiro, Spain; Tania Moreno García, University of Holguín (UHO), Holguín, Cuba jointly with José Pablo Suárez Rivero, University of Las Palmas de Gran Canaria (ULPGC), Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Sugie Lee, John Patton, and Matthew Fox (jointly; students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Jim Wilson, Athens, GA, and the proposer.

- **5165:** *Proposed by Thomas Moore, Bridgewater, MA*

"Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast."

Let $\sigma(n)$ denote the sum of all the different divisors of the positive integer n. Then n is perfect, deficient, or abundant according as $\sigma(n)=2n, \sigma(n)<2n$, or $\sigma(n)>2n$. For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $p_1=2,\ p_2=3,\ p_3=5,\ \ldots$ be the sequence of primes. We show that for any positive integer n, the integer $\prod\limits_{k=1}^{n+10} p_k$ is not the product of two deficient numbers.

Suppose , on the contrary, that $\prod\limits_{k=1}^{n+10} p_k = ab$, where both a and b are deficient numbers. Clearly a and b are relatively prime and so

$$4\left(\prod_{k=1}^{n+10} p_i\right) = 4ab > \sigma(a)\sigma(b) = \sigma(ab) = \sigma\left(\prod_{k=1}^{n+10} p_k\right) = \prod_{k=1}^{n+10}(1+p_k).$$

Hence,

$$4 > \prod_{k=1}^{n+10}\left(1+\frac{1}{p_k}\right) \geq \prod_{k=1}^{11}\left(1+\frac{1}{p_k}\right) = \frac{3822059520}{955049953} = 4.0019\ldots,$$

which is a contradiction. This completes the solution.

Solution 2 by Stephen Chou, Talbot Knighton, and Tom Peller (students at Taylor University), Upland, IN

All negative numbers have the same numerical divisors as their positive counterparts;

however, the negatives also include all the negative forms of those divisors. For instance, -6 has divisor of $1, 2, 3, 6, -1-2, -3, -6$. Therefore $\sigma(n) = 0$ because the divisors will all negate themselves. Knowing that $2n$ of any negative will result in a lower negative, we see that all the negatives are abundant. Since the negatives are all abundant numbers and the only way to have a negative product is to multiply a negative by a positive, then at most a negative number can have only one deficient factor. Therefore, there are infinitely many integers, namely the negatives, that are not the product of two deficient numbers.

Editor's comment: Once again the students have out smarted the professors; the intent of the problem was to find infinitely many *positive* integers that are not the product of two deficient numbers. But the problem wasn't explicitly stated that way, and so the students win; mea culpa.

Solution 3 by Pat Costello, Richmond, KY

Let $n = 2^k \cdot 3780 = 2^{k+2} \cdot 3^3 \cdot 5 \cdot 7 = 2^{k+2} \cdot 945$ for any non-negative integer k. We want to show that for any divisor d of n and pair $(d, n/d)$, one of the two values is either perfect or abundant. Since the σ function is multiplicative, we have

$$\begin{aligned}
\sigma(945) &= \sigma(3^3 \cdot 5 \cdot 7)\\
&= \sigma(3^3) \cdot \sigma(5) \cdot \sigma(7)\\
&= 40 \cdot 6 \cdot 8\\
&= 1920\\
&> 2 \cdot 945.
\end{aligned}$$

So 945 is abundant. Then in the pair $(945, n/945)$, the 945 is abundant.

By multiplicativity,

$$\begin{aligned}
\sigma\left(2^{k+2} \cdot 945\right) &= \sigma\left(2^{k+2}\right) \cdot \sigma(945)\\
&> \sigma\left(2^{k+2}\right) \cdot 2 \cdot 945, \text{by the above}\\
&> 2^{k+2} \cdot 2 \cdot 945, \text{since } \sigma(m) > m \text{ for } m > 1\\
&= 2\left(2^{k+2} \cdot 945\right).
\end{aligned}$$

This means all the n values are themselves abundant so in the pair $(1, n)$, the value n is the abundant value. This argument also shows that in the pair $(2^j, n/2^j)$, the second value is the abundant value.

In the following table, we list the divisors $d > 1$ of 945 and the values of the fractions $\sigma(d)/d$.

d	3	5	7	9	15	21	27	35	45	63	105	135	189	315	945
$\sigma(d)/d$	1.3	1.2	1.14	1.4	1.6	1.5	1.48	1.37	1.73	1.65	1.82	1.77	1.69	1.98	2.03

The key thing we want to see from the table is that the minimum value in the second row corresponds to $d = 7$.

Suppose that d is a divisor of $2^{k+2} \cdot 945$ that is of the form $2^j \cdot m$ where $j \geq 2$ and m is a divisor of 945 greater than 1. The fractions $\sigma(2^j)/2^j$ are easily seen to be strictly increasing with a limit of 2. Then

$$\begin{aligned}
\sigma\left(2^j\cdot m\right)/2^j\cdot m &= \sigma\left(2^j\right)\cdot\sigma(m)/2^j\cdot m\\
&= \sigma(2^j)/2^j\cdot\sigma(m)/m\\
&\geq \frac{7}{4}\cdot\frac{8}{7} \text{ from the table and that } j\geq 2\\
&= 2.
\end{aligned}$$

Hence the divisor $2^j\cdot m$ is perfect or abundant.

Suppose that d is a divisor of 945 and less than 945, say $d=945/m$ for an $m\geq 1$. Then the pair is $(945/m, 2^{k+2}\cdot m)$ and the second value is perfect or abundant.

All pairs $(d, n/d)$ have at least one value which is perfect or abundant. Since k is an arbitrary nonnegative integer, we have the desired infinite set.

Solution 4 by Brian D. Beasley, Clinton, SC

We make use of the following three facts:

(1) 945 is abundant (in fact, it is the smallest odd abundant number);

(2) any nontrivial multiple of a perfect number is abundant;

(3) any multiple of an abundant number is abundant.

Given any integer $k\geq 2$, we show that $n_k=2^k\cdot 945$ is not the product of two deficient numbers. For contradiction, if $n_k=2^k\cdot 3^3\cdot 5\cdot 7=xy$ for deficient numbers x and y, then the perfect number 6 divides neither x nor y, so without loss of generality, we assume that 2^k divides x and 3^3 divides y. Next, we consider cases:

(a) If 5 divides x, then x is abundant, since it is a multiple of the abundant number 20.

(b) If 7 divides x, then x is either perfect or abundant, since it is a multiple of the perfect number 28.

(c) If neither 5 nor 7 divides x, then $y=3^3\cdot 5\cdot 7=945$ is abundant.

Since each case leads to a contradiction, we are done. In fact, it follows that for $k\geq 3$, if $n_k=xy$, then at least one of x or y is abundant.

Addendum. Facts (1) and (2) above may be found in Burton's *Elementary Number Theory* (6th edition) on page 235, while fact (3) follows by applying an argument similar to that used to prove fact (2).

Solution 5 by proposer

A computer program shows that there are 55 such numbers below 10^5, the smallest being 3780. The canonical factorization of these numbers is revealing. One notices that the list includes all numbers of the from $3780p$ were $p\in\{11,13,17,19,23\}$. This suggest that $N=3780p$ is such a number, for all primes $p\geq 11$.

To prove this, lent $N=3780p=ab$ with $1\leq a\leq b\leq N$. Now $3780p=2^2\cdot 3^3\cdot 5\cdot 7\cdot p$ has 96 divisors, many of which are multiples of 12. But 12 is an abundant number and so is any multiple of 12. (More generally, any multiple of an abundant number is also abundant.) So we need only consider factorizations $N = ab$ where neither a nor b is a multiple of 12. We list all these factorizations in the tables below showing the companion factors a and b, along with their type (P: perfect; D: deficient; A: abundant).

a	*type*	b	*type*
1	D	$3780p$	A
2	D	$1890p$	A
4	D	$945p$	A
6	P	$630p$	A
10	D	$378p$	A
14	D	$270p$	A
18	A	$210p$	D
20	A	$189p$	D
27	D	$140p$	A
28	P	$135p$	D
30	A	$126p$	D
42	A	$90p$	A
54	A	$70p$	A

a	*type*	b	*type*
p	D	3780	A
$2p$	D	1890	A
$4p$	D	945	A
$6p$	A	630	A
$10p$	D	378	A
$14p$	D	270	A
$18p$	A	210	D
$20p$	A	189	D
$27p$	D	140	A
$28p$	A	135	D
$30p$	A	126	D
$42p$	A	90	A
$54p$	A	70	A

Also solved by David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and David Stone and John Hawkins (jointly), Statesboro, GA.

- **5166:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be lengths of the sides of a triangle ABC. Prove that

$$\left(3^{a+b}+\frac{c}{b}3^{-b}\right)\left(3^{b+c}+\frac{a}{c}3^{-c}\right)\left(3^{c+a}+\frac{b}{a}3^{-a}\right)\geq 8.$$

Solution by Boris Rays, Brooklyn, NY

By the Arithmetic-Geometric-Mean Inequality for each expression in the parentheses above we have:

$$\begin{aligned}
3^{a+b}+\frac{c}{b}3^{-b} &\geq 2\sqrt{3^{a+b}\cdot\frac{c}{b}3^{-b}}=2\sqrt{\frac{c}{b}3^{a}}\\
3^{b+c}+\frac{a}{c}3^{-c} &\geq 2\sqrt{3^{b+c}\cdot\frac{a}{c}3^{-c}}=2\sqrt{\frac{a}{c}3^{b}}\\
3^{c+a}+\frac{b}{a}3^{-a} &\geq 2\sqrt{3^{c+a}\cdot\frac{b}{a}3^{-a}}=2\sqrt{\frac{b}{a}3^{c}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(3^{a+b}+\frac{c}{b}3^{-b}\right)\left(3^{b+c}+\frac{a}{c}3^{-c}\right)\left(3^{c+a}+\frac{b}{a}3^{-a}\right) &\geq 8\sqrt{\frac{c}{b}\cdot\frac{a}{c}\cdot\frac{b}{a}\cdot 3^{a}\cdot 3^{b}\cdot 3^{c}}\\
&= 8\sqrt{3^{a+b+c}}
\end{aligned}$$

$$= \quad 8 \cdot 3^{\dfrac{a+b+c}{2}}.$$

The factor $3^{(a+b+c)/2}$ is an exponential expression with base 3 $(3 > 1)$ and exponent $(a+b+c)/2 > 0$. Hence, $3^{(a+b+c)/2} > 1$. Therefore,

$$\left(3^{a+b} + \frac{c}{b}3^{-b}\right)\left(3^{b+c} + \frac{a}{c}3^{-c}\right)\left(3^{c+a} + \frac{b}{a}3^{-a}\right) \geq 8.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego Viveiro Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo TX; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Albert Stadler, Herrliberg, Switzerland: Neculai Stanciu, Buzău, Romania; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5167:** *Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy*

Find the maximum of the real valued function

$$f(x,y) = x^4 - 2x^3 - 6x^2y^2 + 6xy^2 + y^4$$

defined on the set $D = \{(x,y) \in \Re^2 : \ x^2 + 3y^2 \leq 1\}$.

Solution 1 by Michael Brozinsky, Central Islip, NY

We note that the given constraint $x^2 + 3y^2 \leq 1$ implies that $-1 \leq x \leq 1$ and $y^2 \leq \dfrac{1}{3}$. Now, $f(-1,0) = 3$, and to show that 3 is the maximum it suffices to show that $f(x,y) \leq 3 \cdot (x^2 + 3y^2)$. That is

$$\begin{aligned} x^4 - 2x^3 - 6x^2y^2 + 6xy^2 + y^4 &\leq 3 \cdot (x^2 + 3y^2) \text{ or equivalently,} \\ x^2 \cdot \Big(x^2 - 2x - 3\Big) + y^2 \cdot \Big(y^2 + 6x\Big) &\leq y^2\Big(6x^2 + 9\Big) \text{ when } (x,y) \text{ is in } D. \qquad (1) \end{aligned}$$

Now $x^2 - 2x - 3 \leq 0$ if $-1 \leq x \leq 3$ and $y^2 + 6x \leq \dfrac{1}{3} + 6x \leq 6x^2 + 9$ for all x, (as the minimum of $6x^2 - 6x + 9$ is 7.5), and so (1) is obvious as $x^2\left(x^2 - 2x - 3\right) \leq 0$ and $y^2 \cdot (y^2 + 6x) \leq y^2 \cdot (x^2 + 9)$ when (x,y) is in D.

Solution 2 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

First we will look for extreme points inside the region, which is the set $\{(x,y) \in \Re^2 : x^2 + 3y^2 < 1\}$, and such points will be the critical points of the function $f(x,y)$. The partial derivatives of the function $f(x,y)$ will be

$$\begin{cases} f_x = 4x^3 - 6x^2 - 12xy^2 + 6y^2 \\ f_y = -12x^2y + 12xy + 4y^3 \end{cases}$$

Solving the system of the equations

$$\begin{cases} 4x^3 - 6x^2 - 12xy^2 + 6y^2 = 0 \\ -12x^2y + 12xy + 4y^3 = 0, \end{cases}$$

we have that the only critical point inside the region will be $(x, y) = (0, 0)$, which will be considered a point where the function might get the maximum value.

Now we will find the extremes on the contour of the region, which is $\{(x, y) \in \Re^2 : x^2 + 3y^2 = 1\}$. For any point on the contour we have $y^2 = \dfrac{1 - x^2}{3}$ and substituting this into the formula of $f(x, y)$ we obtain the function $g(x)$ such that

$$\begin{aligned} g(x) &= x^4 - 2x^3 - 6x^2\frac{1 - x^2}{3} + 6x\frac{1 - x^2}{3} + \left(\frac{1 - x^2}{3}\right)^2 \\ &= \frac{1}{9} + 2x - \frac{20x}{9} - 4x^3 + \frac{28x^4}{9} \end{aligned}$$

so, we have to find the extremes of the function $g(x)$ on the segment $[-1, 1]$.

If $x = \pm 1$ we have $f(-1, 0) = 3$ and $f(1, 0) = 1$, and so far, we shown that a local maximum point is when $(x, y) = (-1, 0)$. Now we much check to see if there is a maximum point inside the segment $[-1, 1]$. Taking the derivative of the function $g(x)$ we obtain

$$g'(x) = 2 - \frac{40x}{9} - 12x^2 + \frac{112x^3}{9}.$$

The equation $g'(x) = 0$ has no solution inside the segment $[-1, 1]$, which implies that there is no extreme point inside this segment. And so we may conclude that 3 is the absolute maximum of the real valued function $f(x, y)$ on the given domain and that it is achieved at the point $(x, y) = (-1, 0)$.

Solution 3 byÁngel Plaza, University of Las Palmas de Gran Canaria, Spain

Function $f(x, y)$ is harmonic. Then, by the maximum principle its maximum (and minimum) is attained at the boundary of compact subset D. Since the boundary of D is an ellipse, by using its parametrization the problem is reduced to a one variable optimization problem.

The parametric equations of the given ellipse are

$$x = \cos t; \; y = \frac{1}{\sqrt{3}}\sin t$$

and the problem yields to maximizing the function

$$g(t) = \cos^4 t - 2\cos^3 t - 2\cos^2 t \sin^2 t + 2\cos^2 t \sin^2 t + \frac{\sin^4 t}{9} = -2\cos^3 t + \cos^4 t + \frac{\sin^4 t}{9}.$$

Since $g'(t) = -\frac{2}{9}\cos t \sin t \left(-27\cos t + 18\cos^2 t - 2\sin^2 t\right)$ it is deduced that the maximum is attained at $t = \pi$ with the value $f(\pi) = 3$.

Also solved by Pat Costello, Richmond, KY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5168:** *Proposed by G. C. Greubel, Newport News, VA*

  Find the value of a_n in the series

$$\frac{7t+2t^2}{1-36t+4t^2} = a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n} + \cdots.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

By direct division, $4t^2-36t+1|\overline{2t^2+7t}$ we see that $a_0 = \dfrac{1}{2}, a_1 = \dfrac{25}{4}$. Moreover, the characteristic equation of the denominator is $1-36r+4r^2=0$, whose roots are $r_1 = \dfrac{9-4\sqrt{5}}{2}$, $r_2 = \dfrac{9+4\sqrt{5}}{2}$, so $a_n = Ar_1^n + Br_2^n$ for some real numbers A and B . Taking $n=0$, we obtain

$$A+B = A\cdot 1 + B\cdot 1 = Ar_1^0 + Br_2^0 = a_0 = \frac{1}{2},$$

and taking $n=1$ we obtain

$$A\frac{9+4\sqrt{5}}{2} + B\frac{9-4\sqrt{5}}{2} = Ar_1^1 + Br_2^1 = a_1 = \frac{25}{4}.$$

So, by solving the system of equations $\begin{cases} A+B = \dfrac{1}{2} \\ 18(A+B)+8\sqrt{5}(A-B) = 25 \end{cases}$ we obtain

$$A = \frac{5-4\sqrt{5}}{20},\ B = \frac{5+4\sqrt{5}}{20}.$$

Hence,

$$a_n = Ar_1^n + Br_2^n = \frac{5-4\sqrt{5}}{20}\left(\frac{9-4\sqrt{5}}{2}\right)^n + \frac{5+4\sqrt{5}}{20}\left(\frac{9+4\sqrt{5}}{2}\right)^n.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that

$$\frac{7t+2t^2}{1-36t+4t^2} = \frac{5+4\sqrt{5}}{20}\left(\frac{1}{1-\dfrac{9+4\sqrt{5}}{2t}}\right) + \frac{5-4\sqrt{5}}{20}\left(\frac{1}{1-\dfrac{9-4\sqrt{5}}{2t}}\right).$$

For $t > \dfrac{9+4\sqrt{5}}{2}$, we have $\dfrac{1}{1-\dfrac{9+4\sqrt{5}}{2}} = \displaystyle\sum_{n=0}^{\infty}\left(\frac{9+4\sqrt{5}}{2t}\right)^n$ and $\dfrac{1}{1-\dfrac{9-4\sqrt{5}}{2t}} = \displaystyle\sum_{n=0}^{\infty}\left(\frac{9-4\sqrt{5}}{2t}\right)^n$. Hence for positive integer n

$$a_n = \frac{5+4\sqrt{5}}{20}\left(\frac{9+4\sqrt{5}}{2}\right)^n + \frac{5-4\sqrt{5}}{20}\left(\frac{9-4\sqrt{5}}{2}\right)^n.$$

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Let $\{f_n\}$ be the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. Also, let $\phi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \dfrac{1-\sqrt{5}}{2}$. Then, we will use Binet's Formula

$$f_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$$

for $n \geq 0$ and the known results

$$\phi^n = f_n\phi + f_{n-1}, \overline{\phi}^n = f_n\overline{\phi} + f_{n-1}, \text{and } \phi\overline{\phi} = -1 \tag{1}$$

for $n \geq 1$.

To begin, make the change of variable $s = \dfrac{1}{t}$ and simplify to get

$$\frac{7t + 2t^2}{1 - 36t + 4t^2} = \frac{7s+2}{s^2 - 36s + 4}.$$

Note that (1) implies that $\phi^6 = f_6\phi + f_5 = 8\phi + 5 = 9 + 4\sqrt{5}$ and similarly, $\overline{\phi}^6 = 9 - 4\sqrt{5}$. Then, the roots of $s^2 - 36s + 4$ are $s = 18 \pm 8\sqrt{5} = 2\phi^6, 2\overline{\phi}^6$ and we have

$$\frac{7s+2}{s^2 - 36s + 4} = \frac{7s+2}{(s - 2\phi^6)\left(s - 2\overline{\phi}^6\right)}.$$

If we perform a partial fraction expansion and use Binet's Formula, (1), and the formula for a geometric series, we obtain

$$\begin{aligned}
\frac{7s+2}{s^2 - 36s + 4} &= \frac{7\phi^6 + 1}{8\sqrt{5}} \frac{1}{s - 2\phi^6} - \frac{7\overline{\phi}^6 + 1}{8\sqrt{5}} \frac{1}{s - 2\overline{\phi}^6} \\
&= \frac{1}{8\sqrt{5}} \left[-\frac{7\phi^6 + 1}{2\phi^6} \frac{1}{1 - \left(\frac{s}{2\phi^6}\right)} + \frac{7\overline{\phi}^6 + 1}{2\overline{\phi}^6} \frac{1}{1 - \left(\frac{s}{2\overline{\phi}^6}\right)} \right] \\
&= \frac{1}{16\sqrt{5}} \left[\left(7 + \phi^6\right) \sum_{n=0}^{\infty} \frac{1}{\left(2\overline{\phi}^6\right)^n} s^n - \left(7 + \overline{\phi}^6\right) \sum_{n=0}^{\infty} \frac{1}{(2\phi^6)^n} s^n \right] \\
&= \frac{1}{16\sqrt{5}} \sum_{n=0}^{\infty} \frac{1}{2^n} \left[\frac{7 + \phi^6}{\overline{\phi}^{6n}} - \frac{7 + \overline{\phi}^6}{\phi^{6n}} \right] s^n \\
&= \frac{1}{16\sqrt{5}} \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{(7 + \phi^6)\,\phi^{6n} - \left(7 + \overline{\phi}^6\right)\overline{\phi}^{6n}}{(-1)^{6n}} s^n \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{n+4}} \left[7\left(\frac{\phi^{6n} - \overline{\phi}^{6n}}{\sqrt{5}} \right) + \left(\frac{\phi^{6n+6} - \overline{\phi}^{6n+6}}{\sqrt{5}} \right) \right] s^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{7f_{6n} + f_{6n+6}}{2^{n+4}} s^n$$

$$= \sum_{n=0}^{\infty} \frac{2f_{6n+1} + 3f_{6n}}{2^{n+2}} s^n$$

$$= \sum_{n=0}^{\infty} \frac{2f_{6n+1} + 3f_{6n}}{2^{n+2}} \frac{1}{t^n}.$$

Also, since $\left|\overline{\phi}\right| < \phi$, the series converges when

$$|s| < \min\left\{2\left|\overline{\phi}\right|^6, 2\phi^6\right\} = 2\left|\overline{\phi}\right|^6,$$

i.e., when

$$|t| > \frac{1}{2\left|\overline{\phi}\right|^6} = \frac{\phi^6}{2}.$$

Therefore,

$$a_n = \frac{2f_{6n+1} + 3f_{6n}}{2^{n+2}}$$

for $n \geq 0$.

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; David E. Manes, Oneonta, NY; Ángel Plaza (University of Las Palmas de Gran Canaria), Spain; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA; Boris Rays, Brooklyn, NY, and the proposer.

- **5169:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \geq 1$ be an integer and let i be such that $1 \leq i \leq n$. Calculate:

$$\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n.$$

Solutions 1 and 2 by Albert Stadler, Herrliberg, Switzerland

1) Let $I_i = \displaystyle\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n$. Then by symmetry,

$I_1 = I_2 = \cdots I_n$. So,

$$I_1 + I_2 + \cdots + I_n = \int_0^1 \cdots \int_0^1 \frac{x_1 + x_2 + \cdots x_n}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n = 1,$$

and $I_i = \dfrac{1}{n}$ for $1 \leq i \leq n$.

2) Another albeit less elegant proof runs as follows:

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1+x_2+\cdots+x_n} dx_1 \cdots dx_n &= \int_0^\infty \int_0^1 \cdots \int_0^1 x_i e^{-t(x_1+x_2+\cdots x_n)} dx_1 \cdots dx_n dt \\
&= \int_0^\infty \frac{\left(1-e^{-t}\right)^{n-1}\left(1-(1+t)e^{-t}\right)}{t^{n+1}} dt \\
&= -\frac{1}{n} \int_0^\infty \frac{d}{dt} \frac{(1-e^{-t})^n}{t^n} dt \\
&= \frac{1}{n} \lim_{t \to 0} \frac{(1-e^{-t})^n}{t^n} = \frac{1}{n}.
\end{aligned}$$

The above is so because:

$$\int_0^1 e^{-tx_j} dx_j = \frac{1-e^{-t}}{t}, \quad \int_0^1 x_i e^{-tx_i} dx_i = \frac{1-(1+t)e^{-t}}{t^2},$$

$$\frac{d}{dt} \frac{(1-e^{-t})^n}{t^n} = -\frac{n(1-e^{-t})^n}{t^{n+1}} + \frac{n(1-e^{-t})^{n-1} e^{-t}}{t^n} = -n \frac{(1-e^{-t})^{n-1}(1-(1+t)e^{-t})}{t^{n+1}}.$$

Also solved by Michael N. Fried, Kibbutz Revivim, Israel; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2012

- **5188:** *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with coordinates $A(-5,0), B(0,12)$ and $C(9,0)$. The triangle has an interior point P such that $\angle APB = \angle APC = 120^\circ$. Find the coordinates of point P.

- **5189:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with integer length sides and with $\angle A = 60^\circ$ and with $(a,b,c) = 1$. Find the lengths of b and c if

$$\begin{aligned} i)\ a &= 13, \text{ and if} \\ ii)\ a &= 13^2 = 169, \text{ and if} \\ iii)\ a &= 13^4 = 28561. \end{aligned}$$

- **5190:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Prove: If x, y and z are positive integers such that $\dfrac{x(y+1)}{x-1} \in \mathrm{N}$, $\dfrac{y(z+1)}{y-1} \in \mathrm{N}$, and $\dfrac{z(x+1)}{z-1} \in \mathrm{N}$, then $xyz \le 693$.

- **5191:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \le 1.$$

- **5192:** *Proposed by G. C. Greubel, Newport News, VA*

  Let $[n] = [n]_q = \dfrac{1-q^n}{1-q}$ be a q number and $\ln_q(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{[n]}$ be a q-logarithm. Evaluate the following series:

$$i) \qquad \sum_{k=0}^{\infty} \frac{q^{mk}}{[mk+1][mk+m+1]} \text{ and}$$

$$ii) \quad \sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]}.$$

- **5193:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let f be a function which has a power series expansion at 0 with radius of convergence R.

  a) Prove that $\displaystyle\sum_{n=1}^{\infty} n f^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} t f'(t)dt, \quad |x| < R.$

  b) Let α be a non-zero real number. Calculate $\displaystyle\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right).$

Solutions

- **5170:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $DEFG$ has coordinates $D(-6,-3)$ and $E(2,12)$. The midpoints of the diagonals are on line l.

  Find the area of the quadrilateral if line l intersects line FG at point $P\left(\dfrac{672}{33}, \dfrac{-9}{11}\right)$.

  Solution 1 by Kee-Wai Lau, Hong Kong, China

  We show that the area of the quadrilateral is 378

  Let H and I be respectively the midpoints of the diagonals DF and EG. Let

$$F = (p, q) \quad \text{and} \quad G = (r, s) \text{ so that}$$

$$H = \left(\frac{p-6}{2}, \frac{q-3}{2}\right) \quad \text{and} \quad I = \left(\frac{r+2}{2}, \frac{s+12}{2}\right).$$

  Using the facts that the points H, I, and P lie on l and that P lies on FG, we obtain respectively the relations

$$(150 + 11s)p + (426 - 11r)\,q = 7590 - 15r + 514s \qquad (1)$$

$$(9 + 11s)\,p + (224 - 11r)\,q = 9r + 224s. \qquad (2)$$

  By the standard formula, we find the area of the quadrilateral to be

$$\frac{(12 - s)p + (r - 2)q + 3r - 6s + 66}{2},$$

  which can be written as

$$\frac{\Big((150 + 11s)p + (426 - 11r)q\Big) - 2\Big((9 + 11s)p + (224 - 11r)q\Big) + 33r - 66s + 726}{22}.$$

By (1) and (2), the last expression equals

$$\frac{(7590-15r+514s)-2(9r+224s)+33r-66s+726}{22}=378,$$

and this completes the solution.

Solution 2 by the proposer

Area of Quadrilateral DEFG

$$\begin{aligned}
&= 2\Big[\text{Area } \triangle\text{DEP}\Big]\\
&= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\\
&= \begin{vmatrix} 2 & 12 & 1 \\ -6 & -3 & 1 \\ \dfrac{224}{11} & -\dfrac{9}{11} & 1 \end{vmatrix} = 378.
\end{aligned}$$

Reference, problem number 5033.

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** showed in their solution that there are infinitely many quadrilaterals satisfying the given conditions of the problem, and that each has an area of 378. Their solution started off by showing that the simplest configuration occurs when the quadrilateral is a parallelogram so that the diagonals coincide. They then exhibited all such parallelograms and showed that each one has the stated area. Their solution of nine pages is too lengthy to reproduce here, but if you would like to see it, please contact me and I will send their solution to you in pdf format.

- **5171:** *Proposed by Kenneth Korbin, New York, NY*

A triangle has integer length sides $x, x+y$, and $x+2y$.

Part I: Find x and y if the inradius $r = 2011$.

Part II: Find x and y if $r=\sqrt{2011}$.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

For convenience, let $a=x$, $b=x+y$, $c=x+2y$ be the sides of the triangle. Then, since a, b, c are positive integers, it follows that x is a positive integer and y is an integer (which is not necessarily positive). The semiperimeter s is given by

$$s=\frac{a+b+c}{2}=\frac{3}{2}(x+y)$$

and we have

$$\begin{aligned} s-a &= s-x = \frac{x+3y}{2} \\ s-b &= s-(x+y) = \frac{x+y}{2} \\ s-c &= s-(x+2y) = \frac{x-y}{2}. \end{aligned} \tag{1}$$

If K is the area of the triangle, then

$$sr = K = \sqrt{s(s-a)(s-b)(s-c)}$$

which reduces to

$$\frac{3}{2}(x+y)r^2 = sr^2 = (s-a)(s-b)(s-c) = \frac{x+3y}{2}\frac{x+y}{2}\frac{x-y}{2},$$

i.e.,

$$(x+3y)(x-y) = 12r^2.$$

Note also that (1) implies that $x+3y$ and $x-y$ are positive integers since x and y are integers and $s-a$ and $s-c$ are positive. Further, if

$$\begin{aligned} x+3y &= k_1 \\ x-y &= k_2 \end{aligned}$$

for positive integers k_1 and k_2 such that $k_1k_2 = 12r^2$, then at least one of k_1, k_2 is even. Finally, since $4y = k_1 - k_2$, it follows that k_1 and k_2 must both be even.

Part I: If $r = 2011$, then $12r^2 = 12(2011)^2$ and the possibilities for k_1 and k_2 are

$$\begin{aligned} (k_1, k_2) \in \{ &(2, 6\cdot 2011^2), (6\cdot 2011^2, 2), (4022, 12066), (12066, 4022), \\ &(6, 2\cdot 2011^2), (2\cdot 2011^2, 6)\}. \end{aligned}$$

If

$$\begin{aligned} x-y &= 2 \\ x+3y &= 6\cdot 2011^2 \end{aligned}$$

then $x = 6,066,183$, $y = 6,066,181$, while if

$$\begin{aligned} x-y &= 6\cdot 2011^2 \\ x+3y &= 2 \end{aligned}$$

then $x = 18,198,545$, $y = -6,066,181$. The steps in the remaining cases are similar and the results are summarized in the following table:

x	y	a	b	c
6,066,183	6,066,181	6,066,183	12,132,364	18,198,545
18,198,545	−6,066,181	18,198,545	12,132,364	6,066,183
2,022,065	2,022,059	2,022,065	4,044,124	6,066,183
6,066,183	−2,022,059	6,066,183	4,044,124	2,022,065
6,033	2,011	6,033	8,044	10,055
10,055	−2,011	10,055	8,044	6,033

.

Part II: If $r = \sqrt{2011}$, then $12r^2 = 12 \cdot 2011$ and the possibilities for k_1, k_2 are

$$(k_1, k_2) \in \{(2, 12066), (12066, 2), (6, 4022), (4022, 6)\}.$$

If we solve the system

$$\begin{aligned} x - y &= k_1 \\ x + 3y &= k_2 \end{aligned}$$

for each of these possibilities, the results are:

x	y	a	b	c
$3,018$	$3,016$	$3,018$	$6,034$	$9,050$
$9,050$	$-3,016$	$9,050$	$6,034$	$3,018$
$1,010$	$1,004$	$1,010$	$2,014$	$3,018$
$3,018$	$-1,004$	$3,018$	$2,014$	$1,010$

Remark: In each situation where the assignments for k_1 and k_2 were reversed, we obtained different values for x and y but the triangle was essentially the same (with the values of a and c reversed).

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** solved the more general problem for a triangle having its sides in the arithmetic progression of $x, x+y$, and $x+2y$ by finding x and y if the inradius $r = p^{m/2}$ where p is an odd prime and $m \geq 1$. For $p \geq 5$ they showed that there are $m+1$ solutions and they described them. For $p = 3$ they showed that there are $\lfloor \frac{m+2}{2} \rfloor$ and also described them.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5172:** *Proposed by Neculai Stanciu, Buzău, Romania*

If a, b and c are positive real numbers, then prove that,

$$\frac{a(b-c)}{c(a+b)} + \frac{b(c-a)}{a(b+c)} + \frac{c(a-b)}{b(c+a)} \geq 0.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We have

$$\frac{a(b-c)}{c(a+b)} + \frac{b(c-a)}{a(b+c)} + \frac{c(a-b)}{b(c+a)} = \frac{a^3b^3 + b^3c^3 + c^3a^3 - a^3b^2c - b^3c^2a - c^3a^2b}{abc(a+b)(b+c)(c+a)}. \qquad (1)$$

By the weighted AM-GM inequality,

$$\frac{2}{3}a^3b^3 + \frac{1}{3}c^3a^3 \quad \geq \quad a^3b^2c,$$

$$\begin{aligned}\frac{2}{3}b^3c^3+\frac{1}{3}a^3b^3 &\geq b^3c^2a,\\ \frac{2}{3}c^3a^3+\frac{1}{3}b^3c^3 &\geq c^3a^2b.\end{aligned}$$

If we these inequalities up we that the numerator of (1) is nonnegative, and the problem statement follows.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Since

$$\frac{a(b-c)}{c(a+b)}+\frac{b(c-a)}{a(b+c)}+\frac{c(a-b)}{b(c+a)}$$

$$= \frac{a^3b^3+b^3c^3+c^3a^3-a^3b^3c-b^3c^2a-c^3a^2b}{abc(a+b)(b+c)(c+a)}$$

$$= \frac{a^3(b-c)^2(2b+c)+b^3(c-a)^2(2c+a)+c^3(a-b)^2(2a+b)}{3abc(a+b)(b+c)(c+a)},$$

the inequality of the problem follows.

Also solved by Arkady Alt, San Jose, CA; Michael Brozinsky, Central Islip, NY; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy, and the proposer.

- **5173:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

Find all triples x, y, z of non-negative real numbers that satisfy the system of equations,

$$\begin{cases} x^2(2x^2+x+2)=xy(3x+3y-z)\\ y^2(2y^2+y+2)=yz(3y+3z-x)\\ z^2(2z^2+z+2)=xz(3z+3x-y)\end{cases}$$

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Assume for the moment that $x\neq 0, y\neq 0, z\neq 0$.
Without loss of generality, we may assume that $x \geq y$. Looking at equations (1) and (2) in the statement of the problem and using the fact that x, y, z are non-negative real numbers, we observe

$$\begin{aligned} x^2(2x^2+x+2)\geq y^2(2y^2+y+2) &\Rightarrow xy(3x+3y-z)\geq yz(3y+3z-x)\\ &\Rightarrow x(3x+3y-z)\geq z(3y+3z-x)\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad 3x^2+3xy-xz \geq 3yz+3z^2-xz\\
&\Rightarrow \quad 3(x-z)(x+y+z) \geq 0\\
&\Rightarrow \quad x \geq z
\end{aligned}$$

Looking at equations (1) and (3) and using the fact that x, y, z are non-negative real numbers, we observe

$$\begin{aligned}
x^2(2x^2+x+2) \geq z^2(2z^2+z+2) \quad &\Rightarrow \quad xy(3x+3y-z) \geq zx(3z+3x-y)\\
&\Rightarrow \quad y(3x+3y-z) \geq z(3z+3x-y)\\
&\Rightarrow \quad 3xy+3y^2-yz \geq 3z^2+3xz-yz\\
&\Rightarrow \quad 3(y-z)(x+y+z) \geq 0\\
&\Rightarrow \quad y \geq z
\end{aligned}$$

Similarly, focusing on equations (2) and (3) and using the fact that x, y, z are non-negative real numbers, we observe

$$\begin{aligned}
y^2(2y^2+y+2) \geq z^2(2z^2+z+2) \quad &\Rightarrow \quad yz(3y+3z-x) \geq zx(3z+3x-y)\\
&\Rightarrow \quad y(3y+3z-x) \geq x(3z+3x-y)\\
&\Rightarrow \quad 3y^2+3yz-xy \geq 3xz+3x^2-xy\\
&\Rightarrow \quad 3(y-x)(x+y+z) \geq 0\\
&\Rightarrow \quad y \geq x.
\end{aligned}$$

This implies that $x = y$. In a similar manner we can prove that $y = z$ and substituting this into equation (1) we obtain

$$x^2(2x^2+x+2) = x^2(3x+3x-x) = 0 \Rightarrow 2(x-1)^2 = 0 \Rightarrow x = 1.$$

So a solution will be $(x, y, z) = (1, 1, 1)$.

Substituting $x = 0$ into equation (3) implies that $z^2(2z^2+z+2) = 0$, so either $z = 0$ or $2z^2+z+2 = 0$. It is easy to see that $2z^2+z+2 = 0$ does not have real roots, so we are left with the option that $z = 0$. Similarly, substituting $z = 0$ into equation (2) gives $y = 0$.

Therefore the set of real valued solutions for the given system is $(x, y, z) = \{(0,0,0), (1,1,1)\}$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu with Titu Zvonaru (jointly), from Buzău and Comănesti, Romania respectively, and the proposer.

- **5174:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a positive integer. Compute:

$$\lim_{n\to\infty} \frac{n^2}{2^n} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, and by integration $\dfrac{(1+x)^{n+1}-1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \dfrac{x^{k+1}}{k+1}$.

Iterating the same technique, it is obtained:

$$\frac{(1+x)^{n+2}-(n+2)x-1}{(n+1)(n+2)} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+2}}{(k+1)(k+2)}.$$

$$\frac{(1+x)^{n+3}-(n+2)(n+3)x^2/2-(n+3)x-1}{(n+1)(n+2)(n+3)} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+3}}{(k+1)(k+2)(k+3)}.$$

Now, multiplying each term of the preceding equation by x, differentiating with respect to x and letting $x=1$, we obtain

$$\frac{(n+5)2^{n+2}-3(n+2)(n+3)/2-2(n+3)-1}{(n+1)(n+2)(n+3)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(k+4)}{(k+1)(k+2)(k+3)}.$$

And therefore, the proposed limit becomes

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{n^2}{2^n} \frac{(n+5)2^{n+2}-3(n+2)(n+3)/2-2(n+3)-1}{(n+1)(n+2)(n+3)} \\ &= \lim_{n\to\infty} \frac{n^2}{2^n} \frac{(n+5)2^{n+2}}{n^3} = 4. \end{aligned}$$

Solution 2 by Anastasios Kotronis, Athens, Greece

For $n \in N$ and and $x \in \Re$ we have $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ so $x^4(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{k+4}$.

Now differentiate to obtain

$$\begin{aligned} 4x^3(1+x)^n + nx^4(1+x)^{n-1} &= \sum_{k=0}^{n} (k+4) \binom{n}{k} x^{k+3}, \quad \text{so} \\ 4(1+x)^n + nx(1+x)^{n-1} &= \sum_{k=0}^{n} (k+4) \binom{n}{k} x^k. \end{aligned}$$

Now integrate on $[0,x]$ to obtain

$$\frac{3(1+x)^{n+1}}{n+1} + x(1+x)^n - \frac{3}{n+1} = \sum_{k=0}^{n} \frac{(k+4)}{k+1} \binom{n}{k} x^{k+1}.$$

Integrating once again gives us

$$\frac{2(1+x)^{n+2}}{(n+1)(n+2)} + \frac{x(1+x)^{n+1}}{n+1} - \frac{3x}{n+1} - \frac{2}{(n+1)(n+2)} = \sum_{k=0}^{n} \frac{(k+4)}{(k+1)(k+2)} \binom{n}{k} x^{k+2}.$$

And by integrating still again gives us

$$\sum_{k=0}^{n} \frac{(k+4)}{(k+1)(k+2)(k+3)} \binom{n}{k} x^{k+3} =$$

$$\frac{(1+x)^{n+3}}{(n+1)(n+2)(n+3)}+\frac{x(1+x)^{n+2}}{(n+1)(n+2)}-\frac{3x^2}{2(n+1)}-\frac{2x}{(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}.$$

Setting $x=1$ above, we easily see that

$$\frac{n^2}{2^n}\sum_{k=0}^{n}\frac{(k+4)}{(k+1)(k+2)(k+3)}\binom{n}{k}\overset{n\to+\infty}{\longrightarrow} 4.$$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5175:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the value of,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{i^2+j^2}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We first note by symmetry that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}=2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}-\sum_{i=1}^{n}\frac{1}{i}. \qquad (1)$$

It is well known that for a sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty}a_n=l$ then $\lim\limits_{n\to\infty}\dfrac{\sum_{i=1}^{n}a_i}{n}=l$ as well. Hence, it follows from (1) that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}\\
&= 2\lim_{n\to\infty}\sum_{j=1}^{n}\frac{n+j}{n^2+j^2}-\lim_{n\to\infty}\frac{1}{n}\\
&= 2\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1+\frac{j}{n}}{n\left(1+\left(\frac{j}{n}\right)^2\right)}\\
&= 2\int_0^1\frac{1+x}{1+x^2}dx
\end{aligned}$$

$$= \left[2\arctan(x) + \ln(1+x^2)\right]\Big|_0^1$$

$$= \frac{\pi}{2} + \ln 2.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy

Answer: $\dfrac{\pi}{2} + \ln 2$

Proof: The limit is actually

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{\dfrac{i}{n} + \dfrac{j}{n}}{\dfrac{i^2}{n^2} + \dfrac{j^2}{n^2}}$$

which is the Riemann–sum of

$$\int\int_{[0,1]^2} \frac{x+y}{x^2+y^2} dxdy = 2\int\int_{[0,1]^2} \frac{x}{x^2+y^2} dxdy \ = \ I$$

$$I = \int_0^1 \left[\left(\ln(x^2+y^2)\right)\Big|_0^1\right] dy = \int_0^1 \left(\ln(1+y^2) - 2\ln y\right) dy.$$

Integrating by parts,

$$\int_0^1 \ln(1+y^2)dy = y\ln(1+y^2)\Big|_0^1 - 2\int_0^1 \frac{y^2}{1+y^2} dy$$

$$= \ \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+y^2}\right) dy$$

$$= \ \ln 2 - 2 + 2\arctan y\Big|_0^1 = \ln 2 - 2 + 2\left(\frac{\pi}{4}\right).$$

Moreover,

$$-2\int_0^1 \ln y dy = -2(y\ln y - y)\Big|_0^1 = 2$$

from which the result follows by summing the two integrals.

Comment by editor: Many of the solvers approached the problem in a similar manner as Paolo, by showing that

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \frac{\dfrac{i}{n} + \dfrac{j}{n}}{\left(\dfrac{i}{n}\right)^2 + \left(\dfrac{j}{n}\right)^2} \Longrightarrow \int_0^1\int_0^1 \frac{x+y}{x^2+y^2} dxdy \text{ as } n \to \infty,$$

but they raised the caveat that we must be careful in applying the limit because the function $\phi(x,y) = \dfrac{x+y}{x^2+y^2}$ is not continuous at $(x,y) = (0,0)$. They then showed that in this case, the limit does indeed hold.

Also solved by Arkady Alt, San Jose, CA; Anastasios Kotronis, Athens, Greece; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, German; David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2012

- **5194:** *Proposed by Kenneth Korbin, New York, NY*

  Find two pairs of positive integers (a, b) such that,

  $$\frac{14}{a} + \frac{a}{b} + \frac{b}{14} = 41.$$

- **5195:** *Proposed by Kenneth Korbin, New York, NY*

  If N is a prime number or a power of primes congruent to 1 (mod 6), then there are positive integers a and b such that $3a^2 + 3ab + b^2 = N$ with $(a, b) = 1$.

  Find a and b if $N = 2011$, and if $N = 2011^2$, and if $N = 2011^3$.

- **5196:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Determine the last six digits of the product $(2010)\left(5^{2014}\right)$.

- **5197:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

  Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 4$. Prove that,

  $$\frac{1}{6 - x^2} + \frac{1}{6 - y^2} + \frac{1}{6 - z^2} \leq \frac{1}{xyz}.$$

- **5198:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let m, n be positive integers. Calculate,

  $$\sum_{k=1}^{2n} \prod_{i=0}^{m} \left(\lfloor k + \frac{1}{2} \rfloor + a + i \right)^{-1},$$

  where a is a nonnegative number and $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

- **5199:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k > 0$ and $n \geq 0$ be real numbers. Calculate,

$$\int_0^1 x^n \ln\left(\sqrt{1+x^k} - \sqrt{1-x^k}\right) dx.$$

Solutions

- **5176:** *Proposed by Kenneth Korbin, New York, NY*

Solve:

$$\begin{cases} x^2 + xy + y^2 = 3^2 \\ y^2 + yz + z^2 = 4^2 \\ z^2 + xz + x^2 = 5^2. \end{cases}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let

$$\begin{cases} A = x^2 + xy + y^2 - 9 \\ B = y^2 + yz + z^2 - 16 \\ C = z^2 + xz + x^2 - 25 \end{cases}$$

By assumption $A = B = C = 0$. So, $0 = A + B - C = xy + yz - xz + 2y^2$ or equivalently $z(x-y) = y(x+2y)$. Obviously $x \neq y$, since if $x = y$ then $0 = B = x^2 + xz + z^2 - 16$ and $0 = C = z^2 + xz + x^2 - 25$ which is a contradiction. So,

$$z = \frac{y\,(x+2y)}{x-y}. \qquad (1)$$

We insert this value of z into the equation $B = 0$ and obtain

$$\begin{aligned} 16 &= y^2 + y \cdot \frac{y(x+2y)}{x-y} + \left(\frac{y(x+2y)}{x-y}\right)^2 \\ &= y^2 \cdot \frac{(x-y)^2 + (x-y)(x+2y) + (x+2y)^2}{(x-y)^2} \\ &= y^2 \cdot \frac{x^2 - 2xy + y^2 + x^2 + xy - 2y^2 + x^2 + 4xy + 4y^2}{(x-y)^2} \\ &= y^2 \cdot \frac{3x^2 + 3xy + 3y^3}{(x-y)^2} = \frac{27y^2}{(x-y)^2}. \end{aligned}$$

So,

$$4(x-y) = \pm 3\sqrt{3}y \qquad \text{or} \quad \text{equivalently,}$$

$$x = \left(1 + \frac{3\sqrt{3}}{4}\right) y \qquad \text{or } x = \left(1 - \frac{3\sqrt{3}}{4}\right) y. \qquad (2)$$

$A=0$ then implies

$$\left\{\left(1\pm\frac{3\sqrt{3}}{4}\right)^2+\left(1\pm\frac{3\sqrt{3}}{4}\right)+1\right\}y^2=9.$$

Taking into account (1) and (2) we conclude that

$$\begin{aligned}(x,y,z)\in\quad&\left\{\left(\frac{9+4\sqrt{3}}{\sqrt{25+12\sqrt{3}}},\frac{4\sqrt{3}}{\sqrt{25+12\sqrt{3}}},\frac{4(4+\sqrt{3})}{\sqrt{25+12\sqrt{3}}}\right),\right.\\[2ex]&\left(-\frac{9+4\sqrt{3}}{\sqrt{25+12\sqrt{3}}},-\frac{4\sqrt{3}}{\sqrt{25+12\sqrt{3}}},-\frac{4(4+\sqrt{3})}{\sqrt{25+12\sqrt{3}}}\right),\\[2ex]&\left(\frac{-9+4\sqrt{3}}{\sqrt{25-12\sqrt{3}}},\frac{4\sqrt{3}}{\sqrt{25-12\sqrt{3}}},\frac{4(-4+\sqrt{3})}{\sqrt{25-12\sqrt{3}}}\right),\\[2ex]&\left.\left(\frac{9-4\sqrt{3}}{\sqrt{25-12\sqrt{3}}},\frac{-4\sqrt{3}}{\sqrt{25-12\sqrt{3}}},\frac{4(4-\sqrt{3})}{\sqrt{25-12\sqrt{3}}}\right)\right\}.\end{aligned}$$

The system of equations in the statement of the problem has an interesting geometric interpretation. Let ABC be a triangle all of whose angles are smaller than $120°$. The Fermat point (or Torricelli point) of the triangle ABC is a point P such that the total distance from the three vertices of the triangle to the point is the minimum possible (see http://en.wikipedia.org/wiki/Fermat point).

Let $AB=c, BC=a, CA=b, AP=x, BP=y, CP=z$. Then

$$\begin{aligned}\angle APB=\angle APC&=\angle BPC=120^\circ\text{ and}\\ x^2+xy+y^2&=c^2,\\ y^2+yz+z^2&=a^2,\\ z^2+xz+x^2&=b^2,\end{aligned}$$

by the law of cosines. So x,y and z are the distances from the three vertices of the triangle to the Fermat point of the triangle.

- **Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain**

  Subtracting the equations term by term, we obtain

$$\begin{array}{c}(x^2-y^2)+z(x-y)=9,\\(x^2-z^2)+y(x-z)=-7,\end{array}\quad\Leftrightarrow\quad\begin{array}{c}(x-y)(x+y+z)=9,\\(x-z)(x+y+z)=-7.\end{array}$$

  Putting $u=x+y+z$, then we obtain $(x-y)u=9$ and $(x-z)u=-7$. Adding both equations yields $(3x-(x+y+z))u=2$ from which follows $x=\dfrac{u^2+2}{3u}$. Likewise, we obtain $y = \dfrac{u^2-25}{3u}$, and $z = \dfrac{u^2+23}{3u}$. Substituting the values of x, y, z into one of the equations of the given system, yields

$$\left(\frac{u^2+2}{3u}\right)^2 + \left(\frac{u^2+2}{3u}\right)\left(\frac{u^2-25}{3u}\right) + \left(\frac{u^2-25}{3u}\right)^2 = 3^2$$

or equivalently,

$$3u^4 - 150u^2 + 579 = 0.$$

Solving the preceding equation, we have the solutions:

$$\pm\sqrt{25-12\sqrt{3}}, \quad \pm\sqrt{25+12\sqrt{3}}.$$

Substituting these values in the expressions of x, y, z yields four triplets of solutions for the system. Namely,

$$\begin{aligned}(x_1, y_1, z_1) &= \left(\frac{27-12\sqrt{3}}{3\sqrt{25-12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25-12\sqrt{3}}}, \frac{48-12\sqrt{3}}{3\sqrt{25-12\sqrt{3}}}\right)\\ &= (1.009086173, -3.374440097, 4.418495493)\end{aligned}$$

$$\begin{aligned}(x_2, y_2, z_2) &= \left(-\frac{27-12\sqrt{3}}{3\sqrt{25-12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25-12\sqrt{3}}}, -\frac{48-12\sqrt{3}}{3\sqrt{25-12\sqrt{3}}}\right)\\ &= (-1.009086173, 3.374440097, -4.418495493)\end{aligned}$$

$$\begin{aligned}(x_3, y_3, z_3) &= \left(\frac{27+12\sqrt{3}}{3\sqrt{25+12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25+12\sqrt{3}}}, \frac{48+12\sqrt{3}}{3\sqrt{25-12\sqrt{3}}}\right)\\ &= (2.354003099, 1.023907822, 3.388521646)\end{aligned}$$

$$\begin{aligned}(x_4, y_4, z_4) &= \left(-\frac{27+12\sqrt{3}}{3\sqrt{25+12\sqrt{3}}}, -\frac{4\sqrt{3}}{\sqrt{25+12\sqrt{3}}}, -\frac{48+12\sqrt{3}}{3\sqrt{25-12\sqrt{3}}}\right)\\ &= (-2.354003099, -1.023907822, -3.388521646)\end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5177:** *Proposed by Kenneth Korbin, New York, NY*

  A regular nonagon $ABCDEFGHI$ has side 1.

  Find the area of $\triangle ACF$.

  Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

  We begin with the following known facts:

  1. Each angle in a regular nonagon is $140°$.
  2. $\cos 140° = \cos(180° - 40°) = -\cos 40°$.
  3. $\cos 100° = -\cos 80°$.
  4. $1 + \cos 2\theta = 2\cos^2\theta$.
  5. $\mathcal{A} = \frac{1}{2}ab\sin C$ in $\triangle ABC$.

  Hence, $\triangle ABC \cong \triangle HIA \cong \triangle HGF$ by SAS. Using Fact 1, since $\angle B = \angle I = \angle G = 140°$, it follows that $\angle BAC = \angle IAH = \angle IHA = \angle GHF = \angle GFH = 20°$. Thus, $\angle AHF = 100°$. Since $\triangle AHF$ is an isosceles triangle, $\angle HAF = \angle HFA = 40°$. Therefore, $\angle CAF = 60°$. In $\triangle ABC$, using the Law of Cosines and Facts 2 and 4,

$$\begin{aligned} AC^2 &= 1 + 1 - 2\cos 140° \\ &= 2(1 - \cos 140°) \\ &= 2(1 + \cos 40°) \\ &= 4\cos^2 20° \quad \text{Then,} \\ AC &= 2\cos 20°. \end{aligned}$$

  Similarly, since $AC = HA = HF = 2\cos 20°$, using the Law of Cosines and Facts 3 and 4 in $\triangle HAF$,

$$\begin{aligned} AF^2 &= (2\cos 20°)^2 + (2\cos 20°)^2 - 2(2\cos 20°)^2\cos 100° \\ &= 8\cos^2 20°(1 - \cos 100°) \\ &= 8\cos^2 20°(1 + \cos 80°) \\ &= 16\cos^2 20°\cos^2 40° \quad \text{Thus,} \\ AF &= 4\cos 20°\cos 40°. \end{aligned}$$

  In $\triangle ACF$, using Fact 5,

$$\begin{aligned} A &= \frac{1}{2}(AC)(AF)\sin 60° \\ &= \frac{1}{2}(2\cos 20°)(4\cos 20°\cos 40°)\left(\frac{\sqrt{3}}{2}\right) \\ &= 2\sqrt{3}\cos^2 20°\cos 40° \\ &\approx 2.343237. \end{aligned}$$

  Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

  Denote the circumcenter and the circumradius of the nonagon by O and r, respectively.

The nonagon can be oriented within the Cartesian plane so that its vertices are

$$A\left(r\cos 0^\circ, r\sin 0^\circ\right) \qquad B\left(r\cos 40^\circ, r\sin 40^\circ\right) \qquad C\left(r\cos 80^\circ, r\sin 80^\circ\right)$$

$$D\left(r\cos 120^\circ, r\sin 120^\circ\right) \quad E\left(r\cos 160^\circ, r\sin 160^\circ\right) \quad F\left(r\cos 200^\circ, r\sin 200^\circ\right)$$

$$G\left(r\cos 240^\circ, r\sin 240^\circ\right) \quad H\left(r\cos 280^\circ, r\sin 2800^\circ\right) \quad I\left(r\cos 320^\circ, r\sin 320^\circ\right).$$

Then,

$$\begin{aligned}
1^2 = \overline{AB}^2 &= \left(r\cos 40^\circ - r\cos 0^\circ\right)^2 + \left(r\sin 40^\circ - r\sin 0^\circ\right)^2 \\
&= r^2\left(\cos^2 40^\circ - 2\cos 40^\circ + 1 + \sin^2 40^\circ\right)^2 \\
&= 2r^2\left(1 - \cos 40^\circ\right) \Rightarrow r^2 = \frac{1}{2\left(1-\cos 40^\circ\right)}.
\end{aligned}$$

The area of $\triangle ACF$ is

$$\begin{aligned}
[\triangle ACF] &= \frac{1}{2}\left| det\begin{pmatrix} 1 & 1 & 1 \\ r & r\cos 80^\circ & r\cos 200^\circ \\ 0 & r\sin 80^\circ & r\sin 200^\circ \end{pmatrix}\right| \\
&= \frac{1}{2}\left|r^2\cos 80^\circ\sin 200^\circ + r^2\sin 80^\circ - r^2\cos 200^\circ\sin 80^\circ - r^2\sin 200^\circ\right| \\
&= \frac{1}{2}\left|r^2\left(\cos 80^\circ\sin 200^\circ - \sin 80^\circ\cos 200^\circ\right) + r^2\sin 80^\circ - r^2\sin 200^\circ\right| \\
&= \frac{r^2}{2}\left|(\sin(200^\circ - 80^\circ) + \sin 80^\circ - \sin 200^\circ\right| \\
&= \frac{1}{4\left(1-\cos 40^\circ\right)}\left|\sin 120^\circ + \sin 80^\circ - \sin 200^\circ\right| \\
&\approx 2.343237.
\end{aligned}$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is easy to check that $\angle BAC = 20^\circ$, $\angle IAF = \angle FAC = 60^\circ$ and $AC = 2\cos 20^\circ$. Suppose that the perpendicular from I to AF meets AF at J, the perpendicular from H to AF meets AF at K, and the perpendicular from I to HK meets HK at L. Then $\angle HIL = 20^\circ$ and

$$AF = 2(AJ + JK) = 2(AJ + IL) = 2(\cos 60^\circ + \cos 20^\circ) = 1 + 2\cos 20^\circ.$$

Hence the area of $\triangle ACF$ equals

$$\frac{(AC)(AF)\sin\angle FAC}{2}$$

$$
\begin{aligned}
&= \frac{\cos 20^\circ(1+2\cos 20^\circ)\sqrt{3}}{2} \\
&= \frac{(1+\cos 20^\circ+\cos 40^\circ)\sqrt{3}}{2} \\
&= \frac{\sqrt{3}(1+\sqrt{3}\cos 10^\circ)}{2} \\
&\approx 2.343237.
\end{aligned}
$$

Solution 4 by proposer

$$
\begin{aligned}
\text{Area of } \triangle ACF &= \frac{\sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ}{2\sin^2 20^\circ} \\
&= \frac{\sqrt{3}}{16}\left[3\tan^2 70^\circ - 1\right] \\
&\approx 2.343237.
\end{aligned}
$$

Comment by editor: Sines and cosines of angles of $10^\circ, 20^\circ, 40^\circ$ and their complements often appear in the above solutions. **David Stone and John Hawkins of Statesboro, GA** noted in their solution that: "It may be possible to express the result $(\sqrt{3}\cos 40^\circ\,(1+\cos 40^\circ))$ in terms of radicals, even though $\cos 40^\circ$ itself cannot be expressed in terms of surds; it (along with $\sin 10^\circ$ and $-\sin 70^\circ$) is a zero of the famous *casus irreducibilis* cubic $8x^3 - 6x + 1 = 0$."

Also solved by Scott H. Brown, Montgomery, AL; Brian D. Beasley, Clinton, SC; Kenneth Day and Michael Thew (jointly, students at Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY, and Albert Stadler, Herrliberg, Switzerland.

- **5178:** *Proposed by Neculai Stanciu, Buzău, Romania*

Prove: If x, y and z are positive real numbers such that $xyz \geq 7+5\sqrt{2}$, then

$$x^2+y^2+z^2-2(x+y+z) \geq 3.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

By the AM-GM inequality, $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz} \geq \sqrt[3]{7+5\sqrt{2}} = 1+\sqrt{2}$. Let $f(x) = x^2-2x-1$. $f(x)$ is a convex function that is monotonically increasing for $x \geq 1$. By Jensen's inequality,

$$x^3+y^3+z^3-2(x+y+z)-3 = f(x)+f(y)+f(z) \geq 3f\left(\frac{x+y+z}{3}\right) \geq 3f\left(1+\sqrt{2}\right) = 0.$$

Solution 2 by David E. Manes, Oneonta, NY

Note that for positive real numbers if $x \geq 1+\sqrt{2}$, then $(x-1)^2 \geq 2$ with equality if and only if $x = 1+\sqrt{2}$. Therefore, if $x, y, z \geq 1+\sqrt{2}$, then $xyz \geq 7+5\sqrt{2}$ and $(x-1)^2+(y-1)^2+(z-1)^2 \geq 6$. Expanding this inequality yields $x^2+y^2+x^2-2(x+y+z) \geq 3$ with equality if and only if $x = y = z = 1+\sqrt{2}$.

Solution 3 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

We know tht $x^2+y^2+z^2 \geq \dfrac{(x+y+z)^2}{3}$ thus the inequality is implied by

$$S^2-6S-9 \geq 0, \qquad S = x+y+z$$

yielding $S \geq 3(1+\sqrt{2})$. Moreover by the AGM we have $S \geq 3(xyz)^{1/3} \geq 3(7+5\sqrt{2})^{1/3}$, thus we need to check that $3(7+5\sqrt{2})^{1/3} \geq 3(1+\sqrt{2})$ or $7+5\sqrt{2} \geq (1+\sqrt{2})^3$ which is actually an equality, and we are done.

Also solvled by Arkady Alt, San Jose California; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY, and the proposer.

- **5179:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Find all positive real solutions $(x_1, x_2, \ldots, x_n)$ of the system

$$\left\{\begin{array}{r} x_1+\sqrt{x_2+11} = \sqrt{x_2+76}, \\ x_2+\sqrt{x_3+11} = \sqrt{x_3+76}, \\ \cdots\cdots\cdots \\ x_{n-1}+\sqrt{x_n+11} = \sqrt{x_n+76}, \\ x_n+\sqrt{x_1+11} = \sqrt{x_1+76}. \end{array}\right.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

If $f(t) = \sqrt{t+76}-\sqrt{t+11}$ on $(0,\infty)$, then

$$f'(t) = \frac{1}{2}\left(\frac{1}{\sqrt{t+76}}-\frac{1}{\sqrt{t+11}}\right), \text{ and hence,}$$

$$\begin{aligned} |f'(t)| &= \frac{1}{2}\left(\frac{1}{\sqrt{t+11}}-\frac{1}{\sqrt{t+76}}\right) \\ &< \frac{1}{2}\frac{1}{\sqrt{t+11}} \\ &< \frac{\sqrt{11}}{22} \\ &< 1 \end{aligned}$$

for $t > 0$. It follows that $f(t)$ is a contraction mapping on $(0,\infty)$ and therefore, $f(t)$ has a unique fixed point $t^* \in (0,\infty)$. Further, it is well-known that for any $\bar{t} \in (0,\infty)$, the sequence defined recursively by $t_1 = \bar{t}$ and $t_{k+1} = f(t_k)$ for $k \geq 1$ must converge to t^*. By trial and error, we find that $t^* = 5$.

In this problem,

$$\begin{aligned} x_1 &= f(x_2), \\ x_2 &= f(x_3), \\ &\vdots \\ x_{n-1} &= f(x_n), \\ x_n &= f(x_1). \end{aligned}$$

If we let $t_1 = x_1$ and define $t_{k+1} = f(t_k)$ for $k \geq 1$, then $(x_1, x_n, \ldots, x_3, x_2)$ is a cycle in the sequence $\{t_k\}$. However, as described above, $t_k \to 5$ as $k \to \infty$. These conditions force $x_1 = x_2 = \cdots = x_n = 5$ and therefore, this must be the unique solution for this system.

Also solved by Arkady Alt, San Jose, CA; Scott H. Brown, Montgomery, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, Buzău Romania, jointly with Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5180:** *Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy*

Let a, b and c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1+a}{bc} + \frac{1+b}{ac} + \frac{1+c}{ab} \geq \frac{4}{\sqrt{a^2+b^2-ab}} + \frac{4}{\sqrt{b^2+c^2-bc}} + \frac{4}{\sqrt{a^2+c^2-ac}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Multiplying both sides of the desired inequality by abc, we see that it is equivalent to

$$1 + a^2 + b^2 + c^2 \geq 4abc\left(\frac{1}{\sqrt{a^2+b^2-ab}} + \frac{1}{\sqrt{b^2+c^2-bc}} + \frac{1}{\sqrt{a^2+c^2-ac}}\right). \tag{1}$$

Since

$$a^2 + b^2 - ab = (a-b)^2 + ab \geq ab, \quad b^2 + c^2 - bc \geq bc, \quad a^2 + c^2 - ac \geq ac,$$

the right hand side of (1) is less than or equal to

$$\begin{aligned} & 4\left(\sqrt{abc} + \sqrt{bca} + \sqrt{cab}\right) \\ \leq\ & 2\left((a+b)c + (b+c)a + (c+a)b\right) \\ =\ & 4\left(ab + bc + ca\right) \\ =\ & 2\left((a+b+c)^2 - a^2 - b^2 - c^2\right) \end{aligned}$$

$$= \quad 2-2\left(a^2+b^2+c^2\right).$$

Now

$$a^2+b^2+c^2=\left(a-\frac{1}{3}\right)^2+\left(b-\frac{1}{3}\right)^2+\left(c-\frac{1}{3}\right)^2+\frac{2(a+b+c)}{3}-\frac{1}{3}\geq\frac{1}{3},$$

so that $1+a^2+b^2+c^2\geq 2-2\left(a^2+b^2+c^2\right)$.

This proves (1) and completes the solution.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By the AM-GM inequality,

$$\begin{aligned}
\frac{1+a}{bc}+\frac{1+b}{ca}+\frac{1+c}{ab} &= \frac{a+a^2+b+b^2+c+c^2}{abc}\\
&= \frac{1+a^2+b^2+c^2}{abc}\\
&= \frac{(a+b+c)^2+a^2+b^2+c^2}{abc}\\
&= \frac{(2a^2+2bc)+(2b^2+2ca)+(2c^2+2ab)}{abc}\\
&\geq 4\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{abc}\\
&= \frac{4}{\sqrt{bc}}+\frac{4}{\sqrt{ca}}+\frac{4}{\sqrt{ab}}.
\end{aligned}$$

The conclusion follows since

$$\frac{1}{\sqrt{xy}}\geq\frac{1}{\sqrt{x^2+y^2-xy}}.$$

(Note that this inequality is equivalent to $x^2+y^2-xy\geq xy$ which is obviously true.)

Also solved by Arkady Alt, San Jose, CA; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5181:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\cdot m}{(n+m)!}.$$

Solution 1 by Anastasios Kotronis, Athens, Greece The summands being all positive we can sum by triangles :

$$\begin{aligned}
\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{nm}{(n+m)!} &= \sum_{k,\ell,n\in\wedge k+\ell=n}\frac{nm}{(n+m)!}=\sum_{n=2}^{+\infty}\frac{\sum_{\ell=1}^{n-1}(n-\ell)\ell}{n!}\\
&= \frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n-1)n(n+1)}{n!}=\frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n+1)}{(n-2)!}\\
&= \frac{1}{6}\sum_{n=0}^{+\infty}\frac{(n+3)}{n!}=\frac{1}{6}\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{dx^{n+3}}{dx}\Big|_{x=1}\\
&= \frac{1}{6}\frac{d}{dx}\left(\sum_{n=0}^{+\infty}\frac{x^{n+3}}{n!}\right)\Big|_{x=1}=\frac{1}{6}\frac{d\left(x^3e^x\right)}{dx}\Big|_{x=1}\\
&= \frac{2e}{3}.
\end{aligned}$$

Solution 2 by Arkady Alt, San Jose, CA

Let $k=m+n$. Then $m=k-n$ and domain of summation $\begin{cases}1\le n\\ 1\le m\end{cases}$ can be represented as $\begin{cases}2\le k\\ 1\le n\le k-1\\ m=k-n\end{cases}$. Hence,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm}{(n+m)!}=\sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\frac{n\,(k-n)}{k!}=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=1}^{k-1}n\,(k-n)=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=1}^{k-1}n\,(k-n)\,.$$

Since

$$\begin{aligned}
\sum_{n=1}^{k-1}n\,(k-n) &= \frac{k^2\,(k-1)}{2}-\frac{(k-1)\,k\,(2k-1)}{6}\\
&= \frac{k}{6}\left(3k^2-3k-2k^2+3k-1\right)\\
&= \frac{k\left(k^2-1\right)}{6},
\end{aligned}$$

then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm}{(n+m)!} = \frac{1}{6}\sum_{k=2}^{\infty}\frac{k+1}{(k-2)!}$$

$$
\begin{aligned}
&= \frac{1}{6}\sum_{k=0}^{\infty}\frac{k+3}{k!} \\
&= \frac{1}{6}\left(\sum_{k=0}^{\infty}\frac{3}{k!}+\frac{k}{k!}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{k!} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}+\frac{1}{6}\right) \\
&= \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{k!} \\
&= \frac{2e}{3}.
\end{aligned}
$$

Solution 3 by the proposer

The series equals $\frac{2e}{3}$. First we note that for $m \geq 0$ and $n \geq 1$ one has that

$$\int_0^1 x^m(1-x)^{n-1}dx = B(m+1,n) = \frac{m!\cdot(n-1)!}{(n+m)!}.$$

Thus,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\cdot m}{(n+m)!} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{(n-1)!}\cdot\frac{1}{(m-1)!}\int_0^1 x^m(1-x)^{n-1}dx \\
&= \int_0^1\left(\sum_{n=1}^{\infty}\frac{n}{(n-1)!}(1-x)^{n-1}\right)\cdot\left(\sum_{m=1}^{\infty}\frac{x^m}{(m-1)!}\right)dx \\
&= \int_0^1\left(1+\sum_{n=2}^{\infty}\frac{n}{(n-1)!}(1-x)^{n-1}\right)\cdot xe^x dx \\
&= \int_0^1\left(1+\sum_{n=2}^{\infty}\frac{(1-x)^{n-1}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{(1-x)^{n-1}}{(n-1)!}\right)\cdot xe^x dx \\
&= \int_0^1\left(1+(1-x)e^{1-x}+e^{1-x}-1\right)\cdot xe^x dx \\
&= e\int_0^1(2-x)x dx = \frac{2e}{3},
\end{aligned}
$$

and the problem is solved.

Also solved by Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and Albert Stadler, Herrliberg, Switzerland.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before May 15, 2012

- **5200:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers (a, b, c, d) such that,

$$(a+b+c+d)^2 = 2\left(a^2+b^2+c^2+d^2\right)$$

  with $a < b < c < d$. Find positive integers x, y and z such that

$$\begin{aligned} x &= \sqrt{ab+ad+bd} - \sqrt{ab+ac+bc}, \\ y &= \sqrt{bc+bd+cd} - \sqrt{bc+ab+ac}, \\ z &= \sqrt{bc+bd+cd} - \sqrt{ac+ad+cd}. \end{aligned}$$

- **5201:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral ABCD with integer length sides where $\left(\overline{AB}, \overline{BC}, \overline{CD}\right) = 1$ and with $\overline{AB} < \overline{BC} < \overline{CD}$.

  The inradius, the circumradius, and both diagonals have rational lengths. Find the possible dimensions of the quadrilateral.

- **5202:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Solve in $\Re^2$,

$$\begin{cases} \ln\left(x+\sqrt{x^2+1}\right) = \ln \dfrac{1}{y+\sqrt{y^2+1}} \\ 2^{y-x}\left(1-3^{x-y+1}\right) = 2^{x-y+1} - 1. \end{cases}$$

- **5203:** *Proposed by Pedro Pantoja, Natal-RN, Brazil*

  Evaluate,

$$\int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{\sin^4 x + \cos^4 x}\right) dx.$$

- **5204:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $f : \Re \to \Re$ be a non-constant function such that,

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$$

  for all $x, y \in \Re$. Show that $-1 < f(x) < 1$ for all $x \in \Re$.

- **5205:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Find the sum,

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln \frac{n+1}{n}.$$

Solutions

- **5182:** *Proposed by Kenneth Korbin, New York, NY*

  Part I: An isosceles right triangle has perimeter P and its Morley triangle has perimeter x. Find these perimeters if $P = x + 1$.

  Part II: An isosceles right triangle has area K and its Morley triangle has area y. Find these areas if $K = y + 1$

  Solution by David E. Manes, Oneonta, NY

  For part I, $P = \dfrac{2(2+\sqrt{2})}{4 - 4\sqrt{2} + 3\sqrt{6}}$ and $x = \dfrac{3\sqrt{2}\left(2 - \sqrt{3}\right)}{4 - 4\sqrt{2} + 3\sqrt{6}}$.

  For part II, $K = \dfrac{4\left(16 + 7\sqrt{3}\right)}{109}$ and $y = \dfrac{28\sqrt{3} - 45}{109}$.

  Denote the isosceles right triangle by ABC with the right angle at vertex C and sides a, b, c opposite the vertices A, B, C respectively. Then $a = b$ and $c = \sqrt{2}a$, whence $P = (2+\sqrt{2})a$. The side length s of the Morley triangle of ABC is given by $s = 8R \sin\dfrac{A}{3} \sin\dfrac{B}{3} \sin\dfrac{C}{3}$ where R is the circumradius of triangle ABC. Then

$$\begin{aligned} R &= \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}} \\ &= \frac{\sqrt{2}a^3}{\sqrt{a(2+\sqrt{2})\left(\sqrt{2}a\right)^2 a\left(2-\sqrt{2}\right)}} \\ &= \frac{\sqrt{2}a}{2}. \end{aligned}$$

  Using the identity $\sin^2\dfrac{z}{2} = \dfrac{1-\cos z}{2}$, one calculates

$$\sin\frac{A}{3} \cdot \sin\frac{B}{3} = \sin^2 15^\circ = \frac{1 - \cos 30^\circ}{2} = \frac{2-\sqrt{3}}{4}.$$

Therefore,

$$s = 8R\sin^2 15^\circ \sin 30^\circ = 8\left(\frac{\sqrt{2}a}{2}\right)\left(\frac{2-\sqrt{3}}{4}\right)\frac{1}{2} = \frac{\sqrt{2}}{2}\left(2-\sqrt{3}\right)a$$

so that the perimeter x of the Morley triangle is given by $x = 3s = \dfrac{3\sqrt{2}}{2}\left(2-\sqrt{3}\right)a$.

The equation $P = x+1$ implies $\left(2+\sqrt{2}\right)a = \dfrac{3\sqrt{2}}{2}\left(2-\sqrt{3}\right)a + 1$ or $a = \dfrac{2}{4-4\sqrt{2}+3\sqrt{6}}$. Hence,

$$\begin{aligned} P &= (2+\sqrt{2})a = \frac{2(2+\sqrt{2})}{4-4\sqrt{2}+3\sqrt{6}} \quad \text{and} \\ x &= \frac{3\sqrt{2}}{2}\left(2-\sqrt{3}\right)a = \frac{3\sqrt{2}\left(2-\sqrt{3}\right)}{4-4\sqrt{2}+3\sqrt{6}}. \end{aligned}$$

In part II,

$$\begin{aligned} K &= \frac{a^2}{2} \text{ and} \\ y &= \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left[\frac{\sqrt{2}}{2}\left(2-\sqrt{3}\right)a\right]^2 \\ &= \frac{\sqrt{3}}{8}\left(7-4\sqrt{3}\right)a^2. \end{aligned}$$

The equation $K = y+1$ implies $\dfrac{a^2}{2} = \dfrac{\sqrt{3}}{8}\left(7-4\sqrt{3}\right)a^2 + 1$; that is, $a^2 = \dfrac{8(16+7\sqrt{3})}{109}$. Hence,

$$K = \frac{4\left(16+7\sqrt{3}\right)}{109} \text{ and } y = \frac{28\sqrt{3}-45}{109}.$$

Also solved by Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5183:** *Proposed by Kenneth Korbin, New York, NY*

A convex pentagon ABCDE, with integer length sides, is inscribed in a circle with diameter $\overline{AE}$.

Find the minimum possible perimeter of this pentagon.

Solution by Kee-Wai Lau, Hong Kong, China

We show that the minimum possible perimeter of this pentagon is 164.

Suppose that O is the center of the circle. Let $AB = p, BC = q, CD = r, DE = s, AE = d, \angle AOB = 2\alpha, \angle BOC = 2\beta, \angle COD = 2\gamma, \angle DOE = 2\delta$, where p, q, r, s, d are positive integers and $\alpha, \beta, \gamma, \delta$ are positive numbers such that $\alpha + \beta + \gamma + \delta = \frac{\pi}{2}$.

We have

$$\begin{aligned}
\sin\alpha &= \frac{p}{d},\ \cos\alpha = \frac{\sqrt{d^2-p^2}}{d},\ \sin\beta = \frac{q}{d},\ \cos\beta = \frac{\sqrt{d^2-q^2}}{d},\\
\sin\gamma &= \frac{r}{d},\ \cos\gamma = \frac{\sqrt{d^2-r^2}}{d},\ \text{and}\ \sin\delta = \frac{s}{d}.
\end{aligned}$$

Since $\sin\delta = \cos(\alpha+\beta+\gamma)$, so

$$d^2 s = \sqrt{d^2-p^2}\sqrt{d^2-q^2}\sqrt{d^2-r^2} - \left(\sqrt{d^2-p^2}\right) qr - \left(\sqrt{d^2-q^2}\right) rp - \left(\sqrt{d^2-r^2}\right) pq.$$

It is not hard to see that if at least one of the $\sqrt{d^2-p^2}\sqrt{d^2-q^2}\sqrt{d^2-r^2}$ is irrational, then $d^2 s$ is also irrational. Hence we seek the primitive Pythagorean triples with $d = m^2+n^2,\ p \in \{2mn, m^2-n^2\}$ such that

$$\frac{\sqrt{d^2-p^2}\sqrt{d^2-q^2}\sqrt{d^2-r^2} - \left(\sqrt{d^2-p^2}\right) qr - \left(\sqrt{d^2-q^2}\right) rp - \left(\sqrt{d^2-r^2}\right) pq}{d^2}$$

is a positive integer. We now find with a computer that the minimum perimeter is 164, given by $(d, p, q, s) = (65, 16, 25, 25, 33)$ and other combinations.

Comment by Editor. **David Stone and John Hawkins of Statesboro, GA** approached the problem in the above manner. They ran a MATLAB program checking all integer combinations for p, q, r, s, and $d \le 5000$, and using the data from this program they showed that they, like Kee-Wai, had actually found the smallest solution. They then went on to list some additional pentagons satisfying the requirements of the problem, but with larger perimeters. (I have substituted Kee-Wai's notation into David and John's matrix.)

d	p	q	r	s	*perimeter*
85	13	36	40	40	214
125	35	35	44	75	314
130	32	50	50	66	328
145	17	24	87	87	360
170	26	72	80	80	428
185	57	60	60	104	466
195	48	75	75	99	492
205	45	45	84	133	512
221	21	85	85	140	552
250	70	70	88	150	628
255	39	108	120	120	642
260	64	100	100	132	656
265	23	96	140	140	664
290	34	48	174	174	720
305	55	55	136	207	758
325	36	80	91	260	792
325	36	80	165	204	810
325	36	91	253	91	796
325	36	91	165	195	812
325	80	80	125	204	814
325	36	125	165	165	816
325	80	125	195	91	816
325	80	125	125	165	820

David and John went on to state that they didn't know if an analytical proof exists (as opposed to a computer one) that shows that the minimum perimeter is given by $(d, p, q, s) = (65, 16, 25, 25, 33)$. They then made the following comments.

There is interesting territory for further exploration of these integer-valued, convex, pentagons inscribed in a semi-circle ("Korbin pentagons"?). We could define (p, q, r, s, d) to be *primitive* if there is no common factor and hope to find a parametric-type formula for generating the primitive ones. Based on our few examples, it appears that in a primitive pentagon, s must be the non-prime hypotenuse of a primitive Pythagorean triple, that d is the product of primes congruent to 1(mod 4) and p, q, r and s must be legs of some right triangle having d as its hypotenuse. It also seems that d must be expressible in more than one way as the sum of squares. It looks like many primitive pentagons exist. For example with $d = 325 = 5^2 \cdot 13$, there are several primitive pentagons (and one multiple of our minimal example). Two of these primitive pentagons even have the same perimeter!

Also solved by David Stone and John Hawkins of Statesboro, GA, the proposer.

- **5184:** *Proposed by Neculai Stanciu, Buzău, Romania*

If x, y and z are positive real numbers, then prove that

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} \geq \frac{8}{9}.$$

Solution 1 by Pedro Pantoja, Natal-RN, Brazil

If x, y, z are positive real numbers then,

$$(x+y)(y+z)(z+x) = (xy+yz+zx)(x+y+z) - xyz.$$

So,

$$\frac{(x+y)(y+z)(z+x)}{(xy+yz+zx)(x+y+z)} = 1 - \frac{xyz}{(xy+yz+zx)(x+y+z)}.$$

By the AM-GM inequality,

$$x+y+z \geq 3\sqrt[3]{xyz} \text{ and } xy+yz+zx \geq 3\sqrt[3]{(xyz)^2},$$

which implies that $(xy+yz+zx)(x+y+z) \geq 9xyz$. And this implies that

$$\begin{aligned}
\frac{xyz}{(xy+yz+zx)(x+y+z)} &\leq \frac{1}{9}. \text{ So,} \\
1 - \frac{xyz}{(xy+yz+zx)(x+y+z)} &\geq 1 - \frac{1}{9} = \frac{8}{9}. \text{ Therefore,} \\
\frac{(x+y)(y+z)(z+x)}{(xy+yz+zx)(x+y+z)} &\geq \frac{8}{9}.
\end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro Spain

$$\begin{aligned}
&\frac{9(x+y)(y+z)(z+x)}{8(x+y+z)(xy+yz+zx)} \\
&= \frac{9(xy+yz+zx+y^2)(z+x)}{8(x+y+z)(xy+yz+zx)} \\
&= 9\frac{(xy+yz+zx)(z+x)+y^2(z+x)}{8(z+y+z)(xy+yz+zx)} \\
&= 9\frac{(x+y+z)(xy+yz+zx)-(xy+yz+zx)y+y^2(z+x)}{8(x+y+z)(xy+yz+zx)} \\
&= 9\frac{(x+y+z)(xy+yz+zx)-(xy^2-y^2z-xyz+y^2z+xy^2)}{8(x+y+z)(xy+yz+zx)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{9(x+y+z)(xy+yz+zx)-9xyz}{8(x+y+z)(xy+yz+zx)}\\
&= \frac{8(x+y+z)(xy+yz+zx)+(x+y+z)(xy+yz+zx)-8xyz-xyz}{8(x+y+z)(xy+yz+zx)}\\
&= \frac{8(x+y+z)(xy+yz+zx)-8xyz+(x+y+z)(xy+yz+zx)-xyz}{8(x+y+z)(xy+yz+zx)}\\
&\geq \frac{8(x+y+z)(xy+yz+zx)-8xyz+3\sqrt[3]{xyz}3\sqrt[3]{xyyzzx}-xyz}{8(x+y+z)(xy+yz+zx)}\\
&= \frac{8(x+y+z)(xy+yz+zx)-8xyz+9xyz-xyz}{8(x+y+z)(xy+yz+zx)}\\
&= \frac{8(x+y+z)(xy+yz+zx)}{8(x+y+z)(xy+yz+zx)}\\
&= 1,
\end{aligned}
$$

which is equivalent to the proposed inequality, where the $AM-GM$ inequality has been applied. We note that equality holds if, and only if, $x=y=z$.

Solution 3 by Paul M. Harms, North Newton, KS

Let $y=tx$ and $z=rx$ where $x,r,$ and t are positive teal numbers. Then the inequality to be proved becomes

$$\frac{(1+t)(t+r)(1+r)}{(1+r+t)(r+rt+t)} \geq \frac{8}{9}.$$

The following inequalities are equivalent to the above inequality.

$$
\begin{aligned}
9(1+t)(t+r)(1+r) &\geq 8(1+r+t)(r+rt+t)\\
9(t+t+2rt+t^2+r^2+rt^2+r^2t) &\geq 8(r+3rt+t+r^2+t^2+rt^2+r^2t\\
r+t-6rt+t^2+r^2+rt^2+r^2t &\geq 0.
\end{aligned}
$$

To prove the last inequality we write the left side of this inequality as follows:

$$(t^2-2rt+r^2)+r(t^2-2t+1)+t(r^2-2r+1)=(t-r)^2+r(t-1)^2+t(r-1)^2.$$

Since all three terms are non-negative, the above inequalities are correct. Thus the inequality in the problem has been proved.

Solution 4 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Normalizing the LHS one can assume that $x+y+z=1$ and so our inequality will become equivalent to proving

$$\frac{(x+y)(y+z)(z+x)}{(xy+yz+zx)} \geq \frac{8}{9}$$

subject to $x+y+z=1$ and the fact that x, y and z are positive real numbers.

We will use the Lagrange Multiplier method to find the minimum of the function $f(x,y,z) = \dfrac{(x+y)(y+z)(z+x)}{(xy+yz+zx)}$ subject to $g(x,y,z) = x+y+z = 1$ and the fact that x, y and z are positive real numbers.

Doing easy manipulations we have that

$$\bigtriangledown f(x,y,z) \quad = \quad < f_x, f_y, f_z >$$

$$< \quad \frac{x(y+z)(xy+2yz+zx}{(xy+yz+zx)^2}, \frac{y(z+x)(xy+yz+2zx}{(xy+yz+zx)^2}, \frac{z(x+y)(2xy+yz+zx}{(xy+yz+zx)^2} >$$

and

$$\bigtriangledown g(x,y,z) =< g_x, g_y, g_z >=< 1, 1, 1 >$$

applying the Lagrange Multiplier method, the extremes will be the solutions of the system of the equations

$$\begin{cases} \bigtriangledown f(x,y,z) = \lambda \bigtriangledown g(x,y,z) \\ g(x,y,z) = 1 \end{cases}$$

which is equivalent to

$$\begin{cases} \dfrac{x(y+z)(xy+2yz+zx)}{(xy+yz+zx)^2} = \lambda \\ \\ \dfrac{y(z+x)(xy+yz+2zx)}{(xy+yz+zx)^2} = \lambda \\ \\ \dfrac{z(x+y)(2xy+yz+zx)}{(xy+yz+zx)^2} = \lambda \\ \\ x+y+z = 1 \end{cases}$$

where λ is a real number.
Solving this easy system of equations we have that the solutions will be

$$(x,y,z,\lambda) = \{(1,1,-1,0), (1,-1,1,0), (-1,1,1,0), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{8}{9}\right)\}$$

Using the fact that x, y and z are positive real numbers, the only point of interest will be

$$(x,y,z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

and the value of the function at that point will be

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{\left(\frac{1}{3}+\frac{1}{3}\right)\left(\frac{1}{3}+\frac{1}{3}\right)\left(\frac{1}{3}+\frac{1}{3}\right)}{\left(\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}\right)} = \frac{8}{9}$$

Getting the value of the function $f(x,y,z)$ at another point, let say $(x,y,z) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ we have

$$f\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) = \frac{\left(\frac{1}{2}+\frac{1}{4}\right)\left(\frac{1}{4}+\frac{1}{4}\right)\left(\frac{1}{4}+\frac{1}{2}\right)}{\left(\frac{1}{2}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{2}\right)} = \frac{9}{10} > \frac{8}{9}$$

we have that the extreme point $(x,y,z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a minimum and this is the end of the proof.

Solution 5 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that

$$\frac{(x+y+)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} = \frac{x(y-z)^2+y(z-x)^2+z(x-y)^2}{9(x+y+z)(xy+yz+zx)} + \frac{8}{9},$$

and the inequality of the problem follows.

Solution 6 by Andrea Fanchini, Cantú Italy

Let $p = x+y+z, q = xy+yz+zx$ and $r = xyz$. Then the given inequality becomes

$$\frac{pq-r}{pq} \geq \frac{8}{9}.$$

I.e.,

$$pq \geq 9r,$$

that we can prove easily using the *AM-GM* inequality,

$$pq = (x+y+z)(xy+yz+zx) \geq 3\sqrt[3]{xyz} \cdot 3\sqrt[3]{x^2y^2z^2} = 9r.$$

So the proposed inequality is proved.

Comment by Albert Stadler of Herrliberg, Switzerland

I tried to generalize this problem to n variables and conjectured the following statement: Let $x_1, x_2, \ldots, x_n$ be n real positive numbers , $n \geq 2$. Then

$$\frac{\prod_{i=1}^{n}(x_i+x_{i+1})}{x_1 \cdot x_2 \cdots x_n \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} \frac{1}{x_i}} \geq \frac{2^n}{n^2}, \text{ with the assumption that } x_{n+1} = x_1.$$

For $n = 2$ this says that $\frac{(x_1+x_2)^2}{(x_1+x_2)^2} \geq \frac{2^2}{2^2}$ which is true.

For $n = 3$ we have the statement of problem 5184.

For $n = 4$ this says that $\frac{(x_1+x_2)(x_2+x_3)(x_3+x_4)(x_4+x_1)}{(x_1+x_2+x_3+x_4)(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)} \geq \frac{2^4}{4^2}$, which is equivalent to $(x_1x_3-x_2x_4)^2 \geq 0$, and this is obviously true.

However it turns out that the statement is false for $n = 5$ as is evidenced by the counterexample $(x_1, x_2, x_3, x_4, x_5) = (8, 3, 1, 2, 8)$.

Also solved by Daniel Lopez Aguayo, Institute of Mathematics (UNAM) Morelia, Mexico; Dionne Bailey, Elsie Campbell and Charles Diminnie, San Angelo, TX; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; David E. Manes, Oneonta NY; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti (two solutions), Department of Mathematics, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5185:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Calculate, without using a computer, the value of

$$\sin\left[\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right].$$

Solution 1 by Andrea Fanchini, Cantú, Italy

Knowing that the argument of the product of complex numbers is the sum of the arguments of the factors, we can see that

$$\theta=\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)$$

is the argument of the following multiplication

$$(3+i)\,(5+i)\,(7+i)\,(11+i)\,(13+i)\,(121+111i)$$

multiplying in the usual way, we obtain the pure imaginary number $2696200i$, so $\theta=\dfrac{\pi}{2}$ and then finally we have $\sin\left(\dfrac{\pi}{2}\right)=1$.

Solution 2 by Anastasios Kotronis, Athens, Greece

The following identities are well known:

$$\arctan a+\arctan b=\begin{cases}\arctan\dfrac{a+b}{1-ab} & ,\ ab<1\\[2ex] \arctan\dfrac{a+b}{1-ab}+\pi & ,\ ab>1\wedge a>0\\[2ex] \arctan\dfrac{a+b}{1-ab}-\pi & ,\ ab>1\wedge a<0\end{cases}$$

$$\arctan a+\arctan\frac{1}{a}=\begin{cases}\dfrac{\pi}{2} & ,\ a>0\\[2ex] -\dfrac{\pi}{2} & ,\ a<0\end{cases}$$

Applying these formulas to the pair $\arctan\left(\dfrac{1}{3}\right)$, $\arctan\left(\dfrac{1}{5}\right)$, and then repeating to $\arctan\left(\dfrac{1}{7}\right)$, $\arctan\left(\dfrac{1}{11}\right)$ and to $\arctan\left(\dfrac{1}{13}\right)$, we obtain that

$$\sin\left(\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right)$$

$$\begin{aligned} &= \sin\left(\arctan\left(\frac{121}{111}\right)+\arctan\left(\frac{111}{121}\right)\right) \\ &= \sin\frac{\pi}{2} \\ &= 1. \end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Dionne Bailey, Elsie Campbell and Charles Diminnie, San Angelo, TX; Michael C. Faleski (two solutions), University Center, MI; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kenneth Korbin, New York, NY; David E. Manes, Oneonta NY; Kee-Wai Lau, Hong Kong, China; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays (two solutions), Brooklyn, NY; Neculai Stanciu, Buzău, Romania with Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins (jointly), Statesboro, GA; Albert Stadler, Herrilberg, Switzerland, and the proposer.

- **5186:** *Proposed by John Nord, Spokane, WA*

  Find k so that $\int_0^k \left(-\frac{b}{a}x+b\right)^n dx = \frac{1}{2}\int_0^a \left(-\frac{b}{a}x+b\right)^n dx.$

Solution by Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

It is clear that if $b=0$, then the equation holds for every k. Assuming that the parameter $b \neq 0$, it may be removed, and the problem then becomes to find k so that

$$\int_0^k \left(\frac{a-x}{a}\right)^n dx = \frac{1}{2}\int_0^a \left(\frac{a-x}{a}\right)^n dx.$$

We may assume that $a>0$. Integrating we obtain:

$$\left(\frac{a-x}{a}\right)^n\Bigg]_0^k = \frac{1}{2}\left(\frac{a-x}{a}\right)^n\Bigg]_0^a \;\Rightarrow\; \left(\frac{a-k}{a}\right)^n - 1 = -\frac{1}{2}.$$

And, therefore $k = a\left(1-\frac{1}{\sqrt[n+1]{2}}\right)$ if n is even, while $k = a\left(1 \pm \frac{1}{\sqrt[n+1]{2}}\right)$ if n is odd.

Comment by David Stone and John Hawkins of Statesboro, GA. When a and b are positive, we have the usual area interpretation of our result. The problem asks us to determine how far along we should move to capture half of the area from 0 to a.

In this case, the graph of $y = \left(\frac{-b}{a}\right)(x-a)^n$ has $y-$intercept $(0,b)$, and drops off to its $x-$intercept $(a,0)$, so the integral $\int_0^a \left(-\frac{b}{a}x+b\right)^n dx$ actually represents the area under the curve.

For n even, the graph bottoms out at $(a, 0)$ and stays above the x-axis and we see that $k = \left(1 - \frac{1}{\sqrt[n+1]{2}}\right) a$ is the "magical" spot where we halve the area.

For n odd, the graph slices through the $x-$axis at $(a, 0)$ and is symmetric about this $x-$intercept, so we have two values of k. The first $k_1 = a - \frac{a}{\sqrt[n+1]{2}}$, actually marks the spot where half of the area from 0 to a is achieved, while the second $k_2 = a + \frac{a}{\sqrt[n+1]{a}}$ marks the spot where the net area once again equals half of the area from 0 to a.

Note that the geometrical interpretation is more complicated when a and/or b is negative, but the same values of k provide the correct area interpretation.

Also solved by Daniel Lopez Aguayo, Institute of Mathematics (UNAM) Morelia, Mexico; Brian D. Beasley, Clinton, SC; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Neculai Stanciu, Buzău, Romania; David Stone and John Hawkins (joinlty), Statesboro, GA and the proposer.

- **5187:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let $f : [0, 1] \to (0, \infty)$ be a continuous function. Find the value of

$$\lim_{n\to\infty} \left(\frac{\sqrt[n]{f(\frac{1}{n})} + \sqrt[n]{f(\frac{2}{n})} + \cdots + \sqrt[n]{f(\frac{n}{n})}}{n} \right)^n.$$

Solution by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

The answer is $e^{\int_0^1 \ln(f(x))dx}$

Proof.

$$\sqrt[n]{f(\frac{k}{n})} = \exp\{\frac{1}{n}\ln[f(\frac{k}{n})]\} = 1 + \frac{1}{n}\ln\left(f(\frac{k}{n})\right) + O(\frac{1}{n^2}).$$

We observe that being the function continuous, it is bounded above and below and the lower bound is positive by the positivity of the function namely $0 < m \le f(x) \le M$ for any $x \in [a, b]$. This allowed us to write $O(1/n^2)$ in the last term regardless the presence of the function $f(x)$. Thus,

$$\frac{\sqrt[n]{f(\frac{1}{n})} + \sqrt[n]{f(\frac{2}{n})} \dots \sqrt[n]{f(\frac{k}{n})}}{n} = 1 + \frac{1}{n^2}\sum_{k=1}^{n} \ln(f(\frac{k}{n})) + O(\frac{1}{n^2}) \doteq 1 + \frac{p_n}{n} + O(\frac{1}{n^2})$$

and (use $\ln(1 + x) = x + O(x^2)$).

$$(1 + \frac{p_n}{n} + O(\frac{1}{n^2}))^n = \exp\{n\ln(1 + \frac{p_n}{n} + O(\frac{1}{n^2})\} = \exp\{p_n + O(\frac{p_n}{n}) + O(\frac{1}{n^2}) + O(\frac{p_n^2}{n^2})\}.$$

The quantity p_n is clearly the Riemann sum of $\int_0^1 \ln(f(x))dx$ and then the last exponential is bounded below and above since $\ln(m) \leq \int_0^1 \ln(f(x))dx \leq \ln(M)$. We obtain

$$\exp\{p_n + O(\frac{p_n}{n}) + O(\frac{1}{n^2}) + O(\frac{p_n^2}{n^2})\} \to \ e^{\int_0^1 \ln(f(x))dx}.$$

Also solved by Kee-Wai Lau, Hong Kong, China; Neculai Stanciu, Buzău, Romania; Albert Stadler, Herrilberg, Switzerland, and the proposer.

Addendum

The name of **Brian D. Beasley of Clinton, SC** was inadvertently left off the list of having solved problem 5176. Sorry Brian, mea culpa.

Also, **Albert Stadler of Herrilberg, Switzerland** noticed two typos in the February, 2012 issue of the column. In his solution to 5176, the fourth line in the first equation array lists the term y^3, but it should be y^2. And in his solution to 5178, the last line should have been $x^2 + y^2 + z^2$ and not $x^3 + y^3 + z^3$. Again, sorry, mea culpa.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2012

- **5206:** *Proposed by Kenneth Korbin, New York, NY*

  The distances from the vertices of an equilateral triangle to an interior point P are $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ respectively, where $a, b,$ and c are positive integers.

  Find the minimum and the maximum possible values of the sum $a+b+c$ if the side of the triangle is 13.

- **5207:** *Proposed by Roger Izard, Dallas, TX*

  Consider the following four algebraic terms:

$$\begin{aligned} T_1 &= a^2(b+c)+b^2(a+c)+c^2(a+b) \\ T_2 &= (a+b)(a+c)(b+c) \\ T_3 &= abc \\ T_4 &= \frac{b+c-a}{a}+\frac{a+c-b}{b}+\frac{a+b-c}{c} \end{aligned}$$

  Suppose that $\dfrac{T_1 \cdot T_2}{(T_3)^2}=\dfrac{616}{9}$. What values would then be possible for T_4?

- **5208:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania*

  Let the sequence of positive real numbers $\{a_n\}_{n\geq 1}$, $N \in Z^+$ be such that $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2\cdot a_n}=b$. Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right).$$

- **5209:** *Proposed by Tom Moore, Bridgewater, MA*

  We noticed that 27 is a cube and 28 is an even perfect number. Find all pairs of consecutive integers such that one is cube and the other is an even perfect number.

- **5210:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be four positive real numbers. Prove that

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>\frac{2\sqrt{3}}{3}.$$

- **5211:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let $n \geq 1$ be a natural number and let

$$f_n(x) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}},$$

  where the number of x's in the definition of f_n is n. For example

$$f_1(x)=x, \quad f_2(x)=x^x, \quad f_3(x)=x^{x^x}, \ldots.$$

  Calculate the limit

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}.$$

Solutions

- **5188:** *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with coordinates $A(-5,0), B(0,12)$ and $C(9,0)$. The triangle has an interior point P such that $\angle APB = \angle APC = 120°$. Find the coordinates of point P.

  Solution 1 by Ercole Suppa, Teramo, Italy

  Let us construct equilateral triangles $\triangle ABD$, $\triangle AEC$ externally on the sides AB, AC of triangle $\triangle BAC$ and denote by ω_1, ω_2 the circumcircles of $\triangle ABD$, $\triangle AEC$. The point P is the intersection point of ω_1, ω_2 different from O. In order to find the coordinates of D, E we use complex numbers. If we denote respectively by $a=-5$, $b=12i$, $c=9$ the affixes of A, B, C we get:

$$\begin{aligned} d &= a+(b-a)e^{\frac{\pi}{3}i} = \frac{-5-12\sqrt{3}}{2}+\frac{12+5\sqrt{3}}{2}i \\ e &= a+(c-a)e^{\frac{\pi}{3}i} = 2-7\sqrt{3}i \end{aligned}$$

  so the coordinates of D, E are $D\left(\dfrac{-5-12\sqrt{3}}{2}, \dfrac{12+5\sqrt{3}}{2}\right)$ and $E\left(2,-7\sqrt{3}\right)$.

  The equations of ω_1, ω_2 are:

$$\begin{aligned} \omega_1 &: \quad 169\sqrt{3}x^2+169\sqrt{3}y^2+\left(2028+845\sqrt{3}\right)x+\left(-845-2028\sqrt{3}\right)y+10140=0 \\ \omega_2 &: \quad 196\sqrt{3}x^2+196\sqrt{3}y^2-784\sqrt{3}x+2744y-8820\sqrt{3}=0 \end{aligned}$$

and, after some calculations, we obtain

$$P=\omega_1\cap\omega_2=\left(-\frac{2\left(-981+112\sqrt{3}\right)}{2353},-\frac{21\left(-896+263\sqrt{3}\right)}{2353}\right).$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The isogonic center P is the point from which we see the sides of ABC under equal angles (that is 120° mod 180°). The isogonic center is the common intersection point of the three circumcircles to the equilateral triangles constructed on the sides of $\triangle ABC$ (Napoleon's theorem).

Let $\triangle AC'B$, $\triangle BA'C$, $\triangle ACB'$ be equilateral triangles constructed on the outside on the edges of $\triangle ABC$, then AA', BB', CC' intersect in the isogonic center P. Of course, it is enough to find two of these lines. In our case,

$$\begin{aligned}B' &= -5+(14,0)\cdot 1_{-\pi/3}=2-7\sqrt{3}i\\ C' &= -5+(5,12)\cdot 1_{\pi/3}=-5/2-6\sqrt{3}+(6+5\sqrt{3}/2)i.\end{aligned}$$

The intersection of lines BB' and CC' gives the solution

$$P=\left(\frac{6(56+33\sqrt{3})}{504+295\sqrt{3}},\frac{21(93+56\sqrt{3})}{504+295\sqrt{3}}\right)=\left(\frac{1962-224\sqrt{3}}{2353},\frac{18816-5523\sqrt{3}}{2353}\right).$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Clearly $AC=14$, and $AB=13$. Consider the circumscribed circle of $\triangle APC$. If we denote its center by O_1 and the midpoint of AC by $M(2,0)$, then, since an inscribed angle is measured by one half of its intercepted arc, and a radius perpendicular to a chord bisects the chord and its arc, it readily follows that $\triangle AO_1M$ is a 30, 60, 90 degree right triangle and so since $MA=7$, the radius is $14\frac{\sqrt{3}}{3}$, $O_1M=7\frac{\sqrt{3}}{3}$, and P lies on the circle

$$(x-2)^2+\left(y+\frac{7}{3}\sqrt{3}\right)^2=\frac{196}{3}. \qquad (1)$$

(Note that since the segment of this circle having minor arc AC contains P, the center O_1 is below AC.)

Similarly, consider the circumscribed circle of $\triangle APB$. If we denote its center by O_2 and the midpoint of AB by $N\left(-\frac{5}{2},6\right)$, then $\triangle AO_2N$ is a 30, 60, 90 degree right triangle and so since $NA=\frac{13}{2}$ the radius is $13\frac{\sqrt{3}}{3}$ and $O_2N=13\frac{\sqrt{3}}{6}$.

The perpendicular bisector of AB is

$$y-6=-\frac{5}{12}\cdot\left(x+\frac{5}{2}\right)\Longrightarrow y=-\frac{5}{12}x+\frac{119}{24}. \qquad (2)$$

If (X,Y) are the coordinates of O_2 we have

$$\left(X+\frac{5}{2}\right)^2+\left(-\frac{5}{12}X+\frac{119}{24}-6\right)^2=\left(13\frac{\sqrt{3}}{6}\right)^2,$$

and thus, $X = -\frac{5}{2} \pm 2\sqrt{3}$. We choose $X = -\frac{5}{2} - 2\sqrt{3}$ since O_2 is to the left of AB, and from (2), $Y = 6 + \frac{5}{6}\sqrt{3}$.

Thus P also lies on the circle

$$\left(x + \frac{5}{2} + 2\sqrt{3}\right)^2 + \left(y - 6 - \frac{5}{6}\sqrt{3}\right)^2 = \frac{169}{3}. \tag{3}$$

If we subtract equation (3) from equation (1) we obtain the line

$$-9x - 36 + \frac{19}{3}y\sqrt{3} - 4x\sqrt{3} - 20\sqrt{3} + 12y = 9 \tag{4}$$

or equivalently,

$$y = -\frac{3}{71}\left(-32 + 9\sqrt{3}\right)(x + 5). \tag{5}$$

This line just found passes through the points of intersections of these two circles and thus P is that point that is interior to $\triangle ABC$. Substituting (1) into (5), solving the resulting quadratic equation and rejecting $x = -5$, gives the (x, y) coordinates of P.

$$(x, y) = \left(\frac{1962}{2353} - \frac{224}{2353}\sqrt{3},\ \frac{18816}{2353} - \frac{5523}{2353}\sqrt{3}\right).$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky (two solutions), Central Islip, NY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5189:** *Proposed by Kenneth Korbin, New York, NY*

Given triangle ABC with integer length sides and with $\angle A = 60^\circ$ and with $(a, b, c) = 1$. Find the lengths of b and c if

$$\begin{aligned} i)\ a &= 13, \text{ and if} \\ ii)\ a &= 13^2 = 169, \text{ and if} \\ iii)\ a &= 13^4 = 28561. \end{aligned}$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By the cosine formula formula, we have $a^2 = b^2 + c^2 - bc$ so that $c = \dfrac{b \pm \sqrt{4a^2 - 3b^2}}{2}$ with $1 \leq b < \dfrac{2\sqrt{3}a}{3}$. A computer search yields the following solutions with $(a, b, c) = 1$:

$$i)\ (b, c) = (7, 15), (8, 15), (15, 7), (15, 8).$$

$$ii)\ (b,c) \;=\; (15,176),(161,176),(176,15),(176,161).$$

$$iii)\ (b,c) \;=\; (5055,30751),(25696,30751),(30751,5055),(30751,25696).$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

(*Editor:* Albert gave two solutions to this problem; in the first solution he used the above computer-aided approach. But in his second solution he used the complex roots of unity.)

A more inspired approach is based on Eisenstein integers (see e.g., http : //en.wikipedia.org/wiki/Eisenstein integer).

Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$. The set of Eisenstein integers $Z[\omega] = \{a + b\omega | a, b \in Z\}$ has the following properties:

- (i) $Z[\omega]$ forms a commutative ring of algebraic integers in the real number field $Q(\omega)$
- (ii) $Z[\omega]$ is an Euclidean domain whose norm N is given by $N(a + b\omega) = a^2 - ab + b^2$. As a result of this $Z[\omega]$ is a factorial ring.
- (iii) The group of units in $Z[\omega]$ is the cyclic group formed by the sixth root of unity in the complex plane. Specifically, they are $\{\pm1, \pm\omega, \pm\omega^2\}$. These are just the Eisenstein integers of norm one.
- (iv) An ordinary prime number (or rational prime) which is 3 or congruent to 1 mod 3 is of the form $x^2 - xy + y^2$ for some integers x, y and may therefore be factored into $(x + y\omega)(x + y\omega^2)$ and because of that it is not prime in the Eisenstein integers. Ordinary primes congruent to 2 mod 3 cannot be factored in this way and they are primes in the Eisenstein integers as well.

Based on the above we find the factorization $13 = (4 + \omega)(4 + \omega^2)$, where $4 + \omega$ and $4 + \omega^2$ are two Eisenstein primes that are conjugate to each other . So $13^n = (4 + \omega)^n(4 + \omega^2)^n$, and this is (up to units) the only factorization into two factors of the form $b + c\omega$ with b and c coprime. We find

$$(4+\omega)^2 = 16 + 8\omega + \omega^2 = 15 + 7\omega,$$

$$(4+\omega)^2(-\omega^2) = -15\omega^2 - 7 = 8 + 15\omega,$$

$$(4+\omega)^4 = (15+7\omega)^2 = 225 + 210\omega + 49\omega^2 = 176 + 161\omega,$$

$$(4+\omega)^4(-\omega^2) = (176+161\omega)(-\omega^2) = -176\omega^2 - 161 = 15 + 176\omega,$$

$$(4+\omega)^8 = (176+161\omega)^2 = 30976 + 56672\omega + 25921\omega^2 = 5055 + 30751\omega$$

$$(4+\omega)^8(-\omega) = (5055+30751\omega)(-\omega) = -5055\omega - 30751\omega^2 = 30751 + 25696\omega.$$

We note that

$$
\begin{aligned}
N((4+\omega)^2) &= N((4+\omega)^2(-\omega^2)) = 13^2 \\
N((4+\omega)^4) &= N((4+\omega)^4(-\omega^2)) = 13^4 \\
N((4+\omega)^8) &= N((4+\omega)^8(-\omega^2)) = 13^8 \\
N(x+y\omega) &= x^2 - xy + y^2,
\end{aligned}
$$

and we get the same solutions as with the exhaustive computer search.

Also solved by Brian D. Beasley, Clinton, SC; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo TX; Bruno Salgueiro Fanego, Viveiro Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5190:** *Proposed by Neculai Stanciu, Buzău, Romania*

Prove: If x, y and z are positive integers such that $\dfrac{x(y+1)}{x-1} \in \mathrm{N}$, $\dfrac{y(z+1)}{y-1} \in \mathrm{N}$, and $\dfrac{z(x+1)}{z-1} \in \mathrm{N}$, then $xyz \leq 693$.

Solution by Kee-Wai Lau, Hong Kong, China

Since two consecutive positive integers are relatively prime, so in fact

$$y+1 = a(x-1),\ z+1 = b(y-1),\ x+1 = c(z-1), \tag{1}$$

where $a, b, c \in N$ and $abc < 1$. Solving (1) for x, y, z we obtain

$$x = \frac{1+2c+2bc+abc}{abc-1},\ y = \frac{1+2a+2ac+abc}{abc-1},\ \frac{1+2b+2ab+abc}{abc-1}. \tag{2}$$

Also, we have from (1) that

$$abc = \left(\frac{x+1}{x-1}\right)\left(\frac{y+1}{y-1}\right)\left(\frac{z+1}{z-1}\right) \leq (3)(3)(3) = 27. \tag{3}$$

Using (3), we check with a computer that x, y, and z of (2) are positive integers if and only if

$$
\begin{aligned}
(a,b,c) \;=\; & (1,1,2),(1,1,3),(1,1,4),(1,1,7),(1,2,1),(1,2,3),(1,3,1),(1,3,5)(1,4,1), \\
& (1,7,1),(2,1,1),(2,2,2,)(2,3,1),(3,1,1),(3,1,2),(3,3,3),(3,5,1),(4,1,1), \\
& (5,1,3),(7,1,1).
\end{aligned}
$$

Correspondingly,

$$
\begin{aligned}
(x,y,z) \;=\; & (11,9,7),(8,6,4),(7,5,3),(6,4,2),(9,7,11),(5,3,3),(6,4,8),(4,2,2),(5,3,7), \\
& (4,2,6),(7,11,9),(3,3,3),(3,3,5),(4,8,6),(3,5,3),(2,2,2),(2,2,4),(3,7,5), \\
& (2,4,2),(2,6,4).
\end{aligned}
$$

Hence, $xyz \leq 693$ as desired.

Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Albert Stadler, Herrilberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5191:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \leq 1.$$

Solution 1 by Paul M. Harms, North Newton, KS

From inequalities of the type $\sqrt{bc} \leq \dfrac{b+c}{2}$ we see that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \;\leq\; a\frac{b+c}{2} + b\frac{c+a}{2} + c\frac{a+b}{2} \;=\; ab + bc + ca.$$

If we can show that the last expression is less than or equal to $(a^4 + b^4 + c^4)$, then the inequality of the problem is correct.

From the condition in the problem $ab + bc + ca = 3$, so we will prove that $\dfrac{3}{(a^4 + b^4 + c^4)} \leq 1.$
Let $a \leq b \leq c$ with $b = ta$ and $c = sa$ where $1 \leq t \leq s$. Then

$$3 \leq a^4 + b^4 + c^4 \;=\; \left(a^2\right)^2 \left(1 + t^4 + s^4\right).$$

$$ab + bc + ca \;=\; 3 \Longrightarrow a^2 = \frac{3}{t + ts + s}.\text{ We must prove that}$$

$$\begin{aligned} 3 &\leq \left[\frac{3}{t + ts + s}\right]^2 \left(1 + t^4 + s^4\right) \text{ or equivalently,} \\ (t + ts + s)^2 &= t^2 + 2st^2 + s^2t^2 + 2st + 2s^2t + s^2 \leq 3\left(1 + t^4 + s^4\right) \text{ for } 1 \leq t \leq s, \text{ i.e.,} \\ 0 &\leq 3t^4 + 3s^4 - t^2 - s^2 - 2t^2s - 2s^2t - s^{t2} - 2st + 3 \text{ for } 1 \leq \; t \leq s. \end{aligned}$$

Let $f(t, s)$ be the right side of the last inequality. We use partial derivatives to help find the minimum of the function in the appropriate domain.

Subtracting the equations $f_t(t, s) = 0$ and $f_s(t, s) = 0$ we obtain:

$$12(t^3 - s^3) - 2(t - s) - 2(s^2 - t^2) - 2st(s - t) - 2(s - t) = 0 = 2(t - s)\left[6t^2 + 7st + 6s^2 + s + t\right].$$

The part in the brackets is clearly positive for $1 \leq t \leq s$ so we must check $t = s$ and other boundary points of the domain for a minimum of the function.

When $t = s$,

$$f(t,s) = f(s,s) = 5s^4 - 4s^3 - 4s^2 + 3 = (s-1)^2\left[5s^2 + 6s + 3\right].$$

The function has a minimum in this case for $t = s = 1$. For the boundary $t = 1$ with t=1$\leq s$,

$$f(1,s) = (s-1)^2\left[3s^2 + 6s + 5\right]$$

which again has a minimum for $t = s = 1$. Since $f(s,t) \geq f(1,1) = 0$ for $1 \leq t \leq s$, the inequality of the problem has been proved.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Using the well-known result $x^2 + y^2 \geq 2xy$, with equality if and only if $x = y$, we obtain

$$\begin{aligned} xy + yz + zx &\leq \frac{1}{2}\left[\left(x^2+y^2\right) + \left(y^2+z^2\right) + \left(z^2+x^2\right)\right] \\ &= x^2 + y^2 + z^2, \qquad (1) \end{aligned}$$

and consequently,

$$\begin{aligned} (x+y+z)^2 &= x^2 + y^2 + z^2 + 2(xy + yz + zx) \\ &= \leq 3\left(x^2 + y^2 + z^2\right). \qquad (2) \end{aligned}$$

Further, equality is attained in (1) or (2) if and only if $x = y = z$.

By (1),

$$\begin{aligned} a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} &= \sqrt{ab}\sqrt{ca} + \sqrt{bc}\sqrt{ab} + \sqrt{ca}\sqrt{bc} \\ &\leq ab + bc + ca \\ &= 3, \qquad (3) \end{aligned}$$

with equality if and only if $\sqrt{ab} = \sqrt{bc} = \sqrt{ca}$, i.e., if and only if $a = b = c = 1$.

Also, since $a, b, c > 0$, (1) and (2) imply that

$$\begin{aligned} 9 &= (ab + bc + ca)^2 \\ &\leq \left(a^2 + b^2 + c^2\right)^2 \\ &\leq 3\left(a^4 + b^4 + c^4\right) \end{aligned}$$

and hence,

$$a^4 + b^4 + c^4 \geq 3. \qquad (4)$$

Once again, equality is achieved in (4) if and only if $a = b = c = 1$.

Therefore, by (3) and (4),

$$\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4} \le \frac{3}{3} = 1,$$

with equality if and only if $a=b=c=1$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

The homogeneous form of this inequality reads as

$$\left(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\right)(ab+bc+ca) \le 3\left(a^4+b^4+c^4\right) \text{ or equivalently as}$$

$$a^2b^{\frac{3}{2}}c^{\frac{1}{2}}+a^{\frac{3}{2}}b^2c^{\frac{1}{2}}+a^{\frac{3}{2}}b^{\frac{3}{2}}c+ab^{\frac{3}{2}}c^{\frac{3}{2}}+a^{\frac{1}{2}}b^2c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{3}{2}}c^2+a^2b^{\frac{1}{2}}c^{\frac{3}{2}}+a^{\frac{3}{2}}bc^{\frac{3}{2}}+a^{\frac{3}{2}}b^{\frac{1}{2}}c^2 \le 3\left(a^4+b^4+c^4\right).$$

By the weighted AM-GM inequality

$$a^{4r}b^{4s}c^{4t} \le ra^4+sb^4+tc^4 \qquad (1)$$

for all tuples (r,s,t) of positive real numbers $r, s,$ and t such that $r+s+t=1$. We write down the nine inequalities that result from (1) by choosing:

$$\begin{aligned}(r,s,t) &= \left(\frac{1}{2},\frac{3}{8},\frac{1}{8}\right), \left(\frac{3}{8},\frac{1}{2},\frac{1}{8}\right), \left(\frac{3}{8},\frac{3}{8},\frac{1}{4}\right),\\ &\left(\frac{1}{4},\frac{3}{8},\frac{3}{8}\right), \left(\frac{1}{8},\frac{1}{2},\frac{3}{8}\right), \left(\frac{1}{8},\frac{3}{8},\frac{1}{2}\right),\\ &\left(\frac{1}{2},\frac{1}{8},\frac{3}{8}\right), \left(\frac{3}{8},\frac{1}{4},\frac{3}{8}\right), \left(\frac{3}{8},\frac{1}{8},\frac{1}{2}\right).\end{aligned}$$

and add them up. The result follows.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ercole Suppa, Teramo, Italy; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5192:** *Proposed by G. C. Greubel, Newport News, VA*

Let $[n]=[n]_q=\dfrac{1-q^n}{1-q}$ be a q number and $\ln_q(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{[n]}$ be a q-logarithm. Evaluate the following series:

$$i) \qquad \sum_{k=0}^{\infty}\frac{q^{mk}}{[mk+1][mk+m+1]} \text{ and}$$

$$ii) \qquad \sum_{k=1}^{\infty}\frac{x^k}{[k][k+m]}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For $0<|q|<1$ and for $0<|x|<1$, we have

i)

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{q^{mk}}{[mk+1][mk+m+1]} \\
= & \frac{1}{q[m]} \sum_{k=0}^{\infty} \left(\frac{1}{[mk+1]} - \frac{1}{[mk+m+1]} \right) \\
= & \frac{1}{q[m]} \left(1 - (1-q)\right) \\
= & \frac{1}{[m]}.
\end{aligned}
$$

and ii)

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]} \\
= & \frac{1}{[m]} \sum_{k=1}^{\infty} \left(\frac{x^k}{[k]} - \frac{q^m x^k}{[k+m]} \right) \\
= & \frac{1}{[m]} \left(\sum_{k=1}^{\infty} \frac{x^k}{[k]} - \left(\frac{q}{x}\right)^m \sum_{k=1}^{\infty} \frac{x^{k+m}}{[k+m]} \right) \\
= & \frac{1}{[m]} \left(\ln_q(x) - \left(\frac{q}{x}\right)^m \ln_q(x) + \left(\frac{q}{x}\right)^m \sum_{k=1}^{m} \frac{x^k}{[k]} \right).
\end{aligned}
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

i)

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{q^{mk}}{[mk+1]\,[mk+m+1]} &= (1-q)^2 \sum_{k=0}^{\infty} \frac{q^{mk}}{(1-q^{mk+1})\,(1-q^{mk+m+1})} \\
&= (1-q)^2 \sum_{k=0}^{\infty} q^{mk} \left(\frac{1}{1-q^{mk+1}} - \frac{1}{1-q^{mk+m+1}} \right) \frac{1}{q^{mk+1} - q^{mk+m+1}} \\
&= \frac{(1-q)^2}{q-q^{m+1}} \sum_{k=0}^{\infty} \left(\frac{1}{1-q^{mk+1}} - \frac{1}{1-q^{mk+m+1}} \right) \\
&= \frac{(1-q)^2}{q-q^{m+1}} \sum_{k=0}^{\infty} \left(\frac{1}{1-q^{mk+1}} - \frac{1}{1-q^{m(k+1)+1}} \right)
\end{aligned}
$$

$$= \frac{(1-q)^2}{q-q^{m+1}} \cdot \left(\frac{(1)}{1-q} - 1\right)$$

$$= \frac{(1-q)}{1-q^m}$$

ii)

$$\sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]} = (1-q)^2 \sum_{k=1}^{\infty} \frac{x^k}{(1-q^k)(1-q^{k+m})}$$

$$= (1-q)^2 \sum_{k=1}^{\infty} x^k \left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+m}}\right) \frac{1}{q^k - q^{k+m}}$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left(\frac{x}{q}\right)^k \left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+m}}\right)$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left[x^k \left(\frac{1}{q^k} + \frac{1}{1-q^k}\right) - x^k q^m \left(\frac{1}{q^{k+m}} + \frac{1}{1-q^{k+m}}\right)\right]$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left[x^k \left(\frac{1}{1-q^k}\right) - x^k q^m \left(\frac{1}{1-q^{k+m}}\right)\right]$$

$$= \frac{(1-q)}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1-q)^2 q^m}{(1-q^m)x^m} \sum_{k=1}^{\infty} x^{k+m} \left(\frac{1}{1-q^{k+m}}\right)$$

$$= \frac{(1-q)}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1-q)^2 q^m}{(1-q^m)x^m} \left[-\frac{x}{1-q} - \frac{x^2}{1-q^2} - \cdots - \frac{x^m}{1-q^m} + \sum_{k=1}^{\infty} x^k \left(\frac{1}{1-q^k}\right)\right]$$

$$= \frac{1-q}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} + \frac{(1-q)^2 q^m}{(1-q^m)x^m} \left[\frac{x}{1-q} + \frac{x^2}{1-q^2} + \cdots + \frac{x^m}{1-q^m}\right] - \frac{(1-q)q^m}{(1-q^m)x^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]}$$

$$= \frac{1-q}{1-q^m} \left(1 - \left(\frac{q}{x}\right)^m\right) \ln_q(x) + \frac{(1-q)q^m}{(1-q^m)} \left[\frac{x^{1-m}}{1} + \frac{x^{2-m}}{1+q} + \cdots + \frac{1}{1+q+q^2+\cdots q^{m-1}}\right].$$

Also solved by Arkady Alt, San Jose, CA, and the proposer.

- **5193:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let f be a function which has a power series expansion at 0 with radius of convergence R.

  a) Prove that $\displaystyle\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} t f'(t)dt, \quad |x| < R.$

b) Let α be a non-zero real number. Calculate $\displaystyle\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right)$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

a) Let $S(x)$ be the sum of the series. Then, by differentiation, and for $|x| < R$,

$$S'(x) = \sum_{n=1}^{\infty} nf^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^{n-1}}{(n-1)!}\right) = S(x) + \sum_{n=1}^{\infty} nf^{(n)}(0) \cdot \frac{x^n}{n!}.$$

It follows that $S'(x) = S(x) + xf'(x)$, and hence

$$S(x) = \int_0^x e^{x-t} tf'(t)dt + Ce^x,$$

where C is a constant of integration. Because $S(0) = 0$, we have $C = 0$ and

$$S(x) = \int_0^x e^{x-t} tf'(t)dt.$$

b) Note that if $f(x) = e^{\alpha x}$ then $f^{(n)}(0) = \alpha^n$, for $n \geq 1$. Hence, by part a), the sum of the given series is $\displaystyle\int_0^x e^{x-t} te^t dt = \frac{x^2 e^2}{2}$ if $\alpha = 1$. If $\alpha \neq 1$, the sum of the series is $\displaystyle\int_0^x e^{x-t} t\alpha e^{\alpha t} dt = \frac{\alpha x e^{\alpha x}}{\alpha - 1} + \frac{\alpha(e^x - e^{\alpha x})}{(\alpha - 1)^2}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

a) From the problem's assumptions we have that

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{and} \quad f'(x) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} x^{n-1} \quad \text{for} \quad |x| < R,$$

so, for $|x| < R$ we obtain

$$\begin{aligned}
\int_0^x e^{x-t} tf'(t)\, dt &= \int_0^x e^{x-t} \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} t^n \, dt \\
&= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \int_0^x t^n e^{-t}\, dt \\
&= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} I_n. \qquad (1)
\end{aligned}$$

Now $I_n = -\displaystyle\int_0^x t^n (e^{-t})'\, dt = -x^n e^{-x} + nI_{n-1}$, so it is easily verified by induction that

$$I_n = -e^{-x}\left(x^n + nx^{n-1} + \cdots + n! x^0\right) + n!$$

With the above, (1) will give

$$
\begin{aligned}
\int_0^x e^{x-t} t f'(t)\,dt &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!}\left(-e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!\right) \\
&= \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!}\left(n!e^x - x^n - nx^{n-1} - \cdots - n!x^0\right) \\
&= \sum_{n=1}^{+\infty} n f^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right).
\end{aligned}
$$

2) From (1) with $f(x) = e^{\alpha x}$ we obtained that

$$
\begin{aligned}
\sum_{n=1}^{+\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right) &= \int_0^x e^{x-t} \alpha t e^{\alpha t}\,dt \\
&= I_\alpha.
\end{aligned}
$$

So,

$$
\begin{cases}
\int_0^x e^{x-t} t e^t \; dt = \dfrac{x^2 e^x}{2}, & \text{for } \alpha = 1 \\
I_\alpha = \alpha e^x \left(\int_0^x t\left(\frac{e^{(\alpha-1)t}}{\alpha-1}\right)\; dt\right), & \text{for} \alpha \neq 1 \\
\quad = \frac{\alpha e^{\alpha x}}{\alpha-1}\left(x - \frac{1}{\alpha-1}\right) + \frac{\alpha e^x}{(\alpha-1)^2}. &
\end{cases}
$$

Solution 3 by Arkady Alt, San Jose, CA

a) Let

$a_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}, n \in N \cup \{0\}$ and $F(x) = \sum\limits_{n=1}^{\infty} n f^{(n)}(0)\, a_n(x)$.

Noting that

$$
\begin{aligned}
a_n'(x) &= e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^{n-1}}{(n-1)!} \\
&= a_{n-1}(x), n \in N
\end{aligned}
$$

we obtain

$$
\begin{aligned}
F'(x) &= \left(\sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n(x)\right)' \\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n'(x) \\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_{n-1}(x).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathrm{F}(x)-F'(x) &= \sum_{n=1}^{\infty} n f^{(n)}(0)\left(a_n(x)-a_{n-1}(x)\right)\\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\left(-\frac{x^n}{n!}\right)\\
&= -\sum_{n=1}^{\infty} f^{(n)}(0)\frac{x^n}{(n-1)!}\\
&= -x\sum_{n=0}^{\infty} f^{(n+1)}(0)\frac{x^n}{n!}\\
&= -xf'(x).
\end{aligned}$$

Multiplying equation $F'(x)-F(x)=xf'(x)$ by e^{-x} we obtain

$$\begin{aligned}
F'(x)e^{-x}-F(x)e^{-x} &= e^{-x}xf'(x) \iff \left(F(x)e^{-x}\right)'\\
&= e^{-x}xf'(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
F(x)e^{-x} &= \int_o^x e^{-t}tf'(t)\,dt\\
&\iff F(x)=\int_o^x e^{x-t}tf'(t)\,dt.
\end{aligned}$$

b) Let $f(x)=e^{\alpha x}$ then $f^{(n)}(0)=\alpha^n$ and, using the result we obtained in part (a) we get,

$$\begin{aligned}
\sum_{n=1}^{\infty} n\alpha^n\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\ldots-\frac{x^n}{n!}\right) &= \int_0^x e^{x-t}t\alpha e^{\alpha t}dt\\
&= \alpha e^x\int_0^x te^{t(\alpha-1)}dt.
\end{aligned}$$

If $\alpha=1$ then $\int_0^x te^{t(\alpha-1)}dt=\frac{x^2}{2}$ and, therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} n\alpha^n\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\ldots-\frac{x^n}{n!}\right) &= \sum_{n=1}^{\infty} n\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\ldots-\frac{x^n}{n!}\right)\\
&= \frac{\alpha e^x x^2}{2}.
\end{aligned}$$

If $\alpha\neq 1$ then

$$\int_0^x te^{t(\alpha-1)}dt = \frac{xe^{(\alpha-1)x}}{\alpha-1}-\frac{e^{(\alpha-1)x}}{(\alpha-1)^2}$$

$$= \frac{e^{(\alpha-1)x}\left(x\left(\alpha-1\right)-1\right)}{\left(\alpha-1\right)^2}.$$

Hence,

$$\sum_{n=1}^{\infty} n\alpha^n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) = \frac{\alpha e^{\alpha x}\left(x\left(\alpha-1\right)-1\right)}{\left(\alpha-1\right)^2}.$$

Solution 4 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

a) We need the two lemmas:

Lemma 1 $m!n! \le (n+m)!$

Proof by Induction. Let m be fixed. If $n=0$ evidently holds true. Let's suppose that the statement is true for any $1 \le n \le r$. For $n = r+1$ we have

$$m!(r+1)! = m!r!(r+1) \le (m+r)!(r+1) \le (m+r)!(m+r+1) = (m+r+1)!$$

which clearly holds for any $m \ge 0$. Since the inequality is symmetric, the induction on m proceeds along the same lines. q.e.d.

Lemma 2 The power series

$$\sum_{n=1}^{\infty} nf^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} nf^{(n)}(0)\sum_{k=n+1}^{\infty}\frac{x^k}{k!}$$

converges for $|x| < R$ and is differentiable.

Proof:

$$\sum_{k=n+1}^{\infty}\frac{x^k}{k!} = \frac{x^{n+1}}{(n+1)!}\sum_{k=n+1}^{\infty} x^{k-n-1}\frac{(n+1)!}{k!}.$$

By using the Lemma 1 we can bound

$$\sum_{k=n+1}^{\infty}|x|^{k-n-1}\frac{(n+1)!}{k!} \le \sum_{k=n+1}^{\infty}\frac{|x|^{k-n-1}}{(k-n-1)!} = \sum_{k=0}^{\infty}\frac{|x|^k}{k!} = e^{|x|} \le e^R.$$

Thus we can write

$$\sum_{n=0}^{\infty} n|f^{(n)}(0)|\sum_{k=n+1}^{\infty}\frac{|x|^k}{k!} \le e^R|x|\sum_{n=0}^{\infty} n|f^{(n)}(0)|\frac{|x|^n}{n!}\frac{n!}{(n+1)!}$$

Since

$$\limsup_{n\to\infty}\left|\frac{f^{(n)}(0)}{n!}\right|^{1/n} = R^{-1} \implies \limsup_{n\to\infty}\left|\frac{f^{(n)}(0)}{n!}\frac{n}{n+1}\right|^{1/n} = R^{-1}$$

the series

$$\sum_{n=1}^{\infty} nf^{(n)}(0)\sum_{k=n+1}^{\infty}\frac{x^k}{k!}$$

converges for any $|x| < R$. Its differentiability is a consequence of the standard theory on power–series so we don't write it here. q.e.d.

The function $\int_0^x e^{x-t} t f'(t) dt$ is also differentiable by the fundamental theorem of calculus and the derivative yields

$$\left(\int_0^x e^{x-t} t f'(t) dt \right)' = x f'(x) + \int_0^x e^{x-t} t f'(t) dt$$

namely it satisfies the ordinary differential equation $Q'(x) = Q(x) + x f'(x)$, $Q(0) = 0$.

The derivative of the series in question a) is

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^{n-1}}{(n-1)!} \right)$$

that is

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} + \sum_{n=1}^{\infty} n f^{(n)}(0) \frac{x^n}{n!}$$

which is in turn equals

$$= \sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} + x f'(x)$$

Moreover $\left(\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right) \Big|_{x=0} = 0$ thus the functions $\int_0^x e^{x-t} t f'(t) dt$ and $\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \ldots - \frac{x^n}{n!} \right)$ satisfy the same differential equation with the same initial condition. By the uniqueness theorem for ODE, they are the same function. This concludes the proof.

b) $\alpha^n = (e^{\alpha x})^{(n)} \Big|_{x=0}$ thus

$$\sum_{n=1}^{\infty} n \alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \ldots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t} t \alpha e^{\alpha t} dt$$

If $\alpha = 1$ we obtain $\int_0^x t e^x \alpha dt = \alpha \frac{x^2}{2} e^x$.
If $\alpha \neq 1$ we obtain integrating by parts

$$\begin{aligned} \alpha e^x \int_0^x t e^{t(\alpha-1)} dt &= \alpha e^x \left(\frac{1}{\alpha - 1} t e^{t(\alpha-1)} \Big|_0^x - \frac{1}{\alpha-1} \int_0^x e^{t(\alpha-1)} dt \right) \\ &= \frac{\alpha x e^{\alpha x}}{\alpha - 1} - \frac{\alpha e^{\alpha x}}{(\alpha-1)^2} + \frac{\alpha e^x}{(\alpha-1)^2}. \end{aligned}$$

Also solved by Dionne T. Bailey, Elsie M. Campbell, Charles Diminnie, and Andrew Siefker, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The name of **Achilleas Sinefakopoulos of Larissa, Greece** was inadvertently omitted in the March issue of the column as having solved problem 5184. I am terrible sorry for this oversight–Ted.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October 15, 2012

- **5212:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation
  $$2x + y - \sqrt{3x^2 + 3xy + y^2} = 2 + \sqrt{2}$$
  if x and y are of the form $a + b\sqrt{2}$ where a and b are positive integers.

- **5213:** *Proposed by Tom Moore, Bridgewater, MA*

  The triangular numbers T_n begin $1, 3, 6, 10, \ldots$ and, in general, $T_n = \dfrac{n(n+1)}{2}$, $n = 1, 2, 3, \ldots$.

  For every positive integer $n > 1$, prove that n^4 is a sum of four triangular numbers.

- **5214:** *Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil*

  Let a, b, c be positive real numbers. Prove that
  $$\frac{a^3(b+c)^2+1}{1+a+2b} + \frac{b^3(c+a)^2+1}{1+b+2c} + \frac{c^3(a+b)^2+1}{1+c+2a} \geq \frac{4abc(ab+bc+ca)+3}{a+b+c+1}.$$

- **5215:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Evaluate the integral
  $$\int_{-1}^{1} \frac{2x^{1004} + x^{3014} + x^{2008}\sin x^{2007}}{1 + x^{2010}} dx.$$

- **5216:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $f : \Re \to \Re^+$ be a function such that for all $a, b \in \Re$
  $$f(ab) = f(a)^b f(b)^{a^2}$$
  and $f(3) = 64$. Find all real solutions to the equation
  $$f(x) + f(x+1) - 3x - 2 = 0.$$

- **5217:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Find the value of:

$$\lim_{n\to\infty}\int_0^1\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dxdy,$$

  where k is a positive real number.

Solutions

- **5194:** *Proposed by Kenneth Korbin, New York, NY*

  Find two pairs of positive integers (a,b) such that,

$$\frac{14}{a}+\frac{a}{b}+\frac{b}{14}=41.$$

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

Multiplying $14/a+a/b+b/14=41$ by the LCM of the denominators, it follows that $14a^2+b(b-574)a+196b=0$.

To get positive integer solutions, $b-574<0$. Using MATLAB, we obtain the solutions $(252,567)$ and $(980,25)$. It is easily checked that these are solutions.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

If one intends to make the search amenable to a manual search then the search space needs to be narrowed down by exploiting divisibility properties of the numbers a and b. Equation (1) $\left(\frac{14}{a}+\frac{a}{b}+\frac{b}{14}=41\right)$ is equivalent to

$$14a^2+196b=ab(574-b). \qquad (2)$$

By (2), $14|ab^2$, which implies firstly that ($2|a$ or $2|b$) and secondly that ($7|a$ or $7|b$).

If $2|b$, then, again by (2), $4|14a^2$, which implies that $2|a$. So $2|a$.

If $7|b$, then, again by (2), $49|14a^2$, which implies that $7|a$. So $7|a$.

So a is a multiple of 14 and we write $a=14c$. (2) then reads as

$$196c^2+14b=bc(574-b). \qquad (3)$$

Let p be a prime different from 2 and 7. Let $p^\beta||b$, $p^\gamma||c$. $\big(p^f||n$ means that $p^f|n$ and $p^{f+1}\not|n$ or in words: f is the exact exponent of p in the prime factorization of n.)

We claim that $\beta=2\gamma$.

If $p^\beta||b$, then by (3), $p^\beta|c^2$. So, $\gamma\ge\lceil\beta/2\rceil$.

If $p^\gamma || c$, then by (3), $p^\gamma | b$. Then, again by (3), $p^{2\gamma} | b$. So $\beta \geq 2\gamma$.

So $\gamma \geq \lceil \beta/2 \rceil \geq \gamma$ which indeed implies that $\beta = 2\gamma$.

So b and c are of the form $b = 2^r 7^s k^2, c = 2^u 7^v k$ (4),
where r, s, u, v are nonnegative integers $0 \leq r \leq 9, 0 \leq u \leq 8, 0 \leq s, v \leq 3$,
$k \in \{1, 3, 5, 9, 11, 13, 15, 17, 19, 23\}$, because $b < 573, c < 421$.

We plug (4) into (3) and get

$$14\left(2^{1+2u}7^{1+2v} + 2^r 7^s\right) = 2^{r+u}7^{s+v}k\left(574 - 2^r 7^s k^2\right). \qquad (5).$$

A manual check reveals that (5) can hold only for $k = 5$ and $k = 9$. They give rise to the two pairs $(b, c) = (25, 70)$ and $(b, c) = (567, 18)$ which in turn yield the two solutions $(a, b) \in \{(980, 25), (252, 567)\}$.

Yet another approach to solve (1) consists in solving (3) for b. We find

$$b = \frac{7\left(-1 + 41c \pm \sqrt{(1 - 41c)^2 - 4c^3}\right)}{c}.$$

Obviously $4c^3 \leq (41c - 1)^2 < (41c)^2$. So $c < 420$ (as above). The term under the root sign equals the square of an integer. We are left with a finite set of values of c for which we need to check this condition. We find that the only values of c are $c = 18$ and $c = 70$. They give rise to the solutions already mentioned.

Comment by editor: When Ken submitted this problem he accompanied it with the following explanation.

Let K be a factor of 14, and let $a = K^2 y$ and let $b = y^2$. Then

$$\frac{K}{a} + \frac{a}{b} + \frac{b}{K} = \frac{1 + K^3 + y^3}{Ky}$$

which is equal to an integer if K is a factor of $y^3 + 1$ and if y is a factor of $K^3 + 1$.

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5195:** *Proposed by Kenneth Korbin, New York, NY*

If N is a prime number or a power of primes congruent to 1 (mod 6), then there are positive integers a and b such that $3a^2 + 3ab + b^2 = N$ with $(a, b) = 1$.

Find a and b if $N = 2011$, and if $N = 2011^2$, and if $N = 2011^3$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

From $2a^2 + 3ab + b^2 = N$, we obtain $b = \dfrac{\sqrt{4N - 3a^2} - 3a}{2}$, so that $a < \sqrt{\dfrac{N}{3}}$.

A computer search yields the following results.

$$\text{For } N \;=\; 2011, \text{ we have } (a, b) = (10, 29)$$

$$\begin{aligned} \text{For } N &= 2011^2, \text{ we have } (a,b) = (880, 541) \\ \text{For } N &= 2011^3, \text{ we have } (a,b) = (46619, 10711) \end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

This problem is best put in the context of Eisenstein integers. Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$. The set of Eisenstein integers $Z[\omega] = \{a + b\omega | a, b \in Z\}$ has the following properties:

- (i) $Z[\omega]$ forms a commutative ring of algebraic integers in the real number field $Q(\omega)$
- (ii) $Z[\omega]$ is an Euclidean domain whose norm N is given by $N(a + b\omega) = a^2 - ab + b^2$. As a result of this $Z[\omega]$ is a factorial ring.
- (iii) The group of units in $Z[\omega]$ is the cyclic group formed by the sixth root of unity in the complex plane. Specifically, they are $\{\pm1, \pm\omega, \pm\omega^2\}$. These are just the Eisenstein integers of norm one.
- (iv) An ordinary prime number (or rational prime) which is 3 or congruent to 1 (mod 3) is of the form $x^2 - xy + y^2$ for some integers x, y and may therefore be factored into $(x + y\omega)(x + y\omega^2)$ and because of that it is not prime in the Eisenstein integers. Ordinary primes congruent to 2 (mod 3) cannot be factored in this way and they are primes in the Eisenstein integers as well.

So based on this, if p is a prime number congruent to 1 (mod 6) then p factors as $p = (c + d\omega)(c + d\omega^2)$ where $c + d\omega$ and $c + d\omega^2$ are two Eisenstein primes that are complex conjugates to each other. Of course $(c, d) = 1$, since $c + d\omega$ and $c + d\omega^2$ are both Eisenstein primes. By assumption $N = p^k$ for some natural number k. Then $N = p^k = (c + d\omega)^k(c + d\omega^2)^k$. Let $(c + d\omega)^k = e + f\omega$. We claim that e and f are coprime.

Assume that there is a prime q that divides both e and f. Then $q|(c + d\omega)^k|(c + d\omega)^k(c + d\omega^2)^k = p^k$. So $q = p$ and therefore $q = (c + d\omega)(c + d\omega^2)$. Then $(c + d\omega^2)|(c + d\omega)^{k-1}$ which implies firstly that $k > 1$, (since an Eisenstein prime cannot divide 1) and secondly that $(c + d\omega^2)|(c + d\omega)$, (since $(c + d\omega^2)$ is an Eisenstein prime). Because $|c + d\omega^2| = |c + d\omega|$ we conclude that there is a unit u such that $(c + d\omega^2) = u(c + d\omega)$. So $c, d \in \{0, \pm1\}$ which cannot be, since $N(c + d\omega) = p \equiv 1 \pmod 6$.

So there is a factorization $N = p^k = (e + f\omega)(e + f\omega^2)$, where e and f are coprime integers. We claim that we can assume in addition that either (i) $0 < e < f < 2e$ or (ii) $0 < e < -f$.

Indeed, since

$$(+1)(+1) = (-1)(-1) = (+\omega)(+\omega^2) = (+\omega^2)(+\omega) = (-\omega)(-\omega^2) = (-\omega^2)(-\omega)$$

we conclude that

$$\begin{aligned} N(e + f\omega) &= N(-e - f\omega) = N(f + e\omega) = N(-f - e\omega) \\ &= N(f + (f - e)\omega) = N(-f + (e - f)\omega) = N(f - e + f\omega) = N(e - f - f\omega). \end{aligned}$$

So if we consider the eight Eisenstein integers

$$e+f\omega,\ -e-f\omega,\ f+e\omega,\ -f-e\omega,\ f+(f-e)\omega,\ -f+(e-f)\omega,\ f-e+f\omega,\ e-f-f\omega$$

there is one among these of the form $g+h\omega$ such that either $0<g<h<2g$ or $0<g<-h$, for if g and h have the same sign we can first assume that $g>0$ and $h>0$ (by replacing, if necessary g by $-g$ and h by $-h$). Next we can assume that $h>g$ (by replacing, if necessary g by h and h by g). Next we can assume that $h<2g$ (by replacing, if necessary, g by $h-g$). If g and h have different signs then we can first assume that $g>0,\ h<0$ (by replacing , if necessary, g by $-g$ and h by $-h$). Next we can assume that $g<-h$ (by replacing, if necessary g by $-h$ and h by $-g$).

In case (i) we define: $a:=f-e>0,\ b:=2e-f>0$. Then

$$\begin{aligned} N=p^k &= (e+f\omega)(e+f\omega^2)=e^2-ef+f^2=(a+b)^2-(a+b)(2a+b)+(2a+b)^2 \\ &= a^2+2ab+b^2-(2a^2+3ab+b^2)+4a^2+4ab+b^2=3a^2+3ab+b^2. \end{aligned}$$

In case (ii) we define: $a=e>0,\ b:=-e-f>0$. Then

$$\begin{aligned} N=p^k &= (e+f\omega)(e+f\omega^2)=e^2-ef+f^2=a^2+a(a+b)+(a+b)^2 \\ &= a^2+a^2+ab+a^2+2ab+b^2=3a^2+3ab+b^2. \end{aligned}$$

We find (upon using that $\omega^3=1, \omega^2+\omega+1=0$),

$$2011 = (10+49\omega)(10+49\omega^2), \tag{1}$$

$$2011^2 = (10+49\omega)^2(10+49\omega^2)^2=(2301+1421\omega)(2301+1421\omega^2), \tag{2}$$

$$2011^3 = (10+49\omega)^3(10+49\omega^2)^3=(46619-57330\omega)(46619-57330\omega^2). \tag{3}$$

We note that $39+49\omega^2$ is an associate of $10+49\omega$ since $-(39+49\omega^2)=-(39-49-49\omega)=10+49\omega$.

So, $2011=(39+49\omega)(39+49\omega^2)$, and $0<39<49<78$. We are in case (i) and find $a=10,\ b=29$. Indeed, if we define $f(a,b)=3a^2+3ab+b^2$, then $f(10,29)=2011$.

We note that $2301+1421\omega^2$ is an associate of $1421+2301\omega$ since $\omega(2301+1421\omega^2)=1421+2301\omega$. So, $2011^2=(1421+2301\omega)(1421+2301\omega^2)$, and $0<1421<2301<2842$. We are in case (i) and find $a=880,\ b=541$. Indeed, $f(880,541)=2011^2$.

We note that $2011^3=(46619-57330\omega)(46619-57330\omega^2)$, and $0<46619<57330$. We are in case (ii) and find $a=46619,\ b=10711$. Indeed, $f(46619,10711)=2011^3$.

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5196:** *Proposed by Neculai Stanciu, Buzău, Romania*

Determine the last six digits of the product $(2010)\,(5^{2014})$.

Solution 1 by Robert Howard Anderson, Chesapeake, VA

To determine the last six digits of a product you must know the last six digits of each number you plan to multiply.

To do this the last six digits of 5^{2014} we need to look at the patterns of the solutions to lower powers.

All power of 5 end in 5, and all even powers of five end in 25, then all even powers greater than 2 end in 625. The 4th digit is either 5 or 0; the digit can be determined by using 2008 (mod 4) as 5.

The 5th digit is either 1,9,6, or 4; the digit can be determined by using 2008 (mod 8) as 1.

The 6th digit is either 3,7,1,5,8,2,6,or 0; the digit can be determined by using 2006 (mod 16) as 5.

The last six digits of 5^{2014} are 515625.

The last six digits of (2010)(515625) are 406250; so the last six digits of $(2010)5^{2014}$ are 406250.

Solution 2 by Ercole Suppa, Teramo, Italy

Clearly the last digit of $N = (2010)\left(5^{2014}\right)$ is 0. Therefore in order to find the last six digits of N it is enough to calculate the last five digits of $(201)\left(5^{2014}\right)$.

Let us first calculate a few powers of 5, and to do it we need to know just the last five digits of the previous power of 5:

$5^1 = 5$	$5^2 = 25$	$5^3 = 25$	$5^4 = 625$
$5^5 = 3125$	$5^6 = 15625$	$5^7 = 78125$	$5^8 = \cdots 90625$
$5^9 = \cdots 53125$	$5^{10} = \cdots 65625$	$5^{11} = \cdots 28125$	$5^{12} = \cdots 40625$
$5^{13} = \cdots 03125$	$5^{14} = \cdots 15625$		

Observe that the last five digits of 5^{14} are the same as those of 5^6. Therefore, starting with 5^6 the last five digits of powers of 5 will repeat periodically:

$$15625, 78125, 90625, 53125, 65625, 28125, 40625, 0325, 15625, \cdots.$$

This means that increasing the exponent of eight does not change the last five digits of powers of 5. Since $2014 = 6 + 8 \cdot 251$, it follows that 5^6 and 5^{2014} have the same last five digits, so

$$201 \cdot 5^{2014} \equiv 201 \cdot 5^6 \equiv 201 \cdot 15625 \equiv 40625 \pmod{10^5}$$

and this implies that the last six digits of $(2010)\left(5^{2014}\right)$ are 406250.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By Fermat's Little Theorem, $5^{\varphi(32)} = 5^{16} \equiv 1(\text{mod } 32)$. So,

$$5^{2009} \equiv 5^{2009-16\cdot 125} \equiv 5^9 \equiv (-3)^3 \equiv 5(\text{mod } 32),$$

which means that there is an integer k such that

$$5^{2009} - 5 = 32k.$$

We multiply this equation by $2010 \cdot 5^5$ and get

$$2010 \cdot 5^{2014} - 2010 \cdot 5^6 = 201 \cdot 10^6 k.$$

But

$$2010 \cdot 5^6 = 31406250.$$

So the last six digits are 406250.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that $(2010)\,(5^{2014}) = \cdots 406250$.

It is easy to check that

$$(2010)\left(5^{2014}\right) = 406250 + \left(2^4\right)\left(5^6\right)\left(5^{2011}-1\right) + (2)\left(5^7\right)\left(5^{2008}-1\right).$$

Hence to prove our result, we need only show that $5^{2011}-1$ is a multiple of 4 and $5^{2008}-1$ is a multiple of 32.

In fact,

$$5^{2011} - 1 \equiv 1^{2011} - 1 \equiv 0 \pmod 4), \text{ and}$$

$$5^{2008} - 1 = 390625^{251} - 1 \equiv 1^{251} - 1 \equiv 0 \pmod{32},$$

and this completes the solution.

Also solved by Daniel Lopez Aguayo, UNAM Morelia, Mexico; Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Bruno Salgueiro Fanego, Viveiro Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5197:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

Let x, y, z be positive real numbers such that $x^2+y^2+z^2=4$. Prove that,

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \le \frac{1}{xyz}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

The inequality is evidently

$$\sum_{\text{cyc}} \frac{1}{2+x^2+y^2} \le \frac{1}{xyz}.$$

$a^2+1 \ge 2|a|$ yields

$$\sum_{\text{cyc}} \frac{1}{2+x^2+y^2} \le \sum_{\text{cyc}} \frac{1}{2x+2y} \le \frac{1}{xyz}$$

and $(\sqrt{x}-\sqrt{y})^2 \ge 0$ yields

$$\sum_{\text{cyc}} \frac{1}{2x+2y} \le \sum_{\text{cyc}} \frac{1}{4\sqrt{xy}} \le \frac{1}{xyz} \quad \Longleftrightarrow \quad \sum_{\text{cyc}} z\sqrt{xy} \le 4$$

which is implied by

$$\sum_{\text{cyc}} z\frac{1}{2}(x+y) \le 4 \iff xy+yz+zx \le 4.$$

But this follows by the well known $xy+yz+zx \le x^2+y^2+z^2$, thus concluding the proof.

Soluiton 2 by David E. Manes, Oneonta, NY

Let $L = \dfrac{1}{6-x^2}+\dfrac{1}{6-y^2}+\dfrac{1}{6-z^2}$. Since $x^2+y^2+z^2=4$, it follows that

$$6-x^2 = 2+y^2+z^2,\ 6-y^2 = 2+x^2+z^2,\ 6-z^2 = 2+x^2+y^2.$$

Therefore,

$$L = \frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} = \frac{1}{2+y^2+z^2}+\frac{1}{2+x^2+z^2}+\frac{1}{2+x^2+y^2}.$$

Using the Arithmetic Mean-Geometric Mean Inequality twice, one obtains

$$\begin{aligned}
L &= \frac{1}{2+(y^2+z^2)}+\frac{1}{2+(x^2+z^2)}+\frac{1}{2+(x^2+y^2)}\\
&\le \frac{1}{2+(2yz)}+\frac{1}{2+(2xz)}+\frac{1}{2+(2xy)}\\
&= \frac{1}{2}\left(\frac{1}{1+yz}+\frac{1}{1+xz}+\frac{1}{1+xy}\right)\\
&\le \frac{1}{2}\left(\frac{1}{2\sqrt{yz}}+\frac{1}{2\sqrt{xz}}+\frac{1}{2\sqrt{xy}}\right)\\
&= \frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{xyz}}\right).
\end{aligned}$$

As a result, to show that $L \le \dfrac{1}{xyz}$ it suffices to show that

$$\begin{aligned}
\frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{xyz}}\right) &\le \frac{1}{xyz}, \text{ if and only if}\\
\frac{1}{4}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) &\le \frac{1}{\sqrt{xyz}}, \text{ if and only if}\\
\frac{1}{4}\left(x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}\right) &\le 1.
\end{aligned}$$

However, the Cauchy-Schwarz inequality, and the inequality $xy+yz+zx \le x^2+y^2+z^2$ (*which also follows from the C-S inequality; editor's comment*) imply that

$$\frac{1}{4}\left(x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}\right) \le \frac{1}{4}\sqrt{x^2+y^2+z^2}\sqrt{yz+xz+xy}$$

$$\leq \quad \frac{1}{4}\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2} = 1.$$

Accordingly, if $x, y, z > 0$, and $x^2+y^2+z^2 = 4$, then

$$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} \leq \frac{1}{xyz}.$$

Solution 3 by Arkady Alt, San Jose, CA

Let $a := \dfrac{x^2}{4}$ $b := \dfrac{y^2}{4},\ c := \dfrac{z^2}{4}$ then inequality becomes

$$\frac{1}{6-4a}+\frac{1}{6-4b}+\frac{1}{6-4c} \leq \frac{1}{8\sqrt{abc}},$$

where $a+b+c = 1$.

Let $E = E(a,b,c) := \ \sqrt{abc}\sum_{cyc}\dfrac{1}{3-2a},\ p := ab+bc+ca, q := abc.$

Since $\sum_{cyc}(3-2b)(3-2c) = \sum_{cyc}(9-6(b+c)+4bc) = 15+4p,$

$(3-2a)(3-2b)(3-2c) = 9+12p-8q$ then $E = \dfrac{(15+4p)\sqrt{q}}{9+12p-8q}.$

Since $q \leq \dfrac{p^2}{3}^{*}$ and E is increasing in q then

$$\begin{aligned}
\frac{E}{\sqrt{3}} &\leq \frac{(15+4p)\,p}{27+36p-8p^2} \\
&\leq \frac{\left(15+4\cdot\frac{1}{3}\right)\cdot\frac{1}{3}}{27+36\cdot\frac{1}{3}-8\cdot\frac{1}{9}} = \frac{1}{7}
\end{aligned}$$

because $\dfrac{(15+4p)\,p}{27+36p-8p^2}$ is increasing in positive p and

$p \leq \dfrac{1}{3} \iff ab+bc+ca \leq \dfrac{(a+b+c)^2}{3}.$

Thus,

$$\begin{aligned}
E \leq \frac{\sqrt{3}}{7} \quad &\iff \quad 4E \leq \frac{4\sqrt{3}}{7} \\
&\iff \quad 8\sqrt{abc}\sum_{cyc}\frac{1}{6-4a} \leq \frac{4\sqrt{3}}{7} \\
&\iff \quad xyz\sum_{cyc}\frac{1}{6-x^2} \leq \frac{4\sqrt{3}}{7}
\end{aligned}$$

$$\iff \quad \sum_{cyc} \frac{1}{6-x^2} \leq \frac{4\sqrt{3}}{7xyz}.$$

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \quad \leq \quad \frac{4\sqrt{3}}{7xyz}.$$

Remark: Since $\dfrac{4\sqrt{3}}{7} < 1$ we have $\dfrac{1}{6-x^2} + \dfrac{1}{6-y^2} + \dfrac{1}{6-z^2} \leq \dfrac{4\sqrt{3}}{7xyz} < \dfrac{1}{xyz}$.

So, the inequality in the formulation of problem could have been stated with the stronger statement

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \quad \leq \quad \frac{4\sqrt{3}}{7xyz}, \text{instead of with the weaker one of}$$

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \quad \leq \quad \frac{1}{xyz}.$$

* *Editor's comment:* The inequality $q \leq \dfrac{p^2}{3}$ is equivalent to $3abc(a+b+c) \leq (ab+bc+ca)^2$ which is equivalent to $abc(a+b+c) \leq a^2b^2+b^2c^2+c^2a^2$ which is implied by adding up $a^2bc \leq 0.5a^2(b^2+c^2)$ and its cyclic variants.

Also solved by Kee-Wai Lau*, Hong Kong, China; Ecole Suppa, Teramo, Italy; Albert Stadler*, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer. (* Observed, specifically stated and proved the stricter inequality.)

- **5198:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let m, n be positive integers. Calculate,

$$\sum_{k=1}^{2n} \prod_{i=0}^{m} \left(\lfloor \frac{k+1}{2} \rfloor + a + i \right)^{-1},$$

where a is a nonnegative number and $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

Solution 1 by Arkady Alt, San Jose, CA

$$\begin{aligned}
&\sum_{k=1}^{2n} \prod_{i=0}^{m} \left(\left\lfloor \frac{k+1}{2} \right\rfloor + a + i \right)^{-1} \\
=\ & \sum_{k=1}^{n} \prod_{i=0}^{m} \left(\left\lfloor \frac{2k-1+1}{2} \right\rfloor + a + i \right)^{-1} + \sum_{k=1}^{n} \prod_{i=0}^{m} \left(\left\lfloor \frac{2k+1}{2} \right\rfloor + a + i \right)^{-1} \\
=\ & 2\sum_{k=1}^{n} \prod_{i=0}^{m} (k+a+i)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{n}\frac{1}{(k+a)(k+1+a)\dots(k+m+a)}\\
&= \frac{2}{m}\sum_{k=1}^{n}\left(\frac{1}{(k+a)(k+1+a)\dots(k+m-1+a)}-\frac{1}{(k+1+a)(k+2+a)\dots(k+m+a)}\right)\\
&= \frac{2}{m}\left(\frac{1}{(1+a)(2+a)\dots(m+a)}-\frac{1}{(n+1+a)(n+2+a)\dots(n+m+a)}\right).
\end{aligned}
$$

Solution 2 by Anastasios Kotronis, Athens, Greece

By a direct calculation, using the identity $\Gamma(x+1)=x\Gamma(x), \quad x>0$ for the Γ function, we can see that

$$
\prod_{i=0}^{m}\frac{1}{b+i}=\frac{\Gamma(b)}{\Gamma(b+m+1)}=\frac{1}{m}\left(\frac{\Gamma(b)}{\Gamma(b+m)}-\frac{\Gamma(b+1)}{\Gamma(b+m+1)}\right) \qquad b>0. \qquad (1)
$$

Now

$$
\begin{aligned}
&\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\left[\frac{k+1}{2}\right]+a+i\right)^{-1}\\
&= \sum_{k=1,3,\dots,2n-1}\prod_{i=0}^{m}\left(\frac{k+1}{2}+a+i\right)^{-1}+\sum_{k=2,4,\dots,2n}\prod_{i=0}^{m}\left(\frac{k}{2}+a+i\right)^{-1}\\
&= 2\sum_{k=1}^{n}\prod_{i=0}^{m}(k+a+i)^{-1}\\
&\overset{(1)}{=} \frac{2}{m}\sum_{k=1}^{n}\left(\frac{\Gamma(a+k)}{\Gamma(a+k+m)}-\frac{\Gamma(a+k+1)}{\Gamma(a+k+m+1)}\right)\\
&= \frac{2}{m}\left(\frac{\Gamma(a+1)}{\Gamma(a+1+m)}-\frac{\Gamma(a+n+1)}{\Gamma(a+n+m+1)}\right).
\end{aligned}
$$

Also solved by Albert Stadler, Herrliberg, Switzerland and the proposer.

- **5199:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k>0$ and $n\geq 0$ be real numbers. Calculate,

$$
\int_0^1 x^n \ln\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)dx.
$$

Solution by Anastasios Kotronis, Athens, Greece

$$
\begin{aligned}
I &= \left.\frac{x^{n+1}\ln\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)}{n+1}\right|_0^1 - \frac{k}{2(n+1)}\int_0^1 \frac{x^{n+k}\left(\frac{1}{\sqrt{1+x^k}}+\frac{1}{\sqrt{1-x^k}}\right)}{\sqrt{1+x^k}-\sqrt{1-x^k}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)}\int_0^1 \frac{x^{n+k}\left(\sqrt{1-x^k}+\sqrt{1+x^k}\right)}{\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)\sqrt{1-x^{2k}}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{4(n+1)}\int_0^1 \frac{x^{n}\left(\sqrt{1-x^k}+\sqrt{1+x^k}\right)^2}{\sqrt{1-x^{2k}}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)}\int_0^1 \left(\frac{x^n}{\sqrt{1-x^{2k}}}+x^n\right)dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{k}{2(n+1)}\int_0^1 \frac{x^n}{\sqrt{1-x^{2k}}}\,dx \\
&\overset{x^{2k}=u}{=\!=\!=} \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{1}{4(n+1)}B\left(\frac{n+1}{2k},\frac{1}{2}\right) \quad (B(u,v) \text{ denotes the Euler beta function}) \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{1}{4(n+1)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2k}\}\right)}{\Gamma\left(\frac{n+k+1}{2k}\right)}.
\end{aligned}
$$

Also solved by Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2012

- **5218:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that,

  $$2x - y - \sqrt{3x^2 - 3xy + y^2} = 2013$$

  with $(x, y) = 1$.

- **5219:** *Proposed by David Manes and Albert Stadler, SUNY College at Oneonta, Oneonta, NY and Herrliberg, Switzerland (respectively)*

  Let k and n be natural numbers. Prove that:

  $$\sum_{j=1}^{n} \cos^k\left(\frac{(2j-1)\pi}{2n+1}\right) = \begin{cases} \dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2} - \dfrac{1}{2}, & k \text{ even} \\ \\ \dfrac{1}{2}, & k \text{ odd.} \end{cases}$$

- **5220:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The pentagonal numbers begin $1, 5, 12, 22\ldots$ and are generally defined by $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The triangular numbers begin $1, 3, 6, 10, \ldots$ and are generally defined by $T_n = \dfrac{n(n+1)}{2}, \forall n \geq 1$. Find the greatest common divisor, $\gcd(T_n, P_n)$.

- **5221:** *Proposed by Michael Brozinsky, Central Islip, NY*

  If x, y and z are positive numbers find the maximum of

  $$\frac{\sqrt{(x+y+z)\cdot xyz}}{(x+y)^2 + (y+z)^2 + (x+z)^2}.$$

- **5222:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Calculate without the aid of a computer the following sum

$$\sum_{n=0}^{\infty}(-1)^n\,(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n, \quad \text{where } i=\sqrt{-1}.$$

- **5223:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  a) Find the value of

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right).$$

  b) More generally, if $x\in(-1,1]$ is a real number, calculate

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}+\frac{x^{n+3}}{n+3}-\cdots\right).$$

Solutions

- **5200:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers (a,b,c,d) such that,

$$(a+b+c+d)^2=2\left(a^2+b^2+c^2+d^2\right)$$

with $a<b<c<d$. Find positive integers x,y and z such that

$$\begin{aligned} x &= \sqrt{ab+ad+bd}-\sqrt{ab+ac+bc},\\ y &= \sqrt{bc+bd+cd}-\sqrt{bc+ab+ac},\\ z &= \sqrt{bc+bd+cd}-\sqrt{ac+ad+cd}. \end{aligned}$$

Solution by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The first equation can be treated as a quadratic in d and solved:

$$d=(a+b+c)\pm 2\sqrt{ab+ac+bc}.$$

The simplest way to force d to be an integer is to find a,b and c such that the discriminate $ab+ac+bc$ is a square. (Note that we must use the $+$ sign, because the negative choice would make $d<c$.) (Note also that we could cast a slightly wider net and look for a,b and c such that $ab+ac+bc$ has the form $n^2/4$.)

Suppose $ab+ac+bc=N^2$, so that $d=(a+b+c)+2N$. Then we need $x=\sqrt{ab+ad+bd}-N$ to be an integer, so $ab+ad+bd$ must be a square. Successively, $bc+bd+cd$ and $ac+ad+cd$ must also be squares.

Thus we look for values of a, b, c and d such that $ab + ac + bc, ab + ad + bd, bc + bd + cd$ and $ac + ad + cd$ are all squares.

Surprisingly, there are many such. We used MATLAB to find a sampling and conjecture that there are infinitely many of them. The smallest set is $a = 1, b = 4, c = 9, d = 28$ which give $x = 5, y = 13, z = 3$.

Editor's note: David and John then listed about 145 different 4-tuplets (a, b, c, d) which produce positive integer values for x, y, z. Listed below is a sampling of the values they listed.

$$\begin{pmatrix}
a & b & c & d & x & y & z \\
1 & 4 & 9 & 28 & 5 & 13 & 3 \\
1 & 4 & 12 & 33 & 5 & 16 & 3 \\
1 & 4 & 28 & 57 & 5 & 32 & 3 \\
1 & 4 & 33 & 64 & 5 & 37 & 3 \\
1 & 4 & 57 & 96 & 5 & 61 & 3 \\
1 & 4 & 64 & 105 & 5 & 68 & 3 \\
1 & 4 & 96 & 145 & 5 & 100 & 3 \\
1 & 9 & 16 & 52 & 10 & 25 & 8 \\
1 & 9 & 28 & 72 & 10 & 37 & 8 \\
1 & 9 & 52 & 108 & 10 & 61 & 8 \\
1 & 9 & 72 & 136 & 10 & 81 & 8 \\
1 & 12 & 24 & 73 & 13 & 36 & 11 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

David and John also asked if there were an infinite number of such integers and **Paul M. Harms of North Newton, KS** answered this affirmatively in his solution by showing that for any positive integer a, $(a, b, c, d) = (a, 4a, 9a, 28a)$ satisfies the conditions of the problem and yields positive integers for x, y, z. Note that Paul's parameterization of the simplest solution $(1, 4, 9, 28)$ produces an infinite number of solutions to the problem, but not all solutions. E.g., there is no integer value of a for which $(a, 4a, 9a, 28a)$ will give $(1, 4, 12, 33,)$, the second tuple in the above listing.

Most solvers showed that if the conditions of the problem are satisfied then two cases exist:

$$\begin{cases} x = d - c,\ y = d - a,\ z = b - a & \text{if } a+b+c-d > 0, \\ x = a + b,\ y = b + c,\ z = b - a & \text{if } a+b+c-d < 0. \end{cases}$$

So the main question becomes: when is $(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2)$ solvable?

Albert Stadler of Herrliberg, Switzerland stated that by labeling the equation $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ as (1), we see that (1) is equivalent to $(a-b)^2+(c-d)^2 = 2(a+b)(c+d)$. So if we choose odd integers u and v such that

$$u^2+v^2 = 2rs \text{ with } r \geq u \text{ and } s \geq v$$

then r and s are both odd and $(a,b,c,d) = ((r-u)/2, (r+u)/2, (s-v)/2, (s+v)/2)$ satisfies (1).

Also solved by Brian D. Beasley, Clinton, SC; Samuel Judge, Justin Wydra and Karen Wydra (jointly, students at Taylor University), Upland, IN; Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5201:** *Proposed by Kenneth Korbin, New York, NY*

Given convex cyclic quadrilateral ABCD with integer length sides where $\left(\overline{AB}, \overline{BC}, \overline{CD}\right) = 1$ and with $\overline{AB} < \overline{BC} < \overline{CD}$.

The inradius, the circumradius, and both diagonals have rational lengths. Find the possible dimensions of the quadrilateral.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

A **Brahmagupta quadrilateral** [1] is a cyclic quadrilateral with integer sides, integer diagonals, and integer area. All Brahmagupta quadrilaterals with sides a, b, c, d, diagonals e, f, area K, and circumradius R can be obtained by clearing denominators from the following expressions involving rational parameters t, u, and v:

$$\begin{aligned}
a &= \Big(t(u+v)+1-uv\Big)\Big(u+v-t(1-uv)\Big),\\
b &= (1+u^2)(v-t)(1+tv),\\
c &= t(1+u^2)(1+v^2),\\
d &= (1+v^2)(u-t)(1+tv),\\
e &= u(1+t^2)(1+v^2),\\
f &= v(1+t^2)(1+u^2),\\
K &= \Big|uv\Big(2t(1-uv)-(u+v)(1-t^2)\Big)\Big(2(u+v)t+(1-uv)(1-t^2)\Big)\Big|,\\
4R &= (1+u^2)(1+v^2)(1+t^2).
\end{aligned}$$

(Source: http://en.wikipedia.org/wiki/Cyclic quadrilateral; we have corrected a minor slip in the formula for K as we must take the absolute value of the defining expression of K.)

The condition $\max\left(0, \dfrac{uv-1}{u+v}\right) < t < \min(u,v)$ ensures that $a > 0, b > 0, c > 0$, and $d > 0$.

If the cyclic quadrilateral is in addition tangential (as in the problem statement) then $a + c = b + d$. So ,

$$t + tu^2 + 2tuv - u^2v + t^2u^2v + tv^2 - uv^2 + t^2uv^2 - tu^2v^2 = 0, \text{or,}$$

$$t = \frac{(uv-1)(uv+1) - (u+v)^2 + \sqrt{(1+u^2)(1+v^2)\Big((1+uv)^2 + (u+v)^2\Big)}}{2uv(u+v)}$$

There are many tuples (u, v) of rational numbers such that

$$\sqrt{(1+u^2)(1+v^2)\Big((1+uv)^2 + (u+v)^2\Big)}$$

is rational. Here are a few examples:

$$\begin{pmatrix}
t & u & v \\
31/384 & 4/3 & 124/957 \\
31/384 & 4/3 & 496/3828 \\
1443/1276 & 4/3 & 1914/248 \\
1443/1276 & 4/3 & 7656/992 \\
93/1924 & 6/8 & 124/957 \\
93/1924 & 6/8 & 496/3828 \\
216/319 & 6/8 & 1914/248 \\
44/273 & 14/48 & 156/133 \\
171/1372 & 14/48 & 266/312 \\
31/384 & 16/12 & 124/957 \\
1443/1276 & 16/12 & 1914/248 \\
2816/3705 & 20/21 & 912/215 \\
93/1924 & 24/32 & 124/957 \\
896/1053 & 24/7 & 156/133 \\
152/231 & 24/7 & 266/312 \\
896/1053 & 24/7 & 624/532 \\
896/1053 & 96/28 & 156/133 \\
2625/1664 & 140/51 & 260/69
\end{pmatrix}$$

In what follows we consider only the first entry in this table.

The triple $(t, u, v) = (31/384, 4/3, 124/957)$ yields the quadruple

$$(a, b, c, d) = \left(\frac{23280625}{17639424}, \frac{13885495975}{101285572608}, \frac{721699375}{3165174144}, \frac{447919225}{317509632}\right).$$

Clearing denominators yields

$$(a, b, c, d) = (143550, 14911, 24800, 153439)$$

which is equivalent to

$$(a^{'}, b^{'}, c^{'}, d^{'}) = (14911, 24800, 153439, 143550).$$

Obviously $a^{'} < b^{'} < c^{'}$ and these numbers are coprime.

We see that $a^{'} + c^{'} = b^{'} + d^{'}$, so the quadrilateral is tangential. We have

$$\begin{aligned}
s &= \frac{(a+b+c+d)}{2} = 168350, \\
K &= \sqrt{(s-a)(s-b)(s-c)(s-d)} = 2853965400 = \sqrt{abcd}, \\
r &= \frac{2K}{a+b+c+d} = \frac{K}{s} = \frac{118668}{7}, \\
R &= \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}} = \frac{3710425}{48}, \\
e &= \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} = 148417, \\
f &= \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}} = \frac{7604641}{193}.
\end{aligned}$$

References: [1] Sastry, K.R.S., "Brahmagupta quadrilaterals" Forum Geometricorum, 2, 2002, 167-173.

Solution 2 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We let $b = \overline{AB}, c = \overline{BC}, d = \overline{CD}, a = \overline{DA}$.

From Wolfram Math World at http://mathworld.wolfram.com/CyclicQuadrilateral.html and http://mathworld.wolfram.com/Bicentri Quadrilateral.html, we find for a bicentric quadrilateral with sides a, b, c, and d (in order around the quadrilateral), having inradius r circumradius R and area A, semiperimeter s, the following conditions must be fulfilled:

$$\begin{aligned}
a+c &= b+d, \\
A &= \sqrt{abcd} = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \\
r &= \frac{\sqrt{abcd}}{s},
\end{aligned}$$

$$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{abcd}} = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}.$$

Diagonal lengths are given by $\sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}$ and $\sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}}$.

We are requiring $b < c < d$ (which also forces $b < a < d$), and $(b, c, d) = 1$, which forces any three sides to be coprime.

Rationalizing the denominator in the expressions for the diagonals, we see that $\sqrt{(ad+bc)(ab+cd)(ac+bd)}$ must be an integer if the diagonals are to have rational length.

Since the circumradius must also be rational, we deduce that the area must also be rational. Since it is the square root of a product of integers, it must be an integer.

Using the two formulas for the area $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ and $A = \sqrt{abcd}$ were s is its semiperimeter, we see that $8abcd + 2(a^4+b^4+c^4+d^4) = (a^2+b^2+c^2+d^2)^2$. Thus the side lengths of the quadrilateral must satisfy the following:

- $8abcd + 2(a^4+b^4+c^4+d^4) = (a^2+b^2+c^2+d^2)^2$,
- $a + c = b + d$,
- the product $abcd$ must be a perfect square,
- $\sqrt{(ad+bc)(ab+cd)(ac+bd)}$ must be an integer.

We wrote a MATLAB program to search through integers b, c, and d where $b < c < d$ from 1 to 4000 where these conditions were satisfied. The results give us the possible dimensions of the cyclic quadrilaterals satisfying the requirements of the problem. We found 7 solutions.

Note that the position of the side can be rearranged as long as opposing pairs have the same sum. In the following table we have re-lettered to let a be the smallest entry.

Results are shown below with rational numbers in lowest terms:

a	b	c	d	$2s$	$Area$	$Inradius$	$circumradius$	$diag1$	$diag2$
21	85	140	204	450	7140	476/15	221/2	195	104
91	36	260	315	702	16380	140/3	325/2	280	125
190	231	399	440	1260	87780	418/3	1885/8	13650/29	377
2397	483	1316	1564	5760	1543668	128639/240	2405/2	22015/13	11544/5
4756	123	1428	3451	9758	1697892	348	7565/2	7743/5	414715/89
3256	629	1080	2805	7770	2490840	71224/111	1628	1653	15973/5
4828	1060	2125	3763	11776	6397100	3400	2414	3025	23551/5
2849	1480	2145	2184	8658	4444440	4070	3145/2	2975	15703/5

Comment: It is helpful to look at the prime-power decomposition of a, b, c and d. For instance,

$$21 = 3 \cdot 7, \quad 85 = 5 \cdot 17, \quad 140 = 2^2 \cdot 5 \cdot 7, \quad \text{and} \quad 204 = 2^2 \cdot 3 \cdot 17.$$

Thus the product $abcd = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 17^2$ is clearly a square. But recognizing such patterns does not help us in generating solutions. In fact, it would seem so difficult to satisfy the required conditions that no solutions could exist.

Conjectures: Each of our solutions consists of two even integers and two odd ones, so that would be a reasonable conjecture. We suspect there are infinitely many solutions.

Also solved by the proposer.

- **5202:** *Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Solve in $\Re^2$,

$$\begin{cases} \ln\left(x + \sqrt{x^2+1}\right) = \ln \dfrac{1}{y + \sqrt{y^2+1}} \\ 2^{y-x}\left(1 - 3^{x-y+1}\right) = 2^{x-y+1} - 1. \end{cases}$$

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

From the first equation, considering the fact that functions $f(x) = \sqrt{x^2+1} + x$ and $g(x) = \sqrt{x^2+1} - x$ are symmetric with respect to the y-axis, one can easily observe that this is satisfied for $x = -y$.

Replacing $x = -y$ in the second equation we have

$$2^{2y}\left(1 - 3^{-2y+1}\right) = 2^{-2y+1} - 1.$$

Let's consider the function $f(y) = 2^{2y}\left(1 - 3^{-2y+1}\right) - 2^{-2y+1} + 1$ and find the roots of $f(y) = 0$. One can easily observe that $f(0.5) = 0$.

If $y > 0.5$ then

$$f(y) = 2^{2y}\left(1 - 3^{-2y+1}\right) - 2^{-2y+1} + 1 > 2^1\left(1 - 3^0\right) - 2^0 + 1 = 0.$$

And if $y < 0.5$ then

$$f(y) = 2^{2y}\left(1 - 3^{-2y+1}\right) - 2^{-2y+1} + 1 < 2^1\left(1 - 3^0\right) - 2^0 + 1 = 0.$$

So the only solution of the system is $(x, y) = (-0.5, 0.5)$ and this is end of the proof.

Solution 2 by Kee-Wai Lau, Hong Kong, China

The simultaneous equations have the unique solution $(x, y) = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

For $s \in \Re$ let $f(s) = 2\left(3^s\right) + 4^s - 2^s - 2$, so that

$$\frac{df(s)}{ds} = 2\ln 3\left(3^s\right) + \ln 4\left(4^s\right) - \ln 2\left(2^s\right).$$

It is easy to check that the second equation of the system is equivalent to $f(1 + x - y) = 0$. We need to show that $f(s) = 0$ if and only if $s = 0$.

Since $f(s) < 2\left(3^{-1}\right) + 4^{-1} - 2 < 0$ for $s < -1$ and $f(0) = 0$, it suffices to show that $f(s)$ is strictly increasing for $s > -1$.

But this follows immediately from the facts that

$$\frac{df(s)}{ds} > 2\ln 3\left(3^{-1}\right) + \ln 4\left(4^{-1}\right) - \ln 2 > 0 \text{ for } -1 < s \le 0, \text{ and } \frac{df(s)}{ds} > 2\ln 3 > 0 \text{ for } s > 0.$$

Hence $1 + x - y = 0$ and the first equation of the system can now be written as

$$x + \sqrt{x^2+1} = \frac{1}{y + \sqrt{y^2+1}} = \sqrt{y^2+1} - y = \sqrt{x^2+2x+2} - x - 1, \text{ or}$$

$$\left(2x + 1 + \sqrt{x^2+1}\right)^2 = x^2 + 2x + 2.$$

Expanding and simplifying the last equation, we obtain $2(2x+1)\left(x + \sqrt{x^2+1}\right) = 0$, so that $x = -\dfrac{1}{2}$ and $y = \dfrac{1}{2}$ as claimed.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\operatorname{Arcsh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\begin{aligned}
&= \ln \frac{1}{y+\sqrt{y^2+1}}\\
&= \ln 1 - \ln\left(y+\sqrt{y^2+1}\right)\\
&= -\text{Arsh}y = \text{Arsh(−y)} \iff\\
x &= -y.
\end{aligned}$$

If $x-y+1<0$, then $1-3^{x-y+1}>1-3^0=0$ and $2^{x-y+1}-1<2^0-1=0$. So, since $2^{y-x}>0$, we have that $0<2^{y-x}\left(1-3^{x-y+1}\right)=2^{x-y+1}-1<0$, which is a contradiction.

And if $x-y+1>0$, then $1-3^{x-y+1}<1-3^0=0$, and $2^{x-y+1}-1>2^0-1=0$, so, since $2^{y-x}>0$, we have $0>2^{y-x}\left(1-3^{x-y+1}\right)=2^{x-y+1}-1>0$, which is a contradiction, so $x-y+1=0$.

Hence the given system is equivalent to

$$\begin{aligned}
x+y &= 0\\
x-y &= -1,
\end{aligned}$$

whose only solution in $\Re^2$ is $(x,y)=(-1/2,1/2)$.

Solution 4 by David Manes, SUNY College at Oneonta, Oneonta, NY

The unique solution for the system of equations is $x=-\dfrac{1}{2}, y=\dfrac{1}{2}$.

Note that $\ln \dfrac{1}{y+\sqrt{y^2+1}}=\ln\left(\sqrt{y^2+1}-y\right)$ and the natural logarithm function is one-to-one.

Therefore, $x+\sqrt{x^2+1}=\sqrt{y^2+1}-y$. Squaring both sides of the equation yields

$$x\sqrt{x^2+1}+y\sqrt{y^2+1}=y^2-x^2.$$

Squaring this equation one obtains

$$x^2+y^2+2x^2y^2=-2xy\sqrt{x^2+1}\sqrt{y^2+1},$$

an equation that also implies that x and y have opposite signs. Finally, squaring this equation, we get

$$\left(x^2-y^2\right)^2=0 \iff |x|=|y|.$$

Therefore, $y=-x$, since $y=x$ is impossible. With $y=-x$, the second equations reduces to

$$\frac{1}{2x}\left(1-3^{2x+1}\right)=2^{2x+1}-1.$$

If $t = 2x+1$, then this equation can be written as $4^t + 2\cdot 3^t - 2^t = 2$, whose only solution is $t = 0$; hence, $x = -\dfrac{1}{2}$ and $y = \dfrac{1}{2}$ as claimed.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro Spain (two solutions); Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata Roma," Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5203: *Proposed by Pedro Pantoja, Natal-RN, Brazil*

Evaluate,

$$\int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{\sin^4 x+\cos^4 x}\right)dx.$$

Solution 1 by Marius Damian, Brăila, "Nicolae Balcescu" College, Brailia, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania

First, we have:

$$1 = \left(\sin^2 x+\cos^2 x\right)^2 = \sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x = \sin^4 x+\cos^4 x+\frac{1}{2}\sin^2 2x,$$

so

$$\sin^4 x+\cos^4 x = 1-\frac{1}{2}\sin^2 2x.$$

Then the integral becomes:

$$\begin{aligned} I = \int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{1-\frac{1}{2}\sin^2 2x}\right)dx &= \int_0^{\pi/4} \ln\left[2\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)\right]dx \\ &= \int_0^{\pi/4}\left[\ln 2+\ln\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)\right]dx \\ &= \frac{\pi\ln 2}{4}+\int_0^{\pi/4}\ln\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)dx. \end{aligned}$$

We denote:

$$J = \int_0^{\pi/4}\ln\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)dx,$$

and we substitute $t = \dfrac{\pi}{4}-x$, therefore we deduce that $J = -J$, so $J = 0$.

Hence $I = \dfrac{\pi \ln 2}{4}$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have
$$\sin^2 x = \frac{1-\cos(2x)}{2},\ \cos^2 x = \frac{1+\cos(2x)}{2}.$$
So
$$\sin^4 x + \cos^4 x = \left(\frac{1-\cos(2x)}{2}\right)^2 + \left(\frac{1+\cos(2x)}{2}\right)^2 = \frac{1+\cos^2(2x)}{2}$$
and
$$\begin{aligned}\int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{\sin^4 x + \cos^4 x}\right) dx &= \int_0^{\pi/4} \ln 2 dx + \int_0^{\pi/4} \ln\left(\frac{1+\sin^2(2x)}{1+\cos^2(2x)}\right) dx \\ &= \frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\pi/2} \ln\left(\frac{1+\sin^2 y}{1+\cos^2 y}\right) dy \\ &= \frac{\pi}{4}\ln 2,\end{aligned}$$

since $\displaystyle\int_0^{\pi/2} \ln\left(\frac{1+\sin^2 y}{1+\cos^2 y}\right) dy = \int_0^{\pi/2} \ln\left(1+\sin^2 y\right) dy - \int_0^{\pi/2} \ln\left(1+\cos^2 y\right) dy = 0.$

Also solved by Arkady Alt, San Jose, CA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; David E, Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy; Luke Sly, Joseph Kasper, and Daniel Crane (jointly, students at Taylor University), Upland, IN, and the proposer.

5204: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $f : \Re \to \Re$ be a non-constant function such that,
$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$$
for all $x, y \in \Re$. Show that $-1 < f(x) < 1$ for all $x \in \Re$.

Solution 1 by Michael Brozinsky, Central Islip, NY

The functional equation implies $f(0) + \Big(f(0)\Big)^3 = 2f(0)$ and so $f(0) = 0, 1$ or -1. The two latter possibilities lead to similar contradictions. For example if $f(0) = 1$ then
$$f(x) = f(x+0) = \frac{f(x)+1}{1+f(x)\cdot 1} = 1, \text{ a constant.}$$

Thus we must have $f(0) = 0$.

Now since $(u+1)^2 \geq 0$ and $(u-1)^2 \geq 0$ we have

$$-1 \leq \frac{2u}{1+u^2} \leq 1 \quad (*)$$

with equalities (on the side) occurring only if $u = 1$ or $u = -1$.

If there exits an x_0 such that $f(x_0) = 1$ then

$$f(x) = f\left((x - x_0) + x_0\right) = \frac{f(x - x_0) + 1}{1 + f(x - x_0)} = 1$$

contrary to the stated condition that $f(x)$ is not constant. A similar contradiction follows if there exits an x_0 such that $f(x_0) = -1$.

Finally, since $f(x) = \dfrac{2f\left(\dfrac{x}{2}\right)}{1 + \left(f\left(\dfrac{x}{2}\right)\right)^2}$ we have the given inequality follows upon setting $u = f\left(\dfrac{x}{2}\right)$, and using $(*)$ and the last two results.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

Since $f(0) = 2f(0)/(1 + (f(0))^2)$, we have $f(0) \in \{0, \pm 1\}$. But $f(0) = 1$ would imply $f(x) = (f(x) + 1)/(1 + f(x)) = 1$ for each real x, contradicting the non-constant condition of the hypothesis. Similarly, $f(0) = -1$ would imply $f(x) = (f(x) - 1)/(1 - f(x)) = -1$ for each real x, another contradiction. Thus $f(0) = 0$. This yields

$$0 = \frac{f(x) + f(-x)}{1 + f(x)f(-x)}$$

and hence $f(-x) = -f(x)$ for each real x. Also, given any real x, we have $f(x) = 2f(x/2)/(1 + (f(x/2))^2)$.

If $f(x) \geq 1$ for some real x, then $2f(x/2) \geq 1 + (f(x/2))^2$, so $0 \geq (f(x/2) - 1)^2$ and thus $f(x/2) = 1$. Then $f(x) = 1$ and $f(2x) = 1$, but $f(-x) = -1$, which would mean that

$$f(x) = \frac{f(2x) + f(-x)}{1 + f(2x)f(-x)}$$

is undefined.

Similarly, if $f(x) \leq -1$ for some real x, then $2f(x/2) \leq -1 - (f(x/2))^2$, so $(f(x/2) + 1)^2 \leq 0$ and thus $f(x/2) = -1$. Then $f(x) = -1$ and $f(2x) = -1$, but $f(-x) = 1$, which would again mean that $f(x)$ is undefined.

Hence $-1 < f(x) < 1$ for each real x.

Solution 3 by Arkady Alt, San Jose, CA

First note that $f(x) \cdot \ f(y) \neq -1$ for any $x, y \in R$.

Since $f(x) = f\left(\dfrac{x}{2} + \dfrac{x}{2}\right) = \dfrac{2f\left(\dfrac{x}{2}\right)}{1 + \ f^2\left(\dfrac{x}{2}\right)} \Rightarrow |f(x)| = \dfrac{2\left|f\left(\dfrac{x}{2}\right)\right|}{1 + \left|\ f\left(\dfrac{x}{2}\right)\right|^2}$

and then we have $\left(\left|f\left(\dfrac{x}{2}\right)\right| - 1\right)^2 \geq 0 \iff \dfrac{2\left|f\left(\dfrac{x}{2}\right)\right|}{1 + \left|\ f\left(\dfrac{x}{2}\right)\right|^2} \leq 1 \iff |f(x)| \leq 1.$

If we suppose $|f(x_0)| = 1,$ for some $x_0,$ then $\left|f\left(\dfrac{x_0}{2}\right)\right| = 1$ and $f(x)$ becomes a constant

function. Indeed, if $f(x_0) = 1,$ then for any $x \in R$ we have $\ f(x + x_0) = \dfrac{f(x) + \ 1}{1 + \ f(x)} = 1,$

because $\ f(x) = f(x) \cdot f(x_0) \neq -1.$

If $f(x_0) = -1,$ then for any $x \in R$ we have $\ f(x + x_0) = \dfrac{f(x) - \ 1}{1 - \ f(x)} = -1,$

because $\ -f(x) = f(x) \cdot f(x_0) \neq -1.$ Thus, $|f(x)| < 1 \iff -1 < f(x) < 1$ for any x.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Statesboro, GA; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Buzău, Romania, and the proposer.

5205: *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the sum,

$$\sum_{n=1}^{\infty}\left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln\frac{n+1}{n}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For each integer $m > 1$, is easy to prove by induction that

$$\sum_{n=1}^{m}\left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n}\right)\ln\frac{n+1}{n}$$

$$= \quad \left(1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}\right)\ln(m+1)+\sum_{n=2}^{m}\frac{(-1)^n\ln n}{n}.$$

Since

$$\left|1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}-\ln 2\right|$$

$$= \quad \frac{1}{m+1}\left(1-\frac{m+1}{m+2}+\frac{m+1}{m+3}-\frac{m+1}{m+4}+\cdots\right)<\frac{1}{m+1},$$

so

$$\lim_{m\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}-\ln 2\right)\ln(m+1)=0.$$

It is known [E. R. Hansen: *A Table of Series and Products,* Prentice-Hall, Inc., 1975, p. 288 entry (44.1.8)] that $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{n}=\gamma\ln 2-\frac{(\ln 2)^2}{2}$, where γ is Euler's constant. Hence the sum of the problem equals $\gamma\ln 2-\dfrac{(\ln 2)^2}{2}=0.1598\ldots.$

Solution 2 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy

By writing $q_n=1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{n}-\ln 2$ the series is

$$\sum_{n=1}^{\infty}q_n\ln\frac{n+1}{n}=\sum_{n=1}^{\infty}\left((q_n\ln(n+1)-q_{n-1}\ln n)+\ln n(q_{n-1}-q_n)\right)$$

$$\sum_{n=1}^{\infty}(q_n\ln(n+1)-q_{n-1}\ln n)=\lim_{n\to\infty}q_n\ln(n+1).$$

The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

is Leibniz and converges to $\ln 2$ thus it satisfies

$$\left|\ln 2-\sum_{n=1}^{r}\frac{(-1)^{n-1}}{n}\right|\leq\frac{1}{r+1}.$$

Since this is a well known property of all Leibniz series present in all books on the subject, we omit it. The immediate consequence is

$$\lim_{n\to\infty}q_n\ln(n+1)=0.$$

We remain with

$$\sum_{n=1}^{\infty} \ln n(q_{n-1} - q_n) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln n = \gamma \ln 2 - \frac{1}{2} \ln^2 2$$

where γ is the Euler–Mascheroni constant. Also $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln n = \gamma \ln 2 - \frac{1}{2} \ln^2 2$ is a well known result. Nevertheless we write it here. For $p \geq 4$,

$$\sum_{k=2}^{2p} (-1)^k \frac{\ln k}{k} = \sum_{k=1}^{p} \frac{\ln 2}{2k} + \sum_{k=1}^{p} \frac{\ln k}{2k} - \sum_{k=1}^{p-1} \frac{\ln(2k+1)}{2k+1}.$$

$$-\sum_{k=1}^{p-1} \frac{\ln(2k+1)}{2k+1} = -\sum_{k=2}^{2p-1} \frac{\ln k}{k} + \sum_{k=1}^{p-1} \frac{\ln(2k)}{2k} = -\sum_{k=2}^{2p-1} \frac{\ln k}{k} + \sum_{k=1}^{p-1} \frac{\ln 2}{2k} + \sum_{k=1}^{p-1} \frac{\ln k}{2k}.$$

By summing we get

$$\sum_{k=1}^{p-1} \frac{\ln 2}{k} + \frac{\ln 2}{2p} + \frac{\ln p}{2p} - \sum_{k=p}^{2p-1} \frac{\ln k}{k}.$$

Now we employ the well known

$\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$. Moreover we observe that

$$\int_p^{2p} \frac{\ln x}{x} dx \leq \sum_{k=p}^{2p-1} \frac{\ln k}{k} = \int_{p-1}^{2p-1} \frac{\ln x}{x} dx,$$

(*Editor's note:* We note that the function $\dfrac{\ln x}{x}$ is decreasing for $x \geq e$. So $\displaystyle\int_k^{k+1} \frac{\ln x}{x} dx \leq \frac{\ln k}{k} \leq \int_{k-1}^{k} \frac{\ln x}{x} dx$. The claimed inequalities follow by summing over k from $k = p$ to $k = 2p - 1$.)

thus

$$\begin{aligned}
\sum_{k=p}^{2p-1} \frac{\ln k}{k} &= \int_p^{2p-1} \frac{\ln x}{x} dx + o(1) = \frac{1}{2}(\ln^2(2p-1) - \ln^2 p) + o(1) \\
&= \frac{\ln^2 2}{2} + \frac{\ln^2 p}{2} + \ln 2 \ln p + \ln(2p) \ln(1 - \frac{1}{2p}) + \frac{1}{2} \ln^2(1 - \frac{1}{2p}) - \frac{\ln^2 p}{2} + o(1) \\
&= \frac{\ln^2 2}{2} + \ln 2 \ln p + o(1).
\end{aligned}$$

We get

$$\sum_{k=1}^{p-1}\frac{\ln 2}{k}+\frac{\ln 2}{2p}+\frac{\ln p}{2p}-\sum_{k=p}^{2p-1}\frac{\ln k}{k}=\ln 2(\ln(p-1)+\gamma)-\frac{\ln^2 2}{2}-\ln 2\ln p+o(1)$$

$$= \gamma\ln 2-\frac{\ln^2 2}{2},\ \text{as } p\to\infty.$$

Solution 3 by Anastasios Kotronis, Athens, Greece

We set

$$f_m(x)=\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right) \qquad x<1,$$

and we wish to find

$$\lim_{m\to+\infty} f_m(-1).$$

For $x<1$ we have

$$\begin{aligned}
f_m'(x) &= \left(\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\right)'\\
&= \sum_{n=1}^{m}\left(-\sum_{k=0}^{n-1}x^k+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&= \sum_{n=1}^{m}\left(-\frac{1-x^n}{1-x}+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&= \frac{1}{1-x}\sum_{n=1}^{m}x^n\left(\ln(n+1)-\ln n\right)\\
&= \frac{1}{1-x}\left(\sum_{n=2}^{m}\left(x^{n-1}-x^n\right)\ln n+x^m\ln(m+1)\right)\\
&= \sum_{n=2}^{m}x^{n-1}\ln n+\frac{x^m}{1-x}\ln(m+1).
\end{aligned}$$

So we integrate from 0 to y, where $y<1$, to get

$$f_m(y)=\sum_{n=2}^{m}\frac{y^n}{n}\ln n+\ln(m+1)\int_0^y\frac{x^m}{1-x}\,dx$$

and set $y=-1$ to get

$$
\begin{aligned}
f_m(-1) &= \sum_{n=2}^{m} \frac{(-1)^n}{n} \ln n + \ln(m+1) \int_0^{-1} \frac{x^m}{1-x}\, dx \\
&\overset{x=-t}{=\!=\!=} \sum_{n=2}^{m} \frac{(-1)^n}{n} \ln n + (-1)^{m+1} \ln(m+1) \int_0^{1} \frac{t^m}{1+t} dt \\
&= A_m + (-1)^{m+1} \ln(m+1) B_m. \qquad (1)
\end{aligned}
$$

Now integrating by parts,

$$
\begin{aligned}
B_m &= \left. \frac{t^{m+1}}{(m+1)(1+t)} \right|_0^1 + \frac{1}{m+1} \int_0^1 \frac{t^{m+1}}{(1+t)^2}\, dt \\
&\leq \frac{1}{2(m+1)} + \frac{1}{m+1} \int_0^1 \frac{1}{(1+t)^2}\, dt \\
&= \frac{1}{m+1} < \frac{1}{m} \qquad (2)
\end{aligned}
$$

and for A_m, since it converges from Leibniz Criterion, (see:
http://mathworld.wolfram.com/Leibniz Criterion.html) we can write

$$
\lim_{m \to +\infty} A_m = \lim_{m \to +\infty} A_{2m}
$$

and

$$
\begin{aligned}
A_{2m} &= \sum_{n=1}^{2m} \frac{(-1)^n}{n} \ln n \\
&= \sum_{n=1}^{m} \frac{\ln 2n}{2n} - \sum_{n=1}^{m} \frac{\ln(2n-1)}{2n-1} \\
&= \frac{\ln 2}{2} \sum_{n=1}^{m} \frac{1}{n} + \frac{1}{2} \sum_{n=1}^{m} \frac{\ln n}{n} - \left(\sum_{n=1}^{2m} \frac{\ln n}{n} - \sum_{n=1}^{m} \frac{\ln 2n}{2n} \right) \\
&= \ln 2 H_m + \sum_{n=1}^{m} \frac{\ln n}{n} - \sum_{n=1}^{2m} \frac{\ln n}{n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln(m+n)}{m+n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln m + \ln(1+n/m)}{m+n} \\
&= \ln 2 H_m - \ln m (H_{2m} - H_m) - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m}
\end{aligned}
$$

$$
\begin{aligned}
&= \quad H_m \ln(2m) - H_{2m} \ln m - \frac{1}{m}\sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \\
&\overset{H_m=\ln m+\gamma+O(1/m)}{=} \quad \gamma \ln 2 + O(1/m) - \frac{1}{m}\sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \qquad (3)
\end{aligned}
$$

Now with (2) and (3), (1) will give

$$f_m(-1) \to \gamma \ln 2 - \int_0^1 \frac{\ln(1+x)}{1+x}\, dx = \gamma \ln 2 - \frac{\ln^2 2}{2}.$$

Comment: In fact, one can easily show that

$$\frac{1}{m}\sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} = \frac{\ln^2 2}{2} + O(1/m), \quad \text{so}$$

$$\sum_{n=1}^{m} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln\left(\frac{n+1}{n}\right) = \gamma \ln 2 - \frac{\ln^2 2}{2} + O\left(m^{-1} \ln m\right).$$

Editor's comment: The sum in (3) is a Riemann sum whose limit as m tends to infinity equals the Riemann integral.

Solution 4 by Arkady Alt, San Jose, CA

Let $h_n = \sum\limits_{k=1}^{n} \frac{1}{k}$, $a_n = \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{k} - \ln 2$, and $S = \sum\limits_{n=1}^{\infty} a_n \ln \frac{n+1}{n}$.

Note that

$$
\begin{aligned}
\sum_{k=1}^{n} a_k \ln \frac{k+1}{k} &= \sum_{k=1}^{n} a_k \left(\ln(k+1) - \ln k\right) \\
&= \sum_{k=1}^{n} a_k \ln(k+1) - \sum_{k=1}^{n} a_k \ln k \\
&= \sum_{k=2}^{n+1} a_{k-1} \ln k - \sum_{k=2}^{n} a_k \ln k \\
&= a_n \ln(n+1) - \sum_{k=2}^{n} (a_k - a_{k-1}) \ln k \\
&= a_n \ln(n+1) - \sum_{k=2}^{n} \frac{(-1)^{k-1} \ln k}{k} \\
&= a_n \ln(n+1) + \sum_{k=2}^{n} \frac{(-1)^{k} \ln k}{k}.
\end{aligned}
$$

First we will prove $\lim\limits_{n\to\infty} a_n \ln(n+1) = 0$.

Since $a_{2n+1} = a_{2n} + \dfrac{1}{2n+1}$ then it suffices to prove

$$\lim_{n\to\infty} a_{2n} \ln(2n+1) = 0.$$

We have $a_{2n} = h_{2n} - h_n - \ln 2$ and, since $\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where $\gamma = \lim\limits_{n\to\infty}(h_n - \ln n)$ is Euler's constant, then

$$\begin{aligned}
& \ln 2n - \ln(n+1) - \ln 2 < a_{2n} < \ln(2n+2) - \ln n - \ln 2 \\
\iff \quad & -\ln\frac{n+1}{n} < a_{2n} < \ln\frac{n+1}{n} \\
\iff \quad & |a_{2n}| < \ln\frac{n+1}{n} < \frac{1}{n} \\
& \left(1+\frac{1}{n}\right)^n < e \iff \ln\frac{n+1}{n} < \frac{1}{n}.
\end{aligned}$$

Hence, $0 < |a_{2n}| \ln(2n+1) < \frac{\ln(2n+1)}{n}$ yields $\lim\limits_{n\to\infty} \frac{\ln(2n+1)}{n} = 0$, and, therefore $\lim\limits_{n\to\infty} a_{2n} \ln(2n+1) = 0$.

Thus, $S = \lim\limits_{n\to\infty} \sum\limits_{k=2}^{n} s_n$, where $s_n := \sum\limits_{k=2}^{n} \frac{(-1)^k \ln k}{k}$.

Since $s_{2n+1} = s_{2n} - \frac{\ln(2n+1)}{2n+1}$ and $\lim\limits_{n\to\infty} \frac{\ln(2n+1)}{2n+1} = 0$ then $S = \lim\limits_{n\to\infty} s_{2n}$.

Let $b_n := \sum\limits_{k=1}^{n} \frac{\ln k}{k}$ then

$$\begin{aligned}
s_{2n} &= \sum_{k=1}^{2n} \frac{(-1)^k \ln k}{k} \\
&= \sum_{k=1}^{n} \frac{\ln 2k}{2k} - \sum_{k=1}^{n} \frac{\ln(2k-1)}{2k-1} \\
&= 2\sum_{k=1}^{n} \frac{\ln 2k}{2k} - \sum_{k=1}^{2n} \frac{\ln k}{k} \\
&= \sum_{k=1}^{n} \frac{\ln 2k}{k} - b_{2n} \\
&= \sum_{k=1}^{n} \frac{\ln 2}{k} + \sum_{k=1}^{n} \frac{\ln k}{k} - b_{2n} \\
&= \ln 2 \cdot h_n + b_n - b_{2n}.
\end{aligned}$$

Consider now two sequences $\left(b_n - \frac{\ln^2(n+1)}{2}\right)_{n\geq 1}$ and $\left(b_n - \frac{\ln^2 n}{2}\right)_{n\geq 1}$.

Since $b_n - \frac{\ln^2(n+1)}{2}$ is increasing and $b_n - \frac{\ln^2 n}{2}$ is decreasing in n then

$$b_1 - \frac{\ln^2 2}{2} \leq b_n - \frac{\ln^2(n+1)}{2} < b_n - \frac{\ln^2 n}{2} \leq b_1$$

and, therefore, both sequences converges to the same limit.

Let $\delta = \lim_{n\to\infty}\left(b_n - \frac{\ln^2(n+1)}{2}\right) = \lim_{n\to\infty}\left(b_n - \frac{\ln^2 n}{2}\right)$ then

$$b_n - \frac{\ln^2(n+1)}{2} < \delta < b_n - \frac{\ln^2 n}{2} \iff \frac{\ln^2 n}{2} + \delta < b_n < \frac{\ln^2(n+1)}{2} + \delta, n \in N.$$

Hence,

$$\begin{aligned} \frac{\ln^2 2n - \ln^2(n+1)}{2} &< b_{2n} - b_n < \frac{\ln^2(2n+2) - \ln^2 n}{2} \iff \\ \beta_n &< b_{2n} - b_n - \ln 2 \cdot \ln n < \alpha_n, \end{aligned}$$

where $\alpha_n = \dfrac{\ln^2(2n+2) - \ln^2 n}{2} - \ln 2 \cdot \ln n$ and $\beta_n = \dfrac{\ln^2 2n - \ln^2(n+1)}{2} - \ln 2 \cdot \ln n$.

Noting that

$$\frac{\ln^2 2n - \ln^2 n}{2} - \ln 2 \cdot \ln n = \frac{\ln 2(\ln 2 + 2\ln n)}{2} - \ln 2 \cdot \ln n = \frac{\ln^2 2}{2}, \text{we obtain}$$

$$\lim_{n\to\infty}\left(\alpha_n - \frac{\ln^2 2}{2}\right) = \lim_{n\to\infty}\left(\frac{\ln^2(2n+2) - \ln^2 2n}{2}\right) = \frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)\ln(4n(n+1)) = 0, \text{ and}$$

$$\lim_{n\to\infty}\left(\beta_n - \frac{\ln^2 2}{2}\right) = \lim_{n\to\infty}\frac{\ln^2 n - \ln^2(n+1)}{2} = -\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)\ln(n(n+1)) = 0.$$

This gives us

$$\lim_{n\to\infty}(b_{2n} - b_n - \ln 2 \cdot \ln n) = \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \frac{\ln^2 2}{2}.$$

Since $\lim_{n\to\infty}(h_n - \ln n) = \gamma$ then

$$\begin{aligned} S &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot h_n) \\ &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot \ln n + \ln 2 \cdot (h_n - \ln n)) \\ &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot \ln n) + \lim_{n\to\infty}\ln 2 \cdot (h_n - \ln n) \end{aligned}$$

$$= \ln 2 \cdot \left(\gamma - \frac{\ln 2}{2}\right).$$

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Editor's comment: Mea Culpa once again. I inadvertently gave credit to David Stone and John Hawkins for having solved problem 5199 when they should have been credited for having solved 5198. And I inadvertently forgot to acknowledge **Achilleas Sinefakopoulos of Larissa, Greece** for having correctly solved 5184.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2013

- **5224:** *Proposed by Kenneth Korbin, New York, NY*

  Let $T_1 = T_2 = 1, T_3 = 2$, and $\mathrm{T}_n = T_{n-1} + T_{n-2} + T_{n-3}$. Find the value of

  $$\sum_{n=1}^{\infty} \frac{T_n}{\pi^n}.$$

- **5225:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Find infinitely many integer squares x that are each the sum of a square and a cube and a fourth power of positive integers a, b, c. That is, $x = a^2 + b^3 + c^4$.

- **5226:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Calculate:

  $$\int_a^b \frac{\sqrt[n]{x-a}\left(1+\sqrt[n]{b-x}\right)}{\sqrt[n]{x-a} + 2\sqrt[n]{-x^2+(a+b)x-ab} + \sqrt[n]{b-x}}dx,$$

  where $0 < a < b$ and $n > 0$.

- **5227:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Compute

  $$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}} \right).$$

- **5228:** *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada*

  Given a random variable X with non-negative integer values. Assume the n^{th} moment of X is given by

  $$E\left(X^n\right) = \sum_{k=1}^{\infty} f_n(k) P\left(X \geq k\right) \qquad n = 1, 2, 3, \cdots,$$

where f_n is a non-negative function defined on N. Find a closed form expression for f_n.

- **5229:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $\beta > 0$ and let $(x_n)_{n\in N}$ be the sequence defined by the recurrence relation

  $$x_1 = a > 0,\ x_{n+1} = x_n + \frac{n^{2\beta}}{x_1 + x_2 \cdots + x_n}.$$

  1) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

  2) Calculate $\lim\limits_{n\to\infty} \dfrac{x_n}{n^\beta}$.

Solutions

- **5206:** *Proposed by Kenneth Korbin, New York, NY*

  The distances from the vertices of an equilateral triangle to an interior point P are $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ respectively, where $a, b,$ and c are positive integers.

  Find the minimum and the maximum possible values of the sum $a+b+c$ if the side of the triangle is 13.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton ,SC

We show that $a+b+c$ has a minimum value of 170 and a maximum value of 296. We

model the given triangle using vertices $A(0,0)$, $B(13,0)$, and $C(13/2, 13\sqrt{3}/2)$. Then the centroid of triangle ABC is $G(13/2, 13\sqrt{3}/6)$. Let $P(x,y)$ be a point interior to $\triangle ABC$. We denote $AP = \sqrt{a}$, $BP = \sqrt{b}$, and $CP = \sqrt{c}$ for positive integers a, b, and c; due to the symmetry of the equilateral triangle, we may assume without loss of generality that $a \le b \le c$. It is then straightforward to verify that

$$a+b+c = AG^2 + BG^2 + CG^2 + 3PG^2 = 169 + 3PG^2.$$

Since $AG^2 = 169/3$ is not an integer, we know $P \ne G$, so the minimum value of $a+b+c$ is greater than 169 and thus must be at least 170. In fact, taking P to be $(6, 2\sqrt{3})$ achieves this minimum value of 170, with $(a,b,c) = (48, 61, 61)$.

Next, we note that $x^2+y^2 = a$ and $(13-x)^2 + y^2 = b$, so $x = (a-b+169)/26$. If $a=1$, then P lies on the circle $x^2+y^2=1$, so $1/2 < x < 1$ and hence $14 \le a-b+169 \le 25$. A quick check produces $(a,b,c) = (1, 147, 148)$ for $x = 23/26$ and $y = 7\sqrt{3}/26$, so the maximum value of $a+b+c$ is no smaller than 296. If $a \ge 2$, then PG is less than the distance from G to either $(\sqrt{2}/2, \sqrt{6}/2)$ or $(\sqrt{2}, 0)$, the intersections of the circle $x^2+y^2=2$ with the triangle. This yields

$$PG^2 < \frac{175}{3} - 13\sqrt{2} < 40,$$

so $a+b+c < 169 + 3(40) = 289$. Hence the maximum value of $a+b+c$ is 296.

Solution 2 by Paul M. Harms, North Newton, KS

Put the equilateral triangle of the problem on a coordinate system with

$$A(-6.5,0),\ B(0,6.5\sqrt{3}),\ C(6.5,0) \text{ with } P(x,y).$$

Then

$$\begin{aligned} a &= (x+6.5)^2+y^2, \\ b &= x^2+(y-6.5\sqrt{3})^2, \\ c &= (x-6.5)^2+y^2. \end{aligned}$$

Let $L=a+b+c$ and, temporarily, consider the domain of L to be the triangle and its interior. Using partial derivatives we find that L has a minimum of 169 at $x=0$, and $y=\dfrac{13\sqrt{3}}{6}$. At this point $a=b=c=\dfrac{169}{3}$. Other extremes may occur along the boundary of the domain. Checking for extremes along AC, we find an absolute maximum of 338 at each vertex and a minimum of 211.25 when $x=0$. The absolute minimum is then 169 and occurs at the one point $\left(0,\dfrac{13\sqrt{3}}{6}\right)$. At this point for a minimum L, the numbers a,b,and c are not integers. Then to satisfy the problem L must be at least 170. Also, the absolute maximum found above occurs at the vertices, and not at a point interior to the triangle, so this maximum will not satisfy the problem.

Consider L along $(0,y)$ where $0<y<6.5\sqrt{3}$. Here

$$\begin{aligned} a &= c=(6.5)^2+y^2 \\ b &= \left(y-6.5\sqrt{3}\right)^2. \end{aligned}$$

Then $y=6.5\sqrt{3}-\sqrt{b}$, so $a=c=4\,(6.5)^2+b^2-13\sqrt{3b}$ with $0<\sqrt{b}<6.5\sqrt{3}$.

We see that a,b and c will be integers when b is three times a perfect square. For these values of b, L is a minimum of 170 when $b=3(16)=48, a=c=61$. For these values of b, L is a maximum of 269, when $b=3(1)=3, a=c=133$. This minimum value of L satisfies the problem since the point is interior to the triangle with integer values for a,b, and c.

To check interior points for a maximum L, we check points close to a vertex, since for the general domain, the maximum occurs at a vertex.

Let us consider circles with radius $\sqrt{b}$ where b is an integer and the center of the circle is B.
For the problem, we only need to consider the portion of the circle interior to the triangle and in the first quadrant. Consider a first quadrant point P, interior to the triangle and on the circle with center at B and radius $\sqrt{b}$. Using the law of cosines for $\triangle ABP$ and $\triangle PBC$, we have

$$\begin{cases} a=13^2+b-2(13)\sqrt{b}\cos\theta \text{ and} \\ c=13^2+b-2(13)\sqrt{b}\cos\left(60^\circ-\theta\right), \text{where } 0<\theta<60^\circ. \end{cases}$$

When $\theta=60^\circ$, we have integers for a,b, and c when b is a perfect square, but P is then on a side of the triangle and not interior to the triangle. When $b=1$, the possible integers for c are $145, 146$, and 147. We find that when $c = 147$, and $b = 1, a = 148$ with $L = 296$. For a fixed positive integer b and $30^\circ < \theta < 60^\circ$, the maximum $(a + c)$ occurs at 60°. Checking other values of b, we find that the maximum L is less than 296 for integers $b > 1$.

Thus for positive integers, a, b and c with P interior to the triangle, the minimum L is 170 and the maximum L is 296.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the minimum is 170 and the maximum is 296. The minimum occurs when $a = 61, b = 48$ and $c = 61$ and the maximum occurs when $a = 148, b = 1$ and $c = 147$.

Denote the triangle by ABC with $PA = \sqrt{a},\ PB = \sqrt{b},\ PC = \sqrt{c}$. Let $\angle PBA = \theta$ and $\angle$PBC$=\phi$. Applying the cosine formula respectively to triangle PBA and PBC we obtain

$$\cos\theta = \frac{169 + b - a}{26\sqrt{b}} \text{ and } \cos\phi = \frac{169 + b - c}{26\sqrt{b}}.$$

Hence $\sin\theta = \dfrac{\sqrt{676b - (169 + b - a)^2}}{26\sqrt{b}}$ and $\sin\phi = \dfrac{\sqrt{676b - (169 + b - c)^2}}{26\sqrt{b}}$.

Since

$$\sin\theta\sin\phi = \cos\theta\cos\phi - \cos(\theta + \phi) = \cos\theta\cos\phi - \frac{1}{2}, \text{ so}$$

$$\left(\sqrt{676b - (169 + b - a)^2}\right)\left(\sqrt{676b - (169 + b - c)^2}\right) = (169 + b - a)(169 + b - c) - 338b.$$

Squaring both sides, expanding and simplifying, we obtain the equation

$$a^2 - (169 + b + c)a + b^2 + c^2 - bc - 169b - 169c + 28561 = 0. \text{ Hence}$$

$$a = \frac{1}{2}\left(169 + b + c \pm \sqrt{3}\sqrt{\left(\sqrt{b} + \sqrt{c} + 13\right)\left(\sqrt{b} + \sqrt{c} - 13\right)\left(\sqrt{b} - \sqrt{c} + 13\right)\left(\sqrt{c} - \sqrt{b} + 13\right)}\right).$$

By considering the special case $a = b = c = \dfrac{169}{3}$, we see that in fact

$$a = \frac{1}{2}\left(169 + b + c - \sqrt{3}\sqrt{\left(\sqrt{b} + \sqrt{c} + 13\right)\left(\sqrt{b} + \sqrt{c} - 13\right)\left(\sqrt{b} - \sqrt{c} + 13\right)\left(\sqrt{c} - \sqrt{b} + 13\right)}\right).$$

We now obtain the minimum and maximum values of $a + b + c$ stated above with the help of a computer. Here we impose the restrictions $1 \le b \le 168,\ b \le c \le a \le 168$ by symmetry, $\sqrt{a} + \sqrt{b} > 13,\ \sqrt{b} + \sqrt{c} > 13, \sqrt{c} + \sqrt{a} > 13$, and that a is a positive integer. This completes the solution.

Solution 4 by Albert Stadler of Herrliberg, Switzerland

Let $\alpha = \angle APB,\ \beta = \angle BPC, \gamma = \angle CPA$. Then by the law of cosines,

$$\cos\alpha = \frac{a + b - 169}{2\sqrt{ab}},\ \cos\beta = \frac{b + c - 169}{2\sqrt{bc}},\ \cos\gamma = \frac{c + a - 169}{2\sqrt{ca}}.$$

Obviously $\alpha + \beta + \gamma = 2\pi$. So

$$\cos\gamma = \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$(\cos\gamma - \cos\alpha\cos\beta)^2 = (1-\cos^2\alpha)(1-\cos^2\beta),$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma - 1 = 0,$$

$$\left(\frac{a+b-169}{2\sqrt{ab}}\right)^2 + \left(\frac{b+c-169}{2\sqrt{bc}}\right)^2 + \left(\frac{c+a-169}{2\sqrt{ca}}\right)^2 - 2\left(\frac{a+b-169}{2\sqrt{ab}}\right)\left(\frac{b+c-169}{2\sqrt{bc}}\right)\left(\frac{c+a-169}{2\sqrt{ca}}\right) = 1,$$

which is equivalent to

$$3\left(a^2+b^2+c^2+13^4\right) = \left(a+b+c+13^3\right)^2, \qquad (1)$$

as is seen when multiplying out.

A computer search on the set $\{(a,b,c)|1 \le a,b,c \le 169\}$ reveals that only the tuples of the table in the appendix satisfy (1) The minimal value of $a+b+c$ is 170 and the maximal value is 296.

Editor's note: Ken Korbin, proposer of the problem, also worked with the formula:

$$3\left(a^2+b^2+c^2+13^4\right) = \left(a+b+c+13^3\right)^2.$$

Albert presented a table listing all possible values satisfying the conditions of the problem. His appendix consisted of a table containing 258 rows for the various values of a, b and c; a few of rows are reproduced below.

David Stone and John Hawkins of Statesboro, GA *noted that it can be shown that the quantity $\sqrt{a}+\sqrt{b}+\sqrt{c}$, the sum of the distances from P to the three vertices, achieves its minimum of $\sqrt{3}s$ at the centroid of the triangle, and it achieves its maximum of $2s$ at any vertex (where "s" is a positive integer representing the side length of the equilateral triangle.)*

They also observed that because it is defined as the sum of the square of the distances to the vertices, the quantity $a+b+c$ can properly be called the **moment of inertia of the triangle about the point** P. *They showed that this moment of inertia of an equilateral triangle is minimized when P is the centroid and maximized at any vertex. The same conclusion holds for a square and, they hypothesize, for any regular polygon.*

Stadler's Table

a	b	c	$a+b+c$
48	61	61	170
49	57	64	170
49	64	57	170
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	147	148	296
147	1	148	296
157	1	144	302
157	144	1	302

Also solved by Farideh Firoozbakht and Jahangeer Kholdi (jointly), Isfahan, Iran; Adrian Naco, Polytechnic University, Tirana, Albania, David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro GA, and the proposer.

- **5207:** *Proposed by Roger Izard, Dallas, TX*

Consider the following four algebraic terms:

$$\begin{aligned} T_1 &= a^2(b+c)+b^2(a+c)+c^2(a+b) \\ T_2 &= (a+b)(a+c)(b+c) \\ T_3 &= abc \\ T_4 &= \frac{b+c-a}{a}+\frac{a+c-b}{b}+\frac{a+b-c}{c} \end{aligned}$$

Suppose that $\dfrac{T_1\cdot T_2}{(T_3)^2}=\dfrac{616}{9}$. What values would then be possible for T_4?

Solution by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We show the possible values of T_4 are $\dfrac{13}{3}$ and $-\dfrac{37}{3}$.

For convenience, let $T_5=\dfrac{a+b}{c}+\dfrac{b+c}{a}+\dfrac{a+c}{b}$.
Note that

$$\begin{aligned} T_4 &= \frac{b+c-a}{a}+\frac{a+c-b}{b}+\frac{a+b-c}{c} \\ &= \frac{b+c}{a}-1+\frac{a+c}{b}-1+\frac{a+b}{c}-1 \end{aligned}$$

$$= \quad T_5 - 3.$$

Now we expand and simplify:

$$\begin{aligned}
T_2 &= (a+b)(a+c)(b+c) = a^2b + abc + ab^2 + b^2c + a^2c + ac^2 + abc + bc^2 \\
&= \frac{a}{c}T_3 + 2T_3 + \frac{b}{c}T_3 + \frac{b}{a}T_3 + \frac{a}{b}T_3 + \frac{c}{b}T_3 + \frac{c}{a}T_3 \\
&= T_3\left(2 + \frac{a+b}{c} + \frac{b+c}{a} + \frac{a+c}{b}\right) \\
&= T_3\,(2 + T_5)\,.
\end{aligned}$$

Therefore, $\dfrac{T_2}{T_3} = T_5 + 2$.

Similarly, $T_1 = T_3T_5$, so $\dfrac{T_1}{T_3} = T_5$.

Therefore, $\dfrac{616}{9} = \dfrac{T_1 \cdot T_2}{(T_3)^2} = \dfrac{T_1}{T_3}\dfrac{T_2}{T_3} = T_5\,(T_5 + 2)$.

Hence,

$$\begin{aligned}
T_5^2 + 2T_5 - \frac{616}{9} &= 0 \\
\left(T_5 + \frac{28}{3}\right)\left(T_5 - \frac{22}{3}\right) &= 0. \text{ Thus,} \\
T_5 = -\frac{28}{3} \quad \text{or} \quad T_5 &= \frac{22}{3}, \text{ so,} \\
T_4 = -\frac{28}{3} - 3 = -\frac{37}{3} \quad \text{or} \quad T_4 &= \frac{22}{3} - 3 = \frac{13}{3}.
\end{aligned}$$

Comment: The question still unanswered—do there exist values of a, b, and c which make all of this happen?

Editor's remark: The above question was answered by **Albert Stadler of Herrliberg, Switzerland.** *In his solution to this problem he stated that both values obtained for T_4 are actually assumed: for instance for $(a,b,c) = \left(1, 1, \dfrac{-17+\sqrt{253}}{6}\right)$ and for $(a,b,c) = \left(1, 1, \dfrac{4+\sqrt{7}}{3}\right)$.*

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Ben Carani, Jordan Melendez, Caleb Stevenson (students at Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Samuel David Judge, Justin Wydra, and Karen Wydra (students at Taylor University), Upland, IN; Kee-Wai Lau, Hong, Kong, China; Adrian Naco,

Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics "Tor Vergata University," Rome, Italy; Jungmin Song, Nate Armstrong and Alex Senyshyn (students at Taylor University), Upland IN; Howard Sporn, Great Neck, NY, and the proposer.

- **5208:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania*

Let the sequence of positive real numbers $\{a_n\}_{n\geq 1},\ N\in Z^+$ be such that $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2\cdot a_n}=b.$ Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right).$$

Solution 1 by Anastasios Kotronis, Athens, Greece

Setting $z_n:=\frac{a_n}{n^{2n}}$, we have

$$\frac{z_{n+1}}{z_n}=\frac{a_{n+1}}{n^2a_n}\left[\left(1+\frac{1}{n}\right)^n\right]^{-2}\left(1+\frac{1}{n}\right)^{-2}\to be^{-2}, \tag{1}$$

and by Cesàro Stolz:

$$\begin{aligned}\lim_{n\to+\infty} z_n^{1/n} &= \exp\left(\lim_{n\to+\infty}\frac{\ln z_n}{n}\right)\\ &= \exp\left(\lim_{n\to+\infty}\ln\frac{z_{n+1}}{z_n}\right)\\ &= \exp\left(\ln\lim_{n\to+\infty}\frac{z_{n+1}}{z_n}\right)\\ &= be^{-2}. \end{aligned}\tag{2}$$

On account of (1) and (2):

$$\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)^n=\left(1+\frac{1}{n}\right)^n\frac{z_{n+1}}{z_n}z_{n+1}^{-\frac{1}{n+1}}\to e,$$

so

$$\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}=z_n^{1/n}\left(\frac{\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}-1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)^n\right)\to be^{-2},$$

since

$$\lim_{n\to+\infty}\frac{\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}-1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}=\lim_{n\to+\infty}\frac{\exp\left(\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)\right)-1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}=1.$$

Solution 2 by proposers

We have

(1) $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{2n+2}}\cdot\frac{n^{2n}}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^2a_n}\cdot\frac{1}{e_n^2}=\frac{b}{e^2}$, where $e_n=\left(1+\frac{1}{n}\right)^n$. (The second equality in the chain follows from the Cauchy-D'Alembert criteria.)

(2) Denote $u_n=\dfrac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\cdot\dfrac{n}{n+1},\forall n\geq 2$ and we deduce that

(3) $\displaystyle\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\cdot\frac{n^2}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n}\right)=\frac{b}{e^2}\cdot\frac{e^2}{b}\cdot 1=1,$ respectively

(4) $\displaystyle\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1.$

(5) $\displaystyle\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^n\right)=$
$\displaystyle\lim_{n\to\infty}\left(\frac{a_{n+1}}{n^2a_n}\cdot\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{1}{e_n}\cdot\left(\frac{n}{n+1}\right)^2\right)=b\cdot\frac{e^2}{b}\cdot\frac{1}{e}\cdot 1=e.$

(6) Denote $\displaystyle x_n=\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right)=\frac{\sqrt[n]{a_n}}{n}\cdot\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\cdot\frac{n}{n+1}-1\right)=$
$\displaystyle\frac{\sqrt[n]{a_n}}{n}(u_n-1)=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{\sqrt[n]{a_n}}{n^2}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n.$

By (1), (4), (5) and (6) we obtain
(7) $\displaystyle L=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right)=\lim_{n\to\infty}x_n=\frac{b}{e^2}\cdot 1\cdot\ln e=\frac{b}{e^2}.$

Also solved by Arkady Alt, San-Jose, CA; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; and Albert Stadler, Herrliberg, Switzerland.

- **5209:** *Proposed by Tom Moore, Bridgewater, MA*

  We noticed that 27 is a cube and 28 is an even perfect number. Find all pairs of consecutive integers such that one is cube and the other is an even perfect number.

Solution by Kee-Wai Lau, Hong Kong, China

We show that 27 and 28 are the only consecutive integers such that one is cube and the other is an even perfect number.

It is well known that every even perfect number is of the form $2^{p-1}(2^p-1)$, where 2^p-1 is a prime. Suppose $2^{p-1}(2^p-1)=a^3+1$, where a is an odd integer, then since

$a^3+1=(a+1)(a^2-a+1)$, we have $a+1=2^{p-1}$ and $a^2-a+1=2^p-1$. Hence $a^2-a+1=2a+1$ or $a=3$. This gives the pair 27 and 28.

Next we suppose that $2^{p-1}\left(2^p-1\right)=b^3-1$, where b is an odd integer, then since $b^3-1=(b-1)\left(b^2+b-1\right)$, we have $b-1=2^{p-1}$ and $b^2+b+1=2^p-1$. Hence $b^2+b+1=2b-3$ or $b^2-b+4=0$, which gives no real solutions.

This completes the solution.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo Sate University, San Angelo TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Farideh Firoozbakht and Jahangeer Kholdi (jointly), Isfahan, Iran; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5210:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be four positive real numbers. Prove that

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>\frac{2\sqrt{3}}{3}.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker, Angelo State University, San Angelo, TX

We will establish the slightly improved inequality

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>\frac{7}{6}.$$

This is a little better than the given result because

$$\frac{7}{6}-\frac{2\sqrt{3}}{3}=\frac{7-4\sqrt{3}}{6}=\frac{1}{6\left(7+4\sqrt{3}\right)}>0.$$

We begin with the following known inequality:
If $x_1, x_2, \ldots, x_n>0$, then

$$\left(x_1+x_2+\ldots+x_n\right)\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)\geq n^2. \qquad (1)$$

This follows from applying the Cauchy-Schwarz Inequality to the vectors
$\mathbf{x}=\left(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_n}\right)$ and $\mathbf{y}=\left(\frac{1}{\sqrt{x_1}}, \frac{1}{\sqrt{x_2}}, \ldots, \frac{1}{\sqrt{x_n}}\right)$.

If we let $x_1=a+b+c$, $x_2=b+c+d$, $x_3=c+d+a$, and $x_4=d+a+b$, then since $a, b, c, d>0$, statement (1) implies that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}$$

$$
\begin{aligned}
&> \frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c} \\
&= \left(\frac{a}{b+c+d}+1\right)+\left(\frac{b}{c+d+a}+1\right)+\left(\frac{c}{d+a+b}+1\right)+\left(\frac{d}{a+b+c}+1\right)-4 \\
&= (a+b+c+d)\left(\frac{1}{a+b+c}+\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b}\right)-4 \\
&= \frac{1}{3}(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)-4 \\
&\geq \frac{16}{3}-4 \\
&= \frac{4}{3}.
\end{aligned}
$$

Therefore,

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right) > 1+\frac{1}{8}\left(\frac{4}{3}\right)=\frac{7}{6},$$

and our proof is complete.

Comments: **Kee-Wai Lau of Hong Kong, China** remarked that D.S. Mitrinović $\Big($Analytic Inequalities, Springer Verlag (1970; p. 132)$\Big)$ and L. J. Mordell $\Big($On the inequality $\sum x_r/(x_{r+}+x_{r+2}) \geq \frac{1}{2}n$ Abh. Math. Sem. Univ. Hamburg 22, (1958; pp 229-241)$\Big)$ shown that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2.$$

So the present problem can be sharpened to

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right) \geq \frac{5}{4}.$$

Albert Stadler of Herrliberg, Switzerland noted that the problem statement is a generalization of Nesbitt's inequality to four variables (see http://en.wikipedia.org/wiki/Nesbitt's_inequality). However this generalization is well known: see e.g., Pham Kim Hung's text "Secrets in Inequalities" (GIL Publishing House 2007.) Albert also noted that the inequality can be sharpened to ≥ 1.25, and he presented the proof in Kim Hung's text.

Prove that for all non-negative real numbers a,b,c,d,

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2.$$

Consider the following expressions

$$
\begin{aligned}
S &= \frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}; \\
M &= \frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}+\frac{a}{a+b}; \\
N &= \frac{c}{b+c}+\frac{d}{c+d}+\frac{a}{d+a}+\frac{b}{a+b};
\end{aligned}
$$

We have $M+N=4$. According to AM-GM, we get

$$\begin{aligned}
M{+}S &= \frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}\geq 4;\\
N{+}S &= \frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{a+c}{d+a}+\frac{b+d}{a+b}\\
&= \frac{a+c}{b+c}+\frac{a+c}{a+d}+\frac{b+d}{c+d}+\frac{b+d}{a+b}\\
&\geq \frac{4(a+c)}{a+b+c+d}+\frac{4(b+d)}{a+b+c+d}=4.
\end{aligned}$$

Therefore, $M+N+2S\geq 8$, and $S\geq 2$. The equality holds if $a=b=c=d$ or $a=c, b=d=0$ or a=c=0, b=d.

Also solved by Arkady Alt, San Jose, CA; D.M. Bătinetu-Giurgiu, Bucharest, Neculai Stanciu Buzău and Titu Zvonaru Comănesti, all from Romania (two solutions); Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.

- **5211:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Let $n\geq 1$ be a natural number and let

$$f_n(x)=x^{x^{\cdot^{\cdot^{\cdot^{x}}}}},$$

where the number of x's in the definition of f_n is n. For example

$$f_1(x)=x,\quad f_2(x)=x^x,\quad f_3(x)=x^{x^x},\ldots.$$

Calculate the limit

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show the limit equals $(-1)^n$. Define $f_0(x)=1$. For $n\geq 2$ and $x>0$, we have $f_n(x)=e^{f_{n-1}(x)\ln x}$. Hence by the mean value theorem, we have

$$f_n(x)-f_{n-1}(x)=\ln x\,(f_{n-1}(x)-f_{n-2}(x))\,e^{\xi},$$

where ξ lies between $f_{n-1}(x)\ln x$ and $f_{n-2}(x)\ln x$.

Since $\lim\limits_{x\to 1}f_{n-1}(x)\ln x=\lim\limits_{x\to 1}f_{n-2}(x)\ln x=0$ and $\lim\limits_{x\to 1}\dfrac{\ln x}{1-x}=-1$, so

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=-\lim_{x\to 1}\frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}.$$

Clearly $\lim_{x\to1}\frac{f_1(x)-f_0(x)}{1-x}=-1$. Hence by induction we have

$$\lim_{x\to1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=(-1)^n,$$

as claimed.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We will use induction to prove that $a_n=\lim_{x\to1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=(-1)^n$.
We have by applying L'Hôpital's rule twice,

$$a_2=\lim_{x\to1}\frac{f_2(x)-f_1(x)}{(1-x)^2}=\lim_{x\to1}\frac{x^x-x}{(1-x)^2}=\lim_{x\to1}\frac{x^x(1+\log x)-1}{-2(1-x)}=\lim_{x\to1}\frac{x^x\left[(1+\log x)^2+\frac{1}{x}\right]}{2}=1.$$

So the assertion holds for $n=2$.

We have $\frac{d}{dx}f_n(x)=\frac{d}{dx}e^{f_{n-1}(x)\log x}=f_n(x)\left(f_{n-1}'(x)\log x+\frac{f_{n-1}(x)}{x}\right)$. In particular,

$$f_n'(1)=f_n(1)\left(f_{n-1}'(1)\log(1)+\frac{f_{n-1}(1)}{1}\right)=1.$$

So, by L'Hôpital's rule,

$$\begin{aligned}
a_n &= \lim_{x\to1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=\lim_{x\to1}\frac{f_n'(x)-f_{n-1}'(x)}{-n(1-x)^{n-1}}\\
&= \lim_{x\to1}\frac{f_n(x)\left(f_{n-1}'(x)\log x+\frac{f_{n-1}(x)}{x}\right)-f_{n-1}(x)\left(f_{n-2}'(x)\log x+\frac{f_{n-2}(x)}{x}\right)}{-n(1-x)^{n-1}}\\
&= \lim_{x\to1}\frac{\left(f_n(x)-(f_{n-1}(x)\right)\left(f_{n-1}'(x)\log x+\frac{f_{n-1}(x)}{x}\right)}{-n(1-x)^{n-1}}\\
&+ \lim_{x\to1}\frac{\left(f_{n-1}(x)\left(f_{n-1}'(x)\log x+\frac{f_{n-1}(x)}{x}-f_{n-2}'(x)\log x-\frac{f_{n-2}(x)}{x}\right)\right.}{-n(1-x)^{n-1}}.
\end{aligned}$$

So

$$\begin{aligned}
a_n &= \lim_{x\to1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\left(1+\frac{\left(f_{n-1}'(x)\log x+\frac{f_{n-1}(x)}{x}\right)(1-x)}{n}\right)\\
&= \lim_{x\to1}\frac{f_{n-1}(x)\left(f_{n-1}'(x)\log x+\frac{f_{n-1}(x)}{x}-f_{n-2}'(x)\log x-\frac{f_{n-2}(x)}{x}\right)}{-n(1-x)^{n-1}}
\end{aligned}$$

$$= \lim_{x\to 1}\frac{f'_{n-1}(x)-f'_{n-2}(x)}{-n(1-x)^{n-2}}\cdot\frac{\log x}{1-x}+\lim_{x\to 1}\frac{f_{n-1}(x)-f_{n-2}(x)}{-n(1-x)^{n-1}x}$$

$$= \frac{n-1}{n}\lim_{x\to 1}\frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}\cdot(-1)+\frac{1}{(-n)}\lim_{x\to 1}\frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}$$

$$= -a_{n-1}=-(-1)^{n-1}=(-1)^n.$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

At first we observe that the function is of the form

$$f_n(x)=x^{f_{n-1}(x)}=e^{f_{n-1}(x)\ln x}$$

and that

$$\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} = \frac{e^{f_{n-1}(x)\ln x}-e^{f_{n-2}(x)\ln x}}{(1-x)^n}=e^{f_{n-2}(x)\ln x}\left(\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{(1-x)^n}\right)$$

$$= e^{f_{n-2}(x)\ln x}\cdot\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\cdot\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}(x)]\ln x}. \quad (2)$$

The function $f_1(x)=x$ is continuous everywhere for $x>0$ and

$$\lim_{x\to 1} f_1(x)=\lim_{x\to 1} x=1.$$

One easily comes to the conclusion that the function $f_n(x)=e^{f_{n-1}(x)\ln x}$ is continuous everywhere for $x>0$ as a composition of a product of two continuous functions $u(x)=f_{n-1}(x)\ln x$ and the exponential function $f_n(x)=e^{u(x)}$ and as a logical result implies that

$$\lim_{x\to 1} f_n(x)=e^{\lim_{x\to 1}[f_{n-1}(x)\ln x]}=e^{\left[\lim_{x\to 1} f_{n-1}(x)\right]\cdot\left[\lim_{x\to 1}\ln x\right]}=e^{1\cdot 0}=1. \quad (3)$$

Using the known limit rule

$$\lim_{\alpha\to 0}\frac{e^\alpha-1}{\alpha}=1\Rightarrow\lim_{x\to 1}\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}(x)]\ln x}=1 \quad (4)$$

since

$$\lim_{x\to 1}\alpha(x) = \lim_{x\to 1}[f_{n-1}(x)-f_{n-2}(x)]\ln x$$

$$= \left[\lim_{x\to 1} f_{n-1}(x)-\lim_{x\to 1} f_{n-2}(x)\right]\left(\lim_{x\to 1}\ln x\right)$$

$$= (1-1)\cdot 0=0$$

So from formula (2) and (4) we derive the inductive result for one step.

$$
\begin{aligned}
&\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\\
=\ &\lim_{x\to 1} e^{f_{n-2}(x)\ln x}\cdot\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\cdot\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}(x)]\ln x}\\
=\ &\left(\lim_{x\to 1} e^{f_{n-2}(x)\ln x}\right)\cdot\left(\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\right)\cdot\left(\lim_{x\to 1}\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}(x)]\ln x}\right)\\
=\ &\left(e^{\lim\limits_{x\to 1} f_{n-2}(x)\ln x}\right)\cdot\left(\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\right)\cdot\left(\lim_{x\to 1}\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x}-1}{[f_{n-1}(x)-f_{n-2}(x)]\ln x}\right)\\
=\ &1\cdot\left(\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n}\right)\cdot 1=\lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln x}{(1-x)^n} \qquad (5)
\end{aligned}
$$

Inductively we derive the general formula

$$
\begin{aligned}
\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} &= \lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\ln^0 x\\
&= \lim_{x\to 1}\frac{[f_{n-1}(x)-f_{n-2}(x)]\ln^1 x}{(1-x)^n}\\
&= \lim_{x\to 1}\frac{[f_{n-2}(x)-f_{n-3}(x)]\ln^2 x}{(1-x)^n}\\
&\quad \cdots\cdots\cdots\\
&= \lim_{x\to 1}\frac{[f_2(x)-f_1(x)]\ln^{n-2} x}{(1-x)^n}=\lim_{x\to 1}\frac{[x^x-x]\ln^{n-2} x}{(1-x)^n}\\
&= \lim_{x\to 1}\frac{\left[e^{x\ln x}-e^{\ln x}\right]\ln^{n-2} x}{(1-x)^n}=\lim_{x\to 1}\frac{e^{\ln x}\left[e^{(x-1)\ln x}-1\right]\ln^{n-2} x}{(1-x)^n}\\
&= \lim_{x\to 1} e^{\ln x}\frac{\left[e^{(x-1)\ln x}-1\right]}{(x-1)\ln x}(x-1)\frac{\ln^{n-1} x}{(1-x)^n}\\
&= (-1)\left(\lim_{x\to 1} e^{\ln x}\right)\left(\lim_{x\to 1}\frac{\left[e^{(x-1)\ln x}-1\right]}{(x-1)\ln x}\right)\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1}\\
&= (-1)\cdot e^0\cdot 1\cdot\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1}=-\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1}.
\end{aligned}
$$

Applying L'Hôpital's rule we have that

$$\begin{aligned}\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} &= -\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1} = -\left[\lim_{x\to 1}\frac{(\ln x)'}{(1-x)'}\right]^{n-1}\\ &= -\left[\lim_{x\to 1}\frac{\frac{1}{x}}{(-1)}\right]^{n-1} = (-1)(-1)^{n-1} = (-1)^n.\end{aligned}$$

Editor's comment: There was a mistake in the statement of the problem when it first appeared on the web. That version asked for the $\lim_{x\to 1}\dfrac{f_n(x)-f_{n-1}(x)}{(1-x)^{n+1}}$. *This mistake was corrected almost immediately but not before a few of the readers started working with the incorrect statement of the problem; although those readers noted the error and corrected it in their solutions, once again, mea culpa. Most all who submitted solutions to this problem approached it with induction.*

Also solved by Arkady Alt, San Jose, CA; Anastasios Kontronis, Athens, Greece; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2013

- **5230:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers x, y, z such that

$$\begin{aligned} x^2 + xy + \frac{y^2}{3} &= 41, \\ \frac{y^2}{3} + z^2 &= 16, \\ x^2 + xz + z^2 &= 25. \end{aligned}$$

  Find the value of $xy + 2yz + 3xz$.

- **5231:** *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy*

  The lengths of the sides of the hexagon $ABCDEF$ satisfy $AB = BC, CD = DE$, and $EF = FA$. Prove that

$$\sqrt{\frac{AF}{CF}} + \sqrt{\frac{CB}{EB}} + \sqrt{\frac{ED}{AD}} > 2.$$

- **5232:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Prove that: If $a, b, c > 0$, then,

$$2\sqrt{\frac{a^2+b^2+c^2}{3}} \cdot \frac{\sin x}{x} + \frac{a+b+c}{3} \cdot \frac{\tan x}{x} > a+b+c,$$

  for any $x \in \left(0, \frac{\pi}{2}\right)$.

- **5233:** *Proposed by Anastasios Kotronis, Athens, Greece*

  Let $x \geq \frac{1+\ln 2}{2}$ and let $f(x)$ be the function defined by the relations:

$$f^2(x) - \ln f(x) = x$$

$$f(x) \quad \geq \quad \frac{\sqrt{2}}{2}.$$

- 1. Calculate $\displaystyle\lim_{x\to+\infty} \frac{f(x)}{\sqrt{x}}$, if it exists.
- 2. Find the values of $\alpha \in \Re$ for which the series $\displaystyle\sum_{k=1}^{+\infty} k^{\alpha}\left(f(k) - \sqrt{k}\right)$ converges.
- 3. Calculate $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x}f(x) - x}{\ln x}$, if it exists.

- **5234:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let $a < b$ be positive real numbers and let $f_1 : [a, b] \to \Re$ $(i = 1, 2)$ be continuous functions in $[a, b]$ and differentiable in (a, b). If f_2 is strictly decreasing then prove that there exists an $\alpha \in (a, b)$ such that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a).$$

- **5235:** *Proposed by Albert Stadler, Herrliberg, Switzerland*

  On December 21, 2012 ("$12 - 21 - 12$") the Mayan Calendar's 13^{th} Baktun cycle will end. On this date the world as we know it will also change (see <http://www.mayan-calendar.org/2012/end-of-the-world.html>). Since every end is a new beginning we are looking for natural numbers n such that the decimal representation of 2^n starts and ends with the digit sequence 122112. Let S be the set of natural numbers n such that $2^n = 122112....122112$. Let $s(x)$ be the number of elements of S that are $\leq x$.

  Prove that $\displaystyle\lim_{x\to\infty} \frac{s(x)}{x}$ exists and is positive. Calculate the limit.

Solutions

- **5212:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation

$$2x + y - \sqrt{3x^2 + 3xy + y^2} = 2 + \sqrt{2}$$

if x and y are of the form $a + b\sqrt{2}$ where a and b are positive integers.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

In a similar way to the published solution to SSM problem 5105, we let $x = a + b\sqrt{2}, y = c + d\sqrt{2}$ and $y = \alpha x$, where a, b, c and d are positive integers and α is a positive real number. Substituting into the given equation gives

$$2+\sqrt{2} = 2x+\alpha x-\sqrt{3x^2 + 3\alpha x^2 + \alpha^2 x^2} = \left(2 + \alpha - \sqrt{3 + 3\alpha + \alpha^2}\right) x = \varphi(\alpha)\left(a + b\sqrt{2}\right),$$

where $\varphi(\alpha) = \dfrac{(2 + \alpha)^2 - \left(\sqrt{3 + 3\alpha + \alpha^2}\right)^2}{2 + \alpha + \sqrt{3 + 3\alpha + \alpha^2}}$ is increasing (*editor's note:*

$\varphi'(\alpha) = 1 - \dfrac{3+2\alpha}{2\sqrt{3+3\alpha+\alpha^2}} > 0$, for $3+3\alpha+\alpha^2 > (1.5+\alpha)^2$.) and such that

$$\lim_{\alpha\to+\infty}\varphi(\alpha) = \lim_{\alpha\to+\infty}\frac{1/\alpha+1}{2/\alpha+1+\sqrt{3/\alpha^2+3/\alpha+1}} = \frac{0+1}{0+1+\sqrt{0+0+1}} = \frac{1}{2}.$$

On the other hand,

$$\varphi(0) \;=\; 2-\sqrt{3},\ \text{so } \varphi(0) \le \varphi(\alpha) < \lim_{\alpha\to+\infty}\varphi(\alpha) \text{ and hence,}$$

$$4+2\sqrt{2} \;<\; \frac{2+\sqrt{2}}{\varphi(\alpha)} \le \frac{2+\sqrt{2}}{2-\sqrt{3}},\ \text{that is,}$$

$$4+2\sqrt{2} \;<\; a+b\sqrt{2} \le \frac{2+\sqrt{2}}{2-\sqrt{3}}.$$

From this it follows that $b \le 9$ and that

$$\begin{cases} \text{if } b{=}0 \text{ then } 7 \le a \le 12, \\ \text{if } b{=}1 \text{ then } 6 \le a \le 11, \\ \text{if } b{=}2 \text{ then } 5 \le a \le 9, \\ \text{if } b{=}3 \text{ then } 3 \le a \le 8, \\ \text{if } b{=}4 \text{ then } 2 \le a \le 7, \\ \text{if } b{=}5 \text{ then } 0 \le a \le 5, \\ \text{if } b{=}6 \text{ then } 0 \le a \le 4, \\ \text{if } b{=}7 \text{ then } 0 \le a \le 2, \\ \text{if } b{=}8 \text{ then } 0 \le a \le 1, \text{and} \\ \text{if } b{=}9 \text{ then } a{=}0. \end{cases}$$

The given equation is equivalent to

$$\left[2x+y-\left(2+\sqrt{2}\right)\right]^2 = \left(\sqrt{3x^2+3xy+y^2}\right)^2,\ \text{that is,}$$

$$4x^2+4xy+y^2-\left(8+4\sqrt{2}\right)x-\left(4+2\sqrt{2}\right)y+4+4\sqrt{2}+2 = 3x^2+3xy+y^2.$$

So,

$$\begin{aligned} c+d\sqrt{2} \;&=\; y = \frac{x^2-\left(8+4\sqrt{2}\right)x+6+4\sqrt{2}}{4-x+2\sqrt{2}} \\ &=\; \frac{a^2+2b^2-8a-8b+6+(2ab-4a-8b+4)\sqrt{2}}{(4-a)+(2-b)\sqrt{2}} \\ &=\; \frac{\left[a^2+2b^2-8a-8b+6+(2ab-4a-8b+4)\sqrt{2}\right]\left[4-a+(b-2)\sqrt{2}\right]}{\left[4-a+(2-b)\sqrt{2}\right]\left[4-a+(b-2)\sqrt{2}\right]} \\ &=\; \frac{-a^3+2ab^2+12a^2-8b^2-8ab-22a+8b+8}{(4-a)^2-2(2-b)^2} + \end{aligned}$$

$$\frac{2b^3 - a^2b + 8ab + 2a^2 - 12b^2 - 4a - 10b + 4}{(4-a)^2 - 2(2-b)^2}\sqrt{2}.$$

So,

$$\begin{aligned} c &= \frac{-a^3 + 2ab^2 + 12a^2 - 8b^2 - 8ab - 22a + 8b + 8}{(4-a)^2 - 2(2-b)^2} \text{ and} \\ d &= \frac{2b^3 - a^2b + 8ab + 2a^2 - 12b^2 - 4a - 10b + 4}{(4-a)^2 - 2(2-b)^2}, \text{ where } c \text{ and } d \text{ are positive integers.} \end{aligned}$$

Restricting a, b, c, and d to be positive integers we see that there are eleven solutions (x, y) to the problem. These are obtained by letting $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$, where

$(a, b) \in \Big\{(6,1), (5,2), (6,2), (7,2), (3,3), (4,3), (5,3), (6,3), (2,4)(6,4), (1,5)\Big\}$ and respectively,

$(c, d) \in \Big\{(28,22), (17,12), (7,6), (3,4), (43,29), (12,8), (5,5), (4,2), (23,13), (1,1), (17,7)\Big\}$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If we re-write the equation in the form

$$2x + y - \left(2 + \sqrt{2}\right) = \sqrt{3x^2 + 3xy + y^2}$$

and then square both sides and simplify, we get successively

$$\begin{aligned} x^2 - 4\left(2+\sqrt{2}\right)x + xy - 2\left(2+\sqrt{2}\right)y + 2\left(3+2\sqrt{2}\right) &= 0 \text{ and} \\ \left[x - 2\left(2+\sqrt{2}\right)\right]^2 + \left[x - 2\left(2+\sqrt{2}\right)\right]y &= 6\left(3+2\sqrt{2}\right). \end{aligned}$$

To simplify further, substitute $w = x - 2\left(2+\sqrt{2}\right)$ to obtain

$$w^2 + wy = 6\left(3 + 2\sqrt{2}\right). \qquad (1)$$

From the given instructions for x and y, we have

$$w = a_1 + b_1\sqrt{2} \text{ and } y = a_2 + b_2\sqrt{2},$$

where a_1, b_1, a_2, b_2 are integers with $a_2, b_2 \geq 1$, $a_1 \geq -3$, and $b_1 \geq -1$. If these are substituted into (1) and we use the fact that for integers a, b, c, d, $a + b\sqrt{2} = c + d\sqrt{2}$ if and only if $a = c$ and $b = d$ we obtain the following system:

$$(b_1 + b_2)\,a_1 + (a_1 + a_2)\,b_1 = 12 \qquad (2)$$

$$(a_1+a_2)\,a_1+2\,(b_1+b_2)\,b_1 \quad = \quad 18. \qquad (3)$$

Note that from the above information about a_1, b_1, a_2, b_2, it follows that $a_1+a_2 \geq -2$ and $b_1+b_2 \geq 0$.

If $b_1+b_2=0$, then we must have $b_1=-1$ and $b_2=1$. Equation (2) becomes $a_1+a_2=-12$, which is clearly impossible. If $b_1+b_2=1$, then either $b_1=-1$ and $b_2=2$ or $b_1=0$ and $b_2=1$. When $b_1=-1$ and $b_1+b_2=1$, equation (2) reduces to $-a_2=12$, which is impossible. When $b_1=0$ and $b_1+b_2=1,$ (2) yields $a_1=12$ and (3) becomes $12\,(a_1+a_2)=18$, which is also impossible. Therefore, we will assume hereafter that $b_1+b_2\geq 2$.

If $a_1+a_2=-2$, then since $a_1\geq -3$ and $a_2\geq 1$, we get $a_1=-3$ and $a_2=1$. Equation (3) becomes $(b_1+b_2)\,b_1=6$. Since $b_1+b_2\geq 2$, it follows that $b_1\geq 1$. Then, (2) is of the form

$$-3\,(b_1+b_2)-2b_1=12,$$

which is clearly impossible with $b_1\geq 1$ and $b_1+b_2\geq 2$.

If $a_1+a_2=-1$, then since $a_2\geq 1$, it follows that $a_1<0$. However, equations (2) and (3) are

$$\begin{aligned}(b_1+b_2)\,a_1-b_1 &= 12\\ -a_1+2\,(b_1+b_2)\,b_1 &= 18\end{aligned}$$

and we get

$$a_1=6\frac{4\,(b_1+b_2)+3}{2\,(b_1+b_2)^2-1}>0$$

(since $b_1+b_2\geq 2$). Hence, this case is impossible .

If $a_1+a_2=0$, (2) and (3) reduce to

$$\begin{aligned}(b_1+b_2)\,a_1 &= 12\\ (b_1+b_2)\,b_1 &= 9.\end{aligned}$$

Since $b_1+b_2\geq 2$, this makes $a_1>0$, which is inconsistent with $a_1+a_2=0$.

If $a_1+a_2=1$, then $a_2\geq 1$ implies that $a_1\leq 0$. However, (2) and (3) become

$$\begin{aligned}(b_1+b_2)\,a_1+b_1 &= 12\\ a_1+2\,(b_1+b_2)\,b_1 &= 18\end{aligned}$$

and hence,

$$a_1=6\frac{4\,(b_1+b_2)-3}{2\,(b_1+b_2)^2-1}>0$$

(since $b_1+b_2\geq 2$). Therefore, this case is also impossible and we may assume in the remainder of this solution that $a_1+a_2\geq 2$.

In (2) and (3), if we treat a_1 and b_1 as coefficients and use Cramer's Rule, we obtain

$$a_1+a_2=6\frac{4b_1-3a_1}{2b_1^2-a_1^2}, \qquad b_1+b_2=6\frac{3b_1-2a_1}{2b_1^2-a_1^2} \quad \text{or}$$

$$a_2 = 6\frac{4b_1 - 3a_1}{2b_1^2 - a_1^2} - a_1, \qquad b_2 = 6\frac{3b_1 - 2a_1}{2b_1^2 - a_1^2} - b_1. \qquad (4)$$

If $a_1 = -3$, then

$$a_2 = 6\frac{4b_1 + 9}{2b_1^2 - 9} + 3 = 3\left(2\frac{4b_1 + 9}{2b_1^2 - 9} + 1\right)$$

and

$$b_2 = 18\frac{b_1 + 2}{2b_1^2 - 9} - b_1 = \frac{-2b_1^3 + 27b_1 + 36}{2b_1^2 - 9}.$$

Using elementary calculus, it is straightforward to show that when $b_1 \geq 5$, $-2b_1^3 + 27b_1 + 36 < 0$ and $2b_1^2 - 9 > 0$, and hence, $b_2 < 0$. Also, by direct substitution, $b_2 < 0$ when $b_1 = 0, \pm 1$, or 2. Therefore, we are left with $b_1 = 3$ or 4. Of these, $b_1 = 4$ yields fractional values for a_2 and b_2, while $b_1 = 3$ gives the solution $a_1 = -3, b_1 = 3, a_2 = 17, b_2 = 7$. Therefore, our first solution is $w = -3 + 3\sqrt{2}$, $x = w + 2\left(2 + \sqrt{2}\right) = 1 + 5\sqrt{2}$, $y = 17 + 7\sqrt{2}$.

If $a_1 = -2$, (4) becomes

$$a_2 = 2\left(3\frac{2b_1 + 3}{b_1^2 - 2} + 1\right) \text{ and } b_2 = \frac{3b_1 + 4}{2b_1^2 - 4} - b_1 = \frac{-b_1^3 + 11b_1 + 12}{b_1^2 - 2}.$$

Proceeding as before, we see that $b_2 < 0$ for $b_1 \geq 4$ and $a_2 < 0$ for $b_1 = 0$ or ± 1. If $b_1 = 3$, then a_2 is a fraction. However, $b_1 = 2$ yields the solution $a_1 = -2, b_1 = 2, a_2 = 23, b_2 = 13$. Therefore, our next solution is $w = -2 + 2\sqrt{2}$, $x = 2 + 4\sqrt{2}$, $y = 23 + 13\sqrt{2}$.

If $a_1 = -1$, (4) reduces to

$$a_2 = 6\frac{4b_1 + 3}{2b_1^2 - 1} + 1 \text{ and } b_2 = 6\frac{3b_1 + 2}{2b_1^2 - 1} - b_1 = \frac{-2b_1^3 + 19b_1 + 12}{2b_1^2 - 1}.$$

If $b_1 \geq 4$, then $b_2 < 0$. Also, if $b_1 = -1$ or 0, $a_2 < 0$. Of the remaining choices, $b_1 = 2$ or 3 give fractional answers for a_2. When $b_1 = 1$, we get the solution $a_1 = -1, b_1 = 1, a_2 = 43, b_2 = 29$. This contributes $w = -1 + \sqrt{2}$, $x = 3 + 3\sqrt{2}$, $y = 43 + 29\sqrt{2}$ to our list of solutions.

If $a_1 = 0$, (4) becomes $a_2 = \dfrac{12}{b_1}$ and $b_2 = \dfrac{9}{b_1} - b_1 = \dfrac{9 - b_1^2}{b_1}$. In this case, $b_1 \neq 0$ and we get $a_2 < 0$ when $b_1 = -1$ and $b_2 \leq 0$ when $b_1 \geq 3$. Also, $b_1 = 2$ yields a fractional value for b_2. Hence, we are left with $b_1 = 1$, which gives the solution $a_1 = 0, b_1 = 1, a_2 = 12, b_2 = 8$ and we add $w = \sqrt{2}$, $x = 4 + 3\sqrt{2}$, $y = 12 + 8\sqrt{2}$ to our solution set.

We can now assume that $a_1 \geq 1$ in the remainder of this solution.

If $b_1 = -1$, (4) is of the form

$$a_2 = 6\frac{3a_1 + 4}{a_1^2 - 2} - a_1 = \frac{-a_1^3 + 20a_1 + 24}{a_1^2 - 2} \text{ and } b_2 = 6\frac{2a_1 + 3}{a_1^2 - 2} + 1.$$

As before, we get $a_2 < 0$ if $a_1 \geq 5$ and $b_2 < 0$ if $a_1 = 1$. When $a_1 = 3$ or 4, we get fractional values for b_2. Finally, $a_1 = 2$ gives the solution $a_1 = 2, b_1 = -1, a_2 = 28, b_2 = 22$, which yields $w = 2 - \sqrt{2}$, $x = 6 + \sqrt{2}$, $y = 28 + 22\sqrt{2}$.

If $b_1 = 0$, (4) becomes

$$a_2 = \frac{18}{a_1} - a_1 = \frac{18 - a_1^2}{a_1} \quad \text{and} \quad b_2 = \frac{12}{a_1}.$$

If $a_1 \geq 5$, we get $a_2 < 0$ and $a_1 = 4$ gives a fractional value for a_2. The remaining values $a_1 = 1, 2, 3$ produce the solutions listed below.

$$\begin{array}{ccccccc} \underline{a_1} & \underline{b_1} & \underline{a_2} & \underline{b_2} & \underline{w} & \underline{x} & \underline{y} \\ 1 & 0 & 17 & 12 & 1 & 5+2\sqrt{2} & 17+12\sqrt{2} \\ 2 & 0 & 7 & 6 & 2 & 6+2\sqrt{2} & 7+6\sqrt{2} \\ 3 & 0 & 3 & 4 & 3 & 7+2\sqrt{2} & 3+4\sqrt{2} \end{array}.$$

Finally, we are down to the situation where $a_1 \geq 1$, $b_1 \geq 1$, $a_1 + a_2 \geq 2$, and $b_1 + b_2 \geq 2$. Then, (2) implies that $1 \leq a_1 \leq 6$ and $1 \leq b_1 \leq 6$. By trying the 36 possibilities this presents for the system consisting of (2) and (3), we find that the remaining solutions are as follows:

$$\begin{array}{ccccccc} \underline{a_1} & \underline{b_1} & \underline{a_2} & \underline{b_2} & \underline{w} & \underline{x} & \underline{y} \\ 1 & 1 & 5 & 5 & 1+\sqrt{2} & 5+3\sqrt{2} & 5+5\sqrt{2} \\ 2 & 1 & 4 & 2 & 2+\sqrt{2} & 6+3\sqrt{2} & 4+2\sqrt{2} \\ 2 & 2 & 1 & 1 & 2+2\sqrt{2} & 6+4\sqrt{2} & 1+\sqrt{2} \end{array}.$$

Our conclusion is that the full solution set for this problem is displayed below. With some algebraic fortitude, it can be checked that all are solutions to the original equation.

$$\begin{array}{cc} \underline{x} & \underline{y} \\ 1+5\sqrt{2} & 17+7\sqrt{2} \\ 2+4\sqrt{2} & 23+13\sqrt{2} \\ 3+3\sqrt{2} & 43+29\sqrt{2} \\ 4+3\sqrt{2} & 12+8\sqrt{2} \\ 5+2\sqrt{2} & 17+12\sqrt{2} \\ 5+3\sqrt{2} & 5+5\sqrt{2} \\ 6+\sqrt{2} & 28+22\sqrt{2} \\ 6+2\sqrt{2} & 7+6\sqrt{2} \\ 6+3\sqrt{2} & 4+2\sqrt{2} \\ 6+4\sqrt{2} & 1+\sqrt{2} \\ 7+2\sqrt{2} & 3+4\sqrt{2} \end{array}.$$

Comments: **David Stone and John Hawkins of Statesboro GA** noted that the solutions (x, y) lie on the hyperbola

$$y = \frac{x^2 - 4\left(2+\sqrt{2}\right)x + \left(2+\sqrt{2}\right)^2}{x - 2\left(2+\sqrt{2}\right)}$$

and it is not evident that there should be only finitely many solutions. However, imposing the specific form $a+b\sqrt{2}$ on x and y forces this to be the case.

And **Ken Korbin** (the proposer of the problem) characterized the solutions as follows: Letting

$$(c,d)=(1,2+\sqrt{2}),(2+\sqrt{2},1),(\sqrt{2},1+\sqrt{2}),(1+\sqrt{2},\sqrt{2})$$

then

$$\begin{cases} x=c(2d+1) \\ y=c(3d^2-1) \quad \text{with } x<y, \end{cases} \quad \textit{and} \quad \begin{cases} x=c(2d+3) \\ y=c(d^2-3) \quad \text{with } \ x<y. \end{cases}$$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Titu Zvonaru and Comănesti Romania, Neculai Stanciu, Buzău, Romania (jointly); David Stone and John Hawkins of Georgia Southern University, Statesboro, GA (jointly), and the proposer.

5213: *Proposed by Tom Moore, Bridgewater, MA*

The triangular numbers T_n begin $1,3,6,10,\ldots$ and, in general, $T_n=\dfrac{n(n+1)}{2},\ n=1,2,3,\ldots$.

For every positive integer $n>1$, prove that n^4 is a sum of four triangular numbers.

Solution by Boris Rays, Brooklyn, NY

$$\begin{aligned} n^4 &= n^4-n^2+n^2=2\frac{n^4-n^2}{2}+2\frac{n^2}{2} \\ &= 2\frac{n^2}{2}+2\frac{n^4-n^2}{2} \\ &= \frac{n^2-n+n^2+n}{2}+2\frac{n^2(n^2-1)}{2} \\ &= \frac{(n-1)n}{2}+\frac{n(n+1)}{2}+\frac{(n^2-1)n^2}{2}+\frac{(n^2-1)n^2}{2} \\ &= T_{n-1}+T_n+T_{n^2-1}+T_{n^2-1}. \end{aligned}$$

Comments: **Albert Stadler of Herrliberg, Switzerland.** A.M. Legendre concluded from formulas in his treatise on elliptic functions [1] that the number of ways in which n is a sum of four triangular numbers equals the sum of the divisors of $2n+1$. As a result of this, every natural number can be represented as a sum of four triangular numbers. Reference: [1] Adrien Marie Legendre, Fonctions elliptiques et des intégrales Eulériennes: avec des tables pour en faciliter le calcul numérique; Vol 3 (1828), 133-134.

David Stone and John Hawkins of Statesboro, GA noted in their solution that n^4 is also the sum of *two* triangular numbers: $n^4=T_{n^2-1}+T_{n^2}$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie Campbell, Dionne Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Tirana, Albania; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Armend Sh. Shabani, (student, University of Prishtina), Republic of Kosova; Howard Sporn, Great Neck, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA (jointly); Titu Zvonaru, Comănesti, Romania and Neculai Stanciu Buzău, Romania (jointly), and the proposer.

5214: *Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3(b+c)^2+1}{1+a+2b}+\frac{b^3(c+a)^2+1}{1+b+2c}+\frac{c^3(a+b)^2+1}{1+c+2a} \geq \frac{4abc(ab+bc+ca)+3}{a+b+c+1}.$$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let $L=\dfrac{a^3(b+c)^2+1}{1+a+2b}+\dfrac{b^3(c+a)^2+1}{1+b+2c}+\dfrac{c^3(a+b)^2+1}{1+c+2a}$. Note that by the $AM-GM$ inequality, $(b+c)^2 \geq 4bc, (c+a) \geq 4ca$, and $(a+b)^2 \geq 4ab$ with equality if and only if $a=b=c$. Therefore,

$$\begin{aligned} L &\geq \frac{4a^3bc+1}{1+a+2b}+\frac{4b^3ca+1}{1+b+2c}+\frac{4c^3ab+1}{1+c+2a} \\ &= 4abc\left(\frac{a^2}{(1+a+2b)}+\frac{b^2}{(1+b+2c)}+\frac{c^2}{(1+c+2a)}\right)+\left(\frac{1^2}{(1+a+2b)}+\frac{1^2}{(1+b+2c)}+\frac{1^2}{(1+c+2a)}\right). \end{aligned}$$

The Cauchy-Schwarz inequality implies

$$\frac{a}{\sqrt{1+a+2b}}\cdot\sqrt{1+a+2b}+\frac{b}{\sqrt{1+b+2c}}\cdot\sqrt{1+b+2c}+\frac{c}{\sqrt{1+c+2a}}\cdot\sqrt{1+c+2a}^2 \leq$$
$$\left(\frac{a^2}{1+a+2b}+\frac{b^2}{1+b+2c}+\frac{c^2}{1+c+2a}\right)(3a+3b+3c+3);$$

hence,

$$\frac{a^2}{1+a+2b}+\frac{b^2}{1+b+2c}+\frac{c^2}{1+c+2a} \geq \frac{(a+b+c)^2}{3(a+b+c+1)}.$$

Similarly,

$$\frac{1^2}{1+a+2b}+\frac{1^2}{1+b+2c}+\frac{1^2}{1+c+2a} \geq \frac{(1+1+1)^2}{3(a+b+c+1)}=\frac{3}{a+b+c+1}.$$

Therefore,

$$L \geq 4abc\left(\frac{(a+b+c)^2}{3(a+b+c+1)}\right)+\frac{3}{a+b+c+1}.$$

Furthermore, the Cauchy-Schwarz inequality also implies $a^2+b^2+c^2 \geq ab+bc+ca$ using vectors $\langle a,b,c\rangle$ and $\langle b,c,a\rangle$. Therefore,

$$\begin{aligned}(a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ca \\ &\geq ab+bc+ca+2(ab+bc+ca) \\ &= 3(ab+bc+ca).\end{aligned}$$

Hence,

$$\frac{(a+b+c)^2}{3(a+b+c+1)} \geq \frac{ab+bc+ca}{a+b+c+1}.$$

Accordingly,

$$\begin{aligned}L &\geq 4abc\left(\frac{(a+b+c)^2}{3(a+b+c+1)}\right)+\frac{3}{a+b+c+1} \\ &\geq \frac{4abc(ab+bc+ca)}{a+b+c+1}+\frac{3}{a+b+c+1} \\ &= \frac{4abc(ab+bc+ca)+3}{a+b+c+1},\end{aligned}$$

with equality if and only if $a=b=c$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We prove that

$$\frac{1}{1+a+2b}+\frac{1}{1+b+2c}+\frac{1}{1+c+2a} \geq \frac{3}{a+b+c+1}$$

and

$$\frac{a^3(b+c)^2}{1+a+2b}+\frac{b^3(c+a)^2}{1+b+2c}+\frac{c^3(a+b)^2}{1+c+2a} \geq \frac{4abc(ab+bc+ca)}{a+b+c+1}$$

By Cauchy–Schwarz

$$\frac{1}{1+a+2b}+\frac{1}{1+b+2c}+\frac{1}{1+c+2a} \geq \frac{(1+1+1)^2}{3+3(a+b+c)}$$

thus we prove

$$\frac{(1+1+1)^2}{3+3(a+b+c)} \geq \frac{3}{a+b+c+1}$$

which is actually an equality. As for the second inequality we have

$$\sum_{\text{cyc}} \frac{a^3(b+c)^2}{1+a+2b} \geq \sum_{\text{cyc}} \frac{a^3 4bc}{1+a+2b} \geq \frac{4abc(ab+bc+ca)}{a+b+c+1}$$

or

$$\sum_{\text{cyc}} \frac{a^2}{1+a+2b} \geq \frac{ab+bc+ca}{a+b+c+1}$$

Cauchy–Schwarz again yields

$$\sum_{\text{cyc}} \frac{a^2}{1+a+2b} \geq \frac{(a+b+c)^2}{3+3(a+b+c)} \geq \frac{ab+bc+ca}{a+b+c+1}$$

or

$$S^3 + S^2 \geq 3P + 3PS, \qquad S = a+b+c, \qquad P = ab+bc+ca$$

Now $S^2 \geq 3P$ since it is equivalent to

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

which is a well known inequality.

Solution 3 by Adrian Naco, Polytechnic University,Tirana, Albania.

Editor's comment: The following is a generalization of the stated problem.

Based on Cauchy-Schwarz inequality, for $a_i, b_i \in R^{*+}$ we have that

$$\begin{aligned}
\left(\sum_{i=1}^n a_i\right)^2 = \left[\sum_{i=1}^n \left(\frac{a_i}{\sqrt{b_i}}\right)(\sqrt{b_i})\right]^2 \quad &\leq \quad \left[\sum_{i=1}^n \left(\frac{a_i}{\sqrt{b_i}}\right)^2\right]\left[\sum_{i=1}^n (\sqrt{b_i})^2\right] \\
&= \quad \left(\sum_{i=1}^n \frac{{a_i}^2}{b_i}\right)\left(\sum_{i=1}^n b_i\right) \\
&\Rightarrow \quad \sum_{i=1}^n \frac{{a_i}^2}{b_i} \geq \frac{\left(\sum_{i=1}^n a_i\right)^2}{\sum_{i=1}^n b_i} \qquad (1)
\end{aligned}$$

where the equality holds for $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_n}{b_n}$.
Let us split the original inequality in two separate inequalities (2) and (3) as follows

$$\sum_{i=1}^{n} \frac{1}{1+x_i+2x_{i+1}} \geq \frac{n^2}{n+3\sum\limits_{i=1}^{n} x_i} \qquad (2)$$

and

$$\sum_{i=1}^{n} \frac{{x_i}^3(x_{i+1}+x_{i+2})^2}{1+x_i+2x_{i+1})} \geq \frac{4\Big[\sum\limits_{i=1}^{n}(x_ix_{i+1}x_{i+2}){x_i}^2+2\sum\limits_{1\leq i<j\leq n}(x_ix_jx_{i+1}x_{j+1}x_{i+2}x_{j+2})^{\frac{1}{2}}x_ix_j\Big]}{n+3\sum\limits_{i=1}^{n} x_i} \qquad (3)$$

Applying the above Cauchy-Schwarz inequality for each of the inequalities (2) and (3) we have that

$$\sum_{i=1}^{n} \frac{1^2}{1+x_i+2x_{i+1}} \geq \frac{\left(\sum\limits_{i=1}^{n} 1\right)^2}{\sum\limits_{i=1}^{n}(1+x_i+2x_{i+1})} = \frac{(\underbrace{1+1++1}_{ntimes})^2}{\sum\limits_{i=1}^{n} 1+\sum\limits_{i=1}^{n} x_i+2\sum\limits_{i=1}^{n} x_{i+1}}$$

$$= \frac{n^2}{n+3\sum\limits_{i=1}^{n} x_i} \qquad (4)$$

where the equality holds for $x_1 = x_2 = \ldots = x_n$. Thus we prove (2). Analogously we have that

$$\sum_{i=1}^{n} \frac{{x_i}^3(x_{i+1}+x_{i+2})^2}{1+x_i+2x_{i+1})} = \sum_{i=1}^{n} \frac{\Big[{x_i}^{\frac{3}{2}}(x_{i+1}+x_{i+2})\Big]^2}{1+x_i+2x_{i+1})}$$

$$\geq \frac{\Big[\sum\limits_{i=1}^{n} {x_i}^{\frac{3}{2}} \overbrace{(x_{i+1}+x_{i+2})}^{x_{i+1}+x_{i+2}\geq 2\sqrt{x_{i+1}x_{i+2}}}\Big]^2}{\sum\limits_{i=1}^{n}(1+x_i+2x_{i+1})}$$

$$\geq \frac{4\Big[\sum\limits_{i=1}^{n} x_i(x_ix_{i+1}x_{i+2})^{\frac{1}{2}}\Big]^2}{\sum\limits_{i=1}^{n} 1+\sum\limits_{i=1}^{n} x_i+2\sum\limits_{i=1}^{n} x_{i+1}}$$

$$= \frac{4\Big[\sum\limits_{i=1}^{n} {x_i}^2(x_ix_{i+1}x_{i+2})+2\sum\limits_{1\leq i<j\leq n}^{n} x_ix_j(x_ix_jx_{i+1}x_{j+1}x_{i+2}x_{j+2})^{\frac{1}{2}}\Big]}{n+3\sum\limits_{i=1}^{n} x_i}$$

$$= \frac{4\Big[S + 3\displaystyle\sum_{1\le i<j\le n}^{n} x_i x_j (x_i x_j x_{i+1} x_{j+1} x_{i+2} x_{j+2})^{\frac{1}{2}}\Big]}{n + 3\displaystyle\sum_{i=1}^{n} x_i}$$

where

$$S = \sum_{i=1}^{n} {x_i}^2 (x_i x_{i+1} x_{i+2}) - \sum_{1\le i<j\le n}^{n} x_i x_j (x_i x_j x_{i+1} x_{j+1} x_{i+2} x_{j+2})^{\frac{1}{2}}$$

The problem proposed is a special case of the above generalized problem for

$$x_1 = a \qquad x_2 = b \qquad x_3 = c.$$

Thus, we have that

$$\begin{aligned} S &= abca^2 + bcab^2 + cabc^2 - \Big[(abbcca)^{\frac{1}{2}} ab + (acbacb)^{\frac{1}{2}} ac + (bccaab)^{\frac{1}{2}} bc\Big] \\ &= abc(a^2 + b^2 + c^2 - ab - ac - bc) = abc\frac{1}{2}\Big[(a-b)^2 + (b-c)^2 + (c-a)^2\Big] \ge 0 \end{aligned}$$

with the equality holding for $a = b = c$. The inequality proposed gets the simplified form

$$\begin{aligned} \sum_{cyc} \frac{a^3(b+c)^2 + 1}{1 + a + 2b)} &\ge \frac{4\Big\{S + 3\Big[(abbcca)^{\frac{1}{2}} ab + (acbacb)^{\frac{1}{2}} ac + (bccaab)^{\frac{1}{2}} bc\Big]\Big\} + 3^2}{3 + 3(a+b+c)} \\ &= \frac{4\Big[S + 3abc\Big(ab + ac + bc\Big)\Big] + 3^2}{3 + 3(a+b+c)} \ge \frac{4\Big[0 + 3abc\Big(ab + ac + bc\Big)\Big] + 3^2}{3 + 3(a+b+c)} \\ &= \frac{4abc\Big(ab + ac + bc\Big) + 3}{1 + a + b + c} \end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo TX; Andrea Fanchini, Cantú, Italy; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu Buzău, Romania (jointly); and the proposer.

5215: *Proposed by Neculai Stanciu, Buzău, Romania*

Evaluate the integral
$$\int_{-1}^{1} \frac{2x^{1004}+x^{3014}+x^{2008}\sin x^{2007}}{1+x^{2010}}dx.$$

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$$\begin{aligned}\int_{-1}^{1} \frac{2x^{1004}+x^{3014}+x^{2008}\sin x^{2007}}{1+x^{2010}}dx &= \int_{-1}^{1} \frac{2x^{1004}+x^{3014}}{1+x^{2010}}dx+\int_{-1}^{1} \frac{x^{2008}\sin x^{2007}}{1+x^{2010}}dx \\ &= \int_{-1}^{1} \frac{2x^{1004}+x^{3014}}{1+x^{2010}}dx\end{aligned}$$

Note that $\displaystyle\int_{-1}^{1} \frac{x^{2008}\sin x^{2007}}{1+x^{2010}}dx=0$, since the integrand function is odd and the interval of integration is centered at the origin.

The remaining integral may be solved using the change of variable $x^{1005}=t$.

$$\begin{aligned}\int_{-1}^{1} \frac{2x^{1004}+x^{3014}}{1+x^{2010}}dx &= \frac{1}{1005}\int_{-1}^{1}\frac{2+t^2}{1+t^2}dt \\ &= \frac{1}{1005}\left(2+\int_{-1}^{1}\frac{1}{1+t^2}dt\right) \\ &= \frac{1}{1005}\left(2+\arctan(1)-\arctan(-1)\right) \\ &= \frac{1}{1005}\left(2+\frac{\pi}{2}\right)=\frac{4+\pi}{2010}.\end{aligned}$$

Also solved by Daniel Lopez Aguayo, Institute of Mathematics, UNAM, Morelia, Mexico; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo Sate University, San Angelo, TX; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Fotini Kotroni and Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, (student, University of Prishtina), Republic of Kosova; Albert Stadler, Herrliberg, Switzerland; Howard Sporn, Great Neck, NY, and the proposer.

5216: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $f : \Re \to \Re^+$ be a function such that for all $a, b \in \Re$

$$f(ab) = f(a)^b f(b)^{a^2}$$

and $f(3) = 64$. Find all real solutions to the equation

$$f(x) + f(x+1) - 3x - 2 = 0.$$

Solution by Armend Sh. Shabani, (Graduate Student) University of Prishtina, Republic of Kosova.

Since $f(a \cdot b) = f(b \cdot a)$ we have

$$f(a)^b \cdot f(b)^{a^2} = f(b)^a \cdot f(a)^{b^2} \Leftrightarrow f(a)^b \cdot f(b)^a \left[f(b)^{a^2-a} - f(a)^{b^2-b} \right] = 0$$

$$\Leftrightarrow f(b)^{a^2-a} - f(a)^{b^2-b} = 0 \Leftrightarrow f(b)^{a^2-a} = f(a)^{b^2-b}.$$

Taking $b = x; a = 3$, one obtains:

$$\begin{aligned} f(x)^{3^2-3} &= f(3)^{x^2-x} \\ f(x)^6 &= (64)^{x^2-x} \\ f(x)^6 &= (4^3)^{x^2-x} \Rightarrow f(x) = 2^{x^2-x} = 2^{x(x-1)}. \end{aligned}$$

Substituting into $f(x) + f(x+1) - 3x - 2 = 0$ we obtain:

$$2^{x^2-x} + 2^{x^2+x} - (3x+2) = 0. \qquad (1)$$

Clearly $x = 0$; $x = 1$ are solutions of equation (1).

We show that there are no other solutions.

Let $g(x) = 2^{x^2-x} + 2^{x^2+x}$. One easily finds that

$$g'(x) = \ln 2 \cdot \left((2x+1) \cdot 2^{x^2+x} + (2x-1) \cdot 2^{x^2-x} \right) \text{ and}$$

$$g''(x) = \ln 2 \cdot \left(2^{x^2+x+1} + 2^{x^2-x+1} \right) + (\ln 2)^2 \cdot \left((2x+1)^2 \cdot 2^{x^2+x} + (2x-1)^2 \cdot 2^{x^2-x} \right).$$

So $g''(x) > 0$, and this means that g is a convex function, So the line $h(x) = 3x + 2$ can meet function g in at most 2 points. Therefore equation (1) has no other solutions. (Note that this can also be seen by drawing the graphs of functions g and h.)

Also solved by Dionne Bailey, Elsie Campbell, and Charles Dominnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University Statesboro, GA (jointly), and the proposer.

5217: *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the value of:

$$\lim_{n\to\infty}\int_0^1\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dxdy,$$

where k is a positive real number.

Solution 1 by Anastasios Kotronis, Athens, Greece

It is easily shown that $\sqrt[n]{(x^n+y^n)^k} \to \begin{cases} x^k, & y\le x \\ y^k, & x<y \end{cases}$ and since $0\le \sqrt[n]{(x^n+y^n)^k}\le 2^k$, by the dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to+\infty}\int_0^1\int_0^1 \sqrt[n]{(x^n+y^n)^k}dydx &= \int_0^1\int_0^1 \lim_{n\to+\infty}\sqrt[n]{(x^n+y^n)^k}dydx \\
&= \int_0^1\int_0^x x^k\,dy\,dx + \int_0^1\int_x^1 y^k\,dy\,dx \\
&= \int_0^1 x^{k+1}\,dx + \int_0^1 \frac{1-x^{k+1}}{k+1}\,dx \\
&= \frac{2}{k+2}.
\end{aligned}$$

Solution 2 by Howard Sporn, Great Neck, NY

We use the fact that for n going to ∞, when $x<y$, the y^n term dominates over x^n, and when $x>y$, the x^n term dominates over y^n.

We break up the inner integral into two integrals, like so:

$$\int_0^1 \sqrt[n]{(x^n+y^n)^k}dx = \int_0^y \sqrt[n]{(x^n+y^n)^k}dx + \int_y^1 \sqrt[n]{(x^n+y^n)^k}dx.$$

Note that for the first integral $x\le y$, and for the second integral $x\ge y$. By factoring,

$$\int_0^1 \sqrt[n]{(x^n+y^n)^k}dx = \int_0^y \sqrt[n]{\left[y^n\left(\left(\frac{x}{y}\right)^n+1\right)\right]^k}dx + \int_y^1 \sqrt[n]{\left[x^n\left(1+\left(\frac{y}{x}\right)^n\right)\right]^k}dx.$$

For the first integral, in which x, we first consider the case $x<y$. In that case, $\left(\frac{x}{y}\right)^n\to 0$ for n$\to\infty$. Then the integrand becomes $\sqrt[n]{[y^n\,(0+1)]^k} = y^k$.

If, on the other hand, $x = y$, then the integrand becomes $\sqrt[n]{(\left[y^n\left(\left(\frac{y}{y}\right)^n+1\right)\right]^k} = \sqrt[n]{\left[y^n(1+1)\right]^k} = y^k\sqrt[n]{2^k}$, which approaches y^k (once again) as $n \to \infty$.

Similarly, the integrand in the second integral approaches x^k.

The quantity we are seeking is now

$$\int_0^1 \left(\int_0^y y^k dx + \int_y^1 x^k dx\right) dy$$

which is straight-forward to compute.

The solution is

$$\int_0^1 \left(\left(y^k x\right)\Big|_{x=0}^y + \frac{x^{k+1}}{k+1}\Big|_{x=y}^1\right) dy$$

$$= \int_0^1 \left(y^{k+1} + \frac{1}{k+1} - \frac{y^{k+1}}{k+1}\right) dy$$

$$= \left(\frac{y^{k+2}}{k+2} + \frac{y}{k+1} - \frac{y^{k+2}}{(k+1)(k+2)}\right)\Big|_0^1$$

$$= \frac{1}{k+2} + \frac{1}{k+1} - \frac{1}{(k+1)(k+2)}$$

$$= \frac{(k+1)+(k+2)-1}{(k+1)(k+2)}$$

$$= \frac{2k+2}{(k+1)(k+2)}$$

$$= \frac{2}{k+2}.$$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

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Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2013

- **5236:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers (a, b, c, x, y, z) such that

$$\begin{aligned} x^2+xy+y^2 &= a, \\ y^2+yz+z^2 &= b, \\ z^2+zx+x^2 &= c. \end{aligned}$$

  Express the value of the sum $x+y+z$ in terms of a, b, and c.

- **5237:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let $0 < R < 1$ and $0 < S < 1$, and define

$$\begin{aligned} a &= \sqrt{-2\sqrt{1-S^2}\sqrt{1-R^2}+2+2RS}, \\ b &= \sqrt{-R-S+1+RS}, \text{ and} \\ c &= \sqrt{R+S+1+RS}. \end{aligned}$$

  Determine whether there is tuple (R, S) such that a, b, and c are sides of a triangle.

- **5238:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  It is fairly well-known that $(1111\ldots1)_9$, a number written in base 9 with an arbitrary number of digits 1, always evaluates decimally to a triangular number. Find another base b and a single digit d in that base, such that $(ddd\ldots d)_b$, using k digits d, has the same property, $\forall k \geq 1$.

- **5239:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Determine all functions $f: \Re - \{-3, -1, 0, 1, 3\} \to \Re$, which satisfy the relation

$$f(x) + f\left(\frac{13+3x}{1-x}\right) = ax+b,$$

  where a and b are given arbitrary real numbers.

- **5240:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let x be a positive real number. Prove that

  $$\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}>\frac{1}{8},$$

  where $[x]$ and $\{x\}$ represent the integral and fractional part of x, respectively.

- **5241:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $\alpha \geq 0$ be a real number. Calculate

  $$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\,dx\right)^n.$$

Solutions

- **5218:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that,

  $$2x-y-\sqrt{3x^2-3xy+y^2}=2013$$

  with $(x,y)=1$.

  Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

  If we re-write the equation in the form $2x-y-2013=\sqrt{3x^2-3xy+y^2}$ (1) and then square both sides and simplify, we get successively

  $$x^2-8052x+(2013)^2+4026y-xy=0,\ \text{and}$$

  $$(x-4026)^2-(x-4026)\,y=3\,(2013)^2.$$

  To simplify further, substitute $w=x-4026$ to obtain

  $$\begin{aligned} w^2-wy &= 3\,(2013)^2 && (2)\\ \text{or } w\,(w-y) &= 3\,(2013)^2. && (3)\end{aligned}$$

  Since w and $w-y$ are integers, the problem can be solved by considering all factorizations of

  $$3\,(2013)^2=3^3 11^2 61^2 \qquad (4)$$

  into a product of two integers. Also, since $y>0$, we have $w-y<w$ in each instance.

  Before proceeding, we note that (2) implies that

  $$y=\frac{w^2-3\,(2013)^2}{w}$$

and we get

$$\begin{aligned} 2x - y - 2013 &= 2(w + 4026) - \frac{w^2 - 3(2013)^2}{w} - 2013 \\ &= \frac{w^2 + 6093w + 3(2013)^2}{w}. \end{aligned}$$

Since

$$w^2 + 6093w + 3(2013)^2 = \left(w + \frac{6039}{2}\right)^2 + \frac{3}{4}(2013)^2 > 0$$

for all w, we end up with $2x - y - 2013 < 0$ when $w < 0$. Hence, when $w - y < w < 0$, (1) implies that we will get extraneous solutions.

Next, suppose that $(w, w - y) > 1$. Then, there is a prime p which is a divisor of both w and $w - y$. Conditions (3) and (4) tell us that $p = 3$, 11, or 61 and hence, p divides 2013. First of all, p divides both w and $w - y$ implies that p divides $w - (w - y) = y$. Also, since p divides both w and 2013, it follows that p divides $w + 2(2013) = x$. As a result, when $(w, w - y) > 1$, we have $(x, y) > 1$ as well. Therefore, we may restrict our work to the case where $(w, w - y) = 1$.

Finally then, we need only consider (3) and (4) with $0 < w - y < w$ and $(w, w - y) = 1$. The full set of solutions is given in the following table.

$w - y$	w	$x = w + 4026$	$y = w - (w - y)$
1	$3^3 11^2 61^2$	$12,160,533$	$12,156,506$
3^3	$11^2 61^2$	$454,267$	$450,214$
11^2	$3^3 61^2$	$104,493$	$100,346$
$3^3 11^2$	61^2	$7,747$	454

With a good software package, it's possible to check that all of these are solutions of (1) with $(x, y) = 1$.

Solution 2 by Adrian Naco, Polytechnic University, Tirana, Albania

The left side of the equation can be transformed to

$$2x - y - \sqrt{(2x - y)^2 - x(x - y)} = 2013 \quad \Rightarrow \quad x(x - y) > 0 \quad \Rightarrow \quad 0 < y < x. \qquad (1)$$

(since x and y are positive integers). Further more,

$$\sqrt{3x^2 - 3xy - y^2} = 2x - y - 2013 \quad \Rightarrow \quad 2x - y - 2013 \geq 0 \quad \Leftrightarrow \quad 2x - y \geq 2013. \qquad (2)$$

Solving the equation we have that

$$\begin{aligned} 3x^2 - 3xy - y^2 = (2x - y - 2013)^2 \quad &\Rightarrow \quad x^2 - xy - 2 \cdot 2013y - 4 \cdot 2013x + 2013^2 = 0 \\ &\Rightarrow \quad y = x - 2 \cdot 2013 - \frac{3 \cdot 2013^2}{x - 2 \cdot 2013}. \qquad (3) \end{aligned}$$

where

$$\frac{3\cdot 2013^2}{x-2\cdot 2013} = r \in Z \quad \Rightarrow \quad x = 2\cdot 2013 + \frac{3\cdot 2013^2}{r}, \tag{4}$$

and since

$$\frac{3\cdot 2013^2}{r} = s \in Z \quad \Rightarrow \quad rs = 3\cdot 2013^2 = 3^3\cdot 11^2\cdot 61^2. \tag{5}$$

Considering (3), (4), (5) we have that,

$$\begin{aligned} x &= 2\cdot 2013 + s \\ y &= s - r \\ rs &= 3\cdot 2013^2 = 3^3\cdot 11^2\cdot 61^2 \quad \text{where} \quad r, s \in Z. \end{aligned}$$

The general structure of r and s is

$$r = 3^{\alpha_1}11^{\alpha_2}61^{\alpha_3} \quad \text{and s} = 3^{\beta_1}11^{\beta_2}61^{\beta_3} \quad \text{where}$$

$$\alpha_1+\beta_1 = 3, \quad \alpha_2+\beta_2 = 2, \quad \alpha_3+\beta_3 = 2.$$

From (1) and (2)

$$\begin{aligned} x > y > 0 \quad &\Rightarrow \quad s > r > -2\cdot 2013 \\ 2x - y - 2013 > 0 \quad &\Rightarrow \quad s + r > -3\cdot 2013 \\ &\Rightarrow \quad s + \frac{3\cdot 2013^2}{s} > -3\cdot 2013 \\ &\Rightarrow \quad \frac{s^2 + 3\cdot 2013s + 3\cdot 2013^2}{s} > 0 \\ &\Rightarrow \quad s > 0 \quad \Rightarrow \quad r > 0. \end{aligned}$$

Furthermore, if $(r, s) = p$ then $p|2013$ and consequently $p|x$ and $p|y$. Since $(x, y) = 1$ then $p = 1$, resulting that there are only eight possible combinations for r and s (since for each combination we have $\alpha_i = 0$ or $\beta_i = 0$, $\forall i \in \{1, 2, 3\}$)

$$r = 3^0 11^0 61^0 \quad \text{and} \quad s = 3^3 11^2 61^2$$

$$r = 3^3 11^0 61^0 \quad \text{and} \quad s = 3^0 11^2 61^2$$

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$$r = 3^3 11^2 61^2 \quad \text{and} \quad s = 3^3 11^2 61^2,$$

and since $s > r$, there are only four possible combinations, each of them generates a solution for the given equation. More concretely the four solutions are

$$\begin{array}{llcll}
r = 3^0 11^0 61^0, & s = 3^3 11^2 61^2 & \Rightarrow & x = 12160533, & y = 12156506 \\
r = 3^3 11^0 61^0, & s = 3^0 11^2 61^2 & \Rightarrow & x = 454267, & y = 450214 \\
r = 3^0 11^2 61^0, & s = 3^3 11^0 61^2 & \Rightarrow & x = 104493, & y = 100346 \\
r = 3^3 11^2 61^0, & s = 3^0 11^0 61^2 & \Rightarrow & x = 7747, & y = 454.
\end{array}$$

Comment by David Stone and John Hawkins of Georgia Southern University, Statesboro, GA. The above four points (x, y) are called *visible* points (i.e., the view from the origin is not blocked by any other lattice point.)

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Albert Stadler Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5219: *Proposed by David Manes and Albert Stadler, SUNY College at Oneonta, Oneonta, NY and Herrliberg, Switzerland (respectively)*

Let k and n be natural numbers. Prove that:

$$\sum_{j=1}^{n} \cos^k\left(\frac{(2j-1)\pi}{2n+1}\right) = \begin{cases} \dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2} - \dfrac{1}{2}, & k \text{ even} \\[2ex] \dfrac{1}{2}, & k \text{ odd.} \end{cases}$$

Solution by Kee-Wai Lau, Hong Kong, China

Since the stated result is not true for $(k, n) = (3, 1), (6, 1)$, we modify it to

$$\sum_{j=1}^{n} \cos^k\left(\frac{(2j-1)\pi}{2n+1}\right) = \begin{cases} \dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2} - \dfrac{1}{2}, & k = 2, 4, 6, \ldots, 4n \\ \\ \dfrac{1}{2}, & k = 1, 3, 5, \ldots, 2n-1. \end{cases}$$

Let $i = \sqrt{-1}$ and $\theta = \theta(j, n) = \dfrac{(2j-1)\pi}{2n+1}$. By the binomial theorem we have

$$\begin{aligned} \sum_{j=1}^{n} \cos^k \theta &= \frac{1}{2}\sum_{j=-n}^{n} \cos^k \theta + \frac{(-1)^{k-1}}{2} \\ &= \frac{1}{2^{k+1}}\sum_{j=-n}^{n}\left(e^{i\theta} + e^{-i\theta}\right) + \frac{(-1)^{k-1}}{2} \\ &= \frac{1}{2^{k+1}}\sum_{j=-n}^{n}\sum_{t=0}^{k}\binom{k}{t}e^{i(k-2t)\theta} + \frac{(-1)^{k-1}}{2} \\ &= \frac{1}{2^{k+1}}\sum_{t=0}^{k}\binom{k}{t}\sum_{j=-n}^{n} e^{i(k-2t)\theta} + \frac{(-1)^{k-1}}{2}. \end{aligned}$$

For $k = 2, 4, 6, \ldots, 4n$ and $t = 0, 1, 2, \ldots, k$, $\dfrac{2(k-2t)}{2n+1}$ is not an integer unless $t = \dfrac{k}{2}$. So for $t = \dfrac{k}{2}$, we have $\displaystyle\sum_{j=-n}^{n} e^{i(k-2)\theta} = 2n+1$ and for $t = 0, 1, 2, \ldots, \dfrac{k-2}{2}, \dfrac{k+2}{2}, \ldots, k$, we have $\displaystyle\sum_{j=-n}^{n} e^{i(k-2t)\theta} = \frac{1 - e^{2(k-2t)\pi i}}{1 - e^{(2(k-2t)\pi i)/(2n+1)}} = 0.$

This proves the first part of the modified statement of the problem.

For $k = 1, 3, 5, \ldots, 2n-1$ and $t = 0, 1, 2, \ldots, k$, $\dfrac{2(k-2t)}{2n+1}$ is not an integer and so $\displaystyle\sum_{j=-n}^{n} e^{i(k-2t)\theta} =$ 0, and this proves the second part of the modified statement of the problem.

Editor's note: David Manes and Anastasios Kotronis, noted the error in the statement of the problem, but the problem had already been posted. Each went on to correct the mistake and each made reference to a general technique for solving such problems that is discussed in a paper by Mircea Merca (of the University of Craiova in Romania) entitled: "A Note on Cosine Power Sums" that appeared in the Journal of Integer Sequences, Vo. 15(2012); Article 12.5.3. Other solvers of 5219 parenthetically referenced the need to modify of the original statement.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Anastasios Kotronis, Athens, Greece; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposers.

5220: *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The pentagonal numbers begin $1, 5, 12, 22\ldots$ and are generally defined by $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The triangular numbers begin $1, 3, 6, 10, \ldots$ and are generally defined by $T_n = \dfrac{n(n+1)}{2}, \forall n \geq 1$. Find the greatest common divisor, $\gcd(T_n, P_n)$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Case 1: If n is even, $n = 2k$ for some $k \geq 1$, so, using properties of the gcd, the Euclidean algorithm, and the fact that $2k+1$ is odd $\Longrightarrow \gcd(2k+1, 4) = 1$, we obtain

$$\begin{aligned}
\gcd(P_n, T_n) &= \gcd\left(k\left(6k-1\right), k\left(2k+1\right)\right) \\
&= k\gcd\left(6k-1, 2k+1\right) \\
&= k\gcd\left(2k+1, -4\right) \\
&= k\gcd\left(2k+1, 4\right) \\
&= k = \frac{n}{2}.
\end{aligned}$$

Case 2: If n is odd, then $\dfrac{n}{4}$ gives a remainder of 1 or 3; so $n \equiv 1\ (mod\ 4)$ or $n \equiv 3\ (mod\ 4)$. We have two cases to consider.

Case 2.1: $n = 4k+1$ for some $k \geq 0$; then

$$\begin{aligned}
\gcd(P_n, T_n) &= \gcd\left(n(2k+1), n\left(6k+1\right)\right) \\
&= n\gcd\left(2k+1, 6k+1\right) \\
&= n\gcd\left(-2, 2k+1\right) \\
&= n\gcd\left(2, 2k+1\right) \\
&= n.
\end{aligned}$$

Case 2.2 $n = 4k+3$ for some $k \geq 0$; then

$$\gcd(P_n, T_n) = \gcd\left(n(6k+4), n\left(2k+2\right)\right)$$

$$
\begin{aligned}
&= 2n\gcd(3k+2,k+1)\\
&= 2n\gcd(k+1,-1)\\
&= 2n\gcd(k+1,1)\\
&= 2n.
\end{aligned}
$$

Hence,

$$
\gcd(P_n,T_n)=\begin{cases}\dfrac{n}{2} & n\ even\\ n & n\equiv 1\ (mod\ 4)\\ 2n & n\equiv 3\ (mod\ 4)\end{cases}
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

If n is even then,

$$
(P_n,T_n)=\frac{n}{2}(3n-1,n+1)=\frac{n}{2}(3n-1-3(n+1),n+1)=\frac{n}{2}(-4,n+1)=\frac{n}{2}.
$$

If $n\equiv 1\ (mod\ 4)$ then,

$$
(P_n,T_n)=n\left(\frac{3n-1}{2},\frac{n+1}{2}\right)=n\left(\frac{3n-1}{2}-3\cdot\frac{n+1}{2},\frac{n+1}{2}\right)=n\left(-2,\frac{n+1}{2}\right)=n.
$$

If $n\equiv 3\ (mod\ 4)$ then

$$
(P_n,T_n)=n\left(\frac{3n-1}{2},\frac{n+1}{2}\right)=n\left(\frac{3n-1}{2}-3\cdot\frac{n+1}{2},\frac{n+1}{2}\right)=n\left(-2,\frac{n+1}{2}\right)=2n.
$$

These three lines can be summarized in one formula by, e.g.,

$$
(P_n,T_n)=\frac{n}{2}\left(2\sin^2\frac{\pi n}{2}-\sin\frac{\pi n}{2}+1\right).
$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Editor's comment: Brian generalized the problem for the nth r-gonal number.

Given integers $n\geq 1$ and $r\geq 3$, the nth r-gonal number is defined by

$$
p_n^r=\tfrac{1}{2}n[(r-2)n-(r-4)].
$$

Find the following greatest common divisors for $a)$ $\gcd(p_n^r,p_n^{r+1})$ $\quad b)$ $\gcd(p_n^r,p_n^{r+2})$, r even, and c) $\gcd(p_n^r,p_n^{r+2})$, r odd.

a) We show that $\gcd(p_n^r, p_n^{r+1}) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$.

If $n = 2m$ for some positive integer m, then $p_n^r = m(2mr - 4m - r + 4)$ and $p_n^{r+1} = m(2mr - 2m - r + 3)$. Since $2mr - 2m - r + 3 = (2mr - 4m - r + 4) + (2m - 1)$, $2mr - 4m - r + 4 = (2m-1)(r-2) + 2$, and $\gcd(2m - 1, 2) = 1$, we also have $\gcd(2mr - 2m - r + 3, 2mr - 4m - r + 4) = 1$. Hence $\gcd(p_n^r, p_n^{r+1}) = m = n/2$.

If $n = 2m + 1$ for some nonnegative integer m, then $p_n^r = (2m + 1)(mr - 2m + 1)$ and $p_n^{r+1} = (2m + 1)(mr - m + 1)$. Since $mr - m + 1 = mr - 2m + 1 + (m)$ and $mr - 2m + 1 = m(r - 2) + 1$, we have $\gcd(mr - m + 1, mr - 2m + 1) = 1$. Hence $\gcd(p_n^r, p_n^{r+1}) = 2m + 1 = n$.

b) We show that for even r, $\gcd(p_n^r, p_n^{r+2}) = n$.

Write $r = 2m$ for some positive integer m. Then $p_n^r = n(mn - m - n + 2)$ and $p_n^{r+2} = n(mn - m + 1)$. Since $mn - m + 1 = (mn - m - n + 2) + (n - 1)$ and $mn - m - n + 2 = (n - 1)(m - 1) + 1$, we have $\gcd(mn - m + 1, mn - m - n + 2) = 1$. Hence $\gcd(p_n^r, p_n^{r+2}) = n$.

c) We show that for odd r, $\gcd(p_n^r, p_n^{r+2}) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n & \text{if } n \equiv 1 \pmod 4 \\ 2n & \text{if } n \equiv 3 \pmod 4 \end{cases}$.

Write $r = 2m + 1$ for some nonnegative integer m. Then $p_n^r = n(2mn - n - 2m + 3)/2$ and $p_n^{r+2} = n(2mn + n - 2m + 1)/2$. Since $2mn + n - 2m + 1 = (2mn - n - 2m + 3) + (2n - 2)$, $2mn - n - 2m + 3 = (2n - 2)(m - 1) + (n + 1)$, and $2n - 2 = (n + 1)(2) - (4)$, we have three cases:

If n is even, then $\gcd(n + 1, 4) = 1$, so $\gcd(p_n^r, p_n^{r+2}) = (n/2)(1) = n/2$.

If $n \equiv 1 \pmod 4$, then $\gcd(n + 1, 4) = 2$, so $\gcd(p_n^r, p_n^{r+2}) = (n/2)(2) = n$.

If $n \equiv 3 \pmod 4$, then $\gcd(n + 1, 4) = 4$, so $\gcd(p_n^r, p_n^{r+2}) = (n/2)(4) = 2n$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David Manes, SUNY College at Oneonta, Oneonta, NY; Melfried Olson, University of Hawaii, Honolulu, HI; Boris Rays, Brooklyn, NY; Neculai Stanciu "George Emil Palade" Secondary School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA, and the proposer.

5221: *Proposed by Michael Brozinsky, Central Islip, NY*

If x, y and z are positive numbers find the maximum of

$$\frac{\sqrt{(x+y+z)\cdot x\cdot y\cdot z}}{(x+y)^2+(y+z)^2+(x+z)^2}.$$

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Normalising the expression, the problem will be equivalent to finding the maximum of

$$\frac{\sqrt{xyz}}{(x+y)^2+(y+z)^2+(x+z)^2}$$

subject to $x+y+z=1$.
Using the AM-GM Inequality we have

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3} = \frac{1}{3} \Rightarrow \sqrt{xyz} \leq \left(\frac{1}{3}\right)^{\frac{3}{2}}$$

and

$$(x+y)^2+(y+z)^2+(x+z)^2 \geq \frac{1}{3}((x+y)+(y+z)+(x+z))^2 = \frac{4}{3}$$

Applying these two results we have

$$\frac{\sqrt{xyz}}{(x+y)^2+(y+z)^2+(x+z)^2} \leq \frac{\left(\frac{1}{3}\right)^{\dfrac{3}{2}}}{\dfrac{4}{3}} = \frac{1}{4\sqrt{3}}.$$

So the maximum value of the required expression is $\dfrac{1}{4\sqrt{3}}$, and this is achieved when $x=y=z$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the expression of the problem by f. We show that the maximum of f is $\dfrac{\sqrt{3}}{12}$.

Since f equals the constant $\dfrac{\sqrt{3}}{12}$ whenever $x=y=z>0$, so it suffices to show that for $x, y, z > 0$, we have

$$f \leq \frac{\sqrt{3}}{12}. \qquad (1)$$

From $f = \dfrac{\sqrt{(x+y+z)\cdot xyz}}{(x-y)^2+(y-z)^2+(x-z)^2+4(xy+yz+zx)} \leq \dfrac{\sqrt{(x+y+z)\cdot xyz}}{4(xy+yz+zx)}$, we see

that (1) will follow from $\dfrac{(x+y+z)xyz}{(xy+yz+zx)^2} \leq \dfrac{1}{3}$, or equivalently

$$(xy+yz+zx)^2 - 3xyz(x+y+z) \geq 0. \qquad (2)$$

But (2) in fact holds because its left side equals

$$\frac{x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2}{2}.$$

This completes the solution.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since $x, y, z > 0$, the Arithmetic - Geometric Mean Inequality implies that

$$xyz \le \left(\frac{x+y+z}{3}\right)^3 = \frac{(x+y+z)^3}{27},$$

with equality if and only if $x = y = z$. Hence,

$$\sqrt{(x+y+z) \cdot xyz} \le \sqrt{\frac{(x+y+z)^4}{27}} = \frac{\sqrt{3}}{9}(x+y+z)^2, \qquad (1)$$

with equality if and only if $x = y = z$.

Next, we use the strict convexity of $f(t) = t^2$ and Jensen's Theorem to get

$$\begin{aligned} (x+y)^2 + (y+z)^2 + (x+z)^2 &\ge 3\left[\frac{(x+y)+(y+z)+(x+z)}{3}\right]^2 \\ &= \frac{4}{3}(x+y+z)^2. \qquad (2) \end{aligned}$$

Here, equality results if and only if $x + y = y + z = x + z$, i.e., if and only if $x = y = z$.

Therefore, by (1) and (2),

$$\frac{\sqrt{(x+y+z) \cdot xyz}}{(x+y)^2 + (y+z)^2 + (x+z)^2} \le \frac{\sqrt{3}}{9} \cdot \frac{3}{4} \cdot \frac{(x+y+z)^2}{(x+y+z)^2} = \frac{\sqrt{3}}{12},$$

with equality if and only if $x = y = z$. It follows that the maximum value of

$$\frac{\sqrt{(x+y+z) \cdot xyz}}{(x+y)^2 + (y+z)^2 + (x+z)^2}$$

is $\dfrac{\sqrt{3}}{12}$ and this is attained precisely when $x = y = z$.

Solution 4 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

We prove that the maximum is $\sqrt{3}/12$. To this end

$$\frac{\sqrt{(x+y+z)xyz}}{(x+y)^2 + (y+z)^2 + (z+x)^2} = \frac{\sqrt{(x+y+z)xyz}}{(x+y+z)^2 + (x^2+y^2+z^2)} \le \frac{\sqrt{3}}{12}$$

and this is implied by

$$\frac{\sqrt{(x+y+z)}\dfrac{(x+y+z)^{3/2}}{3^{3/2}}}{(x+y+z)^2+\dfrac{(x+y+z)^2}{3}} \leq \frac{\sqrt{3}}{12}$$

which is actually an identity and this completes the proof.

Also solved by Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University,Tirana, Albania; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5222: *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Calculate without the aid of a computer the following sum

$$\sum_{n=0}^{\infty}(-1)^n\,(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n, \quad \text{where } i=\sqrt{-1}.$$

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The sum of the series is $\dfrac{164+103\sqrt{2}i}{108}$.

Consider the complex function $f(z)=\dfrac{1}{1+z}$ that is represented by the power series

$$f(z)=\frac{1}{1+z}=\sum_{n=0}^{\infty}(-1)^n z^n$$

on the interior of the unit circle $|z|<1$. Since $\left|\dfrac{1}{1+2\sqrt{2}i}\right|=\dfrac{1}{3}$, the power series and all of its derivatives converge absolutely for $z=\dfrac{1}{1+2\sqrt{2}i}$. For the first derivative

$$f'(z)=\frac{-1}{(1+z)^2}=\sum_{n=1}^{\infty}(-1)^n n z^{n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\,z^n.$$

Therefore,

$$\frac{1}{(1+z)^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)z^n.$$

Differentiating again, one obtains

$$\frac{-2}{(1+z)^3}=\sum_{n=1}^{\infty}(-1)^n(n+1)z^{n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}(n+2)(n+1)z^n.$$

Therefore,

$$\frac{2}{(1+z)^3} = \sum_{n=0}^{\infty}(-1)^n(n^2+3n+2)z^n.$$

Let $z = \dfrac{1}{1+2\sqrt{2}i}$. Then

$$\begin{aligned}
\frac{1}{1+z} &= \frac{1}{1+\frac{1}{1+2\sqrt{2}i}} = \frac{1+2\sqrt{2}i}{2(1+\sqrt{2}i)} = \frac{(1+2\sqrt{2}i)(1-\sqrt{2}i)}{2(1+\sqrt{2}i)(1-\sqrt{2}i)} = \frac{5+\sqrt{2}i}{6}. \\
\frac{1}{(1+z)^2} &= \left(\frac{1}{1+z}\right)^2 = \frac{1}{36}(5+\sqrt{2}i)^2 = \frac{23+10\sqrt{2}i}{36}, \\
\frac{2}{(1+z)^3} &= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(\frac{5+\sqrt{2}i}{3}\right).
\end{aligned}$$

Consequently, if $z = \dfrac{1}{1+2\sqrt{2}i}$, then

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n &= \sum_{n=0}^{\infty}(-1)^n(n^2+3n+2)z^n + \sum_{n=0}^{\infty}(-1)^n(n+1)z^n \\
&= \frac{2}{(1+z)^3} + \frac{1}{(1+z)^2} \\
&= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(\frac{5+\sqrt{2}i}{3}\right) + \left(\frac{23+10\sqrt{2}i}{36}\right) \\
&= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(1+\frac{5+\sqrt{2}i}{3}\right) \\
&= \left(\frac{23+10\sqrt{2}i}{36}\right)\left(\frac{8+\sqrt{2}i}{3}\right) \\
&= \left(\frac{164+103\sqrt{2}i}{108}\right),
\end{aligned}$$

as claimed.

Also solved by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University,Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland;

David Stone and John Hawkins of Georgia Southern University, Statesboro, GA, and the proposer.

5223: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

a) Find the value of
$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right).$$

b) More generally, if $x\in(-1,1]$ is a real number, calculate
$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}+\frac{x^{n+3}}{n+3}-\cdots\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland

We have
$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j\frac{x^{n+1+j}}{n+1+j} &= \sum_{j=0}^{k-1}(-1)^j\int_0^x t^{n+j}dt\\ &= \int_0^x t^n\frac{1-(-t)^k}{1+t}dt\\ &= \int_0^x \frac{t^n}{1+t}dt+O\left(\int_0^x t^{n+k}dt\right)\\ &= \int_0^x \frac{t^n}{1+t}dt+O\left(\frac{1}{n+k+1}\right).\end{aligned}$$

We let k tend to infinity and get
$$\sum_{j=0}^{\infty}(-1)^j\frac{x^{n+1+j}}{n+1+j}=\int_0^x\frac{t^n}{1+t}dt.$$

Then
$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^n\int_0^x\frac{t^n}{1+t}dt &= \int_0^x\frac{1}{1+t}\cdot\frac{1-(-1)^k}{1+t}dt\\ &= \int_0^x\frac{1}{(1+t)^2}dt+O\left(\int_0^x t^k dt\right)\\ &= \left[\frac{-1}{1+t}\right]_0^x+O\left(\frac{1}{k+1}\right).\end{aligned}$$

So,

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{j=0}^{\infty}(-1)^j\frac{x^{n+1+j}}{n+1+j}\right)=\frac{x}{1+x}.$$

Letting $x=1$ implies that the sum of the first series is $\frac{1}{2}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2013

- **5242:** *Proposed by Kenneth Korbin, New York, NY*

  Let N be any positive integer, and let $x = N(N+1)$. Find the value of

$$\sum_{K=0}^{x/2} \binom{x-K}{K} x^K.$$

- **5243:** *Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  If a, b, c are consecutive Pythagorean numbers, then solve in the integers the equation:

$$\frac{x^2+bx}{a^y-1} = c.$$

  (A consecutive Pythagorean triple is a Pythagorean triple that is composed of consecutive integers.)

- **5244:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Let T_a and S_b denote the a^{th} triangular and the b^{th} square number, respectively. Find explicit instances of such numbers to prove that every Fibonacci number F_n occurs among the values $\gcd(T_a, S_b)$.

- **5245:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Determine all functions $f : \Re \rightarrow \Re - \{-2, -\frac{1}{2}, -1, 0, \frac{1}{2}, 2\}$, which satisfy the relation

$$f(x) + f\left(\frac{-x-5}{2x+1}\right) + f\left(\frac{4x+5}{-2x+2}\right) = ax + b$$

  where $a, b, \in \Re$.

- **5246:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $a_1, a_2, \ldots a_n, (n \geq 3)$ be distinct complex numbers. Compute the sum

$$\sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{(-1)^n}{a_j - a_k},$$

where $s_k = \left(\sum_{i=1}^{n} a_i\right) - a_k,\ 1 \leq k \leq n.$

- **5247:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^1 \ln(1+e^x)\ln(1+e^{2x})\cdots\ln(1+e^{nx})\ dx}.$$

Solutions

- **5224:** *Proposed by Kenneth Korbin, New York, NY*

  Let $T_1 = T_2 = 1, T_3 = 2$, and $\mathrm{T}_N = T_{N-1} + T_{N-2} + T_{N-3}$. Find the value of

$$\sum_{N=1}^{\infty} \frac{T_N}{\pi^N}.$$

Solution 1 by Arkady Alt, San Jose, CA

Noting that $\{T_n\}_{n\geq 1}$ is an increasing sequence of positive integers we obtain:

$$\begin{aligned}
\frac{T_{n+1}}{T_n} &= 1 + \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_n} \\
&= 1 + \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_{n-1}} \cdot \frac{T_{n-1}}{T_n} \\
&< 1 + 1 + 1 \cdot 1 = 3,\ n \in N.
\end{aligned}$$

Hence,

$$\frac{T_{n+1}}{T_n} < 3 \iff \frac{T_{n+1}}{3^{n+1}} < \frac{T_n}{3^n},\ n \in N \Longrightarrow \frac{T_n}{3^n} < \frac{T_1}{3^1} \iff T_n < 3^{n-1},\ n \in N.$$

and therefore, by the comparison test for series, $\sum_{i=1}^{n} T_i x^{i-1}$ is convergent for any $x \in \left(0, \frac{1}{3}\right)$ because for such x it is bounded by $\sum_{n=1}^{\infty} (3x)^{n-1} = \frac{1}{1-3x}$.

Since

$$\left(1 - x - x^2 - x^3\right) \sum_{n=1}^{\infty} T_n x^{n-1} = T_1 + x(T_2 - T_1) + x^2(T_3 - T_2 - T_1)$$

$$
\begin{aligned}
&+ \quad \sum_{n=1}^{\infty} x^{n+2}\left(T_{n+3}-T_{n+2}-T_{n+2}-T_{n}\right) \\
&= \quad T_1 + x(1-1) + x^2(2-1-1) + \sum_{n=1}^{\infty} x^{n+2}\cdot 0 = 1
\end{aligned}
$$

then

$$
\sum_{n=1}^{\infty} T_n x^{n-1} \frac{1}{1-x-x^2-x^3} \iff \sum_{n=1}^{\infty} T_n x^n = \frac{x}{1-x-x^2-x^3}
$$

and therefore, for $x = \dfrac{1}{\pi} < 3$, we obtain

$$
\sum_{n=1}^{\infty} \frac{T_n}{\pi^n} = \frac{\frac{1}{\pi}}{1-\frac{1}{\pi}-\frac{1}{\pi^2}-\frac{1}{\pi^3}} = \frac{\pi^2}{\pi^3-\pi^2-\pi-1}.
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that $1 \le T_n \le 2^{n-1}$ for $n \ge 1$. Indeed this is true for $n = 1, 2,$ and 3 and

$1 \le T_n = T_{n-1}+T_{n-2}+T_{n-3} \le 2^{n-2}+2^{n-3}+2^{n-4} < 2^{n-2}+2^{n-3}+2^{n-3} = 2^{n-1}$, as claimed.

So, $S = \displaystyle\sum_{n=1}^{\infty} \frac{T_n}{\pi^n}$ is convergent and

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{T_n}{\pi^n} = \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \sum_{n=1}^{\infty} \frac{T_{n-1}+T_{n-2}+T_{n-3}}{\pi^n} \\
&= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \frac{1}{\pi}\sum_{n=3}^{\infty} \frac{T_n}{\pi^n} + \frac{1}{\pi^2}\sum_{n=2}^{\infty} \frac{T_n}{\pi^n} + \frac{1}{\pi^3}\sum_{n=1}^{\infty} \frac{T_n}{\pi^n} \\
&= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \frac{1}{\pi}\left(S - \frac{1}{\pi} - \frac{1}{\pi^2}\right) + \frac{1}{\pi^2}\left(S - \frac{1}{\pi}\right) + \frac{1}{\pi^3} S \\
&= \frac{1}{\pi} + S\left(\frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3}\right). \ \text{ So,} \\
S &= \frac{\pi^2}{\pi^3-\pi^2-\pi-1}
\end{aligned}
$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

Let us pose, $a_n = \dfrac{T_n}{\pi^n}$, $T_0 = 0$. We prove by induction that, $\ T_n \le T_{n+1} \le 2T_n$.

$$
T_n \le T_{n+1} = T_n + T_{n-1} + T_{n-2} \le 2T_{n-1} + 2T_{n-2} + 2T_{n-3} = 2T_n.
$$

Thus, it implies that,

$$
\forall n \in N : \qquad \frac{1}{\pi} a_n \le a_{n+1} = \frac{T_{n+1}}{\pi^{n+1}} = \frac{1}{\pi}\cdot\frac{T_{n+1}}{T_n}\cdot\frac{T_n}{\pi^n} \le \frac{2}{\pi} a_n,
$$

and by induction it results that

$$\left(\frac{1}{\pi}\right)^n = \left(\frac{1}{\pi}\right)^n a_1 \le a_{n+1} \le \left(\frac{1}{\pi}\right)^n a_1 = \left(\frac{2}{\pi}\right)^n.$$

Thus, the given series converges, and

$$\frac{1}{\pi-1} = \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n \le \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{T_n}{\pi^n} \le \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\right)^n = \frac{1}{\pi-2}.$$

Considering the given difference equation for T_n we transform it to a difference equation for a_n

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \qquad \Leftrightarrow \qquad \frac{T_n}{\pi^n} = \frac{1}{\pi}\cdot\frac{T_{n-1}}{\pi^{n-1}} + \frac{1}{\pi^2}\cdot\frac{T_{n-2}}{\pi^{n-2}} + \frac{1}{\pi^3}\cdot\frac{T_{n-3}}{\pi^{n-3}}$$

$$\Leftrightarrow \qquad a_n = \frac{1}{\pi}\cdot a_{n-1} + \frac{1}{\pi^2}\cdot a_{n-2} + \frac{1}{\pi^3}\cdot a_{n-3}.$$

The respective characteristic equation is the following one, the left side of which is a nonnegative polynomial,

$$p(\lambda) = 0 \qquad \Leftrightarrow \qquad \lambda^3 - \frac{1}{\pi}\cdot\lambda^2 - \frac{1}{\pi^2}\cdot\lambda - \frac{1}{\pi^3} = 0.$$

Studying its derivative, $p'(\lambda) = 3(\lambda + \frac{1}{3})(\lambda - 1)$, we come to the conclusion that the characteristic polynomial has a unique positive real root, $\alpha \in (0;1)$, and two complex conjugate roots, $\beta, \gamma \in C$.

Recall the Theorem for the dominance of the unique positive root of a nonnegative polynomial that states:

Theorem. *If λ_0 is a positive root of a nonnegative polynomial $p(x)$, then λ_0 is a dominant root, in the sense that any other root $\lambda \in C$ satisfies the relation $|\lambda| \le \lambda_0$.* Thus, $0 < |\beta| = |\gamma| < \alpha < 1$.

The general structure of the term a_n is,

$$\forall n = 0, 1, 2, : \quad a_n = c_1\cdot\alpha^n + c_2\cdot\beta^n + c_3\cdot\gamma^n, \quad \textit{where} \quad c_1, c_2, c_3 \in C.$$

To define the constants we consider the initial conditions,

$$a_0 = \quad 0 \quad = \quad c_1\cdot\alpha^0 + c_2\cdot\beta^0 + c_3\cdot\gamma^0$$

$$a_1 = \quad \frac{1}{\pi} \quad = \quad c_1\cdot\alpha^1 + c_2\cdot\beta^1 + c_3\cdot\gamma^1$$

$$a_2 = \frac{1}{\pi^2} \quad = \quad c_1\cdot\alpha^2 + c_2\cdot\beta^2 + c_3\cdot\gamma^2$$

And these imply:

$$c_1 = \frac{(\beta-\gamma)(\beta+\gamma-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)},\ c_2 = \frac{(\gamma-\alpha)(\gamma+\alpha-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)},\ c_3 = \frac{(\alpha-\beta)(\alpha+\beta-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

Since, $\quad \alpha, \beta, \gamma \in \{z \in C : |z| < 1\} \quad \Rightarrow \quad \lim_{n\to\infty} \alpha^n = \lim_{n\to\infty} \beta^n = \lim_{n\to\infty} \gamma^n = 0$

Doing some simple operations and based on Vieta's formulas

$$\alpha\beta\gamma = \frac{1}{\pi^3}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{\pi^2}, \qquad \alpha + \beta + \gamma = \frac{1}{\pi^3}$$

implies that

$$\begin{aligned}
\Rightarrow \quad \sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{\infty} (c_1 \cdot \alpha^n + c_2 \cdot \beta^n + c_3 \cdot \gamma^n) = c_1 \cdot \sum_{n=0}^{\infty} \alpha^n + c_2 \cdot \sum_{n=0}^{\infty} \beta^n + c_3 \cdot \sum_{n=0}^{\infty} \gamma^n \\
&= \frac{c_1}{1-\alpha} + \frac{c_2}{1-\beta} + \frac{c_3}{1-\gamma} = \frac{c_1}{1-\alpha} + \frac{c_2}{1-\beta} + \frac{c_3}{1-\gamma} \\
&= \frac{1}{\pi(1-\alpha)(1-\beta)(1-\gamma)} = \frac{1}{\pi\left(1 - \frac{1}{\pi} - \frac{1}{\pi^2} - \frac{1}{\pi^3}\right)} \\
&= \frac{\pi^2}{\pi^3 - \pi^2 - \pi - 1}
\end{aligned}$$

since $p(1) = (1-\alpha)(1-\beta)(1-\gamma) = 1 - \dfrac{1}{\pi} - \dfrac{1}{\pi^2} - \dfrac{1}{\pi^3}$ is the value of the characteristic polynomial for $\lambda = 1$.

Comment: Let us prove that $a_n = (c_1 \cdot \alpha^n + c_2 \cdot \beta^n + c_3 \cdot \gamma^n) \in R$, even if c_1, c_2, c_3 are complex constants.

The first term $c_1\alpha^n$ is a real number since $c_1 \in R$ and $\alpha \in R$. Indeed,

$$c_1 = \frac{(\beta-\gamma)(\beta+\gamma-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} = \frac{-\alpha}{\pi(\alpha-\beta)(\gamma-\alpha)} = \frac{-\alpha}{\pi[\alpha(\beta+\gamma) - \alpha^2 - \beta\gamma]} \in R$$

since $\dfrac{\alpha}{\pi} \in R$, $(\beta+\gamma) = 2Re\beta \in R$ and $\beta\gamma = |\beta|^2 \in R$.

To prove that the summation of the other two terms in the expression for a_n is a real number, we need to prove by induction in n that $\forall n \in N$,$\dfrac{(\beta^n - \gamma^n)}{(\beta-\gamma)} \in R$.

Indeed, supposing that the given expression is a real number $\forall\, k < n$. Then

$$\begin{aligned}
\frac{(\beta^n - \gamma^n)}{(\beta-\gamma)} &= \frac{(\beta^{n-1} - \gamma^{n-1})(\beta+\gamma) - \beta\gamma(\beta^{n-2} - \gamma^{n-2})}{(\beta-\gamma)} \\
&= (\beta+\gamma)\frac{(\beta^{n-1} - \gamma^{n-1})}{(\beta-\gamma)} - \beta\gamma\frac{(\beta^{n-2} - \gamma^{n-2})}{(\beta-\gamma)} \in R \text{ since} \\
(\beta+\gamma) &= 2Re\beta \in R\ , \quad \beta\gamma = |\beta|^2 \in R.
\end{aligned}$$

Thus,

$$c_2 \cdot \beta^n + c_3 \cdot \gamma^n = \frac{(\gamma-\alpha)(\gamma+\alpha-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} \cdot \beta^n + \frac{(\alpha-\beta)(\alpha+\beta-\frac{1}{\pi})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} \cdot \gamma^n$$

$$= \frac{(\gamma-\alpha)(-\beta)\beta^n + (\alpha-\beta)(-\gamma)\gamma^n}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

$$= \frac{\alpha(\beta^{n+1}-\gamma^{n+1}) - \beta\gamma(\beta^{n-1}-\gamma^{n-1})}{\pi(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

$$= \frac{1}{\pi[\alpha(\beta+\gamma)-\alpha^2-\beta\gamma)]}\left[\alpha\frac{(\beta^{n+1}-\gamma^{n+1})}{(\beta-\gamma)} - \beta\gamma\frac{(\beta^{n-1}-\gamma^{n-1})}{(\beta-\gamma)}\right] \in R$$

$\forall n \in N,\ \dfrac{(\beta^n-\gamma^n)}{(\beta-\gamma)} \in R\ ,\quad \alpha \in R,\quad (\beta+\gamma) = 2Re\beta \in R\ ,\quad \beta\gamma = |\beta|^2 \in R.$

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University, Statesboro, GA** noted in their solution that the π in the statement of the problem is simply a stand in. They found the characteristic equation for the linear recurrence to be $p(x) = x^3 - x^2 - x - 1$. Letting $z, \overline{z}$, and r be the roots of the characteristic polynomial they observed that $\displaystyle\sum_{n=0}^{\infty}\frac{T_n}{\pi^n} = \sum_{n=0}^{\infty}\frac{k_1z^n + k_2(\overline{z})^n + k_3r^n}{\pi^n}$ is the sum of three geometric series, each of which must necessarily converge. They then found the values of $z, \overline{z}$, and r.

$$\begin{aligned}
p(x) &= x^3 - x^2 - x - 1, \text{ and also} \\
&= (x-z)(x-\overline{z})(x-r) \\
&= x^3 - (z+\overline{z}+r)x^2 - (z\overline{z} + zr + \overline{z}r)x - z\overline{z}r, \\
&\quad \text{and by equating coefficients} \\
z+\overline{z} &= 1 - r \text{ and} \\
|z|^2 &= z\overline{z} = \frac{1}{r}.
\end{aligned}$$

Using a calculator they approximated $r \approx 1.87$ so $|z| = |\overline{z}| \approx 0.54$. They went on to say that they could have solved the characteristic equation with Cardan's formula, but all they needed to know about the roots is that each, in absolute value, is smaller than π, which they just saw; so that the three geometric series in $\displaystyle\sum_{n=0}^{\infty}\frac{T_n}{\pi^n}$ converge. By Cardan's formula, the root r equals $\dfrac{1}{3} - \dfrac{C}{3} - \dfrac{4}{3C}$ where $C = \sqrt[3]{3\sqrt{33}-19}$. They calculated $r \approx 1.839286755$.

They then noted that if t is *any* real constant larger than r, the same calculations hold, thus showing

$$\sum_{n=0}^{\infty}\frac{T_n}{t_n} = \frac{t^2}{p(t)} = \frac{t^2}{t^3 - t^2 - t - 1}.$$

For instance, $\displaystyle\sum_{n=0}^{\infty}\frac{T_n}{2^n} = \frac{2^2}{p(t)} = \frac{4}{1}.$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego Viveiro, Spain; Michael N. Fried, Ben-Gurion University, Beer

Sheva, Israel; Noel Evens, Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Enkel Hysnelaj, University of Technology, Sydney, Australia together with Elton Bojaxhiu, Kriftel, Germany; Anastasios Kotronis, Athens, Greece; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA and the proposer.

- **5225:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Find infinitely many integer squares x that are each the sum of a square and a cube and a fourth power of positive integers a, b, c. That is, $x = a^2 + b^3 + c^4$.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

By observation, we conclude that for $n \geq 1$,

$$\begin{aligned}(2n^3)^4 + (2n^2)^3 + 1^2 &= 16n^{12} + 8n^6 + 1 \\ &= (4n^6 + 1)^2.\end{aligned}$$

Also, it can be observed for $n \geq 1$,

$$\begin{aligned}1^4 + (2n^2)^3 + (4n^6)^2 &= 1 + 8n^6 + 16n^{12} \\ &= (4n^6 + 1)^2.\end{aligned}$$

Thus, for $n \geq 1$, $x^2 = (4n^6 + 1)^2$ generates infinitely many integer squares such that $x^2 = a^2 + b^3 + c^4$ where a, b, c are positive integers

Solution 2 by Ángel Plaza, University of Las Palmas de gran, Canaria, Spain

Since $(a + c^2)^2 = a^2 + c^4 + 2ac^2$ it is enough to consider $b = 2a = c$ to obtain infinitely many integer squares $x = (a + c^2)^2 = a^2 + c^4 + 2ac^2 = a^2 + b^3 + c^4$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let m and n be any positive integers. Using the identity

$$\left(4m^3 + 4n^3 + n\right)^2 = \left(4m^3 - 4n^3 + n\right)^2 + (4mn)^3 + (2n)^4,$$

we find infinitely many such x.

Solution 4 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

For each positive integer n, let $a = 2^{3n-2} + 3,\ b = 2^n$, and $c = 2$. Then

$$\begin{aligned}a^2 + b^3 + c^4 &= \left(2^{3n-2} + 3\right)^2 + 2^{3n} + 2^4 \\ &= 2^{6n-4} + 10 \cdot 2^{3n-2} + 25 \\ &= \left(2^{3n-2} + 5\right)^2 = x.\end{aligned}$$

Note that if $b = 2^n, c = 2$ and $x = y^2$, then $y^2 = a^2 + 2^{3n} + 2^4$. Therefore,

$$y^2 - a^2 = 2^{3n} + 2^4 \text{ or } (y + a)(y - a) = 2(2^{3n-1} + 2^3).$$

Let

$$\begin{cases} y + a = 2^{3n-1} + 2^3 & \text{and} \\ y - a = 2 & . \end{cases}$$

The simultaneous solution for this system of equations is $y = 2^{3n-2} + 5$ and $a = 2^{3n-2} + 3$.

Accordingly, the infinitely many integer squares $x = a^2 + b^3 + c^4$ are $x = \left(2^{3n-2} + 5\right)^2$ for each positive integer n.

Solution 5 by Ken Korbin, New York, NY

There are infinitely many pairs of positive integers b and c such that $b + c$ is odd. If $a = \dfrac{b^3 + c^4 - 1}{2}$ then $a^2 + b^3 + c^4 = (a + 1)^2 = x$. Examples:

a	b	c	$x = (a+1)^2$
316	2	5	$(317)^2$
70	5	2	$(71)^2$
128	1	4	$(129)^2$
72	4	3	$(73)^2$

If a, b, and c are positive integers such that $a^2 + b^3 + c^4 = (a + 1)^2$ and if k is a positive integer then

$$\begin{aligned} a^2 \cdot k^{12} + b^3 \cdot k^{12} + c^4 \cdot k^{12} &= (a + 1)^2 \cdot k^{12} \\ &= \left(a \cdot k^6\right)^2 + \left(b \cdot k^4\right)^3 + \left(c \cdot k^3\right)^4 \\ &= \left((a + 1) \cdot k^6\right)^2 = x. \end{aligned}$$

Solution 6 by Brian D. Beasley, Presbyterian College, Clinton, SC

In order to have $x = k^2 = a^2 + b^3 + c^4$ for positive integers k, a, b, and c, we need $b^3 + c^4$ to be expressible as the difference of two squares. As Burton notes (*Elementary Number Theory* , 7th ed., Theorem 13-4, p. 269), a positive integer n has such an expression if and only if n is not congruent to 2 modulo 4. Thus as long as $b^3 + c^4$ is not congruent to 2 modulo 4, we may solve for k and a.

In particular, when $b^3 + c^4$ is odd, we may take $a = (b^3 + c^4 - 1)/2$ and $k = a + 1$, as seen in the following two cases:

One infinite set of solutions occurs when $c = 1$ and $b = 2n$ for any positive integer n, which makes $b^3 + c^4 = 8n^3 + 1$ odd. We then take $a = 4n^3$ to produce $k = 4n^3 + 1$ and hence $x = (4n^3 + 1)^2 = 16n^6 + 8n^3 + 1$.

Another infinite set of solutions occurs when $b = 1$ and $c = 2n$ for any positive integer n, which makes $b^3 + c^4 = 16n^4 + 1$ odd. We then take $a = 8n^4$ to produce $k = 8n^4 + 1$ and hence $x = (8n^4 + 1)^2 = 64n^8 + 16n^4 + 1$.

Also solved by Farideh Firoozbakht and Jahangeer Kholdi University of Isfahan, Khansar, Iran; Enkel Hysnelaj, University of Technology, Sydney, Australia together with Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; Charles McCracken, Dayton, OH; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

- **5226:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

If a and b, $a < b$ are real-valued positive numbers, then calculate:

$$\int_a^b \frac{\sqrt[n]{x-a}\left(1+\sqrt[n]{b-x}\right)}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,dx,$$

where n is a positive integer greater than one, $(n > 1)$.

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania

Let

$$\begin{aligned} I_1 &= \int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx \text{ and} \\ I_2 &= \int_a^b \frac{\sqrt[n]{b-x}(1+\sqrt[n]{x-a})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx. \end{aligned}$$

Setting $y = b + a - x$, we have

$$\begin{aligned} I_1 &= \int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx \\ &= \int_b^a \frac{\sqrt[n]{b-y}(1+\sqrt[n]{y-a})}{\sqrt[n]{y-a}+2\sqrt[n]{y-a}\sqrt[n]{b-y}+\sqrt[n]{b-y}}d(b+a-y) \\ &= \int_a^b \frac{\sqrt[n]{b-y}(1+\sqrt[n]{y-a})}{\sqrt[n]{y-a}+2\sqrt[n]{y-a}\sqrt[n]{b-y}+\sqrt[n]{b-y}}dy = I_2 \end{aligned}$$

So,

$$I_1 + I_2 = \int_a^b dx = b - a, \text{ and therefore,}$$

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{x-a}\sqrt[n]{b-x}+\sqrt[n]{b-x}}dx = \frac{b-a}{2}$$

Solution 2 by Anastasios Kotronis, Athens, Greece

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,dx;\ \text{letting } x=y+\frac{a+b}{2}, \text{we obtain}$$

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{\sqrt[n]{y+\frac{b-a}{2}}\left(1+\sqrt[n]{\frac{b-a}{2}-y}\right)}{\sqrt[n]{y+\frac{b-a}{2}}+2\sqrt[n]{\left(y+\frac{b-a}{2}\right)\left(\frac{b-a}{2}-y\right)}+\sqrt[n]{\frac{b-a}{2}-y}}-\frac{1}{2}+\frac{1}{2}\,dy$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(y)+\frac{1}{2}\,dy$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(y)\,dy+\frac{b-a}{2}.$$

Now it is easy to see that $g(y)$ is odd so the given integral equals $\dfrac{b-a}{2}$.

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $\dfrac{b-a}{2}$

Proof: The integral is actually

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\left(\sqrt[n]{x-a}+\sqrt[n]{b-x}\right)^2}dx=\int_a^b \frac{1}{1+\sqrt[n]{\frac{b-x}{x-a}}}dx$$

Setting $t=(b-x)/(x-a)$ we get

$$(b-a)\int_0^\infty \frac{1}{(1+t)^2}\frac{1}{1+t^{1/n}}dt$$

The further change $t=y^n$ yields

$$(b-a)\int_0^\infty \frac{1}{(1+y^n)^2}\frac{1}{1+y}ny^{n-1}dy$$

Integrating by parts

$$(b-a)\frac{1}{1+y}\frac{1}{1+y^n}\Big|_\infty^0-\int_0^\infty \frac{b-a}{(1+y)^2}\frac{1}{1+y^n}dy=b-a-\int_0^\infty \frac{b-a}{(1+y)^2}\frac{1}{1+y^n}dy.$$

To compute the last integral we set $y=1/z$ and obtain

$$\int_0^\infty \frac{1}{(1+y)^2}\frac{1}{1+y^n}dy=\int_0^\infty \frac{z^2}{(1+z)^2}\frac{z^n}{1+z^n}\frac{1}{z^2}dz=\int_0^\infty \frac{1}{(1+z)^2}\frac{z^n}{1+z^n}dz=$$

$$=\int_0^\infty \frac{1}{(1+z)^2}dz-\int_0^\infty \frac{1}{(1+z)^2}\frac{1}{1+z^n}dz$$

that is,
$$\int_0^\infty \frac{1}{(1+y)^2}\frac{1}{1+y^n}dy = \frac{1}{2}\int_0^\infty \frac{1}{(1+z)^2}dz = \frac{1}{2}.$$

The final result is $\dfrac{1}{2}(b-a.)$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5227:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Compute
$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right).$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Since $\ln(1+x) = x + O\left(x^2\right)$ as $x \longrightarrow 0$, so
$$\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right) = \sum_{k=1}^{n}\frac{1}{n+\sqrt{nk}} + O\left(\frac{1}{n}\right).$$

Hence,
$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\left(1+\sqrt{\frac{k}{n}}\right)} = \int_0^1 \frac{dx}{1+\sqrt{x}}.$$

By the substitution $x = y^2$, we easily evaluate the last integral to be $2(1-\ln 2)$.

Now by exponentiation, we find the limit of the problem to be $\dfrac{e^2}{4}$.

Solution 2 by Arkady Alt, San Jose, CA

First note that for any positive real x we have
$$e^x\left(1-\frac{x^2}{2}\right) < 1+x < e^x. \qquad (1)$$

Indeed, for any positive x we can obtain from the Taylor representation of e^x that:
$$\begin{aligned} 1+x < e^x &= 1+x+\frac{x^2}{2!}+\sum_{n=1}^{\infty}\frac{x^{n+2}}{(n+2)!} \\ &= 1+x+\frac{x^2}{2}\left(1+\sum_{n=1}^{\infty}\frac{2x^n}{(n+2)!}\right) \end{aligned}$$

$$< \quad 1 + x + \frac{x^2}{2}\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right)$$

$$= \quad 1 + x + \frac{x^2 e^x}{2} \text{ and then we have}$$

$$e^x < 1 + x + \frac{x^2 e^x}{2} \iff e^x\left(1 - \frac{x^2}{2}\right) < 1 + x.$$

Applying inequality (1) to $x = \dfrac{1}{n+\sqrt{nk}}, k = 1, 2, ..., n$ we obtain

$$e^{a_{kn}} b_{kn} < 1 + \frac{1}{n+\sqrt{nk}} < e^{a_{kn}}, k = 1, 2, ..., n, \qquad (2)$$

where $a_{kn} = \dfrac{1}{n+\sqrt{nk}}$ and $b_{kn} = 1 - \dfrac{1}{2\left(n+\sqrt{nk}\right)^2}$.

Let $S_n = \sum_{k=1}^{n} a_{kn}$. Hence,

$$e^{S_n} \prod_{k=1}^{n} b_{kn} < \prod_{k=1}^{n} \left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) < e^{S_n}.$$

Note that $\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} b_{kn} = 1$. Indeed, since $n < n + \sqrt{nk} < 2n, k = 1, 2, ..., n$ then

$$1 - \frac{1}{2n^2} < 1 - \frac{1}{2\left(n+\sqrt{nk}\right)^2} < 1 - \frac{1}{8n^2}, k = 1, 2, ..., n$$

and we obtain

$$\left(1 - \frac{1}{2n^2}\right)^n < \prod_{k=1}^{n} b_{kn} < \left(1 - \frac{1}{8n^2}\right)^n < 1.$$

Since

$$\lim_{n\to\infty} \left(1 - \frac{1}{2n^2}\right)^{n^2} = \frac{1}{\sqrt{e}} \text{ then}$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{2n^2}\right)^n = \lim_{n\to\infty} \sqrt[n]{\left(1 - \frac{1}{2n^2}\right)^{n^2}} = 1.$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \sqrt{\frac{k}{n}}}$$

$$= \int_0^1 \frac{1}{1+\sqrt{x}} dx = \left[x = t^2; dx = 2tdt\right]$$

$$= \quad 2\int_0^1 \frac{t}{1+t}dt = 2\left(t - \ln\left(1+t\right)\right)|_0^1 = 2\left(1-\ln 2\right), \text{ then}$$

$$\lim_{n\to\infty} e^{S_n} = \lim_{n\to\infty} e^{S_n}\prod_{k=1}^{n} b_{kn} = e^{2(1-\ln 2)} \ = \frac{e^2}{4}.$$

By the Squeeze Principle we see that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) = \frac{e^2}{4}.$$

Solution 3: by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The proposed limit may be written as $L = \lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(1+\dfrac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)$. So,

$\ln L = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\ln\left(1+\dfrac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)$. Now we expand each of the logs according to its power series and write this as a double sum. Then we change order of summation and sum up by columns. This is allowed because both directions provide convergent sums. So

$$\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right)^3}{3} + \cdots.$$

$$\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right)^3}{3} + \cdots.$$

$$\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right)^3}{3} + \cdots.$$

Note that

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}} &= \int_0^1\frac{1}{1+\sqrt{x}}dx = \ln\left(\frac{e^2}{4}\right), \\
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)^m}{m} &= 0, \ \text{for } m>1.
\end{aligned}$$

From where $\ln L = \ln\left(\dfrac{e^2}{4}\right)$, and therefore $L = \dfrac{e^2}{4}$.

Also solved by Bruno Salgueiro Fanego Viveiro, Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel,

German; Anastasios Kotronis, Athens, Greece; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland;

- **5228:** *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada*

Given a random variable X with non-negative integer values. Assume the n^{th} moment of X is given by

$$E\left(X^n\right) = \sum_{k=1}^{\infty} f_n(k) P\left(X \geq k\right) \qquad n = 1, 2, 3, \cdots,$$

where f_n is a non-negative function defined on N. Find a closed formula for f_n.

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

From the first principle we have

$$E(X^n) = \sum_{k=1}^{\infty} k^n P(X = k)$$

Doing easy manipulations we have

$$\begin{aligned}
E(X^n) &= \sum_{k=1}^{\infty} f_n(k) P(X \geq k) \\
&= f_n(1)P(X \geq 1) + f_n(2)P(X \geq 2) + ... + f_n(k)P(X \geq k) + ... \\
&= f_n(1)(P(X = 1) + P(X = 2) + ...) + f_n(2)(P(X = 2) + P(X = 3) + ...) + ... \\
&\quad + f_n(k)(P(X = k) + P(X = k+1) + ...) + ... \\
&= f_n(1)P(X = 1) + (f_n(1) + f_n(2))P(X = 2) + ... \\
&\quad + (f_n(1) + f_n(2) + ... + f_n(k))P(X = k) + ... \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{k} f_n(i) P(X = k)
\end{aligned}$$

Comparing this with the expression we have from the first principle we have

$$\sum_{i=1}^{k} f_n(i) = k^n$$

for any non-negative integers k and n.

Finally, using the above result implies

$$f_n(k) = \sum_{i=1}^{k} f_n(i) - \sum_{i=1}^{k-1} f_n(i) = k^n - (k-1)^n$$

and this is the end of the proof.

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5229:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $\beta > 0$ be a real number and let $(x_n)_{n\in N}$ be the sequence defined by the recurrence relation

$$x_1 = a > 0,\ x_{n+1} = x_n + \frac{n^{2\beta}}{x_1 + x_2 \cdots + x_n}.$$

1) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

2) Calculate $\lim\limits_{n\to\infty} \frac{x_n}{n^\beta}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

1) By induction, we have $x_n > 0$ for positive integers n. Hence x_n is strictly increasing. Suppose, on the contrary that, $\lim\limits_{x\to\infty} s_n = L$, where $0 < L < \infty$.

Since $0 < x_1 + x_2 + \ldots + x_n < nL$, so, $x_{n+1} > x_n + \frac{n^{2\beta-1}}{L}$.

Hence for any positive integer N, we have $\sum\limits_{n=1}^{N} x_{n+1} > \sum\limits_{n=1}^{N} x_n + \frac{1}{L}\sum\limits_{n=1}^{N} n^{2\beta-1}$, so that

$L > x_{N+1} > a + \frac{1}{L}\sum\limits_{n=1}^{N} n^{2\beta-1}$. Since $\sum\limits_{n=1}^{N} n^{2\beta-1} \longrightarrow \infty$ as $N \longrightarrow \infty$, this is a contradiction. It follows that $\lim\limits_{n\to\infty} x_n = \infty$.

2) To find the leading behavior of x_n as $n \to \infty$, we try

$$x_n \sim kn^\alpha \qquad (1)$$

for some positive constants k and α. We then have $x_1 + x_2 + \ldots + x_n \sim \frac{kn^{\alpha+1}}{\alpha+1}$.

Hence $x_{n+1} - x_n \sim \frac{(\alpha+1)n^{2\beta-\alpha-1}}{k}$. If $\alpha > 2\beta$, then x_{n+1} is bounded, which is not true.

If $\alpha = 2\beta$, then $x_{n+1} \sim \frac{(\alpha+1)\ln n}{k}$, which is inconsistent with (1) . So $0 < \alpha < 2\beta$, and we we have

$$x_{n+1} \sim \frac{(\alpha+1)n^{2\beta-\alpha}}{k(2\beta-\alpha)}.$$

By (1) and (2), we see that $\alpha = 2\beta - \alpha$ and $k = \frac{\alpha+1}{k(2\beta-\alpha)}$. Hence $\alpha = \beta$ and $k = \sqrt{\frac{\beta+1}{\beta}}$. It follows that $\lim\limits_{n\to\infty} \frac{s_n}{n^\beta} = \sqrt{\frac{\beta+1}{\beta}}$.

Solution 2 by proposer

(1) It is easy to see that $x_n > 0$, for all $n \in N$. Also, $x_{n+1} - x_n = \frac{n^{2\beta}}{x_1+x_2+\cdots+x_n} > 0$, and hence the sequence is strictly increasing. By way of contradiction, we assume that $\lim_{n\to\infty} x_n = l$. We have, since (x_n) increases, that $l \neq 0$ and $x_n < l$ for all $n \in N$. Iterating the recurrence relation we get that

$$x_{n+1} \quad = \quad x_1 + \frac{1}{x_1} + \frac{2^{2\beta}}{x_1+x_2} + \cdots + \frac{n^{2\beta}}{x_1+x_2+\cdots+x_n} > x_1 + \frac{1}{l} + \frac{2^{2\beta}}{2l} + \cdots + \frac{n^{2\beta}}{nl}$$

$$= \quad x_1 + \frac{1}{l}\left(1 + 2^{2\beta-1} + \cdots + n^{2\beta-1}\right).$$

Passing to the limit in the preceding inequality we get that $l \geq \infty$, which is a contradiction.

2) The limit equals $\sqrt{(\beta+1)/\beta}$. We apply Cesaro-Stolz Lemma and we have that

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{x_n}{n^\beta} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty} \frac{\frac{n^{2\beta}}{x_1+x_2+\cdots+x_n}}{(n+1)^\beta - n^\beta} \\
&= \lim_{n\to\infty}\left(\frac{n^{\beta+1}}{x_1+x_2+\cdots+x_n}\cdot\frac{n^{\beta-1}}{(n+1)^\beta - n^\beta}\right) \\
&= \frac{1}{\beta}\cdot\lim_{n\to\infty}\left(\frac{n^{\beta+1}}{x_1+x_2+\cdots+x_n}\right) \\
& \quad Cesaro - Stolz\ again = \frac{1}{\beta}\cdot\lim_{n\to\infty}\frac{(n+1)^{\beta+1}-n^{\beta+1}}{x_{n+1}} \\
&= \frac{1}{\beta}\lim_{n\to\infty}\left(\frac{(n+1)^\beta}{x_{n+1}}\cdot\frac{(n+1)^{\beta+1}-n^{\beta+1}}{(n+1)^\beta}\right) \\
&= \frac{(\beta+1)}{\beta\cdot L}.
\end{aligned}$$

Thus, $L = \sqrt{(\beta+1)/\beta}$ and the problem is solved.

Solution 3: by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

1) Since $x_1 = 1 > 0$ it is easy to see that sequence $\{x_n\}_{n\in N}$ is increasing and also that

$$\begin{aligned}
x_{n+1} &= x_1 + \frac{1}{x_1} + \frac{2^{2\beta}}{x_1+x_2} + \cdots + \frac{n^{2\beta}}{x_1+x_2+\cdots+x_n} \\
&> x_1 + \frac{1}{x_n} + \frac{2^{2\beta}}{2x_n} + \cdots + \frac{n^{2\beta}}{nx_n} \\
&= x_1 + \frac{1}{x_n}H_n
\end{aligned}$$

where, $H_n = 1 + 2^{2\beta-1} + \cdots + n^{2\beta-1}$. Since $\{x_n\}_{n\in N}$ is increasing, then either $\{x_n\}_{n\in N}$ is convergent if bounded, or $\lim\limits_{n\to\infty} x_n = \infty$.

Now, since $\lim\limits_{n\to\infty} H_n = \infty$, the hypothesis of $\{x_n\}_{n\in N}$ convergent gives a contradiction with the fact that $x_1 + \frac{1}{x_n}H_n < x_{n+1}$. Therefore $\lim\limits_{n\to\infty} x_n = \infty$.

2. Note that since $x_{n+1} = x_1 + \dfrac{1}{x_1} + \dfrac{2^{2\beta}}{x_1+x_2} + \cdots + \dfrac{n^{2\beta}}{x_1+x_2+\cdots+x_n}$, then, by Stolz-Cezaro criteria

$$L \quad = \quad \lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^\beta} = \lim_{n\to\infty}\frac{\frac{n^{2\beta}}{x_1+x_2+\cdots+x_n}}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty}\frac{\frac{1}{\beta}n^{\beta+1}}{x_1+x_2+\cdots+x_n}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{\beta} \frac{n^{\beta+1}-(n-1)^{\beta+1}}{x_n} = \lim_{n\to\infty} \frac{1}{\beta} \cdot \frac{(\beta+1)n^{\beta}}{x_n} \\
&= \frac{\beta+1}{\beta} \cdot \frac{1}{L},
\end{aligned}
$$

from where $L = \sqrt{\frac{\beta+1}{\beta}}$.

Also solved by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Notes and Comments

From Charles McCracken of Dayton, OH:

In their solution to Problem 5213 David Stone and John Hawkins note that n^4 is always the sum of two triangular numbers. But n^2 is also the sum of two (consecutive) triangular numbers:

$$
\begin{aligned}
T_n + T_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \\
&= \frac{n^2+n+n^2+3n+2}{2} = \frac{2n^2+4n+2}{2} \\
&= n^2+2n+1 = (n+1)^2.
\end{aligned}
$$

Thus, adding the triangular numbers in sequential pairs generates all the squares; which generates all the fourth powers.

Mea Culpa

The names of **Brian D. Beasley of Presbyterian College in Clinton, SC and of Arkady Alt of San Jose, CA** were inadvertently left off the list of having solved problem 5218. Arkady also solved 5220 and 5221, and I missed listing his name for those too. To Brian and Arkardy, mea culpa, sorry.

Additionally, **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA** should receive credit for having solved 5215. I am happy to report that this time the "senior moment" is theirs and not mine; they forgot to send me their solution!

Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2013

- **5248:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with sides (a, a, b) has the same area and the same perimeter as a triangle with sides (c, c, d) where a, b, c and d are positive integers and with

  $$\frac{b^2 + bd + d^2}{b + d} = 7^6.$$

  Find the sides of the triangles.

- **5249:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  (a) Let n be an odd positive integer. Prove that $a^n + b^n$ is the square of an integer for infinitely many integers a and b.

  (b) Prove that $a^2 + b^3$ is the square of an integer for infinitely many integers a and b.

- **5250:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Let $a \in \left(0, \dfrac{\pi}{2}\right)$ and $b, c \in (1, \infty)$. Calculate:

  $$\int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right) \cdot \sin x \cdot dx.$$

- **5251:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Compute the following sum:

  $$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{\cos(m+n)}{(m+n)^2}.$$

- **5252:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3, a_2 = 5$ and for all $n \geq 2, a_{n+1} = \dfrac{1}{2}\left(a_n^2 + 1\right)$. Prove that

$$1 + 2\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 < F_{n+2},$$

where F_n represents the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

- **5253:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
$$\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy} dxdy.$$

Solutions

- **5230:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers x, y, z such that

$$\begin{aligned} x^2 + xy + \frac{y^2}{3} &= 41, \\ \frac{y^2}{3} + z^2 &= 16, \\ x^2 + xz + z^2 &= 25. \end{aligned}$$

Find the value of $xy + 2yz + 3xz$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that the given system is equivalent to

$$\begin{aligned} x^2 - 2x\frac{y}{\sqrt{3}}\cos 150^\circ + \left(\frac{y}{\sqrt{3}}\right)^2 &= \left(\sqrt{41}\right)^2, \\ \left(\frac{y}{\sqrt{3}}^2\right) + z^2 &= 4^2, \\ x^2 + 2xz\cos 120^\circ + z^2 &= 5^2. \end{aligned}$$

Let us take the right triangle ABC with $\angle B = 90^\circ, AB = 4$ and $BC = 5$ and let P be the interior point of ABC obtained as the intersection of the semicircle with diameter AB and the spanning arc of angle 120° (this is the locus of the points from which the segment BC is seen from an angle of 120°. Note that $\angle APB = 90^\circ$, $\angle BPC = 120^\circ$ and $\angle CPA = 150^\circ$. If we denote $x = CP, y = \sqrt{3}AP, z = BP$, we obtain the equations in the given system by applying the law of cosines to triangles ACP, ABP, and BCP.

Denoting the area of a triangle by $[\cdots]$ we have:

$$[ACP]+[ABP]+[BCP]=[ABC], \text{ or equivalently,}$$

$$\left(\frac{1}{2}\cdot PC\cdot PA\sin 150^\circ\right)+\left(\frac{1}{2}\cdot PA\cdot PB\right)+\left(\frac{1}{2}\cdot PC\cdot PB\cdot\sin 120\right)=\frac{1}{2}\cdot AB\cdot BC.$$

That is,

$$\left(\frac{1}{2}\cdot x\cdot\frac{y}{\sqrt{3}}\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{y}{\sqrt{3}}z\right)+\left(\frac{1}{2}\cdot x\cdot z\cdot\frac{\sqrt{3}}{2}\right)=\frac{1}{2}\cdot 4\cdot 5.$$

Multiplying by $4\sqrt{3}$, gives us that

$$xy+2yz+3xz=40\sqrt{3}.$$

Comment by Bruno: Very similar problems to this one are problems #12 of the 1984 All-Soviet Union Mathematical Olympiad and problem # E1 in *Problem Solving Strategies* by Arthur Engel (Springer-Verlag), 1998, pp. 380-381

Solution 2 by Arkady Alt, San Jose, California, USA

Let $S=xy+2yz+3xz$. By replacing y in the original problem with $y\sqrt{3}$ we obtain:

$$\begin{aligned}
x^2+xy\sqrt{3}+y^2 &= a^2+b^2,\\
y^2+z^2 &= a^2, \text{ and}\\
x^2+xz+z^2 &= b^2, \text{where } a=4,\ b=5, \text{ and}\\
S &= xy\sqrt{3}+2\sqrt{3}yz+3xz, \text{ or}\\
x^2+y^2-2\cos\frac{5\pi}{6}xy &= a^2+b^2,\\
y^2+z^2-2\cos\frac{\pi}{2}yz &= a^2,\\
x^2+z^2-2\cos\frac{2\pi}{3}xz &= b^2,\\
\frac{S}{2\sqrt{3}} &= xy\sin\frac{5\pi}{6}+yz\sin\frac{\pi}{2}+zx\sin\frac{2\pi}{3}.
\end{aligned}$$

Consider four points A,B,C,P on a plane such that $PA=x, PB=y, PM=z$ and $\angle APB=\frac{5\pi}{6}, \angle BPC=\frac{\pi}{2}, \angle CPA=\frac{2\pi}{3}$.

Since $\frac{5\pi}{6}+\frac{2\pi}{3}+\frac{\pi}{2}=2\pi$ then, accordingly to the equalities

$$x^2+y^2-2\cos\frac{5\pi}{6}xy = a^2+b^2,$$

$$\begin{aligned} y^2+z^2-2\cos\frac{\pi}{2}yz &= a^2, \\ x^2+z^2-2\cos\frac{2\pi}{3}xz &= b^2, \text{where} \end{aligned}$$

P is the interior point of the right triangle ABC with right angle at C, and sides $BC=a,\ AC=b$.

Then we have $[ABC]=[APB]+[BPC]+[CPA] \iff$

$$\frac{AC\cdot BC}{2}=\frac{PA\cdot PB}{2}\sin\frac{5\pi}{6}+\frac{PB\cdot PC}{2}\sin\frac{\pi}{2}+\frac{PC\cdot PA}{2}\sin\frac{2\pi}{3} \iff$$

$$\begin{aligned} a\cdot b &= xy\sin\frac{5\pi}{6}+yz\sin\frac{\pi}{2}+zx\sin\frac{2\pi}{3} \iff \\ ab &= \frac{S}{2\sqrt{3}} \iff S=2\sqrt{3}ab. \end{aligned}$$

For $a=4$ and $b=5$ we obtain $S=40\sqrt{3}$.

Remark: The original problem is a particular case of a more general problem.

Given positive numbers $x,\ y,\ z,\alpha,\beta,\gamma,a,b,c$ such that $\alpha+\beta+\gamma=2\pi,\ a,b,c$ and

$$\begin{cases} x^2+y^2-2\cos\gamma xy=c^2 \\ y^2+z^2-2\cos\alpha yz=a^2 \\ x^2+z^2-2\cos\beta xz=b^2. \end{cases}$$

Find the value of $|xy\sin\gamma+yz\sin\alpha+zx\sin\beta|$. This problem has a simple vector interpretation.

Indeed, let $\mathbf{x},\mathbf{y},\mathbf{z}$ be three pairwise non-collinear vectors on a plane such that

$$\|\mathbf{x}\|=x,\ \|\mathbf{y}\|=y,\ \|\mathbf{z}\|=z$$

the oriented angles between the pairs of vectors are

$$\angle(\mathbf{x},\mathbf{y})=\gamma, \angle(\mathbf{y},\mathbf{z})=\alpha, \angle(\mathbf{z},\mathbf{x})=\beta.$$

Then according to the conditions of problem, we also have

$$\begin{aligned} \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y}) \\ &= \|\mathbf{x}\|^2-2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2 \\ &= x^2+y^2-2\cos\gamma\cdot xy \\ &= c^2 \text{ and similarly,} \\ \|\mathbf{y}-\mathbf{z}\|^2 &= a^2, \\ \|\mathbf{z}-\mathbf{x}\|^2 &= b^2. \end{aligned}$$

It is easy to see that

$$a+b=\|\mathbf{y}-\mathbf{z}\|+\|\mathbf{z}-\mathbf{x}\|\geq\|\mathbf{x}-\mathbf{y}\|=c,$$

and since $\mathbf{y}-\mathbf{z}$ and $\mathbf{z}-\mathbf{x}$ aren't collinear then $a+b>c$.

Similarly, $b+c>a$ and $c+a>b$. Thus the positive numbers a,b,c define a triangle with area with semi-perimeter s and area $F=\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$.

Definition

For any two vectors $\mathbf{x}=(x_1,x_2)\,,\mathbf{y}=(y_1,y_2)$ we define the "exterior product" of two vectors in the plane as follows:

$$\mathbf{x}\wedge\mathbf{y}=x_1y_2-x_2y_1.$$

From this definition we can immediately obtain the following properties of the exterior product:

- 1. $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$,
- **2.** $\mathbf{x}\wedge\mathbf{x}=\mathbf{0}$,
- **3.** $\mathbf{x}\wedge(\mathbf{y}+\mathbf{z})=\mathbf{x}\wedge\mathbf{y}+\mathbf{x}\wedge\mathbf{z}$ and $(\mathbf{x}+\mathbf{y})\wedge\mathbf{z}=\mathbf{x}\wedge\mathbf{z}+\mathbf{y}\wedge\mathbf{z}$,
- **4.** $(k\,\mathbf{x})\wedge\mathbf{y}=\mathbf{x}\wedge k\mathbf{y}=k\,(\mathbf{x}\wedge\mathbf{y})\,.$

One more property expresses the geometric essence of the exterior product in a plane.

Let

$$\mathbf{e}=(0,1)\,,\varphi=\angle(\mathbf{e},\mathbf{x})\,,\psi=\angle(\mathbf{e},\mathbf{y})\,,\ \angle(\mathbf{x},\mathbf{y})=\psi-\varphi$$

and since

$$\begin{aligned}(x_1,x_2)&=\|\mathbf{x}\|\,(\cos\varphi,\sin\varphi)\,,\\(y_1,y_2)&=\|\mathbf{y}\|\,(\cos\psi,\sin\psi)\,,\ \text{then}\\\mathbf{x}\wedge\mathbf{y}&=x_1y_2-x_2y_1\\&=\|\mathbf{x}\|\,\|\mathbf{y}\|\,(\cos\varphi\sin\psi-\sin\varphi\cos\psi)\\&=\|\mathbf{x}\|\,\|\mathbf{y}\|\sin\angle(\mathbf{x},\mathbf{y})\,.\end{aligned}$$

Hence, $\mathbf{x}\wedge\mathbf{y}$ is the oriented area of the parallelogram defined by $(\mathbf{x},\mathbf{y})$, and $|\mathbf{x}\wedge\mathbf{y}|$ is area of this parallelogram.

Coming back to our problem we obtain

$$\begin{aligned}xy\sin\gamma+yz\sin\alpha+zx\sin\beta&=\|\mathbf{x}\|\,\|\mathbf{y}\|\sin\angle(\mathbf{x},\mathbf{y})+\|\mathbf{y}\|\,\|\mathbf{z}\|\sin\angle(\mathbf{y},\mathbf{z})+\|\mathbf{z}\|\,\|\mathbf{x}\|\sin\angle(\mathbf{z},\mathbf{x})\\&=\mathbf{x}\wedge\mathbf{y}+\mathbf{y}\wedge\mathbf{z}+\mathbf{z}\wedge\mathbf{x}.\end{aligned}$$

Using properties $1-4$ we have

$$(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})=\mathbf{x}\wedge\mathbf{x}-\mathbf{y}\wedge\mathbf{x}-\mathbf{x}\wedge\mathbf{z}+\mathbf{y}\wedge\mathbf{z}=\mathbf{x}\wedge\mathbf{y}+\mathbf{z}\wedge\mathbf{x}+\mathbf{y}\wedge\mathbf{z}.$$

Thus,

$$|xy\sin\gamma + yz\sin\alpha + zx\sin\beta| = |(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})| \text{ and since}$$

$|(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})|$ is the area of the parallelogram defined by vectors $\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z}$ which is equal to $2F$. So, we obtain finally that

$$|xy\sin\gamma + yz\sin\alpha + zx\sin\beta| = 2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $xy + 2yz + 3xz = 40\sqrt{3}$.

Denote the given equations by $(1), (2)$, and (3) in given order. Then $(2)+(3)-(1)$ gives $2z^2 + xz - xy = 0$, so that

$$y = \frac{2z^2}{x} + z. \qquad (4)$$

Substitute y of (4) into (2) and simplifying gives

$$z^4 + xz^3 + x^2z^2 = 12x^2. \qquad (5)$$

From (5) and (3) we have

$$z^2 = \frac{12x^2}{25}. \qquad (6)$$

Substitute z^2 of (6) into (3) and simplifying, we obtain

$$z = \frac{625 - 37x^2}{25x}. \qquad (7)$$

Substitute z of (7) into (6) and simplifying, we obtain

$$1069x^4 - 46250x^2 + 390625 = 0. \qquad (8)$$

Now (8) gives

$$x^2 = \frac{625\left(37 - 10\sqrt{3}\right)}{1069}, \text{ and } \frac{625\left(37 + 10\sqrt{3}\right)}{1069}.$$

If $x^2 = \dfrac{625\left(37+10\sqrt{3}\right)}{1069}$, then by (6), we have $z^2 = \dfrac{300\left(37+10\sqrt{3}\right)}{1069}$. Then using (3), we see that $xz = -\dfrac{250\left(30+37\sqrt{3}\right)}{1069} < 0$, must be rejected. Hence by (6) and (2), we have

$$x^2 = \frac{625\left(37-10\sqrt{3}\right)}{1069},\ z^2 = \frac{300\left(37-10\sqrt{3}\right)}{1069},\ y^2 = \frac{12\left(1501+750\sqrt{3}\right)}{1069}.$$

By (1) and (3) we obtain

$$xy = \frac{50\left(294+65\sqrt{3}\right)}{1069},\ xz = \frac{250\left(-30+37\sqrt{3}\right)}{1069}.$$

Since $yz = \dfrac{(xy)(xz)}{x^2} = \dfrac{60\left(65+98\sqrt{3}\right)}{1069}$, we have $xy + 2yz + 3xz = 40\sqrt{3}$.

Remark: David Stone and John Hawkins, Georgia Southern University, Statesboro GA noted that "the problem poses three nice cylinders in space and asks for their intersection. In the first quadrant, this consists of exactly one point. Perhaps the desired expression has a geometric significance and it is possible to make use of the geometry and compute its value without actually solving for x, y and z. There are other points that satisfy the three given equations. For instance, negating the x, y and z gives us another solution (which produces the identical value for $xy + 2yz + 3xz$). But there are others which produce $xy + 2yz + 3xz = -69.282$"

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5231:** *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy*

The lengths of the sides of the hexagon $ABCDEF$ satisfy $AB = BC, CD = DE$, and $EF = FA$. Prove that

$$\sqrt{\frac{AF}{CF}} + \sqrt{\frac{CB}{EB}} + \sqrt{\frac{ED}{AD}} > 2.$$

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

The inequality will be equivalent to $\sqrt{\dfrac{EF}{CF}} + \sqrt{\dfrac{AB}{EB}} + \sqrt{\dfrac{CD}{AD}} > 2$ Using Ptolemy's Inequality (*http://mathworld.wolfram.com/PtolemyInequality.html*) for quadrilateral $ABCE$ we have

$$AB \cdot CE + BC \cdot AE > EB \cdot AC \Rightarrow \frac{AB}{EB} = \frac{AC}{CE + AE}$$

Using the Ptolemy's Inequality for quadrilateral $EFAC$ and quadrilateral $CDEA$ we obtain

$$\begin{aligned} \frac{EF}{CF} &= \frac{AE}{CA + CE} \\ \frac{CD}{AD} &= \frac{CE}{AE + CA} \end{aligned}$$

Now if $CA = a, CE = b, AE = c$, it is enough to prove that

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2.$$

Normalizing this we can assume that $a + b + c = 1$, so we require to prove

$$\sqrt{\frac{a}{1-a}}+\sqrt{\frac{b}{1-b}}+\sqrt{\frac{c}{1-c}}>2.$$

It is obvious we just need to prove that

$$\sqrt{\frac{a}{1-a}}>2\frac{a}{a+b+c}=2a.$$

Squaring both sides and doing easy manipulations we have

$$\frac{a}{1-a}>4a^2\Rightarrow -4a^2+4a-1<0\Rightarrow -(2a-1)^2<0.$$

which obviously is true for any $a\in(0,1)$.

Finally we have

$$\sqrt{\frac{a}{1-a}}+\sqrt{\frac{b}{1-b}}+\sqrt{\frac{c}{1-c}}>2a+2b+2c=2(a+b+c)=2.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, and the proposer

- **5232:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Prove that: If $a,b,c>0$, then,

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}>a+b+c,$$

for any $x\in\left(0,\frac{\pi}{2}\right)$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $f(x)=2\sin x+\tan x-3x$, then for $x\in\left(0,\frac{\pi}{2}\right)$,

$$\begin{aligned}
f'(x) &= 2\cos x+\sec^2 x-3\\
&= \frac{2\cos^3 x-3\cos^2 x+1}{\cos^2 x}\\
&= \frac{(2\cos x+1)(\cos x-1)^2}{\cos^2 x}\\
&> 0.
\end{aligned}$$

Since $f(x)$ is continuous on $\left[0,\frac{\pi}{2}\right)$ and $f(0)=0$, it follows that $f(x)>0$ for all $x\in\left(0,\frac{\pi}{2}\right)$. Therefore, for all $x\in\left(0,\frac{\pi}{2}\right)$,

$$2\sin x+\tan x>3x$$

or

$$\frac{2\sin x+\tan x}{3x}>1. \qquad (1)$$

By the Arithmetic Mean - Root Mean Square Inequality,

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3} \qquad (2)$$

when $a,b,c>0$. Since $\sin x>0$ on $\left(0,\frac{\pi}{2}\right)$, we may combine (1) and (2) to get

$$\begin{aligned}
&2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}\\
\geq\ &\frac{2\sin x+\tan x}{3x}\cdot(a+b+c)\\
>\ &a+b+c
\end{aligned}$$

for any $x\in\left(0,\frac{\pi}{2}\right)$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

By the Cauchy-Schwarz inequality

$$\sqrt{3}\sqrt{a^2+b^2+c^2}\geq a+b+c.$$

So

$$\begin{aligned}
&\frac{3}{a+b+c}\left(2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}-a-b-c\right)\\
\geq\ &2\frac{\sin x}{x}+\frac{\tan x}{x}-3\\
=\ &\frac{1}{x}(2\sin x+\tan x-3x)\\
=\ &\frac{1}{x}\int_0^x\left(2\cos t+\frac{1}{\cos^2 t}-3\right)dt\\
=\ &\frac{1}{x}\int_0^x\frac{2\cos^3 t-3\cos^2 t+1}{\cos^2 t}dt\\
=\ &\frac{1}{x}\int_0^x\frac{(2\cos t+1)(1-\cos t)^2}{\cos^2 t}dt>0,\ \text{ for any } x\in\left(0,\frac{\pi}{2}\right).
\end{aligned}$$

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}>a+b+c \text{ for any } x\in\left(0,\frac{\pi}{2}\right).$$

Solution 3 by Paul M. Harms, North Newton, KS

A convergent series for $\dfrac{\sin x}{x}$ is $1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\ldots$, and a convergent series for $\dfrac{\tan x}{x}$ is $1+\dfrac{x^2}{3}+\dfrac{2}{15}x^4+\ldots$ for the interval $\left(0,\dfrac{\pi}{2}\right)$.

In this interval, $\dfrac{\sin x}{x}<1-\dfrac{x^2}{6}$ and $\dfrac{\tan x}{x}<1+\dfrac{x^2}{3}$.

The inequality in the problem holds if we can show that

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\left(1-\frac{x^2}{6}\right)+\frac{a+b+c}{3}\left(1+\frac{x^2}{3}\right)-(a+b+c)>0.$$

Let the left hand side of the inequality be $f(x)$. Then

$$f'(x)=2\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\frac{-x}{3}\right)+\frac{a+b+c}{3}\left(\frac{2x}{3}\right).$$

The only place where $f'(x)=0$ on the interval $\left[0,\dfrac{\pi}{2}\right]$ is at $x=0$, if $a=b=c$ is not true as in shown below.

To check where $f'(x)<0$ we check where

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\frac{x}{3}\right)>\frac{a+b+c}{3}\left(\frac{2x}{3}\right).$$

Simplifying we see:

$$\begin{aligned}\frac{a^2+b^2+c^2}{3} &> \left(\frac{a+b+c}{9}\right)^2 \text{ which is equivalent to}\\ 3a^2+3b^2+3c^2-(a+b+c)^2 &= (a-b)^2+(b-c)^2+(c-a)^2>0.\end{aligned}$$

Then $f'(x)<0$ on the interval $\left(0,\dfrac{\pi}{2}\right]$ where $a,b,$ and c are not all the same positive number. If $a=b=c$ is not true, then the inequality will be correct provided $f\left(\dfrac{\pi}{2}\right)>0$. We see that:

$$f\left(\frac{\pi}{2}\right)=\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\frac{24-\pi^2}{12}\right)+\frac{a+b+c}{3}\left(\frac{\pi^2-24}{12}\right).$$

To show that $f\left(\dfrac{\pi}{2}\right)\geq 0$ is suffices to show that

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3}.$$

This last inequality was shown previously. The inequality in the problem then is correct when a,b,c are not all the same positive number.

Now consider the case when $a = b = c > 0$. The inequality of the problem is then equivalent to

$$a\left(\frac{2\sin x}{x} + \frac{\tan x}{x} - 3\right) > 0.$$

We have

$$\begin{aligned}\frac{2\sin x}{x} &= 2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots \text{ and}\\ \frac{\tan x}{x} &= 1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \ldots.\end{aligned}$$

Then the left side of the inequality is

$$a\left[\left(2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots\right) + \left(1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \ldots\right) - 3\right],$$

and the inequality of the problem can be written as

$$a\left[\left(\frac{x^4}{60} - \frac{2x^6}{7!} + \ldots\right) + \left(\frac{2x^4}{15} + \frac{17x^6}{315} + \ldots\right)\right] > 0.$$

On the interval $\left(0, \frac{\pi}{2}\right)$, the alternating series part is a convergent series whose terms in absolute value are decreasing and whose first term is positive. Thus both series inside the brackets are positive and the inequality of the problem is correct for positive numbers a, b, and c for x in the interval $\left(0, \frac{\pi}{2}\right)$.

Also solved by Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata," University,Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Boris Rays, Brooklyn, NY, and the proposer.

- **5233:** *Proposed by Anastasios Kotronis, Athens, Greece*

Let $x \geq \dfrac{1 + \ln 2}{2}$ and let $f(x)$ be the function defined by the relations:

$$\begin{aligned} f^2(x) - \ln f(x) &= x\\ f(x) &\geq \frac{\sqrt{2}}{2}.\end{aligned}$$

- 1. Calculate $\displaystyle\lim_{x\to+\infty} \frac{f(x)}{\sqrt{x}}$, if it exists.
- 2. Find the values of $\alpha \in \Re$ for which the series $\displaystyle\sum_{k=1}^{+\infty} k^{\alpha}\left(f(k) - \sqrt{k}\right)$ converges.
- 3. Calculate $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x}f(x) - x}{\ln x}$, if it exists.

Solution 1 by Arkady Alt, San Jose, CA

1. Since $x \geq \dfrac{1+\ln 2}{2}$ and $f(x) \geq \dfrac{\sqrt{2}}{2}$ then
$\ln f(x) + x \geq x + \ln\left(\dfrac{\sqrt{2}}{2}\right) \geq \dfrac{1+\ln 2}{2} - \dfrac{\ln 2}{2} = \dfrac{1}{2}$
and, therefore, for such x and $f(x)$ we have

$$f^2(x) - \ln f(x) = x \iff$$

$$f(x) = \sqrt{x + \ln f(x)} \text{ and}$$

$$f(x) \geq \sqrt{x + \ln\left(\frac{\sqrt{2}}{2}\right)} = \sqrt{x - \frac{\ln 2}{2}}.$$

$$\text{Hence, } \lim_{\mathrm{x}\to+\infty} \mathrm{f(x)} = \infty$$

Since $f(x) > 0$ then

$$f^2(x) - \ln f(x) = x \iff f(x) = \frac{x}{f(x)} + \frac{\ln f(x)}{f(x)}$$

and, therefore,

$$f(x) \leq \frac{x}{\sqrt{x - \frac{\ln 2}{2}}} + \frac{\ln f(x)}{f(x)}.$$

Hence,

$$\frac{\sqrt{x - \frac{\ln 2}{2}}}{\sqrt{x}} \leq \frac{f(x)}{\sqrt{x}} \leq \frac{\sqrt{x}}{\sqrt{x - \frac{\ln 2}{2}}} + \frac{\ln f(x)}{\sqrt{x} f(x)}.$$

Since

$$\lim_{x\to+\infty} \frac{\sqrt{x - \frac{\ln 2}{2}}}{\sqrt{x}} = 1, \lim_{x\to+\infty} \frac{\sqrt{x}}{\sqrt{x - \frac{\ln 2}{2}}} = 1 \text{ and } \lim_{\mathrm{x}\to+\infty} \frac{\ln f(x)}{f(x)} = 0.$$

Then by the squeeze principle we obtain

$$\lim_{x\to+\infty} \frac{f(x)}{\sqrt{x}} = 1.$$

2. First note that series $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(Let $p > 1$ and $\varepsilon = \dfrac{1-p}{2}$. Since $p - \varepsilon = \frac{3p-1}{2} > 1$ then series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{p-\varepsilon}}$ is convergent.

There is $n_0 \in N$ such that $\ln n < n^{\varepsilon}$ for all $n > n_0$ (because $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^q} = 0$ for any $q > 0$).

Hence,

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^p}=\sum_{k=1}^{n_0}\frac{\ln k}{k^p}+\sum_{n=n_0+1}^{\infty}\frac{\ln n}{n^p}<\sum_{k=1}^{n_0}\frac{\ln k}{k^p}+\sum_{n=n_0+1}^{\infty}\frac{n^{\varepsilon}}{n^p}=\sum_{k=1}^{n_0}\ln k k^p+\sum_{n=n_0+1}^{\infty}\frac{1}{n^{p-\varepsilon}}.$$

If $p\le 1$ then $\sum\limits_{n=3}^{\infty}\frac{\ln n}{n^p}>\sum\limits_{n=3}^{\infty}\frac{1}{n^p}$, where by p test $\sum\limits_{n=3}^{\infty}\frac{1}{n^p}$ is divergent series and, therefore, the series $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^p}$ is divergent.)

Also note that $\lim\limits_{x\to+\infty}\frac{\ln f(x)}{\ln x}=\frac{1}{2}$. Indeed,

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{2\ln f(x)}{\ln x}-1\right) &= 2\lim_{x\to+\infty}\frac{\ln\left(\frac{f(x)}{\sqrt{x}}\right)}{\ln x}\\
&= 2\lim_{x\to+\infty}\frac{1}{\ln x}\cdot\lim_{x\to+\infty}\ln\left(\frac{f(x)}{\sqrt{x}}\right)\\
&= 2\lim_{x\to+\infty}\frac{1}{\ln x}\cdot\ln\left(\lim_{x\to+\infty}\frac{f(x)}{\sqrt{x}}\right)=2\cdot 0\cdot\ln 1=0.
\end{aligned}$$

Since

$$f^2(x)-\ln f(x)=x\iff f(x)-\sqrt{x}=\frac{\ln f(x)}{f(x)+\sqrt{x}},\ \text{then}$$

$n^{\alpha}\left(f(n)-\sqrt{n}\right)=\dfrac{n^{\alpha}\ln f(n)}{f(n)+\sqrt{n}}$ and, therefore,

$$\lim_{n\to\infty}\frac{n^{\alpha}\left(f(n)-\sqrt{n}\right)}{n^{\alpha-1/2}\ln n}=\lim_{n\to\infty}\left(\frac{1}{n^{\alpha-1/2}\ln n}\cdot\frac{n^{\alpha}\ln f(n)}{f(n)+\sqrt{n}}\right)=\lim_{n\to\infty}\frac{\sqrt{n}\ln f(n)}{\left(f(n)+\sqrt{n}\right)\ln n}$$

$$=\lim_{n\to\infty}\frac{\ln f(n)}{\left(\frac{f(n)}{\sqrt{n}}+1\right)\ln n}=\lim_{n\to\infty}\frac{\ln f(n)}{\ln n}\cdot\lim_{n\to\infty}\frac{1}{\left(\frac{f(n)}{\sqrt{n}}+1\right)}=\frac{1}{4}.$$

Thus, by the limit convergency test, both series $\sum\limits_{n=1}^{\infty}n^{\alpha}\left(f(n)-\sqrt{n}\right)$ and $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^{1/2-\alpha}}$ have the same character of convergency.

Since $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^{1/2-\alpha}}$ converges if $1/2-\alpha>1\iff\alpha<-1/2$ and diverges if $1/2-\alpha\le 1\iff -1/2\le\alpha$ we may conclude that series $\sum\limits_{n=1}^{\infty}n^{\alpha}\left(f(n)-\sqrt{n}\right)$ is convergent if $\alpha<-1/2$ and divergent if $-1/2\le\alpha$.

3. Since

$$
\begin{aligned}
f(x)-\sqrt{x} &= \frac{\ln f(x)}{f(x)+\sqrt{x}} \text{ then} \\
\lim_{x\to+\infty}\frac{\sqrt{x}f(x)-x}{\ln x} &= \lim_{x\to+\infty}\frac{\sqrt{x}\,(f(x)-\sqrt{x})}{\ln x} \\
&= \lim_{x\to+\infty}\frac{\sqrt{x}\ln f(x)}{\ln x\,(f(x)+\sqrt{x})} \\
&= \lim_{x\to+\infty}\frac{\sqrt{x}\ln f(x)}{\ln x\,(f(x)+\sqrt{x})} \\
&= \lim_{x\to+\infty}\frac{\ln f(x)}{\ln x}\cdot\lim_{x\to+\infty}\frac{1}{\dfrac{f(x)}{\sqrt{x}}+1} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.
\end{aligned}
$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

• 1. The function $t^2-\ln t$ is strictly increasing for $t\geq 1/\sqrt{2}$ thus the equation $t^2-\ln t = x$ admits a unique solution for any $x\geq(1+\ln 2)/2$. This defines the function $f(x)$ of the problem which is strictly increasing and then it admits the limit L which can be finite of infinite. If L is finite the equation $f^2(x)=\ln f(x)+x$ cannot hold thus $L=+\infty$. Moreover the differentiability of $t^2-\ln t$ assures the differentiability of $f(x)$ and in particular

$$2ff' = \frac{f'}{f}+1 \implies f'(x)=\frac{f}{2f^2-1}$$

whence using l'Höpital

$$\lim_{x\to\infty}\frac{f^2}{x} = \lim_{x\to\infty}2ff' = \lim_{x\to\infty}2f\frac{f}{2f^2-1} = 1 \implies \lim_{x\to\infty}\frac{f}{\sqrt{x}} = 1$$

• 2. We have $f(x)=\sqrt{x}+o(\sqrt{x})$ thus $\ln f(x) = \frac{1}{2}\ln x + \ln(1+o(1)) = \frac{1}{2}\ln x + o(1)$ and

$$f(x)=\sqrt{x+\ln f} = \sqrt{x+\frac{1}{2}\ln x + o(1)} = \sqrt{x}\sqrt{1+\frac{1}{2}\frac{\ln x}{x}+\frac{o(1)}{x}}$$

whence

$$f(x)=\sqrt{x}\left(1+\frac{1}{4}\frac{\ln x}{x}+o\left(\frac{\ln x}{x}\right)\right)$$

and then

$$\sum_{k=1}^{\infty}k^{\alpha}(f(k)-\sqrt{k}) = \sum_{k=1}^{\infty}\left[k^{\alpha-\frac{1}{2}}\frac{\ln k}{4}+k^{\alpha+\frac{1}{2}}o\left(\frac{\ln k}{k}\right)\right] = \sum_{k=1}^{\infty}k^{\alpha-\frac{1}{2}}\frac{\ln k}{4}\left(1+o\left(\frac{1}{k}\right)\right).$$

Thus the series converges if and only if converges the series $\sum_{k=1}^{\infty} k^{\alpha-\frac{1}{2}} \ln k$ and this occurs if and only if $\alpha < -1/2$.
This may be seen for instance by using the Cauchy–condensation test after observing that $k^\alpha \ln k$ decreases definitively in k for $\alpha < 0$. Thus we investigate the convergence of the series

$$\sum_{k=1}^{\infty} 2^k 2^{k(\alpha-\frac{1}{2})} k \frac{\ln 2}{2} = \frac{\ln 2}{2} \sum_{k=1}^{\infty} 2^{k(\alpha+\frac{1}{2})} k$$

Here we can use any of the countless method to study such a series. For instance the ratio test

$$\lim_{n\to\infty} \frac{2^{(k+1)(\alpha+\frac{1}{2})}(k+1)}{2^{k(\alpha+\frac{1}{2})} k} = 2^{\alpha+\frac{1}{2}}$$

If $\alpha + 1/2 < 0$ the series converges. If $\alpha + 1/2 > 0$ the series diverges. If $\alpha = -1/2$ we have the series

$$\frac{\ln 2}{2} \sum_{k=1}^{\infty} 2^{k(\alpha+\frac{1}{2})} k = \frac{\ln 2}{2} \sum_{k=1}^{\infty} k$$

thus diverges.

- 3. By employing $f(x) = \sqrt{x}\left(1 + \frac{1}{4}\frac{\ln x}{x} + o\left(\frac{\ln x}{x}\right)\right)$

$$\lim_{x\to\infty} \frac{\sqrt{x}f(x) - x}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{4}\ln x + o(\ln x)}{\ln x} = \frac{1}{4}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Firstly we have $f(x) = \sqrt{x + \ln f(x)} \geq \sqrt{x - \dfrac{\ln 2}{2}}$.

Using the well-known inequality $e^x \geq 1 + x$ for real x, we obtain $\ln f(x) \leq f(x) - 1$. Hence

$$f^2(x) - 1 + f(x), \text{ so that } \mathrm{f}(x) \leq \frac{1 + \sqrt{4x-3}}{2}.$$

So by the squeezing principle, we have $\lim\limits_{x+\infty} \dfrac{f(x)}{\sqrt{x}} = 1$. This answers part one.

Suppose that $f(x) = \sqrt{x} + g(x)$, where $\lim\limits_{x\to+\infty} \dfrac{(x)}{\sqrt{x}} - 0$. From

$$\left(\sqrt{x} + g(x)\right) - \ln\sqrt{x} - \ln\left(1 + \frac{g(x)}{\sqrt{x}}\right) = x,$$

we see that $g(x) \sim \dfrac{\ln x}{4\sqrt{x}}$ as $x \to +\infty$.

Thus $\sum_{k=1}^{+\infty} k^\alpha \left(f(k) = \sqrt{k}\right)$ converges for $\alpha < \dfrac{-1}{2}$, diverges for $\alpha \geq \dfrac{-1}{2}$ and that

$$\lim_{x\to+\infty} \frac{\sqrt{x}f(x) - x}{\ln x} = \lim_{x\to+\infty} \frac{\sqrt{x}g(x)}{\ln x} = \frac{1}{4}.$$

These answer parts two and three.

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5234:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $a < b$ be positive real numbers and let $f_i : [a, b] \to \Re$ $(i = 1, 2)$ be continuous functions in $[a, b]$ and differentiable in (a, b). If f_2 is strictly decreasing then prove that there exists an $\alpha \in (a, b)$ such that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a).$$

Comment by Editor: **Paolo Perfetti of the Department of Mathematics at Tor Vergata University in Rome, Italy** provided a counter-example to the above statement. The incompleteness of the statement was acknowledged by José Luis and he revised the problem. Following is his solution to the revised statement.

5234 (Revised:) *Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.*

Let $a < b$ be positive real numbers and let $f_i : [a, b] \to \Re$ $(i = 1, 2)$ be continuous functions in $[a, b]$ and differentiable in (a, b). (1) If f_1 and f_2 are strictly decreasing, then prove that there exists $\alpha \in (a, b)$ such that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right)$$

(2) If f_1 is strictly increasing and f_2 is strictly decreasing, then prove that there exists $\alpha \in (a, b)$ such that

$$f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a)$$

Solution by the proposer

Consider the function $f : [a, b] \to \Re$ defined by

$$f(x) = \Big(f_2(a) - f_2(x)\Big)\Big(f_2(b) - f_2(x)\Big)e^{f_1(x)}$$

Since f is continuous in $[a, b]$, differentiable in (a, b), and $f(a) = f(b) = 0$, then on account of Rolle's theorem there exists $\alpha \in (a, b)$ such that $f'(\alpha) = 0$. That is,

$$f_1'(\alpha)e^{f_1(\alpha)}\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big) - f_2'(\alpha)\Big(f_2(a) + f_2(b) - 2f_2(\alpha)\Big)e^{f_1(\alpha)} = 0$$

from which follows

$$\frac{f_2'(\alpha)}{f_1'(\alpha)} = \frac{\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big)}{f_2(a) + f_2(b) - 2f_2(\alpha)}$$

(1) Now we prove the first part of the statement. Indeed, we have that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right)$$

is equivalent to

$$(f_2(b) - f_2(\alpha) < 2\,\frac{\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big)}{f_2(a) + f_2(b) - 2f_2(\alpha)},$$

where $f_2(b) - f_2(\alpha) < 0$, $f_2(a) - f_2(\alpha) > 0$ and $f_2(a) + f_2(b) - 2f_2(\alpha) < 0$ because the RHS of the preceding inequality is positive. Then, after division by $f_2(b) - f_2(\alpha) < 0$ and multiplication by $\dfrac{1}{f_2(a) + f_2(b) - 2f_2(\alpha)} < 0$ yields

$$f_2(a) + f_2(b) - 2f_2(\alpha) < 2\left(f_2(a) - f_2(\alpha)\right)$$

or $f_2(b) < f_2(a)$. The preceding trivially holds because f_2 is strictly decreasing.

(2) To prove the second part of the statement, we have

$$f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a)$$

is equivalent to

$$f_2(a) - f_2(\alpha) < 2\,\frac{\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big)}{f_2(a) + f_2(b) - 2f_2(\alpha)},$$

where $f_2(a) - f_2(\alpha) > 0$, $f_2(b) - f_2(\alpha) < 0$ and $f_2(a) + f_2(b) - 2f_2(\alpha) > 0$ because the RHS of the preceding inequality is negative. Then, after rearranging terms we get

$$2\left(f_2(b) - f_2(\alpha)\right) < f_2(a) + f_2(b) - 2f_2(\alpha)$$

from which follows $f_2(b) < f_2(a)$ that again holds on account that f_2 is strictly decreasing. This completes the proof.

- **5235:** *Proposed by Albert Stadler, Herrliberg, Switzerland*

On December 21, 2012 (“12-21-12”) the Mayan Calendar’s 13th Baktun cycle will end. On this date the world as we know it will also change. Since every end is a new beginning we are looking for natural numbers n such that the decimal representation of 2^n starts and ends with the digit sequence 122112. Let S be the set of natural numbers n such that $2^n = 122112\ldots122112$. Let $s(x)$ be the number of elements of S that are $\leq x$.

Prove that $\displaystyle\lim_{x\to\infty}\frac{s(x)}{x}$ exists and is positive. Calculate the limit.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC

First, we determine the probability that a power of 2 begins with 122112. As noted in [1] and [2], Benford’s Law may be generalized as follows: The probability that the decimal representation of a number begins with the string of digits n is $\log_{10}(1 + 1/n)$. Since the sequence of the powers of 2 satisfies Benford’s Law (see [1]), we conclude that the probability that a power of 2 begins with 122112 is $\log_{10}(1 + 1/122112)$.

Next, we determine the probability that a power of 2 ends with 122112. We start by noting that $2^{89} \equiv 562112 \pmod{10^6}$, which is the first occurrence of a power of 2 that is congruent to 112 modulo 1000. The next occurrence of such a power of 2 is 2^{189}, with each successive occurrence at $2^{100k+89}$. We find that the first power of 2 that is congruent to 122,112 modulo 10^6 is

$$2^{3089} \equiv 122112 \pmod{10^6},$$

and the sequence becomes periodic modulo 10^6 at

$$2^{12589} \equiv 2^{89} \equiv 562112 \pmod{10^6}.$$

Hence every 12500th term of the sequence of powers of 2 is congruent to 122,112 modulo 10^6, so the probability that a power of 2 ends with 122112 is 1/12500.

Finally, we calculate

$$\lim_{x\to\infty} \frac{s(x)}{x} = \frac{1}{12500} \log_{10}\left(1 + \frac{1}{122112}\right) \approx 2.845 \times 10^{-10}.$$

References.

[1] "Benford's Law," Wikipedia web page, http://en.wikipedia.org/wiki/Benford's_law

[2] Theodore P. Hill, The Significant-Digit Phenomenon, *The American Mathematical Monthly,* Vol. 102, No. 4 (Apr. 1995), pp. 322-327

Solution 2 by proposer

We first claim that $2^n \equiv 122112 \pmod{10^6}$ if and only if $n = 3089 \pmod{12500}$.

We first note that $122112 = 2^8 \cdot 3^2 \cdot 53$. Of course $n \geq 6$. So $2^n = 122112 (\bmod\ 10^6)$ is equivalent to $2^{n-6} \equiv 1908 \pmod{5^6}$.

We note that $2^0 \equiv 1 \bmod(5)$, $2^1 \equiv 2 \bmod(5)$, $2^2 \equiv 4 \bmod(5)$, $2^3 \equiv 3 \bmod(5)$, $2^4 \equiv 1 \bmod(5)$.

So $2^n \equiv 3 \bmod(5)$ if and only if $n \equiv 3 \bmod(4)$.

Then $2^3 \equiv 8 \bmod(25)$, $2^7 \equiv 3 \bmod(25)$, $2^{11} \equiv 23 \bmod(25)$, $2^{15} \equiv 18 \bmod(25)$, $2^{19} \equiv 13 \bmod(25)$, $2^{23} \equiv 8 \bmod(25)$.

So $2^n \equiv 8 \bmod(25)$ if and only if $n \equiv 3 \bmod(20)$

Then $2^3 \equiv 8 \bmod(125)$, $2^{23} \equiv 108 \bmod(125)$, $2^{43} \equiv 83 \bmod(125)$, $2^{63} \equiv 58 \bmod(125)$, $2^{83} \equiv 33 \bmod(125)$, $2^{103} \equiv 8 \bmod(125)$

So $2^n \equiv 33 \bmod(125)$ if and only if $n \equiv 83 \bmod(100)$

Then $2^{83} \equiv 33 \bmod(625)$, $2^{183} \equiv 533 \bmod(625)$, $2^{283} \equiv 408 \bmod(625)$, $2^{383} \equiv 283 \bmod(625)$, $2^{4833} \equiv 158 \bmod(625)$, $2^{583} \equiv 33 \bmod(625)$

So $2^n \equiv 33 \bmod(625)$ if and only if $n \equiv 83 \bmod(500)$.

Then $2^{83} \equiv 2533 \bmod(3125)$, $2^{583} \equiv 1908 \bmod(3125)$, $2^{1083} \equiv 1283 \bmod(3125)$, $2^{1583} \equiv 658 \bmod(3125)$, $2^{2083} \equiv 33 \bmod(3125)$, $2^{2583} \equiv 2533 \bmod(3125)$.

So $2^n \equiv 1908 \bmod(3125)$ if and only if $n \equiv 583 \bmod(2500)$.

Then $2^{583} \equiv 5033 \bmod(5^6)$, $2^{3083} \equiv 1908 \bmod(5^6)$, $2^{5583} \equiv 14408 \bmod(5^6)$, $2^{8083} \equiv 11283 \bmod(5^6)$, $2^{10583} \equiv 8158 \bmod(5^6)$, $2^{13083} \equiv 5033 \bmod(5^6)$.

So $2^n \equiv 1908 \bmod(5^6)$ if and only if $n \equiv 3083 \bmod(12500)$.

$2^{n-6} \equiv 1908 \pmod{5^6}$ if and oinly if $n \equiv 3089 \pmod{12500}$.

$2^{n} \equiv 122112 \pmod{10^6}$ if and only if $n \equiv 3089 \pmod{12500}$.

Therefore we can assume that $n = 3089 + 12500k$ for some nonnegative integer k.

The fact that $2^{n=3089+12500k}$ starts with the digits 122112 implies that there is an integer m such that

$$1.22112 \cdot 10^m < 2^{3089+12500k} < 1.22113 \cdot 10^m.$$

This is equivalent to saying that

$$\{3089 + 12500k) \log_{10} 2\} \in (\log_{10} 1.22112, \log_{10} 1.22113),$$

where $\{x\}$ denotes the fractional part of the real number x.

$\log_{10} 2$ is irrational, for the assumption that $\log_{10} 2 = p/q$ for some coprime natural numbers $p \geq 1$ and $q \geq 1$ would imply that $10^p = 2^q$, which cannot be due to the uniqueness of the prime number factorization. Therefore the sequence $\{12500k \log_{10} 2\}$ is equidistributed mod 1, and we conclude that the portion of natural numbers that satisfy the condition $2^n = 122112\ldots122112$ equals

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{x}\left(\sum_{\substack{n\leq x\\ 2^n=122112\ldots122112}} 1\right) &= \lim_{x\to\infty} \frac{1}{x}\left(\sum_{\substack{3089+12500k\leq x\\ \{(3089+12500k)\log_{10} 2\}\in(\log_{10} 1.22112, \log_{10} 1.22113)}} 1\right)\\
&= \lim_{x\to\infty} \frac{1}{x}\left(\sum_{\substack{k\leq \frac{x}{12500}\\ \{12500k\log_{10} 2\}\in(\log_{10} 1.22112\, \log_{10} 1.22113}} 1\right)\\
&= \frac{1}{12500}\lim_{y\to\infty} \frac{1}{y}\left(\sum_{\substack{k\leq y\\ \{12500k\log_{10} 2\}\in(\log_{10} 1.22112, \log_{10} 1.22113)}} 1\right)\\
&= \frac{\log_{10}\frac{122113}{122112}}{12500} = \frac{\log\left(1+\frac{1}{122112}\right)}{12500\log 10} \approx 2.8\cdot 10^{-10}.
\end{aligned}$$

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2013

- **5254:** *Proposed by Kenneth Korbin, New York, NY*

  Five different triangles, with integer length sides and with integer area, each have a side with length 169. The size of the angle opposite 169 is the same in all five triangles. Find the sides of the triangles.

- **5255:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Let n be a natural number. Let $\phi(n), \sigma(n)$ and $\tau(n)$ be the Euler phi-function, the sum of the different divisors of n and the number of different divisors of n, respectively.
  Prove:
  (a) $\forall n \geq 2,\ \exists$ natural numbers a and b such that $\phi(a) + \tau(b) = n$.
  (b) $\forall k \geq 1,\ \exists$ natural numbers a and b such that $\phi(a) + \sigma(b) = 2^k$.
  (c) $\forall n \geq 2,\ \exists$ natural numbers a and b such that $\tau(a) + \tau(b) = n$.
  (d) $\forall k \geq 1,\ \exists$ natural numbers a and b such that $\sigma(a) + \sigma(b) = 2^k$.
  (e) $\forall n \geq 3,\ \exists$ natural numbers a, b and c such that $\phi(a) + \sigma(b) + \tau(c) = n$
  (f) $\exists$ infinitely many natural numbers n such that $\phi(\tau(n)) = \tau(\phi(n))$.

- **5256:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Let a be a positive integer. Compute:
  $$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{na}}\right).$$

- **5257:** *Proposed by Pedro H.O. Pantoja, UFRN, Brazil*

  Prove that:
  $$1 + \frac{1}{2}\cdot\sqrt{1+\frac{1}{2}} + \frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}} + \cdots + \frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} \sim \ln(n),$$
  where $f(x) \sim g(x)$ means $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = 1$.

- **5258:** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Polytechnical University of Catalonia, Barcelona, Spain*

  Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers such that $1 + \sum_{k=1}^{n} \cos^2 \alpha_k = n$. Prove that:

$$\sum_{1 \le i < j \le n} \tan \alpha_i \tan \alpha_j \le \frac{n}{2}.$$

Solutions

- **5236:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers (a, b, c, x, y, z) such that

$$\begin{aligned} x^2 + xy + y^2 &= a, \\ y^2 + yz + z^2 &= b, \\ z^2 + zx + x^2 &= c. \end{aligned}$$

Express the value of the sum $x + y + z$ in terms of a, b, and c.

Solution 1 by David Diminnie, Texas Instruments, Inc., Dallas, TX and Charles R. Diminnie, Angelo State University, San Angelo, TX

From the first two equations, we get

$$\begin{aligned} a - b &= x^2 - z^2 + xy - yz \\ &= (x - z)(x + y + z). \end{aligned}$$

Similarly, combining other pairs of equations yields

$$b - c = (y - x)(x + y + z)$$

and

$$c - a = (z - y)(x + y + z).$$

Hence,

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = (x+y+z)^2 \left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]. \qquad (1)$$

Also, by adding the three equations, we obtain

$$\begin{aligned} a + b + c &= 2\left(x^2 + y^2 + z^2\right) + (xy + yz + zx) \\ &= (x+y+z)^2 + \frac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]. \qquad (2) \end{aligned}$$

Then, by (1) and (2),

$$(a+b+c)^2 = (x+y+z)^4 + (x+y+z)^2 \left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]$$

$$
\begin{aligned}
& +\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 \\
= \quad & (x+y+z)^4+\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\
& +\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 .
\end{aligned}
$$

This in turn implies that

$$
\begin{aligned}
& (a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\
= \quad & (x+y+z)^4-\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\
& +\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 \\
= \quad & (x+y+z)^4-(x+y+z)^2\left[(x-y)^2+(y-z)^2+(z-x)^2\right] \\
& +\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 \\
= \quad & \left[(x+y+z)^2-\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)\right]^2 \\
= \quad & [3(xy+yz+zx)]^2 .
\end{aligned}
$$

Since $x, y, z>0$,

$$
3(xy+yz+zx)=\sqrt{(a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]} .
$$

As a result,

$$
\begin{aligned}
& a+b+c+\sqrt{(a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]} \\
= \quad & \left[2\left(x^2+y^2+z^2\right)+(xy+yz+zx)\right]+3(xy+yz+zx) \\
= \quad & 2\left[\left(x^2+y^2+z^2\right)+2(xy+yz+zx)\right] \\
= \quad & 2(x+y+z)^2 .
\end{aligned}
$$

Finally, since $x, y, z>0$,

$$
x+y+z=\sqrt{\frac{a+b+c+\sqrt{(a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}}{2}} .
$$

Solution 2 by David Diminnie, Texas Instruments, Incorporated, Dallas, TX

By summing the three equations in the problem statement we obtain

$$
2x^2+2y^2+2z^2+xy+yz+zx=a+b+c. \qquad (1)
$$

The cross terms may be eliminated from (1) via the change of variables

$$\begin{aligned} x &= \frac{1}{\sqrt{3}}x' - \frac{1}{\sqrt{2}}y' - \frac{1}{\sqrt{6}}z', \\ y &= \frac{1}{\sqrt{3}}x' + \frac{1}{\sqrt{2}}y' - \frac{1}{\sqrt{6}}z', \\ z &= \frac{1}{\sqrt{3}}x' + \sqrt{\frac{2}{3}}z', \end{aligned}$$

yielding

$$3x'^2 + \frac{3}{2}y'^2 + \frac{3}{2}z'^2 = a + b + c. \tag{2}$$

Note that the sum $x + y + z$ becomes

$$x + y + z = \sqrt{3}x' \tag{3}$$

in the new variables, and since x, y, and z are positive x' must also be positive. We may now rewrite the original problem statement in our new variables:

$$\begin{aligned} x'^2 - \sqrt{2}x'z' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 &= a, \\ x'^2 + \sqrt{\frac{3}{2}}x'y' + \frac{1}{\sqrt{2}}x'z' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 &= b, \\ x'^2 - \sqrt{\frac{3}{2}}x'y' + \frac{1}{\sqrt{2}}x'z' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 &= c. \end{aligned} \tag{4}$$

By subtracting the third equation from the second equation in (4) we obtain an expression for y' in terms of x':

$$\begin{aligned} \sqrt{6}x'y' &= b - c, \quad \text{or} \\ y' &= \frac{b-c}{\sqrt{6}x'}. \end{aligned} \tag{5}$$

Similarly, we may obtain an expression for z' in terms of x' by subtracting half the sum of the second and third equations from the first equation in (4): $-\frac{3}{\sqrt{2}}x'z' = a - \frac{1}{2}(b+c)$, or

$$z' = -\frac{\sqrt{2}}{3x'}\left(a - \frac{1}{2}(b+c)\right). \tag{6}$$

Substituting (5) and (6) into (2), we arrive at an equation for x' in terms of a, b, and c,

$$\frac{a^2 + b^2 + c^2 - (ab + ac + bc)}{3x'^2} + 3x'^2 = a + b + c,$$

or

$$9x'^4 - 3(a + b + c)x'^2 + a^2 + b^2 + c^2 - (ab + ac + bc) = 0. \tag{7}$$

The left side of (7) is quadratic in x'^2, so by applying the quadratic formula (or, if one prefers, by completing the square) we may solve for x'^2:

$$x'^2 = \frac{a + b + c\sqrt{-3\left(a^2 + b^2 + c^2\right) + 6(ab + ac + bc)}}{6}. \tag{8}$$

If we substitute the values of a, b, and c from the original problem statement into (8) and simplify the result, we see that the discriminant is positive (the discriminant simplifies to $9\,(xy+xz+yz)^2$ in the original variables) and that the solution involving the negative radical is spurious (since from (3)

$$x'^2 = \frac{1}{3}(x+y+z)^2 = \frac{1}{3}\left(x^2+y^2+z^2+2xy+2yz+2xz\right),$$

while the offending solution simplifies to

$$\frac{1}{3}\left(x^2+y^2+z^2-xy-xz-yz\right)$$

in the original variables).

We may now solve for x' in a straightforward manner (after rejecting the spurious solution) by taking square roots of both sides of (8):

$$x' = \sqrt{\frac{a+b+c+\sqrt{-3\left(a^2+b^2+c^2\right)+6(ab+ac+bc)}}{6}}, \tag{9}$$

where this time we have rejected the negative branch because x' is positive. (Note that the quantity under the outermost radical is positive because each of its terms is positive.) By substituting (9) into (3) we finally obtain the desired sum,

$$x+y+z = \sqrt{\frac{a+b+c+\sqrt{-3\left(a^2+b^2+c^2\right)+6(ab+ac+bc)}}{2}}.$$

Solution 3 by Brian Beasley and Doug Daniel (jointly), Presbyterian College, Clinton, SC

Adding the three equations produces

$$2(x^2+y^2+z^2)+(xy+yz+zx) = a+b+c.$$

Since $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$, we seek to express $xy+yz+zx$ in terms of a, b, and c. By the Law of Cosines, we note that x, y, and $\sqrt{a}$ may represent the lengths of the three sides of a triangle, with the angle between x and y having measure 120°. Similarly, we have two more triangles containing angles of measure 120°, one with sides of lengths y, z, and $\sqrt{b}$, and the other with sides of lengths z, x, and $\sqrt{c}$. Then we may combine these three triangles to create one triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. By Heron's Formula, this new triangle has area

$$A = \sqrt{s(s-\sqrt{a})(s-\sqrt{b})(s-\sqrt{c})},$$

where $s = (\sqrt{a}+\sqrt{b}+\sqrt{c})/2$. By adding the areas of the three smaller triangles, we also obtain $A = (\sqrt{3}/4)(xy+yz+zx)$. Hence

$$(x+y+z)^2 = \frac{a+b+c-4A/\sqrt{3}}{2} + 2\left(\frac{4A}{\sqrt{3}}\right) = \frac{a+b+c}{2} + 2\sqrt{3}A,$$

so

$$x+y+z = \sqrt{\frac{a+b+c}{2} + 2\sqrt{3}A}$$

with A as given previously as a function of $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$.

Editor's comments: **David Stone and John Hawkins** approached the problem as in solution 3 above, and made the following comments about the problem and its solution.

The common vertex of our three interior triangles is often referred to the as the Steiner Point of the large triangle. The sides x, y, z form a minimal Spanning Tree (MST) of the large triangle, so the sum $x+y+z$ is the length of the MST. One would think that the length of this MST (in terms of the sides of the larger triangle is common knowledge, but we could not find it referenced.

We know that the larger triangle is actually the union of three interior triangles because we know the x, y, z and a, b, c are *all* given to satisfy the original equations. If we were simply given a, b, c then we might not have a triangle (or a solution x, y, z), or the Steiner point might be exterior to the triangle formed.

Also solved by Arkady Alt (two solutions), San Jose, CA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain; Adrian Naco, Polytechnic University, Tirana, Albania; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5237:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let $0 < R < 1$ and $0 < S < 1$, and define

$$\begin{aligned} a &= \sqrt{-2\sqrt{1-S^2}\sqrt{1-R^2}+2+2RS}, \\ b &= \sqrt{-R-S+1+RS}, \text{ and} \\ c &= \sqrt{R+S+1+RS}. \end{aligned}$$

Determine whether there is tuple (R, S) such that a, b, and c are sides of a triangle.

Solution 1 by Ed Gray, Highland Beach, FL

Consider the squares of a, b, and c.

1) $c^2 = 1+RS+R+S = (1+R)(1+S)$

2) $b^2 = 1+RS-R-S = (1-R)(1-S)$, so

3) $b^2+c^2 = 2+2RS$

4) $a^2 = 2+2RS-2\sqrt{1-S^2)}\sqrt{1-R^2)}$

5) $b^2c^2 = (1-R)(1-S)(1+R)(1+S) = (1-R^2)(1-S^2)$

6) $bc = \sqrt{(1-R^2)(1-S^2)}$. So combining (3), (4), (6);

7) $a^2 = b^2+c^2-2bc = (c-b)^2$, since $c > b$. Then

8) $a = c - b$ or

9) $c = a + b$

So there can be no triangle since the sum of two legs of a triangle is greater than the third.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that no such tuples exist. Suppose, on the contrary, that there is a tuple (R, S) such that a, b, and c are the sides of a triangle. By the triangle inequality, we have $a > c - b > 0$. Hence,

$$a^2 > c^2 + b^2 - 2cb$$

$$\Longrightarrow -2\sqrt{1-S^2}\sqrt{1-R^2} + 2 + 2RS > -2\sqrt{R+S+1+RS}\sqrt{-R-S+1+RS} + 2 + 2RS$$

$$\Longrightarrow \sqrt{(1-S)(1+S)}\sqrt{(1-R)(1+R)} < \sqrt{(1+R)(1+S)}\sqrt{(1-R)(1-S)},$$

which is not true. Thus we obtain a contradiction and complete the solution.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain; Adrian Naco, Polytechnic University, Tirana, Albania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5238:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

It is fairly well-known that $(1111\ldots1)_9$, a number written in base 9 with an arbitrary number of digits 1, always evaluates decimally to a triangular number. Find another base b and a single digit d in that base, such that $(ddd\ldots d)_b$, using k digits d, has the same property, $\forall k \geq 1$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We begin by noting that triangular numbers are of the form $T(m) = \dfrac{m(m+1)}{2}$ for integers $m \geq 1$. Also, for decimally evaluating a base b number $(ddd\ldots d)_b$, with k digits, we use the formula for a geometric sum to get

$$(ddd\ldots d)_b = d + d\cdot b + d\cdot b^2 + \cdots + d\cdot b^{k-1} = d\cdot\frac{b^k-1}{b-1}. \qquad (1)$$

Further, for $n \geq 1$,

$$(2n+1)^2 - T(n) \quad = \quad 4n(n+1) + 1 - \frac{1}{2}n(n+1)$$

$$= \frac{7}{2}n(n+1)+1$$

$$> 0.$$

Hence, for $n \geq 1$, we may consider $T(n)$ as a digit in base $(2n+1)^2$.

Then, there are an infinite number of choices for b and d which have the desired property for all $k \geq 1$. For $n \geq 1$, choose $d_n = T(n)$ and $b_n = (2n+1)^2$. Since $(2n+1)$ is odd, (1) implies that when k digits are used, with $k \geq 1$, we have

$$\begin{aligned}
(T(n)\,T(n)\,T(n)\ldots T(n))_{(2n+1)^2} &= T(n)\cdot\frac{(2n+1)^{2k}-1}{(2n+1)^2-1} \\
&= \frac{n(n+1)}{2}\cdot\frac{\left[(2n+1)^k-1\right]\left[(2n+1)^k+1\right]}{4n(n+1)} \\
&= \frac{\left[(2n+1)^k-1\right]\left[(2n+1)^k-1+2\right]}{8} \\
&= \frac{1}{2}\left[\frac{(2n+1)^k-1}{2}\right]\left[\frac{(2n+1)^k-1}{2}+1\right] \\
&= T\left(\frac{(2n+1)^k-1}{2}\right).
\end{aligned}$$

E. g., when $n = 1,\ 2,\ 3$, this yields

$$\begin{aligned}
(111\ldots1)_9 &= T\left(\frac{3^k-1}{2}\right), \\
(333\ldots3)_{25} &= T\left(\frac{5^k-1}{2}\right), \\
(666\ldots6)_{49} &= T\left(\frac{7^k-1}{2}\right),
\end{aligned}$$

when k digits are used in each situation.

Solution 2 by Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain

We have:

$$(1111...1)_9 = 1 + 1\cdot 9 + ... + 1\cdot 9^{k-1} = 1\cdot\frac{9^k-1}{8} = \frac{1}{2}\cdot\frac{3^k-1}{2}\cdot\frac{3^k+1}{2} = \frac{m(m+1)}{2}$$

Thus, just search b, d such that $b = x^2$, $x \in Z^+$, and $(b-1) = 8d$, i.e., $x \in Z^+$ such that $(x^2-1) \equiv 0\ (mod\ 8)$. But, $(x^2-1) \equiv 0\ (mod\ 8) \Leftrightarrow x$ is odd.
Therefore, $\forall x = 2n+1, n \in Z^+$, $b = x^2$ and $d = \dfrac{b-1}{8}$ satisfy the property.

Examples: $(333...3)_{25}$, $(666...6)_{49}$.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Given a positive integer k, we seek a base b, a digit d in base b, and a positive integer n such that

$$(ddd\ldots d)_b = d\left(\frac{b^k-1}{b-1}\right) = \frac{n(n+1)}{2}.$$

Solving the resulting quadratic for n yields a discriminant of $(b-1)^2+8d(b-1)(b^k-1)$, and taking $d=(b-1)/8$ reduces this expression to $b^k(b-1)^2$. To make this a perfect square and to ensure that d is an integer, we let b be an odd square. Given any integer $m>1$, we may take $b=(2m-1)^2$, so that $d=m(m-1)/2$. Then

$$(ddd\ldots d)_b = d\left(\frac{b^k-1}{b-1}\right) = \frac{b^k-1}{8} = \frac{n(n+1)}{2},$$

where $n=[(2m-1)^k-1]/2$. In particular, letting $m=3$ produces $b=25$, $d=3$, and

$$(333\ldots 3)_{25} = \frac{25^k-1}{8} = \frac{n(n+1)}{2}$$

for $n=(5^k-1)/2$; also, letting $m=4$ produces $b=49$, $d=6$, and

$$(666\ldots 6)_{49} = \frac{49^k-1}{8} = \frac{n(n+1)}{2}$$

for $n=(7^k-1)/2$.

Also solved by Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5239:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

Determine all functions $f:\Re-\{-3,-1,0,1,3\}\to\Re$, which satisfy the relation

$$f(x)+f\left(\frac{13+3x}{1-x}\right)=ax+b,$$

where a and b are given arbitrary real numbers.

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania

If we let, $g(x)=\dfrac{13+3x}{1-x}$, then we have that,

$$(g\circ g)(x) \;=\; g(g(x))=\frac{13+3g(x)}{1-g(x)}=\frac{x-13}{x+3} \qquad (1)$$

$$\textit{and}\qquad (g\circ g\circ g)(x) \;=\; g(g(g(x)))=\frac{g(x)-13}{g(x)+3}=x \qquad (2)$$

Considering the above and the given relation, it implies that,

$$f(x)+(f\circ g)(x) \;=\; ax+b, \qquad (3)$$

$$
\begin{aligned}
(f\circ g)(x)+(f\circ g\circ g)(x) &= ag(x)+b, &(4)\\
(f\circ g\circ g)(x)+(f\circ g\circ g\circ g)(x) &= a(g\circ g)(x)+b,
\end{aligned}
$$

The last relation is simplified to

$$(f\circ g\circ g)(x)+f(x)=a(g\circ g)(x)+b, \qquad (5)$$

Adding equations (3) and (4) to (5) results that,

$$f(x)+(f\circ g)(x)+(f\circ g\circ g)(x)=\frac{a}{2}[x+g(x)+(g\circ g)(x)]+\frac{3b}{2}. \qquad (6)$$

Finally, if we subtract equation (4) from equation (6), then,

$$
\begin{aligned}
f(x) &= \frac{a}{2}[x-g(x)+(g\circ g)(x)]+\frac{b}{2} \quad\Rightarrow\\
f(x) &= \frac{a}{2}\left[x-\frac{13+3x}{1-x}+\frac{x-13}{x+3}\right]+\frac{b}{2} \quad\Rightarrow\\
f(x) &= \frac{a}{2}\cdot\frac{x^3+6x^2+5x+52}{(x-1)(x+3)}+\frac{b}{2}
\end{aligned}
$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the given relationship by (1). Replacing x by $\dfrac{13+3x}{1-x}$ and $\dfrac{x-13}{x+3}$ in (1), we obtain respectively

$$f\left(\frac{13+3x}{1-x}\right)+f\left(\frac{x-13}{x+3}\right)=a\left(\frac{13+3x}{1-x}\right)+b \qquad (2)$$

and

$$f\left(\frac{x-13}{x+3}\right)+f(x)=a\left(\frac{x-13}{x+3}\right)+b. \qquad (3)$$

Now $(1)-(2)+(3)$ gives

$$2f(x)=(ax+b)-\left(a\left(\frac{13+3x}{1-x}\right)+b\right)+\left(a\left(\frac{x-13}{x+3}\right)+b\right).$$

Simplifying, we obtain

$$f(x)\frac{ax^3+(6a+b)x^2+(5a+2b)x+52a-3b}{2(x-1)(x+3)}.$$

Also solved by Arkady Alt, San Jose, CA; David Diminnie, Texas Instruments, Inc., Dallas TX and Charles Diminnie, Angelo State

University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5240:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let x be a positive real number. Prove that

$$\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}>\frac{1}{8},$$

where $[x]$ and $\{x\}$ represent the integral and fractional part of x, respectively.

Solution 1 by Armend Sh. Shabani, University of Prishtina, Republic of Kosova

Let $[x]=k$. Since $x=[x]+\{x\}$ we have that $\{x\}=x-k$, therefore we need to prove that $\dfrac{xk}{(2x-k)^2}+\dfrac{x(x-k)}{(x+k)^2}>\dfrac{1}{8}$, which is equivalent to

$$8kx(x+k)^2+8x(x-k)(2x-k)^2>(2x-k)^2(x+k)^2.$$

After calculations one obtains:

$$28x^4+59x^2k^2-60x^3k+2xk^3-k^4>0$$

which can be written as:

$$27x^4+59x^2k^2-60x^3k+2xk^3+x^4-k^4>0.$$

Clearly $x^4-k^4\geq 0$ and $2xk^3\geq 0$.

Consider the function

$$f(k)=59x^2k^2-60x^3k+27x^4.$$

Since $59x^2>0$ and $(60x^3)^2-4\cdot 59x^2\cdot 27x^4=-2772x^6<0$ we conclude that $f(k)>0$ for all k, which completes the proof.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

We prove the stronger inequality

$$\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}>\frac{2}{5}$$

Rewrite it as

$$\frac{(x[x])^2}{x[x](x+\{x\})^2}+\frac{(x\{x\})^2}{x\{x\}(x+[x])^2}>\frac{2}{5}$$

Cauchy–Schwarz yields

$$\frac{(x[x])^2}{x[x](x+\{x\})^2}+\frac{(x\{x\})^2}{x\{x\}(x+[x])^2}\geq\frac{(x[x]+x\{x\})^2}{x[x](x+\{x\})^2+x\{x\}(x+[x])^2}>\frac{2}{5}$$

Clearing the denominators and taking into account that $[x]+\{x\}=x$ we come to

$$5[x]^2+5\{x\}^2>2x^2$$

and this follows by

$$5[x]^2+5\{x\}^2\geq\frac{5}{2}([x]+\{x\})^2=\frac{5}{2}x^2>2x^2$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

If $x\in(0;1)$ then we have that $[x]=0$ and $x=\{x\}$. Thus the left side of the given inequality is valued by 1, and as a result, the inequality is true.

Suppose that $x\geq 1$. Then $[x]\geq 1$ and $\{x\}\in[0;1)$. Let $\{x\}=q[x]$ where $q\in[0;1)$, then $x=(1+q)[x]$

Since, $q+2>2q+1$, then the left side of the inequality is transformed to

$$\begin{aligned} S &= \frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}=\frac{q+1}{(2q+1)^2}+\frac{q(q+1)}{(q+2)^2} \\ &\geq \frac{q+1}{(q+2)^2}+\frac{q(q+1)}{(q+2)^2}=\left(\frac{q+1}{q+2}\right)^2\geq\left(\frac{1}{2}\right)^2>\frac{1}{8}. \end{aligned}$$

Editor's note: Most of the solvers noted that the right hand side of the inequality $\frac{1}{8}$ can be raised to $\frac{4}{9}$. Adrain Naco (see solution above) restated the problem as follows:

Let x be a positive number. Prove that

a) $$\inf_{x>0}\left\{\frac{1+[x]}{(1+2[x])^2}+\frac{[x]\,(1+[x])}{(2+[x])^2}\right\}=\inf_{x>0}\left\{\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}\right\}=\frac{4}{9},$$

b) $$\sup_{x>0}\left\{\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}\right\}=1$$

where $[x]$ and $\{x\}$ represent the integral and fractional part of x, respectively.

Following are two additional proofs of the restated problem.

Solutions 4 and 5 by David Stone and John Hawkins of Georgia Southern University, Statesboro GA

For convenience we let $E(x)=\dfrac{x[x]}{(x+\{x\})^2}+\dfrac{x\{x\}}{(x+[x])^2}$.

Note that $E(n)=1+0=1$, for any integer $n\geq 1$ and $E(x)=0+1=1$, for any x with $0<x<1$.

We can describe precisely how the function E behaves: on the interval $[n,n+1), n\geq 1$, it *descends* strictly from a height of 1 towards the height $\dfrac{(n+1)n}{(n+2)^2}+\dfrac{n+1}{2n+1)^2}$. Thus the infimum on this interval is $\dfrac{(n+1)n}{n+2)^2} + \dfrac{n+1}{2n+1)^2}$. As n increases, these greatest lower bounds grow, so the smallest of these, $\dfrac{4}{9}$, occurs on the first interval, $[1,2)$.

Thus, $E(x) > \dfrac{4}{9}$ for all positive x, and the lower bound is sharp because $\displaystyle\lim_{x\to 2^-} E(x) = \frac{4}{9}$. Note that $E(x)$ barely dips below height 1 for large x.

To verify these claims, let $n < x < n+1$, with $x = n+f, n \geq 1, 0 < f < 1$.
Then $E(x) = \dfrac{(n+f)n}{(n+f+f)^2} + \dfrac{(n+f)f}{n+f+f)^2} = \dfrac{(n+f)n}{(n+2f)^2} + \dfrac{(n+f)f}{(2n+f)^2}$.

By letting $f \longrightarrow 1$ from the left, we see that $E(x) = \dfrac{(n+1)n}{(n+2)^2} + \dfrac{(n+1)}{2n+1)^2}$.

In particular, $\displaystyle\lim_{x\to 2^-} E(x) = \frac{(1+1)}{(1+2)^2} + \frac{(1+1)}{2+1)^2} = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$.

Also, $\displaystyle\lim_{x\to 3^-} E(x) = \frac{(2+1)2}{(2+2)^2} + \frac{(2+1)}{2\cdot 2+1)^2} = \frac{3}{8} + \frac{3}{25} = \frac{99}{200} = 0.495 > \frac{4}{9}$.

To verify that the function E decreases for $0 < f < 1$, we compute the derivative $\dfrac{dE}{df}$.

$$
\begin{aligned}
\frac{dE}{df} &= \frac{(n+2f)^2 - (n+f)n \cdot 2(n+2f)\cdot 2}{(n+2f)^4} + \frac{(2n+f)^2(n+2f) - (n+f)\cdot f \cdot 2(2n+f)}{(n+2f)^4} \\
&= \frac{n(2n+3f)}{(2n+f)^3} - \frac{n(3n+2f)}{(n+2f)^3} \\
&= n\frac{(2n+3f)(n+2f)^3 - (3n+2f)(2n+f)^3}{(2n+f)^3(n+2f)^3}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{dE}{df} &< 0 \\
&\Longleftrightarrow (2n+3f)(n+2f)^3 - (3n+2f)(2n+f)^3 < 0 \\
&\Longleftrightarrow (2n+3f)(n+2f)^3 < (3n+2f)(2n+f)^3 \\
&\Longleftrightarrow \frac{2n+3f}{3n+2f)} < \frac{(2n+f)^3}{(n+2f)^3} \\
&\Longleftrightarrow 1 - \frac{n-f}{n+2f} < \left(1 + \frac{n-f}{n+2f}\right)^3.
\end{aligned}
$$

But $n \geq 1$ and $0 < f < 1$, so $n - f$ is positive. Hence the expression on the left of our inequality is less than 1 and the expression the right is larger than 1, so the final inequality is true.

Finally, we verify that the interval infima, $\dfrac{(n+1)n}{(n+2)^2} + \dfrac{(n+1)}{(2n+1)^2}$, form an increasing sequence (with limit 1):
regarded as a function of n, $\dfrac{(n+1)n}{(n+2)^2} + \dfrac{(n+1)}{(2n+1)^2}$, has derivative

$$\frac{3n+2}{(n+2)^3} - \frac{2n+3}{(2n+1)^3} = \frac{(n-1)(n+1)(22n^2+37n+22)}{(n+2)^3(2n+1)^3} \geq 0, \text{ for } n \geq 1.$$

Solution 5

This method verifies the proposed inequality, although it does not reveal as much information about the given expressions as does the preceding solution.
Recognizing that the expression of the left equals 1 when $0 < x < 1$ or when x is an integer, we consider $x > 1$ and write x in terms of its integral and fractional parts: let $n < x < n+1$ with $x = n + f,\ n \geq 1,\ 0 < f < 1.$ Then we want to show

$$\begin{aligned} \frac{(n+f)n}{(n+2f)^2} + \frac{(n+f)f}{(n+2f)^2} \quad & > \quad \frac{4}{9} \\ & \Longleftrightarrow \quad \Big\{(n+f)\, n\, (2n+f)^2 + (n+f)\, f\, (n+2f)^2\Big\} \\ & > \quad 4(n+2f)^2(2+2f)^2. \end{aligned}$$

Upon division by n^4, this becomes an equivalent inequality in a single variable:

$$\Longleftrightarrow\ 9\left[\left(1+\frac{f}{n}\right)\left(2+\frac{f}{n}\right)^2 + \left[\frac{f}{n} + \left(\frac{f}{n}\right)^2\right]\left(1+2\frac{f}{n}\right)^2\right\} > 4\left(1+2\frac{f}{n}\right)^2\left(2+\frac{f}{n}\right)^2.$$

Letting $t = \dfrac{f}{n}$, so that $0 < t < \dfrac{f}{n} < 1$, we have more equivalent inequalities:

$$\begin{aligned} & \Longleftrightarrow \quad 9\Big\{(1+t)\,(2+t)^2 + \Big[t+t^2\Big]\,(1+2t)^2\Big\} > 4(1+2t)^2(2+t)^2 \\ & \Longleftrightarrow \quad 9\Big\{4t^4 + 9t^3 + 10t^2 + 9t + 4\Big\} > 4\Big\{4t^4 + 20t^3 + 33t^2 + 20t + 4\Big\} \\ & \Longleftrightarrow \quad 20t^4 + t^3 - 42t^2 + t + 20 > 0 \\ & \Longleftrightarrow \quad (t-1)^2(20t^2 + 41t + 20) > 0, \end{aligned}$$

which is certainly true.

Also solved by Arkady Alt, San Jose, CA; Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; David Diminnie, Texas Instruments, Incorporated, Dallas, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica,Campus de Bellaterra, Barcelona, Spain; Boris Rays, Brooklyn NY, and the proposer.

- **5241:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $\alpha \geq 0$ be a real number. Calculate

$$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria , Spain

For $x \in [0,1]$, $\alpha \leq x^n+\alpha$. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{\alpha}\ dx\right)^n &\leq \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n\\ \alpha &\leq \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n.\end{aligned}$$

On the other hand, since function $y=x^n$ is convex for $n \geq 1$, by Jensen's inequality

$$\begin{aligned}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n &\leq \int_0^1 (x^n+\alpha)\ dx\\ \lim_{n\to\infty}\int_0^1 x^n+\alpha\ dx &\leq \lim_{n\to\infty}\frac{1}{n+1}+\alpha=\alpha.\end{aligned}$$

So, $\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n=\alpha$.

Solution 2 by Arkady Alt, San Jose, CA

Let $a_n=\int_0^1 \sqrt[n]{x^n+\alpha}dx$. Note that $\lim_{n\to\infty} a_n=1$.

Indeed, we have

$\sqrt[n]{\alpha}=\int_0^1 \sqrt[n]{\alpha}dx \leq a_n \leq \int_0^1 \sqrt[n]{1+\alpha}dx=\sqrt[n]{1+\alpha}$ and $\lim_{n\to\infty}\sqrt[n]{\alpha}=\lim_{n\to\infty}\sqrt[n]{1+\alpha}=1$.

Since $\lim_{n\to\infty} a_n^n=\lim_{n\to\infty} e^{n\ln a_n}$ we will find $\lim_{n\to\infty} n\ln a_n$.

Since

$$\begin{aligned}\lim_{n\to\infty}(a_n-1) &= 0 \text{ we have}\\ \lim_{n\to\infty} n\ln a_n &= \lim_{n\to\infty} n\ln\left(1+(a_n-1)\right)\\ &= \lim_{n\to\infty}\left(n\left(a_n-1\right)\cdot\frac{\ln\left(1+(a_n-1)\right)}{(a_n-1)}\right)\end{aligned}$$

$$\begin{aligned} &= \lim_{n\to\infty} n\,(a_n - 1) \text{ because} \\ &\quad \lim_{n\to\infty} \frac{\ln(1+(a_n-1))}{(a_n-1)} = 1. \end{aligned}$$

Thus, it suffices to find $\lim_{n\to\infty} n\,(a_n - 1)$.

Since

$$\begin{aligned} n\,(a_n - 1) &= n\left(\int_0^1 \sqrt[n]{x^n+\alpha}\,dx - 1\right) \\ &= n\int_0^1 \left(\left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right) + \left(\sqrt[n]{\alpha} - 1\right)\right) dx \\ &= n\left(\sqrt[n]{\alpha} - 1\right) + n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right) dx \text{ and} \\ \lim_{n\to\infty} n\left(\sqrt[n]{\alpha} - 1\right) &= \lim_{n\to\infty} n\left(e^{\ln\alpha n} - 1\right) \\ &= \ln\alpha \end{aligned}$$

then it remains to find

$$\lim_{n\to\infty} n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right) dx.$$

By the Mean Value Theorem

$$\frac{\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}}{x^n} = \frac{1}{n\sqrt[n]{\theta^{n-1}}} \text{ where } \theta \in (\alpha, x^n+\alpha).$$

Hence,

$$\frac{\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}}{x^n} < \frac{1}{n\sqrt[n]{\alpha^{n-1}}} \iff \sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha} < \frac{x^n}{n\sqrt[n]{\alpha^{n-1}}}$$

and, therefore,

$$0 < n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right) dx < n\int_0^1 \frac{x^n}{n\sqrt[n]{\alpha^{n-1}}}dx = \frac{1}{(n+1)\sqrt[n]{\alpha^{n-1}}}.$$

Since

$$\lim_{n\to\infty} \frac{1}{(n+1)\sqrt[n]{\alpha^{n-1}}} = 0,$$

by the Squeeze Principle,

$$\lim_{n\to\infty} n\int_0^1 \left(\sqrt[n]{x^n+\alpha} - \sqrt[n]{\alpha}\right) dx = 0.$$

Thus,

$$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}dx\right)^n = e^{\ln\alpha} = \alpha.$$

Solution 3 by Anastasios Kotronis, Athens, Greece

1. For $a=0$ the limit is trivially $0=a$.
2. For $a>0$. We set $I_n^n = \left(\int_0^1 \sqrt[n]{x^n+a}\,dx\right)^n = \exp\left(n\ln\left(\int_0^1 \sqrt[n]{x^n+a}\,dx\right)\right) = e^{A_n}$.

   Now, considering that $n\in[1,+\infty)$, since $0<\sqrt[n]{x^n+a}\le 1+a$ and $\sqrt[n]{x^n+a} \overset{n\to+\infty}{\longrightarrow} 1$ for $x\in[0,1]$, by dominated convergence theorem we get that $I_n\to 1$, thus $\ln I_n \to 0$.

   Furthermore, by Leibniz's rule we have that for $n\ge 1$

$$\frac{\partial I_n}{\partial n} = \int_0^1 \frac{\partial}{\partial n}\sqrt[n]{x^n+a}\,dx = \int_0^1 (x^n+a)^{\frac{1-n}{n}}\left(\frac{nx^n\ln x-(x^n+a)\ln(x^n+a)}{n^2}\right)dx.$$

   We also have that

$$\begin{aligned}\left|(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a)-nx^n\ln x\right)\right| &\le \frac{1+a}{a}\left(|(x^n+a)\ln(x^n+a)|+|nx^n\ln x|\right.\\ &\le \frac{1+a}{a}\left(\max\{e^{-1},(1+a)\ln(1+a)\}+e^{-1}\right)\end{aligned}$$

   and since

$$(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a)-nx^n\ln x\right)\to\begin{cases}\ln(1+a), & \text{if } x=1\\ \ln a, & \text{if } x\in[0,1)\end{cases}$$

   by the dominated convergence theorem it is $-n^2\dfrac{\partial I_n}{\partial n}\to\ln a$.

   Now applying De l' Hospital's rule we get

$$\lim_{n\to+\infty}A_n = \lim_{n\to+\infty}\frac{\ln I_n}{n^{-1}} = \lim_{R\ni n\to+\infty}\frac{\ln I_n}{n^{-1}} \overset{0/0}{=} \lim_{n\to+\infty} I_n^{-1}\cdot\left(-n^2\frac{\partial I_n}{\partial n}\right)\to\ln a,$$

so the required limit in each case is a.

Solution 4 by Adrian Narco, Polytechnic University, Tirana, Albania

The function, $f(x)=\sqrt[n]{x^n+\alpha}=(x^n+\alpha)^{\frac{1}{n}}$, is strictly increasing and everywhere continuous on $[0;1]$, thus we can apply the mean value theorem for integral, that is,

$$\exists c\in(0;1): \int_0^1 \sqrt[n]{x^n+\alpha}\mathrm{dx} = f(c)(1-0) = (c^n+\alpha)^{\frac{1}{n}}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\mathrm{dx}\right)^{\mathrm{n}}$$

$$= \quad \lim_{n\to\infty}\left((c^n+\alpha)^{\frac{1}{n}}\right)^n = \lim_{n\to\infty}(c^n+\alpha) = \alpha$$

since $c \in (0,1)$ and $c^n \overset{n\to+\infty}{\longrightarrow} 0$.

Also solved by Kee-Wai Lau, Hong Kong, China; Carl Libis (two solutions; one alone and one with Tom Dunion), Ivy Bridge College of Tiffin University, Toledo, OH and Bentley University, Waltham, MA (respectively); Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

Mea Culpa

The names of Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany were inadvertently not listed as having solved problem 5232.

The featured solutions to Problem 5229 have turned out to be in error, or perhaps more correctly stated, incomplete. Following is a note received from **Arkady Alt of San Jose, CA.**

I'm writing you about problem 5229. I think that there are some issues with the proposed solutions and I wanted to give a few arguments to prove this point. Also, below, I'm attaching my solution that I have not posted after realizing that it is not complete, although I did obtain the desired limit.

There are two main approaches to finding limits. Both are in two steps.

The first way is to prove that limit exists and then find it;

The second way is to find the value of the limit assuming that it exists, and then prove that the obtained value is indeed a limit.

The second way isn't complete without such a proof, because there are counterexamples of sequences which have no limit, but when assuming that it exists we can obtain a value.

For example: let $a_1 = 1$ and $a_{n+1} = a_n^2 + 3a_n + 1, n \geq 1$ then obviously $\lim\limits_{n\to\infty} a_n = \infty$.

But assuming that $(a_n)_{n\geq 1}$ is convergent and denoting $a = \lim\limits_{n\to\infty} a_n$ we immediately obtain

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(a_n^2+3a_n+1\right) = \lim_{n\to\infty} a_n^2 + 3\lim_{n\to\infty} a_n + 1 = a^2+3a+1a^2+2a+1 = 0 \iff a = -1.$$

Also, the Stolz Theorem cannot be inverted.

Example:

Let $a_n = \sum_{k=1}^{n} \sin k$, then $\dfrac{a_{n+1} - a_n}{n+1-n} = \sin(n+1)$ and the sequence $(\sin n)_{n \in N}$ isn't convergent, but

since $\sum_{k=1}^{n} \sin k = \dfrac{\sin\left(\dfrac{n+1}{2}\right)\sin\dfrac{n}{2}}{\sin\dfrac{1}{2}}$ ($2a_n \sin\dfrac{1}{2} = \sum_{k=1}^{n}\left(\cos\left(k-\dfrac{1}{2}\right) - \cos\left(k+\dfrac{1}{2}\right)\right) =$
$\cos\dfrac{1}{2} - \cos\left(n+\dfrac{1}{2}\right) = 2\sin\left(\dfrac{n+1}{2}\right)\sin\dfrac{n}{2}$) then $\lim\limits_{n\to\infty}\dfrac{a_n}{n} = 0$ because
$\left|\sin\left(\dfrac{n+1}{2}\right)\sin\dfrac{n}{2}\right| \leq 1.$

Here is my solution, which I decided not to send because it is missing the crucial "proof" points that are mentioned above and it is only based on an assumption. (Note that the published solutions 2 and 3 for problem 5229 are incomplete for the same reason).

Solution 1 is also incomplete (for another reason) because it is based on an unproved assumption about the asymptotic behavior of $(x_n)_{n\geq 1}$, namely that $x_n \sim kn^{\alpha}$, for some k and α.

This assumption is basically equivalent to the problem statement.

I have a slight suspicion that a "simple" solution from the proposer was originally the rationale for the publication of this problem.

So, in my opinion this problem has not been solved as of yet.

5229. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Let $\beta, a > 0$ be a real numbers and let $\{x_n\}_{n\in N}$ be the sequence defined by the recurrence
relation

$$x_1 = a, \text{ and } x_{n+1} = x_n + \tfrac{n^{2\beta}}{x_1 + x_2 + \ldots + x_n} \quad \text{for } n \geq 1.$$

1. Prove that $\lim\limits_{n\to\infty} x_n = \infty$;
2. Calculate $\lim\limits_{n\to\infty} \dfrac{x_n}{n^{\beta}}$.

Solution by Arkady Alt, San Jose ,CA

1. Let $S_n := x_1 + x_2 + \ldots + x_n, n \in N$. It is easy to see (by Math. Induction) that $x_n > 0$ for all $n \in N$.

Also, note that sequence $\{x_n\}_{n\in N}$ is increasing, since

$x_{n+1} - x_n = \dfrac{n^{2\beta}}{S_n} > 0 \iff x_{n+1} > x_n$, $n \in N$.

Then $x_{n+1}^2 - x_n^2 = \dfrac{n^{2\beta}\left(x_n + x_{n+1}\right)}{S_n} > \dfrac{2n^{2\beta}x_n}{nx_n} = 2n^{2\beta-1}, n \in N$ and, therefore,

$x_{n+1}^2 - x_1^2 = \sum_{k=1}^{n}\left(x_{k+1}^2 - x_k^2\right) > 2\sum_{k=1}^{n} k^{2\beta-1} > \dfrac{n^{2\beta}}{\beta} x_{n+1}^2 > a + \dfrac{n^{2\beta}}{\beta} > \dfrac{n^{2\beta}}{\beta} x_n > \dfrac{(n-1)^{\beta}}{\sqrt{\beta}}.$

Thus, $\lim\limits_{n\to\infty} x_n = \infty$.

We can prove that sequence $\dfrac{x_n}{n^\beta}$ has an upper bound.

Indeed, since $x_n > \dfrac{(n-1)^\beta}{\sqrt{\beta}}$ then $S_n > \displaystyle\sum_{k=1}^{n} \frac{(k-1)^\beta}{\sqrt{\beta}} > \frac{1}{\sqrt{\beta}} \sum_{k=1}^{n-1} k^\beta > \frac{(n-1)^{\beta+1}}{(\beta+1)\sqrt{\beta}}$

and, therefore ,

$x_{n+1} - x_n = \dfrac{n^{2\beta}}{S_n} < \dfrac{n^{2\beta}(\beta+1)\sqrt{\beta}}{(n-1)^{\beta+1}} = n^{\beta-1} \cdot \left(1 + \dfrac{1}{n-1}\right)^{\beta+1} (\beta+1)\sqrt{\beta} < Kn^{\beta-1}$,

where

$K = e(\beta+1)\sqrt{\beta}$, because $\left(1 + \dfrac{1}{n-1}\right)^{\beta+1} < \left(1 + \dfrac{1}{n-1}\right)^{n-1} < e$ for any n bigger then some $n_0 > 0$.

Then $x_{n+1} - x_{n_0} < \dfrac{K(n+1)^\beta}{\beta} \dfrac{x_n}{n^\beta} < \dfrac{x_{n_0}}{n^\beta} + \dfrac{K}{\beta}, n \geq n_0$.

If I can prove that $\dfrac{x_n}{n^\beta}$ is increasing, then we can conclude that $\displaystyle\lim_{n\to\infty} \frac{x_n}{n^\beta}$ exists.

Attempts to do so failed.

Or, assuming that $\left(\dfrac{x_n}{n^\beta}\right)_{n\in N}$ is convergent we can try to find $L = \displaystyle\lim_{n\to\infty} \frac{x_n}{n^\beta}$, but later we must prove that the obtained value is really the desired limit. Value of L can be obtained repeatedly using Stolz Theorem:

Indeed, using* $\displaystyle\lim_{n\to\infty} \frac{(n+1)^\alpha - n^\alpha}{\alpha n^{\alpha-1}} = 1, \alpha > 0$ we obtain

L$= \displaystyle\lim_{n\to\infty} \frac{x_n}{n^\beta} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{\beta n^{\beta-1}} = \lim_{n\to\infty} \frac{n^{2\beta}}{\beta n^{\beta-1} S_n} = \frac{1}{\beta} \lim_{n\to\infty} \frac{n^{\beta+1}}{S_n} =$

$\displaystyle\frac{1}{\beta} \lim_{n\to\infty} \frac{(n+1)^{\beta+1} - n^{\beta+1}}{S_{n+1} - S_n} = \frac{1}{\beta} \lim_{n\to\infty} \frac{(n+1)^{\beta+1} - n^{\beta+1}}{S_{n+1} - S_n} = \frac{\beta+1}{\beta} \lim_{n\to\infty} \frac{n^\beta}{x_{n+1}} =$

$\displaystyle\frac{\beta+1}{\beta} \lim_{n\to\infty} \left(\frac{x_n}{x_{n+1}} \cdot \frac{n^\beta}{x_n}\right) = \frac{\beta+1}{\beta} \lim_{n\to\infty} \frac{n^\beta}{x_n} = \frac{\beta+1}{\beta} . \frac{1}{L} L = \sqrt{\frac{\beta+1}{\beta}}$.

(here, the chain of equalities according to Stolz Theorem works from the right to the left).

But attempts to prove that $\displaystyle\lim_{n\to\infty} \frac{x_n}{n^\beta} = \sqrt{\frac{\beta+1}{\beta}}$ failed as well.

(*) By Mean Value Theorem $(n+1)^\alpha - n^\alpha = \alpha c_n^{\alpha-1}$, where $c_n \in (n, n+1)$

and, therefore, $\alpha \min\left\{n^{\alpha-1}, (n+1)^{\alpha-1}\right\} < (n+1)^\alpha - n^\alpha < \alpha \max\left\{n^{\alpha-1}, (n+1)^{\alpha-1}\right\}$.

Hence, $\alpha \min\left\{1, \dfrac{(n+1)^{\alpha-1}}{n^{\alpha-1}}\right\} < \dfrac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}} < \alpha \max\left\{1, \dfrac{(n+1)^{\alpha-1}}{n^{\alpha-1}}\right\}$.

Editor again: I sent Arkady's comments to Ovidiu (proposer of the problem), and he answered as follows:

"I have read Prof. Alt's comments on problem 5229 and he is right, namely the applicability of the Stolz-Cesaro lemma is valid provided that $\lim \dfrac{x_{\{n+1\}} - x_n}{(n+1)^\beta - n^\beta}$ exists, which I failed to prove. It seems hard to establish the existence of this limit. It appears that the solution of this problem is incomplete, as Prof. Alt has observed."

Ovidiu went on to say that he had communicated the above to some of his colleagues, but to date, they had not been able to solve, or circumvent the glitch.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
October 15, 2013

- **5259:** *Proposed by Kenneth Korbin, New York, NY*

  Find a, b, and c such that with $a < b < c$,

  $$\begin{cases} ab + bc + ca = -2 \\ a^2b^2 + b^2c^2 + c^2a^2 = 6 \\ a^3b^3 + b^3c^3 + c^3a^3 = -11. \end{cases}$$

- **5260:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Find all primes p and q such that $a^{pq-1} \equiv a \pmod{pq}$, for all a relatively prime to pq.

- **5261:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Show without calculus or trigonometric functions that the shortest focal chord of an ellipse is the latus rectum.

- **5262:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

  Prove that the equation $\varphi(10x^2) + \varphi(30x^3) + \varphi(34x^4) = y^2 + y^3 + y^4$ has infinitely many solutions for $x, y \in N$ where $\varphi(x)$ is the Euler-φ function.

- **5263:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let a, b, c be positive numbers lying in the interval $(0, 1]$. Prove that

  $$a \cdot \sqrt{\frac{bc}{1 + c + ab}} + b \cdot \sqrt{\frac{ca}{1 + a + bc}} + c \cdot \sqrt{\frac{ab}{1 + b + ca}} \le \sqrt{3}.$$

- **5264:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

  Let x, y, z, α be positive real numbers. Show that if

  $$\sum_{cyclic} \frac{(n+1)x^3 + nx}{x^2 + 1} = \alpha$$

then

$$\sum_{cyclic} \frac{1}{x} > \frac{3n}{\alpha} + \frac{(2n-1)\alpha}{3n} + \frac{3n\alpha}{9n^2+\alpha^2}$$

where n is a positive integer. Cyclic means the cyclic permutation of x, y, z (and not x, y, z and α).

Solutions

- **5242:** *Proposed by Kenneth Korbin, New York, NY*

Let N be any positive integer, and let $x = N(N+1)$. Find the value of

$$\sum_{K=0}^{x/2} \binom{x-K}{K} x^K.$$

Solution 1 by Anastasios Kotronis, Athens, Greece,

Using m instead of x for notation convenience we compute the generating function of $\sum_{k=0}^{m/2} \binom{m-k}{k} y^k$:

$$\begin{aligned}
\sum_{m\geq 0}\sum_{k=0}^{m/2} \binom{m-k}{k} y^k t^m &= \sum_{k\geq 0} y^k \sum_{m\geq 2k} \binom{m-k}{k} t^m \\
&= \sum_{k\geq 0} y^k \sum_{m\geq 0} \binom{m+k}{k} t^{m+2k} \\
&= \sum_{k\geq 0} y^k \sum_{m\geq 0} \binom{m+k}{m} t^{m+2k} \\
&= \sum_{k\geq 0} (yt^2)^k \sum_{m\geq 0} \binom{-k-1}{m} (-t)^m \\
&= \sum_{k\geq 0} (yt^2)^k (1-t)^{-k-1} \\
&= \frac{1}{1-t} \sum_{k\geq 0} \left(\frac{yt^2}{1-t}\right)^k \\
&= \frac{1}{1-t-yt^2}
\end{aligned}$$

It is easily shown, decomposing into partial fraction and expanding the geometric series, that if ax^2+by+c has two distinct non negative roots ρ_1, ρ_2, then

$$\frac{1}{ax^2+by+c} = \sum_{m\geq 0} \frac{1}{a(\rho_1-\rho_2)} \left(\rho_2^{-m-1} - \rho_1^{-m-1}\right) x^m,$$

so

$$\sum_{m\geq 0}\sum_{k=0}^{m/2} \binom{m-k}{k} y^k t^m = \sum_{m\geq 0} \frac{1}{\sqrt{1+4y}} \left(\left(\frac{-2y}{1-\sqrt{1+4y}}\right)^{m+1} - \left(\frac{-2y}{1+\sqrt{1+4y}}\right)^{m+1} \right) t^m$$

and hence

$$\sum_{k=0}^{m/2} \binom{m-k}{k} y^k = \frac{1}{\sqrt{1+4y}} \left(\left(\frac{-2y}{1-\sqrt{1+4y}}\right)^{m+1} - \left(\frac{-2y}{1+\sqrt{1+4y}}\right)^{m+1} \right).$$

Putting m in the place of y and then $N(N+1)$ in the place of m in the above relation, and since $N(N+1)+1$ is odd, we get

$$\sum_{K=0}^{N(N+1)/2} \binom{N(N+1)-K}{K} (N(N+1))^K = \frac{1}{2N+1} \left((N+1)^{N^2+N+1} + N^{N^2+N+1}\right).$$

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will attack this problem in four steps.

1. If $q>0$, let

$$x_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k$$

for $n \geq 1$. Then, $x_1 = 1$, $x_2 = 1+q$, and for $n \geq 1$,

$$\begin{aligned} x_{n+1} + q x_n &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} q^k + q \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} q^k + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^{k+1} \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} q^k + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor+1} \binom{n-k+1}{k-1} q^k. \end{aligned}$$

Note that if n is odd, then $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n+1}{2}$, while if n is even, then $\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n}{2}$ and $\left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2}+1$. It follows that if n is odd,

$$x_{n+1} + q x_n = \sum_{k=0}^{\frac{n+1}{2}} \binom{n+1-k}{k} q^k + \sum_{k=1}^{\frac{n+1}{2}} \binom{n-k+1}{k-1} q^k$$

$$
\begin{aligned}
&= 1 + \sum_{k=1}^{\frac{n+1}{2}} \left[\binom{n+1-k}{k} + \binom{n+1-k}{k-1} \right] q^k \\
&= 1 + \sum_{k=1}^{\frac{n+1}{2}} \binom{n+2-k}{k} q^k \\
&= \sum_{k=0}^{\frac{n+1}{2}} \binom{n+2-k}{k} q^k \\
&= \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-k}{k} q^k \\
&= x_{n+2}
\end{aligned}
$$

while if n is even,

$$
\begin{aligned}
x_{n+1} + qx_n &= \sum_{k=0}^{\frac{n}{2}} \binom{n+1-k}{k} q^k + \sum_{k=1}^{\frac{n}{2}+1} \binom{n-k+1}{k-1} q^k \\
&= 1 + \sum_{k=1}^{\frac{n}{2}} \left[\binom{n+1-k}{k} + \binom{n+1-k}{k-1} \right] q^k + q^{\frac{n}{2}+1} \\
&= 1 + \sum_{k=1}^{\frac{n}{2}} \binom{n+2-k}{k} q^k + q^{\frac{n}{2}+1} \\
&= \sum_{k=0}^{\frac{n}{2}+1} \binom{n+2-k}{k} q^k \\
&= \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-k}{k} q^k \\
&= x_{n+2}.
\end{aligned}
$$

Therefore, $\{x_n\}$ can also be described by the recursive definition $x_1 = 1$, $x_2 = 1 + q$, and $x_{n+2} = x_{n+1} + qx_n$ for all $n \geq 1$.

2. We can now find a closed form formula for $\{x_n\}$ by following the usual method for solving homogeneous linear difference equations with constant coefficients. This entails considering solutions of the form $x_n = t^n$ for some $t \neq 0$. Then, the recurrence relation $x_{n+2} = x_{n+1} + qx_n$ becomes

$$t^{n+2} = t^{n+1} + qt^n$$

or

$$t^2 = t + q \qquad (1)$$

since $t \neq 0$. Further, $q > 0$ guarantees that (1) has two distinct real solutions

$$t_1 = \frac{1+\sqrt{1+4q}}{2} \quad \text{and} \quad t_2 = \frac{1-\sqrt{1+4q}}{2}.$$

In this situation, the general solution is

$$x_n = c_1 t_1^n + c_2 t_2^n$$

for some constants c_1 and c_2. Finally, the initial conditions $x_1 = 1$ and $x_2 = 1 + q$ imply that

$$c_1 = \frac{t_1}{\sqrt{1+4q}} \quad \text{and} \quad c_2 = \frac{-t_2}{\sqrt{1+4q}}.$$

As a result, we have

$$x_n = \frac{t_1^{n+1} - t_2^{n+1}}{\sqrt{1+4q}}$$

for $n \geq 1$.

3. By Parts 1 and 2,

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k \\ = \quad &\frac{1}{\sqrt{1+4q}} \left[\left(\frac{1+\sqrt{1+4q}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{1+4q}}{2} \right)^{n+1} \right] \qquad (2) \end{aligned}$$

for all $n \geq 1$. In particular, since $n(n+1)$ is always even, we have

$$\left\lfloor \frac{n(n+1)}{2} \right\rfloor = \frac{n(n+1)}{2}$$

and (2) yields

$$\begin{aligned} &\sum_{k=0}^{\frac{n(n+1)}{2}} \binom{n(n+1)-k}{k} q^k \\ = \quad &\frac{1}{\sqrt{1+4q}} \left[\left(\frac{1+\sqrt{1+4q}}{2} \right)^{n(n+1)+1} - \left(\frac{1-\sqrt{1+4q}}{2} \right)^{n(n+1)+1} \right] \qquad (3) \end{aligned}$$

for $n \geq 1$.

4. Finally, if we substitute $q = n(n+1)$ in (3), then $\sqrt{1+4q} = 2n+1$ and for all $n \geq 1$, we get

$$\sum_{k=0}^{\frac{n(n+1)}{2}} \binom{n(n+1)-k}{k} [n(n+1)]^k \quad = \quad \frac{(n+1)^{n(n+1)+1} - (-n)^{n(n+1)+1}}{2n+1}$$

$$= \frac{(n+1)^{n(n+1)+1} + n^{n(n+1)+1}}{2n+1}$$

(since $n(n+1)+1$ is odd for all $n \geq 1$).

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

Based on the *Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX. to problem 4919, SSMA, February 2007* ,for $n \in Z^{*+}$ and $0 \leq k \leq n+2$, we have that

$$\binom{2n+4-k}{k} = \binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2}.$$

Let $S(n) = \sum_{k=0}^{n} \binom{2n-k}{k} z^k$, (z is constant) $\forall n \geq 1$. Then we have that,

$$\begin{aligned}
S(n+2) &= \sum_{k=0}^{n+2} \binom{2n+4-k}{k} z^k = 1 + (2n+3)z + \sum_{k=2}^{n+1} \binom{2n+4-k}{k} z^k + z^{n+2} \\
&= 1 + (2n+3)z + \sum_{k=2}^{n+1} \left[\binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2} \right] z^k + z^{n+2} \\
&= 2z + \sum_{k=0}^{n+1} \binom{2n+2-k}{k} z^k + 2\sum_{k=2}^{n+1} \binom{2n+3-k}{k-1} z^k - \sum_{k=2}^{n+1} \binom{2n+2-k}{k-2} z^k + z^{n+2} \\
&= 2z + S(n+1) + 2\sum_{k=1}^{n} \binom{2n+2-k}{k} z^{k+1} - \sum_{k=0}^{n-1} \binom{2n-k}{k} z^{k+2} + z^{n+2} \\
&= 2z + S(n+1) + 2z\sum_{k=1}^{n} \binom{2n+2-k}{k} z^{k} - z^2\sum_{k=0}^{n-1} \binom{2n-k}{k} z^{k} + z^{n+2} \\
&= 2z + S(n+1) + 2z\left[S(n+1) - 1 - z^{n+1}\right] - z^2\left[S(n) - z^n\right] + z^{n+2} \\
&= (1+2z)S(n+1) - z^2 S(n).
\end{aligned}$$

As a result, we get the following homogeneous linear difference equation with constant coefficients,

$$S(n+2) - (1+2z)S(n+1) + z^2 S(n) = 0.$$

Solving the respective characteristic equation (considering z as constant),

$$r^2 - (1+2z)r + z^2 = 0$$

we get the solutions

$$r_1 = \frac{(1+2z) + \sqrt{1+4z}}{2}, \qquad \text{and} \qquad r_2 = \frac{(1+2z) - \sqrt{1+4z}}{2}.$$

The general formula for $S(n)$ is

$$S(n) = C_1 r_1^n + C_2 r_2^n, \qquad n \in Z^{*+}.$$

Considering the fact that $S(1) = 1 + z$ and $S(2) = 1 + 3z + z^2$ we have that

$$\begin{aligned} S(1) &= C_1 r_1 + C_2 r_2 = 1 + z = \sum_{k=0}^{1} \binom{2-k}{k} z^k \quad \text{and} \\ S(2) &= C_1 r_1^2 + C_2 r_2^2 = 1 + 3z + z^2 = \sum_{k=0}^{2} \binom{4-k}{k} z^k \end{aligned}$$

from where it implies that

$$C_1 = \frac{(1+3z+z^2) - r_2(1+z)}{r_1(r_1 - r_2)} \qquad \text{and} \qquad C_2 = \frac{(1+3z+z^2) - r_1(1+z)}{r_2(r_2 - r_1)}.$$

Finally,

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} \binom{2n-k}{k} z^k = C_1 \cdot r_1^n + C_2 \cdot r_2^n \\ &= \frac{(1+z)\sqrt{1+4z} + (1+3z)}{2\sqrt{1+4z}} \cdot r_1^{n-1} + \frac{(1+z)\sqrt{1+4z} - (1+3z)}{2\sqrt{1+4z}} \cdot r_2^{n-1} \\ &= \frac{1}{2\sqrt{1+4z}} \Big\{ [(1+z)\sqrt{1+4z} + (1+3z)] r_1^{n-1} + [(1+z)\sqrt{1+4z} - (1+3z)] r_2^{n-1} \Big\}. \end{aligned}$$

Thus, the general formula is,

$$S(n) = \frac{1}{2\sqrt{1+4z}} \Big\{ [(1+z)\sqrt{1+4z} + (1+3z)] r_1^{n-1} + [(1+z)\sqrt{1+4z} - (1+3z)] r_2^{n-1} \Big\}.$$

Applying the above formula for $z = x = 2n = N(N+1)$, (since $N(N+1)$ is an even number for $N \in Z^{*+}$), and after making some manipulations, we have that,

$$r_1 = (N+1)^2, \quad r_2 = N^2, \quad C_1 = \frac{N+1}{2N+1}, \quad C_2 = \frac{N}{2N+1}, \quad N = \frac{\sqrt{1+4x} - 1}{2}$$

$$\Rightarrow \sum_{k=0}^{x/2} \binom{x-k}{k} x^k = \frac{N+1}{2N+1} \cdot (N+1)^{N(N+1)} + \frac{N}{2N+1} \cdot N^{N(N+1)}$$

$$\Rightarrow \sum_{k=0}^{x/2} \binom{x-k}{k} x^k = \frac{1}{2N+1} \Big[(N+1)^{N(N+1)+1} + N^{N(N+1)+1} \Big]$$

or related to $x, (x = N(N+1)),$ we get the formula,

$$\Rightarrow \quad \sum_{k=0}^{x/2} \binom{x-k}{k} x^k = \frac{1}{2^{x+1}\sqrt{1+4x}} \Big[(\sqrt{1+4x}+1)^{x+1} + (\sqrt{1+4x}-1)^{x+1} \Big].$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5243:** *Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

If a, b, c are consecutive Pythagorean numbers, then solve in the integers the equation:

$$\frac{x^2+bx}{a^y-1} = c.$$

(A consecutive Pythagorean triple is a Pythagorean triple that is composed of consecutive integers.)

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

There are no solutions to the equation for a consecutive Pythagorean triple.

Assume that a is a positive integer and $b = a+1, c = a+2$ so that a, b, c is a consecutive Pythagorean triple. Then $a^2 + (a+1)^2 = (a+2)^2$ reduces to the quadratic equation $a^2 - 2a - 3 = 0$ whose only positive integer solution is $a = 3$. Therefore $a = 3, b = 4, c = 5$ is the only positive consecutive Pythagorean triple and the given equations becomes

$$\frac{x^2+4x}{3^y-1} = 5.$$

Note that if $y = 0$ the the equation is undefined. If $y < 0$, then $y = -n$ for some positive integer n. The equation then reduces to $3^n(x^2+4x) = 5(1-3^n)$. Since 3 is a prime, it follows that either 3 divides 5 or 3 divides $1-3^n$, both contradictions.

Hence, $y > 0$ and $x^2 + 4x = 5(3^y - 1)$ or $x^2 + 4x + 5 = 3^y = 5$. Let $p(x) = x^2 + 4x + 5$. If $x \equiv 0 \pmod 3$, then $p(x) \equiv 2 \pmod 3$. Therefore, $p(x) = x^2 + 4x + 5$ is never congruent to 0 module 3 for any integer x. However, $3^y 5 \equiv 0 \pmod 3$ for each integer $y > 0$. Hence, there are no nonzero solutions, where $y \neq 0$ to the equation $x^2 + 4x + 5 = 3^y 5$ and this completes the solution.

Editor's comment: Some readers gave $(0,0)$ and $(-4,0)$ as solutions to the equation $x^2 + 4x + 5 = 3^y 5$. This certainly true, but the expression $x^2 + 4x + 5 = 3^y 5$ was obtained from the original statement of the problem under the assumption that $y \neq 0$.

$$\left(\frac{x^2+4x}{3^y-1} = 5\right) \iff \left(x^2 + 4x + 5 = 3^y 5\right) \text{if, and only if } y \neq 0.$$

In this case, multiplication by the denominator is not valid. Stated otherwise, the equation $\dfrac{x^2+4x}{3^y-1} = 5$ has no solution, but the equation $x^2 + 4x + 5 = 3^y 5$ has two integer solutions, $(0,0)$ and $(-4,0)$. The two equations are not equivalent to one another because they have different domains of definition.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5244:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let T_a and S_b denote the a^{th} triangular and the b^{th} square number, respectively. Find explicit instances of such numbers to prove that every Fibonacci number F_n occurs among the values $\gcd(T_a, S_b)$.

Solution 1 by David Diminnie, Texas Instruments, Inc., Dallas, TX

Recall that $T_a = \dfrac{a\,(a+1)}{2}$ and $S_b = b^2$. If we set $a = 2F_n$ and $b = F_n$ then by applying the identity $\gcd(p,\,q) = \gcd(p-q,\,q)$, $p > q$ we may evaluate $\gcd(T_a,\,S_b)$ as follows:

$$\begin{aligned}
\gcd\left(T_{2F_n},\,S_{F_n}\right) &= \gcd\left(\frac{2F_n\,(2F_n+1)}{2},\,F_n^2\right)\\
&= \gcd\left(2F_n^2+F_n,\,F_n^2\right)\\
&= \gcd\left(F_n^2+F_n,\,F_n^2\right)\\
&= \gcd\left(F_n,\,F_n^2\right)\\
&= F_n.
\end{aligned}$$

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, (jointly), Angelo State University, San Angelo, TX

More generally, we will show that for every positive integer n, $\gcd(T_{2n}, S_{2n}) = n$. The desired result then follows as an easy application of this property. To do so, we will use the following elementary results from number theory.

Lemma 1. If m and n are positive integers and d is a positive common divisor of m and n such that $\gcd\left(\dfrac{m}{d}, nd\right) = 1$, then $d = \gcd(m,n)$.

Proof. Since $\gcd(md, nd) = 1$, there are integers a and b such that

$$1 = a\left(\frac{m}{d}\right) + b\left(\frac{n}{d}\right)$$

or

$$d = am + bn.$$

Then, any positive common divisor of m and n must also divide d and it follows that $d = \gcd(m,n)$.

Lemma 2. For every positive integer n, $\gcd(2n+1, 4n) = 1$.

Proof. If $d = \gcd(2n+1, 4n)$, then d divides $(2n+1)$ and hence, d is odd. Further, since d is odd and d divides $4n$, d must divide n. Finally, d is a common divisor of n and $(2n+1)$ implies that d divides $(2n+1) - 2n = 1$. Therefore, $d = 1$.

For any positive integer n,

$$T_{2n} = \frac{2n\,(2n+1)}{2} = n\,(2n+1) \text{ and } S_{2n} = 4n^2.$$

Then, n is a positive common divisor of T_{2n} and S_{2n} and Lemma 2 implies that

$$\gcd\left(\frac{T_{2n}}{n}, \frac{S_{2n}}{n}\right) = \gcd\,(2n+1, 4n) = 1.$$

By Lemma 1, we have $\gcd\,(T_{2n}, S_{2n}) = n$ and our solution is complete.

Solution 3 by Paul M. Harms, North Newton, KS

We have $T_a = a(a+1)/2$ and $S_b = b^2$. When the Fibonacci number F_n is an odd integer let $a = F_n = b$. Then $a+1$ is even and the number $a = F_n$ does not have any common factor (except 1) with $a+1$ or $(a+1)/2$.

With $S_b = b^2 = F_n^2$, the $\gcd(T_a, S_b) = \gcd\big(F_n\,(F_n+1)\,, F_n^2\big) = F_n$. When F_n is an even integer let $a = 2F_n$ and $b = F_n$. Then $a+1$ is odd and has no common factors with $a/2 = F_n$. Again we have $\gcd(T_a, S_b) = \gcd\big(F_n\,(2F_n+1)\,, F_n^2\big) = F_n$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA and the proposer.

- **5245:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

Determine all real valued functions $f : \Re - \{-2, -\frac{1}{2}, -1, 0, \frac{1}{2}, 1, 2\} \to \Re$, which satisfy the relation

$$f(x) + f\left(\frac{-x-5}{2x+1}\right) + f\left(\frac{4x+5}{-2x+2}\right) = ax + b$$

where $a, b, \in \Re$.

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania

If we let, $g(x) = \dfrac{-x-5}{2x+1}$, $h(x) = \dfrac{4x+5}{-2x+2}$, then we have that,

$$\begin{aligned} (g\circ g)(x) &= x && (1)\\ (g\circ h)(x) &= (h\circ g)(x) && (2)\\ \text{and}\qquad (h\circ h)(x) &= g(x) && (3) \end{aligned}$$

Thus the given problem can be expressed as,

$$f(x) + (f\circ g)(x) + (f\circ h)(x) = ax + b, \quad (4)$$

Considering equation (4) and applying for $g(x)$, it implies that,

$$\begin{aligned}(f\circ g)(x)+[f\circ(g\circ g)](x)+(f\circ h\circ g)(x) &= ag(x)+b, \text{ or equvalently}\\ (f\circ g)(x)+f(x)+(f\circ h\circ g)(x) &= ag(x)+b, \qquad (5)\end{aligned}$$

Considering typos (4) and applying for $h(x)$, it implies that,

$$\begin{aligned}(f\circ h)(x)+[f\circ(g\circ h)](x)+(f\circ h\circ h)(x) &= ah(x)+b, \text{ or equivalently}\\ (f\circ h)(x)+[f\circ(g\circ h)](x)+(f\circ g)(x) &= ah(x)+b, \qquad (6)\end{aligned}$$

Considering equation (5) and applying for $h(x)$, it implies that,

$$\begin{aligned}(f\circ g\circ h)(x)+(f\circ h)(x)+(f\circ h\circ g\circ h)(x) &= a(g\circ h)(x)+b, \text{ or equivalently}\\ (f\circ g\circ h)(x)+(f\circ h)(x)+f(x) &= a(g\circ h)(x)+b, \qquad (7)\end{aligned}$$

Adding (simultaneously) side by side equations in (4), (5), and (6) to equation (7), results in,

$$3[f(x)+(f\circ g)(x)+(f\circ h)(x)+(f\circ g\circ h)(x)] = ax+ag(x)+ah(x)+a(g\circ h)(x)+4b,$$

$$f(x)+(f\circ g)(x)+(f\circ h)(x)+(f\circ g\circ h)(x) = \frac{1}{3}[ax+ag(x)+ah(x)+a(g\circ h)(x)+4b], \qquad (8)$$

Finally, if we subtract equation (6) from equation (8), then,

$$\begin{aligned} f(x) &= \frac{1}{3}a[x+g(x)-2h(x)+(g\circ h)(x)]+\frac{1}{3}b\\ \Leftrightarrow \qquad f(x) &= \frac{1}{3}a\left[x+\frac{-x-5}{2x+1}-2\frac{4x+5}{-2x+2}+\frac{2x-5}{2x+4}\right]+\frac{1}{3}b\\ \Leftrightarrow \qquad f(x) &= \frac{1}{3}a\left[x-\frac{x+5}{2x+1}+\frac{4x+5}{x-1}+\frac{2x-5}{2(x+2)}\right]+\frac{1}{3}b\\ \Leftrightarrow \qquad f(x) &= \frac{a}{3}\cdot\frac{4x^4+24x^3+30x^2+59x+45}{2(2x^3+3x^2-3x-2)}+\frac{1}{3}b\end{aligned}$$

Solution 2 by David Diminnie, Texas Instruments, Inc., Dallas, TX, and Charles Diminnie, Angelo State University, San Angelo, TX

The restrictions on the domain and range in the problem statement appear to be swapped, and the domain restriction appears to be both overly stringent and missing a critical value. For the discussion below we will assume that $f : \Re - \left\{-2,\, -\frac{1}{2},\, 1\right\} \to \Re$ satisfies

$$f(x) + f\left(\frac{-x-5}{2x+1}\right) + f\left(\frac{4x+5}{-2x+2}\right) = ax + b \quad (1)$$

for given $a, b \in \Re$. 8pt
Consider the function $g : \Re - \left\{-2,\, -\frac{1}{2},\, 1\right\} \to \Re$ with definition

$$g(x) = \frac{4x+5}{-2x+2}.$$

Since $g(x) \neq -2,\, -\frac{1}{2},\, 1$ when $x \neq -2,\, -\frac{1}{2},\, 1$ it follows that

$$g^2(x) = (g \circ g)(x) = \frac{4g(x)+5}{-2g(x)+2} = \frac{-x-5}{2x+1}.$$

Similarly, $g^2(x) \neq -2,\, -\frac{1}{2},\, 1$ when $x \neq -2,\, -\frac{1}{2},\, 1$ and we see that

$$g^3(x) = (g \circ g \circ g)(x) = \frac{-g(x)-5}{2g(x)+1} = \frac{2x-5}{2x+4}.$$

Finally, $g^3(x) \neq -2,\, -\frac{1}{2},\, 1$ when $x \neq -2,\, -\frac{1}{2},\, 1$ implies that

$$g^4(x) = (g \circ g \circ g \circ g)(x) = \frac{2g(x)-5}{2g(x)+4} = x.$$

As a result, we can see (by Comment [1]) that $g^n(x) = g^{n \bmod 4}(x)$ may therefore be re-expressed as

$$f(x) + f\left(g^2(x)\right) + f\left(g(x)\right) = ax + b. \quad (2)$$

If we substitute $g(x)$, $g^2(x)$, $g^3(x)$ into (2), taking advantage of the fact that $g^{i+j}(x) = g^{i+j \bmod 4}(x)$ (with $g^0(x) \equiv x$), we obtain the following additional relations (respectively):

$$\begin{aligned}
f\left(g(x)\right) + f\left(g^3(x)\right) + f\left(g^2(x)\right) &= ag(x) + b \quad (3)\\
f\left(g^2(x)\right) + f\left(x\right) + f\left(g^3(x)\right) &= ag^2(x) + b \quad (4)\\
f\left(g^3(x)\right) + f\left(g(x)\right) + f(x) &= ag^3(x) + b. \quad (5)
\end{aligned}$$

By adding (2), (4), and (5) and subtracting two times (3) from the result (again, with $x \neq -2,\, -\frac{1}{2},\, 1$), we may find an expression for $f(x)$:

$$3f(x) \;=\; a\left(x + g^2(x) + g^3(x) - 2g(x)\right) + b$$

$$
\begin{aligned}
f(x) &= \frac{a}{3}\Big(x+g^2(x)+g^3(x)-2g(x)\Big)+\frac{b}{3} \\
f(x) &= \frac{a}{3}\left(x-\frac{x+5}{2x+1}+\frac{2x-5}{2x+4}-\frac{4x+5}{-x+1}\right)+\frac{b}{3}. \qquad (6)
\end{aligned}
$$

To verify (6) is a solution, note that

$$
\begin{aligned}
f\left(\frac{4x+5}{-2x+2}\right) &= \frac{a}{3}\left(\frac{4x+5}{-2x+2}+\frac{2x-5}{2x+4}+x+\frac{2x+10}{2x+1}\right)+\frac{b}{3} \\
f\left(\frac{-x-5}{2x+1}\right) &= \frac{a}{3}\left(\frac{-x-5}{2x+1}+x+\frac{4x+5}{-2x+2}-\frac{2x-5}{x+2}\right)+\frac{b}{3}
\end{aligned}
$$

and therefore

$$
f(x)+f\left(\frac{4x+5}{-2x+2}\right)+f\left(\frac{-x-5}{2x+1}\right)=ax+b.
$$

Comment [1]. Note that $\left\{x,\ \frac{4x+5}{-2x+2},\ \frac{-x-5}{2x+1},\ \frac{2x-5}{2x+4}\right\}=\{g^0(x),\ g(x),\ g^2(x),\ g^3(x)\}$ forms a cyclic group of order 4 under function composition, with generator $g(x)$: Function composition is an associative operation, the identity element is $g^0(x)=g^4(x)=x$ (and hence $g^n(x)=g^{n \bmod 4}(x)$, as claimed above, so the set is closed under composition), and $g^k\circ g^{4-k}(x)=g^{4-k}\circ g^k(x)=x$ for $k=0,\,1,\,2,\,3$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Let $L(x)=ax+b, h(x)=\dfrac{-x-5}{2x+1}=-\dfrac{x+5}{2x+1}$ and $k(x)=\dfrac{4x+5}{-2x+2}=-\dfrac{4x+5}{2(x-1)}$.

Then the given condition becomes

$$
(1)\quad f(x)+f\left(h(x)\right)+f\left(k(x)\right)=L(x).
$$

Suppressing the argument x and adopting concatenation to represent composition, this becomes a functional condition:

$$
(1a)\quad f+fh+fk=L.
$$

Straightforward computation shows that $h^2(x)=h(h(x))=x$, that

$$
k^2(x)=h(x) \text{ and } h(k(x))=k(h(x))=\frac{2x-5}{2(x+2)}.
$$

That is, with i denoting the identity function,

$$
(2)\quad h^2=i, k^2=h \text{ and } kh=hk.
$$

It follows that $k^4=i$ and $khk=hk^2=hk=i$.

Applying both sides of $(1a)$ to $h(x)$ yields $fh+fh^2+fkh=Lh$, or

$$
(3)\quad fh+f+fkh=Lh.
$$

Applying both sides of $(1a)$ to $k(x)$ yields $fk + fhk + fkk = Lk$ or

$$(4)\quad fk + fhk + fh = Lk.$$

Finally, applying both sides of (3) to $k(x)$ yields $fhk + fk + fkhk = Lhk$, or

$$(5)\quad fhk + fk + f = Lhk.$$

Thus we have a system of 4 equations in the 4 unknowns, f, fh, fk, fhk:

$$\begin{cases} f + fh + fk \qquad = L \\ f + fh \qquad + fhk = Lh \\ \qquad fh + fk + fhk = Lk \\ f \qquad + fk + fhk = Lhk \end{cases}$$

Calculations reveal that

$$(6)\quad f = \frac{1}{3}\{L + Lh + Lhk - 2Lk\}.$$

That is,

$$\begin{aligned} f(x) &= \frac{1}{3}\{ax + b + ah(x) + b + ah(k(x)) + b - 2ak(x) - 2b\} \\ &= \frac{a}{3}\Big\{x + h(x) + h(k(x)) - 2k(x)\Big\} + \frac{b}{3} \\ &= \frac{a}{3}\left\{x + \frac{-x-5}{2x+1} + \frac{2x-5}{2(x+2)} - 2\frac{4x+5}{-2x+2}\right\} + \frac{b}{3} \\ &= \frac{a}{3}\left\{\frac{4x^4 + 24x^3 + 30x^3 + 59x + 45}{2(2x+1)(x-1)(x+2)}\right\} + \frac{b}{3}. \end{aligned}$$

Comment 1. More generally, note that if h and k are any two functions such that h has order $2, k^2 = h$ and h commute with k, then (6) gives the function f satisfying (1).

Comment 2. We believe that he domain and codomain of f, as stated in in the problem, are a typo. The conditions on the domain and codomain of f (and h and k and kh) are probably best summarized as "for all x for which everything makes sense." The domain of f consists of all reals except the obvious ones: $1, -2$ and $-\dfrac{1}{2}$.

Then fh is well defined because h is defined for all x except $-\dfrac{1}{2}$ and does not map any real to $-\dfrac{1}{2}$. Similarly fk is defined because k is defined for all reals except 1 and has range all reals except -2.

The composed functions $kh = hk$ both map from $\Re - \{-2\}$ to $\Re - \{1\}$ despite the technical concerns with domains.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Boris Rays, Brooklyn, NY, and the proposers.

- **5246:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $a_1, a_2, \ldots a_n, (n \geq 3)$ be distinct complex numbers. Compute the sum

$$\sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{(-1)^n}{a_j - a_k},$$

where $s_k = \left(\sum_{i=1}^{n} a_i\right) - a_k,\ 1 \leq k \leq n.$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $f : C \to C$ be the function defined by $f(a) = \left(\sum_{i=1}^{n} a_j\right) - a, a \in C$. Then f is a polynomial function such that $f(a_k) = s_k, 1 \leq k \leq n$. It is also known that there is only one polynomial function $p : C \to C$ with degree less than n and such that $p(a_k) = s_k, 1 \leq k \leq n$ which can be obtained for example with the Lagrange interpolation formula:

$$p(a) = \sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{a - a_j}{a_k - a_j} = \sum_{k=1}^{n} s_k \frac{\prod_{j \neq k} (a - a_j)}{(-1)^{n-1} \prod_{j \neq k} (a_j - a_k)} = \sum \frac{(-1)^{n-1} s_k}{\prod_{j \neq k} (a_j - a_k)} \prod_{j \neq k} (a - a_j).$$

So, both polynomial functions p and f, must be equal; in particular, their respective leading coefficients must coincide, that is $\sum_{k=1}^{n} \frac{(-1)^{n-1} s_k}{\prod_{j \neq k} (a_j - a_k)} = 0$. Thus, the required sum is

$$\sum_{k=1}^{n} \frac{(-1)^n}{\prod_{j \neq k} (a_j - a_k)} = 0.$$

Solution 2 by Paul M. Harms, North Newton, KS

Consider the polynomial

$$\begin{aligned} P(x) &= \frac{(x-a_2)(x-a_3)\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + \frac{(x-a_1)(x-a_3)\cdots(x-a_n)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} \\ &+ \cdots \frac{(x-a_1)(x-a_2)\cdots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})} - 1. \end{aligned}$$

We see that the degree of $p(x)$ is $n-1$. Note that $0 = p(a_1) = p_(a_2) = \ \cdots \ = p(a_n)$.

Since n different complex number have a polynomial value of zero for the $n-1$ degree polynomial, the polynomial must be identically zero.

If $p(x)$ (given above is expanded, then all coefficients of the different powers of x must be zero. Consider the coefficient of x^{n-2}. From the first fraction of $p(x)$ the coefficient of x^{n-2} is

$$\frac{-(a_2 + a_3 + \cdots + a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} = \frac{-s_1}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}.$$

We see that the coefficient of x^{n-2} for $p(x)$ is

$$\frac{-s_1}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}+\frac{-s_2}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}+\cdots$$

$$+\quad \frac{-s_n}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}=0.$$

The left side of the last equality is equal to or the negative of the summation in the problem. Thus the summation in the problem is zero.

Also solved by the proposer.

- **5247:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\ln(1+e^{2x})\cdots\ln(1+e^{nx})\,dx}.$$

Solution 1 by Anastasios Konronis, Athens, Greece

For $n\in N,\ x\in(0,1]$ we have

$$\begin{aligned}\ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx}) &= n!x^n\prod_{k=1}^{n}\left(1+\frac{\ln(1+e^{-kx})}{kx}\right)=n!x^n\prod_{k=1}^{n}\left(1+\mathcal{O}\left(\frac{e^{-kx}}{kx}\right)\right)\\ &= n!x^n\left(1+\mathcal{O}\left(\frac{e^{-x}}{x^n}\right)\right)\\ &= n!\left(x^n+\mathcal{O}\left(e^{-x}\right)\right)\end{aligned}$$

so

$$\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})=\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right).$$

Now from the above and taking into account that, from Stirling's formula,

$$\ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

we get that

$$\begin{aligned}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})\,dx} &= \frac{1}{n}\exp\left(\frac{1}{n}\ln\left(\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right)\right)\right)\\ &= \frac{1}{n}\exp\left(\ln n-1+\mathcal{O}\left(\frac{\ln n}{n}\right)\right)=e^{-1}+\mathcal{O}\left(\frac{\ln n}{n}\right)\to e^{-1}\end{aligned}$$

Solution 2 by Arkady Alt, San Jose, California, USA.

Let $f_n(x) = \prod\limits_{k=1}^{n} \ln\left(1 + e^{kx}\right)$. Since $f_n(x) > \prod\limits_{k=1}^{n} \ln\left(e^{kx}\right) = x^n n!$ then

$$\int_0^1 f_n(x)\,dx > n! \int_0^1 x^n dx = \frac{n!}{n+1}.$$

On the other hand, since $f_n(x) f_n(1) \le 1$ we have

$$\int_0^1 f_n(x)\,dx \le f_n(1) \int_0^1 dx = f_n(1).$$

Thus,

$$\frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} < \frac{1}{n}\sqrt[n]{\int_0^1 f_n(x)\,dx} \le \frac{1}{n}\sqrt[n]{f_n(1)}.$$

Let $a_n = \dfrac{f_n(1)}{n^n}$.
Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{a_{n-1}} &= \lim_{n\to\infty} \left(\frac{f_n(1)}{n^n} \cdot \frac{(n-1)^{n-1}}{f_{n-1}(1)}\right) \\
&= \lim_{n\to\infty} \left(\left(1 - \frac{1}{n}\right)^{n-1} \cdot \frac{\ln(1+e^n)}{n}\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{n-1} \cdot \lim_{n\to\infty} \frac{\ln(1+e^{-n}) + n}{n} \\
&= e^{-1} \cdot 1 = e^{-1}
\end{aligned}$$

then by *, the Multiplicative Stolz Theorem $\lim\limits_{n\to\infty} \dfrac{1}{n}\sqrt[n]{f_n(1)} = \lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}} = e^{-1}$.

Also we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} &= \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{1}{\sqrt[n]{n+1}} \\
&= \lim_{n\to\infty} \sqrt[n]{n!}n \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} \\
&= e^{-1} \cdot 1 = e^{-1}.
\end{aligned}$$

(Note: $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = e^{-1}$. Indeed,

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+1}{e}\right)^n (n+1) \Rightarrow$$

$$\frac{1}{e} < \frac{\sqrt[n]{n!}}{n} < \frac{1}{e} \cdot \frac{n+1}{n} \cdot \sqrt[n]{n+1},$$

or again, applying the Multiplicative Stolz Theorem to $\sqrt[n]{\dfrac{n!}{n^n}}$).

Then by the squeeze principle,

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{f_n(1)} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e^{-1} = e^{-1} \text{ yields}$$

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\int_0^1 f_n(x)\,dx} = e^{-1}.$$

* We use the Multiplicative Stolz Theorem in the following form:
If the sequence $\left(\dfrac{a_{n+1}}{a_n}\right)_{n\geq 1}$ has a limit then the sequence $\left(\sqrt[n]{a_n}\right)_{n\geq 1}$ has a limit and

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}.$$

Solution 3 by Kee-Wai, Hong Kong, China

We show that the limit equals $\dfrac{1}{e}$.

Denote the integrand by $f(x)$. Since $f(x) > (x)(2x)\cdots(nx) = (n!)x^n$, so

$$\int_0^1 f(x)dx > \frac{n!}{n+1}. \qquad (1)$$

For $0 \leq x \leq 1$ and $k = 1, 2, \cdots, n$, we have

$$1 + e^{kx} \leq 1 + e^k < 2e^k < e^{1+k}, \text{so that}$$

$$f(x) < (n+1)! \text{ and}$$

$$\int_0^1 f(x)dx < (n+1)!. \qquad (2)$$

By Stirling's formula for $n!$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} = \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{(n+1)!} = \frac{1}{e}.$$

Now by $(1), (2)$ and the squeezing principle, we obtain the result we claimed.

Also solved by Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University, Tirana, Albania and the proposer.

Mea Culpa (yet again)

Featured solution 5241(3) that appeared in the April 2013 issue of the column was submitted jointly by **Anastasios Kotronis and Konstantinos Tsouvalas, University of Athens, Athens, Greece.** I inadvertently forgot to list Konstantinos' name. Sorry.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
December 15, 2013

- **5265:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that
  $$2x - y - \sqrt{3x^2 - 3xy + y^2} = 2014,$$
  with $(x, y) = 1$.

- **5266:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The pentagonal numbers begin $1, 5, 12, 22, \cdots$ and in general satisfy $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The positive Jacobsthal numbers, which have applications in tiling and graph matching problems, begin $1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 1$. Prove that there exists infinitely many pentagonal numbers that are the sum of three Jacobsthal numbers.

- **5267:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

  Let n be a positive integer. Prove that
  $$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_n L_{n+1}}\right) L_{n+2},$$
  where F_n and L_n represents the n^{th} Fibonacci and Lucas Numbers defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$; and $L_0 = 2, L_1 = 1$, and for all $n \geq 0$, $L_{n+2} = L_{n+1} + L_n$, respectively.

- **5268:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

  Let $N = 121^a + a^3 + 24$. Determine all positive integers a for which

  a) N is a perfect square.

  b) N is a perfect cube.

- **5269:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $\{a_n\}_{n\geq 1}$ be the sequence defined by

$$a_1 = 1,\ a_2 = 5,\ a_{n-1}^2 - a_n a_{n-2} + 4 = 0.$$

Show that all of the terms of the sequence are integers.

- **5270:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Calculate

$$\int_0^1 \int_0^1 (x+y)^k \, (-1)^{\left\lfloor \frac{1}{x} - \frac{1}{y} \right\rfloor} \, dxdy,$$

where$\lfloor x \rfloor$ denotes the integer part of x.

Solutions

- **5248:** *Proposed by Kenneth Korbin, New York, NY*

A triangle with sides (a, a, b) has the same area and the same perimeter as a triangle with sides (c, c, d) where a, b, c and d are positive integers and with

$$\frac{b^2 + bd + d^2}{b + d} = 7^6.$$

Find the sides of the triangles.

Solution 1 by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

First, note that the condition

$$\frac{b^2 + bd + d^2}{b + d} = 7^6 \qquad (1)$$

implies that $b \neq d$ and thus, we may assume that $b > d$. Further, the required equality of the perimeters and areas of the triangles yields

$$\begin{aligned} 2a + b &= 2c + d, \text{and} && (2) \\ b^2(4a^2 - b^2) &= d^2(4c^2 - d^2) && (3) \end{aligned}$$

By (1), (2), and (3),

$$\begin{aligned} b^2(2a - b) &= d^2(2c - d) \\ \Rightarrow b^2(2a + b) - 2b^3 &= d^2(2c + d) - 2d^3 \\ \Rightarrow (b^2 - d^2)(2a + b) &= 2(b^3 - d^3) \end{aligned}$$

$$\begin{aligned}\Rightarrow (b+d)(2a+b) &= 2(b^2+bd+d^2) \\ \Rightarrow 2a+b &= 2c+d = 2(7^6). \qquad (4)\end{aligned}$$

It follows that b and d must be even.

Condition (1) can be re-written in the form

$$b^2 + (d-7^6)b + d^2 - 7^6 d = 0$$

and hence,

$$\begin{aligned} b &= \frac{7^6 - d \pm \sqrt{(d-7^6)^2 - 4(d^2 - 7^6 d)}}{2} \\ &= \frac{7^6 - d \pm \sqrt{(d+7^6)^2 - 4d^2}}{2}. \qquad (5)\end{aligned}$$

Since b and d are even integers, there must exist an odd positive integer k such that

$$\begin{aligned}(d+7^6)^2 - 4d^2 &= k^2, \text{ or} \\ (d+7^6)^2 &= 4d^2 + k^2.\end{aligned}$$

Using known properties of Pythagorean Triples, there are positive integers s, m, and n such that $m > n$, $(m,n) = 1$, $m - n \equiv 1 \pmod 2$, and

$$d + 7^6 = s(m^2+n^2), \quad k = s(m^2 - n^2), \quad \text{and} \quad 2d = s(2mn). \qquad (6)$$

Note that since k and $m^2 - n^2$ are odd, s must also be odd. Then (6) implies that s divides d and s divides $d + 7^6$ and hence, s divides 7^6. Therefore, $s = 7^r$ for some $r \in \{0, 1, 2, \ldots 6\}$.

Next it follows from (6) that

$$\begin{aligned} 7^r(m^2+n^2) &= d + 7^6 = 7^r(mn) + 7^6, \text{ or} \\ m^2 - mn + n^2 &= 7^{6-r}. \qquad (7)\end{aligned}$$

Using

$$m^2 + n^2 = \frac{1}{2}\left[(m+n)^2 + (m-n)^2\right] \quad \text{and } mn = \frac{1}{4}\left[(m+n)^2 - (m-n)^2\right],$$

(7) can be re-written

$$(m+n)^2 + 3(m-n)^2 = 4 \cdot 7^{6-r}. \qquad (8)$$

Also, (5) and (6) imply that

$$\begin{aligned} b &= \frac{7^r(m-n)^2 \pm 7^r(m^2-n^2)}{2} \\ &= 7^r m(m-n) \quad \text{or} \quad 7^r n(n-m). \end{aligned}$$

Since $m > n$, we have $b = 7^r m(m-n)$ to go with $d = 7^r mn$ (from (6)). The fact that b is even now forces m to be even and n to be odd.

We can now solve (8) for m and n and thereby solve for b and d. Our work is reduced by the facts that $m+n$ and $m-n$ are odd, $(m+n, m-n) = (m,n) = 1$, and

$$\begin{aligned} 4 \cdot 7^{6-r} &= (m+n)^2 + 3(m-n)^2 > 4(m-n)^2, \text{i.e.,} \\ m-n &< \sqrt{7^{6-r}}. \end{aligned}$$

Using these and some help from MuPAD, there are only two feasible solutions for (8), namely

r	m	n	b	d
0	360	37	116,280	13,320
4	8	3	96,040	57,624.

Then, (4) may be employed to get the final solutions

a	b	c	d
59,509	116,280	110,989	13,320
69,629	96,040	88,837	57,624.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

Without loss of generality, we assume $b < d$. Since $2a + b = 2c + d$, we have $d - b = 2(a-c)$, so b and d have the same parity. But if both b and d are odd, then $b^2 + bd + d^2$ is odd and $7^6(b+d)$ is even, a contradiction. Thus both b and d are even. Letting $b = 2x$ and $d = 2y$ for positive integers x and y, we obtain $y - x = a - c$ and $2(x^2 + xy + y^2) = 7^6(x+y)$. Then

$$y = \frac{7^6 - 2x \pm \sqrt{(6x+7^6)(7^6-2x)}}{4}.$$

A quick search via computer program yields two possible solutions with $x < y$:

$$(x,y) = (6660, 58140) \text{ or } \ (x,y) = (28812, 48020).$$

Next, since the areas of the two isosceles triangles must be equal, we have

$$\begin{aligned} \frac{1}{2}b\sqrt{a^2 - \frac{1}{4}b^2} &= \frac{1}{2}d\sqrt{c^2 - \frac{1}{4}d^2} \ \text{ and thus} \\ b^2(4a^2 - b^2) &= d^2(4c^2 - d^2). \end{aligned}$$

Since $2a+b=2c+d$, we obtain $b^2(2a-b)=d^2(2c-d)$, or $x^2(a-x)=y^2(c-y)$. Then $cy^2-ax^2=y^3-x^3$, so

$$c(y+x)(y-x)+x^2(c-a)=(y-x)(x^2+xy+y^2).$$

Applying $y-x=a-c$ and $x^2+xy+y^2=7^6(x+y)/2$, we have

$$\begin{aligned} c(y+x)-x^2 &= 7^6(x+y)/2 \text{ and hence} \\ c &= x^2/(x+y)+7^6/2. \end{aligned}$$

Similarly,

$$a=y^2/(x+y)+7^6/2.$$

In particular, we note that this implies

$$2a+b=\frac{2(x^2+xy+y^2)}{x+y}+7^6=2\cdot 7^6,$$

the perimeter of all such triangles.

Finally, we verify that the two possible solutions for (x,y) yield the required triangles:

$$(x,y)=(6660,58140) \quad\Rightarrow\quad (a,b,c,d)=(110989,13320,59509,116280).$$
$$(x,y)=(28812,48020) \quad\Rightarrow\quad (a,b,c,d)=(88837,57624,69629,96040).$$

In the first solution, both triangles have area 737,854,740. In the second solution, both triangles have area 2,421,216,420. Also, the second solution may be written in the form $(a,b,c,d)=7^4(37,24,29,40)$.

Editor's comments: **David Stone and John Hawkins** stated in their solution that it would be nice if an analytical solution for b and d in the following could be found.

$$\begin{aligned} \frac{b^2+bd+d^2}{b+d} &= 7^6, \Longrightarrow \\ b+\frac{d^2}{b+d} &= 7^6, \text{ and } \frac{b^2}{b+d}+d=7^6, \text{ and } b+d-\frac{bd}{b+d}=7^6. \end{aligned}$$

This allowed them to put some conditions onto b and d. But then they stated: "we see no path towards a complete solution. Finding integers b and d whose sum divides their product seems to be a difficult problem."

Ed Gray of Highland Beach, FL also reached the equation $u^2+uv+v^2=7^6$ and found that the general solution to

$$x^2+xy+y^2=z^2$$

has been characterized parametrically by J. Neuburg and G.B. Mathews (See L. E. Dickson's, History of History of The Theory of Numbers, vol.II, 2005, Dover Books on Mathematics, p.406). Specifically,

$$\begin{cases} x=p^2-q^2 \\ y=2pq+q^2 \\ z=x^2+pq+q^2. \end{cases}$$

He then applied this generic solution to the problem by solving

$$x^2 + xy + y^2 = (7^3)^2 = 343^2.$$

There are two positive integer solutions to this equation: $(x, y) = (18, 1)$ and $(x, y) = (14, 7)$. With these solutions it was possible for him, by retracing his steps, to obtain two sets, wherein each set contains two isosceles triangles with sides (a, a, b) and (c, c, d), and for which the triangles have the same perimeter, the same area, and for which $\dfrac{b^2 + bd + d^2}{b + d} = 7^6$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro GA, and the proposer.

- **5249:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

(a) Let n be an odd positive integer. Prove that $a^n + b^n$ is the square of an integer for infinitely many integers a and b.

(b) Prove that $a^2 + b^3$ is the square of an integer for infinitely many integers a and b.

Solution 1 by Arkady Alt, San Jose, CA

(a) Let $a = x\left(x^n + y^n\right), b = y\left(x^n + y^n\right)$ where $x, y \in \mathcal{N}$ then

$$a^n + b^n = x^n\left(x^n + y^n\right)^n + y^n\left(x^n + y^n\right)^n = \left(x^n + y^n\right)^{n+1}$$

and, since $n = 2m - 1, m \in \mathcal{N}$ then

$$a^n + b^n = \left(\left(x^n + y^n\right)^m\right)^2.$$

(b) We will show that equation $a^2 + b^3 = c^2$ have infinitely many solutions in integers. Assuming that $c = 2a$ we obtain $b^3 = 3a^2$. Let $a = 3t^3, t \in \mathcal{Z}$ then

$$b^3 = 3 \cdot 9t^6 \iff b = 3t^2.$$

Thus, for $(a, b) = \left(3t^3, 3t^2\right)$, where t is any integer we have

$$a^2 + b^3 = 9t^6 + 27t^6 = 36t^6 = \left(6t^3\right)^2.$$

Solution 2 by Pat Costello, Eastern Kentucky University, Richmond, KY

(a) Let n be an odd positive integer. Let $a = 2 \cdot 2^{2j}$ and $b = 2 \cdot 2^{2j}$ for an arbitrary positive integer j. Then

$$\begin{aligned} a^n + b^n &= (2 \cdot 2^{2j})^n + (2 \cdot 2^{2j})^n \\ &= 2^n \cdot 2^{2nj} + 2^n \cdot 2^{2nj} \\ &= 2 \cdot \left(2^n \cdot 2^{2nj}\right) \\ &= 2^{n+1} \cdot 2^{2nj} \end{aligned}$$

$$= \left(2^{(n+1)/2} \cdot 2^{nj}\right)^2,$$

the square of an integer since n is odd.

(b) Let $a = 2^{3n}$ and $b = 2 \cdot 2^{2n}$ for an arbitrary positive integer n. Then

$$\begin{aligned} a^2 + b^3 &= \left(2^{3n}\right)^2 + \left(2 \cdot 2^{2n}\right)^3 \\ &= 2^{6n} + 8 \cdot 2^{6n} \\ &= 9 \cdot 2^{6n} \\ &= \left(3 \cdot 2^{3n}\right)^2, \end{aligned}$$

the square of an integer.

Solution 3 by David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, Ga

(a) Let $n = 2k+1$ for $k \geq 0$. Then let $a = b = 2m^2$, for any $m \geq 1$. Then

$$a^n + b^n = 2a^n = 2\left(2m^2\right)^{2k+1} = 2^{2k+2}m^{2(2k+1)} = \left(2^{k+1}m^{2k+1}\right)^2, \text{ which is square.}$$

Of course, there is also a trivial solution; let a be any square and $b = 0$.

(b) Let $a = m^2\left(16m^2 - 1\right)$ and $b = 4m^2$, for any integer m. Then

$$\begin{aligned} a^2 + b^3 &= m^4\left(16m^2 - 1\right)^2 + \left(4m^2\right)^3 \\ &= m^4\left(256m^4 - 32m^2 + 1\right) + 64m^6 \\ &= 256m^8 + 32m^4 + 1 \\ &= \left(16m^4 + m^2\right)^2; \text{ a square.} \end{aligned}$$

In addition to the trivial solution, let a be any square and $b = 0$, there is also a "semi-trivial" solution: For any $c, m \geq 1$, let $a = c^{3m}, b = -c^{2m}$, so that

$$a^2 + b^3 = \left(c^{3m}\right)^2 + \left(-c^{2m}\right)^3 = c^{6m} - c^{6m} = 0; \text{ a square.}$$

Solution 4 by Ken Korbin, New York, NY

(a) Let $a = N^2, b = 2N^2$ where N is a positive integer. Then $a^3 + b^3 = \left(3N^3\right)^2$, and it follows that for n odd, $a^n + b^n$ is a perfect square.

(b) Let $a = 4N^3 + 6N^2 + 3N$, and $b = 2N + 1$, where N is a positive integer. Then $a^2 + b^3 = \left(4N^3 + 6N^2 + 3N + 1\right)^2$.

Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian

College, Clinton, SC; Roberto de la Cruz Moreno, Centre de Recerca Matematica, Campus de Bellaterra, Barcelona, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Charles McCracken, Dayton, OH, and the proposer.

- **5250:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let $a \in \left(0, \dfrac{\pi}{2}\right)$ and $b, c \in (1, \infty)$. Calculate:

$$\int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right) \cdot \sin x \cdot dx.$$

Solution 1 by Anastasios Kotronis, Athens, Greece

We have

$$\begin{aligned} I = \int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right) \sin x \, dx \quad &\overset{y=-x}{=} \quad -\int_{-a}^{a} \ln\left(b^{-\sin^3 x} + c^{-\sin^3 x}\right) \sin x \, dx \\ &= \quad \int_{-a}^{a} \ln\left(\frac{(bc)^{\sin^3 x}}{b^{\sin^3 x} + c^{\sin^3 x}}\right) \sin x \, dx \\ &= \quad \ln(bc) \int_{-a}^{a} \sin^4 x \, dx - I. \end{aligned}$$

So
$$I = \frac{\ln(bc)}{2} \int_{-a}^{a} \sin^4 x \, dx = \frac{\ln(bc)}{2} \left(\frac{3a}{4} - \frac{\sin(2a)}{2} + \frac{\sin(4a)}{16}\right).$$

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Tor Vergata Roma, Rome, Italy

Answer: $\frac{1}{2}\left(\frac{3}{4}a - \frac{1}{2}\sin(2a) + \frac{1}{16}\sin(4a)\right)\ln(bc)$

Proof: We observe

$$\begin{aligned} \ln\left(b^{\sin^3(-x)} + c^{\sin^3(-x)}\right)\sin(-x) \quad &= \quad -\ln\left(b^{-\sin^3 x} + c^{-\sin^3 x}\right)\sin x \\ &= \quad -\ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right)\sin x + \ln\left((bc)^{\sin^3 x}\right)\sin x \end{aligned}$$

thus

$$\begin{aligned} 2\int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right)\sin x \, dx \quad &= \quad \int_{-a}^{a} \sin^4 x dx \ln(bc) \\ &= \quad \int_{-a}^{a} \sin^2 x(1 - \cos^2 x) dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{x-\sin x\cos x}{2}\Big|_{-a}^{a} - \int_{-a}^{a}\frac{1}{4}(\sin^2(2x))dx \\
&= a - \frac{1}{2}\sin(2a) - \int_{-2a}^{2a}\frac{1}{8}(\sin^2 x)dx \\
&= a - \frac{1}{2}\sin(2a) - \frac{a}{4} + \frac{1}{16}\sin(4a)
\end{aligned}
$$

Also solved by Arkady Alt, San Jose, CA; Roberto de la Cruz Moreno, Centre de Recerca Matematica, Campus de Bellaterra, Barcelona, Spain; Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Boris Rays, Brooklyn, NY, and the proposers.

- **5251:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

Compute the following sum:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\cos(m+n)}{(m+n)^2}.$$

Solution 1 by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

The sum equals

$$\frac{\pi^2}{12} - \frac{1}{4} - \ln\left(2\cos\frac{1}{2}\right).$$

The problem is a particular case of the following theorem (see the first citation below [Theorem 1, p. 2]).

Theorem 1. *Suppose that both series*

$$\sum_{k=1}^{\infty} a_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} k a_k$$

converge and let σ *and* $\widetilde{\sigma}$ *denote their sums, respectively. Then, the iterated series*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n+m},$$

converges and its sum s *equals* $\widetilde{\sigma} - \sigma$.

The following two formulae are well-known (see citation 2, [Formula 1.441(4), p. 44])

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos kx}{k} = \ln\left(2\cos\frac{x}{2}\right), \quad -\pi < x < \pi$$

and (citation 2 [Formula 1.443(4), p. 45])

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos kx}{k^2}=\frac{\pi^2}{12}-\frac{x^2}{4}, \quad -\pi\le x\le\pi.$$

Now, we apply the Theorem in citation 1 with $a_k=(-1)^k\cdot\dfrac{\cos k}{k^2}$, and we have that

$$\sigma=\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}(-1)^k\frac{\cos k}{k^2}=\frac{1}{4}-\frac{\pi^2}{12}$$

and

$$\widetilde{\sigma}=\sum_{k=1}^{\infty}ka_k=\sum_{k=1}^{\infty}(-1)^k\frac{\cos k}{k}=-\ln\left(2\cos\frac{1}{2}\right).$$

It follows, based on the Theorem in citation 1, that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{n+m}=\widetilde{\sigma}-\sigma=\frac{\pi^2}{12}-\frac{1}{4}-\ln\left(2\cos\frac{1}{2}\right).$$

Citations:

1) Ovidiu Furdui and Tiberiu Trif, *On the Summation of Certain Iterated Series*, Journal of Integer Sequences, Vol. 14, 2011, Issue 6, article 11.6.1, article available online at https://cs.uwaterloo.ca/journals/JIS/VOL14/Furdui/furdui3.pdf

2) I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* Sixth Edition, Academic Press, 2000

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the double sum by S. We show that

$$S=\frac{\pi^2-3-6\ln\left(2(1+\cos 1)\right)}{12}=0.00990\ldots.$$

Let m and M be positive integers with $m\le M$. We have

$$\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\cos(m+n)}{(m+n)^2}=\sum_{k=m+1}^{\infty}(-1)^k\frac{\cos k}{k^2}=\sum_{k=m+1}^{M^2+1}(-1)^k\frac{\cos k}{k^2}+r,$$

where $|r|\le\displaystyle\sum_{k=M^2+2}^{\infty}\frac{1}{k^2}<\frac{1}{M^2}$. Hence,

$$\sum_{m=1}^{M}\sum_{n=1}^{\infty}(-1)^{m+n}\,\frac{\cos(m+n)}{(m+n)^2}=\sum_{k=2}^{M^2+1}(-1)^k\,\frac{(k-1)\cos k}{k^2}+R,$$

where $|R|<\dfrac{1}{M}$. By taking the limit as M tends to infinity, we have

$$S=\sum_{k=2}^{\infty}(-1)^k\frac{(k-1)\cos k}{k^2}=\sum_{k=2}^{\infty}(-1)^k\frac{\cos k}{k}-\sum_{k=2}^{\infty}(-1)^k\frac{\cos k}{k^2}.$$

For $-\pi < x < \pi$, is known ([1], formula 17.2.6, p.239) that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k} = -\frac{\ln\left(2\left(1+\cos x\right)\right)}{2},$$

and ([1] formula 17.2.9, p.239) that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2} = \frac{3x^2-\pi^2}{12}.$$

By putting $x=1$, we obtain our result for S.

Reference: 1. E.R. Hansen: *A Table of Series and Products,* Prentice-Hall, Inc. (1975).

Also solved by Ed Gray, Highland Beach FL; Anastasios Kotronis, Athens, Greece; Paolo Perfetti, Department of Mathematics, University of Tor Vergata Roma, Rome, Italy, and the proposers.

- **5252:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1=3, a_2=5$ and for all $n\geq 2, a_{n+1}=\dfrac{1}{2}\left(a_n^2+1\right)$. Prove that

$$1+2\left(\sum_{k=1}^{n}\sqrt{\frac{F_k}{1+a_k}}\right)^2 < F_{n+2},$$

where F_n represents the n^{th} Fibonacci number defined by $F_1=F_2=1$ and for $n\geq 3, F_n=F_{n-1}+F_{n-2}$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

By the Cauchy-Schwarz inequality

$$\left(\sum_{k=1}^{n}\sqrt{\frac{F_k}{1+a_k}}\right)^2 \leq \left(\sum_{k=1}^{n}F_k\right)\left(\sum_{k=1}^{n}\frac{1}{1+a_k}\right),$$

and since $\displaystyle\sum_{k=1}^{n}F_k = F_{n+2}-1$, it is enough to prove that $\displaystyle\sum_{k=1}^{n}\frac{1}{1+a_k} < \frac{1}{2}$.

We will prove by induction that $\displaystyle\sum_{k=1}^{n}\frac{1}{1+a_k} = \frac{\frac{a_{n+1}-1}{2}-1}{a_{n+1}-1}$, which is less than $\frac{1}{2}$.

Clearly it is true for $n=1$. Let us suppose it holds for n. Then, for $n+1$ we have

$$\begin{aligned}
\sum_{k=1}^{n+1}\frac{1}{1+a_k} &= \sum_{k=1}^{n}\frac{1}{1+a_k}+\frac{1}{1+a_{n+1}} \\
&= \frac{\frac{a_{n+1}-1}{2}-1}{a_{n+1}-1}+\frac{1}{1+a_{n+1}} \quad \text{by hypotesis of induction} \\
&= \frac{\frac{a_{n+1}^2-1}{2}-a_{n+1}-1+a_{n+1}-1}{a_{n+1}^2-1} = \frac{\frac{a_{n+1}^2-1}{2}-2}{a_{n+1}^2-1}
\end{aligned}$$

$$= \frac{\frac{2a_{n+2}-2}{2} - 2}{2a_{n+2} - 2} \quad \text{by the definition of sequence } \{a_n\}$$
$$= \frac{\frac{a_{n+2}-1}{2} - 1}{a_{n+2} - 1}.$$

And, therefore, the conclusion follows.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

By the Cauchy-Schwarz inequality applied to the vectors $\left(\sqrt{F_1}, \cdots, \sqrt{F_n}\right)$ and $\left(\frac{1}{\sqrt{1+a_1}}, \cdots, \frac{1}{\sqrt{1+a_n}}\right)$, we have for $n \geq 1$

$$\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 = \left(\sum_{k=1}^{n} \sqrt{F_k} \frac{1}{\sqrt{1+a_k}}\right)^2 \leq \left(\sum_{k=1}^{n} F_k\right)\left(\sum_{k=1}^{n} \frac{1}{1+a_k}\right) \qquad (1)$$

The sequence $\{a_n\}_{n\geq 1}$ is related to the Sylvester sequence $\{b_n\}_{n\geq 1}$ defined by $b_1 = 2$ and for $n \geq 1, b_{n+1} = b_n^2 - b_n + 1$, by the equality $b_n = \frac{1}{2}(a_n + 1)$, and it is known that the sum of the reciprocals of the Sylvester sequence is 1. So for $n \geq 1$, we have that

$$\sum_{k=1}^{n} \frac{1}{1+a_k} < \sum_{k=1}^{n} \frac{1}{1+a_k} = \sum_{k=1}^{\infty} \frac{1}{2b_k} = \frac{1}{2}. \qquad (2)$$

From (1) and (2) and the property $1 + \sum_{k=1}^{n} F_k = F_{n+2}$, it follows that, for $n \geq 1$

$$1 + 2\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 \leq 1 + 2\left(\sum_{k=1}^{n} F_k\right)\left(\sum_{k=1}^{n} \frac{1}{1+a_k}\right) < 1 + \sum_{k=1}^{n} F_k = F_{n+2}.$$

Solution 3 by Roberto de la Cruz Moreno, Centre de Recerca Matematica, Campus de Bellaterra, Barcelona, Spain

Lemma. Let $\{b_n\}_{n\geq 1}$ be the sequence of real numbers defined by $b_1 = 5$ and for all $n \geq 1$, $b_{n+1} = \frac{1}{2}(b_n^2 + 1)$. Then:

$$\sum_{k=1}^{m} \frac{1}{b_k + 1} = \frac{1}{4} - \frac{1}{b_{m+1} - 1}, \quad \forall m \in \mathcal{Z}^+$$

Proof. By induction:
$m = 1$:

$$\frac{1}{b_1 + 1} = \frac{1}{6} = \frac{1}{4} - \frac{1}{12} = \frac{1}{4} - \frac{1}{b_2 - 1}$$

$m \Rightarrow m + 1$:

$$\sum_{k=1}^{m+1} \frac{1}{b_k + 1} = \sum_{k=1}^{m} \frac{1}{b_k + 1} + \frac{1}{b_{m+1} + 1} = \frac{1}{4} - \frac{1}{b_{m+1} - 1} + \frac{1}{b_{m+1} + 1}$$
$$= \frac{1}{4} - \frac{2}{b_{m+1}^2 - 1} = \frac{1}{4} - \frac{1}{b_{m+2} - 1}$$

Corollary. $\sum_{k=1}^{m} \frac{1}{a_k+1} < \frac{1}{2}, \ \forall m \in \ \mathcal{Z}^+$

By Cauchy-Schwarz inequality:

$$\begin{aligned} 1+2\left(\sum_{k=1}^{n}\sqrt{\frac{F_k}{1+a_k}}\right)^2 &\leq 1+2\left(\sum_{i=1}^{n}F_i\right)\left(\sum_{j=1}^{n}\frac{1}{1+a_j}\right) \\ &= 1+2(F_{n+2}-1)\left(\sum_{j=1}^{n}\frac{1}{1+a_j}\right) < F_{n+2} \end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Adrian Naco, Polytechnic University, Tirana, Albania, and the proposer.

- **5253:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy} dxdy.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the double integral equals $\frac{\pi^4}{30}$.

For $s, t \geq 0$ we have

$$\begin{aligned} \int_0^1 \int_0^1 \frac{x^{s+t}y^s}{1-xy} dxdy &= \int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^{k+s+t}y^{k+s} dxdy \\ &= \sum_{k=0}^{\infty} \int_0^1 \int_0^1 x^{k+s+t}y^{k+s} dxdy \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+s+t+1)(k+s+1)}. \end{aligned}$$

Differentiating with respect to t, then setting $t = 0$, we obtain

$$\int_0^1 \int_0^1 \frac{\ln x . x^s y^s}{1-xy} dxdy = \sum_{k=0}^{\infty} \frac{-1}{(k+s+1)^3}$$

Differentiating with respect to s, then setting $s = 0$, we obtain

$$\int_0^1 \int_0^1 \frac{\ln x . \ln(xy)}{1-xy} dxdy = 3\sum_{k=0}^{\infty} \frac{1}{(k+1)^4}.$$

Now it is well known that the sum $\sum_{k=0}^{\infty} \frac{1}{(k+1)^4}$ equals $\frac{\pi^4}{90}$. Hence the result.

Solution 2 by Anastasios Kotronis, Athens, Greece

We have

$$
\begin{aligned}
\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}\,dx\,dy \quad & \overset{1}{=} \quad \int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}\,dy\,dx = \int_0^1 \int_0^1 \sum_{k\geq 0} (xy)^k \ln x \ln(xy)\,dy\,dx \\
& \overset{xy=u}{=} \quad \int_0^1 \frac{\ln x}{x} \int_0^x \sum_{k\geq 0} u^k \ln u\,du\,dx \\
& \overset{2}{=} \quad \int_0^1 \frac{\ln x}{x} \sum_{k\geq 0} \int_0^x u^k \ln u\,du\,dx \\
& = \quad \int_0^1 \frac{\ln x}{x} \sum_{k\geq 0} \left(\frac{u^{k+1}}{k+1} \ln u \Big|_0^x - \frac{1}{k+1} \int_0^x u^k\,du \right) dx \\
& = \quad \int_0^1 \sum_{k\geq 0} \frac{x^k}{k+1} \ln^2 x\,dx - \int_0^1 \sum_{k\geq 0} \frac{x^k}{(k+1)^2} \ln x\,dx \\
& \overset{3}{=} \quad \sum_{k\geq 0} \int_0^1 \frac{x^k}{k+1} \ln^2 x\,dx - \sum_{k\geq 0} \int_0^1 \frac{x^k}{(k+1)^2} \ln x\,dx
\end{aligned}
$$

But integrating by parts twice and once respectively,

$$\int_0^1 x^k \ln^2 x\,dx = \frac{2}{(k+1)^3} \quad \text{and} \quad \int_0^1 x^k \ln x\,dx = -\frac{1}{(k+1)^2},$$

so

$$\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}\,dx\,dy = 3\sum_{k\geq 0} \frac{1}{(k+1)^4} = 3\zeta(4) = \frac{\pi^4}{30}.$$

Notes:
[1]From Fubini's theorem $< http://on.wikipedia.org/wiki/Fubini\#27 >$,
since the integrand doesn't change sign.
[2] Again from Fubini's theorem
[3] Again from Fubini's theorem

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, University of Tor Vergata Roma, Rome, Italy, and the proposer.

Mea Culpa

Enkel Hysnelaj of the University of Technology in Sydney Australia and Elton Bojaxhiu of Kriftel, Germany were inadvertently omitted from the list of those having solved problem 5232 that appeared in the April issue this column. Once again, sorry.

Problems
Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2014

- **5271:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with $\overline{AB} = x, \overline{BC} = y$, and $\overline{BD} = 2\overline{AD} = 2\overline{CD}$.

  Express the radius of the circum-circle in terms of x and y.

- **5272:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The Jacobsthal numbers begin $0, 1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 0$. Prove that there are infinitely many Pythagorean triples like $(3, 4, 5)$ and $(13, 84, 85)$ that have "hypotenuse" a Jacobsthal number.

- **5273:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Solve in the positive integers the equation $abcd + abc = (a+1)(b+1)(c+1)$.

- **5274:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

  Let x, y, z, α be real positive numbers. Show that if

  $$\sum_{cyclic} \frac{(n+1)\,x^3 + nx}{x^2+1} = \alpha$$

  then

  $$\sum_{cyclic} \frac{1}{x} > \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2}$$

  where n is a natural number.

- **5275:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations

$$\left.\begin{array}{r}
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n}=x_2\sqrt{2}, \\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n}=x_3\sqrt{2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n}=x_n\sqrt{2}, \\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n}=x_1\sqrt{2},
\end{array}\right\}$$

where $n \geq 2$.

- **5276:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $a \in (0,1]$ be a real number. Calculate

$$\int_0^1 a^{\left\lfloor \frac{1}{x} \right\rfloor} dx,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

(b) Calculate

$$\int_0^1 2^{-\left\lfloor \frac{1}{x} \right\rfloor} dx.$$

Solutions

- **5254:** *Proposed by Kenneth Korbin, New York, NY*

Five different triangles, with integer length sides and with integer area, each have a side with length 169. The size of the angle opposite 169 is the same in all five triangles. Find the sides of the triangles.

Solution 1 by Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Iran

Let a, b and c be the lengths of three sides of the triangles, A is the measure of the angle opposite the side of length 169, and S is the area of triangle. Note that, given the conditions in the hypothesis, $\cos A$ must be a rational number based on the Law of Cosines. We found eleven such triangles $(S, \cos A, a, b, c)$, where $S = \sqrt{(p(p-a)(p-b)(p-c)}$ and $p = \dfrac{a+b+c}{2}$. They are as follows:

1. (2184, 84/85, 105, 169, 272)

2. (8580, 84/85, 169, 264, 425)

3. (18720, 84/85, 169, 425, 576)

4. (26364, 84/85, 169, 520, 663)

5. (30030, 84/85, 169, 561, 700)

6. (62244, 84/85, 169, 855, 952)

7. (65910, 84/85, 169, 884, 975)

8. (73554, 84/85, 169, 943, 1020)

9. (83694, 84/85, 169, 1020, 1073)

10. (90090, 84/85, 169, 1071, 1100)

11. (92274, 84/85, 169, 1092, 1105)

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will assume that $a = 169$ and b and c are the other two sides, with $b \leq c$. Since b and c are to be integers, the Law of Cosines dictates that $\cos A$ is to be rational. Also, the requirement that each triangle is to have integral area insures that $\sin A$ must be rational (using the formula $Area = \frac{1}{2}bc\sin A$). One way to achieve both and still satisfy $\sin^2 A + \cos^2 A = 1$ is to make

$$\cos A = \frac{x}{z} \quad \text{and} \quad \sin A = \frac{y}{z}$$

for some Pythagorean triple (x, y, z). After experimenting with several triples, we had the best results by choosing

$$\cos A = \frac{84}{85} \quad \text{and} \quad \sin A = \frac{13}{85}.$$

Then, the Law of Cosines yields

$$\begin{aligned} 169^2 &= b^2 + c^2 - 2bc\left(\frac{84}{85}\right) \\ &= c^2 - \frac{168}{85}bc + \left(\frac{84}{85}b\right)^2 + b^2 - \left(\frac{84}{85}b\right)^2 \end{aligned}$$

$$= \left(c - \frac{84}{85}b\right)^2 + \left(\frac{13}{85}b\right)^2,$$

which reduces to

$$(85c - 84b)^2 + (13b)^2 = [(169)(85)]^2.$$

(Note that the assumption $b \leq c$ makes $85c - 84b > 0$.)

We now know that $(13b, 85c - 84b, (169)(85))$ must be a Pythagorean triple and hence, there are positive integers k, m, n such that

$$m > n,\ \gcd(m, n) = 1,\ m \not\equiv n \pmod 2,\ \text{and}\ (169)(85) = k\left(m^2 + n^2\right).$$

Then, either $13b = 2kmn$ and $85c - 84b = k(m^2 - n^2)$ or $13b = k(m^2 - n^2)$ and $85c - 84b = 2kmn$. E. g., when $k = (13)(85) = 1,105$, we get $m = 3$ and $n = 2$. When we set $13b = 2(1,105)(3)(2)$ and $85c - 84b = 1,105(3^2 - 2^2)$, we obtain $b = 1,020$ and $c = 1,073$ while the assignment $13b = 1,105(3^2 - 2^2)$ and $85c - 84b = 2(1,105)(3)(2)$ yields $b = 425$ and $c = 576$. Proceeding in this way, we found 11 feasible values for the sides b and c. Each presented an integral area for the triangle and each resulted in $\cos A = \dfrac{84}{85}$ (by the Law of Cosines). Since $\cos x$ is injective on $[0, \pi]$, each of our solutions produced the same value for $\angle A$. Our results are summarized in the following table.

k	m	n	a	b	c	*Area*
13	33	4	169	264	425	$8,580$
13	32	9	169	943	$1,020$	$73,554$
$5 \cdot 13$	14	5	169	855	952	$62,244$
$5 \cdot 13$	11	10	169	105	272	$2,184$
13^2	7	6	169	$1,092$	$1,105$	$92,274$
$13 \cdot 17$	8	1	169	$1,071$	$1,100$	$90,090$
$13 \cdot 17$	7	4	169	561	700	$30,030$
$5 \cdot 13^2$	4	1	169	520	663	$26,364$
$5 \cdot 13 \cdot 17$	3	2	169	$1,020$	$1,073$	$83,694$
$5 \cdot 13 \cdot 17$	3	2	169	425	576	$18,720$
$13^2 \cdot 17$	2	1	169	884	975	$65,910$

Comment by editor: **David Stone and John Hawkins of Georgian Southern University in Statesboro GA** exhibited two families of triangles satisfying the conditions of the problem. The first family contained 11 triangles with the angle opposite the side of length 169 having a common value of $\cos^{-1}\left(\dfrac{84}{85}\right) = 8.7974°$. The triangles that they obtained for this family are exhibited in the above solutions. But in their second family they listed 5 additional triangles for which the angle opposite the side of length 169 have a common value of $\cos^{-1}\left(\dfrac{1517}{1525}\right) = 5.8713°$

They obtained their triangles by denoting the sides of the triangles as $(a, b, 169)$ with $a \geq b$ and the angle θ opposite 169, and then they used the following tools:

1) Law of cosines, $\cos\theta = \dfrac{a^2 + b^2 - 169^2}{2ab}$.

2) Triangle Inequality: $-169 \leq b - a \leq 169$; thus, for any given value of a it must be that $a - 169 \leq a$.

3) Heron's formula: with $s = \dfrac{a+b+169}{2}$, and where $s(s-a)(s-b)(s-169)$ is a perfect square. That is, where $\left[(a+b)^2 - 169^2\right]\left[169^2 - (a+b)^2\right]$ is a perfect square.

4) MATLAB and Excel. They coded nested loops to find values of a and b which satisfy (2) and (3) and then computed $\cos\theta$ by (1). Then they put the results into an Excel file and sorted by $\cos\theta$. From there they said: it was easy to see the families sharing a common angle.

They wrote: For $a, b \leq 40,000$ we found 262 triangles with integer sides and integer area and having 169 as a side. In our table, we have only listed the two families containing five or more elements with a common angle opposite 169. For each triangle we also show its area. They then listed the above table and made observations on it. They wrote: the last triangle in the family $(a = 1105, b = 1092, c = 169)$ is a (13,84,85) right triangle magnified by 13. They also noted that two triangles have sides 169 and 425, while two others have two sides of 169 and 1020 (an appearance of the SSA or Ambiguous case from Trigonometry!).

They then listed their second table and made the following comments on it.

a	b	c	*Area*
350	183	169	3276
1037	900	169	47736
1525	1452	169	113256
1582	1525	169	123396
1625	1586	169	131820

Empirically, the common angle (opposite 169) equals

$$\cos^{-1}\left[\frac{350^2 + 183^2 - 169^2}{2\,(350)\,(183)}\right] = \cos^{-1}\left(\frac{1517}{1525}\right) \approx 5.8713^\circ.$$

Comment 1: We did not have complete confidence in trusting floating point arithmetic to give us triangles with an identical angle. For instance, to see that $(272, 105, 169)$ and $(425, 264, 169)$ have the same angle opposite the side of length 169, we must have

$$\frac{272^2 + 105^2 - 169^2}{2(272)(105)} = \frac{425^2 + 264^2 - 169^2}{2(425)(264)}.$$

Cross-multiplying, we can check this with **integer** arithmetic:

$$(425)\,(264)\left[272^2 + 105^2 - 169^2\right] = 6333465600 - (272)\,(105)\left[425^2 + 262^2 - 169^2\right].$$

In each of our families, we checked the first entry again each other triangle to verify true equality of angles.

Comment 2: Our MATLAB file ran a and b up to 40,000, but found no solutions near this peak value. We do not believe that there are any more such triangles (other than the 262 we found.)

Comment 3: There is a nice geometric way to visualize each family of triangles. We explain by focusing on the first group of 11 triangles. Let two rays OA and OB emanate from a vertex O, separated by our angle $\approx 8.7974°$. Starting at O, mark off the "a values" along OA and the "b values" along OB. For instance, designate A_1 as the point 272 units along OA and B_1 the points 105 units along OB. We have drawn our first triangle –the distance A_1 to B_1 across the "wedge" is 169. Similarly we have A_2= 425 and B_2=264, and the distance A_2 to B_2 across the "wedge" is 169.

Eventually, we will draw all eleven of our triangles in the wedge in nested fashion. Because the distance across the wedge will eventually surpass 169, no more triangles are possible. So we have a nice geometric argument that any such family of triangles must be finite. (In fact, by trigonometry, the maximum value for a (and b) to form an isosceles triangle with this angle and bridge 169 is approximately 1101.75. Note that the largest triangle in this family is near this limiting size.)

Finally, note that each of the quadrilaterals $A_iA_jB_jB_i,\ 1 \le i < j \le 11$ has integer sides and integer area and a pair of opposing sides equal to 169. For instance, the quadrilateral $A_1A_2B_2B_1$ has sides $\left(\overline{A_1A_2}, \overline{A_2B_2}, \overline{B_1B_2}, \overline{B_1A_1}\right) = (A_2 - A_1, 169, B_2 - B_1, 169) = (153, 169, 159, 169)$ and area $Area(\triangle A_2OB_2) - (\triangle A_1OB_1) = 8580 - 2184 = 6396$. An almost unimaginable family of 55 such quadrilaterals.

Also solved by Brian E. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5255:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let n be a natural number. Let $\phi(n), \sigma(n)$ and $\tau(n)$ be the Euler phi-function, the sum of the different divisors of n and the number of different divisors of n, respectively.
Prove:
(a) $\forall n \ge 2,\ \exists$ natural numbers a and b such that $\phi(a) + \tau(b) = n$.
(b) $\forall k \ge 1,\ \exists$ natural numbers a and b such that $\phi(a) + \sigma(b) = 2^k$.
(c) $\forall n \ge 2,\ \exists$ natural numbers a and b such that $\tau(a) + \tau(b) = n$.
(d) $\forall k \ge 1,\ \exists$ natural numbers a and b such that $\sigma(a) + \sigma(b) = 2^k$.
(e) $\forall n \ge 3,\ \exists$ natural numbers a, b and c such that $\phi(a) + \sigma(b) + \tau(c) = n$
(f) $\exists$ infinitely many natural numbers n such that $\phi(\tau(n)) = \tau(\phi(n))$.

Solution 1 by Brian D. Beasley, Presbyterian Colleg, Clinton, SC

(a) Given $n \ge 2$, let $a = 1$ and $b = 2^{n-2}$. Then $\phi(a) = 1$ and $\tau(b) = n - 1$. (Note that we may take $b = p^{n-2}$ for any prime p.)

(b) Given $k \ge 1$, let $a = 1$ and $b = 2^{k-1}$. Then $\phi(a) = 1$ and
$$\sigma(b) = 1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1.$$

(c) We may use the same a and b as in part (a), since $\tau(1) = \phi(1) = 1$.

(d) We may use the same a and b as in part (b), since $\sigma(1) = \phi(1) = 1$.

(e) Given $n \ge 3$, let $a = b = 1$ and $c = 2^{n-3}$. Then $\phi(a) = \sigma(b) = 1$ and $\tau(c) = n - 2$. (Note that we may take $c = p^{n-3}$ for any prime p.)

(f) Let p be a prime and take $n = 2^{p-1}$. Then
$$\phi(\tau(n)) = \phi(p) = p - 1 \quad \text{and} \quad \tau(\phi(n)) = \tau(2^{p-2}) = p - 1.$$

Since there are infinitely many primes, the result follows.

Solution 2 by Kee-Wai Lau, Hong Kong, China

(a) $\phi(n) + \tau\left(2^{n-\phi(n)-1}\right) = n.$

(b) $\phi(2) + \sigma\left(2^{k-1}\right) = 2^k.$

(c) $\tau(1) + \tau(1) = 2$ and $\tau(2) + \tau\left(2^{n-3}\right) = n$ for $n \geq 3$.

(d) $\sigma(1) + \sigma\left(2^{k-1}\right) = 2^k.$

(e) $\phi(n-1) + \sigma(1) + \tau(2^{n-2-\phi(n-1)}) = n.$

(f) $\phi(\tau(2^{p-1})) = \tau(\phi(2^{p-1}) = p - 1$ for any odd prime p.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We will make use of some well-known rules, where p denotes a prime.

$$\phi(p^m) = p^{m-1}(p-1),\ \sigma(2^m) = 2^{m+1} - 1,\ \text{and } \tau(p^m) = m + 1.$$

(a) For any prime p, $\phi(n) + \tau\left(p^{n-\phi(n)-1}\right) = \phi(n) + [n - \phi(n)] = n.$

(b) $\phi(1) + \sigma(2^{k-1}) = 1 + \left[2^k - 1\right] = 2^k.$

(c) For any pirme p, and any m with $2 \leq m \leq n$, we have

$$\tau(p^{n-m}) + \tau(p^{m-2}) = (n - m + 1) + (m - 2 + 1) = n.$$

(d) $\sigma(2^{k-1}) + \sigma(1) = \left(2^k - 1\right) + 1 = 2^k.$

(e) For any prime p, $\phi(1) + \sigma(1) + \tau(p^{n-3}) = 1 + 1 + (n-2) = n.$

(f) For any prime p, $\phi(\tau(2^{p-1})) = \phi(p) = p - 1$ and
$\tau(\phi(2^{p-1})) = \tau(2^{p-2}) = (p-2) + 1 = p - 1.$ So $\phi(\tau(2^{p-1})) = \tau(\phi(2^{p-1})).$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo Sate University, San Angelo, TX; Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Iran; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5256:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let a be a positive integer. Compute:

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{na}}\right).$$

Solution 1 by Ángel Plaza and Kisin Sadarangani, University de Las Palmas, de Gran Canaria, Spain

Let H_n be the nth harmonic number, that is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Note first that $e^{\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{na}} \to a$ when n tends to infinity, because

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{na} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}} + \frac{\frac{1}{n}}{1+\frac{2}{n}} + \ldots + \frac{\frac{1}{n}}{1+\frac{(a-1)n}{n}} \\
&= \int_0^{a-1} \frac{1}{1+x}\,dx = \ln a.
\end{aligned}$$

The proposed limit may be obtained as follows:

$$\begin{aligned}
\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{na}}\right) &= \lim_{n\to\infty} -an\left(e^{H_{an}-H_n-\ln a} - 1\right) \\
&= \lim_{n\to\infty} -an\left(H_{an} - H_n - \ln a\right) \\
&= \lim_{n\to\infty} -an \cdot \frac{1-a}{2an} = \frac{a-1}{2}.
\end{aligned}$$

Where we have used that $H_n = \ln n + \gamma + \dfrac{1}{2n} - \dfrac{1}{12n^2} + \cdots$, being γ is the Euler-Mascheroni constant. Hence $H_{an} - H_n \sim \ln a + \frac{1}{2an} - \frac{1}{2n} + o\left(\frac{1}{n^2}\right)$.

Solution 2 by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

The limit equals -2 if $a = 1$ and $\frac{a-1}{2}$ if $a > 1$. First we consider the case when $a = 1$. We have,

$$\lim_{n\to\infty} n\left(1 - e^{\frac{1}{n+1}+\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{1-\exp\left(\frac{2n+1}{n(n+1)}\right)}{\frac{2n+1}{n(n+1)}} \cdot \frac{2n+1}{n+1}\right) = -2.$$

Now we consider the case when $a > 1$. We will be using, in our analysis, the following asymptotic expansion for the nth harmonic number (see 1, [Entry 23 p. 59])

$$1 + \frac{1}{n} + \cdots + \frac{1}{n} = \gamma + \ln n + \frac{1}{2n} - \frac{1}{8n^2} + \frac{15}{2n^4} - \cdots \quad (n \to \infty).$$

Equivalently,

$$1 + \frac{1}{n} + \cdots + \frac{1}{n} = \gamma + \ln n + \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

It follows that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{na} = H_{na} - H_n = \ln a + \frac{1-a}{2na} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

Thus

$$\begin{aligned}
n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{na}}\right) &= n\left(a - a \cdot e^{\frac{1-a}{2na}+O\left(\frac{1}{n^2}\right)}\right) \\
&= a \cdot \frac{1-\exp\left(\frac{1-a}{2na} + O\left(\frac{1}{n^2}\right)\right)}{\frac{1-a}{2na} + O\left(\frac{1}{n^2}\right)} \cdot \left(\frac{1-a}{2a} + O\left(\frac{1}{n}\right)\right),
\end{aligned}$$

which in turn implies that

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{na}}\right) = \frac{a-1}{2}.$$

The problem is solved.

[1] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, 2001.

Solution 3 by Ed Gray, Highland Beach, FL

1) Let $S = e^{\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{na}}$

2) $\ln(S) = \dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{an}$

3) $\ln(S) = \displaystyle\sum_{k=1}^{na}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k}.$

We use the Euler's approximation for the partial sum of the harmonic series. That is

4) $T_m = \displaystyle\sum_{k=1}^{m}\frac{1}{k} = \ln(m)+\gamma+\frac{1}{2m}-\frac{1}{12m^2}+\frac{1}{120m^4}-\cdots$, where γ is the Euler-Mascheroni constant $0.577\cdots$

In our approximation, we will only keep the term $\dfrac{1}{2m}$ to avoid unnecessary computations. Then from (3) and (4),

5) $\ln(S) = \ln(na)+\gamma+\dfrac{1}{2na}-\left(\ln(n)+\gamma+\dfrac{1}{2n}\right)$ or

6) $\ln(S) = \ln(na)-\ln(n)+\dfrac{1}{2na}-\dfrac{1}{2n}$

7) $\ln(S) = \ln\left(\dfrac{na}{n}\right)+\dfrac{1}{2na}-\dfrac{1}{2n}$

8) $\ln(S) = \ln a+\dfrac{1}{2na}-\dfrac{1}{2n}$

9) $S = e^{\ln a}\cdot e^{\frac{1}{2na}}\cdot e^{-\frac{1}{2n}}$, or

10) $S = a\cdot e^{\frac{1}{2na}}\cdot e^{-\frac{1}{2n}}$

For large n the exponents are small, and we keep only the first two terms in the expansion for e^y

11) $e^{\frac{1}{2na}} = 1+\dfrac{1}{2na}$

12) $e^{-\frac{1}{2na}} = 1 - \dfrac{1}{2n}$

13) The product is: $1 - \dfrac{1}{2n} + \dfrac{1}{2na} - \dfrac{1}{4an^2}$, and step 10 becomes

14) $S = a - \dfrac{a}{2n} + \dfrac{1}{2n} - \dfrac{1}{4n^2}$. Then

15) $a - S = \dfrac{a}{2n} - \dfrac{1}{2n} + \dfrac{1}{4n^2}$

16) $n(a-S) = \dfrac{a}{2} - \dfrac{1}{2} + \dfrac{1}{4n}$

So the limit as n approaches infinity is $\dfrac{a-1}{2}$.

Solution 4 by Paul M. Harms, North Newton, KS

When m is a positive integer $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{m} = \ln m + \gamma + R(m)$ where γ is the Euler-Mascheroni constant and $R(m)$ is an error term that approaches $\dfrac{1}{2}m$ as m gets large. Let a be a positive integer greater than one. We have

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+1} + \cdots \frac{1}{na} &= 1 + \frac{1}{2} + \cdots \frac{1}{na} - \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right) \\
&= \ln na + \gamma + R(na) - (\ln n + \gamma + (n)) \\
&= \ln a + R(na) - R(n).
\end{aligned}$$

Then the limit in the problem involves

$$n\left(a - e^{\ln a} e^{R(na) - R(n)}\right) = na\left(1 - e^{R(na)-R(n)}\right).$$

For large n this can be approximated by

$$\frac{a\left(1 - 2^{\frac{1}{2}na - \frac{1}{2}n}\right)}{\frac{1}{n}}.$$

Thinking of n as a continuous variable and using L'Hôpital's Rule, the limit of the last expression is the limit of $\left(ae^{\frac{1}{2}na - \frac{1}{2}n}\left(\frac{1}{2}a - \frac{1}{2}\right)\right)$ as $n \to \infty$. The result is $\dfrac{a-1}{2}$.

Solution 5 by G. C. Greubel, Newport News, VA

We are asked to evaluate the limit

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{na}}\right).$$

The primary difficulty is reducing the exponential to some aspect easier to work with. With this in mind consider the series of the exponential. This is given by

$$\phi_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{na}.$$

This can be quickly be seen as

$$\begin{aligned}\phi_n &= \sum_{k=1}^{an}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k}\\ &= H_{an} - H_n\end{aligned}$$

where H_n is the Harmonic number. With this there is a basis to expand upon. In order to proceed further the expansion of a Harmonic number in terms of factors of $1/n$ is required. The required expansion is obtained from Wolfram Mathworld Harmonic numbers site[1] and is given by

$$H_n \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \mathcal{O}\left(\frac{1}{n^8}\right).$$

where γ is Euler's constant. When use of this is made the result becomes

$$\begin{aligned}\phi_n &= \ln(an) - \ln(n) + \frac{1}{2n}\left(\frac{1}{a} - 1\right) - \frac{1}{12n^2}\left(\frac{1}{a^2} - 1\right)\\ &+ \frac{1}{120n^4}\left(\frac{1}{a^4} - 1\right) - \mathcal{O}\left(\frac{1}{n^6}\right)\\ &= \ln a + \frac{(1-a)}{2an} - \frac{(1-a^2)}{12a^2n^2} + \frac{(1-a^4)}{120a^4n^4} - \mathcal{O}\left(\frac{1}{n^6}\right).\end{aligned}$$

Now that a valid approximation for large values of n has been obtained it can be used to reduce the exponential portion of the limit. With this in mind the result becomes

$$\begin{aligned}e^{\phi_n} &= 1 + \phi_n + \frac{1}{2}\phi_n^2 + \cdots\\ &\approx 1 + \left[\ln a + \frac{1-a}{2an} - \frac{1-a^2}{12a^2n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] + \frac{1}{2}\left[\ln^2 a + \frac{2(1-a)\ln a}{4a^2n}\right.\\ &\left.\mathcal{O}\left(\frac{1}{n^2}\right)\right] + \frac{1}{3!}\left[\ln^3 a + \frac{3(1-a)}{2an}\ln^2 a + \mathcal{O}\left(\frac{1}{n^2}\right)\right] + \cdots\\ &\approx e^{\ln a} + \frac{1-a}{2an}\left(1 + \frac{\ln a}{1!} + \frac{\ln^2 a}{2!} + \cdots\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\end{aligned}$$

[1]The Wolfram Mathworld site for Harmonic numbers is found at http://mathworld.wolfram.com/HarmonicNumber.html and is stated as equation (13).

$$\approx \quad e^{\ln a} + \frac{1-a}{2an} e^{\ln a} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$e^{\phi_n} \quad \approx \quad a + \frac{1-a}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

With this result it can now be seen that

$$a - e^{\phi_n} \approx \frac{a-1}{2n} - \mathcal{O}\left(\frac{1}{n^2}\right) \text{ and}$$
$$n\left(a - e^{\phi_n}\right) \approx \frac{a-1}{2} - \mathcal{O}\left(\frac{1}{n}\right).$$

Now the limit is easy to compute and is given by

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{an}}\right) = \frac{a-1}{2}.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Sagueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

5257: *Proposed by Pedro H.O. Pantoja, UFRN, Brazil*

Prove that:

$$1 + \frac{1}{2}\cdot\sqrt{1+\frac{1}{2}} + \frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}} + \cdots + \frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} \sim \ln(n),$$

where $f(x) \sim g(x)$ means $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.

Solution 1 by Arkady Alt, San Jose, CA

Let $S_n = 1 + \frac{1}{2}\cdot\sqrt{h_2} + \frac{1}{3}\cdot\sqrt[3]{h_3} + ... + \frac{1}{n}\cdot\sqrt[n]{h_n}$,where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$.

Since $\dfrac{1}{k+1} < \ln(k+1) - \ln k < \dfrac{1}{k} \left(\iff \left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1} \right)$ then

$\sum\limits_{k=1}^{n} (\ln(k+1) - \ln k) < h_n \iff \ln(n+1) < h_n$ and $h_k - 1 < \sum\limits_{k=2}^{n} (\ln k - \ln(k-1)) \iff h_n < 1 + \ln n$

and, therefore, $\dfrac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = \dfrac{\frac{\sqrt[n]{h_n}}{n}}{\ln\left(1+\frac{1}{n-1}\right)} =$

$$\frac{\sqrt[n]{h_n}}{\ln\left(1+\frac{1}{n-1}\right)^n} \in \left(\frac{\sqrt[n]{\ln(n+1)}}{\ln\left(1+\frac{1}{n-1}\right)^n}, \frac{\sqrt[n]{\ln n + 1}}{\ln\left(1+\frac{1}{n-1}\right)^n}\right).$$

Since $\lim_{n\to\infty} \sqrt[n]{\ln(n+1)} = 1$, $\lim_{n\to\infty} \sqrt[n]{1+\ln n} = 1$, $\lim_{n\to\infty} \ln\left(1+\frac{1}{n-1}\right)^n = 1$ then

$\lim_{n\to\infty} \frac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = 1$ and by Stolz Theorem we obtain

$$\lim_{n\to\infty} \frac{S_n}{\ln n} = \lim_{n\to\infty} \frac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = 1.$$

Solution 2 by Ángel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain

Let L be the $\lim_{n\to\infty} \frac{1+\frac{1}{2}\cdot\sqrt{1+\frac{1}{2}}+\frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}}+\cdots+\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln(n)}$.

Since $\lim_{n\to\infty} \ln(n) = \infty$, by the Stolz-Cesàro theorem,

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln(n)-\ln(n-1)} \\ &= \lim_{n\to\infty} \frac{\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln\left(\frac{n}{n-1}\right)^n}. \end{aligned}$$

Note that $\lim_{n\to\infty} \sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{1+\frac{1}{2}+\cdots+\frac{1}{n-1}} = 1$, by the Stolz-Cesàro theorem, and also that $\lim_{n\to\infty} \ln\left(\frac{n}{n-1}\right)^n = 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that $1+\frac{1}{2}+\cdots+\frac{1}{n} = \ln n + O(1)$ as $n\to\infty$ and $\ln(1+x) = x + O\left(x^2\right)$, $e^x = 1 = x + O\left(x^2\right)$ as $x \to 0$. Hence

$$\frac{\ln\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{n} = \frac{\ln\ln n}{n} + O\left(\frac{1}{n\ln n}\right)$$

and

$$\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} = \frac{1}{n}e^{\ln\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)/n} = \frac{1}{n}\left(1+\frac{\ln\ln n}{n} + O\left(\frac{1}{n\ln n}\right)\right).$$

Since $\sum_{n=3}^{\infty} \frac{\ln\ln n}{n^2}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converge, so

$$1+\frac{1}{2}\cdot\sqrt{1+\frac{1}{2}}+\frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}}+\cdots+\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}=\ln(n)+O(1),$$

and we are done.

Editor's comment: **D. M. Bătinetu-Giurgiu, of the "Matei Basarab" National College in Bucharest, Romania and Neculai Stanciu, of George Emil Palade School in Buzău, Romania,** submitted two solutions to the problem. Their first solution was similar in approach to the second solution presented above, but in their second solution they generalized the problem as follows:

If $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences of positive real numbers such that:

- $\{y_n\}_{n\geq 1}$ is increasing and unbounded,
- $\exists t \in \Re_+$ such that $\lim_{n\to\infty} n^t\{y_{n+1}-y_n\} = a \in \Re_+$,
- $\lim_{n\to\infty} n^t x_n = a$ exists $\in \Re_+$, and $z_n = \sum_{k=1}^{n} x_k$, then

  $\{y_n\}_{n\geq 1} \sim \{z_n\}_{n\geq 1}$. I.e., $\lim_{n\to\infty} \frac{z_n}{y_n}$.

Proof. By the Cesaro-Stolz theorem we have:

$$\lim_{n\to\infty}\frac{z_n}{y_n}=\lim_{n\to\infty}\frac{z_{n+1}-z_n}{y_{n+1}-y_n}=\lim_{n\to\infty}\frac{x_{n+1}}{y_{n+1}-y_n}=\lim_{n\to\infty}\frac{(n+1)^t x_{n+1}}{\left(\frac{n+1}{n}\right)^t n^t(y_{n+1}-y_n)}=\frac{a}{1\cdot a}=1.$$

Remark: If we take $y_n=\ln n$, $h_n=\sum_{k=1}^{n}\frac{1}{k}$, $x_n=\frac{1}{n}\sqrt[n]{h_n}$, and $z_n=\sum_{k=1}^{n}x_k$, then by the above we obtain $\{y_n\}_{n\geq 1}\sim\{z_n\}_{n\geq 1}$ which is problem 5257.

Also solved by Bruno Sagueiro Fanego, Viveiro, Spain; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy, and the proposer.

5258: *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers such that $1+\sum_{k=1}^{n}\cos^2\alpha_k = n$. Prove that:

$$\sum_{1\leq i<j\leq n}\tan\alpha_i\tan\alpha_j \leq \frac{n}{2}.$$

Solution 1 by Arkady Alt, San Jose, CA

Let $x_i = \tan^2 \alpha_i, i = 1, 2, ..., n$ then $x_i \geq 0, i = 1, 2, ..., n, 1 + \sum\limits_{k=1}^{n} \cos^2 \alpha_k = n \iff$

$\sum\limits_{k=1}^{n} \dfrac{1}{1+x_i} = n - 1$ and, since $\sum\limits_{1 \leq i < j \leq n}^{n} \tan \alpha_i \tan \alpha_j \leq \sum\limits_{1 \leq i < j \leq n}^{n} |\tan \alpha_i| \, |\tan \alpha_j| = \sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{x_i x_j}$,

then it is sufficient to prove $\sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{x_i x_j} \leq \dfrac{n}{2}$.

Let $a_i = \dfrac{x_i}{1+x_i}, 1, 2, .., n$ then $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} \left(1 - \dfrac{1}{1+x_i}\right) = n - \sum\limits_{i=1}^{n} \dfrac{1}{1+x_i} = 1$ and,

since $x_i = \dfrac{a_i}{1-a_i}, 1, 2, ..., n$ our problem is:

Prove inequality $\sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{\dfrac{a_i a_j}{(1-a_i)(1-a_j)}} \leq \dfrac{n}{2}$, for $a_i \geq 0, i = 1, 2, ..., n$ such

that $\sum\limits_{k=1}^{n} a_i = 1$.

We have $\sum\limits_{1 \leq i < j \leq n}^{n} \sqrt{\dfrac{a_i a_j}{(1-a_i)(1-a_j)}} \leq \sum\limits_{1 \leq i < j \leq n}^{n} \dfrac{1}{2} \left(\dfrac{a_j}{1-a_i} + \dfrac{a_i}{1-a_j}\right) =$

$$\frac{1}{2}\left(\sum_{1 \leq i < j \leq n}^{n} \frac{a_j}{1-a_i} + \sum_{1 \leq i < j \leq n}^{n} \frac{a_i}{1-a_j}\right) = \frac{1}{2}\left(\sum_{i=1}^{n-1} {}_{j=i+1}^{\,n} \frac{a_j}{1-a_i} + \sum_{j=2}^{n} {}_{i=1}^{\,j} \frac{a_i}{1-a_j}\right) =$$

$$\frac{1}{2}\left(\sum_{i=1}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j + \sum_{j=2}^{n} \frac{1}{1-a_j} \sum_{i=1}^{j-1} a_i\right) = \frac{1}{2} \cdot \frac{1}{1-a_1} \sum_{j=2}^{n} a_j + \frac{1}{2} \sum_{i=2}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j +$$

$$\sum_{j=2}^{n-1} \tfrac{1}{1-a_j} \sum_{i=1}^{j-1} a_i + \frac{1}{1-a_n} \sum_{i=1}^{n-1} a_i = 1 + \frac{1}{2}\left(\sum_{i=2}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j + {}_{j=2}^{n-1} \frac{1}{1-a_j} \sum_{i=1}^{j-1} a_i\right) =$$

$$1 + \frac{1}{2}\left(\sum_{i=2}^{n-1} \frac{1}{1-a_i} \sum_{j=i+1}^{n} a_j + \sum_{i=2}^{n-1} \frac{1}{1-ai} \sum_{j=1}^{i-1} a_i\right) = 1 + \frac{1}{2}\left(\sum_{i=2}^{n-1} \frac{1}{1-a_i}\left(\sum_{j=i+1}^{n} a_j + \sum_{j=1}^{i-1} a_i\right)\right) =$$

$$1 + \frac{1}{2} \sum_{i=2}^{n-1} \frac{1-a_i}{1-a_i} = 1 + \frac{n-2}{2} = \frac{n}{2}.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy

Proof: We first note that if

$\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$, $\alpha_n = \pi/2$, the constraints of the problem are satisfied, but

$$\sum_{1 \leq i < j \leq n} \tan \alpha_i \tan \alpha_j$$

is undefined; so we add the assumption $\alpha_i \neq \pi/2 + 2k\pi$, $k \in Z$,

$i = 1, \ldots, n$. Both $\cos^2 x$ and $\tan x$ are π–periodic so we can assume

$\alpha_i \in (-\pi/2, \pi/2)$ and set $\alpha_i = \arctan a_i$. This yields

$$1 + \sum_{k=1}^{n} \frac{1}{1+a_k^2} = n \implies \sum_{1 \le i<j \le n} a_i a_j \le n.$$

By defining $a_k = \sqrt{b_k}$ the inequality becomes

$$2 \sum_{1 \le i<j \le n} \sqrt{b_i b_j} \le n \qquad \text{whenever} \qquad \sum_{i=1}^{n} \frac{1}{b_i+1} = n-1.$$

By convexity of $1/(1+x)$ for $x > 0$ we have

$$n-1 = \sum_{i=1}^{n} \frac{1}{b_i+1} \le \frac{n}{1+\dfrac{b_1+\ldots+b_n}{n}},$$

that is, $b_1 + \ldots + b_n \le n/(n-1)$. Now

$$2 \sum_{1 \le i<j \le n} \sqrt{b_i b_j} \le \sum_{1 \le i<j \le n} (b_i + b_j) = (n-1)(b_1 + \ldots + b_n) \le (n-1)n/(n-1) = n,$$

and we are done.

Solution 3 Adrian Naco, Polytechnic University, Tirana, Albania.

Let $x_i = \tan \alpha_i, \forall i \in \{1, 2, \ldots.., n\}$. Applying the trigonometric formula, $\cos^2 \alpha_i = \dfrac{1}{1+\tan^2 \alpha_i}$, the condition and the initial inequality give respectively,

$$1 + \sum_{1}^{n} \frac{1}{1+x_i^2} = n \qquad \text{and} \qquad \sum_{1 \le i<j \le n} x_i x_j \le \frac{n}{2}.$$

Let us assume

$$a_i = \frac{1}{(n-1)(x_i^2+1)} \quad \Rightarrow \quad x_i^2 = \tfrac{1-(n-1)a_i}{(n-1)a_i} \quad \text{and} \quad \sum_{i=1}^{n} a_i = 1$$

$$y_{i,j} = \frac{1-(n-1)a_i}{(n-1)a_j} \quad \Rightarrow \quad y_{i,j} y_{j,i} = x_i^2 x_j^2 \quad \text{and} \quad y_{i,j} \ge 0, \forall i, j$$

Thus we have that

$$2 \sum_{1 \le i<j \le n} x_i x_j \quad \le \quad 2 \sum_{1 \le i<j \le n} |x_i||x_j| = 2 \sum_{1 \le i<j \le n} \sqrt{x_i^2 x_j^2}$$

$$
\begin{aligned}
&\le 2\sum_{1\le i<j\le n}\frac{1}{2}\Big(x_i^2+x_j^2\Big)=\sum_{1\le i<j\le n}\Big(x_i^2+x_j^2\Big)\\
&= \sum_{1\le i<j\le n}\left[\frac{1-(n-1)a_i}{(n-1)a_j}+\frac{1-(n-1)a_j}{(n-1)a_i}\right]\\
&= \frac{1}{n-1}\sum_{1\le i<j\le n}\left[\frac{1}{a_j}+\frac{1}{a_i}\right]-\sum_{1\le i<j\le n}\left[\frac{a_j}{a_i}+\frac{a_i}{a_j}\right]\\
&= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{1}{a_j}+\frac{1}{a_i}\right]-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{a_j}{a_i}+\frac{a_i}{a_j}\right]\\
&= \frac{1}{n-1}\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{a_j}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{a_i}\right]-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \frac{1}{n-1}\left[\sum_{i=1}^{n-1}(n-i)\frac{1}{a_i}+\sum_{i=2}^{n}(i-1)\frac{1}{a_i}\right]-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \frac{1}{n-1}\sum_{i=1}^{n}(n-1)\frac{1}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\frac{1}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{n}a_j\right)}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}+\sum_{i=1}^{n}\frac{a_i}{a_i}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_j}{a_i}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{a_i}{a_j}\\
&= \sum_{i=1}^{n}\frac{a_i}{a_i}=\sum_{i=1}^{n}1=n.
\end{aligned}
$$

Finally we have that

$$\sum_{1\le i<j\le n}\tan\alpha_i\tan\alpha_j=\sum_{1\le i<j\le n}x_ix_j\le\frac{n}{2}.$$

The equality holds for $x_i = \tan\alpha_i = \tan\alpha_j = x_j, 1 \le i < j \le n$ or equivalently for $\alpha_i = k\pi + \alpha_j, 1 \le i < j \le n, k \in Z$.

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that $n = 1 + \sum_{k=1}^{n} \cos^2\alpha_k = 1 + \sum_{k=1}^{n}(1 - \sin^2\alpha_k) = 1 + \sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \sin^2\alpha_k = 1 + n - \sum_{k=1}^{n} \sin^2\alpha_k \iff \sum_{k=1}^{n} \sin^2\alpha_k = 1 \iff \sum_{k=1}^{n} \frac{\tan^2\alpha_k}{1 + \tan^2\alpha_k} = 1$, and that the inequality to prove is equivalent to $\left(\sum_{k=1}^{n} \tan\alpha_k\right)^2 - \sum_{k=1}^{n} \tan^2\alpha_k = 2 \sum_{1 \le i < j \le n} \tan\alpha_i \tan\alpha_j \le n = \sum_{k=1}^{n} 1 \iff \left(\sum_{k=1}^{n} \tan\alpha_k\right)^2 \le \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \tan^2\alpha_k = \sum_{k=1}^{n} \left(1 + \tan^2\alpha_k\right) \iff \frac{\left(\sum_{k=1}^{n} \tan\alpha_k\right)^2}{\sum_{k=1}^{n} \left(1 + \tan^2\alpha_k\right)} \le 1 = \sum_{k=1}^{n} \frac{\tan^2\alpha_k}{+\tan^2\alpha_k}$ which is just Bergström′s inequality $\frac{\left(\sum_{k=1}^{n} a_k\right)^2}{\sum_{k=1}^{n} b_k} \le \sum_{k=1}^{n} \frac{a_k^2}{b_k}$ applied to $a_k = \tan\alpha_k \in \Re$ and $b_k = 1 + \tan^2\alpha_k \in \Re;\ 1 \le k \le n$.

Equality occurs if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$, that is if and only if $\frac{1}{2}\sin(2\alpha_1) = \frac{1}{2}\sin(2\alpha_2) = \cdots = \frac{1}{2}\sin(2\alpha_n)$, and $\sum_{k=1}^{n} \sin^2\alpha_k = 1$.

Also solved by the proposers.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2014

- **5277:** *Proposed by Kenneth Korbin, New York, NY*

  Find x and y if a triangle with sides $(2013, 2013, x)$ has the same area and the same perimeter as a triangle with sides $(2015, 2015, y)$.

- **5278:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The triangular numbers $6 = (2)(3)$ and $10 = (2)(5)$ are each twice a prime number. Find all triangular numbers that are twice a prime.

- **5279:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzu, Romania*

  Let $f : \Re_+ \longrightarrow \Re_+$ be a convex function on $\Re_+$, where $\Re_+$ stands for the positive real numbers. Prove that

  $$3\left(f^2(x)+f^2(y)_+f^2(z)\right)-9f^2\left(\frac{x+y+z}{3}\right) \geq (f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2.$$

- **5280:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a \geq b \geq c$ be nonnegative real numbers. Prove that

  $$\frac{1}{3}\left(\frac{(a+b)(c+a)}{2+\sqrt{a+b}}+\frac{(c+a)(b+c)}{2+\sqrt{c+a}}+\frac{(b+c)(a+b)}{2+\sqrt{b+c}}\right) \leq \frac{(a+b)^2}{2+\sqrt{b+c}}.$$

- **5281:** *Proposed by Arkady Alt, San Jose, CA*

  For the sequence $\{a_n\}_{n\geq 1}$ defined recursively by $a_{n+1} = \dfrac{a_n}{1+a_n^p}$ for $n \in \mathcal{N}, a_1 = a > 0$, determine all positive real p for which the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

- **5282:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate
$$\int_0^1 x \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)dx.$$

Solutions

- **5259:** *Proposed by Kenneth Korbin, New York, NY*

Find a, b, and c such that with $a<b<c$, $\begin{cases} ab+bc+ca & =-2 \\ a^2b^2+b^2c^2+c^2a^2 & = \ \ \ 6 \\ a^3b^3+b^3c^3+c^3a^3 & =-11. \end{cases}$

Solution 1 by Arkady Alt, San Jose, CA

Let $s=a+b+c$, $p=ab+bc+ca$, and $q=abc$. Then a,b,c are the roots of the equation $x^3-sx^2+px-q=0$. Since,

$$\begin{aligned} 6 &= a^2b^2+b^2c^2+c^2a^2=p^2-2sq=4-2sq \text{ and} \\ -11 &= a^3b^3+b^3c^3+c^3a^3=3q^2+p^3-3spq=3q^2-8+6sq, \text{ then} \\ sq &= -1 \text{ and } q^2=1 \iff q=1 \text{ or } q=-1. \end{aligned}$$

Thus we obtain $(s,p,q)=(-1,-2,1),(1,-2,-1)$ and, respectively, the two equations

$$x^3+x^2-2x-1=0 \quad\text{and}\quad x^3-x^2-2x+1=0.$$

Since,

$$\begin{aligned} (-x)^3+(-x)^2-2(-x)-1=0 &\iff x^3-x^2-2x+1=0, \text{ and} \\ x^3+x^2-2x-1=0 &\iff x=1.2470,\ -0.44504,\ -1.8019, \end{aligned}$$

we see that,

$$(a,b,c) = (-1.8019,\ -0.44504,\ 1.2470),\ (-1.2470,\ 0.44504,\ 1.8019).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

As in problem 5135, let $x=ab$, $y=bc$ and $z=ca$, so that
$x+y+z=-2$, $x^2+y^2+z^2=6$, and $x^3+y^3+z^3=-1$. We have

$$abc(a+b+c)=xy+yz+zx=\frac{(x+y+z)^2-x^2-y^2-z^2}{2}=\frac{(-2)^3-6}{2}=-1, \text{ and}$$

$$(abc)^3=xyz=\frac{x^3+y^3+z^3-(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}{3}=\frac{-11+2(6+1)}{3}=1.$$

Hence, either $\begin{cases} a+b+c &= -1 \\ ab+bc+ca &= 2 \\ abc &= 1 \end{cases}$ or $\begin{cases} a+b+c &= 1 \\ ab+bc+ca &= 2 \\ abc &= -1. \end{cases}$

In the former case a, b, and c are the roots of the polynomial t^3+t^2-2t-1, and in the latter case, the roots of the polynomial t^3-t^2-2t+1. By the trigonometric method to find the roots of a cubic polynomial equation, we obtain respectively

$$\begin{aligned} a &= \frac{2\sqrt{7}}{3}\cos\left(\frac{\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)+2\pi}{3}\right)-\frac{1}{3} \approx -1.80194, \\ b &= \frac{2\sqrt{7}}{3}\cos\left(\frac{\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)+4\pi}{3}\right)-\frac{1}{3} \approx -0.445042, \text{and} \\ c &= \frac{2\sqrt{7}}{3}\cos\left(\frac{\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)}{3}\right)-\frac{1}{3} \approx 1.24698 \end{aligned}$$

$a \approx -1.24698,\ b \approx 0.445042$, and $c \approx 1.80194$.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, label the equations as follows: $\begin{cases} ab+bc+ca &= -2 \quad (1) \\ a^2b^2+b^2c^2+c^2a^2 &= 6 \quad (2) \\ a^3b^3+b^3c^3+c^3a^3 &= -11. \quad (3) \end{cases}$

Then, by (1) and (2),

$$\begin{aligned} 4 &= (ab+bc+ca)^2 \\ &= a^2b^2+b^2c^2+c^2a^2+2\left(ab^2c+bc^2a+ca^2b\right) \\ &= 6+2abc\,(a+b+c) \ \text{ and hence,} \\ abc\,(a+b+c) &= -1. \quad (4) \end{aligned}$$

Next, use (1), (2), (3), and (4) to obtain

$$\begin{aligned} -12 &= (ab+bc+ca)\left(a^2b^2+b^2c^2+c^2a^2\right) \\ &= a^3b^3+b^3c^3+c^3a^3+ab^3c^2+a^3bc^2+a^2b^3c \\ &\quad +a^2bc^3+a^3b^2c+ab^2c^3 \\ &= -11+abc\,[ab\,(a+b)+bc\,(b+c)+ca\,(c+a)] \\ &= -11+abc\,[(ab+bc+ca)\,(a+b+c)-3abc] \end{aligned}$$

$$
\begin{aligned}
&= -11 + abc\left[-2\left(a+b+c\right)\right] - 3\left(abc\right)^2 \\
&= -9 - 3\left(abc\right)^2 \text{ or} \\
\left(abc\right)^2 &= 1. \qquad (5)
\end{aligned}
$$

It follows from (4) and (5) that either $abc = 1$ and $a+b+c = -1$ or $abc = -1$ and $a+b+c = 1$. Since

$$\left(x-a\right)\left(x-b\right)\left(x-c\right) = x^3 - \left(a+b+c\right)x^2 + \left(ab+bc+ca\right)x - abc,$$

a, b, c must be the solutions of either

$$x^3 + x^2 - 2x - 1 = 0 \qquad (6)$$

or

$$x^3 - x^2 - 2x + 1 = 0 \qquad (7)$$

We will utilize a method for solving (6) described on pg. 59 of [1]. The solutions of (7) can then be found by making an appropriate adjustment in this method. Let $R = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. Then, as a 7^{th} root of unity, R has several useful properties:

- 1. Since $R^7 = 1$, we have

$$1 + R + R^2 + R^3 + R^4 + R^5 + R^6 = \frac{R^7 - 1}{R - 1} = 0.$$

- 2. For $k = 1, \ldots, 7$,

  a) $\frac{1}{R^k} = R^{7-k}$

  b) $R^k = R^{7+k}$

  c) $R^k + \frac{1}{R^k} = 2\text{Re}\left(R^k\right)$.

Pair the powers of R as follows:

$$
\begin{aligned}
x_1 &= R + R^6 = R + \frac{1}{R} = 2\cos\frac{2\pi}{7}, \\
x_2 &= R^2 + R^5 = R^2 + \frac{1}{R^2} = 2\cos\frac{4\pi}{7} = -2\cos\frac{3\pi}{7}, \\
x_3 &= R^3 + R^4 = R^3 + \frac{1}{R^3} = 2\cos\frac{6\pi}{7} = -2\cos\frac{\pi}{7}.
\end{aligned}
$$

Then, since

$$x_1 + x_2 + x_3 = R + R^2 + R^3 + R^4 + R^5 + R^6 = -1,$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \left(R^3 + R^6 + R^8 + R^{11}\right) + \left(R^5 + R^6 + R^8 + R^9\right)$$

$$
\begin{aligned}
&\quad + \left(R^4 + R^9 + R^5 + R^{10}\right) \\
&= \left(R^3 + R^6 + R + R^4\right) + \left(R^5 + R^6 + R + R^2\right) \\
&\quad + \left(R^4 + R^2 + R^5 + R^3\right) \\
&= 2\left(R + R^2 + R^3 + R^4 + R^5 + R^6\right) \\
&= -2, \text{ and} \\
x_1x_2x_3 &= \left(R + R^6\right)\left(R^5 + R^6 + R + R^2\right) \\
&= R^6 + R^7 + R^2 + R^3 + R^{11} + R^{12} + R^7 + R^8 \\
&= 2 + R + R^2 + R^3 + R^4 + R^5 + R^6 \\
&= 1,
\end{aligned}
$$

x_1, x_2, x_3 must be the solutions of (6). The condition $a < b < c$ then implies that one possible solution of our system is $a = -2\cos\frac{\pi}{7},\ b = -2\cos\frac{3\pi}{7}$, and $c = 2\cos\frac{2\pi}{7}$.

Similarly, if

$$
\begin{aligned}
y_1 &= -x_1 = -2\cos\frac{2\pi}{7}, \\
y_2 &= -x_2 = 2\cos\frac{3\pi}{7}, \text{ and} \\
y_3 &= -x_3 = 2\cos\frac{\pi}{7}, \text{ then,} \\
y_1 + y_2 + y_3 &= -(x_1 + x_2 + x_3) = 1, \\
y_1y_2 + y_2y_3 + y_3y_1 &= x_1x_2 + x_2x_3 + x_3x_1 = -2, \text{ and} \\
y_1y_2y_3 &= -x_1x_2x_3 = -1.
\end{aligned}
$$

Therefore, y_1, y_2, y_3 are the solutions of (7). Again, since $a < b < c$, the remaining possible solution of our system is $a = -2\cos\frac{2\pi}{7},\ b = 2\cos\frac{3\pi}{7}$, and $c = 2\cos\frac{\pi}{7}$.

To show that neither solution is extraneous, we note first that since

$$y_1y_2 + y_2y_3 + y_3y_1 = x_1x_2 + x_2x_3 + x_3x_1 = -2,$$

we have

$$ab + bc + ca = -2$$

in both cases. Further, the conditions

$$x_1 + x_2 + x_3 = -1, \quad x_1x_2x_3 = 1$$

and

$$y_1 + y_2 + y_3 = 1, \quad y_1 y_2 y_3 = -1$$

imply that

$$(abc)^2 = 1 \quad \text{and} \quad abc\,(a+b+c) = -1$$

in both cases. It follows that both solutions also satisfy

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (ab+bc+ca)^2 - 2abc\,(a+b+c) \\ &= 4+2 \\ &= 6 \end{aligned}$$

and

$$\begin{aligned} a^3b^3 + b^3c^3 + c^3a^3 &= (ab+bc+ca)\left(a^2b^2+b^2c^2+c^2a^2\right) \\ &\quad -abc\,(ab+bc+ca)\,(a+b+c) + 3\,(abc)^2 \\ &= (-2)\,(6) - (-1)\,(-2) + 3 \\ &= -11. \end{aligned}$$

Hence, our solutions for (6) and (7) both satisfy the original system as well.

Reference:

[1] Benjamin Bold, *Famous Problems of Geometry and How to Solve Them*, Dover Publications, Inc., 1969.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, Fl; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5260:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Find all primes p and q such that $a^{pq-1} \equiv a \pmod{pq}$, for all a relatively prime to pq.

Solution 1 by Ken Korbin, New York, NY

Let $p = 2$ and q be any odd prime.

$$\begin{aligned} \phi(pq) &= \phi(2q) = q-1 \\ (a, pq) &= 1, \text{ therefore} \end{aligned}$$

$$\begin{aligned}
a^{\phi(pq)} &\equiv 1\,(\text{mod } pq)\\
a^{q-1} &\equiv 1\,(\text{mod } pq)\\
\left[a^{q-1}\right]\cdot\left[a^{q-1}\right] &\equiv 1\cdot 1\,(\text{mod } pq)\\
a^{2q-2} &\equiv 1\,(\text{mod } pq)\\
a\cdot a^{2q-2} &\equiv a\cdot 1\,(\text{mod } pq)\\
a^{2q-1} &\equiv a\,(\text{mod } pq)\,, \text{therefore}\\
a^{pq-1} &\equiv a\,(\text{mod } pq)\,,\ \text{if } p\ = 2 \text{ and } q \text{ is any odd prime.}
\end{aligned}$$

Solution 2 by Kee-Wai Lau,Hong Kong, China

We show that primes p and q satisfy $a^{pq-1} \equiv a \pmod{pq}$ for all a relatively prime to pq, if and only if at least one of them is 2.

We need only that

I. For any prime q, $a^{2q-1} \equiv a \pmod{2q}$, for all a relatively prime to $2q$.

II. If $p \le q$ are odd primes, then $a^{pq-1} \not\equiv a \pmod{pq}$ if $a > 1$ is a primitive root modulo q.

If $(a, 2q) = 1$, then $a^{q-1} + 1$ is even and by Fermat's little theorem, we have
$a^{q-1} - 1 \equiv 0 \pmod{2q}$. Hence

$$a^{2q-1} - a = a(a^{q-1} + 1)(a^{q-1} - 1) \equiv 0 \pmod{2q}.$$

This proves I. We now prove II.

Suppose, on the contrary, that $a > 1$ is a primitive root modulo q such that

$$a^{pq-1} \equiv a \pmod{pq}. \qquad (1)$$

By Fermat's little theorem we have

$$\begin{aligned}
a^{pq-1} &= a^{p-1}(a^{q-1})^p\\
&= a^{p-1}(1 + kq)^p\\
&= a^{p-1}\sum_{j=0}^{p}\binom{p}{j}(kq)^j \text{ for some positive integer } k.
\end{aligned}$$

(3)

It is well known that p divides $\binom{p}{j}$ for $j = 1, 2, \cdots, p-1$. Hence

$$a^{pq-1} \equiv a^{p-1}(1 + k^p q^p) \pmod{pq}. \qquad (2)$$

From (1) and (2), we see that

$$a^{p-1} \equiv a \pmod{q}. \qquad (3)$$

Since a is a primitive root modulo q, so $a^r \not\equiv a \pmod{q}$ for $r = 2, 3, \cdots, q-1$.

Since $p > 2$, so by (3) we have $p - 1 \geq q$, which contradicts the fact tht $p \leq q$. This proves II and completes the solution.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5261:** *Proposed by Michael Brozinsky, Central Islip, NY*

Show without calculus or trigonometric functions that the shortest focal chord of an ellipse is the latus rectum.

Solution 1 by Paul M. Harms, North Newton, KS

Any ellipse can be placed on a coordinate system so that the equation of the ellipse is $\dfrac{(x+c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > b$. One focal point is at $(0,0)$. I will consider the focal chords through $(0,0)$.

Focal chords with slope m are on the line $y = mx$. The x values of the points of intersection of the ellipse and the line $y = mx$ come from the equation $\dfrac{(x+c)^2}{a^2} + \dfrac{m^2x^2}{b^2} = 1$ which yields the quadratic equation $\left(a^2m^2 + b^2\right)x^2 + 2b^2cx - b^4 = 0$, where $b^4 = b^2\left(a^2 - c^2\right)$.

If $H = \sqrt{b^4c^2 + (a^2m^2 + b^2)b^4}$, the x solutions are $\dfrac{-b^2c + H}{a^2m^2 + b^2}$ and $\dfrac{-b^2c - H}{a^2m^2 + b^2}$.

Let the intersection points of the focal chord and the ellipse be (x_1, y_1) and (x_2, y_2). To determine the shortest focal chord, I will look for the minimum of the square of the distance L between (x_1, y_1) and (x_2, y_2).

Here $L = (y_2 - y_1)^2 + (x_2 - x_1)^2$. Since the points are on $y = mx$ we have $y_2 - y_1 = m(x_2 - x_1)$ and $L = (x_2 - x_1)^2(m^2 + 1)$. The points x_1 and x_2 are the two solutions of the quadratic equation given above.

We have

$$\begin{aligned}
(x_2 - x_1)^2 &= \left(\frac{2H}{a^2m^2 + b^2}\right)^2 \text{ and } L = (x_2 - x_1)^2(m^2 + 1) \\
&= \frac{4b^4(c^2 + a^2m^2 + b^2)}{(a^2m^2 + b^2)^2}(m^2 + 1) \\
&> \frac{4b^4(a^2m^2 + b^2)(m^2 + 1)}{(a^2m^2 + b^2)^2}
\end{aligned}$$

$$= \frac{\frac{4b^4}{a^2}\left(m^2+1\right)}{m^2+\left(\frac{b}{a}\right)^2}$$

$$> \frac{4b^4}{a^2}(1).$$

The last inequality occurs since $0 < \frac{b}{a} < 1$.

Thus any focal chord with slope m has the square of its length greater than $\frac{4b^4}{a^2}$, which is the square of the length of the vertical chord and the latus rectum. The conclusion of the problem follows.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let F be one of the foci, d the directrix closest to F, e the eccentricity, and M, N, L points on the ellipse such that MN is a focal chord (that is, $F \in MN$) and L is one of the endpoints of the latus rectum ($LF||d$) and M', N', L', F' the respective projections of M, N, L, on d.

We want to prove that the length of the focal chord MN is greater or equal to the length of the latus rectum that is, that $MN \geq 2LF$.

Since the distance of any point on the ellipse to F is equal to e times its distance to d, we have that $MN = MF + FN = eMM' + eNN' = e(MM' + NN')$ and $LF = eLL'$, so we want to prove that $MM' + NN' \geq 2LL'$.

By Thales' theorem $\frac{MM'}{FF'} = \frac{FF'}{NN'}$ that is $MM' \cdot NN' = (FF')^2$. So by the arithmetic mean-geometric mean inequality

$$MM' + NN' \geq 2\sqrt{MM' \cdot NN'} = 2FF'$$

with equality if, and only if, $MM' = NN'$, that is if, and only if, MN coincides with the latus rectum, as we wanted to prove.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5262:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

Prove that the equation $\varphi(10x^2) + \varphi(30x^3) + \varphi(34x^4) = y^2 + y^3 + y^4$ has infinitely many solutions for $x, y \in N$ where $\varphi(x)$ is the Euler-φ function.

Solution by Tom Moore, Bridgewater State University, Bridgewater, MA

Let $x = 2^k$. Then,

$$\varphi(10x^2) = \varphi\left(5 \cdot 2^{2k+1}\right) = \varphi(5)\varphi\left(2^{2k+1}\right) = 4 \cdot 2^{2k} = 2^{2k+2} = \left(2^{k+1}\right)^2.$$

$$\begin{aligned}\varphi(30x^3) &= \varphi\left(2\cdot 5\cdot 6\cdot 2^{3k}\right)=\varphi(5)\varphi(3)\varphi\left(2^{3k}\right)=8\cdot 2^{3k}=2^{2k+3}=\left(2^{k+1}\right)^3.\\ \varphi(34x^4) &= \varphi\left(2\cdot 17\cdot 2^{4k}\right)=\varphi(17)\varphi\left(2^{4k}\right)=16\cdot 2^4k=2^{4k+4}=\left(2^{k+1}\right)^4.\end{aligned}$$

So, we have infinitely many solutions$(x,y)=\left(2^k,2^{k+1}\right),\ k\geq 0$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Ken Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5263:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let a,b,c be positive numbers lying in the interval $(0,1]$. Prove that

$$a\cdot\sqrt{\frac{bc}{1+c+ab}}+b\cdot\sqrt{\frac{ca}{1+a+bc}}+c\cdot\sqrt{\frac{ab}{1+b+ca}}\leq\sqrt{3}.$$

Solution 1 by Ed Gray, Highland Beach, FL

Consider the function $f(x,y,z)=x\sqrt{\dfrac{y}{1+z+xy}}$. Each term in the problem is a representation of f by assigning a,b,c appropriately. Maximizing any term in the problem is equivalent to maximizing f.

Write f as $\sqrt{\dfrac{(x^2)\,yz}{1+z+xy}}$. Define $u=xy$ and f becomes $\sqrt{\dfrac{xuz}{1+z+u}}$. Note that u is in $(0,1]$.

Since x appears alone in the numerator and we wish to maximize the function, we assign to x its largest value possible: that is, $x=1$. The problem now becomes to maximize $\dfrac{uz}{1+z+u}$, for then its square root will attain its maximum.

Define $z+u=2t$, where t is in $(0,1]$. It is well know that the maximum of the product zu is t^2. Since if

$$\begin{aligned} r &= zu=u(2t-u)=2tu-u^2.\\ \frac{dr}{du} &= 2t-2u=0\Longrightarrow u=t,\ \text{and } z=t.\\ &\frac{uz}{1+z+u}\text{ becomes }\frac{t^2}{1+2t}.\end{aligned}$$

Since the derivative of this last term is greater than zero, it attains its maximum for $t=1$ and is $\dfrac{1}{3}$.

Therefore the maximum of the left hand side of the statement of the problem is

$$3\sqrt{\frac{1}{3}}=3\sqrt{\frac{3}{9}}=\frac{3}{3}\sqrt{3}\leq\sqrt{3}.\ \text{ Q.E.D.}$$

Solution 2 by Adrian Naco, Polytechnic University,Tirana, Albania.

Considering the left side of the last inequality and applying the wellknown AM-GM inequality we have that

$$a\cdot\sqrt{\frac{bc}{1+c+ab}}+b\cdot\sqrt{\frac{ca}{1+a+bc}}+c\cdot\sqrt{\frac{ab}{1+b+ca}}=$$

$$=\sqrt{abc}\left[\frac{\sqrt{a}}{\sqrt{1+c+ab}}+\frac{\sqrt{b}}{\sqrt{1+a+bc}}+\frac{\sqrt{c}}{\sqrt{1+b+ca}}\right]\leq$$

$$\leq\sqrt{abc}\left[\frac{\sqrt{a}}{\sqrt{3}\sqrt[6]{abc}}+\frac{\sqrt{b}}{\sqrt{3}\sqrt[6]{abc}}+\frac{\sqrt{c}}{\sqrt{3}\sqrt[6]{abc}}\right]$$

$$=\frac{\sqrt[3]{abc}}{\sqrt{3}}\left[\sqrt{a}+\sqrt{b}+\sqrt{c}\right]\leq\frac{\sqrt[3]{1}}{\sqrt{3}}\left[\sqrt{1}+\sqrt{1}+\sqrt{1}\right]=\sqrt{3}$$

since

$$\begin{aligned}
1+c+ab\geq 3\sqrt[3]{1\cdot c\cdot ab}=3\sqrt[3]{abc} \quad &\Rightarrow \quad \frac{1}{\sqrt{1+c+ab}}\leq\frac{1}{\sqrt{3}\sqrt{\sqrt[3]{abc}}}\\
1+a+bc\geq 3\sqrt[3]{1\cdot a\cdot bc}=3\sqrt[3]{abc} \quad &\Rightarrow \quad \frac{1}{\sqrt{1+a+bc}}\leq\frac{1}{\sqrt{3}\sqrt{\sqrt[3]{abc}}}\\
1+b+ca\geq 3\sqrt[3]{1\cdot b\cdot ca}=3\sqrt[3]{abc} \quad &\Rightarrow \quad \frac{1}{\sqrt{1+b+ca}}\leq\frac{1}{\sqrt{3}\sqrt{\sqrt[3]{abc}}}
\end{aligned}$$

The equality holds for $a=b=c=1$

Solution 3 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\left(\sum_{\text{cyclic}} a\cdot\sqrt{\frac{bc}{1+c+ab}}\right)^2 &\leq \left(\sum_{\text{cyclic}} a^2\right)\left(\sum_{\text{cyclic}}\frac{bc}{1+c+ab}\right)\\
&\leq 3\left(\sum_{\text{cyclic}}\frac{bc}{ac+bc+ab}\right)=3.
\end{aligned}$$

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The concavity of $\sqrt{x}$ yields

$$\sum_{\text{cyc}} a\sqrt{\frac{bc}{1+c+ab}}=(a+b+c)\sum_{\text{cyc}}\frac{a}{a+b+c}\sqrt{\frac{bc}{1+c+ab}}\leq$$

$$\leq (a+b+c)\sqrt{\sum_{\text{cyc}} \frac{a}{a+b+c}\frac{bc}{1+c+ab}} \leq \sqrt{3}.$$

Squaring we get

$$(abc)(a+b+c)\sum_{\text{cyc}} \frac{1}{1+c+ab} \leq 3.$$

Now define $x = 1/a \geq 1$, $y = 1/b \geq 1$, $z = 1/c \geq 1$. We have

$$\frac{xy+yz+zx}{xyz}\sum_{\text{cyc}} \frac{1}{z+xy+xyz} \leq 3,$$

and moreover

$$\frac{xy+yz+zx}{xyz}\sum_{\text{cyc}} \frac{1}{z+xy+xyz} \leq \frac{xy+yz+zx}{xyz}\sum_{\text{cyc}} \frac{1}{3} \leq 3 \iff 3xyz \geq xy+yz+zx,$$

which follows by $x, y, z \geq 1$.

Solution 5 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since $a, b, c > 0$, the Arithmetic - Geometric Mean Inequality implies that

$$1+c+ab \geq 3\sqrt[3]{abc}.$$

Then, because $0 < a, b, c \leq 1$, we have

$$\begin{aligned}
a \cdot \sqrt{\frac{bc}{1+c+ab}} &= \sqrt{a} \cdot \sqrt{\frac{abc}{1+c+ab}} \\
&\leq \sqrt{a} \cdot \sqrt{\frac{abc}{3\sqrt[3]{abc}}} \\
&= \frac{\sqrt{a} \cdot \sqrt{(abc)^{\frac{2}{3}}}}{\sqrt{3}} \\
&= \frac{\sqrt{a}\sqrt[3]{abc}}{\sqrt{3}} \\
&\leq \frac{1}{\sqrt{3}},
\end{aligned}$$

with equality if and only if $a = b = c = 1$.

Similarly,

$$b \cdot \sqrt{\frac{ca}{1+a+bc}} \leq \frac{1}{\sqrt{3}} \quad \text{and} \quad c \cdot \sqrt{\frac{ab}{1+b+ca}} \leq \frac{1}{\sqrt{3}},$$

with equality in each case if and only if $a = b = c = 1$.

Therefore,

$$
\begin{aligned}
a \cdot \sqrt{\frac{bc}{1+c+ab}} + b \cdot \sqrt{\frac{ca}{1+a+bc}} + c \cdot \sqrt{\frac{ab}{1+b+ca}} \\
\leq 1\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\
= \sqrt{3},
\end{aligned}
$$

with equality if and only if $a = b = c = 1$.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5264:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

Let x, y, z, α be positive real numbers. Show that if

$$\sum_{cyclic} \frac{(n+1)x^3 + nx}{x^2+1} = \alpha$$

then

$$\sum_{cyclic} \frac{1}{x} > \frac{3n}{\alpha} + \frac{(2n-1)\alpha}{3n} + \frac{3n\alpha}{9n^2+\alpha^2}$$

where n is a positive integer. Cyclic means the cyclic permutation of x, y, z (and not x, y, z and α).

Solution by proposer

Doing easy manipulations we have

$$\alpha = \sum_{cycl} \frac{(n+1)x^3 + nx}{x^2+1} = \sum_{cycl} \frac{1}{x} + \sum_{cycl} \frac{-1 + (n-1)x^2 + (n+1)x^4}{x(x^2+1)}.$$

Let $f(x) = \dfrac{-1 + (n-1)x^2 + (n+1)x^4}{x(x^2+1)}$. One easily observes that

$$
\begin{aligned}
f'(x) &= \frac{1 + (n+2)x^2 + (2n+4)x^4 + (n+1)x^6}{x^2(1+x^2)^2} \\
f''(x) &= -\frac{2(1+3x^2+2x^6)}{x^3(1+x^2)^3}.
\end{aligned}
$$

It is obvious that $f'(x) > 0$ and $f''(x) < 0$ for any x that is a positive real number, which implies that the function $f(x)$ is an increasing and concave function in the positive real domain. Applying Jensen's inequality we have

$$\sum_{cycl} \frac{-1 + (n-1)x^2 + (n+1)x^4}{x(x^2+1)} = \sum_{cycl} f(x) \leq 3f\left(\frac{\sum_{cycl} x}{3}\right).$$

Doing easy manipulations, one easily observes that

$$\alpha = \sum_{cycl} \frac{(n+1)x^3 + nx}{x^2+1} = \sum_{cycl} nx + \sum_{cycl} \frac{x^3}{x^2+1} > n \sum_{cycl} x \Longrightarrow \sum_{cycl} x < \frac{\alpha}{2n}.$$

Finally, using the above results we have

$$\begin{aligned}
\sum_{cycl} \frac{1}{x} &= \alpha - \sum_{cycl} \frac{-1 + (n-1)x^2 + (n+1)x^4}{x(x^2+1)} \\
&\geq \alpha - 3f\left(\frac{\sum\limits_{cycl} x}{3}\right) \\
&> \alpha - 3f\left(\frac{\frac{\alpha}{n}}{3}\right) \\
&= \alpha - 3f\left(\frac{\alpha}{3n}\right) \\
&= \frac{3n}{\alpha} + \frac{(2n-1)\alpha}{3n} + \frac{3n\alpha}{9n^2+\alpha^2}
\end{aligned}$$

and this completes the proof.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2014

- **5283:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles that both have perimeter 162 and area 1008.

- **5284:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Prove:

  a) $3^{3^n} + 1 \equiv 0 \text{ mod } 28, \ \forall \text{n} \geq 1$,

  b) $3^{3^n} + 1 \equiv 0 \text{ mod } 532, \ \forall \text{n} \geq 2$,

  c) $3^{3^n} + 1 \equiv 0 \text{ mod } 19684, \ \forall \text{n} \geq 3$,

  d) $3^{3^n} + 1 \equiv 0 \text{ mod } 3208492, \ \forall \text{n} \geq 4$.

- **5285:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

  Let $\{a_n\}_{n\geq 1}$, and $\{b_n\}_n \geq 1$ be positive sequences of real numbers with $\lim\limits_{n\to\infty}(a_{n+1}-a_n) = a \in \Re_+$ and $\lim\limits_{n\to\infty}\dfrac{b_{n+1}}{nb_n} = b \in \Re_+$.
  For $x \in \Re$, calculate

  $$\lim_{n\to\infty}\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right).$$

- **5286:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, an ant is assigned a specific equilateral triangle EFG and three distinct positive numbers $0 < a < b < c$. The ant's job is to find a unique point $P(x, y)$ such that the distances from P to the vertices E, F and G of his triangle are proportionate to $a : b : c$ respectively. Some ants are eternally doomed to an impossible search. Find a relationship between a, b and c that guarantees eventual success; i.e., that such a unique point P actually exists.

- **5287:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let u, v, w, x, y, z be complex numbers. Prove that

$$2\text{Re}(ux + vy + zw) \le 3\left(|u|^2 + |v|^2 + |w|^2\right) + \frac{1}{3}\left(|x|^2 + |y|^2 + |z|^2\right).$$

- **5288:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $a, b, c \ge 0$ be real numbers. Find the value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + ai + bj + c}}.$$

Solutions

- **5265:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = 2014,$$

  with $(x, y) = 1$.

Solution 1 by G. C. Greubel, Newport News, VA

The process to be considered, for a slightly general class of values, can be seen as follows. Consider the equation

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = a \qquad (1)$$

for which rearranging terms and squaring both sides leads to

$$\begin{aligned} 3x^2 - 3xy + y^2 &= (2x - y - a)^2 \\ 3x^2 - 3xy + y^2 &= 4x^2 + y^2 + a^2 + 2(-2xy - 2ax + ay) \end{aligned}$$

or

$$\begin{aligned} y &= \frac{x^2 - 4ax + a^2}{x - 2a} \\ &= \frac{(x^2 - 4ax + 4a^2) - 3a^2}{x - 2a} \\ &= \frac{(x - 2a)^2 - 3a^2}{x - 2a} \\ y &= x - 2a - \frac{3a^2}{x - 2a}. \end{aligned}$$

This equation provides y in terms of x for a given x. The relations for x and y can be put into a "parametric" form by making the substitution

$$u = \frac{3a^2}{x - 2a} \quad \text{and} \quad v = \frac{3a^2}{u}.$$

From this it can now be seen that

$$\begin{aligned} x &= v + 2a \\ y &= v - u \\ uv &= 3a^2. \end{aligned}$$

It is readily seen that the possible factors of a are the primary values used in u and v. This is to say that if a is a product of four integers, say $\{a_i\}_{1 \le i \le 4}$, rasied to powers b_i then

$$uv = 3_1^{a2b_1} a_2^{2b_2} a_3^{2b_3} a_4^{2b_4}$$

and leads to the forms of u and v being of the form

$$u = 3^{\alpha_1} a_1^{\alpha_2} a_2^{\alpha_3} a_3^{\alpha_4} a_4^{\alpha_5} \quad \text{and} \quad v = 3^{\beta_1} a_1^{\beta_2} a_2^{\beta_3} a_3^{\beta_4} a_4^{\beta_5}, \tag{2}$$

where $\alpha_1 + \beta_1 = 1$ and $\alpha_i + \beta_i = 2b_{i-1}$ for $2 \le i \le 5$.

Now, returning to equation (1) it can also be seen in the form

$$(2x - y) - \sqrt{(2x - y)^2 - x(x - y)} = a.$$

Invoking the conditions x and y be positive integers leads to the following conditions. If $x - y = 0$ then this reduces to $0 = a$ which is invalid for all $a \neq 0$. In the case $x - y < 0$ the reduction is seen to be

$$x - |x - y| - \sqrt{(x - |x - y|)^2 + x|x - y|} = a.$$

This equation is also invalid for $a > 0$. The remaining condition $x > y$ is the only option for $a > 0$, $x > 0$ and $y > 0$. In order to be completely valid the statement should be $x > y > 0$ for $a > 0$.

Also by rearranging the equation into the form

$$\sqrt{3x^2 - 3xy + y^2} = 2x - y - a$$

which, for positive integer values x and y, leads to the square root being positive and the condition $2x - y - a \ge 0$ or $2x - y \ge 0$. The conditions $x > y > 0$ and $2x - y \ge 0$ can also be stated as $v > u$ and $u + v + 3a \ge 0$.

Introducing the additional condition $(x, y) = p$ then $p|a$, $p|x$ and $p|y$, or p is the divisor of a, x and y. This condition leads to only relatively prime solutions are considered as solutions of this particular problem.

$$\mathbf{a = 2014}.$$

With $a = 2014$ it is quickly seen that the factors are 2, 19, and 53, i.e., $a = 2 \cdot 19 \cdot 53$ and $3a^2 = 3 \cdot 2^2 \cdot 19^2 \cdot 53^2$. The possible factors from this factorable set, in view of equation (2), is seen by:

factors of $uv = 3(2014)^2$

u	v	u	v
$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$
$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^2$
$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$
$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$
$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$
$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$
$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$

Invoking the condition $v > u$ then the possible values are preceded by an asterisk $*$:

factors of $uv = 3(2014)^2$

u	v	u	v
$*3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$*3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$*3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$*3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$
$*3^0 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$*3^1 \cdot 2^0 \cdot 19^2 \cdot 53^2$	$*3^1 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$*3^0 \cdot 2^0 \cdot 19^2 \cdot 53^2$
$*3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$*3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$*3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$*3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$
$*3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$*3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$
$*3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$*3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$
$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$
$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$

These eight value pairs for u and v lead to the eight value pairs of x and y, with $(x, y) = 1$, being

x	y
12,172,616	12,168,587
4,060,224	4,056,193
3,046,175	3,042,143
1,018,077	1,014,037
37,736	33,347
15,264	10,153
12,455	6,983
8,360	1,523.

a = 2015

With $a = 2015$ it is quickly seen that the factors are $5, 13$, and 31, i.e., $a = 5 \cdot 13 \cdot 31$ and $3a^2 = 3 \cdot 5^2 \cdot 13^2 \cdot 31^2$. The possible factors from this factorable set, in view of equation (2), is seen by:

factors of $uv = 3(2015)^2$

u	v	u	v
$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^2$
$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^0$	$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^2$
$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^2$
$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^0$
$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^2$
$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^0$
$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^0$

Invoking the condition $v > u$ then the possible values are preceded by an asterisk $*$:,

factors of $uv = 3(2015)^2$

u	v	u	v
$*3^0\cdot 5^0\cdot 13^0\cdot 31^0$	$*3^1\cdot 5^2\cdot 13^2\cdot 31^2$	$*3^1\cdot 5^0\cdot 13^0\cdot 31^0$	$*3^0\cdot 5^2\cdot 13^2\cdot 31^2$
$3^0\cdot 5^2\cdot 13^0\cdot 31^0$	$*3^1\cdot 5^0\cdot 13^2\cdot 31^2$	$*3^1\cdot 5^2\cdot 13^0\cdot 31^0$	$*3^0\cdot 5^0\cdot 13^2\cdot 31^2$
$3^0\cdot 5^0\cdot 13^2\cdot 31^0$	$*3^1\cdot 5^2\cdot 13^0\cdot 31^2$	$*3^1\cdot 5^0\cdot 13^2\cdot 31^0$	$*3^0\cdot 5^2\cdot 13^0\cdot 31^2$
$3^0\cdot 5^0\cdot 13^0\cdot 31^2$	$3^1\cdot 5^2\cdot 13^2\cdot 31^0$	$*3^1\cdot 5^0\cdot 13^0\cdot 31^2$	$*3^0\cdot 5^2\cdot 13^2\cdot 31^0$
$3^0\cdot 5^2\cdot 13^2\cdot 31^0$	$3^1\cdot 5^0\cdot 13^0\cdot 31^2$	$3^1\cdot 5^2\cdot 13^2\cdot 31^0$	$3^0\cdot 5^0\cdot 13^0\cdot 31^2$
$3^0\cdot 5^2\cdot 13^0\cdot 31^2$	$3^1\cdot 5^0\cdot 13^2\cdot 31^0$	$3^1\cdot 5^2\cdot 13^0\cdot 31^2$	$3^0\cdot 5^0\cdot 13^2\cdot 31^0$
$3^0\cdot 5^2\cdot 13^2\cdot 31^2$	$3^1\cdot 5^0\cdot 13^0\cdot 31^0$	$3^1\cdot 5^2\cdot 13^2\cdot 31^2$	$3^0\cdot 5^0\cdot 13^0\cdot 31^0$

These eight value pairs for u and v lead to the eight value pairs of x and y, with $(x, y) = 1$, being

x	y
12,184,705	12,180,674
4,064,255	4,060,222
491,257	487,202
166,439	162,334
76,105	71,906
28,055	23,518
8,255	1,342

Solution 2 by Ercole Suppa, Teramo, Italy

The given equation is equivalent to

$$\begin{aligned}(2x-y-2014)^2 &= 3x^2-3xy+y^2 \quad \Leftrightarrow\\ x^2-xy-8056x+4028y+2014^2 &= 0 \quad \Leftrightarrow\\ y &= x-4028-\frac{3\cdot 2014^2}{x-4028} \qquad (1)\end{aligned}$$

where x, y are positive integers such that $\gcd(x, y) = 1$ and $2x - y \geq 2014$.

Since y is integer, we have that $x = 4028 + d$ where d is a divisor of $3 \cdot 2014^2$

Furthermore, since $y > 0$ we have

$$(x-4028)^2 > 3\cdot 2014^2 \quad \Leftrightarrow \quad x > 4028 + 2014\sqrt{3}.$$

Therefore $d > 2014\sqrt{3}$ and the possible values of x are:

$$\begin{aligned}x \in \{&8056, 8360, 9646, 10070, 12455, 15264, 16112, 20882, 23161, 37736,\\ &42294, 57399, 61427, 80560, 110770, 118826, 164141, 217512, 233624, 324254,\\ &644480, 1018077, 2032126, 3046175, 4060224, 6088322, 12172616\}. \qquad (2)\end{aligned}$$

By using (1) and (2), a simple check shows that the only pairs (x, y) such that $2x - y \geq 2014$ and $\gcd(x, y) = 1$ are:

$$\begin{aligned}&\{(8360, 1523), (12455, 6983), (15264, 10153),\\ &(37736, 33347), (1018077, 1014037), (3046175, 3042143)\\ &(4060224, 4056193), (12172616, 12168587)\}.\end{aligned}$$

Solution 3 by Brian D. Beasley Presbyterian College, Clinton, SC

We seek to solve the equation $2x - y - \sqrt{3x^2 - 3xy + y^2} = c$ for any positive integer c. Examining this equation for various values of c, we note the following two patterns of solutions:

(1) Let $x = 3c^2 + 2c$ and $y = 3c^2 - 1$. It is then straightforward to verify that

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = 3c^2 + 4c + 1 - \sqrt{(3c^2 + 3c + 1)^2} = c.$$

Next, let $d = \gcd(x, y)$. If $d > 1$, then there is a prime p such that p divides d. Thus p divides $c(3c + 2)$, so either p divides c or p divides $3c + 2$. But p also divides $3c^2 - 1$, so p cannot divide c. Hence p divides $3c + 2$, but p also divides $x - y = 2c + 1$ and thus divides $2(3c + 2) - 3(2c + 1) = 1$, a contradiction. We therefore conclude that $\gcd(x, y) = 1$.

(2) Let $x = c^2 + 2c$ and $y = c^2 - 3$. (To keep $y > 0$, we assume $c > 1$ here.) It is then straightforward to verify that

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = c^2 + 4c + 3 - \sqrt{(c^2 + 3c + 3)^2} = c.$$

Next, we note that if 3 divides c, then $\gcd(x, y) \geq 3$, so we assume that 3 does not divide c in this case. Let $d = \gcd(x, y)$. If $d > 1$, then there is a prime p such that p divides d. Thus p divides $c(c + 2)$, so either p divides c or p divides $c + 2$. But p also divides $c^2 - 3$, so p cannot divide c, since $p \neq 3$ in this case. Hence p divides $c + 2$, but p also divides $x - y = 2c + 3$ and thus divides $2(c + 2) - (2c + 3) = 1$, a contradiction. We therefore conclude that $\gcd(x, y) = 1$.

Since $c = 2014$ for the given equation and 3 does not divide 2014, this approach produces two solutions:

$$x = 12,172,616 \quad \text{and} \quad y = 12,168,587;$$
$$x = 4,060,224 \quad \text{and} \quad y = 4,056,193.$$

Addendum. This approach generates at least one solution for each value of c, with at least two solutions when 3 does not divide c (and when $c > 1$). However, it does not find all solutions, and it does not necessarily find the smallest solution.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, NY; David Stone and John Hawkins, Southern Georgia University, Statesborogh,GA, and the proposer.

- **5266:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The pentagonal numbers begin $1, 5, 12, 22, \cdots$ and in general satisfy $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The positive Jacobsthal numbers, which have applications in tiling and graph matching problems, begin $1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 1$. Prove that there exists infinitely many pentagonal numbers that are the sum of three Jacobsthal numbers.

Solution 1 by Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie,

Angelo State University, San Angelo, TX

For $n \geq 1$, let $k_n = \frac{2}{3}\left(2^{2n-1}+1\right) = 2J_{2n-1}$. Then,

$$\begin{aligned}
P_{k_n} &= \frac{k_n\left(3k_n-1\right)}{2} \\
&= \frac{1}{2}\cdot\frac{2}{3}\left(2^{2n-1}+1\right)\left[2\left(2^{2n-1}+1\right)-1\right] \\
&= \frac{\left(2^{2n-1}+1\right)\left(2^{2n}+1\right)}{3}, \quad \text{while} \\
J_{2n-1}+J_{2n}+J_{4n-1} &= \frac{1}{3}\left[\left(2^{2n-1}+1\right)+\left(2^{2n}-1\right)+\left(2^{4n-1}+1\right)\right] \\
&= \frac{2^{2n-1}+1+2^{4n-1}+2^{2n}}{3} \\
&= \frac{\left(2^{2n-1}+1\right)+2^{2n}\left(2^{2n-1}+1\right)}{} \\
&= \frac{\left(2^{2n-1}+1\right)\left(2^{2n}+1\right)}{3}.
\end{aligned}$$

Therefore, for all $n \geq 1$,

$$J_{2n-1}+J_{2n}+J_{4n-1} = P_{2J_{2n-1}}.$$

Solution 2 by Ed Gray, Highland Beach, FL

The sum of two consecutive Jacobsthal numbers is a power of two since

$$\frac{2^x-(-1)^x}{3}+\frac{2^{x+1}-(-1)^{x+1}}{3} = \frac{1}{3}\left(2^x+2^{x+1}\right) = \frac{1}{3}\left(2^x\right)(1+2) = 2^x.$$

Therefore we need to prove that

$$(1) \quad 2^x + \frac{\left(2^a-(-1)^a\right)}{3} = \frac{n(3n-1)}{2}$$

has infinitely many solutions.

Let a be odd so that $a+1=2L$
Multiplying (1) by 6 gives us

$$(2) \qquad 6(2^x)+2^{a+1}+2 = 3n(3n-1), \text{ or}$$

$$(3) \qquad 9n^2-3n-2^{a+1}-2-6(2^x) = 0.$$

This is a quadratic in n whose solution is by the quadratic formula :

$$(4) \qquad 18n = 3+\sqrt{9+36\left(6(2^x)+2^{a+1}+2\right)}$$

The discriminate D is given by

(5) $D^2 = 81 + 36(2^{a+1}) + 216(2^x))$

(6) Consider$D = 9 + 6(2^L)$. Recall that $a + 1 = 2L$

(7) $D^2 = 81 + 108(2^L) + 36(2^{2L})$

(8) Let $108(2^L) = 216(2^x)$

(9) $2^L = 2^{(x+1)}$

(10) $L = x + 1, 2L = 2x + 2 = a + 1$

Then (4) becomes :

(11) $18n = 3 + 9 + 6(2^L) = 12 + 6(2^L)$

Dividing by 6,

(12) $3n = 2 + 2^L$

Since $2 \equiv -1 \pmod 3$

$2^L \equiv -1^L \equiv\ 1$ if L is even.

Letting $L = 2y$ we obtain $n = \dfrac{1}{3}(2 + 2^{2y})$.

Solution 3 by David E. Manes, SUNY at Oneonta, Oneonta, NY

We will show if $k \geq 0$ and $n\dfrac{2\left(2^{2k+1}+1\right)}{3}$, then

$$P_n = J_{4k+3} + J_{2k+2} + J_{2k+1},$$

from which the result follows.

Observe that if k is a nonnegative integer, the modulo 3

$$2\left(2^{2k+1}+1\right) \equiv 2\left((-1)^{2k+1}+1\right) \equiv 0 \pmod 3.$$

Therefore, $n = \dfrac{2\left(2^{2k+1}+1\right)}{3}$ is a positive integer for each $k \geq 0$. Moreover,

$$\begin{aligned} P_n &= \frac{\frac{2(2^{2k+1}+1)}{3}\left[2(2^{2k+1}+1)-1\right]}{2} \\ &= \left(\frac{2^{2k+1}+1}{3}\right)\left(2^{2k+2}+1\right). \end{aligned}$$

If $k \geq 0$, then

$$\begin{aligned} J_{4k+3} + J_{2k+2} + J_{2k+1} &= \frac{\left[(2^{4k+3}+1) + (2^{2k+2}-1)\right] + (2^{2k+1}+1)}{3} \\ &= \frac{2^{2k+2}(2^{2k+1}+1) + (2^{2k+1}+1)}{3} \\ &= \left(\frac{2^{2k+1}+1}{3}\right)\left(2^{2k+2}+1\right) \\ &= P_{\frac{2(2^{2k+1}+1)}{3}} = P_n. \end{aligned}$$

Hence, there exists infinitely many pentagonal numbers $P_{\frac{2(2^{2k+1}+1)}{3}}$ (k$\geq$ 0), that are the sum of three Jacobsthal numbers; namely

$$J_{4k+3} + J_{2k+2} + J_{2k+1}.$$

Also solved by Brian D. Beasley Presbyterian College, Clinton, SC; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Southern Georgia University, Statesborogh, GA, and the proposer.

- **5267:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

Let n be a positive integer. Prove that

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_n L_{n+1}}\right) L_{n+2},$$

where F_n and L_n represents the nth Fibonacci and Lucas Numbers defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$; and $L_0 = 2, L_1 = 1$, and for all $n \geq 0$, $L_{n+2} = L_{n+1} + L_n$, respectively.

Solution by G. C. Greubel, Newport News, VA

The inequality to be shown valid is that of

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_n L_{n+1}}\right) L_{n+2}. \qquad (1)$$

Using the AM-GM inequality then it can be seen that

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} \geq 2\left[\frac{F_n F_{n+1}}{F_{n+3} L_{n+1}}\right]^{1/2} L_{n+2} L_{n+3}. \qquad (2)$$

It can be shown that

$$\frac{1}{3} \geq \left[\frac{F_n F_{n+1}}{F_{n+3} L_{n+1}}\right]^{1/2} \geq \frac{1}{4}, \qquad (3)$$

which is valid for $n \geq 1$, for which its use in equation (2) leads to

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} \geq \frac{1}{2} L_{n+2} L_{n+3}. \qquad (4)$$

By making use of this on the left-hand side of (1) it is now left to show that

$$\frac{1}{2} L_{n+2} L_{n+3} + (L_n + L_{n+2})^2 \geq 2\sqrt{6 L_n L_{n+2}}\, L_{n+2}. \qquad (5)$$

Multiplying both sides by 2 yields

$$L_{n+2} L_{n+3} + 2\,(L_n + L_{n+2})^2 \geq 4\sqrt{6 L_n L_{n+2}}\, L_{n+2}. \qquad (6)$$

It is with little difficulty to show that

$$L_{n+2} L_{n+3} + 2\,(L_n + L_{n+2})^2 = 2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 18 L_{n+2}^2 \qquad (7)$$

which, when used in (6), leads to

$$2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 18 L_{n+2}^2 \geq 4\sqrt{6 L_n L_{n+2}}\, L_{n+2}. \qquad (8)$$

Now consider

$$2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 8 L_{n+2}^2$$

which, when use of the AM-GM inequality is made,[1] namely $L_{n+2} \geq 2\sqrt{L_n L_{n+1}}$, becomes

$$\begin{aligned}
2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 8 L_{n+2}^2 &\geq 8 L_{n+1} L_{n+2} - 11 L_{n+2} L_{n+3} + 32 L_n L_{n+1} \\
&\geq 32 L_n L_{n+1} - 22 L_n L_{n+1} - 3 L_{n+1} L_{n+2} \\
&\geq 10 L_n L_{n+1} - 3 L_{n+1} L_{n+2} \\
&\geq L_{n+1}\,(7L_n - 3L_{n+1}) \\
&\geq 7L_n^2 + L_n L_{n-1} \\
&\geq L_n\,(L_{n+2} + 5L_n) \geq 0. \qquad (9)
\end{aligned}$$

From this it is then seen that, when (9) is used in (8),

$$\begin{aligned}
2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 18 L_{n+2}^2 &= \left(2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 8 L_{n+2}^2\right) + 10 L_{n+2}^2 \\
&\geq 10 L_{n+2}^2 \\
&\geq 20\sqrt{L_n L_{n+1}}\, L_{n+2}. \qquad (10)
\end{aligned}$$

Since this represents the left-hand side of the inequality (8) then it is seen that

$$20\sqrt{L_nL_{n+1}}L_{n+2} \geq 4\sqrt{6L_nL_{n+1}}L_{n+2} \qquad (11)$$

and leads to the result $20 \geq 4\sqrt{6}$ which reduces to $5 \geq \sqrt{6}$. Since this is a valid inequality the original statement holds. For the case $n = 0$ equation (3) can be stated as

$$\frac{1}{3} \geq \left[\frac{F_nF_{n+1}}{F_{n+3}L_{n+1}}\right]^{1/2} \geq 0. \qquad (12)$$

Then by following a similar pattern the statement leads to the same result.
Thus, for $n \geq 0$,

$$\frac{F_nL_{n+2}^2}{F_{n+3}} + \frac{F_{n+1}L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_nL_{n+1}}\right)L_{n+2}. \qquad (13)$$

[1] It is seen that $L_{n+2} = L_{n+1} + L_n \geq 2\sqrt{L_nL_{n+1}}$.

Also solved by Ed Gray, Highland Beach, FL, and the proposers.

- **5268:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

Let $N = 121^a + a^3 + 24$. Determine all positive integers a for which

a) N is a perfect square.

b) N is a perfect cube.

Solution 1 by Ed Gray, Highland Beach, FL

(a) The answer to the first part of the question is that there are none, other than the trivial solution of $\mathbf{a = 0}$. We will now show why this is the case.

(1) Let $121^a = (11^2)^a = 11^{2a} = (11^a)^2$, so

(2) $N = (11^a)^2 + a^3 + 24$. Suppose $N = m^2$, so,

(3) $m^2 = (11^a)^2 + a^3 + 24$. Clearly, $m > (11)^a$. Let

(4) $m = (11)^a + b$

(5) $m^2 = (11^a)^2 + 2b(11)^a + b^2$. Equating (2) to (5)

(6) $(11^a)^2 + a^3 + 24 = (11^a)^2 + 2b(11^a) + b^2$. Simplifying gives

(7) $a^3 + 24 = 2b(11)^a + b^2$.

Note that for every positive integer a, $(11)^a > a^3$, since $a(\ln(11)) > 3\ln(a)$, dividing by $3a$ gives $\dfrac{\ln(11)}{3} > \dfrac{\ln(a)}{a}$.

The maximum value of $\dfrac{\ln(a)}{a}$ is when its derivative equals zero, or
$\dfrac{a \cdot (\frac{1}{a}) - \ln(a)}{a^2} = \dfrac{1 - \ln(a)}{a^2} = 0$, which implies that $a = e$.

So the maximum value of $\dfrac{\ln(a)}{a} = \dfrac{\ln(e)}{e} = \dfrac{1}{e} = 0.3678$, and $\dfrac{\ln(a)}{a}$ is monotonically decreasing for $a > e$.

Now $\dfrac{\ln(11)}{3} = 0.7993$, so $(11)^a > a^3$. We note for $a = 2$, the equation in (7) becomes: $32 = 242b + b^2$, which is clearly impossible, and the situation only gets worse for $a > 2$. For $a = 1$ the equation in (7) becomes:

(8) $25 = 22b + b^2$ which clearly has no interger solution. So, the only solution is the trivial one, i.e., when $a = 0$.

(b) The answer to the second part of the question is no; N can never be a perfect cube. By (2) we have:

(9) $N = (11)^{2a} + a^3 + 24$. First, suppose that a is of the form $3y$ and $N = m^3$. Then,

(10) $m^3 = (11)^{6y} + 27y^3 + 24$, or

(11) $m^3 = (11^{2y})^3 + 27y^3 + 24$. Then,

(12) $m > (11)^{2y}$. Letting $m = 11^{2y} + b$

(13) $m^3 = (11)^{6y} + 3(11)^{4y}b + 3(11)^{2y}b^2 + b^3$. Equating (11) and (13),

(14) $(11)^6y + 27y^3 + 24 = (11)^{6y} + 3b(11)^{4y} + 3b^2(11)^{2y} + b^3$. Canceling the term $(11)^{6y}$,

(15) $27y^3 + 24 = 3b(11)^{4y} + 3b^2(11)^{2y} + b^3$.

As before, we show that $(11)^{4y} > 27y^3$ since $4y(\ln(11)) > \ln(27) + 3\ln(y)$ or $9.591y > 3.2958 + 3\ln(y)$ or $1 > \dfrac{0.3436}{y} + \dfrac{0.3128\ln(y)}{y}$.

We have seen the maximum value of $\dfrac{\ln y}{y} = 0.3678$ when $y = e$.

If $y = e$, $\dfrac{0.3436}{2.71828} + (0.3128)(0.3678) = 0.1264 + 0.115 = 0.241$.

For $y = 1$, $1 > 0.3436$ and the right hand side is monotonically decreasing. Notice that we have not used the coefficient $3b$, the additional term $3b^2(11)^2y$, or b^3. The smallest we can make the right hand side is for $y = b = 1$, and the value is $(3)(1)(14641) + (3)(10)(121) + 1 = 132133$, while the right hand side has the value of 51.

There was nothing special about the parameter y and we would get these wildly different values on different sides of the equation for $a = 3y, 3(y+1), 3(y+2)\cdots$. By continuity any value of a sandwiched between any of the above numbers will suffer the same fate. In summary, there can never be an integer cube.

Solution 2 by Kee Wai Lau, Hong Kong, China

We show that for all positive integers a, N is neither a perfect square nor a perfect cube.

a) We first show that for $a = 2, 3, 4, \cdots$,

$$a^3 + 24 < 11^a. \qquad (1)$$

Clearly (1) hold for $a = 2$. Suppose (12) hold for $a \geq 2$. Then

$$(k+1)^3 + 24 < 8k^3 + 24 < 8(k^3 + 24) < 8(11^k) < 11^{k+1}.$$

so (1) is true for $a = k + 1$ an so for $a = 2, 3, 4\cdots$. Now suppose, on the contrary, that $N = n^2$, where n is a positive integer. Then

$$a^3 + 24 = (n + 11^a)(n - 11^a > 11^a.$$

By (1), $a = 1$, so that $n = \sqrt{146}$, which is a contradiction. Thus N is never a perfect square.

b) It can be proved readily by induction that for positive integers m

$$\begin{cases} N \equiv 2 (\text{mod } 9), & a = 3m - 2 \\ N \equiv 3 (\text{mod } 9), & a = 3m - 1 \\ N \equiv 7 (\text{mod } 9), & a = 3 \end{cases}$$

However, the cube of a positive integer is aways congruent either to 0 or 1 or 8(mod 9). It follows that N is never a perfect cube.

Solution 3 by David Stone and John Hawkins of Georgia Southern University, Statesboro, GA and Chuck Garner, Rockdale Magnet School, Conyers, GA.

There are no such integers a in either (a) or (b).

When $a = 1,\ N = 146$, which is neither a square nor a cube. Now assume $a \geq 2$.

For part (a), we can show that N is trapped between consecutive squares, so cannot itself be a square.

$$21^{2a} < N = 11^{2a} + a^3 + 24 < (11^a + 1)^2 = 11^{2a} + 2 \cdot 11^a + 1.$$

The first inequality is clear.

The second, $N = 11^{2a} + a^3 + 24 < (11^a + 1)^2 = 11^{2a} + 2 \cdot 11^a + 1$ is equivalent to $a^3 + 23 < 2 \cdot 11^a$, which can be verified by a straightforward induction argument.

For part (b), we take advantage of the fact that a cube cannot take on many values 9. Namely, only $0, 1$ and 8. note,

$$\text{mod } 9,\ 11^{2\text{a}} \equiv \begin{cases} 1, & \text{if } a \equiv 0 \text{ mod } 3 \\ 4, & \text{if } a \equiv 1 \text{ mod } 3 \\ 7, & \text{if } a \equiv 2 \text{ mod } 3, \quad \text{and} \end{cases}$$

$$\text{mod } 9,\ \text{a}^3 \equiv \begin{cases} 0, & \text{if } a \equiv 0 \text{ mod } 3 \\ 1, & \text{if } a \equiv 1 \text{ mod } 3 \\ 8, & \text{if } a \equiv 2 \text{ mod } 3. \end{cases}$$

Thus $\text{mod } 9,\ \text{N} = 11^{2\text{a}} + \text{a}^3 + 24 \ \equiv \begin{cases} 1 + 0 + 6 \equiv 7, & \text{if } a \equiv 0 \text{ mod } 3 \\ 4 + 1 + 6 \equiv 2, & \text{if } a \equiv 1 \text{ mod } 3\ . \\ 7 + 8 + 6 \equiv 3, & \text{if } a \equiv 2 \text{ mod } 3 \end{cases}$

That is, N is congruent to 2, 3 or 7, and never congruent to $0, 1$, or 8, N cannot be a cube.

Comment : Numerical evidence suggests that the power of 11 is so dominant that N also lies between identifiable consecutive cubes $m^3 < N < (m + 1)^3$, where m=$\left[11^{2a/3}\right]$.

Also solved by David E. Manes, SUNY College at Oneonta, NY, and the proposer.

- **5269:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $\{a_n\}_{n \geq 1}$ be the sequence defined by

$$a_1 = 1,\ a_2 = 5,\ a_{n-1}^2 - a_n a_{n-2} + 4 = 0.$$

Show that all of the terms of the sequence are integers.

Solution 1 by Ercole Suppa, Teramo, Italy

From the given recurrence we get

$$a_n a_{n-2} = a_{n-1}^2 + 4 \qquad (1)$$

$$a_{n+1} a_{n-1} = a_n^2 + 4 \qquad (2)$$

Now subtracting (1) and (2) from each other, we find that for every $n \in N$:

$$\begin{aligned} a_n a_{n-2} - a_{n+1}a_{n-1} &= (a_{n-1} - a_n)(a_{n-1} + a_n) \quad \Leftrightarrow \\ a_n a_{n-2} - a_{n+1}a_{n-1} &= a_{n-1}^2 - a_n^2 \quad \Leftrightarrow \\ a_n(a_{n-2} + a_n) &= a_{n-1}(a_{n+1} + a_{n-1}) \quad \Leftrightarrow \\ \frac{a_{n-2} + a_n}{a_{n-1}} &= \frac{a_{n+1} + a_{n-1}}{a_n} \qquad (3) \end{aligned}$$

Therefore the expression $\dfrac{a_{n-2} + a_n}{a_{n-1}}$ is constant. From the initial conditions we obtain

$$\begin{aligned} \frac{a_{n-2} + a_n}{a_{n-1}} &= \frac{a_3 + a_1}{a_2} = \frac{29 + 1}{5} = 6 \Rightarrow \\ a_n &= 6a_{n-1} - a_{n-2}, \quad \forall n \geq 3. \qquad (4) \end{aligned}$$

By using (4) a simple induction on n show that all the terms of the sequence are integers.

Solution 2 Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since

$$a_{n-1}^2 - a_n a_{n-2} + 4 = 0$$

for $n \geq 3$, we have

$$a_n a_{n-2} = a_{n-1}^2 + 4 \geq 4.$$

Therefore, $a_{n-2} \neq 0$ for all $n \geq 3$ and we may write the recursive formula for $\{a_n\}$ in the form

$$a_n = \frac{a_{n-1}^2 + 4}{a_{n-2}}$$

for all $n \geq 3$, or equivalently

$$a_{n+2} = \frac{a_{n+1}^2 + 4}{a_n} \qquad (1)$$

for all $n \geq 1$.

When we evaluate the first six terms using (1) and the initial values $a_1 = 1$ and $a_2 = 5$, we obtain

$$a_1 = 1, \quad a_2 = 5, \quad a_3 = 29, \quad a_4 = 169, \quad a_5 = 985, \quad \text{and} \quad a_6 = 5741.$$

These entries suggest the following alternative recursive definition for $\{a_n\}$:

$$a_1 = 1, \quad a_2 = 5, \quad \text{and} \quad a_{n+2} = 6a_{n+1} - a_n \text{ for } n \geq 1. \;\; 2 \tag{2}$$

We will establish (2) by Mathematical Induction. Let $P(n)$ be the statement

$$a_{n+2} = 6a_{n+1} - a_n.$$

Then, the conditions

$$a_3 = \frac{a_2^2 + 4}{a_1} = 29 = 6a_2 - a_1$$

and

$$a_4 = \frac{a_3^2 + 4}{a_2} = 169 = 6a_3 - a_2$$

imply that $P(1)$ and $P(2)$ are true. If we assume that $P(1), P(2), \ldots, P(n)$ are true for some $n \geq 2$, then in particular, $a_{n+2} = 6a_{n+1} - a_n$ and $a_{n+1} = 6a_n - a_{n-1}$. It follows that

$$\begin{aligned} a_{n+3} &= \frac{a_{n+2}^2 + 4}{a_{n+1}} \\ &= \frac{(6a_{n+1} - a_n)^2 + 4}{a_{n+1}} \\ &= 36a_{n+1} - 12a_n + \frac{a_n^2 + 4}{a_{n+1}} \\ &= 6(6a_{n+1} - a_n) - 6a_n + \frac{a_{n+1}a_{n-1}}{a_{n+1}} \\ &= 6a_{n+2} - (6a_n - a_{n-1}) \\ &= 6a_{n+2} - a_{n+1} \end{aligned}$$

and hence, $P(n+1)$ is true also. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$, i.e., $a_{n+2} = 6a_{n+1} - a_n$ for all $n \geq 1$.

As a result, the conditions

$$a_1 = 1, \quad a_2 = 5, \quad \text{and} \quad a_{n+2} = 6a_{n+1} - a_n \text{ for } n \geq 1$$

and a trivial Mathematical Induction argument imply that a_n is an integer for all $n \geq 1$.

Additionally, this new description affords us a method for finding a formula for the sequence $\{a_n\}$. Using the customary technique for solving homogeneous linear difference equations, we look for solutions of the form $a_n = \lambda^n$, with $\lambda \neq 0$. Then, the formula

$$a_{n+2} = 6a_{n+1} - a_n$$

simplifies to

$$\lambda^2 = 6\lambda - 1$$

whose solutions are $\lambda = 3 \pm 2\sqrt{2}$. The general solution is of the form

$$a_n = c_1 \left(3 + 2\sqrt{2}\right)^n + c_2 \left(3 - 2\sqrt{2}\right)^n$$

for some constants c_1 and c_2. Further, the initial values $a_1 = 1$ and $a_2 = 5$ yield

$$c_1 = \frac{2 - \sqrt{2}}{4} \quad \text{and} \quad c_2 = \frac{2 + \sqrt{2}}{4}.$$

Finally, since

$$3 \pm 2\sqrt{2} = \frac{\left(2 \pm \sqrt{2}\right)^2}{2},$$

we get

$$\begin{aligned}
a_n &= 2-\sqrt{2}4\left(3+2\sqrt{2}\right)^n + \frac{2+\sqrt{2}}{4}\left(3-2\sqrt{2}\right)^n \\
&= \frac{1}{4}\left[\left(2-\sqrt{2}\right)\frac{\left(2+\sqrt{2}\right)^{2n}}{2^n} + \left(2+\sqrt{2}\right)\frac{\left(2-\sqrt{2}\right)^{2n}}{2^n}\right] \\
&= \frac{(2)\left(2+\sqrt{2}\right)^{2n-1} + (2)\left(2-\sqrt{2}\right)^{2n-1}}{2^{n+2}} \\
&= \frac{\left(2+\sqrt{2}\right)^{2n-1} + \left(2-\sqrt{2}\right)^{2n-1}}{2^{n+1}}
\end{aligned}$$

for all $n \geq 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

For positive integers n, let $b_n = \dfrac{(2-\sqrt{2})(3+2\sqrt{2})^n + (2+\sqrt{2})(3-2\sqrt{2})^n}{4} > 0$. It is easy to check that $b_1 = 1, b_2 = 54$ and for $n \geq 3, b_n = b_{n-1} - b_{n-2}$. Hence b_n are always positive integers.

Using the equation $a_n a_{n-2} = a_{n-1}^2 + 4$, we prove readily by induction that
$a_n = \dfrac{(2-\sqrt{2})(3+2\sqrt{2})^n + (2+\sqrt{2})(3-2\sqrt{2})^n}{4}$ as well.

Thus, $a_n = b_n$ are positive integers.

Editor's comment: David Stone and John Hawkins of Georgia Southern University in Statesboro, GA also solved the problem by generating a few terms of the given sequence, and then finding a recursive definition for these initial terms that was different from the given recursion in the statement of the problem. Then, using induction, they showed that the new recursive definition satisfied the recursion in the statement of the problem. Essentially, their solution path was that used in Solution 1 above.

They also commented that the problem can also be solved as it is in Solution 3 above, where one finds an explicit formula for the Fibonacci sequence. They continued on the following way:

Comment 1: Other Fibonacci-like properties can be derived. For instance, the ratio of consecutive terms, $\dfrac{a_{n+1}}{a_n}$ approaches $\alpha = 3 + 2\sqrt{2} \approx 5.8284$.

Comment 2: In the proposed problem the true nature of the $\{an\}_{n\geq 1|}$ was cleverly disguised by an unfamiliar recurrence relation: $a_n = \dfrac{a_{n-1}^2 + 4}{a_{n-2}}$. Perhaps there is a similar relation fot he Fibonacci numbers.

Comment 3: The sequence $\{a_n\}_{n\geq 1|}$ is A001653 at the Online Encyclopedia of Integer Sequences. Several interesting properties and applications are given: the recurrence relation $a_n = 6a_{n-1]-a_{n-2}}$ is given. We do not see (in this encyclopedia) the recurrence relation that was given in the problem statement (so perhaps it is heretofore unknown).

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kenneth Korbin, New York, NY; Carl Libis and Junhua Wu, Lane College, Jackson, TN; Carl Libis (a second solution), Lane College, Jackson, TN; David E. Manes, SUNY College at Oneonta, NY; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposer.

- **5270:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Calculate

$$\int_0^1 \int_0^1 (x+y)^k \, (-1)^{\left\lfloor \frac{1}{x} - \frac{1}{y} \right\rfloor} \, dxdy,$$

where$\lfloor x \rfloor$ denotes the floor of x.

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For any point $(x, y) \in [0, 1]$, also $(y, x) \in [0, 1]$. Note that for (x, y) such that $\dfrac{1}{x} - \dfrac{1}{y} \in (m, m+1)$ with $m \in$, then $\left\lfloor \dfrac{1}{x} - \dfrac{1}{y} \right\rfloor = m$, but for the corresponding point (y, x) also in the domain $[0, 1]$ we have that $\dfrac{1}{y} - \dfrac{1}{x} \in (-(m+1), -m)$ and therefore $\left\lfloor \dfrac{1}{y} - \dfrac{1}{x} \right\rfloor = -(m+1)$. Since $(-1)^m = -(-1)^{-(m+1)}$ and $(x+y)^k = (y+x)^k$ the proposed integral is 0.

Solution 2 by Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Let $A = \{(x, y) \colon 0 \leq x \leq 1, 0 \leq y \leq 1,\ y \geq x\}$ and $B = \{(x, y) \colon 0 \leq x \leq 1, 0 \leq y \leq 1,\ y \leq x\}$. By doing $(x, y) \to (y, x)$ we get

$$\int\int_A (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy = \int\int_B (y+x)^k (-1)^{\lfloor \frac{1}{y} - \frac{1}{x} \rfloor} dxdy.$$

Moreover because of $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ we get

$$\int\int_B (x+y)^k (-1)^{-1-\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy = -\int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor},$$

and then

$$\begin{aligned}
\int_0^1 \int_0^1 (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy &= \int\int_A (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy + \int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy \\
&= -\int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy + \int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy = 0.
\end{aligned}$$

Solution 3 by the proposer

The integral equals 0. We have, based on symmetry reasons, that

$$\int_0^1 \int_0^1 x\,(x+y)^{k-1} \,(-1)^{\left\lfloor \frac{1}{x} - \frac{1}{y} \right\rfloor} \, dxdy = \int_0^1 \int_0^1 y(x+y)^{k-1} \,(-1)^{\left\lfloor \frac{1}{y} - \frac{1}{x} \right\rfloor} \, dxdy.$$

On the other hand, for all real numbers x that are not integers, one has

$$\lfloor x \rfloor + \lfloor -x \rfloor = -1.$$

It follows that,

$$\begin{aligned}\int_0^1\int_0^1 x\,(x+y)^{k-1}\,(-1)^{\left\lfloor \frac{1}{x}-\frac{1}{y}\right\rfloor}\,dxdy &= \int_0^1\int_0^1 y(x+y)^{k-1}\,(-1)^{\left\lfloor \frac{1}{y}-\frac{1}{x}\right\rfloor}\,dxdy\\ &= -\int_0^1\int_0^1 y(x+y)^{k-1}\,(-1)^{-\left\lfloor \frac{1}{x}-\frac{1}{y}\right\rfloor}\,dxdy\\ &= -\int_0^1\int_0^1 y(x+y)^{k-1}\,(-1)^{\left\lfloor \frac{1}{x}-\frac{1}{y}\right\rfloor}\,dxdy,\end{aligned}$$

and the result follows.

Also solved by Paul M. Harms, North Newton, KS, and by Ed Gray, Highland Beach, FL.

Mea Culpa; once again

My sincerest apologies to David Stone and to John Hawkins of Georgia Southern University, for inadvertently forgetting to mention that they had correctly solved problems 5260, 5261, and 5262; and also to Brian D. Beasley, of Presbyterian College in Clinton, South Carolina for his solution to 5262.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2014

- **5289:** *Proposed by Kenneth Korbin, New York, NY*

  Part 1: Thirteen different triangles with integer length sides and with integer area each have a side with length 1131. The angle opposite 1131 is *Arcsin* $\left(\frac{3}{5}\right)$ in all 13 triangles.

  Find the sides of the triangles.

  Part 2: Fourteen different triangles with integer length sides and with integer area each have a side with length 6409. The size of the angle opposite 6409 is the same in all 14 triangles.

  Find the sides of the triangles.

- **5290:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Someone wrongly remembered the description of an even perfect number as:
  $N = 2^p\left(2^{p-1}-1\right)$, where p is a prime number. Classify these numbers correctly. Which are deficient and which are abundant?

  (If n and d are positive integers, $d \neq n$, but $d \Big| n$, then d is called a proper divisor of n. The integer n is called *perfect* if the sum of its proper divisors is equal to n. The number n is called *deficient* if the sum of its proper divisors is less than n; and if the sum of its proper divisors is greater than n, then n is called an *abundant* number. E.g., The proper divisors of 6 are $1, 2,$ and 3. Their sum is 1+2+3=6, and so 6 is a perfect number; all prime numbers are deficient, and the proper divisors of 12 are $1, 2, 4,$ and 6. So 12 is an abundant number.)

- **5291:** *Arkady Alt, San Jose, CA*

  Let $m_a m_b$ be the medians of a triangle with side lengths a, b, c. Prove that:

  $$m_a m_b \leq \frac{2c^2+ab}{4}.$$

- **5292:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School,* *Buzău, Romania*

Let a and b be real numbers with $a < b$, and let c be a positive real number. If $f : R \longrightarrow R_+$ is a continuous function, calculate:

$$\int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}} + e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx.$$

- **5293:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let ABC be a triangle. Prove that

$$\sqrt[4]{\sin A \cos^2 B} + \sqrt[4]{\sin B \cos^2 C} + \sqrt[4]{\sin C \cos^2 A} \leq 3\sqrt[8]{\frac{3}{64}}.$$

- **5294:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

a) Calculate $\displaystyle\sum_{n=2}^{\infty}\left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)\right)$.

b) More generally, for $k \geq 2$ an integer, find the value of the multiple series

$$\sum_{n_1,n_2,\cdots,n_k=1}^{\infty}\left(n_1+n_2+\cdots+n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1+n_2+n_3+\cdots+n_k)\right),$$

where ζ denotes the Riemann Zeta function.

Solutions

- **5271:** *Proposed by Kenneth Korbin, New York, NY*

Given convex cyclic quadrilateral $ABCD$ with $\overline{AB} = x, \overline{BC} = y$, and $\overline{BD} = 2\overline{AD} = 2\overline{CD}$.

Express the radius of the circum-circle in terms of x and y.

Solution 1 by Andrea Fanchini, Cantú, Italy

Method I

In a cyclic quadrilateral with successive vertices A, B, C, D and sides $a = \overline{AB}, b = \overline{BC}, c = \overline{CD}, d = \overline{DA}$, the length of the diagonal $q = \overline{BD}$ can be expressed in terms of the sides as:

$$q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$$

Let $t=\overline{AD}=\overline{CD}$. Then in our case we have

$$2t=\sqrt{\frac{(xt+yt)\left(xy+t^2\right)}{xt+yt}}\Rightarrow t=\sqrt{\frac{xy}{3}}$$

Let $p=\overline{AC}$ and according Ptolemy's theorem

$$p=\frac{ac+bd}{q}=\frac{x+y}{2}$$

Then we denote $\angle ABD=\angle DBC=\beta$, so $\angle ABC=2\angle ABD=2\angle DBC=2\beta$.
Furthermore, from the angle at the center theorem $\angle AOD=\angle ABC=2\beta$.
Now with the Carnot's theorem at the side $\overline{AC}$ of the $\triangle ABC$, we have

$$p^2=x^2+y^2-2xy\cos 2\beta\Rightarrow\cos 2\beta=\frac{3x^2+3y^2-2xy}{8xy}$$

Using another time Carnot's theorem at the side $\overline{AD}$ of the $\triangle AOD$, we obtain

$$t^2=R^2+R^2-2R^2\cos 2\beta$$

from which, we finally obtain, the radius R of the circum-circle in terms of x and y

$$R=\frac{2xy}{\sqrt{3\left(10xy-3x^2-3y^2\right)}}$$

Method II

Applying Parameshvara's formula, a cyclic quadrilateral with successive sides a,b,c,d and semiperimeter s has the circumradius R given by

$$R=\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

In our case we have $a=x,b=y$ and $c=d=\sqrt{\dfrac{xy}{3}}$. Substituting, we obtain the formula requested.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Let $\overline{BD}=2z$ and $\angle BAD=\theta=\pi-\angle BCD$. Applying the cosine formula to triangles BAD and BCD respectively, we obtain,

$$\cos\theta=\frac{x^2-3z^2}{2xz}\text{ and }-\cos\theta=\cos(\pi-\theta)=\frac{y^2-3z^2}{2yz}.$$

Hence,

$$z=\sqrt{\frac{xy}{3}},\quad\cos\theta=\frac{\sqrt{3}(x-y)}{2\sqrt{xy}},\text{and }\sin\theta=\frac{1}{2}\sqrt{\frac{(3x-y)(3y-x)}{xy}}.$$

It is easy to check that $\sin\theta$ is a positive real number not exceeding 1 if and only if $\dfrac{1}{3}<\dfrac{x}{y}<3$. Subject to this condition, we obtain

that the radius of the circum-cirlce $=\dfrac{\overline{BD}}{2\sin\theta}=\dfrac{2xy}{\sqrt{3(3x-y)(3y-x)}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.

- **5272:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The Jacobsthal numbers begin $0, 1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 0$. Prove that there are infinitely many Pythagorean triples like $(3, 4, 5)$ and $(13, 84, 85)$ that have "hypotenuse" a Jacobsthal number.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

For $n \geq 1$, $\left(2^{2n}-1, 2^{n+1}, 2^{2n}+1\right)$ is a primitive Pythagorean triple since $\gcd\left(2^{2n}-1, 2^{n+1}\right) = 1$ and

$$\begin{aligned}\left(2^{2n}-1\right)^2 + \left(2^{n+1}\right)^2 &= 2^{4n} - 2^{2n+1} + 1 + 2^{2n+2} \\ &= 2^{4n} + 2^{2n+1} + 1 \\ &= \left(2^{2n}+1\right)^2.\end{aligned}$$

It follows that for any positive integer m, $\left(\left(2^{2n}-1\right)m, 2^{n+1}m, \left(2^{2n}+1\right)m\right)$ is also a Pythagorean triple. In particular, when $n \geq 1$, $\left(\left(2^{2n}-1\right)J_{2n}, 2^{n+1}J_{2n}, \left(2^{2n}+1\right)J_{2n}\right)$ is a Pythagorean triple with

$$\begin{aligned}\left(2^{2n}+1\right)J_{2n} &= \left(2^{2n}+1\right)\cdot\frac{2^{2n}-1}{3} \\ &= \frac{2^{4n}-1}{3} \\ &= J_{4n}.\end{aligned}$$

Hence, for $n \geq 1$, $\left(\left(2^{2n}-1\right)J_{2n}, 2^{n+1}J_{2n}, J_{4n}\right)$ is a Pythagorean triple whose "hypotenuse" is a Jacobsthal number.

Solution 2 by Ed Gray, Highland Beach, FL

1) $2^2 \equiv (-1) \pmod 5$
2) $2^{2k} \equiv (-1)^k \pmod 5$
3) If k is even, $2^{2k} - 1 \equiv 0 \pmod 5$
4) If k is odd, $2^{2k} + 1 \equiv 0 \pmod 5$, in either case
5) $\left(2^{2k}-1\right)\left(2^{2k}+1\right) \equiv 0 \pmod 5$, or
6) $2^{4k} - 1 \equiv 0 \pmod 5$.

Suppose
7) $n = 4k$.
Then
8) $J_n = J_{4k} = \dfrac{2^{4k}-1}{3} \equiv 0 \pmod 5$ by (6).

Therefore,
9) If $n = 4k$, let $J_n = J_{4k} = r(2^2 + 1^2)$.
Let this be the "hypotenuse." The formulae for a Pythagorean triple are:
10) $x = r(2ab), y = r(a^2 - b^2), z = r(a^2 + b^2)$.
From (9), let $a = 2, b = 1$.
Then (10) becomes:
11) $x = r(2ab),\ y = r(a^2 - b^2),\ z = r(a^2 + b^2)$, or
12) $x = 4r,\ y = 3r,\ z = 5r$, where r is defined by (9).
13) Hence $x^2 + y^2 = z^2$.

Solution 3 by Kenneth Korbin, New York, NY

If a positive integer is a multiple of 5, then it is the length of the hypotenuse of at least one Pythagorean triangle.

In the J series, every fourth term is a multiple of 5.
For example, $J_4 = 5,\ J_8 = 85,\ J_{12} = 1365$, and in general $J_{4n} = 16J_{4(n-1)} + 5$.

We have

$$\begin{aligned} J_n &= \frac{2^n - (-1)^n}{3}. \text{ Then,} \\ J_{4n} &= \frac{2^{4n} - (-1)^{4n}}{3} = \frac{16^n - 1^n}{3}. \\ 16^n - 1 &\equiv 15 \pmod{15} \\ \frac{16^n - 1}{3} &\equiv 5 \pmod{5}. \end{aligned}$$

The J sequence (mod 10) is

$$(1, 1, 3, 5, 1, 1, 3, 5, \ldots, 1, 1, 3, 5, \ldots)$$

If a and b are positive integers and if $a|b$, then $J_{4a}|J_4b$.

Also solved by Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Carl Libis, Lane College, Jackson, TN; Bob Sealy, Sackville, NB, Canada; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5273:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

Solve in the positive integers the equation $abcd + abc = (a + 1)(b + 1)(c + 1)$.

Solution 1 by Adrian Naco, Polytechnic University,Tirana, Albania.

We have that,

$$2 \leq d + 1 = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \leq \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{1}\right) = 8, \text{ or } 1 \leq \text{d} \leq 7.$$

Let us suppose that $1 \leq c \leq b \leq a$, then,

$$2 \leq (d+1) = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \leq \left(1+\frac{1}{c}\right)^3$$
$$\Rightarrow \qquad \sqrt[3]{2} \leq 1+\frac{1}{c} \qquad \Rightarrow \qquad c \leq \frac{1}{\sqrt[3]{2}-1}$$
$$\Rightarrow \qquad c \in \{1,2,3\}$$

Case 1. $c = 1$. Thus,

$$ab(d+1) = 2(a+1)(b+1) \quad \Rightarrow \quad d+1 = 2\left(\frac{a+1}{a}\right)\left(\frac{b+1}{b}\right) > 2$$

Thus, we have that $2 \leq d \leq 7$.
a) If $a = b$, then it implies that,

$$d = 1 + 2 \cdot \frac{2a+1}{a^2} \quad \Rightarrow \quad a = 1 = b,\ d = 7$$

b) If $a \geq b+1$, then,

$$\begin{aligned} 3ab \leq ab(d+1) = 2(a+1)(b+1) \quad &\Rightarrow \quad 3ab \leq 2ab + 2a + 2b + 2 \\ &\Rightarrow \quad ab \leq 2a + 2b + 2 \\ &\Rightarrow \quad b \leq 2 + \frac{2(b+1)}{a} \leq 2 + 2 = 4 \\ &\Rightarrow \quad b \in \{1,2,3,4\} \end{aligned}$$

Thus, we have the following solutions

$$\begin{gathered} b = 1,\ a = 2,\ d = 5 \\ b = 1,\ a = 4,\ d = 4 \\ b = 3,\ a = 8,\ d = 2 \\ b = 4,\ a = 5,\ d = 2 \end{gathered}$$

Case 2. If $c = 2$, then,

$$2ab(d+1) = 3(a+1)(b+1).$$

a) If $a = b$, then it implies that,

$$2a^2(d+1) = 2(a+1)^2 \quad \Rightarrow \quad a^2/3 \quad \Rightarrow \quad a = 1 < 2 = c \leq a \quad \Rightarrow \quad a < a!$$

b) If $a \geq b+1$, then,

$$\begin{aligned} 4ab \leq 2ab(d+1) = 3(a+1)(b+1) \quad &\Rightarrow \quad 4ab \leq 3ab + 3a + 3b + 3 \\ &\Rightarrow \quad ab \leq 3a + 3b + 3 \\ &\Rightarrow \quad b \leq 3 + 3\frac{(b+1)}{a} \leq 3 + 3 = 6 \\ &\Rightarrow \quad b \in \{2, 3, 4, 5, 6\} \end{aligned}$$

Thus, we have the following solutions

$$\begin{aligned} &b = 2,\ a = 3,\ \ d = 2 \\ &b = 4,\ a = 15,\ d = 1 \\ &b = 6,\ a = 7,\ \ d = 1. \end{aligned}$$

Case 3. If $c = 3$, then,

$$\begin{aligned} 6ab \leq 3ab(d+1) = 4(a+1)(b+1) \quad &\Rightarrow \quad 6ab \leq 4ab + 4a + 4b + 4 \\ &\Rightarrow \quad ab \leq 2a + 2b + 2 \\ &\Rightarrow \quad b \leq 2 + 2\frac{b+1}{a} \leq 2 + 2 = 4 \\ &\Rightarrow \quad b \in \{3, 4\} \end{aligned}$$

Thus, we have the following solutions

$$\begin{aligned} &b = 3,\ a = 8,\ d = 1 \\ &b = 4,\ a = 5,\ d = 1. \end{aligned}$$

Finally, the solutions (a, b, c, d), of the given equality are,

$$\begin{aligned} &Case\ 1: \ (1,1,1,8) \\ &(1,1,2,5), (1,2,1,5), (2,1,1,5) \\ &(1,1,4,4), (1,4,1,4), (4,1,1,4) \\ &(1,3,8,2), (1,8,3,2), (3,1,8,2), (3,8,1,2), (8,1,3,2), (8,3,1,2) \end{aligned}$$

$(1,4,5,2),(1,5,4,2),(4,1,5,2),(4,5,1,2),(5,1,4,2),(5,4,1,2).$

$Case\ 2:$ $(2,2,3,2),(2,3,2,2),(3,2,2,2)$
$(2,4,15,1),(2,15,4,1),(4,2,15,1),(4,15,2,1),(15,2,4,1),(15,4,2,1)$
$(2,6,7,1),(2,7,6,1),(6,2,7,1),(6,7,2,1),(7,2,6,1),(7,6,2,1).$

$Case\ 3:$ $(3,3,8,1),(3,8,3,1),(8,3,3,1)$
$(3,4,5,1),(3,5,4,1),(4,3,5,1),(4,5,3,1),(5,3,4,1),(5,4,3,1).$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the solutions are given by

$$\begin{aligned}(a,b,c,d) \quad = \quad & (1,1,1,7),(1,1,2,5),\ (1,1,4,4),\ (1,2,3,3),\ (1,3,8,2),\ (1,4,5,2),\\ & (2,2,3,2),\ (2,4,15,1),\ (2,5,9,1),\ (2,6,7,1),\ (3,3,8,1),\ (3,4,5,1).\end{aligned}$$

together with solutions obtained by permutations of entries a,b,c.

Clearly it suffices to consider the case $a \le b \le c$. We have

$$1 \le d = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) - 1 \le \left(1+\frac{1}{a}\right)^3 - 1 \text{ so that } a \le \frac{1}{2^{\frac{1}{3}}-1} < 4.$$

Hence, for $a = 1,2,3,$ we have respectively $1 \le d \le 7,\ 1 \le d \le 2,\ d = 1.$ We then obtain the following table readily:

a	d	c in terms of b	Solutions (b,c) in positive integers with $a \leq b \leq c$
1	1	-b -1	No solutions
	2	$2+\dfrac{6}{b-2}$	(3,8),(4,5)
	3	$1+\dfrac{2}{b-1}$	(2,3)
	4	$1+\dfrac{4-b}{3b-2}$	(1,4)
	5	$1+\dfrac{2-b}{2b-1}$	(1,2)
	6	$1+\dfrac{4-3b}{5b-2}$	No solutions
	7	$1+\dfrac{2(b-1)}{3b-1}$	(1,1)
2	1	$3+\dfrac{12}{b-3}$	(4,15), (5,9),(6,7)
	2	$1+\dfrac{2}{b-1}$	(2,3)
3	1	$2+\dfrac{6}{b-2}$	(3,8), (4,5)

Also solved by Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, NY, NY, and by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

- **5274:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

Let x, y, z, α be real positive numbers. Show that if

$$\sum_{cycl} \frac{(n+1)\,x^3 + nx}{x^2+1} = \alpha$$

then

$$\sum_{cycl} \frac{1}{x} > \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2}$$

where n is a natural number.

Solution by proposer

Doing easy manipulations we have

$$\alpha = \sum_{cycl} \frac{(n+1)x^3+nx}{x^2+1} = \sum_{cycl} \frac{1}{x} + \sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}$$

Let $f(x) = \dfrac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}$. One can easy observe that

$$\begin{aligned} f'(x) &= \frac{1+(n+2)x^2+(2n+4)x^4+(n+1)x^6}{x^2(1+x^2)^2} \\ f''(x) &= -\frac{2(1+3x^2+2x^6)}{x^3(1+x^2)^3} \end{aligned}$$

It is obvious that $f'(x) > 0$ and $f''(x) < 0$ for any real positive number x, which implies that the function $f(x)$ is an increasing and concave function in the real positive domain. Applying Jensen's inequality we have

$$\sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)} = \sum_{cycl} f(x) \le 3f\left(\frac{1}{3}\sum_{cycl} x\right)$$

Doing easy manipulations, one can easy observe that

$$\alpha = \sum_{cycl} \frac{(n+1)x^3+nx}{x^2+1} = \sum_{cycl} nx + \sum_{cycl} \frac{x^3}{x^2+1} > n\sum_{cycl} x$$

Finally, using the above results we have

$$\begin{aligned} \sum_{cycl} \frac{1}{x} &= \alpha - \sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)} \\ &\ge \alpha - 3f\left(\frac{1}{3}\sum_{cycl} x\right) \\ &> \alpha - 3f\left(\frac{\frac{\alpha}{n}}{3}\right) \\ &= \alpha - 3f\left(\frac{\alpha}{3n}\right) \\ &= \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2} \end{aligned}$$

and this is the end of the proof.

- **5275:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations

$$\left.\begin{array}{r} \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n} = x_2\sqrt{2}, \\ \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n} = x_3\sqrt{2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n} = x_n\sqrt{2}, \\ \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n} = x_1\sqrt{2}, \end{array}\right\}$$

where $n \geq 2$.

Solution by Arkady Alt, San Jose, CA

Let $h(x) := \sqrt{2+x}$. Then $h(x)$ is a function defined on $[-2,\infty)$ with range $[0,\infty)$.

Since $h : [-2,\infty) \longrightarrow [0,\infty)$ then for any $n \in N$ we can define recursively $n-$iterated function $h_n : [-2,\infty) \longrightarrow [0,\infty)$, namely $h_1 = h$ and $h_{n+1} = h \circ h_n, n \geq 1$.

Let $f(x) := \dfrac{h_n(x) + \sqrt{2 - h_{n-1}(x)}}{\sqrt{2}}$ for $x \in [-2,\infty)$ such that $h_{n-1}(x) \leq 2$.

Since $h_{n-1}(x) \leq 2 \iff h_{n-1}^2(x) \leq 4 \iff h_{n-2}(x) \leq 2 \iff .... \iff h_1(x) \leq 2 \iff x \leq 2$

then $Dom(f) = [-2,2]$. Moreover, applying inequality $\dfrac{a+b}{\sqrt{2}} \leq \sqrt{a^2+b^2}$ to $a = h_n(x)$ and $b = \sqrt{2 - h_{n-1}(x)}$ we obtain $f(x) \leq 2$ and since by definition $f(x) \geq 0$ for $x \in Dom(f)$
then $range(f) \subset [0,2]$.

Using f we can rewrite original system as follow:

(1) $\begin{cases} x_{k+1} = f(x_k), k = 1,2,...,n-1 \\ x_1 = f(x_n) \end{cases}$.

Since $x_k \in [0,2]$, $k = 1,2,...,n$ then by setting $t_k := \cos^{-1}\left(\dfrac{x_k}{2}\right), k = 1,2,...,n$

we obtain $t_k \in \left[0, \dfrac{\pi}{2}\right]$, $x_k = 2\cos t_k$, $k = 1,2,...,n$.

Noting that $h(2\cos t) = 2\cos t2$ for $t \in \left[0, \dfrac{\pi}{2}\right]$ by Math. Induction we obtain

$h_k(2\cos t) = 2\cos\dfrac{t}{2^k}, k = 1,2,...,...$and, therefore, $f(2\cos t) =$

$$\frac{1}{\sqrt{2}}\left(2\cos\frac{t}{2^n} + \sqrt{2 - 2\cos\frac{t}{2^{n-1}}}\right) = 2\left(\frac{1}{\sqrt{2}}\cos\frac{t}{2^n} + \frac{1}{\sqrt{2}}\sin\frac{t}{2^n}\right) = 2\cos\left(\frac{\pi}{4} - t2^n\right).$$

Since $\dfrac{\pi}{4} - \dfrac{t}{2^n} \in \left[0, \dfrac{\pi}{2}\right]$ for $t \in \left[0, \dfrac{\pi}{2}\right]$ then $\dfrac{\pi}{4} - \dfrac{t_k}{2^n} \in \left[0, \dfrac{\pi}{2}\right]$ as well as $t_k \in \left[0, \dfrac{\pi}{2}\right]$ for any $k = 1,2,...,n$ and, therefore, **(1)**

$$\iff \begin{cases} 2\cos t_{k+1} = 2\cos\left(\dfrac{\pi}{4} - \dfrac{t_k}{2^n}\right), k = 1,2,...,n-1 \\ 2\cos t_1 = 2\cos\left(\dfrac{\pi}{4} - \dfrac{t_n}{2^n}\right) \end{cases} \iff$$

(2) $\begin{cases} t_{k+1} = \dfrac{\pi}{4} - \dfrac{t_k}{2^n}, k = 1,2,...,n-1 \\ t_1 = \dfrac{\pi}{4} - t_n 2^n \end{cases}$.

Lemma:

Let a, b be real numbers such that $|a| \neq 1$. Then system of equations

$$\begin{cases} t_{k+1} = b + at_k, k = 1,2,...,n-1 \\ t_1 = b + at_n \end{cases}$$

have only solution $t_1 = t_2 = ... = t_n = \dfrac{b}{1-a}$.

Proof: Noting that $\dfrac{b}{1-a} = b + a \cdot \dfrac{b}{1-a}$ and denoting $c := \dfrac{b}{1-a}$ we obtain
$t_{k+1} = b + at_k \iff t_{k+1} - c = a\left(t_k - c\right), k = 1, 2, ...n-1$
and $t_1 = b + at_n \iff t_1 - c = a\left(t_n - c\right)$. Since $t_k - c, k = 1, 2, ...$
is geometric sequence we have $t_k - c = a^{k-1}\left(t_1 - c\right), k = 1, 2, ...n-1$ and therefore,

$t_1 - c = a \cdot a^{n-1}\left(t_1 - c\right) \iff t_1 - c = a^n\left(t_1 - c\right) \iff \left(t_1 - c\right)\left(1 - a^n\right) = 0 \iff t_1 = c.$

That yield $t_k - c = a^{k-1}\left(t_1 - c\right) = 0 \iff t_k = c, k = 2, ..., n.$

Thus, $t_1 = t_2 = ... = t_n = c = \dfrac{b}{1-a}$.

Applying the Lemma with $a = -\dfrac{1}{2^n}$ and $b = \dfrac{\pi}{4}$ we obtain the only solution of **(2)**,

$t_1 = t_2 = ... = t_n = \dfrac{2^{n-2}\pi}{2^n + 1}$ and then $x_1 = x_2 = ... = x_n = 2\cos\left(\dfrac{2^{n-2}\pi}{2^n+1}\right)$ is the only solution of original system.

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; and the proposer.

- **5276:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $a \in (0, 1]$ be a real number. Calculate

$$\int_0^1 a^{\left\lfloor \frac{1}{x} \right\rfloor} dx,$$

where $\lfloor x \rfloor$ denotes the floor of x.

(b) Calculate

$$\int_0^1 2^{-\left\lfloor \frac{1}{x} \right\rfloor} dx.$$

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

(a) Using the substitution $1/x = y$, the integral becomes $I = \int_1^\infty a^{\lfloor y \rfloor}/y^2 dy$. For any positive integer k and $y \in [k, k+1)$ we have $\lfloor y \rfloor = k$. Then

$$\begin{aligned} I &= \sum_{k=1}^\infty \int_k^{k+1} a^k/y^2 dy = \sum_{k=1}^\infty a^k \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \sum_{k=1}^\infty \frac{a^k}{k} - \sum_{k=1}^\infty \frac{a^k}{k+1} (\text{ since both series are absolutely convergent}) \\ &= -\ln(1-a) + \frac{\ln(1-a) + a}{a}. \end{aligned}$$

Since $\sum_{k=1}^\infty a^k = \dfrac{1}{1-a}$, and $\dfrac{a^k}{k} = \int_0^a x^{k-1} dx$ for $k \geq 1$.

(b) Since $2^{-\lfloor \frac{1}{x} \rfloor} = \left(\frac{1}{2}\right)^{\lfloor \frac{1}{x} \rfloor}$, then by part (a) we have

$$\int_0^1 2^{-\lfloor \frac{1}{x} \rfloor} dx = -\ln(1/2) + 2\ln(1/2) + 1 = 1 - \ln 2.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Proof (a). We change $y = 1/x$.

$$\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = \int_1^\infty \frac{a^{\lfloor y \rfloor}}{y^2} dy = \sum_{k=1}^\infty \int_k^{k+1} \frac{a^k}{y^2} dy = \sum_{k=1}^\infty a^k \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

If $a = 1$ we have telescoping

$$\sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1.$$

If $a < 1$ we

$$\sum_{k=1}^\infty a^k \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^\infty \int_0^a y^{k-1} dy - \frac{1}{a} \sum_{k=1}^\infty \int_0^a y^k dy$$

$$= e = \int_0^a \frac{dy}{1-y} - \frac{1}{a} \int_0^a \frac{y}{1-y} dy = \int_0^a \frac{dy}{1-y} + \frac{1}{a} \int_0^a dy - \frac{1}{a} \int_0^a \frac{1}{1-y} dy$$

$$= -\ln(1-a) + a + \frac{1}{a} \ln(1-a) = 1 + \frac{1-a}{a} \ln(1-a).$$

(b). If $a = 1/2$ we have $1 - \ln 2$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro,GA

The solutions:

$$(a) \quad \int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = \begin{cases} 1, & \text{if } a = 1 \\ 1 + \dfrac{1-a}{a} \ln(1-a), & \text{if } \ 0 < a < 1 \end{cases}$$

$$\text{(b)} \quad \int_0^1 2^{\lfloor \frac{1}{x} \rfloor} dx = 1 - \ln 2.$$

For part (a), note first that if $a = 1$, then $\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = 1$.

Henceforth, we assume $0 < a < 1$.

We shall use the following sums, for $x \in (0, 1]$.

By integrating $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ and re-indexing, we have the well-known sum:

(1) $\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x)$.

Then, by some algebraic manipulations, we have

(2) $\sum_{k=1}^{\infty} \frac{1}{k+1} x^k = -1 - \frac{1}{x} \ln(1-x)$.

If we partition the interval $(0, 1]$ into subintervals $\left(\frac{1}{k+1}, \frac{1}{k}\right]$, our integral can be written as a sum: $\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = \sum_{k=1}^{\infty} \int_{1/k+1}^{1/k} a^{\lfloor \frac{1}{x} \rfloor} dx$.

We see that

$$
\begin{aligned}
& \frac{1}{k+1} < x \leq \frac{1}{k} \\
\iff \quad & \frac{1}{k+1} < x \text{ and } x \leq \frac{1}{k} \\
\iff \quad & \frac{1}{x} < k+1 \text{ and } k \leq \frac{1}{x} \\
\iff \quad & k \leq \frac{1}{x} < k+1 \\
\iff \quad & \left\lfloor \frac{1}{x} \right\rfloor = k.
\end{aligned}
$$

Thus

$$\int_{1/k+1}^{1/k} a^{\lfloor \frac{1}{x} \rfloor} dx = \int_{1/k+1}^{1/k} a^k dx = a^k \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Therefore, summing and applying (1) and (2),

$$
\begin{aligned}
\int_{1/k+1}^{1/k} a^{\lfloor \frac{1}{x} \rfloor} dx &= \sum_{k=1}^{\infty} \frac{a^k}{k} - \sum_{k=1}^{\infty} \frac{a^k}{k+1} \\
&= -\ln(1-a) - \left\{ -1 - \frac{1}{a} \ln(1-a) \right\} \\
&= -\ln(1-a) + 1 + \frac{1}{a} \ln(1-a)
\end{aligned}
$$

$$= \quad 1 + \frac{1-a}{a}\ln(1-a).$$

For part (b), note that $\displaystyle\int_0^1 2^{-\left\lfloor \frac{1}{x}\right\rfloor} dx = \int_0^1 \left(\frac{1}{2}\right)^{\left\lfloor \frac{1}{x}\right\rfloor} dx.$

Applying the result for (a), this equals

$$1 + \frac{1-\frac{1}{2}}{\frac{1}{2}}\ln\left(1-\frac{1}{2}\right) = 1 + \ln\left(\frac{1}{2}\right) = 1 - \ln 2.$$

Also solved by Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Adrian Naco, Polytechnic University,Tirana, Albania, and the proposer.

Mea Culpa (once again)

When Enkel Hysnelaj of the University of Technology in Sydney, Australia submitted problem 5264, it came to me in several versions, with the successor version correcting an error he noticed in the previous version. Foolishly I kept all versions of the problem, and when I posted 5264, I posted an incorrect version of it. Problem 5274 is the corrected statement of the problem. Thanks to Ed Gray for coming up with a counter-example to 5264, and to Enkel for setting things straight in 5274.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before May 15, 2014

- **5295:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic hexagon has sides

  $$\left(5,\ 7\sqrt{17},\ 23\sqrt{13},\ 25\sqrt{13},\ 25\sqrt{17},\ 45\right).$$

  Find the diameter of the circumcircle and the area of the hexagon.

- **5296:** *Proposed by Roger Izard, Dallas, TX*

  Consider the "Star of David," a six pointed star made by overlapping the triangles ABC and FDE. Let

  $$\overline{AB}\cap\overline{DF}=G, \text{ and } \overline{AB}\cap\overline{DE}=H,$$
  $$\overline{AC}\cap\overline{DF}=L, \text{ and } \overline{AC}\cap\overline{FE}=K,$$
  $$\overline{BC}\cap\overline{DE}=I, \text{ and } \overline{BC}\cap\overline{FE}=J,$$

  in such a way that:

  $$\frac{CK}{AC}=\frac{EI}{DE}=\frac{BI}{BC}=\frac{GD}{DF}=\frac{AG}{AB}=\frac{FK}{EF} \text{ and}$$

  $$\frac{AL}{AC}=\frac{DH}{DE}=\frac{BH}{AB}=\frac{EJ}{EF}=\frac{FL}{DF}=\frac{CJ}{CB}.$$

  Let $r=\dfrac{CK}{AC}$ and let $p=\dfrac{AL}{AC}$. Prove that $r+p=\dfrac{3pr+1}{2}$.

- **5297:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Let $s_n=n^2,\ t_n=\dfrac{n(n+1)}{2},\ p_n=\dfrac{n(3n-1)}{2}$, for positive integers n, be the square, triangular and pentagonal numbers respectively. Prove, independently of each other, that

$$\begin{aligned} i)\ \ t_a + p_b &= t_c \\ ii)\ \ t_a + s_b &= p_c \\ iii)\ \ p_a + s_b &= s_c, \end{aligned}$$

for infinitely many positive integers, a, b, and c.

- **5298:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Let $(a_n)_{n\geq 1}$ be an arithmetic progression and m a positive integer. Calculate:

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_k} - me\right)n\right).$$

- **5299:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the aid of a computer, show that

$$\ln^2 2\int_0^1 \frac{x^{3/2}2^x \sin x}{(1+x\ln 2)^2}\,dx \ \geq\ \frac{1-\ln 2}{1+\ln 2}\int_0^1 \sqrt{x}\ \sin x\ dx.$$

- **5300:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n \geq 1$ be an integer. Prove that

$$\int_{\pi/4}^{\pi/2}\frac{dx}{\sin^{2n} x} = \sum_{k=0}^{n-1}\binom{n-1}{k}\cdot\frac{1}{2n-2k-1}.$$

Solutions

- **5277:** *Proposed by Kenneth Korbin, New York, NY*

  Find x and y if a triangle with sides $(2013, 2013, x)$ has the same area and the same perimeter as a triangle with sides $(2015, 2015, y)$.

Solution 1 by Carl Libis, Lane College, Jackson, TN

The perimeter of $(2013, 2013, x)$ equals the perimeter of $(2015, 2015, y)$ implies that $x = y + 4$.

Also, the altitude h_1 of $(2013, 2013, y + 4)$ bisects $y + 4$.

Use the Pythagorean Theorem on right triangle $(2013, h_1, (y+4)/2)$ to obtain $h_1 = \sqrt{2013^2 - (2+y/2)^2}$. Similarly for altitude h_2 of $(2015, 2015, y)$ we obtain $h_2 = \sqrt{2015^2 - (y/2)^2}$.

Equal areas implies that

$$\left(2+\frac{y}{2}\right)\sqrt{2013^2 - \left(2+\frac{y}{2}\right)^2} = \frac{y}{2}\sqrt{2015^2 - \left(\frac{y}{2}\right)^2}.$$

Square both sides, simplify, and then factor to obtain

$$\begin{aligned} 0 &= y^3 + 2020y^2 - 81043224y - 16208660 \\ &= (y+4030)(y^2 - 2010y - 4022) \\ &= (y+4030)(y^2 - 2010y - 4022) \\ &= (y+4030)\left(y - 1005 - \sqrt{1014047}\right)\left(y - 1005 + \sqrt{1014047}\right). \end{aligned}$$

The only positive solution of the three solutions is $y = 1005 + \sqrt{1014047} \approx 2012$.

Thus the values are: $y \approx 2012$ and $x \approx 2016$.

Solution 2 by proposer

The method to obtain x and y is to solve the system of equations:

$$\begin{cases} \dfrac{2y^2 + 8y + 12}{y+2} & = 2013{+}2015, \text{ and} \\ x = y + 4. & \end{cases}$$

If a triangle with sides (a, a, b) has the same area and the same perimeter as a triangle with sides (c, c, d), where a, b, c and d are positive integers, then the value of the area and the perimeter can be expressed in terms of b and d. Namely,

$$\begin{aligned} \text{Area} &= \frac{bd\sqrt{b^2 + bd + d^2}}{2b + 2d} \\ \text{Perimeter} &= \frac{2b^2 + 2bd + 2d^2}{b + d}. \end{aligned}$$

Comment by **David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.** More generally, if we let $k > 2$ be some positive constant and enforce the same "equal-area and equi-perimeter" condition on the two triangles (k, k, x) and $(k+2, k+2, y)$, we find the single solution

$$y = \frac{k - 3 + \sqrt{(k+1)^2 - 8}}{2} \text{ and } x = y + 4 = \frac{k + 5 + \sqrt{(k+1)^2 - 8}}{2}.$$

Also solved by Dionne Bailey, Elsie Camjpbell, and Charles Diminnie, Angelo State University, TX; Brian D. Beasely, Presbyterian College, Clinton, SC; D. M. Batinetu-Giurgiu, Bucharest, Romania, Neculai Stanciu, Buza, Romania, and Titu Zvonaru, Comanesi, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Michael Fried, Ben-Gurion University, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposer.

- **5278:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The triangular numbers $6 = (2)(3)$ and $10 = (2)(5)$ are each twice a prime number. Find all triangular numbers that are twice a prime.

Solution 1 by Ed Gray, Highland Beach, FL

The triangular numbers are given by: (1) $T_n = \dfrac{n(n+1)}{2}$, so if a triangular number is double a prime p, we must have the following equation: (2) $\dfrac{n(n+1)}{2} = 2p.$

First, suppose n is an even integer. Then $n = 2k$ for some integer k, and $\dfrac{n(n+1)}{2}$ becomes $\dfrac{2k(2k+1)}{2} = k(2k+1)$. If $k(2k+1) = 2p$, then k must be even, say $k = 2r$ and $k(2k+1) = 2r(4r+1) = 2p$. So, $r(4r+1) = p$. But p is prime and this implies that $r = 1, k = 2, n = 4$ and $\dfrac{(n)(n+1)}{2} = 10.$

Second, If n is odd, let $n = 2k+1$; then

$$\frac{n(n+1)}{2} = \frac{(2k+1)(2k+2)}{2} = (2k+1)(k+1) = 2p.$$

Here, $k+1$ must be even, say $k+1 = 2r$, and $(2k+1)(k+1) = 2r(4r-1) = 2p$. Since p is prime, $r = 1, k = 1, n = 3$ and $\dfrac{n(n+1)}{2} = 6$. Hence, all relevant triangular numbers were given in the statement of the problem.

Solution 2 by Paul M. Harms, North Newton, KS

Triangular numbers have the form $\dfrac{n(n+1)}{2}$ where n is a positive integer. For each positive integer n either n or $n+1$ has a factor of 2. When n is a positive integer greater then 4, the number $n, (n+1), \dfrac{n}{2}$, and $\dfrac{n+1}{2}$ are all greater than 2.

When $n > 4$, and an even integer, then $\dfrac{n}{2}$, is a prime number greater than 2 or a product of prime numbers, and $n+1$ is also a prime number greater than 2 or a product of prime numbers. In this case, $\dfrac{n}{2}(n+1)$ cannot be two times one prime number.

Similarly, when $n > 4$ and an odd number, n as well as $\dfrac{n+1}{2}$ are prime numbers greater than 2 or are a product of prime numbers. Then $n\dfrac{(n+1)}{2}$ cannot be two times one prime number.

The triangular numbers that are twice a prime must come from positive integers n which are not greater than 4. We see that the triangular numbers 6 when $n = 3$ and 10 when $n = 4$ are the only triangular numbers which are twice a prime number.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasely, Presbyterian College, Clinton, SC; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Neculai Stanciu and Titu Zvonaru, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5279:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let $f : \Re_+ \longrightarrow \Re_+$ be a convex function on $\Re_+$, where $\Re_+$ stands for the positive real numbers. Prove that

$$3\left(f^2(x)+f^2(y)+f^2(z)\right)-9f^2\left(\frac{x+y+z}{3}\right) \geq (f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2 .$$

Solution 1 by Arkady Alt, San Jose, CA

Since

$$\begin{aligned} & 3\left(f^2(x)+f^2(y)+f^2(z)\right)-(f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2 \\ = \ & (f(x)+f(y)+f(z))^2 , \end{aligned}$$

the original inequality is equivalent to

$$(f(x)+f(y)+f(z))^2 \geq 9f^2\left(\frac{x+y+z}{3}\right) \iff \frac{f(x)+f(y)+f(z)}{3} \geq f\left(\frac{x+y+z}{3}\right),$$

where the latter inequality is Jensen's Inequality for the convex function $f : \Re_+ \longrightarrow \Re_+$.

Solution 2 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Since f is convex, then $f\left(\dfrac{x+y+z}{3}\right) \leq \dfrac{f(x)+f(y)+f(z)}{3}$ and the left-hand side of the given inequality is

$$\begin{aligned} LHS &\geq 3\left(f^2(x)+f^2(y)+f^2(z)\right)-(f(x)+f(y)+f(z))^2 \\ &= 2\left(f^2(x)+f^2(y)+f^2(z)\right)-(2f(x)f(y)+2f(y)f(z)+2f(z)f(x)) \\ &= (f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2 . \end{aligned}$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Since f is convex we know that if $a \leq b$ and $0 < t < 1$ that

$$f\left(t \cdot a + (1-t) \cdot b\right) \leq t \cdot f(a) + (1-t) \cdot f(b).$$

(See, for example, the Chord Theorem in Calculus with Analytic Geometry (1978) by Flanders and Price, pages 153-154.)

Without loss of generality, let $0 < x \leq y \leq z$ and since $x \leq \dfrac{y+z}{2}$, we have, using the above result twice that:

$$\begin{aligned} f\left(\frac{x+y+z}{3}\right) = f\left(\frac{1}{3} \cdot x + \frac{2}{3} \cdot \left(\frac{y+z}{2}\right)\right) &\leq \frac{1}{3} \cdot f(x) + \frac{2}{3} \cdot \left(\frac{y+z}{2}\right) \\ &\leq \frac{1}{3} \cdot f(x) + \frac{2}{3} \cdot \left(\frac{1}{2} \cdot f(z) + \frac{1}{2} \cdot f(z)\right) \\ &= \frac{f(x)+f(y)+f(z)}{3}. \end{aligned}$$

Hence, $f(x)+f(y)+f(z) \geq 3 \cdot f\left(\dfrac{x+y+z}{3}\right)$ where the right hand side is positive by definition of f.

Squaring both sides gives

$$f^2(x)+f^2(y)+f^2(z)+2 \cdot f(x) \cdot f(y)+2 \cdot f(x) \cdot f(z)+2 \cdot f(y) \cdot f(z)-9 \cdot f^2\left(\frac{x+y+z}{3}\right) \geq 0,$$

which is clearly equivalent to the inequality to be proved.

Also solved by Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Titu Zvonaru, Comănesti, Romania, and the proposers.

- **5280:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a \geq b \geq c$ be nonnegative real numbers. Prove that

$$\frac{1}{3}\left(\frac{(a+b)(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}}\right) \leq \frac{(a+b)^2}{2+\sqrt{b+c}}.$$

Solution 1 by Greg Cook, Student, Angelo State University, San Angelo,TX

First, since $a \geq b \geq c \geq 0$, then $(a+b)(c+a) \leq (a+b)^2$ and
$2+\sqrt{a+b} \geq 2+\sqrt{b+c}$. Then,

$$\frac{(a+b)\,(c+a)}{2+\sqrt{a+b}} \leq \frac{(a+b)^2}{2+\sqrt{b+c}}. \qquad (1)$$

Again since $a \geq b \geq c \geq 0$, then $(c+a)\,(b+c) \leq (a+b)^2$ and
$2+\sqrt{c+a} \geq 2+\sqrt{b+c}$. Then,

$$\frac{(c+a)\,(b+c)}{2+\sqrt{c+a}} \le \frac{(a+b)^2}{2+\sqrt{b+c}}. \qquad (2)$$

Also, since $a \ge b \ge c \ge 0$, then $(b+c)(a+b) \le (a+b)^2$. Then,

$$\frac{(b+c)(a+b)}{2+\sqrt{b+c}} \le \frac{(a+b)^2}{2+\sqrt{b+c}}. \qquad (3)$$

Combining (1), (2), and (3),

$$\frac{(a+b)\,(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)\,(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}} \le 3\left(\frac{(a+b)^2}{2+\sqrt{b+c}}\right).$$

Finally,

$$\frac{1}{3}\left(\frac{(a+b)\,(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)\,(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}}\right) \le \frac{(a+b)^2}{2+\sqrt{b+c}}.$$

Solution 2 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

The inequality is a consequence of the Chebyshev's sum inequality. Note that sequences $(a+b)(c+a),\ (c+a)(b+c),\ (b+c)(a+b)$ and $\dfrac{1}{2+\sqrt{a+b}},\ \dfrac{1}{2+\sqrt{c+a}},\ \dfrac{1}{2+\sqrt{b+c}}$ are oppositely sorted. Therefore, the left-hand side of the given inequality LHS is bounded as

$$\begin{aligned} LHS &\le \frac{1}{3}\left((a+b)(c+a)+(c+a)(b+c)+(b+c)(a+b)\right) \\ &\quad \frac{1}{3}\left(\frac{1}{2+\sqrt{a+b}}+\frac{1}{2+\sqrt{c+a}}+\frac{1}{2+\sqrt{b+c}}\right) \\ &\le (a+b)(c+a)\frac{1}{2+\sqrt{b+c}} \\ &\le \frac{(a+b)^2}{2+\sqrt{b+c}}. \end{aligned}$$

Solution 3 by Arkady Alt, San Jose, CA

Note that:

1. $c \le b \iff c+a \le a+b \iff \dfrac{(a+b)\,(c+a)}{2+\sqrt{a+b}} \le \dfrac{(a+b)^2}{2+\sqrt{a+b}}$ and

$c \le a \iff 2+\sqrt{b+c} \le 2+\sqrt{a+b} \iff \dfrac{(a+b)^2}{2+\sqrt{a+b}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}}$ yields $\dfrac{(a+b)\,(c+a)}{2+\sqrt{a+b}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}}$;

2. $\begin{cases} a+b \ge c+a \\ a+b \ge b+c \end{cases}$ $\dfrac{(c+a)\,(b+c)}{2+\sqrt{c+a}} \le \dfrac{(a+b)^2}{2+\sqrt{c+a}}$ and $2+\sqrt{c+a} \ge 2+\sqrt{b+c}$ yields $\dfrac{(c+a)\,(b+c)}{2+\sqrt{c+a}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}}$;

3. $\dfrac{(b+c)\,(a+b)}{2+\sqrt{b+c}} \leq \dfrac{(a+b)^2}{2+\sqrt{b+c}} \iff b+c \leq a+b \iff c \leq a.$

Then $\dfrac{1}{3}\left(\dfrac{(a+b)\,(c+a)}{2+\sqrt{a+b}} + \dfrac{(c+a)\,(b+c)}{2+\sqrt{c+a}} + \dfrac{(b+c)\,(a+b)}{2+\sqrt{b+c}}\right) \leq$
$\dfrac{1}{3}\cdot 3\ frac(a+b)^2 2+\sqrt{b+c} = \dfrac{(a+b)^2}{2+\sqrt{b+c}}.$

Solution 4 by Michael Brozinsky, Central Islip, NY

Denote the left hand side and right hand side of the given inequality by L and R respectively. The inequality will be established if we can show the maximum value of L and the minimum value of R are equal to one another. Specifically, we will show that max L = min $R = \dfrac{4a^2}{2+2\sqrt{2a}}$, and that this occurs when $a=b=c$.

If we differentiate L, with respect to b we obtain

$$\frac{\partial}{\partial b}\left(\frac{1}{3}\left(\frac{(a+b)\cdot(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)\cdot(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)\cdot(a+b)}{2+\sqrt{b+c}}\right)\right) = \frac{1}{3}\cdot(A+B) \text{ where}$$

$$\begin{aligned} A &= \frac{c+a}{2+\sqrt{a+b}} - \frac{1}{2}\frac{\sqrt{a+b}\,(c+a)}{(2+\sqrt{a+b})^2} + \frac{c+a}{2+\sqrt{a+b}} \\ &= \frac{1}{2}\frac{(c+a)\left(16+4\sqrt{c+a}+10\sqrt{a+b}+\sqrt{a+b}\sqrt{c+a}+2a+2b\right)}{\left(2+\sqrt{a+b}\right)^2\left(2+\sqrt{c+a}\right)} \end{aligned}$$

and

$$\begin{aligned} B &= \frac{a+b}{2+\sqrt{b+c}} + \frac{b+c}{2+\sqrt{b+c}} - \frac{1}{2}\frac{\sqrt{b+c}\,(a+b)}{(2+\sqrt{b+c})^2} \\ &= \frac{1}{2}\frac{4a+a\sqrt{b+c}+8b+3b\sqrt{b+c}+4c+2c\sqrt{b+c}}{\left(2+\sqrt{b+c}\right)^2}. \end{aligned}$$

Since A and B are clearly non-negative and since $a \geq b \geq c$ we have L increases with b and so has its maximum when $b=a$.

Replacing b by a in L (call this expression M) and differentiating with respect to c gives

$$\frac{\partial}{\partial c}(M) = \frac{\partial}{\partial c}\left(\frac{1}{3}\left(\frac{2a(c+a)}{2+\sqrt{2a}} + \frac{(c+a)^2}{2+\sqrt{c+a}} + \frac{2(c+a)a}{2+\sqrt{c+a}}\right)\right)$$

$$= \frac{2}{3}\left(\frac{a}{2+\sqrt{2a}}\right) + \frac{2}{3}\left(\frac{c+a}{2+\sqrt{c+a}}\right) - \frac{1}{6}\frac{(c+a)\sqrt{c+a}}{\left(2+\sqrt{c+a}\right)^2}$$

$$+\quad \frac{2}{3}\left(\frac{a}{2+\sqrt{c+a}}\right)-\frac{1}{3}\frac{\sqrt{c+a}\;a}{(2+\sqrt{c+a})^2}$$

which simplifies to

$$\frac{1}{6}\;\frac{1}{(2+\sqrt{2a})(2+\sqrt{c+a})^2}\Big(48a+26\sqrt{c+a}\;a+4ac+4a^2+16c+6c\sqrt{c+a}$$
$$+8c\sqrt{2a}+3c\sqrt{2a}\sqrt{c+a}+16a\sqrt{2a}+5a\sqrt{2a}\sqrt{c+a}\Big).$$

Since this derivative is clearly nonnegative, M increases with c and since $a \geq c$, M is maximized when $c = a$. So, L is maximized when b and c are both a. This value is $\dfrac{4a^2}{2+\sqrt{2a}}$.

Now if R is differentiated with respect to a we obtain.

$$\frac{\partial}{\partial a}\left(\frac{(a+b)^2}{2+\sqrt{b+c}}\right)=\frac{2(a+b)}{2+\sqrt{b+c}}$$

which is clearly nonnegative and so R increases with a and since $a \geq b$ is minimized when $a = b$.

Replacing a by b in R (call this expression N) we have

$$\frac{\partial}{\partial b}(N)=\frac{\partial}{\partial b}\left(\frac{(2b)^2}{2+\sqrt{b+c}}\right)=\frac{2b\left(8\sqrt{b+c}+3b+4c\right)}{\left(2+\sqrt{b+c}\right)^2\sqrt{b+c}}$$

which is clearly nonnegative. So, N increases with b, and since $b \geq c$ is minimized when $b = c$, R is minimized when $a = b = c$, and has value of $\dfrac{4a^2}{2+\sqrt{2a}}$.

Editor's Comment: **D. M. Bătinetu-Giurgiu, Neuclai Stanciu and Titu Zvonaru, all of Romania,** jointly constructed and proved a generalization of Problem 5280. Their generalization follows:

Let $n \in N,\ n \geq 3,\ a = x_1 \geq b = x_2 \geq x_3 \geq \ldots \geq c = x_{n-1} \geq d = x_n > 0$ and $u, v \in R_+ = (0, \infty)$.

If $x_{n+1} = x_1, x_{n+2} = x_2$, then

$$\sum_{k=1}^{n}\frac{(x_k+x_{k+1})(x_k+x_{k+2})}{u+v\sqrt{x_{k+1}+x_{k+2}}}\leq\frac{n(a+b)^2}{u+v\sqrt{c+d}}.$$

Letting $n = 3,\ x_1 = a,\ x_2 = b,\ x_3 = c$ and $u = 2,\ v = 1$, they showed that the inequality holds.

Also solved by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College Bucharest, Neuclai Stanciu, "George Emil Palade" School, Buzău, and Titu Zvonaru, Comănesti, Romania; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland

Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

- **5281:** *Proposed by Arkady Alt, San Jose, CA*

For the sequence $\{a_n\}_{n\geq 1}$ defined recursively by $a_{n+1} = \dfrac{a_n}{1+a_n^p}$ for $n \in \mathcal{N}, a_1 = a > 0$, determine all positive real p for which the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $p < 1$.

Proof: Since $a_{n+1} < a_n$, $a_n \to 0$.
It follows that

$$a_{n+1} = a_n - a_n^{p+1} + a_n^{2p+1} + O(a_n^{3p+1})$$

We employ the standard result of the exercise num.174 at page 38 of the book by G. Pólya, G. Szegö, *Problems and Theorems in Analysis, I.*

Assume that $0 < f(x) < x$ and $f(x) = x - ax^k + bx^l + x^l\varepsilon(x)$, $\lim_{x\to 0}\varepsilon(x) = 0$, for $0 < x < x_0$ where $1 < k < l$ and a, b both positive. The sequence x_n defined by $x_{n+1} = f(x_n)$ satisfies

$$\lim_{n\to\infty} n^{1/(k-1)} x_n = (a(k-1))^{-1/(k-1)}.$$

In our case we have $a = 1$, $k = p+1$, $b = 1$, $l = 2p+1$. Thus the sequence satisfies

$$a_n = p^{-1/p} n^{-1/p} + o(n^{-1/p})$$

and then the series converges if and only if $p < 1$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if $0 < p < 1$ and divergent if ≥ 1.

We assume in what follows that $n \in N$. Clearly $a_n > 0$ and by the given recursive relation, we have $a_{n+1} < a_n$. Therefore $L = \lim\limits_{n\to\infty} a_n$ exists and from $L = \dfrac{L}{1+L^p}$, we see that $L = 0$. Inductively, we have

$$a_{n+1} = \frac{a}{\displaystyle\prod_{k=1}^{n}(1+a_k^p)}. \qquad (1)$$

By making use of the well-known inequality $1 + x < e^x$ for $x > 0$, we deduce from (1) that $a_{n+1} > ae^{-\sum_{k=1}^{n} a_k^p} > 0$. Since $\lim\limits_{n\to\infty} a_{n+2} = 0$, so $\displaystyle\sum_{k=1}^{n} a_k^p$ is divergent. Now there exits $k_0 \in N$, depending at most on a and p, such that $a_k < 1$ whenever $k > k_0$. Hence if $p \geq 1$, then for any integer $M > k_0$, we have $\displaystyle\sum_{k=k_0+1}^{M} a_k \geq \sum_{k=k_0+1}^{M} a_k^p$. Thus $\displaystyle\sum_{k=+1}^{\infty} a_k$ is divergent.

We next consider the case $0 < p < 1$. Let $m = \left\lfloor \dfrac{1}{1-p} \right\rfloor + 1$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. By (1), for any $n > m$, we have

$$0 < a_{n+1} \leq \frac{a}{(1+a_n^p)^n} < \frac{a}{(1+a_{n+1}^p)^n} < \frac{a}{\dbinom{n}{m} a_{n+1}^{mp}},$$

so that

$$0 < a_{n+1} < \left(\frac{am!}{\prod_{k=0}^{m-1}(n-k)}\right)^{1/(1+mp)} \leq \left(\frac{am!}{(n-m+1)^m}\right)^{1/(1+mp)}.$$

It is easy to check that $\dfrac{m}{1+mp} > 1$, and so $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

This completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5282:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate
$$\int_0^1 x \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) \ln\left(\sqrt{1+x} + \sqrt{1-x}\right) dx.$$

Solution 1 by Anastasios Kotronis, Athens, Greece

Using the identity
$$ab = \frac{1}{4} \cdot a + b^2 - a - b^2,$$
with $a = \ln \sqrt{1+x} - \sqrt{1-x}$ and $b = \ln \sqrt{1+x} + \sqrt{1-x}$ we have

$$\begin{aligned}
I &= \int_0^1 x \ln \sqrt{1+x} - \sqrt{1-x} \ln \sqrt{1+x} + \sqrt{1-x}\, dx \\
&= \frac{1}{4} \int_0^1 x\ln^2(2x) - \ln^2 \frac{1 - \sqrt{\dfrac{1-x}{1+x}}}{1 + \sqrt{\dfrac{1-x}{1+x}}}\, dx \\
&= \frac{1}{4} \int_0^1 x \ln^2(2x)\, dx - \frac{1}{4} \int_0^1 x \ln^2 \frac{1 - \sqrt{\dfrac{1-x}{1+x}}}{1 + \sqrt{\dfrac{1-x}{1+x}}}\, dx \\
&= I_1 - I_2.
\end{aligned}$$

Integrating by parts twice we easily get that

$$I_1 = \frac{\ln^2 2}{8} - \frac{\ln 2}{8} + \frac{1}{16}. \qquad (1)$$

In order to calculate I_2, we first note that

$$\begin{aligned}
\int \frac{u(1-u^2)}{(1+u^2)^3}\,du \quad u^2 = y \quad & \frac{1}{2}\int \frac{1-y}{(1+y)^3}\,dy \\
= \quad & \int \frac{1}{(1+y)^3}\,dy - \frac{1}{2}\int \frac{1}{(1+y)^2} \\
= \quad & \frac{u^2}{2(1+u^2)^2} + c,
\end{aligned}$$

so, letting $\sqrt{\dfrac{1-x}{1+x}} = y$ and letting $\dfrac{1-y}{1+y} = u$ we have

$$\begin{aligned}
\frac{1}{4}\int x \ln^2 \frac{1-\sqrt{\dfrac{1-x}{1+x}}}{1+\sqrt{\dfrac{1-x}{1+x}}}\,dx &= \int \frac{y(1-y^2)}{(1+y^2)^3} \ln^2 \frac{1-y}{1+y}\,dy \\
&= \int \frac{u(1-u^2)}{(1+u^2)^3} \ln^2 u\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} - \int \frac{u}{2(1+u^2)^2} \ln u\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} - \int -\frac{1}{2(1+u^2)}' \ln u\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} + \frac{\ln u}{2(1+u^2)} - \frac{1}{2}\int \frac{1}{u} - \frac{u}{1+u^2}\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} + \frac{\ln u}{2(1+u^2)} - \frac{\ln u}{2} + \frac{\ln(1+u^2)}{4} + \\
&= A(x) + c
\end{aligned}$$

which yields

$$I_2 = A(x)\Big|_0^1 = \lim_{x\to 0+} A(x) - \lim_{x\to 1-} A(x) = \frac{\ln 2}{4}, \qquad (2)$$

and hence, from (1) and (2), $I = \dfrac{\ln^2 2}{8} - \dfrac{\ln 8}{8} + \dfrac{1}{16}$.

Solution 2 by Arkady Alt, San Jose, CA

Solution A.

Let $I=\int\limits_0^1 x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)dx.$

Then $4I=\int\limits_0^1 x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)^2\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)^2 dx=\int\limits_0^1 xu(x)\,v(x)\,dx,$

where $u(x)=\ln\left(2+2\sqrt{1-x^2}\right), v(x)=\ln\left(2-2\sqrt{1-x^2}\right).$

Since $u(x)+v(x)=\ln\left(4x^2\right)=2\ln(2x)$ then

$u^2(x)+v^2(x)+2u(x)\,v(x)=4\ln^2(2x)\iff u(x)\,v(x)=2\ln^2(2x)-\dfrac{u^2(x)+v^2(x)}{2}$

and, therefore, $4I=2\int\limits_0^1 x\ln^2(2x)\,dx-\dfrac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx+\int\limits_0^1 xv^2(x)\,dx\right).$

1. Using substitution and integration by parts we obtain

$2\int\limits_0^1 x\ln^2(2x)\,dx=[t=2x;dt=2dx]=\dfrac{1}{2}\int\limits_0^2 t\ln^2(t)\,dt=\ln^2 2-\dfrac{1}{2}\int\limits_0^2 t\ln t\,dt=$

$\ln^2 2-\ln 2+\dfrac{1}{2}.$

2. Let $t=2+2\sqrt{1-x^2}$. Since $xdx=-\dfrac{(t-2)\,dt}{4}$ then

$\int\limits_0^1 xu^2(x)\,dx=\dfrac{1}{4}\int\limits_4^2-(t-2)\ln^2 tdt=\dfrac{1}{4}\int\limits_2^4(t-2)\ln^2 tdt.$

3. Let $t=2-2\sqrt{1-x^2}$. Since $xdx=\dfrac{(2-t)\,dt}{4}$ then

$\int\limits_0^1 xv^2(x)\,dx=\dfrac{1}{4}\int\limits_0^2(2-t)\ln^2 tdt=-\dfrac{1}{4}\int\limits_0^2(t-2)\ln^2 tdt.$

Hence $\dfrac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx+\int\limits_0^1 xv^2(x)\,dx\right)=\dfrac{1}{8}\left(\int\limits_2^4(t-2)\ln^2 tdt-\int\limits_0^2(t-2)\ln^2 tdt\right)=$

$\dfrac{1}{8}\left(\int\limits_0^4(t-2)\ln^2 tdt-2\int\limits_0^2(t-2)\ln^2 tdt\right).$

Using integration by parts twice we obtain

$\int(t-2)\ln^2 tdt=\begin{bmatrix} p'=t-2;p=\dfrac{t^2}{2}-2t\\ q=\ln^2 t;q'=\dfrac{2\ln t}{t}\end{bmatrix}=\left(\dfrac{t^2}{2}-2t\right)\ln^2 t-\int(t-4)\ln tdt=$

$\left(\dfrac{t^2}{2}-2t\right)\ln^2 t-\left(\dfrac{t^2}{2}-4t\right)\ln t+\dfrac{t^2}{4}-4t.$

Since $\int_0^4 (t-2)\ln^2 t dt = \left(\left(\frac{t^2}{2}-2t\right)\ln^2 t - \left(\frac{t^2}{2}-4t\right)\ln t + \frac{t^2}{4} - 4t\right)_0^4 = 8\ln 4 - 12$
and
$\int_0^2 (t-2)\ln^2 t dt = \left(\left(\frac{t^2}{2}-2t\right)\ln^2 t - \left(\frac{t^2}{2}-4t\right)\ln t + \frac{t^2}{4} - 4t\right)_0^2 = 6\ln 2 - 2\ln^2 2 - 7$
then $\frac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx + \int\limits_0^1 xv^2(x)\,dx\right) = \frac{1}{8}\left(8\ln 4 - 12 - 2\left(6\ln 2 - 2\ln^2 2 - 7\right)\right) =$
$\frac{1}{2}\ln 2 + \frac{1}{2}\ln^2 2 + \frac{1}{4}$.
Therefore, $4I = \ln^2 2 - \ln 2 + \frac{1}{2} - \left(\frac{1}{2}\ln 2 + \frac{1}{2}\ln^2 2 + \frac{1}{4}\right) = \frac{1}{2}\ln^2 2 - \frac{3}{2}\ln 2 + \frac{1}{4}$
$I = \frac{1}{8}\ln^2 2 - \frac{3}{8}\ln 2 + \frac{1}{16} \approx -0.137\,37$

Solution B.

Let $u(x) = \ln\left(\sqrt{1+x}+\sqrt{1-x}\right)$, $v(x) = \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)$ and
$I = \int\limits_0^1 xu(x)v(x)\,dx.$

Since $u(x)+v(x) = \ln\left(\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2\right) = \ln(2x)$ then

$$u(x)v(x) = \frac{\ln^2(2x) - u^2(x) - v^2(x)}{2}$$

and, therefore, $2I = \int\limits_0^1 x\ln^2(2x)\,dx - \int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx.$

Calculation of $\int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx.$

1. Let $t = \sqrt{1+x}+\sqrt{1-x}$. Then $u^2(x) = \ln^2 t$ and
$t^2 = 2 + 2\sqrt{1-x^2} \iff \frac{t^2-2}{2} = \sqrt{1-x^2}$
yield $tdt = \frac{-xdx}{\sqrt{1-x^2}} \iff xdx = -\frac{t(t^2-2)}{2}dt.$
Hence, $\int\limits_0^1 xu^2(x)\,dx = -\int\limits_2^{\sqrt{2}} \frac{t(t^2-2)}{2}\ln^2 t dt = \frac{1}{2}\int\limits_{\sqrt{2}}^2 t\left(t^2-2\right)\ln^2 t dt;$
2. Let $t = \sqrt{1+x}-\sqrt{1-x}$. Then $v^2(x) = \ln^2 t$ and
$t^2 = 2 - 2\sqrt{1-x^2} \iff \frac{2-t^2}{2} = 2\sqrt{1-x^2}$
yield $-tdt = \frac{-x}{\sqrt{1-x^2}}dx \iff xdx = \frac{t(2-t^2)}{2}dt$. Hence,

$$\int\limits_0^1 xu^2(x)\,dx = \int\limits_0^{\sqrt{2}} \frac{t(2-t^2)}{2}\ln^2 t dt = -\frac{1}{2}\int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt$$

and we obtain $\int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx = \frac{1}{2}\int\limits_{\sqrt{2}}^{2} t\left(t^2-2\right)\ln^2 t dt - \frac{1}{2}\int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt =$
$\frac{1}{2}\int\limits_0^2 t\left(t^2-2\right)\ln^2 t dt - \int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt.$

Using integration by parts twice we obtain we obtain

$\int t\left(t^2-2\right)\ln^2 t dt = \left[\begin{array}{c} p'=t^3-2t;\ p=\frac{t^4}{4}-t^2 \\ q=\ln^2 t;\ q'=\frac{2\ln t}{t} \end{array}\right] =$
$\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \int\left(\frac{t^3}{2}-2t\right)\ln t dt =$
$\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \left(\frac{t^4}{8}-t^2\right)\ln t + \int\left(\frac{t^3}{8}-t\right)dt =$
$\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \left(\frac{t^4}{8}-t^2\right)\ln t + \left(\frac{t^4}{32}-\frac{t^2}{2}\right).$

Hence,
$\int\limits_0^2 t\left(t^2-2\right)\ln^2 t dt = \left(\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \left(\frac{t^4}{8}-t^2\right)\ln t + \left(\frac{t^4}{32}-\frac{t^2}{2}\right)\right)_0^2 = 2\ln 2 - \frac{3}{2},$
$\int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt = \left(\frac{\sqrt{2}^4}{4}-\sqrt{2}^2\right)\ln^2\sqrt{2} - \left(\frac{\sqrt{2}^4}{8}-\sqrt{2}^2\right)\ln\sqrt{2} + \left(\frac{\sqrt{2}^4}{32}-\sqrt{2}^2 2\right) =$
$\frac{3}{4}\ln 2 - \frac{1}{4}\ln^2 2 - \frac{7}{8}$ and, therefore,
$\int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx = \frac{1}{2}\left(2\ln 2-\frac{3}{2}\right) - \left(\frac{3}{4}\ln 2 - \frac{1}{4}\ln^2 2 - \frac{7}{8}\right) =$
$\frac{1}{4}\left(\ln^2 2 + \ln 2 + \frac{1}{2}\right).$

Since (using integration by parts again)
$\int\limits_0^1 x\ln^2(2x)dx = \frac{1}{4}\int\limits_0^1 2x\ln^2(2x)\cdot 2dx = \frac{1}{4}\int\limits_0^2 t\ln^2 t dt = \frac{1}{4}\left(\frac{t^2}{2}\left(\ln^2 t - \ln t + \frac{1}{2}\right)\right)_0^2 =$
$\frac{1}{2}\left(\ln^2 2 - \ln 2 + \frac{1}{2}\right)$ then $I = \frac{1}{2}\left(\frac{1}{2}\left(\ln^2 2 - \ln 2 + \frac{1}{2}\right) - \frac{1}{4}\left(\ln^2 2 + \ln 2 + \frac{1}{2}\right)\right) =$
$\frac{1}{8}\left(\ln^2 2 - 3\ln 2 + \frac{1}{2}\right) \approx -0.137\,37.$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the integral of the problem by I. We show that

$$I = \frac{2\ln^2 2 - 6\ln 2 + 1}{16}. \qquad (1)$$

Let $I_1 = \int_0^1 x\ln^2(2x)dx,\quad I_2 = \int_0^1 x\ln^2\left(\sqrt{1+x}-\sqrt{1-x}\right)dx$ and

$I_3 = \int_0^1 x \ln^2\left(\sqrt{1+x} - \sqrt{1-x}\right) dx$. Using the identity $ab = \dfrac{(a+b)^2 - a^2 - b^2}{2}$ with $a = \ln\left(\sqrt{1+x} - \sqrt{1-x}\right)$ and $b = \ln\left(\sqrt{1+x} + \sqrt{1-x}\right)$, we see that

$$I = \frac{1}{2}\left(I_1 - I_2 - I_3\right). \qquad (2)$$

To evaluate I_1, I_2, and I_3, we need the known result, readily proved by differentiation, that for nonnegative integer n,

$$\int x^n \ln^2 x dx = x^{n+1}\left(\frac{\ln^2 x}{n+1} - \frac{2\ln x}{(n+1)^2} + \frac{2}{(n+1)^3}\right) + \text{constant} \qquad (3)$$

Since $I_1 = \dfrac{1}{4}\displaystyle\int_0^2 x \ln^2 x dx$, so by (3) we have

$$I_1 = \frac{2\ln^2 - 2\ln 2 + 1}{4}. \qquad (4)$$

Since $(\sqrt{1+x} - \sqrt{1-x})^2 = 2(1 - \sqrt{1-x^2})$, so

$$I_2 = \frac{1}{4}\int_0^1 x \ln^2\left(2\left(1 - \sqrt{1-x^2}\right)\right) = \frac{1}{8}\int_0^1 \ln^2\left(2\left(1 - \sqrt{1-x}\right)\right) dx.$$

By the substitution $y = 2(1 - \sqrt{1-x})$, so that $x = y - \dfrac{y^2}{4}$, we obtain $I_2 = \dfrac{1}{16}\displaystyle\int_0^2 (2-y)\ln^2 y dy$. By (3) we have

$$I_2 = \frac{2\ln^2 2 - 6\ln 2 + 7}{16}. \qquad (5)$$

By using the identity $\left(\sqrt{1+x} + \sqrt{1-x}\right)^2 = 2\left(1 + \sqrt{1-x^2}\right)$, we obtain

$$I_3 = \frac{1}{4}\int_0^1 x \ln^2\left(2\left(1 + \sqrt{1-x^2}\right)\right) dx = \frac{1}{8}\int_0^1 x \ln^2\left(2\left(1 + \sqrt{1-x}\right)\right) dx.$$

By the substitution $y = 2\left(1 + \sqrt{1-x}\right)$, so that $x = y - \dfrac{y^2}{4}$, we obtain

$I_3 = \dfrac{1}{16}\displaystyle\int_2^4 (y-2)\ln^2 y \, dy$. By (3),we have

$$I_3 = \frac{2\ln^2 2 + 10\ln 2 - 5}{16}. \qquad (6)$$

Now by $(2), (4), (5)$ and (6), we obtain (1) and this completes the solution.

Editor's comment: **Ed Gray of Highland Beach, FL** transformed the given integral into

$$\frac{1}{4}\int_2^{\sqrt{2}} \left(2y - y^3\right) \ln y \left(\ln(2-y) + \ln(2+y)\right) dy$$

and then he converted the various ln functions into series expansions to obtain a polynomial in y. This gave the approximate value of the integral as listed above.

Also solved (in closed form) by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

Comment by the proposer, Ovidiu Furdui: It is worth mentioning this logarithmic integral is missing from the book by Gradshteyn and Ryzhik, *Tables of Integrals, Series and Products,* Sixth Edition, Academic Press, 2000.

Late Solutions

Late solutions to 5271 and to 5273 were received by **Paul M. Harms of North Newton, KS** and from **David E. Manes, SUNY College at Oneonta, NY.** Their solutions were mailed on time but they got caught up in the Christmas rush mail, and arrived on my desk after the solutions to these problems had been published.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2014

- **5301:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic quadrilateral with integer length sides is such that its area divided by its perimeter equals 2014.

  Find the maximum possible perimeter.

- **5302:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  If n is an even perfect number, $n > 6$, and $\phi(n)$ is the Euler phi-function, then show that $n - \phi(n)$ is a fourth power of an integer. Find infinitely many integers n such that $n - \phi(n)$ is a fourth power.

- **5303:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let a, b, c, d be positive real numbers. Prove that

$$a^4 + b^4 + c^4 + d^4 + 4 \;\geq\; 4\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4}.$$

- **5304:** *Proposed by Michael Brozninsky, Central Islip, NY*

  Determine whether or not there exist nonzero constants a and b such that the conic whose polar equation is

$$r = \sqrt{\frac{a}{\sin(2\theta) - b\cdot\cos(2\theta)}}$$

  has a rational eccentricity.

- **5305:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let x be a positive real number. Prove that

$$\frac{[x]}{2x+\{x\}} + \frac{[x]\{x\}}{3x^2} + \frac{\{x\}}{2x+[x]} \;\leq\; \frac{1}{2},$$

  where $[x]$ is the greatest integer function and $\{x\}$ is the fractional part of the real number. I.e., $\{x\} = x - [x]$.

- **5306:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate:
$$\int_0^1 \frac{\ln\left(1-x+x^2\right)}{x-x^2}dx.$$

Solutions

- **5283:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles that both have perimeter 162 and area 1008.

  Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

  To begin, we will let the isosceles triangle be designated with sides (a, a, x) and height h. With given perimeter 162,
$$\begin{aligned} x &= 162-2a &(1)\\ \frac{x}{2} &= 81-a, &(2)\end{aligned}$$
  and, using the Pythagorean Theorem and (2),
$$\begin{aligned} h^2+\left(\frac{x}{2}\right)^2 &= a^2\\ h^2+(81-a)^2 &= a^2\\ h &= 9\sqrt{2a-81}.\end{aligned}$$
  Thus, with given area 1008, (1), and (3),
$$\begin{aligned} \frac{1}{2}(162-2a)(9\sqrt{2a-81}) &= 1008\\ \frac{112}{81-a} &= \sqrt{2a-81}\\ 2\,a^3-405\,a^2+26,244\,a-543,985 &= 0.\end{aligned}$$
  Using Mupad, the solutions are
$$a=\frac{275-7\sqrt{177}}{4},\quad 65,\quad \frac{7\sqrt{177}+275}{4}.$$
  Using (1), $a=\dfrac{7\sqrt{177}+275}{4}$ does not yield a triangle with perimeter 162. Hence, using (1), when $a=\dfrac{275-7\sqrt{177}}{4}$, $x=\dfrac{49+7\sqrt{177}}{2}$, and when $a=65$, $x=32$. Therefore, the isosceles triangles are $\left(\dfrac{275-7\sqrt{177}}{4}, \dfrac{275-7\sqrt{177}}{4}, \dfrac{49+7\sqrt{177}}{2}\right)$ and (65, 65, 32).

With some persistence, these solutions can be verified to yield an isosceles triangle with perimeter 162 and area 1008.

Solution 2 by Arkady Alt, San Jose, CA

Let b be length of the lateral sides and a be half of length of the base.

Then $\begin{cases} 2a+2b=162 \\ a\sqrt{b^2-a^2}=1008 \end{cases} \iff \begin{cases} a+b=81 \\ a\sqrt{b-a}=112 \end{cases} \iff \begin{cases} b=81-a \\ a\sqrt{81-2a}=112 \end{cases}$

We have $a\sqrt{81-2a}=112 \iff \begin{cases} 0<a\leq 81/2 \\ a^2(81-2a)=112^2 \end{cases}$ and the equation

$a^2(81-2a)=16^2\cdot 49 \iff 2a^3-81a^2+112^2=0.$

Since $2a^3-81a^2+112^2=(a-16)\left(2a^2-49a-784\right)$ and the quadratic equation

$2a^2-49a-784=0$ have only one positive root $a=\dfrac{49+7\sqrt{177}}{4}$ then we obtain two different isosceles triangles with side-lengths

$(b,2a,b)=(65,32,65)\,, \quad \left(\dfrac{275-7\sqrt{177}}{4},\dfrac{49+7\sqrt{177}}{2},\dfrac{275-7\sqrt{177}}{4}\right).$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let the sides of the isosceles triangles be $a, a, 162-2a$. By Heron's formula for the area of a triangle we obtain

$$(81-a)\sqrt{2a-81}=112,$$

or

$$(81-a)^2(2a-81)-12544=0,$$

or

$$(a-65)((2a^2-275a+8369)=0.$$

Hence $a=65,\ \dfrac{275-7\sqrt{177}}{4}$. So the sides of the isosceles triangles are $65,\ 65,\ 32$ and $\dfrac{275-7\sqrt{177}}{4}, \dfrac{275-7\sqrt{177}}{4}, \dfrac{7(7+\sqrt{177})}{2}$.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Any such triangle has sides with lengths x, x, and $162-2x$, where $81/2<x<81$. Heron's formula then implies

$$1008^2=81(81-x)^2(2x-81),$$

which in turn is equivalent to

$$2x^3-405x^2+26244x-543985=(x-65)(2x^2-275x+8369)=0.$$

We find three real solutions to this equation, namely $x=65$ and $x=(275\pm 7\sqrt{177})/4$; however, one of these yields $x\approx 92.032$, which is outside the necessary domain. Hence we obtain two triangles, corresponding to $x=65$ and $x\approx 45.468$:

$$(65,65,32); \qquad \left(\frac{275-7\sqrt{177}}{4},\frac{275-7\sqrt{177}}{4},\frac{49+7\sqrt{177}}{2}\right)\approx(45.468,45.468,71.064).$$

Question. In general, if we seek all isosceles triangles of the form $(x, x, P-2x)$ that have perimeter P and area A, then we obtain the equation

$$16Px^3 - 20P^2x^2 + 8P^3x - (P^4 + 16A^2) = 0.$$

The given values $P = 162$ and $A = 1008$ produce exactly two such triangles. For what values of P and A would we find no triangles, one triangle, two triangles, or three triangles?

Also solved by Bruno Salgueiro Fanego, Viveriro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Angel Plaza, Universidad de Las Palmas, de Gran Canaria, Spain; Michael Thew, Student, St. George's School, Spokane, WA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5284:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Prove:

a) $3^{3^n} + 1 \equiv 0 \text{ mod } 28,\ \forall \text{n} \geq 1,$

b) $3^{3^n} + 1 \equiv 0 \text{ mod } 532,\ \forall \text{n} \geq 2,$

c) $3^{3^n} + 1 \equiv 0 \text{ mod } 19684,\ \forall \text{n} \geq 3,$

d) $3^{3^n} + 1 \equiv 0 \text{ mod } 3208492,\ \forall \text{n} \geq 4.$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Note the following congruences:

$$\begin{aligned} 3 &\equiv -1 \text{ mod } 4,\quad 3^3 \equiv -1 \text{ mod } 7,\quad 3^9 \equiv -1 \text{ mod } 19 \\ 3^{27} &\equiv -1 \text{ mod } 37,\quad 3^{81} \equiv -1 \text{ mod } 63 \end{aligned}$$

.
Therefore,

(1) $3^{3^n} + 1 \equiv (-1)^{3^n} + 1 \equiv -1 + 1 \equiv 0 \pmod 4\ \ \forall n \geq 1,$

(2) $3^{3^n} + 1 \equiv \left(3^3\right)^{3^{n-1}} + 1 \equiv (-1)^{3^{n-1}} + 1 \equiv -1 + 1 \equiv 0 \pmod 7\ \ \forall n \geq 1,$

(3) $3^{3^n} + 1 \equiv \left(3^{3^2}\right)^{3^{n-2}} + 1 \equiv (-1)^{3^{n-2}} + 1 \equiv -1 + 1 \equiv 0 \pmod{19}\ \ \forall n \geq 2,$

(4) $3^{3^n} + 1 \equiv \left(3^{3^3}\right)^{3^{n-3}} + 1 \equiv (-1)^{3^{n-3}} + 1 \equiv -1 + 1 \equiv 0 \pmod{37}\ \ \forall n \geq 3,$

(5) $3^{3^n} + 1 \equiv \left(3^{3^4}\right)^{3^{n-4}} + 1 \equiv (-1)^{3^{n-4}} + 1 \equiv -1 + 1 \equiv 0 \pmod{163}\ \ \forall n \geq 4.$

Recall the elementary property of congruences : if $a \equiv b$ mod m and $a \equiv b$ mod n and $\gcd(m, n) = 1$, then $a \equiv b$ mod $m \cdot n$

Therefore,

(a) since $\gcd(4, 7) = 1$, it follows from (1) and (2) that $3^{3^n} + 1 \equiv 0$ mod 28 $\forall n \geq 1$,

(b) since $\gcd(19,28)=1$, it follows from (a) and (3) that $3^{3^n}+1\equiv 0 \bmod 532 = 19\cdot 28$ $\forall n\geq 2$,

(c) since $\gcd(37,532)=1$, it follows from (b) and (4) that $3^{3^n}+1\equiv 0 \bmod 19684$ $\forall n\geq 3$,

(d) since $\gcd(163,19684)=1$, it follows from (c) and (5) that $3^{3^n}+1\equiv 0 \bmod$ 3208492 $\forall n\geq 4$. This completes the solution.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have
$28{=}2^2\times 7,\ 532=2^2\times 7\times 19,\ 19684=2^2\times 7\times 19\times 37,\ 3208492=2^2\times 7\times 19\times 37\times 163,$
$3^3\equiv -1(\bmod\ 28), 3^9\equiv -1(\bmod\ 19),\ 3^{27}\equiv -1(\bmod\ 37),\ 3^{81}\equiv -1(\bmod\ 163).$

Statement a) is true fore $n=1$, statement b) is true for $n=2$, statement c) is true for $n=3$, statement d) is true for $n=4$.

The general statment then follows by induction: If $3^{3^n}\equiv -1 \pmod a$ where $(a,3)=1$ then $3^{3^{n+1}}\equiv \left(3^{3^n}\right)^3\equiv(-1)^3\equiv -1(\bmod a)$.

Also solved by Arkady Alt, San Jose, CA; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5285:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzu, Romania*

Let $\{a_n\}_{n\geq 1}$, and $\{b_n\}_n\geq 1$ be positive sequences of real numbers with $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a\in\Re_+$ and $\lim\limits_{n\to\infty}\dfrac{b_{n+1}}{nb_n}=b\in\Re_+$.

For $x\in\Re$, calculate

$$\lim_{n\to\infty}\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x}-\left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since the $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a$, then by the Stolz Theorem $\lim\limits_{n\to\infty}\dfrac{a_n}{n}=a$. Also note that

$$\lim_{n\to\infty}\frac{\dfrac{b_{n+1}}{(n+1)!}}{\dfrac{b_n}{n!}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)\,b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)\,b_n}\cdot\frac{n+1}{n}=\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b.$$

By the Multiplicative Stolz Theorem $\lim\limits_{n\to\infty} \dfrac{\frac{b_{n+1}}{(n+1)!}}{\frac{b_n}{n!}} = b$ yields $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{b_n}{n!}} = b$.

Let $c_n = \dfrac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \dfrac{\sqrt[n+1]{\dfrac{b_{n+1}}{(n+1)!}}}{\sqrt[n]{\dfrac{b_n}{n!}}} \cdot \dfrac{\dfrac{\sqrt[n+1]{(n+1)!}}{n+1}}{\dfrac{\sqrt[n]{n!}}{n}} \cdot \dfrac{n+1}{n}$.

Since $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$, $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{b_n}{n!}} = b$, $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$ then $\lim\limits_{n\to\infty} c_n = 1$, and, therefore, $\lim\limits_{n\to\infty} \dfrac{c_n^{\cos^2 x} - 1}{\ln\left(c_n^{\cos^2 x}\right)} = 1$.

Hence, $\lim\limits_{n\to\infty} n\left(c_n^{\cos^2 x} - 1\right) = \lim\limits_{n\to\infty} \left(n \ln\left(c_n^{\cos^2 x}\right) \cdot \dfrac{c_n^{\cos^2 x} - 1}{\ln\left(c_n^{\cos^2 x}\right)}\right) = \lim\limits_{n\to\infty} n \ln\left(c_n^{\cos^2 x}\right) =$

$\cos^2 x \lim\limits_{n\to\infty} n \ln c_n = \cos^2 x \ln\left(\lim\limits_{n\to\infty} c_n^n\right) = \cos^2 x \ln\left(\lim\limits_{n\to\infty} \dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n}\right)$.

Since $\dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n} = \dfrac{b_{n+1}}{nb_n} \cdot \dfrac{1}{\sqrt[n+1]{\dfrac{b_{n+1}}{(n+1)!}}} \cdot \dfrac{n}{\sqrt[n+1]{(n+1)}}$, then $\lim\limits_{n\to\infty} \dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n} = b \cdot \dfrac{1}{b} \cdot e = e$

and, therefore, $\lim\limits_{n\to\infty} n\left(c_n^{\cos^2 x} - 1\right) = \cos^2 x$.

And since $a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right) =$

$\left(\dfrac{a_n}{n}\right)^{\sin^2 x} \cdot \left(\sqrt[n]{\dfrac{b_n}{n!}}\right)^{\cos^2 x} \cdot \left(\dfrac{\sqrt[n]{n!}}{n}\right)^{\cos^2 x} \cdot n\left(\left(\dfrac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x} - 1\right)$ then

$\lim\limits_{n\to\infty} \left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right) =$

$\mathrm{a}^{\sin^2 x} b^{\cos^2 x} e^{-\cos^2 x} \lim\limits_{n\to\infty} n\left(c_n^{\cos^2 x} - 1\right) = a^{\sin^2 x} b^{\cos^2 x} e^{-\cos^2 x} \cos^2 x$.

Solution 2 by Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy

$$\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right)$$

$$= \left(\frac{a_n}{n}\right)^{\sin^2 x} n^{\sin^2 x} b_n^{\frac{\cos^2 x}{n}} \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x} - 1\right)$$

$$= \left(\frac{a_n}{n}\right)^{\sin^2 x} n \left(\frac{b_n^{\frac{1}{n}}}{n}\right)^{\cos^2 x} \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^{\cos^2 x} - 1 \right)$$

$$= \left(\frac{a_n}{n}\right)^{\sin^2 x} \left(\frac{b_n^{\frac{1}{n}}}{n}\right)^{\cos^2 x} \frac{\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^{\cos^2 x} - 1}{\ln \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^{\cos^2 x} \right)} \ln \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^{n \cos^2 x} \right).$$

By Cesaro-Stolz,

$$\lim_{n\to\infty} \frac{b_n^{1/n}}{n} = \lim_{n\to\infty} \left(\frac{b_n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{b_{n+1}}{n b_n} \frac{n^{n+1}}{(n+1)^{n+1}} = \frac{b}{e}$$

and

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} (a_{n+1} - a_n).$$

Now

$$\lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}}{\left(b_n\right)^{\frac{\cos^2 x}{n}}} = \lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}}{(n+1)^{\cos^2 x}} \frac{n^{\cos^2 x}}{(n+1)^{\cos^2 x}} \frac{n^{\cos^2 x}}{\left(b_n\right)^{\frac{\cos^2 x}{n}}} = \frac{b^{\cos^2 x}}{e^{\cos^2 x}} \cdot 1 \cdot \frac{e^{\cos^2 x}}{b^{\cos^2 x}} = 1.$$

Moreover,

$$\lim_{n\to\infty} \left(\frac{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}}{\left(b_n\right)^{\frac{\cos^2 x}{n}}} \right)^n = \lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\cos^2 x}}{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}} \frac{1}{\left(b_n\right)^{\cos^2 x}}$$

$$= \lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\cos^2 x}}{n^{\cos^2 x}\left(b_n\right)^{\cos^2 x}} \frac{n^{\cos^2 x}}{(n+1)^{\cos^2 x}} \frac{(n+1)^{\cos^2 x}}{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}} = b^{\cos^2 x} \cdot 1 \cdot \frac{e^{\cos^2 x}}{b^{\cos^2 x}} = e^{\cos^2 x}$$

The limit is thus

$$a^{\sin^2 x} \cdot \frac{b^{\cos^2 x}}{e^{\cos^2 x}} \cdot 1 \cdot \cos^2 x.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By assumption $a_{n+1} - a_n = a + o(1)$, $\frac{b_{n+1}}{n b_n} = b e^{o(1)}$, as $n \to \infty$. So,

$$a_n = a_1 + \sum_{j=2}^{n} (a_j - a_{j-1}) = na + o(n), \; b_n = \frac{n! b_1}{n} \prod_{j=2}^{n} \frac{b_j}{(j-1) b_{j-1}} = n! b^n e^{o(n)} = n^n e^{-n} b^n e^{o(n)}, \text{ as } n \to \infty.$$

We have used a weak form of Stirling's formula, namely $n! = n^n e^{-n+o(n)}$ as $n \to \infty$.

We conclude

$$
\begin{aligned}
&\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right) = \\
= \quad & n^{\sin^2 x}\left(a+o(1)\right)^{\sin^2 x}\left(\left((n+1)\, be^{-1+o(1)}\right)^{\cos^2 x} - \left(nbe^{-1+o(1)}\right)^{\cos^2 x}\right) = \\
= \quad & n^{\sin^2 x+\cos^2 x}\left(a+o(1)\right)^{\sin^2 x}\left(be^{-1+o(1)}\right)^{\cos^2 x}\left(\left(1+\frac{1}{n}\right)^{\cos^2 x} - 1\right) = \\
= \quad & n\left(a+o(1)\right)^{\sin^2 x}\left(be^{-1+o(1)}\right)^{\cos^2 x}\left(\frac{\cos^2 x}{n} + O\left(\frac{1}{n^2}\right)\right) \\
= \quad & \left(a+o(1)\right)^{\sin^2 x}\left(be^{-1+o(1)}\right)^{\cos^2 x}\left(\cos^2 x + O\left(\frac{1}{n}\right)\right) \\
\to \quad & a^{\sin^2 x} b^{\cos^2 x} e^{-\cos^2 x}\cos^2 x \text{ as } n \to \infty.
\end{aligned}
$$

Comment by Bruno Salgueiro Fanego, Viveiro, Spain

A more general question can be seen in problem 75 from the journal *Mathproblems,* available at < http://mathproblems-ks.com/?wpfb_d1=11> (see page 2) and solved at < http://mathproblems-ks.com/?wpfb_d1=17>(see pages 6-8)>

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5286:** *Proposed by Michael Brozinsky, Central Islip, NY*

In Cartesianland, where immortal ants live, an ant is assigned a specific equilateral triangle EFG and three distinct positive numbers $0 < a < b < c$. The ant's job is to find a unique point $P(x,y)$ such that the distances from P to the vertices E, F and G of his triangle are proportionate to $a : b : c$ respectively. Some ants are eternally doomed to an impossible search. Find a relationship between a, b and c that guarantees eventual success; i.e., that such a unique point P actually exists.

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let s be the length of the side of $\triangle EFG$ and suppose we are given three distinct positive integers $0 < a < b < c$ such that $a + b > c, b + c > a$ and $c + a > b$.

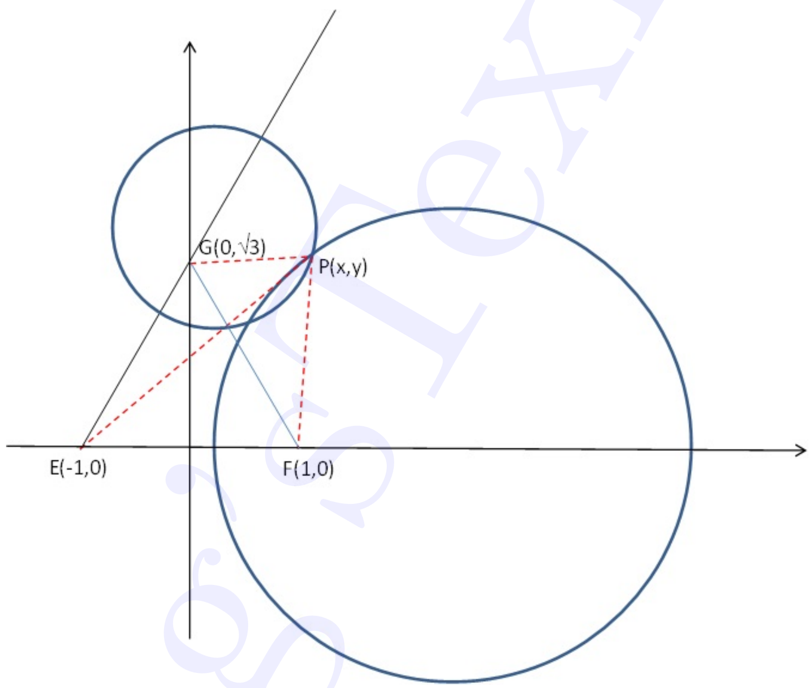
Recall the following: the symmetric equation

$$3\left(x^4 + y^4 + z^4 + w^4\right) = \left(x^2 + y^2 + z^2 + w^2\right)$$

relates the size of an equilateral triangle ABC to the distances of a point from its three vertices. Substituting a, b and c for x, y and z respectively and solving for w then gives the triangle's side (say $w = s'$) and the existence of a point P'. By Pompeiu's Theorem, if P' is an arbitrary point an equilateral triangle ABC, then there exists a triangle with sides of length $P'A, P'B, P'C$. Moreover, the theorem remains valid for any point P' in the plane of triangle ABC and that the triangle is degenerate if and only if P' lies on the circumcircle of $\triangle ABC$. Therefore, $a+b>c, b+c>a$ and $c+a>b$. Finally using a dilation transformation from $\triangle ABC$ to $\triangle EFG$ with a dilation factor of $\frac{s}{s'}$, it follows that there exists a point $P = P'\left(\frac{s}{s'}\right)$whose distances from the three vertices are $PE = a\left(\frac{s}{s'}\right), PF = b\left(\frac{s}{s'}\right)$ and $PG = c\left(\frac{s}{s'}\right)$. Hence, $\frac{PE}{a} = \frac{PF}{b} = \frac{PG}{c} = \frac{s}{s'}$ so that the distancs from P to the vertices E, F and G are proportionate to $a : b : c$ respectively.

Solution 2 by Michael Fried, Ben Gurion University, Beer-Sheva, Israel

Since this is Cartesianland, we might as well place the equilateral triangle in the Cartesian plane and give the vertices convenient coordinates, say, $E = (-1, 0)$, $F = (1, 0)$, and $G = (0, \sqrt{3})$ (see figure below.)



Let us set $\alpha = b/c = PE/PF$, $\beta = a/c = PG/PF$, and $\gamma = a/b = PG/PE$.

Then the locus of points P with $PE/PF = \alpha$ is the Apollonius circle:

$$\alpha^2\left((x-1)^2+y^2\right) - \left((x+1)^2+y^2\right) = 0$$

Similarly, the locus of points P with $PG/PE = \gamma$ is the Apollonius circle:

$$\gamma^2\left((x+1)^2+y^2\right) - \left(x^2+(y-\sqrt{3})^2\right) = 0$$

The condition that the system of equations,

$$\alpha^2\left((x-1)^2+y^2\right) - \left((x+1)^2+y^2\right) = 0$$

$$\gamma^2\left((x+1)^2+y^2\right) - \left(x^2+(y-\sqrt{3})^2\right) = 0$$

has a solution, that is, that the two Apollonius circles have an intersection is (after some messy but routine algebra) is:

$$\Delta = 16\left[\left(\gamma^2\gamma^2 + (\alpha^2+1)\right)^2 - \left((2\alpha^2\gamma^2 - (\alpha^2+1))^2 - 3(\alpha^2-1)^2\right)\right] \geq 0$$

After some further manipulation, this come down to the inequality:

$$\left(\alpha^2\gamma^2 - (\alpha+1)^2\right)\left(\alpha^2\gamma^2 - (\alpha-1)^2\right) \leq 0$$

From which we have the condition:

$$\left(1 - \frac{1}{\alpha}\right)^2 \leq \gamma^2 \leq \left(1 + \frac{1}{\alpha}\right)^2$$

Or going back to the definition $\alpha = b/c, \gamma = a/b$, we have:

$$\left(1 - \frac{c}{b}\right)^2 \leq \frac{a^2}{b^2} \leq \left(1 + \frac{c}{b}\right)^2$$

So that,

$$(b-c)^2 \leq a^2 \leq (b+c)^2$$

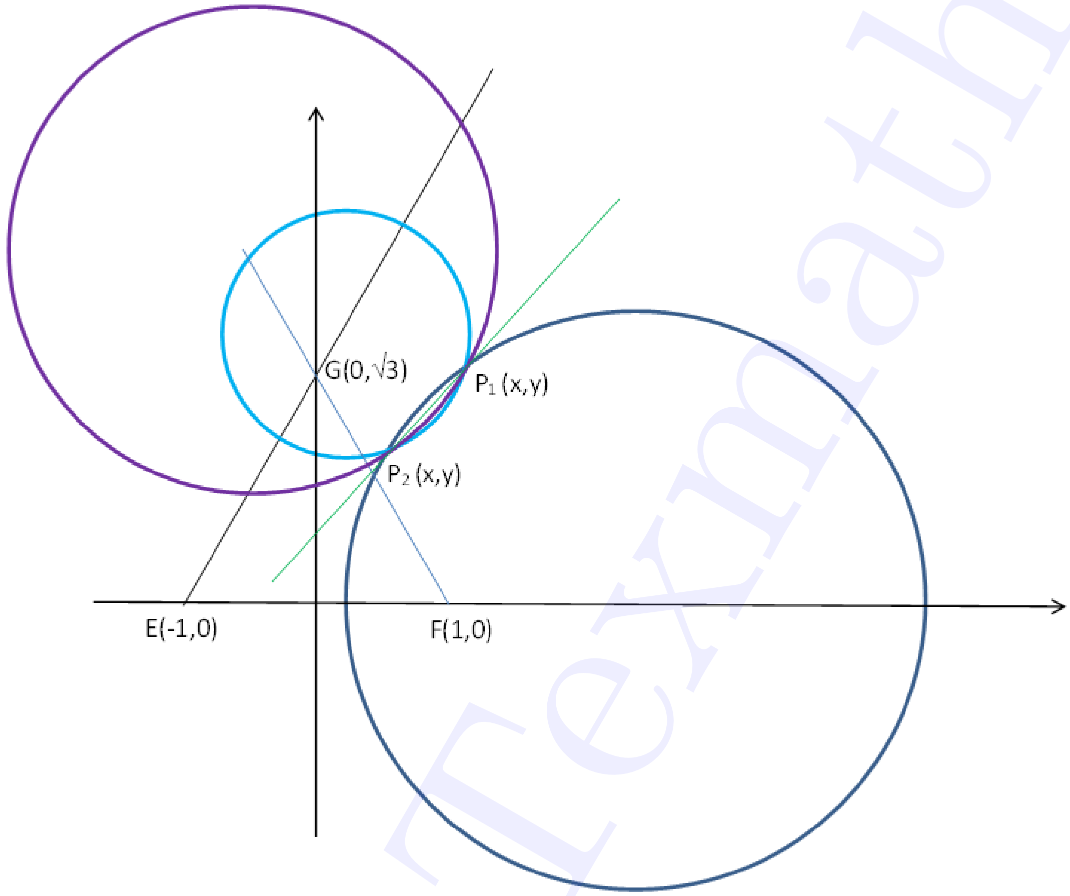
Since a, b, c are positive numbers, and since this must be true no matter which Apollonius circle ratio we begin with, we have the triangle-like inequalities:

$$a \leq b + c$$

$$b \leq a + c$$

$$c \leq a + b$$

One should note that if the circles $\alpha^2\left((x-1)^2 + y^2\right) - \left((x+1)^2 + y^2\right) = 0$ and $\gamma^2\left((x+1)^2 + y^2\right) - \left(x^2 + (y-\sqrt{3})^2\right) = 0$ intersect, they will generally intersect in *two* points P_1 and P_2, where both P_1G/PF and $P_2G/PF = a/c$, and a single Apollonius circle with respect to G and F will pass through these points. Observe too, the three circles then have the same radical axis, namely, P_1P_2 (see figure below).



Comments

1. Ken Korbin, New York, NY

Given $0 < a < b < c$. If it is possible to construct a triangle with sides (a, b, c) in which each of the angles is less than $120°$, them there is a unique point P.

2. Bruno Salgueiro Fanego, Viveiro, Spain

In the article by Oene Bottema *On the distances of a point to the vertices of a triangle.* journal *Crux Mathematicorum,* $1984, 10(8), 242-246$, it is proved (among other things) the following relationship between the lengths of the sides
$\alpha_1 = \angle A_2A_1A_3, \alpha_2 = \angle A_3A_2A_1, \alpha_3 = \angle A_1A_3A_2$ and any point P in the plane of $\triangle A_1A_2A_3$ with distances to the vertices $d_1 = PA_1, d_2 = PA_2, d_3 = PA_3$, then:

$$a_1^2d_1^4 + a_2^2d_2^4 + a_3^2d_3^4 - 2a_2a_3\cos\alpha_1 d_2^2d_3^2 - 2a_3a_1\cos\alpha_2 d_3^2d_1^2 - 2a_1a_2\cos\alpha_3 d_1^2d_2^2 -$$

$$2a_1^2a_2a_3\cos\alpha_1 d_1^2 - 2a_1a_2^2a_3\cos\alpha_2 d_2^2 - 2a_1a_2a_3^2\cos\alpha_3 d_3^2 + a_1^2a_2^2a_3^2 = 0$$

called identity (6) and reciprocally. That is, that if d_1, d_2, d_3 are positive numbers satisfying identity (6) then there is a unique point P such that
$PA_1 = d_1, PA_2 = d_2, PA_3 = d_3$.

This implies that identity(6) is the relationship which solves a problem more generally than the one proposed.

Note: In particular, if we suppose that $A_1A_2A_3$ is the equilateral triangle EFG of the statement of the problem, with sides $e = a_1 = a_2 = a_3$ and k is the constant of proportionality such that $d_1 = ka, d_2 = kb, d_3 = kc$ then identity (6), when divided by e^2 becomes

$$k^4\left(a^4+b^4+c^4\right)+e^4-k^4\left(a^2b^2+a^2c^2+b^2c^2\right)-k^2\left(a^2+b^2+c^2\right)e^2+e^4=0,$$

which is the required relationship in the original statement of the problem.

On the other hand, if we suppose that a point P exists and k is the constant of proportionality, such aht $PE = ka, PF = kb$, and $PG = kc$, using the identity which appears in the editor's comment of SSM problem 5140, or its equivalent,

$$PE^4+PF^4+PG^4+EF^4 = PE^2PF^2+PE^2PG^2+PF^2PG^2+PE^2EF^2+PF^2EF^2+PG^2EF^2,$$

we obtain directly the relationship which is required in the problem, that is,

$$k^4\left(a^4+b^4+c^4\right)+e^4=k^4\left(a^2b^2+a^2c^2+b^2c^2\right)+k^2\left(a^2+b^2+c^2\right)e^2,$$

which is also equivalent to equality (4) in the published solution #2 to 5140.

Also solved by Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS, and the proposer.

- **5287:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let u, v, w, x, y, z be complex numbers. Prove that

$$2\mathrm{Re}(ux+vy+zw) \le 3\left(|u|^2+|v|^2+|w|^2\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$0 \le \left|\sqrt{3}u \ -\frac{1}{\sqrt{3}}\overline{x}\right|^2=\left(\sqrt{3}u \ -\frac{1}{\sqrt{3}}\overline{x}\right)\left(\sqrt{3}\overline{u} \ -\frac{1}{\sqrt{3}}x\right)=3|u|^2+\frac{1}{3}|x|^2-2\mathrm{Re}(ux).$$

So, $2\mathrm{Re}(ux) \le 3|u|^2+\frac{1}{3}|x|^2$.

Similarly, $2\mathrm{Re}(vy) \le 3|v|^2+\frac{1}{3}|y|^2$, and $2\mathrm{Re}(zw) \le 3|w|^2+\frac{1}{3}|z|^2$.

The statement follows by adding these inequalities.

Solution 2 by David Diminnie and Tatyana Savchuk, Texas Instruments, Inc., Dallas, TX

We will prove the equivalent statement

$$0 \le 3\left(|u|^2+|v|^2+|w|^2\right)-2\mathrm{Re}\left(ux+vy+zw\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right). \quad (1)$$

Let u_1, u_2 denote the real and imaginary parts of u, respectively, and similarly for v, w, x, y, z. Then the right side of (1) becomes

$$3\left(u_1^2+u_2^2+v_1^2+v_2^2+w_1^2+w_2^2\right)-2\left(u_1x_1-u_2x_2+v_1y_1-v_2y_2+w_1z_1-w_2z_2\right)$$
$$+\frac{1}{3}\left(x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2\right),$$

which we rewrite as

$$\left(3u_1^2-2u_1x_1+\frac{1}{3}x_1^2\right)+\left(3u_2^2+2u_2x_2+\frac{1}{3}x_2^2\right)+\left(3v_1^2-2v_1y_1+\frac{1}{3}y_1^2\right)+\left(3v_2^2+2v_2y_2+\frac{1}{3}y_2^2\right)$$
$$+\left(3w_1^2-2w_1z_1+\frac{1}{3}z_1^2\right)+\left(3w_2^2+2w_2z_2+\frac{1}{3}z_2^2\right). \qquad (2)$$

Noting that $3a^2+2ab+\frac{1}{3}b^2$ and $3a^2-2ab+\frac{1}{3}b^2$ may be rewritten as $\left(\sqrt{3}a+\dfrac{1}{\sqrt{3}}b\right)^2$ and $\left(\sqrt{3}a-\dfrac{1}{\sqrt{3}}b\right)^2$, respectively, (2) becomes

$$\left(\sqrt{3}u_1-\frac{1}{\sqrt{3}}x_1\right)^2+\left(\sqrt{3}u_2+\frac{1}{\sqrt{3}}x_2\right)^2+\left(\sqrt{3}v_1-\frac{1}{\sqrt{3}}y_1\right)^2+\left(\sqrt{3}v_2+\frac{1}{\sqrt{3}}y_2\right)^2$$
$$+\left(\sqrt{3}w_1-\frac{1}{\sqrt{3}}z_1\right)^2+\left(\sqrt{3}w_2+\frac{1}{\sqrt{3}}z_2\right)^2. \qquad (3)$$

Since (3) is a sum of squares of real numbers the expression must be nonnegative, and therefore (1) holds.

Solution 3 by Paul M. Harms, North Newton, KS

We know that the real part of a finite sum of complex numbers is less than or equal to the modulus of the sum which is less than or equal to the sum of the moduli. Also the modulus of a finite product of complex numbers equals the product of the moduli.

We have $0 \le \left(3\,|u|-|x|\right)^2+\left(3\,|v|-|y|\right)^2+\left(3\,|w|-|z|\right)^2$. After squaring the three parts, moving terms and dividing by 3, we can obtain,

$$2\left(|u|\,|x|+|v|\,|y|+|z|\,|w|\right)\le 3\left(|u|^2+|v|^2+|w|^2\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right).$$

From what was said and shown above,

$$2\mathrm{Re}(ux+vy+zw)\le 2\left(|u|\,|x|+|v|\,|y|+|z|\,|w|\right)\le 3\left(|u|^2+|v|^2+|w|^2\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right).$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We have

$$3|u|^2+\frac{1}{3}|x|^2-2\mathrm{Re}(ux)\ge 3|u|^2+\frac{1}{3}|x|^2-2|u||x|=\frac{1}{3}\left(3|u|-|x|\right)^2\ge 0,$$

and similarly,

$$3|v|^2+\frac{1}{3}|y|^2-2\mathrm{Re}(vy)\ge 0,\quad 3|\mathrm{z}|^2+\frac{1}{3}|\mathrm{w}|^2-2\mathrm{Re}(zw)\ge 0.$$

The inequality of the problem follows by adding up the three inequalities above.

Solution 5 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

$$2\text{Re}(ux + vy + zw) \le 2(|ux| + |vy| + |zw|) = 2(|u| \cdot |x| + |v| \cdot |y| + |z| \cdot |w|)$$

and

$$|z| \cdot |w| \le 3|z|^2 + \frac{1}{3}|w|^2$$

is simply the AGM.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL and the proposer.

- **5288:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c \ge 0$ be real numbers. Find the value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + ai + bj + c}}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $2\ln(\sqrt{2}+1)$

Proof: We show that the limit is independent on a, b, c allowing us to set $a = b = c = 0$ for evaluating it. If $Q = [0,1] \times [0,1]$, the limit becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2}} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{\sqrt{\frac{i^2}{n^2} + \frac{j^2}{n^2}}} = \int\int_Q \frac{1}{\sqrt{x^2+y^2}} dxdy.$$

By writing the integral as $2\int_0^1 \left(\int_0^x \frac{1}{\sqrt{x^2+y^2}} dy \right) dx$ and passing to polar coordinates we have

$$2\int_{\pi/4}^{\pi/2} \left(\int_0^{1/\sin\theta} \frac{\rho}{\rho} d\rho \right) d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1}{\sin\theta} d\theta = 2\ln\tan\frac{\theta}{2}\Big|_{\pi/4}^{\pi/2} = 2\ln(\sqrt{2}+1).$$

To show that the limit is independent by a, b, c, we prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + ai + bj + c}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + a'i + b'j + c'}}$$

for any a, b, c, a', b', c'. We introduce a number of positive constants C_k, $k = 0, 1, \ldots$.

Since $i|a' - a| + j|b' - b| + |c' - c| \le C_0(i + j)$ and $i^2 + j^2 + ai + bj + c \le C_1(i^2 + j^2)$ we have the bound

$$\left|\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}-\frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}\right|=$$

$$\left|\frac{(a'-a)+j(b'-b)+c'-c}{(i^2+j^2+ai+bj+c)(i^2+j^2+a'i+b'j+c')}\right|\times$$

$$\times\left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}+\frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}\right)^{-1}\le$$

$$\le\frac{C_0(i+j)}{(i^2+j^2)^2}\frac{\sqrt{i^2+j^2}}{C_1}=C_2\frac{i+j}{(i^2+j^2)^{3/2}}$$

Thus

$$\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{(i^2+j^2)^{3/2}}\le\frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=1}^{n}\frac{i}{(2ij)^{3/2}}+\frac{1}{n}\sum_{i=1}^{\infty}\sum_{j=1}^{n}\frac{j}{(2ij)^{3/2}}\le C_3/\sqrt{n}$$

and it follows that for any a,b,c,a',b',c'

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}-\frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}\right)=0.$$

In particular we can take $a'=b'=c'=0$ and write

$$\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}=\left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}-\frac{1}{\sqrt{i^2+j^2}}\right)+\frac{1}{\sqrt{i^2+j^2}}$$

The conclusion is that for any a,b,c the limit assumes the same value $2\ln(\sqrt{2}+1)$.

Solution 2 by Ed Gray, Highland Beach, FL

Consider the integral

$$\lim_{n\to\infty}\frac{1}{n}\int_{x=1}^{x=n}\int_{y=1}^{y=n}\frac{dxdy}{\sqrt{x^2+y^2}}.$$

(*Editor's comment: Ed used intuition in moving from the double summation to the double integral by reasoning that the linear terms in the summation wouldn't contribute much to the summation for very large values of n. His intuition was right on target, as seen in Paolo's solution above. Ed evaluated the double integral in the usual manner, by first integrating the inside integral with respect to x treating y as a constant, and then integrating that answer with respect to y, treating x as a constant.*

$$\int_{x=1}^{n}\frac{dx}{\sqrt{x^2+y^2}} \quad = \quad \ln\left(\sqrt{x^2+y^2}+x\right)-\ln y\Big|_{x=1}^{n}$$

$$= \ln\left(\sqrt{n^2+y^2}+n\right) - \ln y - \ln\left(\sqrt{1^2+y^2}+1\right) + \ln y$$

$$= \ln\left(\sqrt{n^2+y^2}+n\right) - \ln\left(\sqrt{1^2+y^2}+1\right)$$

And now we compute:

$$\int_{y=1}^{n} \ln\left(\sqrt{n^2+y^2}+n\right) dy - \int_{y=1}^{n} \ln\left(\sqrt{1^2+y^2}+1\right) dy.$$

$$\int_{y=1}^{n} \ln\left(\sqrt{n^2+y^2}+n\right) dy = y\ln\left(\sqrt{n^2+y^2}+n\right) + n\ln\left(\sqrt{n^2+y^2}+y\right) - y\Big|_{y=1}^{n}$$

$$\left[n\ln\left(\sqrt{n^2+n^2}+n\right) + n\ln\left(\sqrt{n^2+n^2}+n\right) - n\right] - \left[(1)\ln\left(\sqrt{n^2+1}+n\right) + n\ln\left(\sqrt{n^2+1}+1\right) - 1\right]$$

Let's called this **A.** And evaluating

$$\int_{y=1}^{n} \ln\left(\sqrt{y^2+1}+1\right) dy = y\ln\left(\sqrt{y^2+1}+1\right) - y + \ln\left(y+\sqrt{1+y^2}\right)\Bigg|_{y=1}^{n}$$

we obtain

$$n\left[\ln(\sqrt{n^2+1}+1)\right] - n + \ln\left(n+\sqrt{n^2+1}\right) - \left[(1)\left(\ln\left(\sqrt{2}+1\right)\right) - 1 + \ln(1+\sqrt{2})\right].$$

And let's call this **B.**

We now evaluate $\frac{1}{n}\lim_{n\to\infty} \mathbf{A} - \frac{1}{n}\lim_{n\to\infty} \mathbf{B}$. Doing this gives us $2\ln(\sqrt{2}+1)$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the limit equal $2\ln(1+\sqrt{2})$, independent of a, b, c.

We first note that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\sqrt{i^2+j^2+ai+bj+c}} - \sum_{i=1}^{n}\sum_{j=1^n} \frac{1}{\sqrt{i^2+j^2}}\right|$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{ai+bj+c}{\left(\sqrt{i^2+j^2+ai+bj+c}\right)\left(\sqrt{i^2+j^2}\right)\left(\sqrt{i^2+j^2+ai+bj+c}+\sqrt{i^2+j^2}\right)}$$

$$\leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{ai+bj+c}{(i^2+j^2)^{3/2}}$$

$$= O\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{ai+bj+c}{(ij)^{3/2}}\right)$$

$$= O\left(\sum_{i=1}^{n}\frac{1}{i^{1/2}}\sum_{j=1}^{n}\frac{1}{j^{3/2}}\right)+O\left(\sum_{i=1}^{n}\frac{1}{i^{3/2}}\sum_{j=1}^{n}\frac{1}{j^{1/2}}\right)+O\left(\sum_{i=1}^{n}\frac{1}{i^{3/2}}\sum_{j=1}^{n}\frac{1}{j^{3/2}}\right)$$

$$= O(\sqrt{n}).$$

The constants implied by O depend at most on a, b, and c. It follows that the limit of the problem in fact equals $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2}}$. Now the last limit equals

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{\left(\frac{i}{n}\right)^2+\left(\frac{j}{n}\right)^2}}=\int_0^1\int_0^1\frac{dydx}{\sqrt{x^2+y^2}},$$

which we are going to evaluate. It is easy to check that

$$\frac{d}{dy}\left(\ln\left(y+\sqrt{x^2+y^2}\right)\right)=\frac{1}{\sqrt{x^2+y^2}}$$

and

$$\frac{d}{dx}\left(\ln\left(x+\sqrt{x^2_+1}\right)+x\ln\left(1+\sqrt{x^2+1}\right)-\ln x\right)=\ln\left(1+\sqrt{x^2+y^2}\right)-\ln x.$$

Hence

$$\int_0^1\int_0^1\frac{dydx}{\sqrt{x^2+y^2}}=\int_0^1\left(\ln\left(1+\sqrt{x^2+1}\right)-\ln x\right)dx=2\ln\left(1+\sqrt{2}\right),$$

where we have used the fact that $\lim_{x\to 0^+}(x\ln x)=0$.
This completes the solution.

Solution 4 by Anastasios Kotronis, Athens, Greece

Let

$$a_n=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}.$$

We have

$$a_{n+1}-a_n \quad = \quad \sum_{i=1}^{n+1}\frac{1}{\sqrt{i^2+(n+1)^2+ai+b(n+1)+c}}+\sum_{j=1}^{n+1}\frac{1}{\sqrt{(n+1)^2+j^2+a(n+1)+bj+c}}$$

$$
-\frac{1}{\sqrt{2(n+1)^2+(a+b)(n+1)+c}}
$$

$$
= \quad b_{n+1}+c_{n+1}-d_{n+1}
$$

But

$$
\begin{aligned}
b_n &= \sum_{i=1}^{n}\frac{1}{\sqrt{i^2+n^2+ai+bn+c}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(i/n)^2+1+ai/n^2+b/n+c/n^2}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(i/n)^2+1}}1+\mathcal{O}(n^{-1}) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(i/n)^2+1}}+\mathcal{O}(n^{-1}) \\
&\to \int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx = \ln(1+\sqrt{2})
\end{aligned}
$$

and by symmetry, the same holds for c_n. Since clearly $d_n \to 0$, by Cezàro Stolz

$$
\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} \to 2\ln(1+\sqrt{2}).
$$

Comment by Bruno Salgueiro Fanego, Viveiro, Spain

This problem and its solution appeared as challenge exercise $U114$ in the journal *Mathematical Reflections.* See:
< https://www.awesomemath.org/wp-ontent/uploads/reflections/2009_2/MR_2_2009_Solutions.pdf >. Pages 36-38.

The required value is $2\ln\left(\sqrt{2}+1\right)$.

Also solved by Arkady Alt, San Jose, CA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The name of Michael Thew, a student at St. George's School in Spokane, WA was inadvertently omitted from the list of those who had solved 5277 and 5279.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October 15, 2014

- **5307:** *Proposed by Haishen Yao and Howard Sporn, Queensborough Community College, Bayside, NY*

  Solve for x:
  $$\sqrt{x^{15}} = \sqrt{x^{10}-1} + \sqrt{x^5-1}.$$

- **5308:** *Proposed by Kenneth Korbin, New York, NY*

  Given the sequence
  $$t = (1, 7, 41, 239, \ldots)$$
  with $t_n = 6t_{n-1} - t_{n-2}$. Let (x, y, z) be a triple of consecutive terms in this sequence with $x < y < z$.

  Part 1) Express the value of x in terms of y and express the value of y in terms of x.

  Part 2) Express the value of x in terms of z and express the value of z in terms of x.

- **5309:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Consider the expression $3^n + n^2$ for positive integers n. It is divisible by 13 for $n = 18$ and $n = 19$. Prove, however, that it is never divisible by 13 for three consecutive values of n.

- **5310:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Let $a > 0$ and a sequence $\{E_n\}_{n\geq 0}$, be defined by $\mathrm{E}_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$. Evaluate:
  $$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right).$$

- **5311:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let x, y, z be positive real numbers. Prove that

$$\sum_{cyclic} \sqrt{\left(\frac{x^2}{3} + 3y^2\right)\left(\frac{2}{xy} + \frac{1}{z^2}\right)} \geq 3\sqrt{10}.$$

- **5312:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate:

$$\int_0^1 \ln|\sqrt{x} - \sqrt{1-x}|dx.$$

Solutions

- **5289:** *Proposed by Kenneth Korbin, New York, NY*

  Part 1: Thirteen different triangles with integer length sides and with integer area each have a side with length 1131. The angle opposite 1131 is $Arcsin\left(\frac{3}{5}\right)$ in all 13 triangles.

  Find the sides of the triangles.

  Part 2: Fourteen different triangles with integer length sides and with integer area each have a side with length 6409. The size of the angle opposite 6409 is the same in all 14 triangles.

  Find the sides of the triangles.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Part 1: If $\alpha = Arc\sin\left(\frac{3}{5}\right)$, then $\sin\alpha = \frac{3}{5}$ and $0 < \alpha < \frac{\pi}{2}$. It follows that

$$\cos\alpha = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

Suppose x and y are the other sides of the triangle with $x \geq y$. The Law of Cosines implies that

$$\begin{aligned}(1131)^2 &= x^2 + y^2 - 2xy\cos\alpha \\ &= x^2 + y^2 - \frac{8}{5}xy.\end{aligned}$$

If we complete the square in x and simplify, we get

$$(5655)^2 = (5x - 4y)^2 + (3y)^2$$

and hence, $(5x - 4y, 3y, 5655)$ is a Pythagorean Triple. To solve for x and y, we must find all such triples and assign $5x - 4y$ and $3y$ to the sides of each triple. E.g., for the triple $(2175, 5220, 5655)$, setting

$$5x - 4y = 2175$$

$$3y = 5220$$

yields $x = 1827$ and $y = 1740$, while

$$\begin{aligned} 5x - 4y &= 5220 \\ 3y &= 2175 \end{aligned}$$

yields $x = 1624$ and $y = 725$. Some other triples give only one integral solution for x and y and a few give no integral solutions. In all, we found 14 solutions which are listed in the following table. (Repeated triples indicate multiple solutions as above.)

Pythagorean Triple	x	y
$(3393, 4524, 5655)$	1885	1508
$(2175, 5220, 5655)$	1827	1740
$(2175, 5220, 5655)$	1624	725
$(3900, 4095, 5655)$	1872	1365
$(3900, 4095, 5655)$	1859	1300
$(936, 5577, 5655)$	1365	312
$(663, 5616, 5655)$	1300	221
$(2280, 5175, 5655)$	1836	1725
$(2280, 5175, 5655)$	1643	760
$(2025, 5280, 5655)$	1813	1760
$(2025, 5280, 5655)$	1596	675
$(2772, 4929, 5655)$	1725	924
$(3009, 4788, 5655)$	1760	1003
$(2871, 4872, 5655)$	1740	957

It should be noted that in each case, the values of x, y, and 1131 satisfy the required triangle inequalities for the sides of a non-degenerate triangle. Also, the area of each triangle is $\frac{1}{2}xy\sin\alpha = \frac{3xy}{10}$. Since xy is a multiple of 10 in each case, the resulting triangle has integral area as well.

Part 2: If we once again use $\alpha = Arc\sin\left(\frac{3}{5}\right)$ for the angle opposite 6409, then by the same steps as described in Part 1, the remaining sides x and y (with $x \geq y$) must satisfy the equation

$$(32,045)^2 = (5x - 4y)^2 + (3y)^2 .$$

Following the same procedure as in Part 1, we found the 22 solutions listed in the following table. As before, each satisfies the required inequalities for the sides of a triangle and each yields an integral area.

Pythagorean Triple	x	y
$(15916, 27813, 32045)$	10600	9271
$(22244, 23067, 32045)$	10600	7689
$(8283, 30956, 32045)$	8400	2761
$(2277, 31964, 32045)$	7000	759
$(2400, 31955, 32045)$	7031	800
$(21000, 24205, 32045)$	10441	7000
$(19795, 25200, 32045)$	10679	8400
$(10192, 30381, 32045)$	10140	10127
$(18291, 26312, 32045)$	10140	6097
$(15708, 27931, 32045)$	9775	5236
$(7656, 31117, 32045)$	8265	2552
$(8580, 30875, 32045)$	8463	2860
$(12920, 29325, 32045)$	10404	9775
$(11475, 29920, 32045)$	9044	3825
$(20300, 24795, 32045)$	10672	8265
$(3045, 31900, 32045)$	7192	1015
$(5304, 31603, 32045)$	7735	1768
$(13572, 29029, 32045)$	9425	4524
$(22100, 23205, 32045)$	10608	7735
$(15080, 28275, 32045)$	10556	9425
$(12325, 29580, 32045)$	10353	9860
$(16269, 27608, 32045)$	9860	5423

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, (part 1); David E. Manes, SUNY at Oneonta, Oneonta, NY, and the proposer.

- **5290:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Someone wrongly remembered the description of an even perfect number as:
$N = 2^p\left(2^{p-1}-1\right)$, where p is a prime number. Classify these numbers correctly. Which are deficient and which are abundant?

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta NY

We will show that if p is a prime, then $N = 2^p\left(2^{p-1}-1\right)$ is abundant except when $p = 2$ in which case N is deficient.

If $\sigma(n)$ is the sum of the positive divisors of n, then n is deficient when $\sigma(n) - n < n$ and abundant if $\sigma(n) - n > n$. If $p = 2$, then $N = 2^p\left(2^{p-1}-1\right) = 4$ and $\sigma(4) - 4 = 7 - 4 = 3$. Therefore $N = 4$ is deficient. If p is an odd prime, then $\gcd\left(2^p,\ 2^{p-1}-1\right) = 1$ implies

$$\sigma(N) = \sigma\left(2^p\left(2^{p-1}-1\right)\right) = \sigma\left(2^p\right)\sigma\left(2^{p-1}-1\right)$$

since σ is a multiplicative function. Moreover $\sigma\left(2^p\right) = 2^{p+1} - 1$ and
$\sigma\left(2^{p-1}-1\right) > \left(2^{p-1}-1\right) + 1 = 2^{p-1}$. Thus, $\sigma\left(N\right) > \left(2^{p+1}-1\right)2^{p-1}$. Therefore,

$$\sigma(N) - N \quad > \quad \left(2^{p+1}-1\right)2^{p-1} - 2^p\left(2^{p-1}-1\right)$$

$$
\begin{aligned}
&= \left(2^{p-1}\right)\left(2^{p+1}-1-2\left(2^{p-1}-1\right)\right)\\
&= \left(2^{p-1}\right)\left(2^{p+1}-2^{p}+1\right)\\
&= 2^{p-1}\left(2^{p}+1\right)\\
&> \left(2^{p-1}-1\right)2^{p}=N.
\end{aligned}
$$

Hence, N is an abundant integer.

Solution 2 by Paul M. Harms, North Newton, KS

I will use the theorem stating that proper multiples of perfect numbers and abundant numbers are abundant numbers.

When $p=2, N=4$ which is a deficient number.

When $p=3, N=2^2\left(2\left(2^2-1\right)\right)=4(6)=24$ which is 4 times the perfect number 6 and thus is an abundant number.

Consider p a prime number, $p \geq 3$. Then
$\left(2^{p-1}-1\right)=\left(2^2-1\right)\left(2^{p-3}+2^{p-5}+\ldots+2^2+1\right)$.

We now have $N=2^p\left(2^{p-1}-1\right)=\left(2^{p-1}\left(2^{p-3}+2^{p-5}+\ldots+2^2+1\right)\right)\left(2\left(2^2-1\right)\right)$. Since N is a proper multiple of the perfect number $2\left(2^2-1\right)=6,$ N is an abundant number.

In conclusion, N is a deficient number when $p=2$, but an abundant number for prime numbers $p>2$.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

We first establish that every nontrivial multiple of a perfect number is abundant (this result appears in most number theory texts, such as Burton's *Elementary Number Theory*). Given any positive integer n, we denote the sum of its positive divisors (including n itself) by $\sigma(n)$. The key observation is that for any positive integer n, we may sum over its positive divisors d to obtain

$$\sigma(n)=n\sum_{d|n}\frac{1}{d}.$$

Thus if n is perfect and m is a nontrivial multiple of n, then $\sigma(m)/m>\sigma(n)/n=2$, so m is abundant. (In general, if we denote the *abundancy index* of n by $I(n)=\sigma(n)/n$, then the above observation establishes that $I(n)\leq I(m)$ whenever n divides m.)

Next, we solve the original problem based on the parity of the prime p. If $p=2$, then $N=4$ is deficient. If p is odd, then $2^{p-1}-1$ is divisible by 3, since $p-1$ is even and 2 raised to any even power is congruent to 1 modulo 3. Thus in this case N is a nontrivial multiple of 6, so N is abundant.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern

University, Statesboro, GA, and the proposer.

- **5291:** *Proposed by Arkady Alt, San Jose, CA*

Let $m_a m_b$ be the medians of a triangle with side lengths a, b, c. Prove that:

$$m_a m_b \leq \frac{2c^2 + ab}{4}.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

We wish to prove that

$$\begin{aligned} 2c^2 + ab - 4m_a m_b &\geq 0 \text{ or equivalently,} \\ \left(2c^2 + ab + 4m_a m_b\right)\left(2c^2 + ab - 4m_a m_b\right) &\geq 0, \text{that is,} \\ \left(2c^2 + ab\right)^2 - 16m_a^2 m_b^2 &\geq 0. \end{aligned}$$

Since $m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, and $m_b = \dfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$ we obtain:

$$(2c^2 + ab)^2 - 16m_a^2 m_b^2 = \left(2c^2 + ab\right)^2 - \left(2b^2 + 2c^2 - a^2\right)\left(2c^2 + 2a^2 - b^2\right)$$

$$\begin{aligned} &= 4c^4 + 4abc^2 + a^2b^2 - \left(4b^2c^2 + 4a^2b^2 - 2b^4 + 4c^4 + 4c^2a^2 - 2b^2c^2 - 2c^2a^2 - 2a^4 + a^2b^2\right) \\ &= 4abc^2 - 4a^2b^2 - 2b^2c^2 - 2c^2a^2 + 2a^4 + 2b^4 \\ &= 2a^4 + 2b^4 - 4a^2b^4 - 2b^2c^2 - 2c^2a^2 + 4abc^2 \\ &= 2\left(\left(a^2 - b^2\right)^2 - (bc - ca)^2\right) \\ &= 2\left((a + b)^2 (a - b)^2\right) - c^2 (b - a)^2 \\ &= 2 (a - b)^2 \left((a + b)^2 - c^2\right) \\ &= 2 (a - b)^2 (a + b + c) (a + b - c) \geq 0 \end{aligned}$$

By the triangle inequality $a + b - c > 0$, with equality if and only if $a = b$, that is , if and only if the triangle is isosceles with equal side lengths a and b.

Solution 2 by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Since the length of the medians of any triangle ABC with side lengths a, b, and c are given by the expression

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{(cyclic)},$$

as it is well-known, then the inequality claimed becomes

$$\left(\frac{1}{2}\sqrt{2b^2+2c^2-a^2}\right)\left(\frac{1}{2}\sqrt{2c^2+2a^2-b^2}\right)\le\frac{2c^2+ab}{4}$$

or

$$\sqrt{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)}\le 2c^2+ab$$

Squaring both sides of the above inequality and after canceling terms, we obtain

$$2a^4+2b^4-4c^2ab-4a^2b^2-2b^2c^2-2c^2a^2\ge 0$$

or equivalently,

$$2(a-b)^2(a+b+c)(a+b-c)\ge 0$$

which is true on account that in any non degenerate triangle ABC is $a+b>c$. Equality holds when $a=b$. That is when $\triangle ABC$ is isosceles, and we are done.

Also solved by D. M. Bătinetu-Giurgiu,"Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania, and Titu Zvonaru, Comănesti, Romania; Ed Gray, Highland Beach, FL; Kenneth Korbin, New York, NY; Paul M. Harms, North Newton, KS, Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Ecole Suppa, Teramo, Italy,and the proposer.

- **5292:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let a and b be real numbers with $a<b$, and let c be a positive real number. If $f:R\longrightarrow R_+$ is a continuous function, calculate:

$$\int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

If $f(x)=e^{f(x-a)}(f(x-a))^{\frac{1}{c}}$ and $g(x)=e^{f(b-x)}(f(b-x))^{\frac{1}{c}}$, then for $x\in(a,b)$, $f(x)=g(b-x+a)$ and hence the proposed integral, say I is equal to

$$I=\int_a^b \frac{e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}+e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}\ dx,$$

and so $I=\dfrac{b-a}{2}$.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

By letting $y=\dfrac{x-a}{b-a}$, the integral is equal to

$$I=(b-a)\int_0^1 \frac{F((b-a)y)}{F((b-a)y)+F((b-a)(1-y))}dy$$

$$= (b-a)\int_0^1 dy - \frac{1}{b-a}\int_0^1 \frac{F((b-a)(1-y))}{F((b-a)y)+F((b-a)(1-y))}dy.$$

Letting $t = 1-y$ we obtain

$$I = (b-a) - (b-a)\int_0^1 \frac{F((b-a)(1-y))}{F((b-a)y)+F((b-a)(1-y))}dy$$

$$= (b-a) - (b-a)\int_0^1 \frac{F((b-a)t)}{F((b-a)(1-t))+F((b-a)t)}dy.$$

It follows that $2I = b-a \implies I = \dfrac{b-a}{2}$.

Solution 3 by Paul M. Harms, North Newton, KS

Let $A(x) = e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}$ and $B(x) = e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}$. We see that

$$\int_a^b \frac{A(x)+B(x)}{A(x)+B(x)}dx = b-a = \int_a^b \frac{A(x)}{A(x)+B(x)}dx + \int_a^b \frac{B(x)}{A(x)+B(x)}dx.$$

For the definite integral from a to b of $\dfrac{B(x)}{A(x)+B(x)}$ consider the change of variables $x = a+b-u$. Then

$$f(x-a) = f(b-u)$$

$$f(b-u) = f(u-a)$$

$$B(x) = A(u) \text{ and}$$

$$A(x) = B(u).$$

With this change of variables,

$$\int_a^b \frac{B(x)}{A(x)+B(x)}dx = \int_b^a \frac{A(u)}{B(u)+A(u)}(-1)du = \int_a^b \frac{A(u)}{A(u)+B(u)}du.$$

Thus $\displaystyle\int_a^b \frac{A(x)}{A(x)+B(x)}dx$ and $\displaystyle\int_a^b \frac{B(x)}{A(x)+B(x)}dx$ have the same value. Since their sum is $(b-a)$, the value of $\displaystyle\int_a^b \frac{A(x)}{A(x)+B(x)}dx$ is $\dfrac{b-a}{2}$.

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5293:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let ABC be a triangle. Prove that

$$\sqrt[4]{\sin A\cos^2 B}+\sqrt[4]{\sin B\cos^2 C}+\sqrt[4]{\sin C\cos^2 A}\le 3\sqrt[8]{\frac{3}{64}}.$$

Comment: **Michael Brozinsky of Central Islip, NY and Kee-Wai Lau of Hong Kong China each** noticed that if $\triangle ABC$ has an obtuse angle, then the above inequality does not hold. This oversight can be corrected by restricting the statement of the problem to acute triangles.

Solution 1 by Michael Brozinsky of Central Islip, NY

The given inequality is proved for acute triangles. Without loss of generality let the diameter of the circumcircle be 1 so that by the law of sines, the sides corresponding to angle A, B, and C satisfy the following:

$$a=\sin A, \qquad b=\sin B,\ c=\sin(\pi-(A+B))=\sin(A+B),$$

$$\begin{aligned}\cos^2 C &= (-\cos(A+B))^2=\cos^2(A+B)\\ \cos^2 B &= (-\cos(A+C))^2=\cos^2(A+C) \text{ and}\\ \cos^2 A &= (-\cos(C+B))^2=\cos^2(C+B).\end{aligned}$$

We shall also use the identity $\cos(x+y)=\cos x\cos y-\sin x\sin y$ $(*)$.

We may also assume $A\le B\le C$ so that $a\le b\le c<1$ and by acuteness

$$\frac{\pi}{2}<A+B\le A+C\le B+C,\ \text{since } A+B+C=\pi.$$

We have using $(*)$ that
$\sin(A)\cdot\cos^2(B)=\sin(A)\cdot\cos^2(A+C)=a\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-c^2}-a\cdot c\right))^2.$

Now $\dfrac{\partial}{\partial a}\left(a\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-c^2}-a\cdot c\right)^2\right)=$

$\left(\sqrt{1-a^2}\cdot\sqrt{1-c^2}-ac\right)^2+2a\left(\sqrt{1-a^2}\sqrt{1-c^2}-ac\right)\left(-\dfrac{\sqrt{1-c^2}a}{\sqrt{1-a^2}}-c\right)$ is clearly positive when one notes that factor $\sqrt{1-a^2}\sqrt{1-c^2}-ac$ is negative being $\cos(A+C)$ where $A+C$ is obtuse. Hence the radicand in the first term on the left hand side of the given inequality increases with a and since $a\le b\le c$ has it maximum value when $a=b$.

Similarly we have using $(*)$ that
$\sin(B)\cdot\cos^2(C)=\sin(B)\cdot\cos^2(A+B)=b\cdot\left(\sqrt{1-a^2}\sqrt{1-b^2}-ab\right)^2.$

Now $\dfrac{\partial}{\partial b}\left(b\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-b^2}-a\cdot b\right)^2\right)=$

$\left(\sqrt{1-a^2}\cdot\sqrt{1-b^2}-ab\right)^2+2b\left(\sqrt{1-a^2}\sqrt{1-b^2}-ab\right)\left(-\dfrac{\sqrt{1-a^2}b}{\sqrt{1-b^2}}-a\right)$ is clearly positive when one notes that factor $\sqrt{1-a^2}\sqrt{1-b^2}-ab$ is negative being $\cos(A+B)$

where $A+B$ is obtuse. Hence the radicand in the second term on the left hand side of the given inequality increases with b and since $a \leq b \leq c$ has it maximum value when $b=c$.

And similarly we have using $(*)$ that
$\sin(C)\cdot\cos^2(A)=\sin(C)\cdot\cos^2(C+B)=c\cdot\left(\sqrt{1-c^2}\sqrt{1-b^2}-cb\right)^2$ and
$\frac{\partial}{\partial b}\left(c\cdot\left(\sqrt{1-c^2}\cdot\sqrt{1-b^2}-c\cdot b\right)^2\right)=$
$2c\left(\sqrt{1-b^2}\cdot\sqrt{1-c^2}-bc\right)^2+2b\left(-\frac{\sqrt{1-c^2}b}{\sqrt{1-b^2}}-c\right)$ is clearly positive when one notes that factor $\sqrt{1-b^2}\sqrt{1-c^2}-b\cdot c$ is negative being $\cos(C+B)$ where $C+B$ is obtuse. Hence the radicand in the third term on the left hand side of the given inequality increases with b and since $a\leq b\leq c$ has its maximum value with $b=c$.

Thus the first three radicands are maximized simultaneously when $a=b=c$ and since A, B and C are acute, we have $A=B=C=\frac{\pi}{3}$ and the left hand side of the given inequality has its maximum value $3\cdot\sqrt[4]{\left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{1}{2}\right)^2}=3\cdot\sqrt[4]{\frac{\sqrt{3}}{8}}=3\cdot\sqrt[8]{\frac{3}{64}}$ as was to be shown.

Solution 2 by Arkady Alt, San Jose, CA

Since by AM-GM Inequality

$$\sqrt[4]{\frac{1}{2}\cdot\frac{\sin A}{\sqrt{3}}\cdot\cos^2 B} \quad\leq\quad \frac{\frac{1}{2}+\frac{\sin A}{\sqrt{3}}+2\cos B}{4} \text{ then}$$

$$\frac{1}{\sqrt[8]{12}}\sum_{cyc}\sqrt[4]{\sin A\cos^2 B} = \sum_{cyc}\sqrt[4]{\frac{1}{2}\cdot\frac{\sin A}{\sqrt{3}}\cdot\cos^2 B}\leq\sum_{cyc}\frac{\frac{1}{2}+\frac{\sin A}{\sqrt{3}}+2\cos B}{4}$$
$$=\frac{3}{8}+\frac{1}{\sqrt{3}}\left(\sin A+\sin B+\sin C\right)+2\left(\cos A+\cos B+\cos C\right).$$

Since $R\geq 2r$ (Euler Inequality) we have $\cos A+\cos B+\cos C=1+\frac{r}{R}\leq\frac{3}{2}$.
Also, since $\sin x$ is concave down on $[0,\pi]$ then

$$\frac{\sin A+\sin B+\sin C}{3} \leq \sin\frac{A+B+C}{3}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}\iff\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}.$$

Thus,

$$\frac{1}{\sqrt[8]{12}}\sum_{cyc}\sqrt[4]{\sin A\cos^2 B} \quad\leq\quad \frac{1}{4}\left(\frac{3}{2}+\frac{1}{\sqrt{3}}\cdot\frac{3\sqrt{3}}{2}+2\cdot\frac{3}{2}\right)=\frac{3}{2}$$

$$\iff \sum_{cyc} \sqrt[4]{\sin A \cos^2 B} \leq \frac{3}{2} \cdot \sqrt[8]{12} = 3\sqrt[8]{\frac{3}{64}}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; D.M. Bătinetu−Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5294:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

a) Calculate $\displaystyle\sum_{n=2}^{\infty} (n - \zeta(2) - \zeta(3) - \cdots - \zeta(n))$.

b) More generally, for $k \geq 2$ an integer, find the value of the multiple series

$$\sum_{n_1,n_2,\cdots,n_k=1}^{\infty} (n_1 + n_2 + \cdots + n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1 + n_2 + n_3 + \cdots + n_k)),$$

where ζ denotes the Riemann Zeta function.

Solution 1 by Anastasios Kotronis, Athens, Greece

We will answer b) which answers both questions. At first, it is rather straightforward using induction and the sum of geometric series that for $k \geq 1$ and $m \geq 2$ integers we have

$$\sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \frac{1}{m^{n_1+n_2+\cdots+n_k}} = \frac{1}{(m-1)^k}.$$

Now with the change of the summation order, whenever takes place, being justified by the constant sign of the summands, we have

$$\sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} (n_1 + n_2 + \cdots + n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1 + n_2 + \cdots + n_k))$$

$$= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \left(1 - \sum_{k=2}^{n_1+n_2+\cdots+n_k} \sum_{m\geq 2} \frac{1}{m^k}\right)$$

$$= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \left(1 - \sum_{m\geq 2} \sum_{k=2}^{n_1+n_2+\cdots+n_k} \frac{1}{m^k}\right)$$

$$
\begin{aligned}
&= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \left(1 - \sum_{m\geq 2}\left(\frac{1}{m-1} - \frac{1}{m}\right) + \sum_{m\geq 2}\frac{1}{m-1}\cdot\frac{1}{m^{n_1+n_2+\cdots+n_k}}\right) \\
&= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \sum_{m\geq 2}\frac{1}{m-1}\cdot\frac{1}{m^{n_1+n_2+\cdots+n_k}} \\
&= \sum_{m\geq 2}\frac{1}{m-1}\sum_{n_1,n_2,\ldots,n_k=1}^{+\infty}\frac{1}{m^{n_1+n_2+\cdots+n_k}} \\
&= \sum_{m\geq 2}\frac{1}{(m-1)^{k+1}} = \zeta(k+1).
\end{aligned}
$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

For $k \geq 2$, we have

$$
\begin{aligned}
&\sum_{n_1,n_2,\ldots n_k=1}^{\infty} \left(n+1+n_2+\cdots+n_k - \zeta(2) - \zeta(3) - \cdots - \zeta\left(n_1+n_2+\cdot+n_k\right)\right) \\
&= \sum_{n_1,n_2,\ldots n_k=1}^{\infty}\left(n_1+n_2+\cdots+n_k - \sum_{s=2}^{n_1+n_2+\cdots+n_k}\sum_{m=1}^{\infty}\frac{1}{m^s}\right) \\
&= \sum_{n_1,n_2,\ldots n_k=1}^{\infty}\left(1 - \sum_{m=2}^{\infty}\sum_{s=2}^{n_1+n_2+\cdots+n_k}\frac{1}{m^s}\right) \\
&= \sum_{n_1,n_2,\ldots n_k=1}^{\infty}\left(1 - \sum_{m=2}^{\infty}\left(\frac{1}{(m-1)m} - \frac{1}{(m-1)m^{n_1+n_2+\cdot+n_k}}\right)\right) \\
&= \sum_{n_1,n_2,\ldots n_k=1}^{\infty}\left(1 - \sum_{m=2}^{\infty}\left(\frac{1}{m-1} - \frac{1}{m}\right) + \sum_{m+2}^{\infty}\frac{1}{(m-1)m^{n+1+n_2+\cdots+n_k}}\right) \\
&= \sum_{n_1,n_2,\ldots n_k=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{(m-1)m^{n+1+n_2+\cdots+n_k}} \\
&= \sum_{m=2}^{\infty}\frac{1}{m-1}\left(\sum_{n_1=1}^{\infty}\frac{1}{m^{n_1}}\right)\left(\sum_{n_2=1}^{\infty}\frac{1}{m^{n_2}}\right)\cdots\left(\sum_{n_k=1}^{\infty}\frac{1}{m^{n_k}}\right) \\
&= \sum_{m=2}^{\infty}\frac{1}{(m-1)^{k+1}}.
\end{aligned}
$$

So the answer to (b) is $\zeta(k+1)$. From the steps above, we see that the sum in (a) equals

$$\begin{aligned} & \sum_{m=2}^{\infty} \frac{1}{m-1} \sum_{n=2}^{\infty} \frac{1}{m^n} \\ = & \sum_{m=2}^{\infty} \frac{1}{(m-1)^2 m} \\ = & \sum_{m=2}^{\infty} \frac{1}{(m-1)^2} - \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) \\ = & \frac{\pi^2}{6} - 1. \end{aligned}$$

Solution 3 by G.C. Greubel, Newport News, VA

First note that

$$\sum_{k=2}^{n} x^k = \frac{x(x-x^n)}{1-x}. \qquad (1)$$

Now, the first series to consider is that of

$$S_1 = \sum_{n=2}^{\infty} \left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n) \right). \qquad (2)$$

The Zeta function is given by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (3)$$

and helps lead the series S_1 to the form

$$\begin{aligned} S_1 &= \sum_{n=2}^{\infty} \left[n - \sum_{k=2}^{n} \zeta(k) \right] \\ &= \sum_{n=2}^{\infty} \left[n - \sum_{r=1}^{\infty} \left(\sum_{k=2}^{n} \frac{1}{k^r} \right) \right] \\ &= \sum_{n=2}^{\infty} \left[n - \sum_{r=1}^{\infty} \frac{1}{r-1} \left(\frac{1}{r} - \frac{1}{r^n} \right) \right], \qquad (4) \end{aligned}$$

where (1) was used. It is seen that the first term of the series summed by r is problematic. To handle the difficulty consider the limit of the terms as $r \to 1$. This limit is

$$\lim_{r \to 1} \left\{ \frac{1}{r-1} \left(\frac{1}{r} - \frac{1}{r^n} \right) \right\} \to \frac{0}{0}. \qquad (5)$$

Use of L'Hospital's rule applies and leads to

$$\lim_{r\to1}\left\{\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^n}\right)\right\}=\lim_{r\to1}\left\{\frac{-1}{s^2}+\frac{n}{s^{n+1}}\right\}=n-1. \quad (6)$$

With this term the series of (4) now becomes

$$\begin{aligned}
S_1 &= \sum_{n=2}^{\infty}\left[1-\sum_{r=2}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^n}\right)\right] \\
&= \sum_{n=2}^{\infty}\left[1-\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r}-\frac{1}{r^n(r-1)}\right)\right] \\
&= \sum_{n=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r^n(r-1)} \\
&= \sum_{r=2}^{\infty}\frac{1}{r-1}\cdot\sum_{n=2}^{\infty}\frac{1}{r^n} \\
&= \sum_{r=2}^{\infty}\frac{2r-1}{r(r-1)^2} \\
&= \sum_{r=2}^{\infty}\left(\frac{1}{(r-1)^2}-\frac{1}{r(r-1)}\right) \\
&= \zeta(2)-\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r}\right) \\
S_1 &= \zeta(2)-1. \qquad (7)
\end{aligned}$$

This is the value of the first series in question.

The second series to consider is that of

$$S_2=\sum_{n_1,n_2,\cdots,n_k=1}^{\infty}\left(\sum_{p=1}^{k}n_p-\sum_{s=2}^{n_1+n_2+\cdots+n_k}\zeta(s)\right). \qquad (8)$$

In a similar manor as in the evaluation of the first series the second follows here.

$$\begin{aligned}
S_2 &= \sum_{n_k=1}^{\infty}\left[\sum_{p=1}^{k}n_p-\sum_{s=2}^{n_1+\cdots n_k}\sum_{r=1}^{\infty}\frac{1}{r^s}\right] \\
&= \sum_{n_k=1}^{\infty}\left[\sum_{p=1}^{k}n_p-\sum_{r=1}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^{n_1+\cdots+n_k}}\right)\right]. \qquad (9)
\end{aligned}$$

As in the case before the first term of the series summed over r is problematic and is dealt with by use of L'Hospital's rule and leads to the result

$$\lim_{r\to 1}\left\{\frac{1}{r-1}\left(\frac{1}{r}+\frac{1}{r^{n_1+\cdots+n_k}}\right)\right\}=\sum_{p=1}^{k} n_p-1. \qquad (10)$$

This then leads to

$$\begin{aligned}
S_2 &= \sum_{n_k=1}^{\infty}\left[1-\sum_{r=2}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^{n_1+\cdots+n_k}}\right)\right]\\
&= \sum_{n_k=1}^{\infty}\left[1-\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r}\right)+\sum_{r=2}^{\infty}\frac{1}{(r-1)r^{n_1+\cdots+n_k}}\right]\\
&= \sum_{n_k=1}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}\right)^{n_1+\cdots+n_k}\\
&= \sum_{r=2}^{\infty}\frac{1}{r-1}\left(\sum_{n=1}^{\infty}\frac{1}{r^n}\right)^{k}\\
&= \sum_{r=2}^{\infty}\frac{1}{(r-1)^{k+1}}\\
S_2 &= \zeta(k+1). \qquad (11)
\end{aligned}$$

This is the desired value of the second series.

Solution 4 by the proposer

First, we prove that

$$S_n=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^n}=n-\zeta(2)-\zeta(3)-\cdots-\zeta(n).$$

We have, since

$$\frac{1}{k(k+1)^n}=\frac{1}{k(k+1)^{n-1}}-\frac{1}{(k+1)^n},$$

that

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)^n}=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^{n-1}}-\sum_{k=1}^{\infty}\frac{1}{(k+1)^n},$$

and hence, $S_n=S_{n-1}-(\zeta(n)-1)$. Iterating this equality we obtain that

$$S_n=S_1-(\zeta(2)+\zeta(3)+\cdots+\zeta(n)-(n-1)),$$

and, since $S_1=\sum_{k=1}^{\infty}1/(k(k+1))=1$, we get that $S_n=n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)$. Now we are ready to solve the problem.

a) The series equals $\zeta(2)-1$. We have,

$$\begin{aligned}
\sum_{n=2}^{\infty}(n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)) &= \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(k+1)^n}\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{n=2}^{\infty}\frac{1}{(k+1)^n}\right)\\
&= \sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\\
&= \frac{\pi^2}{6}-1,
\end{aligned}$$

and the first part of the problem is solved.

b) The series equals $\zeta(k+1)$. Let T_k be the value of the multiple series. We have,

$$\begin{aligned}
T_k &= \sum_{n_1,n_2,\cdots,n_k=1}^{\infty}\left(\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{n_1+n_2+\cdots+n_k}}\right)\\
&= \sum_{p=1}^{\infty}\frac{1}{p}\left(\left(\sum_{n_1=1}^{\infty}\frac{1}{(p+1)^{n_1}}\right)\cdots\left(\sum_{n_k=1}^{\infty}\frac{1}{(p+1)^{n_k}}\right)\right)\\
&= \sum_{k=1}^{\infty}\frac{1}{p}\left(\sum_{m=1}^{\infty}\frac{1}{(p+1)^m}\right)^k\\
&= \sum_{k=1}^{\infty}\frac{1}{p^{k+1}}\\
&= \zeta(k+1),
\end{aligned}$$

and the problem is solved.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed. Gray, Highland Beach, FL (part a), and Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy.

Comments

Kenneth Korbin's problem 5283 challenged us to find the sides of two different isosceles triangles for which each has a perimeter of 162 and an area 1008.

Brian D. Beasley's solution was one of those featured in the April issue of the column and in it he stated: "In general, if we seek all isosceles triangles of the form $(x, x, P-2x)$ that have perimeter P and area A, then we obtain the equation

$$16Px^3-20P^2x^2+8P^3x-(P^4+16A^2)=0.$$

The given values $P = 162$ and $A = 1008$ produce exactly two such triangles. For what values of P and A would we find no triangles, one triangle, two triangles, or three triangles?"

Ken Korbin answered this question.

- If $A > \dfrac{P^2\sqrt{3}}{36}$, then no triangle is possible.
- If $A = \dfrac{P^2\sqrt{3}}{36}$, the exactly one triangle is possible and that triangle is equilateral.
- If $0 < A < \dfrac{P^2\sqrt{3}}{36}$ then exactly two different isosceles triangles have perimeter $=P$, and area $=A$.

Late Solutions

G. C. Greubel of Newport News, VA solved 5283.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
December 15, 2014

- **5313:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles if they both have perimeter 256 and area 1008.

- **5314:** *Proposed by Roger Izard, Dallas TX*

  A biker and a hiker like to workout together by going back and forth on a road which is ten miles long. One day, at 8 AM, at the starting end of the road, they went out together. The biker soon got far past the hiker, reached the end of the road, reversed his direction, and soon passed by the hiker at 9:06 AM. Then, the biker got down to the beginning part of the road, reversed his direction, and got back to the hiker at 9:24 AM. The biker and the hiker were, then, going in the same direction. Calculate in miles per hour the speeds of the hiker and the biker.

- **5315:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The hexagonal numbers have the form $H_n = 2n^2 - n,\ n = 1, 2, 3, \ldots$. Prove that infinitely many hexagonal numbers are the sum of two hexagonal numbers.

- **5316:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $\{u_n\}_{n\geq 0}$ be a sequence defined recursively by

  $$u_{n+1} = \sqrt{\frac{u_n^2 + u_{n-1}^2}{2}}.$$

  Determine $\lim\limits_{n\to\infty} u_n$ in terms of u_0, u_1.

- **5317:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_k, b_k > 0,\ 1 \leq k \leq n$, be real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that

  $$\frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \leq \sum_{k=1}^{n} \frac{b_k^5}{a_k}.$$

- **5318:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Prove that $(1+x)^x \le 1+x^2$ for $0 \le x \le 1$.

Solutions

- **5295:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic hexagon has sides
  $$\left(5,\ 7\sqrt{17},\ 23\sqrt{13},\ 25\sqrt{13},\ 25\sqrt{17},\ 45\right).$$
  Find the diameter of the circumcircle and the area of the hexagon.

  Solution by Kee-Wai Lau, Hong Kong, China

  We show that diameter of the circumcircle is 125 and the area of the hexagon is $8(86\sqrt{34}+81\sqrt{39})$.

  Let O be the center and d be the diameter of the circumcircle, which we denote by C. It is easy to see that the anlgle subtended at O by a side of the hexagon with length s equals $2\sin^{-1}\left(\dfrac{s}{d}\right)$. We first suppose that O lies inside the hexagon, so that
  $$f(d)=\pi, \qquad (1)$$
  where
  $$f(d)=\sin^{-1}\left(\frac{5}{d}\right)+\sin^{-1}\left(\frac{7\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{23\sqrt{13}}{d}\right)+2\sin^{-1}\left(\frac{25\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{4.}{d}\right).$$
  $$\begin{aligned} a &= \sin^{-1}\left(\frac{23\sqrt{13}}{125}\right)+\sin^{-1}\left(\frac{\sqrt{13}}{5}\right)+\sin^{-1}\left(\frac{\sqrt{1}}{25}\right) \text{ and} \\ b &= \sin^{-1}\left(\frac{7\sqrt{17}}{125}\right)+\sin^{-1}\left(\frac{\sqrt{17}}{5}\right)+\sin^{-1}\left(\frac{\sqrt{9}}{25}\right) \end{aligned}$$
  Then $f(125)=a+b$. Since $a=\sin^{-1}\left(\dfrac{4\sqrt{39}}{25}\right)+\sin^{-1}\left(\dfrac{1}{25}\right)=\sin^{-1}\ 1=\dfrac{\pi}{2}$ and

  $b=\sin^{-1}\left(\dfrac{4\sqrt{34}}{25}\right)+\sin^{-1}\left(\dfrac{9}{25}\right)=\sin^{-1}\ 1=\dfrac{\pi}{2}$ so (1) holds if and only if $d=125$.

  Now the distances from O to the sides $\left(5,\ 7\sqrt{17},\ 23\ \sqrt{13},\ 25\sqrt{13},\ 25\sqrt{17},\ 45\right)$ are $\left(10\sqrt{39},\ 43\sqrt{2},\ 27\sqrt{3},\ 25\sqrt{3},\ 25\sqrt{2},\ 10\sqrt{34}\right)$. So the area of the hexagon equals
  $$\frac{1}{2}\left(50\sqrt{39}+301\sqrt{34}+621\sqrt{39}+625\sqrt{39}+10\sqrt{39}+625\sqrt{34}+450\sqrt{34}\right)$$

$$= \quad 8\left(86\sqrt{34}+81\sqrt{39}\right).$$

We next suppose that O lies on or is outside the hexagon. Since the longest side of the hexagon is $25\sqrt{17}$, so $d \geq 25\sqrt{17}$. Moreover,

$$\sin^{-1}\left(\frac{5}{d}\right)+\sin^{-1}\left(\frac{7\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{23\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{17}}{d}\right)+\sin^{-1}\left(\frac{45}{d}\right)$$

$$= \quad \sin^{-1}\left(\frac{25\sqrt{17}}{d}\right),$$

and hence,

$$\sin^{-1}\left(\frac{23\sqrt{13}}{d}\right)+\sin^{-1}\left(\frac{25\sqrt{13}}{d}\right) < \sin^{-1}\left(\frac{25\sqrt{17}}{d}\right). \qquad (2)$$

If $d < \sqrt{15002} = \sqrt{(2)(13)(577)}$, then by (2)

$$\sin^{-1}\left(\frac{25\sqrt{17}}{d}\right) > \sin^{-1}\left(\frac{23}{\sqrt{1154}}\right)+\sin^{-1}\left(\frac{25}{\sqrt{1154}}\right) = \sin^{-1}1 = \frac{\pi}{2} \quad \text{which is false.}$$

If $d \geq \sqrt{15002}$, then the left hand side of (2) equals

$$\begin{aligned}
\sin^{-1}\left(\frac{25\sqrt{13}}{d}\sqrt{1-\frac{6877}{d^2}}+\frac{23\sqrt{13}}{d}\sqrt{1-\frac{8125}{d^2}}\right) \quad &\geq \quad \sin^{-1}\left(\frac{25\sqrt{13}}{d}\sqrt{1-\frac{6877}{15002}}+\frac{23\sqrt{13}}{d}\sqrt{1-\frac{8125}{15002}}\right)\\
&= \quad \sin^{-1}\left(\frac{\sqrt{15002}}{d}\right)\\
&> \quad \sin^{-1}\left(\frac{25\sqrt{17}}{d}\right),
\end{aligned}$$

which is also false. Thus we conclude that O must lie inside the hexagon, and this completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5296:** *Proposed by Roger Izard, Dallas, TX*

Consider the "Star of David," a six pointed star made by overlapping the triangles ABC and FDE. Let

$$\overline{AB}\cap\overline{DF}=G, \text{ and } \overline{AB}\cap\overline{DE}=H,$$
$$\overline{AC}\cap\overline{DF}=L, \text{ and } \overline{AC}\cap\overline{FE}=K,$$
$$\overline{BC}\cap\overline{DE}=I, \text{ and } \overline{BC}\cap\overline{FE}=J,$$

in such a way that:

$$\frac{CK}{AC}=\frac{EI}{DE}=\frac{BI}{BC}=\frac{GD}{DF}=\frac{AG}{AB}=\frac{FK}{EF} \text{ and}$$

$$\frac{AL}{AC}=\frac{DH}{DE}=\frac{BH}{AB}=\frac{EJ}{EF}=\frac{FL}{DF}=\frac{CJ}{CB}.$$

Let $r=\dfrac{CK}{AC}$ and let $p=\dfrac{AL}{AC}$. Prove that $r+p=\dfrac{3pr+1}{2}$.

Solution by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We construct a drawing of the figure and determine lengths of some of the sides in terms of r,p and the sides of the given triangles.
The following shows that the side lengths of the smaller triangle based on

$$r=\frac{CK}{AC}=\frac{EI}{DE}=\frac{BI}{BC}=\frac{GD}{DF}=\frac{AG}{AB}=\frac{FK}{EF} \text{ and}$$

$$p=\frac{AL}{AC}=\frac{DH}{DE}=\frac{BH}{AB}=\frac{EJ}{EF}=\frac{FL}{DF}=\frac{CJ}{CB}.$$

We see that $AC+AL+LK+KC=pAC+LK+rAC$, so $LK=(1-r-p)AC$.
Similarly,

$$\begin{aligned} HI &= (1-r-p)DE\\ KJ &= (1-r-p)EF\\ GH &= (1-r-p)AB\\ IJ &= (1-r-p)BC\\ GL &= (1-r-p)DF. \end{aligned}$$

We apply the Law of Cosines to the two triangles having A as principal vertex.
In $\triangle ABC,\ AC^2+AB^2-2AC\cdot AB\cos A=BC^2$, and in
$\triangle AGL,\ (pAC)^2+(rAB)^2-2prAC\cdot AB\cos A=GL^2=(1-r-p)^2DF^2$.
Solving each equation for $2AC\cdot AB\cos A$ and equating the results, we have

$$2AC\cdot AB\cos A=AC^2+AB^2-BC^2=\frac{p^2AC^2+r^2AB^2-(1-r-p)^2DF^2}{pr}.$$

Clearing fractions yields

$$prAC^2+prAB^2-prBC^2=p^2AC^2+r^2AB^2-(1-r-p)^2DF^2$$

so

$$(pr-pr)AC^2+(pr-r^2)AB^2-prBC^2+(1-r-p)^2DF^2=0.$$

By considering the other vertices B, C, D, E, F in turn, we obtain analogous equations:

$$\begin{aligned}
(pr-p^2)AB^2+(pr-r^2)BC^2-prAC^2+(1-r-p)^2DE^2 &= 0\\
(pr-p^2)BC^2+(pr-r^2)AC^2-prAB^2+(1-r-p)^2FE^2 &= 0\\
(pr-p^2)DE^2+(pr-r^2)DF^2-prFE^2+(1-r-p)^2AB^2 &= 0\\
(pr-p^2)EF^2+(pr-r^2)DE^2-prDF^2+(1-r-p)^2BC^2 &= 0\\
(pr-p^2)DF^2+(pr-r^2)EF^2-prDE^2+(1-r-p)^2AC^2 &= 0.
\end{aligned}$$

Summing these six equations and letting $S = AB^2+AC^2+BC^2+DE^2+DF^2+EF^2$ yields a very nice result:

$$\begin{aligned}
(pr-p^2)S+(pr-r^2)S-prS+(1-r-p)^2S &= 0, \text{ or}\\
\left\{(pr-p^2)+(pr-r^2)-pr+(1-r-p)^2\right\}S &= 0.
\end{aligned}$$

Because S is not zero, this gives

$$(pr-p^2)+(pr-r^2)-pr+(1-r-p)^2=0.$$

Expanding the trinomial and collecting like terms gives us

$$\begin{aligned}
3pr-2r-2p+1 &= 0. \text{ So,}\\
2(r+p) &= 1+3pr. \text{ Thus,}\\
r+p &= \frac{3pr+1}{2}.
\end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5297:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let $s_n = n^2,\ t_n = \dfrac{n(n+1)}{2},\ p_n = \dfrac{n(3n-1)}{2}$, for positive integers n, be the square, triangular and pentagonal numbers respectively. Prove, independently of each other, that

$$\begin{aligned} i) \ \ t_a + p_b &= t_c \\ ii) \ \ t_a + s_b &= p_c \\ iii) \ \ p_a + s_b &= s_c, \end{aligned}$$

for infinitely many positive integers, a, b, and c.

Solution by Carl Libis, Lane College, Jackson, TN

$$\begin{aligned} i) \quad t_n + p_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)(3n+2)}{2} = \frac{n^2+n+3n^2+5n+2}{2} \\ &= \frac{4n^2+6n+2}{2} = \frac{(2n+1)(2n+2)}{2} = t_{2n+1} \end{aligned}$$

$$ii) \quad s_n + t_{n-1} = n^2 + \frac{(n-1)n}{2} = \frac{2n^2}{2} + \frac{n^2-n}{2} = \frac{3n^2-n}{2} = \frac{n(3n-1)}{2} = p_n$$

$$\begin{aligned} iii) \quad p_{4n+1} + s_n &= \frac{(4n+1)(12n+2)}{2} + n^2 = \frac{48n^2+20n+2}{2} + \frac{2n^2}{2} \\ &= \frac{50n^2+20n+2}{2} = 25n^2+10n+1 = (5n+1)^2 = s_{5n+1} \end{aligned}$$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Becca Rousseau, Ellie Erehart, and Davis Weerheim (jointly), students at Taylor University, Upland, IN; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5298:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Let $(a_n)_{n\geq 1}$ be an arithmetic progression and m a positive integer. Calculate:

$$\lim_{n\to\infty} \left(\left(\sum_{k=1}^{m} \left(1+\frac{1}{n}\right)^{n+a_k} - me \right) n \right).$$

Solution by Anastasios Kotronis, Athens, Greece

Let $a_n = a_1 + (n-1)d$ where a_1 is the initial term and d is the common difference of successive terms. Then

$$\begin{aligned}
\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_k} &= \sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_1+(k-1)d} = \left(1+\frac{1}{n}\right)^{n+a_1-d}\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{kd} \\
&= \exp\left((n+a_1-d)\ln\left(1+\frac{1}{n}\right)\right)\sum_{k=1}^{m}\exp\left(kd\ln\left(1+\frac{1}{n}\right)\right) \\
&= \exp\left((n+a_1-d)\left(\frac{1}{n}-\frac{1}{2n^2}+\mathcal{O}(n^{-3})\right)\right)\sum_{k=1}^{m}\exp\left(kd\left(\frac{1}{n}+\mathcal{O}(n^{-2})\right)\right) \\
&= \left(e+\frac{e(a_1-d-1/2)}{n}+\mathcal{O}(n^{-2})\right)\sum_{k=1}^{m}\left(1+\frac{kd}{n}+\mathcal{O}(n^{-2})\right) \\
&= \left(e+\frac{e(a_1-d-1/2)}{n}+\mathcal{O}(n^{-2})\right)\left(m+\frac{dm(m+1)}{2n}+\mathcal{O}(n^{-2})\right) \\
&= em+\frac{em\,(d(m-1)+2a_1-1)}{2n}+\mathcal{O}(n^{-2}) = em+\frac{em\,(a_m+a_1-1)}{2n}+\mathcal{O}(n^{-2})
\end{aligned}$$

so the desired limit is $\dfrac{em\,(a_m+a_1-1)}{2}$.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Kee-Wai Lau, Hong Kong, China, and the proposers.

- **5299:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the aid of a computer, show that

$$\ln^2 2\int_0^1 \frac{x^{3/2}2^x\sin x}{(1+x\ln 2)^2}\,dx \;\geq\; \frac{1-\ln 2}{1+\ln 2}\int_0^1 \sqrt{x}\,\sin x\,dx.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University Rome, Italy

The two functions $\sqrt{x}\sin x$ and $\dfrac{x2^x}{(1+x\ln 2)^2}$ are both increasing in $[0,1]$. Indeed, $\dfrac{1}{2\sqrt{x}}\sin x+\sqrt{x}\cos x$ and $\dfrac{2^x(1+x\ln 2+x^2\ln^2 2)}{(1+x\ln 2)^2}$ are the derivatives respectively of the first and the second function.

Chebyshev's inequality yields

$$\ln^2 2 \int_0^1 \frac{x^{3/2} 2^x \sin x}{(1+x\ln 2)^2} dx \geq \ln^2 2 \int_0^1 \frac{x 2^x}{(1+x\ln 2)^2} dx \int_0^1 \sqrt{x} \sin x dx.$$

Moreover,

$$\ln^2 2 \int_0^1 \frac{x 2^x}{(1+x\ln 2)^2} dx = \frac{2^x}{1+x\ln 2}\Big|_0^1 = \frac{1-\ln 2}{1+\ln 2},$$

hence, the result.

Solution 2 by Ed Gray, Highland Beach, FL

The method will be to increase the integral on the right to get a function that is integrable, and decrease the integral on the left to get a function which is integral in such a way that the inequality is maintained. We will also evaluate $\frac{1}{(\ln(2))^2} \cdot \frac{1-\ln 2}{1+\ln 2}$, and use its value as a coefficient on the right hand side of the inequality.

For $0 \leq x \leq 1$,

$$\sin(x) \leq x, \sqrt{x}\sin(x) \leq x^{3/2}.$$

So,

$$\int_0^1 \sqrt{(x)} \sin(x) dx < \int_0^1 x^{3/2} dx = \frac{2}{5} x^{5/2}\Big|_0^1 = 0.4.$$

Also,

$$\frac{1}{(\ln(2))^2} = 2.08136898, \ \frac{1-\ln(2)}{1+\ln(2)} = 0.181232218, \text{ and } \frac{1}{(\ln(2))^2} \cdot \left(\frac{1-\ln(2)}{1+\ln(2)}\right) = 0.3772111.$$

- (1) $\int_0^1 \left(\frac{x^{3/2}(2^x)\sin(x)}{(1+x\ln(2))^2}\right) dx \geq (0.4)(0.3772111) = 0.150884$. We need to reduce the value of the integral to get an approximation that still satisfies the inequality.
- (2) Consider $1 + x > 1 + x\ln(2)$. Squaring,
- (3) $1 + 2x + x^2 > \left(1 + x\ln(2)\right)^2$, and
- (4) $1 + 2x > \left(1 + x\ln(2)\right)^2$. This inequality holds since both functions are monotonically increasing, and the relationship holds for $x = 1$.

Then:

- (5) $\frac{1}{1+2x} < \frac{1}{(1+x\ln(2))^2}$. So,
- (6) $\frac{x^{3/2}\left(2^x\right)\sin(x)}{1+2x} < \left(\frac{x^{3/2}\left(2^x\right)\sin(x)}{(1+x\ln 2)^2}\right)$. For $0 \leq x \leq 1$,
- (7) $\sin(x) > x - \frac{x^3}{6}$, so,
- (8) $\frac{x^{5/2}\left(2^x\right)\left(1-\frac{x^3}{6}\right)}{1+2x} < \frac{x^{3/2} 2^x \sin(x)}{\left(1+x\ln(2)\right)^2}$, or

- (9) $\dfrac{x^{5/2}\,(2^x)\left(1-\dfrac{x^3}{6}\right)}{1+2x} < \dfrac{x^{3/2}\,(2^x\sin(x)}{(1+2x)} < \dfrac{x^{3/2}\,(2^x\sin(x))}{(1+x\ln(2))^2}$

We now express $\dfrac{2^x}{1+2x}$ in a Taylor series expansion about 0.5.

- (10) $f(x) = f(.5) + f'(.5)(x-.5) + \dfrac{f''(.5)}{2!}(x-.5)^2 + \dfrac{f'''(.5)}{3!}(x-.5)^3 + \ldots.$

As one can imagine, the derivatives get quite messy, so let's bring in Bing to compute them for us, (which does not violate the spirit of not using a computer because it is not evaluating the integral, just saving time. In any case, the series out to $(x-.5)^5$ is

$$f(x) = \frac{2^x}{1+2x} \approx 0.7071 - .2169(x-.5) + .3868(x-.5)^2 - .3475(x-.5)^3 + .3543(x-.5)^4 - .3534(x-.5)^5$$

The following table gives a "feel" for the goodness of fit for the approximation over the range of $0 \le x \le 1$.

x	$2^x/(1+2x)$	*Approximate value*
0	1.0	.988951
0.1	.893144	.890731
0.2	.820499	.820131
0.3	.769465	.769412
0.4	.733060	.733060
0.5	.707107	.7072107
0.6	.688962	.688960
0.7	.675877	.676858
0.8	.669654	.669457
0.9	.666452	.665419
1.0	.666667	.662985

Not only is this a good fit, but if we define the expansion by $f(x)$, we see that $f(x) < \dfrac{2^x}{1+2x}$ and the equation in step (9) becomes

- (11) $x^{5/2)}\left(1-\dfrac{x^2}{6}\right)f(x) < x^{5/2}\left(1-\dfrac{x^2}{6}\right)\left(\dfrac{2^x}{1+2x}\right) < x^{3/2}\dfrac{2^x\sin x}{1+2x} < \dfrac{x^{3/2}2^x\sin(x)}{(1+x\ln(2))^2}.$

We now need to write the series expansion of $f(x)$, to obtain a polynomial in x. Then by multiplying by $(x^{5/2})\left(1-\dfrac{x^2}{6}\right)$ we will obtain a polynomial in x for which we can easily perform the integration from 0 to 1. We save the reader the details. The integrand is:

- (12) $.058909x^{19/2} - .20633866x^{17/2} - 0301165x^{15/2} + .924424x^{13/2} - 1.7479965x^{11/2} + 1.7168252x^{9/2} - 1.1521715x^{7/2} + .988952x^{5/2}$. Integrating with respect to x gives us
- (13) $.00561x^{21/2} - .0217198x^{19/2} - .003543x^{17/2} + .123256x^{15/2} - .268923x^{13/2} + .31215x^{11/2} - .256038x^{9/2} + .282558x^{7/2}$.

So, returning to the equation in (1), we see that $.1733502 > .150884$, and this proves the inequality.

Solution 3 by Kee-Wai Lau, Hong Kong, China

For $0 \le x \le 1$, let $f(x) = \dfrac{2^x}{(1 + x\ln 2)^2}$ so that $\dfrac{df(x)}{dx} = \dfrac{(\ln 2)2^x(x\ln 2 - 1)}{(1 + x\ln 2)^2} < 0$,

and $f(x) \ge f(1) = \dfrac{2}{(1+\ln 2)^2}$. Hence, $\displaystyle\int_0^1 \frac{x^{3/2}2^x \sin x}{(1 + x\ln 2)^2}dx \ge \frac{2}{(1+\ln 2)^2}\int 0^1 x^{3/2}\sin x dx$.

By the substitution $x = y^{3/5}$, we obtain

$$\int_0^1 x^{3/2}\sin x dx = \frac{3}{5}\int_0^1 \sqrt{y}\sin(y^{3/5})dy \ge \frac{3}{5}\int_0^1 \sqrt{y}\sin y dy.$$

Hence to prove the inequality of the problem, we need only show that

$\dfrac{6\ln^2 2}{5(1+\ln 2)} \ge 1 - \ln 2$, or equivalently $\ln^2 2 \ge \dfrac{5}{11}$. Since $\left(\dfrac{17}{25}\right)^2 = \dfrac{289}{625} > \dfrac{5}{11}$,

so it suffices to show that $\ln 2 > \dfrac{17}{25}$, or $e^{-17/25} > \dfrac{1}{2}$. But this follows from the fact that $e^{-17/25} > 1 - \displaystyle\sum_{n=1}^{5} \frac{(-1)^{n-1}}{n!}\left(\frac{17}{25}\right)^n = \frac{148386317}{292986750} > \frac{1}{2}$.

Remark: If we use the rapidly convergent series $\ln 2 = \dfrac{2}{3}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)9^k}$, as listed in

"Natural logarithm of 2–Wikipedia" in the internet, we obtain easily

$\ln 2 > \dfrac{2}{3}\left(1 + \dfrac{1}{27}\right) = \dfrac{56}{81} > \dfrac{17}{25}$.

Also solved by the proposer.

- **5300:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n \ge 1$ be an integer. Prove that

$$\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} = \sum_{k=0}^{n-1}\binom{n-1}{k}\cdot\frac{1}{2n-2k-1}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $n = 1$,

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x} &= \int_{\pi/4}^{\pi/2} \csc^2 x \, dx \\
&= -\cot x]_{\pi/4}^{\pi/2} \\
&= 1 \\
&= \sum_{k=0}^{0} \binom{0}{k} \frac{1}{2-2k-1}.
\end{aligned}$$

Hence, the statement is true when $n = 1$.

If $n \geq 2$, then we use the standard calculus approach for evaluating

$$\int \csc^{2m} x \, dx.$$

To begin,

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} &= \int_{\pi/4}^{\pi/2} \csc^{2n} x \, dx \\
&= \int_{\pi/4}^{\pi/2} \left(1 + \cot^2 x\right)^{n-1} \left(\csc^2 x \, dx\right).
\end{aligned}$$

If we substitute $u = \cot x$ and simplify, we get

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} &= -\int_{1}^{0} \left(1 + u^2\right)^{n-1} \, du \\
&= \int_{0}^{1} \left(1 + u^2\right)^{n-1} \, du.
\end{aligned}$$

Finally, by the Binomial Theorem,

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} &= \int_0^1 \sum_{k=0}^{n-1} \binom{n-1}{k} u^{2(n-1-k)} \, du \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^1 u^{2n-2k-2} \, du \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \left. \frac{u^{2n-2k-1}}{2n-2k-1} \right]_0^1 \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{1}{2n-2k-1}.
\end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$
\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x} &= \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sin^2 x}\right)^n dx \\
&= \int_{\pi/4}^{\pi/2} \left(1+\frac{1}{\tan^2 x}\right)^n dx \\
&\underset{(t=1/\tan x)}{=} \int_1^0 \left(1+t^2\right)^n \frac{-dt}{1+t^2} \\
&= \int_0^1 \left(1+t^2\right)^{n-1} dt \\
&= \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot 1^k \cdot \left(t^2\right)^{n-1-k}\right) dt \\
&= \sum_{k=0}^{n-1} \int_0^1 \binom{n-1}{k} \cdot t^{2n-2k-2} dt \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \left.\frac{t^{2n-2k-1}}{2n-2k-1}\right|_{t=0}^{t=1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{1}{2n-2k-1}.
\end{aligned}
$$

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Letting $t=\sin x$ yields $\displaystyle\int_{\{1/\sqrt{2}\}}^1 \frac{1}{t^{2n}} \cdot \frac{1}{\sqrt{1-t^2}} dt.$

Moreover, $y=\sqrt{\dfrac{1}{t^2}-1}$ yields

$$
\int_1^0 (y^2+1)^n \frac{\sqrt{1+y^2}}{y} \frac{-y}{(1+y^2)^{3/2}} dy = \int_0^1 (1+y^2)^{n-1} dy.
$$

Therefore,

$$
\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{2n-2k-1} \quad = \quad \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^1 t^{2n-2k-2} dt
$$

$$
\begin{aligned}
&= \int_0^1 t^{2n-2} \sum_{k=0}^{n-1} \binom{n-1}{k} t^{-2k} dt \\
&= \int_0^1 t^{2n-2} \left(1 + \frac{1}{t^2}\right)^{n-1} \\
&= \int_0^1 (1+t^2)^{n-1} dt.
\end{aligned}
$$

and this concludes the proof.

Also solved by Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China, and the proposer.

Problems   Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2015

- **5319:** *Proposed by Kenneth Korbin, New York, NY*

  Let N be an odd integer greater than one. Then there will be a Primitive Pythagorean Triangle with perimeter equal to $\left(N^2+N\right)^2$. For example, if $N=3$, then the perimeter equals $\left(3^2+3\right)^2=144$.

  Find the sides of the PPT for perimeter $\left(15^2+15\right)^2$ and for perimeter $\left(99^2+99\right)^2$.

- **5320:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  It is fairly well known that if (a,b,c) is a Primitive Pythagorean Triple (PPT), then the product abc is divisible by 60. Find infinitely many PPT's (a,b,c) such that the sum $(a+b+c)$ is also divisible by 60.

- **5321:** *Proposed by Lawrence M. Lesser, University of Texas at El Paso, TX*

  On pop quizzes during the fall semester, Al gets 1 out of 3 questions correct, while Bob gets 3 of 8 correct. During the spring semester, Al gets 3/5 questions correct, while Bob gets 2/3 correct. So Bob did better each semester ($3/8>1/3$ and $2/3>3/5$) but worse for the overall academic year ($5/11<4/8$). The total number of questions involved in the above example was $3+8+5+3=19$, and the author conjectures (in his chapter in the 2001 Yearbook of the National Council of Teachers of Mathematics) that this is smallest dataset with nonzero numerators in which this reversal (Simpson's Paradox) happens. If we allow zeros, the smallest dataset is conjectured to be nine: $0/1<1/4$ and $2/3<1/1$, but $2/4>2/5$.

  Prove these conjectures or find counterexamples.

- **5322:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

If $\lim\limits_{n\to\infty}\left(-\frac{3}{2}\sqrt[3]{n^2}+\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)=a>0$, then compute $\lim\limits_{n\to\infty}\left(\frac{-\frac{3}{2}\sqrt[3]{n^2}+\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}}{a}\right)^{\sqrt[3]{n}}$.

- **5323:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers greater than or equal to one. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{-2}+\left(\frac{1}{n^2}\prod_{k=1}^{n}a_k^{-2}\right)\left(\sum_{k=1}^{n}\left(a_k^2-1\right)^{1/2}\right)^2\leq 1.$$

- **5324:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right).$$

Solutions

- **5301:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic quadrilateral with integer length sides is such that its area divided by its perimeter equals 2014.

  Find the maximum possible perimeter.

Solution 1 by Proposer

- The figure is an isosceles trapezoid. Let b_1, b_2 be the bases, h the height, l the non-parallel sides, and let $N = 2014$.
- The bases are $b_1 = 2$ and $b_2 = 8N^2$.
- Each leg is equal to the arithmetic mean of the bases,

$$l=\frac{b_1+b_2}{2}=4N^2+1.$$

- The altitude h is equal to the geometric mean of the bases.

$$h=\sqrt{b_1b_2}=4N.$$

- The area equals,

$$\frac{1}{2}h\left(b_1+b_2\right)=hl=\frac{(b_1+b_2)(\sqrt{b_1b_2}}{2}=16N^3+4N.$$

- Perimeter $= b_1 + b_2 + 2l = 4l = 16N^2 + 4$.
- $\dfrac{\text{Area}}{\text{Perimeter}} = \dfrac{16N^3 + 4N}{16N^2 + 4} = N$.
- $l^2 - h^2 = (l-2)^2$, (sides of a PPP.)

Letting the sides be (a, b, c, d) and letting $a = 2$ and $\sqrt{ac} = 4 \cdot 2014 = 8056$ gives $c = 32{,}449{,}568$.

Letting $b = d = \dfrac{a+c}{2} = 16{,}224{,}785$.

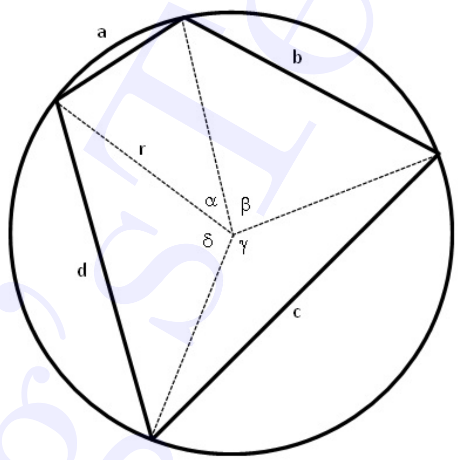
Then, Perimeter $= 4b = 4d$ and $\sqrt{bd} = b = d$.

Area $= K = \sqrt{abcd} = \sqrt{ac}\sqrt{bd} = 8056b$

$\dfrac{\text{Area}}{\text{Perimeter}} = \dfrac{8056b}{4b} = 2014$.

So, Perimeter$= P = 64{,}899{,}140$.

Solution 2 and Comments, jointly posted by Michael N. Fried of Kibbutz Revivim, Israel and Edwin Gray, Highland Beach, FL



We can begin to approach this problem in an obvious way. Let the sides be a, b, c, d, the area A, and the perimeter P. Let the quadrilateral be inscribed in a circle of radius r, and let the sides subtend the angles at the center $\alpha, \beta, \gamma, \delta$ (see figure). Then, we have:

$$A = \frac{1}{2}r^2(\sin\alpha + \sin\beta + \sin\gamma + \sin\delta)$$

And,

$$P = 2r\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}\right)$$

So that,

$$\frac{A}{P} = \frac{1}{4}\frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\delta}{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}}r$$

Or, in terms of P,

$$\frac{A}{P} = \frac{1}{8}\frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\delta}{\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}\right)^2}P = \sigma P$$

where σ is a function of α, β, γ (since $\delta = 360 - (\alpha + \beta + \gamma)$).

It is easy to see that σ tends to 0 as $\alpha, \beta, \gamma, \delta$ tend to 0. Consider a sequence of $\alpha, \beta, \gamma, \delta$ where $\alpha = \beta = \gamma$ and $\delta = 360 - 3\alpha$. For this sequence, we have:

$$\sigma(\alpha) = \frac{1}{8} \frac{3 \sin \alpha - \sin 3\alpha}{\left(3 \sin \frac{\alpha}{2} + \sin \frac{3\alpha}{2}\right)^2}$$

We can see clearly that (1) $\sigma(\alpha)$ is continuous in a right neighborhood of 0, and (2) if we write the Taylor expansions of the numerator and denominator of $\sigma(\alpha)$ we observe that the lowest power of α in the numerator is 3 while the lowest power is 2 in the denominator, so that $\sigma(\alpha)$ is $o(\alpha)$.

Therefore, if $P(\alpha)$ is the perimeter of the cyclic quadrilateral corresponding to $\alpha, \beta, \gamma, \delta$ where $\alpha = \beta = \gamma$ and $\delta = 360 - 3\alpha$ then since $\dfrac{2014}{\sigma(\alpha)} = P(\alpha)$, we find that $P(\alpha)$ increases without bound as α tends to 0.

This does not solve the problem; however, it does show that *if* the problem has a solution it depends entirely on the fact that the sides *each* have integer length (the situation is analogous to the fact that if (x, y) is an *integer* point on the hyperbola $x^2 - y^2 = 81$ then the sum $x + y$ has maximum, while if (x, y) is *any* point on the hyperbola then the sum $x + y$ has no maximum).

Now, Ken Korbin has shown the existence of a cyclic quadrilateral with integer sides and $\dfrac{A}{P} = 2014$, which he claims to be maximal. He maintains this is an isosceles trapezoid (which it must be if it is to be cyclic) with one base equal to 2 and the other 8×2014^2. He sets the remaining sides equal to the arithmetic mean of these values and asserts that the height must then be the geometric mean of the bases.

From this, he shows easily enough that this quadrilateral with sides 2, $8 \times 2014^2 = 32,449,568$, and 16,224,785 taken twice satisfies the condition that $\dfrac{A}{P} = 2014$. But of course this does not prove that the perimeter is maximal (even if it is). I might also mention that the sides of the equilateral trapezoid can be permuted without making the resulting quadrilateral non-cyclic or changing the perimeter and area, being an equilateral trapezoid is not essential

Ed Gray, however, has explained clearly why the equal sides should be the arithmetic mean of the other sides when we take the height to be the geometric mean and why, in this special case, Ken's solution is maximal. Ed writes as follows:

I have looked at Ken's solution to the problem, and while the answer may be correct, I don't see any proof that the answer is a maximum. It is easy to buy into the shape of an isosceles trapezoid, and we shall do that in general terms.

Let the trapezoid have an "upper" base of a, a "lower" base of c, with $c > a$. Let the trapezoid have equal lateral sides be b and d, with b on the right, d on the left, so the figure is $abcd$ reading clockwise.

From the right-most end of a, we drop an altitude h perpendicular to a down to c, where it also meets at right angles. Call the intersection point F. Since $c > a$, there is a part of c to the right of $F = \dfrac{c-a}{2}$ or $\dfrac{c}{2} - \dfrac{a}{2}$. We now have a right triangle with hypotenuse b and legs h and $\dfrac{c}{2} - \dfrac{a}{2}$.

By the Pythagorean Theorem,

$$b^2 = h^2 + (c/2 - a/2)^2 \tag{1}$$

$$b^2 = h^2 + c^2/4 - ac/2 + a^2/4. \tag{2}$$

By letting $h^2 = ac$, we have

$$b^2 = ac + c^2/4 - ac/2 + a^2/4 = c^2/4 + ac/2 + a^2/4 = (c/2 + a/2)^2 \tag{3}$$

or

$$b = (a + c)/2 \tag{4}$$

The area is:

$$A = (1/2)(a + c)\sqrt{ac} \tag{5}$$

The perimeter is:

$$P = a + c + 2(a + c)/2 = 2(a + c) \tag{6}$$

By hypothesis,

$$A = 2014P \tag{7}$$

Substituting (5) and (6) into (7),

$$(1/2)(a + c)\sqrt{ac} = 2014(2a + 2c) = 4028(a + c) \tag{8}$$

Multiplying by $2/(a + c)$,

$$\sqrt{ac} = 8056 \tag{9}$$

Squaring,

$$ac = 8056^2 \tag{10}$$

Side a must be even in order for b to be an integer. Since $b = d = (c + a)/2$, to maximize the perimeter $P = 2(a + c)$, we should like a to be the smallest integer possible (this is because ac is constant). Since it must also be even, let $a = 2$. Then (10) becomes:

$$2c = 8056^2 \tag{11}$$

So that, $c = 32,449,568$ and $b = d = (c + a)/2 = 16,224,785$. Thus the largest perimeter in this case is:

$$p = 2 + 16,224,785 + 32,449,568 + 16,224,785 = 64,899,140$$

Q.E.D.

Michael continues on as follows:

I would only add one clarification to Ed's explanation. It is that seemingly arbitrary assumption that $h^2 = ac$. The point is this. Since $A = 2014P$, A is an integer and h is rational. On the other hand if we multiply equation (1) by 4, we obtain $(2b)^2 = (2h)^2 + (c - a)^2$. From this it follows that $2h$ is an integer and $2b$, $c - a$, and $2h$ are a Pythagorean triple. Accordingly, $2b = k(m^2 + n^2)$, $c - a = k(m^2 - n^2)$ and $2h = 2kmn$ or $h = kmn$. Thus, taking $c = km^2$ and $a = kn^2$, we have, $h^2 = k^2m^2n^2 = km^2kn^2 = ac$.

- **5302:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  If n is an even perfect number, $n > 6$, and $\phi(n)$ is the Euler phi-function, then show that $n - \phi(n)$ is a fourth power of an integer. Find infinitely many integers n such that $n - \phi(n)$ is a fourth power.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC

(i) If n is an even perfect number with $n > 6$, then $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both odd primes. Since ϕ is multiplicative, we have $\phi(n) = 2^{p-2}(2^p - 2)$, which implies

$$n - \phi(n) = 2^{p-1}(2^p - 1) - 2^{p-2}(2^p - 2) = 2^{2p-1} - 2^{2p-2} = 2^{2p-2} = (2^{(p-1)/2})^4,$$

where $2^{(p-1)/2}$ is an integer since p is odd.

(ii) One trivial solution is to let n be any prime. Then $n - \phi(n) = 1$. A less trivial solution is to take $n = 2^{4k+1}$ for any nonnegative integer k. Then

$$n - \phi(n) = 2^{4k+1} - 2^{4k} = 2^{4k} = (2^k)^4.$$

Solution 2 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

We will begin with the following facts for the phi-function:
1. If p is prime, $\phi(p) = p - 1$.
2. If p is prime and a is a positive integer, $\phi(p^a) = p^{a-1}(p-1)$.
3. If the gcd (a,b)=1, $\phi(ab) = \phi(a)\phi(b)$.

We also note that an even perfect number $n > 6$ can be written in the form $n = 2^{k-1}(2^k - 1)$, where k and 2^{k-1} are prime and $k > 2$. Then, since $\gcd(2^{k-1}, 2^k - 1) = 1$ and $2^k - 1$ is prime,

$$\begin{aligned} \phi(n) &= \phi[2^{k-1}(2^k - 1)] \\ &= \phi(2^{k-1})\phi(2^k - 1) \\ &= 2^{k-2}(2^k - 2), \\ &= 2^{k-1}(2^{k-1} - 1). \end{aligned}$$

Further, since k must be an odd prime,

$$\begin{aligned} n - \phi(n) &= 2^{k-1}(2^k - 1) - 2^{k-1}(2^{k-1} - 1) \\ &= 2^{k-1}(2^k - 2^{k-1}) \\ &= 2^{k-1}[2^{k-1}(2 - 1)] \\ &= 2^{2(k-1)} \\ &= (2^{k-1})^2 \\ &= \left[2^{2\left(\frac{k-1}{2}\right)}\right]^2 \\ &= \left(2^{\frac{k-1}{2}}\right)^4. \end{aligned}$$

Therefore, $n - \phi(n)$ is a fourth power of an integer. If $k = 4m + 1$ for $m \geq 1$, and p is an arbitrary prime,

$$\begin{aligned}
\phi(n) &= \phi(p^{4m+1}) \\
&= p^{4m}(p-1) \\
&= p^{4m+1} - p^{4m}.
\end{aligned}$$

Then,

$$\begin{aligned}
n - \phi(n) &= p^{4m+1} - \phi(p^{4m+1}) \\
&= p^{4m+1} - (p^{4m+1} - p^{4m}) \\
&= p^{4m} \\
&= (p^m)^4.
\end{aligned}$$

Since there are an infinite number of choices for p and m, this provides an example of infinitely many integers n such that $n - \phi(n)$ is a fourth power.

Also solved by Pat Costello, Eastern Kentucky University, Richmond. KY; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5303:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let a, b, c, d be positive real numbers Prove that

$$a^4 + b^4 + c^4 + d^4 + 4 \ \geq \ 4\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4}.$$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY By the Arithmetic Mean-Geometric Mean inequality,

$$\Bigg(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\Bigg)^{1/4} \leq \frac{a^2b^2+b^2c^2+c^2d^2+d^2a^2+4}{4}$$

with equality if and only if $a = b = c = d$. Therefore,

$$4\Bigg(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\Bigg)^{1/4} \leq a^2b^2+b^2c^2+c^2d^2+d^2a^2+4.$$

Define vectors $\vec{u}$ and $\vec{v}$ such that $\vec{u} = \Big\langle a^2, b^2, c^2, d^2 \Big\rangle$ and $\vec{v} = \Big\langle b^2, c^2, d^2, a^2 \Big\rangle$,

Then the Cauchy-Schwarz inequality implies $\vec{u} \bullet \vec{v} = ||\vec{u}|| \cdot ||\vec{v}||$ so that

$$\begin{aligned}
a^2b^2+b^2c^2+c^2d^2+d^2a^2 &\leq \sqrt{a^4+b^4+c^4+d^4}\sqrt{b^4+c^4+d^4+a^4} \\
&= a^4+b^4+c^4+d^4.
\end{aligned}$$

Hence,

$$4\Bigg(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\Bigg)^{1/4} \leq a^4+b^4+c^4+d^4+4$$

with equality if and only if $a = b = c = d$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$
\begin{aligned}
a^4+b^4+c^4+d^4+4 &= \left(\left(a^2\right)^2+\left(b^2\right)^2+\left(c^2\right)^2+\left(d^2\right)^2\right)^{1/2}\left(\left(a^2\right)^2+\left(b^2\right)^2+\left(c^2\right)^2+\left(d^2\right)^2\right)^{1/2}+4 \\
&\geq a^2b^2+b^2c^2+c^2d^2+d^2a^2+4 \\
&= \left(a^2b^2+1\right)+\left(b^2c^2+1\right)+\left(c^2d^2+1\right)+\left(d^2a^2+1\right) \\
&\geq 4\Big((a^2b^2+1)(b^2c^2+1)(c^2d^2+1)(d^2a^2+1)\Big)^{1/4},
\end{aligned}
$$

where we have used the Cauchy-Schwarz and the arithmetic mean-geometric mean inequalities.

Equality occurs if, and only if, it occurs in both inequalities, that is if, and only if, $a^2/b^2 = b^2/c^2 = c^2/d^2 = d^2/a^2$ and $a^2b^2+1 = b^2c^2+1 = c^2d^2+1 = d^2a^2+1$.

That is, inequality holds if, and only if, $a = b = c = d$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By the AM-GM inequality,

$$
\begin{aligned}
\frac{a^4+b^4}{2} &\geq a^2b^2 \\
\frac{b^4+c^4}{2} &\geq b^2c^2 \\
\frac{c^4+d^4}{2} &\geq c^2d^2 \\
\frac{d^4+a^4}{2} &\geq d^2a^2.
\end{aligned}
$$

Adding these inequalities we obtain

$$a^4+b^4+c^4+d^4+4 \geq \left(a^2b^2+1\right)+\left(b^2c^2+1\right)+\left(c^2d^2+1\right)+\left(d^2a^2+1\right).$$

We apply once more the AM-GM inequality to obtain

$$\left(a^2b^2+1\right)+\left(b^2c^2+1\right)+\left(c^2d^2+1\right)+\left(d^2a^2+1\right) \geq 4\Big(\left(a^2b^2+1\right)+\left(b^2c^2+1\right)+\left(c^2d^2+1\right)+\left(d^2a^2+1\right)\Big)^{1/4},$$

and the claimed statement follows.

Comment by editor: **Titu Zvonaru, Comăesti, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania** jointly solved the problem in the manner of solution 3, and noted that the statement of the problem can be made stronger for it also holds for all real numbers, not just the positive ones.

Also solved by Arkady Alt, San Jose, CA; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

- **5304:** *Proposed by Michael Brozninsky, Central Islip, NY*

Determine whether or not there exist nonzero constants a and b such that the conic whose polar equation is

$$r = \sqrt{\frac{a}{\sin(2\theta) - b \cdot \cos(2\theta)}}$$

has a rational eccentricity.

Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

We will begin with the use of the transformation formulas and the following trigonometric identities to change the polar form into the rectangular form of the hyperbola:

$$\begin{aligned} x &= r\cos\theta) && (1)\\ y &= r\sin\theta && (2)\\ \sin(2\theta) &= 2\sin\theta\cos\theta, && (3)\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta. && (4) \end{aligned}$$

Then, using (1), (2), (3), and (4),

$$\begin{aligned} r &= \sqrt{\frac{a}{\sin(2\theta) - b\cos(2\theta)}}\\ r^2 &= \frac{a}{\sin(2\theta) - b\cos(2\theta)}\\ 2r^2\sin\theta\cos\theta - br^2(\cos^2\theta - \sin^2\theta) &= a\\ 2(r\sin\theta)(r\cos\theta) - b(r\cos\theta)^2 + b(r\sin\theta)^2 &= a\\ 2xy - bx^2 + by^2 &= a\\ x^2 - \frac{2}{b}xy - y^2 - \frac{a}{b} &= 0. \end{aligned}$$

With the general form of the hyperbola being

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we have $A = 1$, $B = -\dfrac{2}{b}$, $C = -1$, and $F = -\dfrac{a}{b}$. The usual methods of rotation of axes in analytic geometry can be used to ascertain the eccentricity of the hyperbola, or the following formula **[1]** gives the eccentricity in a straightforward manner.

$$e = \sqrt{\frac{2\sqrt{(A-C)^2 + B^2}}{\eta(A+C) + \sqrt{(A-C)^2 + B^2}}} \quad , \qquad (12)$$

where $\eta = 1$ if the determinant of the 3x3 matrix

$$\begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}$$

is negative, or $\eta = -1$ if the determinant is positive. Thus, using (6)

$$\begin{aligned} e &= \sqrt{\frac{2\sqrt{4+\dfrac{4}{b^2}}}{\eta(0)+\sqrt{4+\dfrac{4}{b^2}}}} \\ &= \sqrt{2}. \end{aligned}$$

Thus, the eccentricity is irrational for all values of a and b.

Reference:

[1]Ayoub, Ayoub B., "The Eccentricity of a Conic Section," *The College Mathematics Journal* 34(2), March 2003, 116-121.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5305:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x be a positive real number. Prove that

$$\frac{[x]}{2x+\{x\}} + \frac{[x]\{x\}}{3x^2} + \frac{\{x\}}{2x+[x]} \;\le\; \frac{1}{2},$$

where $[x]$ is the greatest integer function and $\{x\}$ is the fractional part of the real number. I.e., $\{x\} = x - [x]$.

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Since, $x = \lfloor x \rfloor + \{x\}$, then $x^2 = \lfloor x \rfloor^2 + \{x\}^2 + 2\lfloor x \rfloor \{x\}$. Now,

$$\frac{\lfloor x \rfloor}{2x+\{x\}} + \frac{\{x\}}{2x+\lfloor x \rfloor} = \frac{2x^2+\lfloor x \rfloor^2+\{x\}^2}{6x^2+\lfloor x \rfloor \{x\}} = \frac{3x^2-2\lfloor x \rfloor \{x\}}{6x^2+\lfloor x \rfloor \{x\}}.$$

Therefore, the left-hand side of the proposed inequality, LHS is

$$\begin{aligned} LHS &= \frac{3x^2-2\lfloor x \rfloor \{x\}}{6x^2+\lfloor x \rfloor \{x\}} + \frac{\lfloor x \rfloor \{x\}}{3x^2} \\ &= \frac{3A-2B}{6A+B} + \frac{B}{3A} = \frac{9A^2-B^2}{18A^2+3AB} \\ &\le \frac{1}{2} \end{aligned}$$

where $A = x^2$ and $B = \lfloor x \rfloor \{x\}$.

Solution 2 by Titu Zvonaru, Comănesti, and Neculai Stanciu "George Emil Palade" School, Buzău, Romania

We denote $a = [x]$ and $b = \{x\}$, so $a \geq 0,\ b \geq 0$ and $x = a + b$.

Because
$$(2a+3b)(3a+2b) = 6a^2 + 13ab + 6b^2 \geq 6(a+b)^2,$$
we have
$$\begin{aligned}
\frac{a}{2a+3b} + \frac{ab}{3(a+b)^2} + \frac{b}{3a+2b} &= \frac{3a^2+4ab+3b^2}{(2a+3b)(3a+2b)} + \frac{ab}{3(a+b)} \\
&\leq \frac{3a^2+4ab+3b^2}{6(a+b)^2} + \frac{ab}{3(a+b)^2} \\
&= \frac{3a^2+6ab+3b^2}{6(a+b)^2} \\
&= \frac{1}{2}.
\end{aligned}$$

Because we only used the inequality $ab \geq 0$, we obtain that equality holds if, and only if $ab = 0$, i.e., if, and only if x is an integer or if $x \in (0,1)$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For convenience, let $n \leq x \leq n+1$, so $[x] = n$ and $\{x\} = x - n$.

Then the given inequality becomes $\dfrac{n}{2x+(x-n)} + \dfrac{n(x-n)}{3x^2} + \dfrac{x-n}{2x+n} \leq \dfrac{1}{2}$.

Upon clearing fractions and simplifying, this becomes $0 \leq n\left(3x^3 - 5nx^2 + 4n^2x - 2n^3\right)$.

Further algebra simplifies the inequality:
$$\begin{aligned}
n(x-n)\left(3x^2 - 2nx + 2n^2\right) &\geq 0 \\
n(x-n)\left((x-n)^2 + 2x^2 + n^2\right) &\geq 0.
\end{aligned}$$

Because $x \geq n \geq 0$, this is certainly true.

The final version of the inequality also reveals that equality holds if and only if $n = 0$ (that is, $0 \leq x < 1$ so$\{x\} = x$) or $x = n = [x]$ (that is, x is an integer.)

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

- **5306:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate: $\displaystyle\int_0^1 \frac{\ln\left(1-x+x^2\right)}{x-x^2}dx.$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln(1-x+x^2)}{x}dx \\ I_2 &= \int_0^1 \frac{\ln(1-x+x^2)}{1-x}dx \\ I_3 &= \int_0^1 \frac{\ln(1+x^3)}{x}dx \text{ and} \\ I_4 &= \int_0^1 \frac{\ln(1+x)}{x}dx. \end{aligned}$$

Clearly, $I = I_1 + I_2$ and $I_1 = I_3 - I_4$.

By the substitution $x = 1-y$ into I_2, we easily see that $I_2 = I_1$.

By the substitution $x = y^{1/3}$ into I_3, we obtain $I_3 = \dfrac{1}{3}I_4$.

It follows that $I = 2I_1 = 2(I_3 - I_4) = \dfrac{-4}{3}I_4$. But I_4 is a well-known integral with value $\dfrac{\pi^2}{12}$ and so $I = \dfrac{-\pi^2}{9}$.

Solution 2 by Albert Stadler, Herrliberg Switzerland

We have

$$\int_0^1 \frac{\ln\left(1-x+x^2\right)}{x-x^2}dx = \int_0^1 \left(\frac{1}{x} - \frac{1}{1-x}\right)\ln(1-x+x^2)dx$$

$$\begin{aligned} &= \int_0^1 \frac{\ln\left(1-x+x^2\right)}{x}dx + \int_0^1 \frac{\left(\ln\left(1-(1-x)+(1-x)^2\right)\right.}{x}dx \\ &= 2\int_0^1 \frac{\ln(1-x+x^2)}{x}dx \\ &= 2\int_0^1 \frac{\ln\left(\dfrac{1+x^3}{1+x}\right)}{x}dx \\ &= 2\int_0^1 \frac{\ln(1+x^3)}{x}dx - 2\int_0^1 \frac{\ln(1+x)}{x}dx \\ &= 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\int_0^1 x^{3k-1}dx - 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\int_0^1 x^{k-1}dx \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{3k^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} \\
&= -\frac{4}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} \\
&= -\frac{4}{3}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - 2\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right) \\
&= -\frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{k^2} = -\frac{2}{3}\cdot\frac{\pi^2}{6} = -\frac{\pi^2}{9}.
\end{aligned}
$$

The interchange of summation and integration is permitted because of uniform convergence of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^{3k-1}$ and $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^{k-1}$ in the interval $[0,1]$.

Addendum: It is noteworthy to mention that the famous relation $\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}} = \frac{\zeta(2)}{3} = \frac{\pi^2}{18}$ is easily derived from the above integral (see for instance http://en.wikipedia.org/wiki/Ap%C3%A9ry's theorem for reference). Indeed,

$$
\begin{aligned}
\frac{\pi^2}{9} = -\int_0^1 \frac{\ln(1-x+x^2)}{x-x^2}dx &= \sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 \left(x-x^2\right)^{k-1}dx \\
&= \sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}\left(1-x\right)^{k-1}dx = \sum_{k=1}^{\infty}\frac{1}{k}\frac{\Gamma(k)\Gamma(k)}{\Gamma(2k)} \\
&= \sum_{k=1}^{\infty}\frac{1}{k}\frac{(k-1)!(k-1)!}{(2k-1)!} = 2\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{k!k!}{(2k)!} = 2\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.
\end{aligned}
$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposer.

Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
February 15, 2015

- **5325:** *Proposed by Kenneth Korbin, New York, NY*

  Given the sequence $x = (1, 7, 41, 239, 1393, 8119, \ldots)$, with $x_n = 6x_{n-1} - x_{n-2}$.
  Let $y = \dfrac{x_{2n} + x_{2n-1}}{x_n}$. Find an explicit formula for y expressed in terms of n.

- **5326:** *Proposed by Armend Sh. Shabani, University of Prishtina, Republic of Kosova*

  Find all positive integer solutions to $m! + 2^{4k-1} = l^2$.

- **5327:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Show that in any triangle ABC, with the usual notations, that

  $$\left(\frac{ab}{a+b}\right)^2 + \left(\frac{bc}{b+c}\right)^2 + \left(\frac{ca}{c+a}\right)^2 \geq 9r^2.$$

- **5328:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the aid of a computer, find the positive solutions of the equation

  $$2^{x+1}\left(1 - \sqrt{1 + x^2 + 2^x}\right) = \left(x^2 + 2^x\right)\left(1 - \sqrt{1 + 2^{x+1}}\right).$$

- **5329:** *Proposed by Arkady Alt, San Jose, CA*

  Find the smallest value of $\dfrac{x^3}{x^2+y^2} + \dfrac{y^3}{y^2+z^2} + \dfrac{z^3}{z^2+x^2}$ where real $x, y, z > 0$ and $xy + yz + zx = 1$.

- **5330:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $B(x) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$ and let $n \geq 2$ be an integer.

Calculate the matrix product

$$B(2)B(3)\cdots B(n).$$

Solutions

- **5307:** *Proposed by Haishen Yao and Howard Sporn, Queensborough Community College, Bayside, NY*

  Solve for x:

$$\sqrt{x^{15}} = \sqrt{x^{10}-1} + \sqrt{x^5-1}.$$

Solution 1 by Arkady Alt, San Jose, CA

Let $a = \sqrt{x^{10}-1}$ and $b = \sqrt{x^5-1}$ then

$$\begin{aligned}
x^5 &= b^2+1,\ x^{10} = a^2+1, \\
x^{15} &= x^{10}\cdot x^5 = \left(a^2+1\right)\left(b^2+1\right) \text{ and therefore,} \\
\sqrt{\left(a^2+1\right)\left(b^2+1\right)} &= a+b \iff \\
\left(a^2+1\right)\left(b^2+1\right) &= (a+b)^2 \iff \\
(ab-1)^2 &= 0 \iff \\
ab = 1. &
\end{aligned}$$

Also we have

$$x^{10} = \left(x^5\right)^2 \implies a^2+1 = \left(b^2+1\right)^2 \iff b^4+2b^2 = a^2 \iff b^6+2b^4 = a^2b^2.$$

Since $ab=1$ then

$$\begin{aligned}
b^6+2b^4-1 &= 0 \iff \\
\left(b^2+1\right)\left(b^4+b^2-1\right) &= 0 \iff \\
b^4+b^2-1 &= 0 \iff \\
b^2 &= \frac{-1+\sqrt{5}}{2}. \text{ Hence,} \\
x^5 &= b^2+1 \\
&= \frac{-1+\sqrt{5}}{2}+1 \\
&= \frac{1+\sqrt{5}}{2} \iff x = \sqrt[5]{\frac{1+\sqrt{5}}{2}}.
\end{aligned}$$

Solution 2 by Charles McCracken, Dayton, OH

Let $\mu = x^5$ then $\sqrt{\mu^2} = \sqrt{\mu^2 - 1} + \sqrt{\mu - 1}$.

It is readily seen that $1 < x < 2$. A few successive approximations give $\mu \approx 1.618$. So we try $\mu = \phi = \dfrac{1+\sqrt{5}}{2}$, also known as, the Golden Ratio.

The equation then becomes

$$\begin{aligned}
\sqrt{\phi}^3 &= \sqrt{\phi^2 - 1} + \sqrt{\phi - 1} \\
\phi\sqrt{\phi} &= \sqrt{\phi + 1 - 1} + \sqrt{\frac{1}{\phi}} \\
\phi\sqrt{\phi} &= \sqrt{\phi} + \sqrt{\frac{1}{\phi}} \\
\phi^2 &= \phi + 1. \text{ A well known identity.}
\end{aligned}$$

Since $\phi = \mu,\ x = \sqrt[5]{\phi} \approx 1.101025882$.

Solution 3 by Becca Rousseau, Ellie Erehart, and David Weerheim (jointly, students at Taylor University), Upland, IN

The common domain of definition for $\sqrt{x^{15}}, \sqrt{x^{10}-1}$, and $\sqrt{x^5 - 1}$ is $x \geq 1$. We now solve for x:

$$\begin{aligned}
x^{15} &= (x^{10} - 1) + 2\sqrt{(x^{10}-1)(x^5-1)} + (x^5 - 1) \\
x^{15} - x^{10} - x^5 + 2 &= 2\sqrt{x^{15} - x^{10} - x^5 + 1} \\
x^5\left(x^{10} - x^5 - 1\right) + 2 &= 2\sqrt{x^5\left(x^{10} - x^5 - 1\right) + 1}.
\end{aligned}$$

Letting $u = x^5\left(x^{10} - x^5 - 1\right)$, we obtain

$$\begin{aligned}
u + 2 &= 2\sqrt{u+1} \\
u^2 + 4u + 4 &= 4(u+1) \\
u^2 + 4u + 4 &= 4u + 4 \\
u^2 &= 4u + 4 - 4u - 4 \\
u^2 &= 0,\ u = 0.
\end{aligned}$$

Substituting $x^5(x^{10} - x^5 - 1)$for u see that

$$x^5\left(x^{10} - x^5 - 1\right) = 0, \text{ so}$$

$$x^5 = 0 \quad \text{or} \qquad x^{10} - x^5 - 1 = 0.$$

$$x^5 = 0 \qquad x^5 = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore, $x = 0,\ x = \sqrt[5]{\dfrac{1-\sqrt{5}}{2}}$, or $x = \sqrt[5]{\dfrac{1+\sqrt{5}}{2}}$.

The first two roots must be discarded, because they are outside the domain of definition of x, as noted above.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

It is not specified whether x is real or not, so let's assume $x \in R$. The domain of x is $[1, \infty)$ since both $x^{10} - 1$ and $x^5 - 1$ are the product of $(x-1)$ for a positive polynomial respectively of order 9 and 4. Let $x^5 = y$. Squaring we get

$$y^3 = y^2 - 1 + y - 1 + 2(y-1)\sqrt{y+1} \iff y^3 - y^2 + 1 - y + 1 = 2(y-1)\sqrt{y+1}.$$

The r.h.s. is nonnegative for $y \geq 1$. Moreover for $y \geq 0$

$$\frac{y^3}{2} + \frac{y^3}{2} + \frac{1}{2} \geq \frac{3}{2}y^2, \qquad \frac{1}{2}y^2 + \frac{1}{2} \geq y$$

and then

$$y^3 - y^2 - y + 2 \geq 1 + y^2 + y > y^2 + y.$$

We square both sides again getting

$$y^2(y^2 - y - 1)^2 = 0 \iff y = (1+\sqrt{5})/2$$

and then $x = \left((1+\sqrt{5})/2\right)^{1/5}$.

Comment: Brian D. Beasley, Presbyterian College, Clinton, SC, Moti Levy of Rehovot, Israel, Michael Thew (student at Saint George's School), Spokane, WA, Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănesti, Romania, and David Stone and John Hawkins of Georgia Southern University, Statesboro, Georgia noted in their solutions that if complex roots are allowed, the full set of roots is:

$$x = 0, x_k = \left((1+\sqrt{5})/2\right)^{1/5}\left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right), k = 0,1,2,3,4, \text{and}$$

$$x_m = \left((1-\sqrt{5})/2\right)^{1/5}\left(\cos\frac{2m\pi}{5} + i\sin\frac{2m\pi}{5}\right), m = 0,1,2,3,4.$$

David Stone and John Hawkins also noted that if we let $y_1 = \sqrt{x^{15}}$ and $y_2 = \sqrt{x^{10}-1} + \sqrt{x^5 - 1}$, the graphs of these two functions intersect at the real root, and at this point the graphs are tangent to one another.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain;

Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; Kelley McKaig, Madison Thompson, and Melanie Schmocker, (Students at Taylor University), Upland, IN; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposers.

- **5308:** *Proposed by Kenneth Korbin, New York, NY*

Given the sequence

$$t = (1, 7, 41, 239, \ldots)$$

with $t_n = 6t_{n-1} - t_{n-2}$. Let (x, y, z) be a triple of consecutive terms in this sequence with $x < y < z$.

Part 1) Express the value of x in terms of y and express the value of y in terms of x.

Part 2) Express the value of x in terms of z and express the value of z in terms of x.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

From the recursive formula

$$t_{n+2} = 6t_{n+1} - t_n \tag{1}$$

and the initial conditions $t_1 = 1$ and $t_2 = 7$, we can find a closed form expression for t_n by using the customary techniques for solving homogeneous linear difference equations. If we consider solutions of the form $t_n = \lambda^n$, with $\lambda \neq 0$, equation (1) provides us with the auxiliary equation

$$\lambda^2 = 6\lambda - 1$$

whose solutions are

$$\lambda = 3 \pm 2\sqrt{2}.$$

Then, there are constants c_1, c_2 such that

$$t_n = c_1 \left(3 + 2\sqrt{2}\right)^n + c_2 \left(3 - 2\sqrt{2}\right)^n$$

for all $n \geq 1$. The initial conditions $t_1 = 1$ and $t_2 = 7$ give

$$c_1 = \frac{\sqrt{2} - 1}{2} \quad \text{and} \quad c_2 = -\frac{\sqrt{2} + 1}{2}$$

and we have

$$t_n = \frac{\sqrt{2} - 1}{2} \left(3 + 2\sqrt{2}\right)^n - \frac{\sqrt{2} + 1}{2} \left(3 - 2\sqrt{2}\right)^n.$$

Finally, since

$$\left(3 + 2\sqrt{2}\right) = \left(\sqrt{2} + 1\right)^2 \quad \text{and} \quad \left(3 - 2\sqrt{2}\right) = \left(\sqrt{2} - 1\right)^2,$$

we conclude that

$$\begin{aligned} t_n &= \frac{\sqrt{2} - 1}{2} \left(\sqrt{2} + 1\right)^{2n} - \frac{\sqrt{2} + 1}{2} \left(\sqrt{2} - 1\right)^{2n} \\ &= \frac{\left(\sqrt{2} + 1\right)^{2n-1} - \left(\sqrt{2} - 1\right)^{2n-1}}{2} \end{aligned} \tag{2}$$

for all $n \geq 1$.

Equation (2) shows that $t_n > 0$ for all n and then an elementary Mathematical Induction argument using (1) establishes that $t_{n+1} > t_n$ for all n. Therefore, if (x, y, z) is a triple of consecutive terms in this sequence with $x < y < z$, we must have $x = t_n$, $y = t_{n+1}$, and $z = t_{n+2}$ for some $n \geq 1$.

For Part 1), we note that

$$\begin{aligned}
y &= t_{n+1} \\
&= \frac{1}{2}\left[\left(\sqrt{2}+1\right)^{2n+1} - \left(\sqrt{2}-1\right)^{2n+1}\right] \\
&= \frac{1}{2}\left[\left(3+2\sqrt{2}\right)\left(\sqrt{2}+1\right)^{2n-1} - \left(3-2\sqrt{2}\right)\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \frac{3+2\sqrt{2}}{2}\left[\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}\right] + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(3+2\sqrt{2}\right) t_n + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(3+2\sqrt{2}\right) x + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
x &= \frac{1}{3+2\sqrt{2}}\left[y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(3-2\sqrt{2}\right)\left[y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(3-2\sqrt{2}\right) y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(3-2\sqrt{2}\right) y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n+1}.
\end{aligned}$$

For Part 2), equation (1) and Part 1) imply that

$$\begin{aligned}
z &= t_{n+2} \\
&= 6t_{n+1} - t_n \\
&= 6\left[\left(3+2\sqrt{2}\right) x + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] - x \\
&= \left(17+12\sqrt{2}\right) x + 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x &= \frac{1}{17+12\sqrt{2}}\left[z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(17-12\sqrt{2}\right)\left[z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(17-12\sqrt{2}\right) z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{4}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(17-12\sqrt{2}\right) z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n+3}.
\end{aligned}$$

Remark. On page 253 of Recreations in the Theory of Numbers by A. H. Beiler (Dover Publications, Inc., 1966), it is shown that the sequence $\{t_n\}$ provides the solutions for x in the Pell Equation $x^2 - 2y^2 = -1$. The corresponding y solutions satisfy the recursive formula $y_{n+2} = 6y_{n+1} - y_n$ with $y_1 = 1$ and $y_2 = 5$. This yields

$$y_n = \frac{\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}}{2\sqrt{2}}$$

for $n \geq 1$.

Solution 2 by Moti Levy, Rehovot, Israel

The solution of this type of recurrence formulas is

$$t_n = a\alpha^n + b\beta^n,$$

where α and β are the roots of $r^2 - 6r + 1$.

Here,

$$t_n = a\alpha^n - (a+1)\,\alpha^{-n}; \quad a = \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\right); \ \alpha = 3 + 2\sqrt{2}.$$

Part 1):

$$x = a\alpha^n - (a+1)\,\alpha^{-n}$$
$$y = a\alpha^{n+1} - (a+1)\,\alpha^{-n-1}$$

Solving for α^n in terms of x, we get,

$$\alpha^n = \left(\sqrt{2}+1\right)\left(x + \sqrt{x^2+1}\right),$$

$$y = 3x + 2\sqrt{2}\sqrt{x^2+1}.$$

Solving for α^n in terms of y, we get,

$$\alpha^n = \left(\sqrt{2}-1\right)\left(y + \sqrt{y^2+1}\right),$$

$$x = 3y - 2\sqrt{2}\sqrt{y^2+1}.$$

Part 2):

$$z = a\alpha^{n+2} - (a+1)\,\alpha^{-n-2},$$

$$z = 17x + 12\sqrt{2}\sqrt{x^2+1}.$$

Solving for α^n in terms of z, we get,

$$\alpha^n = \left(5\sqrt{2}-7\right)\left(z + \sqrt{z^2+1}\right),$$

$$x = 17z - 12\sqrt{2}\sqrt{z^2+1}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania, and the proposer.

- **5309:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Consider the expression $3^n + n^2$ for positive integers n. It is divisible by 13 for $n = 18$ and $n = 19$. Prove, however, that it is never divisible by 13 for three consecutive values of n.

Solution 1 by Bruno Salgueiro Fanego Viveiro, Spain

Let n be an integer such that $n \geq 1$. We argue by contradiction. If, for the three consecutive values $n-1, n$, and $n+1$ the expressions $3^{n-1} + (n-1)^2, 3^n + n^2$, and $3^{n+1} + (n+1)^2$ are each divisible by 13, then their sum, $\left(1+3+3^2\right) \cdot 3^{n-1} + 3n^2 + 2$ is divisible by 13, or equivalently, the expression $3n^2 + 2$ is divisible by 13.

If we divide n by 13, we obtain an integer quotient c and remainder r, $0 \leq r < 13$, such that $n = 13c + r$, so $3n^2 + 2 = 3\,(13c + r)^2 + 2 = 13 \cdot \left(39c^2 + 2cr\right) + 3r^2 + 2$, which is divisible by 13, so $3r^2 + 2$ is also divisible by 13.

Since $0 \leq r \leq 12,\ 3r^2 + 2 \in \{5, 14, 29, 50, 77, 110, 149, 194, 245, 302, 365, 434\}$ and hence $3r^2 + 2$ is not divisible by 13 (because each remainder of the division of 5, 14, 29, 50, 77, 110, 149, 194, 245, 302, 365, and 434 by 13 is not zero. The remainders are, respectively, 5, 1, 3, 11, 12, 6, 6, 12,11, 3, 1, and 5. Thus we have a contradiction showing that the expressions $3^{n-1} + (n-1)^2,\ 3^n + n^2$, and $3^{n+1} + (n+1)^2$ cannot all be divisible by 13.

Solution 2 by Ed Gray, Highland Beach, FL

Suppose there were three consecutive integers, say, $n, n+1$ and $n+2$ for which $3^n + n^2$ is divisible by 13. Then we have the three congruences:

$$\begin{aligned}
&(1)\quad 3^n + n^2 \equiv 0 \pmod{13}\\
&(2)\quad 3^{n+1} + n^2 + 2n + 1 \equiv 0 \pmod{13}\\
&(3)\quad 3^{n+2} + n^2 + 4n + 4 \equiv 0 \pmod{13}
\end{aligned}$$

Multiple (1) by 9, multiply (2) by 1 and multiply (3) by 3. Then

$$\begin{aligned}
&(4)\quad 9 \cdot 3^n + 9n^2 \equiv 0 \pmod{13}\\
&(5)\quad 3 \cdot 3^n + n^2 + 2n + 1 \equiv 0 \pmod{13}\\
&(6)\quad 27 \cdot 3^n + 3n^2 + 12n + 12 \equiv 0 \pmod{13}
\end{aligned}$$

Adding the three congruences:

$$(7)\quad 39 \cdot 3^n + 13n^2 + 14n + 13 \equiv 0 \pmod{13} \Longrightarrow 13 \Big| n,$$

which is equivalent to saying $n \equiv 0 \pmod{13}$. Therefore, if it were possible to have three consecutive integers such that $3^n + n^2$ were divisible by 13, then 13 would have to divide n and this implies (in eq. 1) that 13 divides 3^n , but this is impossible because the only divisors of 3^n are multiples of 3.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Suppose that there are positive integers n, x, y, z such that

$$3^n + n^2 = 13x, \quad 3^{n+1} + (n+1)^2 = 13y, \quad \text{and} \quad 3^{n+2} + (n+2)^2 = 13z.$$

Then,

$$
\begin{aligned}
13z &= 3\cdot 3^{n+1} + (n+1)^2 + (2n+3) \\
&= \left[3^{n+1} + (n+1)^2\right] + 2\cdot 3^{n+1} + 2n + 3 \\
&= 13y + 6\cdot 3^n + 2n + 3 \\
&= 13y + 6\left(13x - n^2\right) + 2n + 3 \\
&= 13\left(y + 6x\right) - 6n^2 + 2n + 3.
\end{aligned}
$$

Hence,

$$13\left(6x + y - z\right) = 6n^2 - 2n - 3$$

which implies that

$$6n^2 - 2n - 3 \equiv 0 \pmod{13}.$$

However, as shown in the following table, this is impossible.

$n \pmod{13}$	$6n^2 - 2n - 3 \pmod{13}$
0	10
1	1
2	4
3	6
4	7
5	7
6	6
7	4
8	1
9	10
10	5
11	12
12	5

Therefore, no such n, x, y, z exist and $3^n + n^2$ is never divisible by 13 for three consecutive values of n.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Suppose the contrary, that $3^m + m^2,\ 3^{m+1} + (m+1)^2, 3^{m+2} + (m+2)^2$ are divisible by 13 for some positive integer m. Hence their sum

$$13(3^m) + 3m^2 + 6m + 5$$

is also divisible by 13. However this contradicts the fact that $3m^2 + 6m + 5$ is congruent to $5, 1, 3, 11, 12, 6, 6, 12, 11, 3, 1, 5, 2$ modulo 13 according as m is congruent to $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ modulo 13. Hence the assertion of the problem.

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasely, Presbyterian College, Clinton, SC; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins of Georgia Southern University, Statesboro, Georgia; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania, and the proposer.

- **5310:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let $a > 0$ and a sequence $\{E_n\}_{n\geq 0}$, be defined by $E_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$. Evaluate:

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right).$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We know that

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{c}{(n+1)!}, \quad c = e^{\xi}, \quad 0 \leq \xi < 1.$$

It follows that $\sqrt[n]{E_n} \to 1$ and then $\left(a^{\sqrt[n]{E_n}-1} - 1\right)/(\sqrt[n]{E_n} - 1) \to \ln a$, as well as

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right) = \lim_{n\to\infty} \sqrt[n]{n!}(\sqrt[n]{E_n} - 1)\ln a.$$

Moreover,

$$\lim_{n\to\infty} n\left(E_n^{1/n} - 1\right) = 1,$$

and then

$$\lim_{n\to\infty} \sqrt[n]{n!}(\sqrt[n]{E_n} - 1)\ln a = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}\ln a.$$

Finally, the Cesaro–Stolz theorem yields

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}\ln a = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}}\ln a = \ln a \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \ln a \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{\ln a}{e}.$$

Solution 2 by Ed Gray, Highland Beach, FL

We first show that the limit to be evaluated is of the form $\infty \cdot 0$, and then we use L'Hospital's rule to evaluate it.

$$\lim_{n\to\infty} \sqrt[n]{E_n} = \lim_{n\to\infty} (E_n)^{1/n} = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}\right)^{1/n} = \lim_{n\to\infty} \left\{\left(1 + \frac{1}{n}\right)^{n}\right\}^{1/n} = \lim_{n\to\infty} (1 + 1/n) = 1, \text{so,}$$

$$\lim_{n\to\infty} \left(a^{\sqrt[n]{E_n}-1} - 1\right) = 0.$$

Let $y = \lim\limits_{n\to\infty} (n!)^{1/n}$. Then

$$\ln(y) \quad = \quad (1/n)\ln(n!) \to$$

$$
\begin{aligned}
\ln(y) &= (1/n)\ln\left(1\cdot 2\cdot 3\cdots n\right) \text{ and} \\
\ln(y) &= (1/n)\Big(\ln(1)+\ln(2)+\ln(3)+\cdots+\ln(n)\Big) \\
\ln(y) &\approx (1/n)\int_{x=1}^{x=n}\ln(x)dx \\
\ln(y) &= (1/n)\Big(x\ln(x)-x\Big)\Big|_{x=1}^{n} \\
\ln(y) &= (1/n)\Big(n\left(\ln(n)\right)-n\Big) \\
\ln(y) &= \ln(n)-1 \\
\ln(y) &= \ln(n)-\ln e \\
\ln(y) &= \ln\left(\frac{n}{e}\right) \\
y &= \frac{n}{e}
\end{aligned}
$$

So we see that our problem, to evaluate $\lim\limits_{n\to\infty}\sqrt[n]{n!}\left(a^{\sqrt[n]{E}_{n-1}}-1\right)$, is of the form $\infty\cdot 0$, and this allows us to use L'Hospital's rule, to differentiate the numerator and denominator separately with respect to n.

For the numerator, let $u = a^{1/n}-1$.

$$
\begin{aligned}
u &= a^{1/n}-1 \\
(u+1)^n &= a \\
n\ln(u+1) &= \ln(a) \\
\lim_{n\to\infty}\ln(u+1) &= \lim_{n\to\infty}(1/n)\ln(a) \\
\lim_{n\to\infty}\frac{1}{u+1}\frac{du}{dn} &= \lim_{n\to\infty}-\frac{\ln(a)}{n^2} \\
\frac{du}{dn} &= \frac{-(u+1)\ln(a)}{n^2} = -\left(\frac{a^{1/n}\ln(a)}{n^2}\right)
\end{aligned}
$$

For the denominator, $\dfrac{d}{dn}(e/n) = -\dfrac{e}{n^2}$.

So,

$$
\lim_{n\to\infty}\frac{\dfrac{-a^{1/n}\ln(a)}{n^2}}{-\dfrac{e}{n^2}} = \lim_{n\to\infty}\frac{a^{1/n}\ln(a)}{e}
$$

$$= \frac{\ln a}{e}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is known that for real x tending to zero, we have $e^x = 1 + x + O\left(x^2\right)$.

Since $\lim\limits_{n\to\infty} E_n = e$, so $\sqrt[n]{E_n} - 1 = e^{\frac{\ln E_n}{n}} - 1 = \dfrac{\ln E_n}{n} + O\left(\dfrac{1}{n^2}\right)$, and

$a^{\sqrt[n]{E_n}-1} - 1 = e^{\left(\sqrt[n]{E_n}-1\right)\ln a} - 1 = \dfrac{(\ln E_n)(\ln a)}{n} + O\left(\dfrac{1}{n^2}\right)$, where the last constant implied by O depends at most on a. Hence, by Stirling's formula

$n! = \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n\left(1 + O\left(\dfrac{1}{n}\right)\right)$ as n tends to infinity, we obtain

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right) = \frac{\ln a}{e}.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel, and the proposers.

- **5311:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers. Prove that

$$\sum_{cyclic} \sqrt{\left(\frac{x^2}{3} + 3y^2\right)\left(\frac{2}{xy} + \frac{1}{z^2}\right)} \geq 3\sqrt{10}.$$

Solution 1 by Arkady Alt, San Jose, CA

Since by AM-GM Inequality $\dfrac{x^2}{3} + 3y^2 = \dfrac{x^2 + 9y^2}{3} \geq \dfrac{1}{3}\cdot 10\sqrt[10]{x^2\cdot\left(y^2\right)^9} = \dfrac{10}{3}\sqrt[5]{xy^9}$ and

$\dfrac{2}{xy} + \dfrac{1}{z^2} \geq 3\sqrt[3]{\left(\dfrac{1}{xy}\right)^2\cdot\dfrac{1}{z^2}} = \dfrac{3}{\sqrt[3]{x^2y^2z^2}}$ then

$\sqrt{\left(\dfrac{x^2}{3} + 3y^2\right)\left(\dfrac{2}{xy} + \dfrac{1}{z^2}\right)} \geq \sqrt{\dfrac{10}{3}\sqrt[5]{xy^9}\cdot\dfrac{3}{\sqrt[3]{x^2y^2z^2}}} \iff \dfrac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot x^{\frac{1}{10}}y^{\frac{9}{10}}$ and,

therefore,

using again AM-GM Inequality we obtain

$$\sum_{cyclic}\sqrt{\left(\tfrac{x^2}{3} + 3y^2\right)\left(\tfrac{2}{xy} + \tfrac{1}{z^2}\right)} \geq \frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot\sum_{cyclic} x^{\frac{1}{10}}y^{\frac{9}{10}} \geq$$

$$\frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot 3\sqrt[3]{x^{\frac{1}{10}}y^{\frac{9}{10}}\cdot y^{\frac{1}{10}}z^{\frac{9}{10}}\cdot z^{\frac{1}{10}}x^{\frac{9}{10}}} = \frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot 3\sqrt[3]{xyz} = 3\sqrt{10}.$$

Equality holds if $x = y = z$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

By the AM-GM inequality

$\frac{x^3}{3}+3y^2=\frac{x^2}{3}+\overbrace{\frac{y^2}{3}+\ldots+\frac{y^2}{3}}^{9\text{ times}}\geq 10\sqrt[10]{\frac{x^2}{3}+\overbrace{\frac{y^2}{3}+\ldots+\frac{y^2}{3}}^{9\text{ times}}}=\frac{\sqrt[5]{10^5xy^9}}{3}$ with equality iff $\frac{x^2}{3}=\frac{y^2}{3}$, that is, iff $x=y$, and

$\frac{2}{xy}+\frac{1}{z^2}=\frac{1}{xy}+\frac{1}{xy}+\frac{1}{z^2}\geq 3\sqrt[3]{\frac{1}{xy}\cdot\frac{1}{xy}\cdot\frac{1}{z^2}}=\frac{3}{\sqrt[3]{x^2y^2z^2}}$ with equality iff $1xy=\frac{1}{z^2}$, that is, iff $xy=z^2$.

Hence,

$$\sqrt{\left(\frac{x^2}{3}+3y^2\right)\left(\frac{2}{xy}+\frac{1}{z^2}\right)}\geq\sqrt{\frac{\sqrt[5]{10^5xy^9}}{3}\frac{3}{\sqrt[3]{x^2y^2z^2}}}=\sqrt[30]{\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}}$$

with equality iff $x=y=z$, and cyclically. This and the AM-GM inequality prove the inequality, because

$$\begin{aligned}\sum_{cyclic}\sqrt{\left(\frac{x^2}{3}+3y^2\right)\left(\frac{2}{xy}+\frac{1}{z^2}\right)} &\geq \sum_{cyclic}\sqrt[30]{\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}}\\ &\geq 3\sqrt[3]{\prod_{cyclic}\sqrt[30]{\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}}}=3\sqrt[3]{\sqrt[30]{\frac{10^{45}x^{30}y^{30}z^{30}}{x^{30}y^{30}z^{30}}}}=3\sqrt{10},\end{aligned}$$

with equality iff $x=y=z$ and $\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}=\frac{10^{15}y^3z^{27}}{x^{10}y^{10}z^{10}}=\frac{10^{15}z^3x^{27}}{x^{10}y^{10}z^{10}}$, that is iff $x=y=z$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania, and the proposer.

- **5312:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate:

$$\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx = \int_{1/2}^1 \ln(\sqrt{x}-\sqrt{1-x})dx + \int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx.$$

Moreover,

$$\int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx \underbrace{=}_{1-x=y} \int_{1/2}^1 \ln(\sqrt{y}-\sqrt{1-y})dy$$

and then,

$$\begin{aligned}
\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx &= 2\int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx \\
&= \int_0^{1/2} \ln(1-x)dx + 2\int_0^{1/2} \ln\left(1-\sqrt{\frac{x}{1-x}}\right)dx = \left[x=t^2/(1+t^2)\right] \\
&= \int_{1/2}^1 \ln x dx + 2\int_0^1 \ln(1-t)\frac{2t}{(1+t^2)^2}dt \\
&= (x\ln x - x)\Big|_{1/2}^1 + \lim_{a\to 1} 2\frac{t^2}{1+t^2}\ln(1-t)\Big|_0^a + \lim_{a\to 1}\int_0^a \frac{2t^2}{1+t^2}\frac{1}{1-t}dt. \qquad (*)
\end{aligned}$$

$$\begin{aligned}
2\int_0^a \frac{t^2}{1+t^2}\frac{1}{1-t}dt &= 2\int_0^a \left(\frac{1}{1-t}-\frac{1}{(1+t^2)(1-t)}\right)dt \\
&= \int_0^a \left(\frac{2}{1-t}-\frac{1}{1-t}-\frac{1+t}{1+t^2}\right)dt \\
&= \left(-\ln(1-t)-\arctan t-\frac{1}{2}\ln(1+t^2)\right)\Big|_0^a \\
&= -\ln(1-a)-\arctan a-\frac{1}{2}\ln(1+a^2).
\end{aligned}$$

The quantity (*) becomes

$$\frac{1}{2}\ln 2-\frac{1}{2}+\lim_{a\to 1}\ln(1-a)\left(\frac{2a^2}{1+a^2}-1\right)-\frac{\pi}{4}-\frac{\ln 2}{2} = -\frac{1}{2}-\frac{\pi}{4}.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

By the substitution $x=\sin^2(\theta/2)$ we have

$$\int_0^1 \ln\left|\sqrt{x}-\sqrt{1-x}\right|dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 \ln\left(\left(\sqrt{x}-\sqrt{1-x}\right)^2\right)dx \\
&= \frac{1}{2}\int_0^1 \ln\left(1-2\sqrt{x(1-x)}\right)dx \\
&= \frac{1}{4}\int_0^\pi \ln(1-\sin\theta)\sin\theta d\theta \\
&= \frac{-1}{4}\left[\ln(1-\sin\theta)\cos\theta\right]_0^\pi - \frac{1}{4}\int_0^\pi \frac{\cos^2\theta}{1-\sin\theta}d\theta \\
&= \frac{-1}{4}\int_0^\pi (1+\sin\theta)d\theta \\
&= \frac{-1}{4}\left[\theta-\cos\theta\right]\Big|_0^\pi \\
&= -\frac{\pi+2}{4}.
\end{aligned}
$$

Solution 3 by Moti Levy, Rehovot, Israel

Using the symmetry of the integrand and substituting $u = 2x$,

$$
\begin{aligned}
\int_0^1 \ln\left|\sqrt{x}-\sqrt{1-x}\right|dx &= 2\int_0^{\frac{1}{2}} \ln\left(\sqrt{\frac{1}{2}+x}-\sqrt{\frac{1}{2}-x}\right)dx \\
&= -1+\ln\sqrt{2}-\int_0^1 \ln\left(\sqrt{1+u}+\sqrt{1-u}\right)du. \qquad (1)
\end{aligned}
$$

To evaluate the integral in (1), we substitute $u = \cos 2x$, integrate by parts and use the trigonometric equality,

$$
\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\cos 2x = 1-\sin 2x.
$$

$$
\begin{aligned}
&\int_0^1 \ln\left(\sqrt{1+u}+\sqrt{1-u}\right) du \\
&= 2\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{1+\cos 2x}+\sqrt{1-\cos 2x}\right)\sin 2x dx \\
&= 2\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2}\left(\cos x+\sin x\right)\right)\sin 2x dx \\
&= -\ln\left(\sqrt{2}\left(\cos x+\sin x\right)\right)\cos 2x\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\cos 2x dx \\
&= \ln\sqrt{2} + \int_0^{\frac{\pi}{4}} \left(1-\sin 2x\right) dx \\
&= \ln\sqrt{2} + \frac{\pi}{4} - \frac{1}{2}. \qquad (2)
\end{aligned}
$$

By (1) and (2), we obtain,

$$\int_0^1 \ln\left|\sqrt{x}-\sqrt{1-x}\right| dx = -\frac{\pi}{4}-\frac{1}{2}.$$

Solution 4 by Brian D. Beasely, Presbyterian College, Clinton, SC

We denote the given integral by I and let $A=\int_0^{1/2}\ln(\sqrt{1-x}+\sqrt{x})dx$ and $B=\int_0^{1/2}\ln(\sqrt{1-x}-\sqrt{x})dx$. We then show that $A+B=-1/2$ and $A-B=\pi/4$, so we conclude that

$$I=2B=-1/2-\pi/4.$$

Using L'Hopital's Rule, we have

$$A+B=\int_0^{1/2}\ln(1-2x)dx=\frac{(1-2x)\ln(1-2x)-(1-2x)}{-2}=-\frac{1}{2}.$$

Next, we integrate by parts to calculate $A-B$:

$$
\begin{aligned}
\int \ln\left(\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\right) dx &= \left(x-\frac{1}{2}\right)\ln\left(\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\right)+\int\frac{1}{2\sqrt{x(1-x)}}dx \\
&= \left(x-\frac{1}{2}\right)\ln\left(\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\right)+\sin^{-1}(\sqrt{x})+C.
\end{aligned}
$$

Using L'Hopital once again, we conclude $A-B=\pi/4$ as needed.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2015

- **5331:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral $\triangle ABC$ with cevian $\overline{CD}$. Triangle ACD has inradius $3N+3$ and $\triangle BCD$ has inradius N^2+3N where N is a positive integer.

  Find lengths $\overline{AD}$ and $\overline{BD}$.

- **5332:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Inspired by the prime number 1000000000000066600000000000001, known as *Belphegor's prime* where there are thirteen consecutive zeros to the left and right of 666, we consider the numbers $100\ldots0201500\ldots01$ where there are $k-$zeros left and right of 2015. For $k<28$ only $k=9$ and $k=27$ yield prime numbers.

  (a) Prove that the sequence $120151,\ 10201501,\ 1002015001,\ldots$ has an infinite subsequence of all composite numbers.

  (b) Find the next prime in both the sequences $100\ldots066600\ldots01$ and $100\ldots0201500\ldots01$, after the ones noted above.

- **5333:** *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy*

  Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{\left(\ln\left(1+\tan x+\tan^2 x\right)\right)^2}{1+\sin x\cos x}dx.$

- **5334:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $x_{ij},\ (1\le i\le m, 1\le j\le n)$ be nonnegative real numbers. Prove that

$$\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}\frac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}\frac{1}{1+\sqrt{x_{ij}}}\right)\ge 1.$$

- **5335:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any real $p > 1$ and $x > 1$ that

$$\frac{\ln x}{\ln(x+p)} \leq \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p.$$

- **5336:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Caculate:
  $$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Solutions

- **5313:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles if they both have perimeter 256 and area 1008.

  Solution by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

  Let $s = \dfrac{2a+b}{2}$ be the semiperimeter of the triangle. By Heron's formula for the area we also have: $A = 1008 = \sqrt{s(s-a)^2(s-b)}$. Solving the system we obtain $(a,b) = (65, 126)$ and $(a,b) = \left(\dfrac{255-\sqrt{253}}{2}, 1+\sqrt{253}\right)$.

  Also solved by Corneliu Manescu-Avram, Transportation High School, Ploiesti, Romania; Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport, News, VA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5314:** *Proposed by Roger Izard, Dallas TX*

  A biker and a hiker like to workout together by going back and forth on a road which is ten miles long. One day, at 8 AM, at the starting end of the road, they went out together. The biker soon got far past the hiker, reached the end of the road, reversed his direction, and soon passed by the hiker at 9:06 AM. Then, the biker got down to the beginning part of the road, reversed his direction, and got back to the hiker at 9:24 AM. The biker and the hiker were, then, going in the same direction. Calculate in miles per hour the speeds of the hiker and the biker.

Solution 1 by Jerry Chu (student at Saint George's School), Spokane, WA

Let the speed of the biker be x (mph) and let the speed of the hiker be y (mph).

Because it takes $\frac{11}{10}$ hours (8 am to 9:06 am) to meet we have

$$biker - - - - - - - - - - - - - - > - - -\downarrow$$
$$hiker - - - - - - - - - - >< - - - - - -\downarrow$$

$\frac{11(x+y)}{10} = 20$, together they made 20 miles.

And because it takes $\frac{7}{5}$ hours for them to meet again, we have

$$biker - - - - - - - - - - - - - - > - - -\downarrow$$
$$\downarrow - - - - - - - - - - < - - - - - - -\downarrow$$
$$\downarrow - - - - - - - - - - >$$
$$hiker - - - - - - - - >$$

The difference in the distances they traveled is $\frac{7(x-y)}{5} = 20$. Solving the system of equations

$$\begin{aligned} 11(x+y) &= 200 \\ 7(x-y) &= 100, \end{aligned}$$

we obtain $x = \frac{1250}{77}$mph and $y = \frac{150}{77}$ mph, for the biker and hiker respectively.

Solution 2 by Michael Thew (student at Saint George's School), Spokane WA

We are given that the entire length of the road is 20 miles. At their first meeting, the biker has already hit the ten mile mark and started his way back to the starting line. He passes the hiker (who is still traveling away from the starting line) after a total time of 1.1 hours has elapsed. Let the distance from the 10 mile mark to this meeting point be x. Therefore, the biker has traveled $10 + x$, and the hiker has traveled $10 - x$. Letting h and b be the speed in mph of the hiker and the biker respectively, we have, by the $distance = (rate)(time)$ equation, that $10 - x{=}(h)(1.1)$ and $10 + x = (b)(1.1)$. If we add these two equations and cancel the $x's$, we obtain: $20 = (1.1)(h + b)$.
The two continue moving until they end up meeting again after a total of 1.4 hours has elapsed (from the beginning). Therefore, the biker has finished the 10 mile return to the starting line and has reversed his direction again. The hiker was still traveling in the same direction (away from the starting line). Labeling y as the distance from the starting line to this second meeting point, we obtain $y = h(1.4)$ and $20 + y = b(1.4)$ Subtracting the first equation from the second equation and canceling the $y's$ gives : $20 = 1.4(b - h)$.
Once knowing that

$$20 = 1.1(b + h)20 = 1.4(b - h)$$

we solve for b and h obtaining that $b = 16.234$ mph and $h = 1.948$ mph.

Also solved by Adnan Ali (student at A.E.C.S-4), Mumbai, India; Harold Don Allen, Brossard, Quebec, Canada; Brian D. Beasley, Presbyterian College, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Guy Preskill, Butler University, Indianapolis, IN; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5315:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The hexagonal numbers have the form $H_n = 2n^2 - n,\ n = 1, 2, 3, \ldots$. Prove that infinitely many hexagonal numbers are the sum of two hexagonal numbers.

Solution 1 by Susan Abernathy, Dionne Bailey, Elsie Campbell, Charles Diminnie, and Jesse Taylor (jointly), Angelo State University, San Angelo, TX

Suppose that

$$H_{n+j} = H_n + H_k$$

with $j \geq 1$. Then, we have

$$\begin{aligned} 2(n+j)^2 - (n+j) &= 2n^2 - n + 2k^2 - k, \\ 4nj &= 2k^2 - k - 2j^2 + j \\ &= [2k + (2j-1)](k-j). \end{aligned}$$

One possibility is to let $k - j = 4j$ or $k = 5j$. Then,

$$n = 2(5j) + 2j - 1 = 12j - 1$$

and

$$n + j = 13j - 1.$$

To check whether these assignments are feasible, note that

$$\begin{aligned} H_{13j-1} &= 2(13j-1)^2 - (13j-1) \\ &= 338j^2 - 65j + 3, \\ H_{12j-1} &= 2(12j-1)^2 - (12j-1) \\ &= 288j^2 - 60j + 3, \end{aligned}$$

and

$$H_{5j} = 2(5j)^2 - 5j = 50j^2 - 5j.$$

It is now clear that

$$H_{5j} + H_{12j-1} = 338j^2 - 65j + 3 = H_{13j-1}$$

for all $j \geq 1$. The first five solutions of this type are shown in the following table:

j	$5j$	$12j-1$	$13j-1$	H_{5j}	H_{12j-1}	H_{13j-1}
1	5	11	12	45	231	276
2	10	23	25	190	1035	1225
3	15	35	38	435	2415	2850
4	20	47	51	780	4371	5151
5	25	59	64	1225	6903	8128

Solution 2 by Jerry Chu (student at Saint George's School), Spokane, WA

The difference between two consecutive hexagonal numbers is

$$H(n+1) - H(n) = \big(2(n+1)^2 - (n+1)\big) - \big(2n^2 - n\big) = 4n + 1.$$

This is to say that any hexagonal number of the form $4n+1$ is the difference between $H(n)$ and $H(n+1)$. So we look for hexagonal numbers of the form $4m+k$, where $k = 0, 1, 2, 3$

$$H(4m+k)\ (k = 0,1,2,3) = 32m^2 + 16mk + 2k^2 - 4m - k = 2k^2 - k \pmod 4.$$

Only $k = 1$ satisfies this equation. Therefore, hexagonal numbers of the form $H(4m+1) = 32m^2 + 12m + 1$, can be expressed as the difference between $H(8m^2+3m)$ and $H(8m^2+3m+1)$. So there are an infinite number of hexagonal numbers of the form $H(8m^2+3m+1)$ that can be expressed as the sum of two hexagonal numbers.

Comment by Editor: **William J. O'Donnell of Centennial, CO** *mentioned in his solution that:* It can further be shown that infinitely many hexagonal numbers are the sum and difference of two hexagonal numbers, specifically,

$$\begin{aligned} H_{128n^2+12n+1} &= H_{16n+1} + H_{128n^2+12n} \\ &= H_{8192n^4+1536n^3+168n^2+9n+1} - H_{8192n^4+1536n^3+168n^2+9n}, \text{ for } n \geq 1. \end{aligned}$$

For more detail, see O'Donnell, W.J., Two theorems concerning hexagonal numbers *Fibonacci Quarterly* 1979, 17(1), 77-79. Similar results have also been published for triangular, pentagonal, and octagonal numbers. See:

−Hansen, R.T. Arithmetic of pentagonal numbers. *Fibonacci Quarterly,* 1970, 8, 83-87.

− O'Donnell, W.J. A theorem concerning octagonal numbers. *Journal of Recreational Mathematics,* 1979-80, 12(4), 271-272.

−Sierpinski W. Un Théorème sur les nombres triangulaires *Elemente der Mathematik,* 1968, 23, 31-32.

Also solved by Adnan Ali (student at A.E.C.S-4), Mumbai, India; Arkady Alt, San Jose, CA; Corneliu Manescu-Avram,Transportation High School, Ploiesti, Romania; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer

Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; William J. O'Donnell, Centennial, CO; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Bob Sealy, Sackville, New Brunswick, Canada; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5316:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let $\{u_n\}_{n\geq 0}$ be a sequence defined recursively by

$$u_{n+1} = \sqrt{\frac{u_n^2 + u_{n-1}^2}{2}}.$$

Determine $\lim\limits_{n\to\infty} u_n$ in terms of u_0, u_1.

Solution 1 by Moti Levy, Rehovot, Israel

Let $v_n = u_n^2$, then the sequence $\{v_n\}_{n\geq 0}$ follows the linear recurrence

$$2v_{n+1} = v_n + v_{n-1}.$$

The closed form of the sequence $\{v_n\}_{n\geq 0}$ is $v_n = \alpha + \beta\left(-\frac{1}{2}\right)^n$, with initial conditions:

$$u_0^2 = \alpha + \beta,$$
$$u_1^2 = \alpha - \frac{\beta}{2}.$$
$$u_n^2 = \frac{1}{3}u_0^2 + \frac{2}{3}u_1^2 + \left(\frac{2}{3}u_0^2 - \frac{2}{3}u_1^2\right)\left(-\frac{1}{2}\right)^n.$$
$$\lim_{n\to\infty} u_n = \sqrt{\frac{1}{3}u_0^2 + \frac{2}{3}u_1^2}.$$

Solution 2 by Jerry Chu (student at Saint George's School), Spokane, WA

First some observations:

$$u_2 = \sqrt{\frac{u_0^2 + u_1^2}{2}}$$

$$u_3 = \sqrt{\frac{u_0^2 + 3u_1^2}{4}}$$

$$u_4 = \sqrt{\frac{3u_0^2 + 5u_1^2}{8}}.$$

This suggests that the general of the sequence is

$$u_n = \sqrt{\frac{\left(2^{n-2} \pm \frac{1}{2}\right)u_0^2 + \left(2^{n-1} \mp \frac{1}{2}\right)u_1^2}{3\cdot 2^{n-2}}},$$

which is true by induction. So

$$\lim_{n\to\infty} u_n = \sqrt{\frac{1}{3}u_0^2 + \frac{2}{3}u_1^2}$$

Solution 3 by Henry Ricardo, New York Math Circle, NY

In [1], the authors provide three ways of determining convergence and limiting values for linear mean recurrences. In particular, they prove that given a sequence $\{x_n\}$ such that $x_n := (x_{n-1} + x_{n-2} + \cdots + x_{n-m})/m$ for $n \geq m+1$, where $x_1, x_2, \ldots, x_m$ are given real numbers, we can conclude that

$$\lim_{n\to\infty} x_n = \frac{2}{m(m+1)} \sum_{n=1}^{m} n x_n.$$

The substitution $U_k = u_k^2$ converts our given relation to the linear mean recurrence

$$U_n = \frac{U_{n-1} + U_{n-2}}{2} \qquad \text{for } n = 2, 3, \ldots.$$

Then the paper cited above provides the formula

$$\lim_{n\to\infty} U_n = \frac{2}{2(3)} \sum_{n=0}^{1} (n+1) U_n = \frac{U_0 + 2U_1}{3},$$

giving us $u_n^2 \to (u_0^2 + 2u_1^2)/3$, or $u_n \to \sqrt{(u_0^2 + 2u_1^2)/3}$.

Reference

[1] D. Borwein, J. M. Borwein, B. Sims, *On the Solution of Linear Mean Recurrences*, Amer. Math. Monthly **121** 2014, pp. 486-498.

Editor's comment: Henry Ricardo submitted three solutions to problem 5316. Taken together, the above solutions represent the different ways for determining the convergence of linearly stated recursion sequences, as pointed out in the reference Henry cited.

Also solved by Susan Abernathy, Dionne Bailey, Elsie Campbell, Charles Diminnie, and Jesse Taylor (jointly), Angelo State University, San Angelo, TX; Arkady Alt, San Jose, CA; Corneliu Manescu-Avram,Transportation High School, Ploiesti, Romania; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5317:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_k, b_k > 0,\ 1 \leq k \leq n$, be real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that

$$\frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \leq \sum_{k=1}^{n} \frac{b_k^5}{a_k}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

By the Hölder inequality,

$$\left(\sum_{k=1}^{n} 1\right)^{p-q-1} \sum_{k=1}^{n} \frac{b_k^p}{a_k^q}\left(\sum_{k=1}^{n} a_k\right)^q \geq \left(\sum_{k=1}^{n} 1^{\frac{p-q-1}{p}} \frac{b_k^{\frac{p}{p}}}{a_k^{\frac{q}{p}}} a_k^{\frac{q}{p}}\right)^p = \left(\sum_{k=1}^{n} b_k\right)^p.$$

The choice $p = 5$ and $q = 1$ gives the result.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will need two preliminary results:

1. Let $f(x) = x^{\frac{5}{2}}$ on $(0, \infty)$. Since $f''(x) = \dfrac{15}{4} x^{\frac{1}{2}} > 0$ for $x > 0$, it follows that $f(x)$ is strictly convex on $(0, \infty)$. Then, Jensen's Theorem implies that

$$f\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} f(b_k),$$

i.e.,

$$\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^{\frac{5}{2}} \leq \frac{1}{n}\sum_{k=1}^{n} b_k^{\frac{5}{2}}. \qquad (1)$$

Further, equality is attained in (1) if and only if $b_1 = b_2 = \ldots = b_n$.

2. If we apply the Cauchy - Schwarz Inequality to the vectors

$$\overrightarrow{X} = \left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\right) \quad \text{and} \quad \overrightarrow{Y} = \left(\sqrt{\frac{b_1^5}{a_1}}, \sqrt{\frac{b_2^5}{a_2}}, \cdots, \sqrt{\frac{b_n^5}{a_n}}\right),$$

we get

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{b_k^5}{a_k}\right) &= \left\|\overrightarrow{X}\right\|^2 \left\|\overrightarrow{Y}\right\|^2 \\ &\geq \left(\overrightarrow{X} \cdot \overrightarrow{Y}\right)^2 \\ &= \left(\sum_{k=1}^{n} b_k^{\frac{5}{2}}\right)^2. \qquad (2)\end{aligned}$$

Since $\sum_{k=1}^{n} a_k = 1$, we use (1) and (2) to obtain

$$\begin{aligned}
\sum_{k=1}^{n} \frac{b_k^5}{a_k} &= \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{b_k^5}{a_k}\right) \\
&\geq \left(\sum_{k=1}^{n} b_k^{\frac{5}{2}}\right)^2 \\
&= n^2 \left(\frac{1}{n}\sum_{k=1}^{n} b_k^{\frac{5}{2}}\right)^2 \\
&\geq n^2 \left[\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^{\frac{5}{2}}\right]^2 \\
&= n^2 \cdot \frac{1}{n^5}\left(\sum_{k=1}^{n} b_k\right)^5 \\
&= \frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 .
\end{aligned}$$

Also, by the criterion for equality in (1), equality results above if and only if $b_1 = b_2 = \ldots = b_n$.

Solution 3 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

The proposed inequality is homogeneous in b_k, so we may assume in addition that also $b_1 + b_2 + \cdots + b_n = 1$, and the inequality is equivalent to prove that $\frac{1}{n^3} \leq \sum_{k=1}^{n} \frac{b_k^5}{a_k}$. We use the Cauchy-Schwarz inequality in Engel's form which states that

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n},$$

for all real numbers x_i and positive real numbers y_i.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{b_k^5}{a_k} &= \sum_{k=1}^{n} \frac{\left(b_k^{5/2}\right)^2}{a_k} \text{ (which by Engel's inequality)} \\
&\geq \frac{\left(\sum_{k=1}^{n} b_k^{5/2}\right)^2}{\sum_{k=1}^{n} a_k} = \left(\sum_{k=1}^{n} b_k^{5/2}\right)^2 .
\end{aligned}$$

Now, by the power-mean arithmetic-mean inequality

$$\frac{\sum_{k=1}^{n} b_k^{5/2}}{n} \geq \left(\frac{\sum_{k=1}^{n} b_k}{n}\right)^{5/2} = \frac{1}{n^{5/2}}, \quad \text{and so,} \ \sum_{k=1}^{n} b_k^{5/2} \geq \frac{1}{n^{3/2}}.$$

Therefore, $\displaystyle\sum_{k=1}^{n} \frac{b_k^5}{a_k} \geq \left(\sum_{k=1}^{n} b_k^{5/2}\right)^2 \geq \frac{1}{n^3}$ and we are done.

Solution 4 by Adrian Naco, Polytechnic University,Tirana, Albania

Let's prove a more general inequality, that is,

$$\frac{1}{n^s}\left(\sum_{k=1}^{n} b_k\right)^{s+2} \leq \sum_{k=1}^{n} \frac{b_k^{s+2}}{a_k}, \quad (1)$$

Based on the well known Chebyschev inequality we get that

$$\sum_{k=1}^{n} \frac{b_k^{s+2}}{a_k} \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} b_k^{s+2}\right) \quad (2)$$

Furthermore if we apply the same inequality s times recursively then,

$$\sum_{k=1}^{n} \frac{b_k^{s+2}}{a_k} \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} b_k^{s+2}\right) \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{k=1}^{n} b_k^{s+1}\right)$$

$$\geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\frac{1}{n}\right)^2\left(\sum_{k=1}^{n} b_k\right)^2\left(\sum_{k=1}^{n} b_k^{s}\right) \geq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\frac{1}{n}\right)^{s+1}\left(\sum_{k=1}^{n} b_k\right)^{s+1}\left(\sum_{k=1}^{n} b_k\right)$$

$$= \left(\frac{1}{n}\right)^{s+2}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} b_k\right)^{s+2} \geq \left(\frac{1}{n}\right)^{s+2} n^2\left(\sum_{k=1}^{n} b_k\right)^{s+2}$$

$$= \left(\frac{1}{n}\right)^{s}\left(\sum_{k=1}^{n} b_k\right)^{s+2}$$

since,

$$\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) \geq \frac{n}{\sqrt[n]{\prod_{k=1}^{n} a_k}} \geq \frac{n^2}{\left(\sum_{k=1}^{n} a_k\right)} = n^2$$

Finally, the given inequality to prove is a special case of the general inequality (1) taken for $s = 5$.

Solution 5 by Adnan Ali (Student at A.E.C.S-4), Mumbai, India

From Holder's Inequality, one notices that

$$n^3\left(\sum_{k=1}^{n}\frac{b_k^5}{a_k}\right) = \left(\sum_{k=1}^{n}\frac{b_k^5}{a_k}\right)\left(\sum_{k=1}^{n} a_k\right)\left(\underbrace{1+\cdots+1}_{n \text{ times}}\right)\left(\underbrace{1+\cdots+1}_{n \text{ times}}\right)\left(\underbrace{1+\cdots+1}_{n \text{ times}}\right) \geq \left(\sum_{k=1}^{n} b_k\right)^5$$

whence the result immediately follows and so, $\dfrac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \leq \left(\sum_{k=1}^{n}\dfrac{b_k^5}{a_k}\right)$.

Solution 6 by Nicusor Zlota. "Traian Vula" Technical College, Focsani, Romania

We shall prove the following more general inequality.

If $a_k b_k > 0,\ 1 \leq k \leq n$ and $a, b, \in \Re$, such that $a - b \geq 1$, then

$$\sum_{k=1}^{n}\frac{b_k^q}{a_k^b} \geq n\frac{\left(\dfrac{1}{n}\sum_{k=1}^{n} b_k\right)^a}{\left(\dfrac{1}{n}\sum_{k=1}^{n} a_k\right)^b} \qquad (*)$$

Proof: Using the Radon and Jensen inequalities, we have

$$\sum_{k=1}^{n}\frac{b_k^q}{a_k^b} = \sum_{k=1}^{n}\frac{\left(b_k^{\frac{a}{b+1}}\right)^{b+1}}{a_k^b} \underset{Radon}{\geq} \sum_{k=1}^{n}\frac{\left(b_k^{\frac{a}{b+1}}\right)^{b+1}}{a_k^b} \underset{Jensen}{\geq} \frac{\left[n\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^{\dfrac{a}{b+1}}\right]^{b+1}}{\left(\sum_{k=1}^{n} a_k\right)^b} = n\frac{\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^a}{\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^b}.$$

If in $(*)$, $a = 5, b = 1$ and $\sum_{k=1}^{n} a_k = 1$, then we obtain the inequality of the problem.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Henry

Ricardo, New York Math Circle, NY; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "George Emil Palade School," Buzău, Romania, and the proposer.

- **5318:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that $(1+x)^x \leq 1+x^2$ for $0 \leq x \leq 1$.

Solution 1 by Arkady Alt, San Jose, California, USA.

The inequality $(1+x)^x \leq 1+x^2, 0 \leq x \leq 1$ immediately follows from the Bernoulli inequality:

$$(1+t)^\alpha \geq 1+\alpha t, t > -1, a \geq 1. \qquad \mathbf{(1)}$$

Indeed, for $0 < x \leq 1 \iff \dfrac{1}{x} \geq 1,$ and by **(1)** we have

$$\left(1+x^2\right)^{\frac{1}{x}} \geq 1+x^2 \cdot \frac{1}{x} = 1+x \iff 1+x^2 \geq (1+x)^x.$$

For $x=0$ the original inequality is obvious.

Another way to prove the inequality $(1+x)^x \leq 1+x^2$ is based on using the Weighted AM-GM Inequality: $u^p v^q \leq pu + qv$ where $u, v, p, q \geq 0$ and $p+q=1$.

Indeed, for $u = 1+x, v=1,\ p=x,\ q=1-x$ we have

$$(1+x)^x \cdot 1^{1-x} \leq (1+x)\,x + 1\cdot(1-x) \iff (1+x)^x \leq 1+x^2.$$

$(1+x)^a \leq 1+ax,$ where $x > -1$ and $0 \leq a \leq 1.$ **(1)**

Applying inequality **(1)** to $a = x$ we obtain $(1+x)^x \leq 1+x^2$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have equalityfor $x=0$ and $x=1$.

We assume that $0<x<1$. We expand $(1+x)^x$ into a binomial series and get

$$\begin{aligned}(1+x)^x &= \sum_{n=0}^{\infty}\binom{x}{n}x^n = 1+x^2+\sum_{j=1}^{\infty}\left(\frac{x(x-1)\cdots(x-2j+1)}{(2j)!}x^{2j}+\frac{x(x-1)\cdots(x-2j)}{(2j+1)!}x^{2j+1}\right)\\ &= 1+x^2-\sum_{j=1}^{\infty}\underbrace{\frac{x(1-x)\cdots(2j-1-x)}{(2j)!}x^{2j}}_{>0}\underbrace{\left(1-\frac{2j-x}{2j+1}x\right)}_{>0} < 1+x^2.\end{aligned}$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Consider $g(u)$=(1+u)$^{\frac{1}{u}}$ on $(0,1)$. It is decreasing since

$$g'(u) = (1+u)^{\frac{1}{u}} \cdot \frac{\left(\dfrac{u}{1+u} - \ln(1+u)\right)}{u^2}$$

.

Note that $(1+u)^{\frac{1}{u}}$ and u^2 are positive and $f(u) = \dfrac{u}{1+u} - \ln(1+u)$ is negative on $(0,1)$ since and $f(0) = 0$ and $\dfrac{d}{du}f(u) = \dfrac{u}{(1+u)^2}$. Hence since $x^2 < x$ if $0 < x < 1$ we have $g(x^2) > g(x)$ i.e.

$$(1+x^2)^{\frac{1}{x^2}} > (1+x)^{\frac{1}{x}} \quad (*).$$

Now if $0 < a < b$ then $\dfrac{b}{a} > 1$ and so if $t > 0$ then $\left(\dfrac{b}{a}\right)^t > 1$ and so $b^t > a^t$. $\quad (**)$

The desired inequality then follows from (**) upon setting $b = (1+x^2)^{\frac{1}{x^2}},\ a = (1+x)^{\frac{1}{x}}$ and $t = x^2$, i.e., raising both sides of (*) to the x^2 power.

Solution 4 by Ed Gray, Highland Beach, FL

Consider the the general binomial theorem.

$$(a+b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 + \cdots .$$

Since $a = 1$, we suppress its appearance after the first term. Letting $b = x = n$, and substituting in the above we obtain

$$(1+x)^x = 1 + x^2 + \frac{x(x-1)x^2}{2!} + \frac{x(x-1)(x-2)x^3}{3!} + \frac{x(x-1)(x-2)(x-3)x^4}{4!} + \cdots .$$

Clearly, for $x = 0$ and $x = 1$ the expression on the left hand side of the equality sign is the same as the expression on the right hand side of the equality sign.

If $0 < x < 1$ we note that starting with the third term, which is negative, we have an alternating decreasing series that approaches zero. Since each term is numerically less than its precedent and the third term is negative, it is clear that the series must be less than the sum of the first two terms, $1 + x^2$, therefore, for $0 < x < 1$, it must be that $(1+x)^x < 1 + x^2$.

Solution 5 by Kee-Wai Lau, Hong Kong, China

For $0 \le x \le 1$, let $f(x) = x\ln(1+x) - \ln(1+x^2)$. We have

$$f'(x) = \ln(1+x) + \frac{x(x^2-2x-1)}{(1+x)(1+x^2)} \ \text{ and } f''(x) = \frac{x(x^2+2x+3)\left(x^2+2x-1\right)}{(1+x)^2(x^2+1)^2}.$$

Hence for $0 < x < \sqrt{2}-1$, we have $f''(x) < 0$. Since $f'(0) = 0$, so $f'(x) < 0$ for $0 < x \le \sqrt{2}-1$. Since $f(0) = 0$, so $f(x) < 0$ for $0 < x < \sqrt{2}-1$ as well.

Now for $\sqrt{2}-1 < x \le 1$, we have $f''(x) > 0$ so that $f(x)$ is convex. Since $f(\sqrt{2}-1) < 0$ and $f(1) = 0$, so $f(x) \le 0$ for $\sqrt{2}-1 < x \le 1$. It follows that $f(x) \le 0$ for $0 \le x \le 1$.

The inequality of the problem follows by exponentiation.

Also solved by Adnan Ali (Student at A.E.C.S-4), Mumbai, India; Corneliu Manescu-Avram,Transportation High School, Ploiesti, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Moti Levy, Rehovot, Israel; Adrian Naco, Polytechnic University, Tirana, Albania; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

Mea Culpa

Adnan Ali (Student at A.E.C.S-4 in Mumbai, India) submitted solutions to problems 5307 and 5309. Unfortunately these solutions were unintentionally not acknowledged in the previous issue of the column. **And similarly for Hatef I. Arshagi of Guilford Technical Community College in Jamestown, NC** for his solutions to problems 5307, 5309 and 5312. Once again I plead mea culpa to them both, and also to **William J. O'Donnell, Centennial, CO** for not acknowledging his solution to 5213.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2015

- **5337:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex quadrilateral $ABCD$ with sides,

  $$\begin{aligned} \overline{AB} &= 1+3\sqrt{2} \\ \overline{BC} &= 6+4\sqrt{2} \text{ and} \\ \overline{CD} &= -14+12\sqrt{2}. \end{aligned}$$

  Find side $\overline{AD}$ so that the area of the quadrilateral is maximum.

- **5338:** *Proposed by Arkady Alt, San Jose, CA* Determine the maximum value of

  $$F(x,y,z) = \min\left\{\frac{|y-z|}{|x|}, \frac{|z-x|}{|y|}, \frac{|x-y|}{|z|}\right\},$$

  where x, y, z are arbitrary nonzero real numbers.

- **5339:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

  Calculate: $\displaystyle\int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}dx.$

- **5340:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

  Let a, b and c be the side-lengths, and s the semi-perimeter of a triangle. Show that

  $$\frac{a^2+b^2}{(s-c)^2} + \frac{b^2+c^2}{(s-a)^2} + \frac{c^2+a^2}{(s-b)^2} \geq 24.$$

- **5341:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $z_1, z_2, \cdots, z_n$, and $w_1, w_2, \cdots, w_n$ be sequences of complex numbers. Prove that

  $$Re\left(\sum_{k=1}^{n} z_k w_k\right) \leq \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2 + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_k|^2.$$

- **5342:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c, \alpha > 0$ be real numbers. Study the convergence of the integral

$$I(a,b,c,\alpha) = \int_1^\infty \left(a^{1/x} - \frac{b^{1/x}+c^{1/x}}{2}\right)^\alpha dx.$$

The problem is about studying the conditions which the four parameters, a, b, c, and α, should verify such that the improper integral would converge.

Solutions

- **5319:** *Proposed by Kenneth Korbin, New York, NY*

Let N be an odd integer greater than one. Then there will be a Primitive Pythagorean Triangle with perimeter equal to $\left(N^2+N\right)^2$. For example, if $N=3$, then the perimeter equals$\left(3^2+3\right)^2 = 144$.

Find the sides of the PPT for perimeter $\left(15^2+15\right)^2$ and for perimeter $\left(99^2+99\right)^2$.

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The Primitive Pythagorean Triangle (a,b,c) with perimeter $(15^2+15)^2$ is $(6975, 24832, 25793)$, and the PPT with perimeter $(99^2+99)^2$ is $(1950399, 48010000, 48049601)$. One may easily verify that these triangles satisfy the conditions of the problem.

If $m > n$ are relatively prime positive integers of opposite parity, then they generate a PPT

$$(a,b,c) = (m^2-n^2, 2mn, m^2+n^2),$$

with perimeter $P = 2m(m+n)$. If P is a square, then $m = 2q^2$ and $m+n = p^2$ for some positive integers p and q. Therefore,

$$(m,n) = (2q^2, p^2-2q^2)$$

and

$$\begin{aligned}
a &= m^2-n^2 = p^2(4q^2-p^2),\\
b &= 2mn \qquad = 4q^2(p^2-2q^2),\\
c &= m^2+n^2 = p^4-4p^2q^2+8q^4.
\end{aligned}$$

Note that p is odd, $\sqrt{2}q < p < 2q$ since $4q^2-p^2 > 0$ and $p^2-2q^2 > 0$, and $\gcd(p,q) = 1$. Furthermore, the perimeter P is $4p^2q^2 = (2pq)^2$.

If $P = (15^2+15)^2$, then $2pq = 240$. Therefore $pq = 120$ and the only factors of 120 that statisfy p as being odd and $\sqrt{2}q < p < 2q$ are $p = 15$ and $q = 8$. For these values of p and q,

$$\begin{aligned}
a &= 15^2\left(4\cdot 8^2 - 15^2\right) = 6975,\\
b &= 4\cdot 8^2\left(15^2 - 2\cdot 8^2\right) = 24832,\\
c &= 15^4 - 4\cdot 15^2\cdot 8^2 + 8\cdot 8^4 = 25793.
\end{aligned}$$

If $P = (99^2+99)^2$, then $2pq = 99^2 + 99 = 9900$. Therefore $pq = 4950$ and the only factors of 4950 that satisfy p as being odd and $\sqrt{2}q < p < 2q$ are $p = 99$ and $q = 50$. Then

$$\begin{aligned}
a &= 99^2\left(4\cdot 50^2 - 99^2\right) = 1950399,\\
b &= 4\cdot 50^2\left(99^2 - 2\cdot 50^2\right) = 48010000,\\
c &= 99^4 - 4\cdot 99^2\cdot 50^2 + 8\cdot 50^4 = 480449601.
\end{aligned}$$

Also solved by Ashland University Undergraduate Problem Solving Group, Ashland, OH; Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5320:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

It is fairly well known that if (a, b, c) is a Primitive Pythagorean Triple (PPT), then the product abc is divisible by 60. Find infinitely many PPT's (a, b, c) such that the sum $(a + b + c)$ is also divisible by 60.

Solution 1 by Bruno Salgueiro Fanego,Viveiro Spain

It is know that a, b and c are the respective legs and hypothenuse of a PPT if and only if $a = m^2 - n^2,\ b = 2mn$, and $c = m^2 + n^2$ for some positive integers m and n such that $m > n$ and $\gcd(m, n) = 1$ and $m - n$ is odd.

Hence, the perimeter, $a + b + c = 2m(m + n)$, will be divisible by 60 if, for example, m is divisible by 30 because in that case, $2m$ and hence $2m(m + n)$ would each be divisible by 60.

Thus, we can find infinitely many PPT's $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ such that the sum $a + b + c$ is also divisible by 60, if we take $m = 30k$, with k being a positive integer and, for example, $n = 1$ because in that case, $m = 30k > 1$, $\gcd(m, 1) = 1$, and $m - n = 30k - 1$ is odd. A possible infinite set of PPT's is given by

$$(a, b, c) = (900k^2 - 1, 60k, 900k^2 + 1), \text{ where } k \text{ is a positive integer.}$$

Solution 2 by Paul M. Harms, North Newton, KS

Consider the Pythagorean Triangle $\{n^2 + 1, n^2 - 1, 2n\}$ where n is a positive even integer. Then the odd integers $(n^2 + 1)$ and $(n^2 - 1)$, do not have 2 as a factor. Since their difference is 2 units, these two integers have no common prime factor greater than one. Thus the triple $(n^2 + 1, n^2 - 1, 2n)$ represents the sides of a PPT when n is a positive even integer. The sum of the three side is $2n^2 + 2n = 2n(n + 1)$. Let $n = 30K$ where K is a positive integer. Then n is a positive even integer and the sum of the three sides is divisible by 60. Using different $K's$ we see that there are infinitely many PPT's satisfying the problem whose sides have the form $(n^2 + 1, n^2 - 1, 2n)$ and $n = 30K$. In these cases the sum of the three sides is $2n(n + 1) = 60K(30K + 1)$.

Solution 3, a generalization by Brian D. Beasley, Presbyterian College, Clinton, SC

We may generalize the given problem as follows: Given any positive integer m, find infinitely many PPT's (a, b, c) such that the sum $(a + b + c)$ is divisible by m.
Fix any positive integer m. If m is even, then for each positive integer k, we let $s = mk$ and $t = 1$ to produce the PPT

$$(a, b, c) = (m^2k^2 - 1, 2mk, m^2k^2 + 1),$$

for which $a + b + c = 2mk(mk + 1)$. If m is odd, then for each positive integer k, we let $s = 2mk$ and $t = 1$ to produce the PPT

$$(a, b, c) = (4m^2k^2 - 1, 4mk, 4m^2k^2 + 1),$$

for which $a + b + c = 4mk(2mk + 1)$.

Also solved by Adnan Ali (Student in A.E.C.S-4), Mumbai, India; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Moti Levy, Rehovot, Israel; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5321:** *Proposed by Lawrence Lesser, University of Texas at El Paso, TX*

On pop quizzes during the fall semester, Al gets 1 out of 3 questions correct, while Bob gets 3 of 8 correct. During the spring semester, Al gets 3/5 questions correct, while Bob gets 2/3 correct. So Bob did better each semester ($3/8 > 1/3$ and $2/3 > 3/5$) but worse for the overall academic year ($5/11 < 4/8$). The total number of questions involved in the above example was $3+8+5+3=19$, and the author conjectures (in his chapter in the 2001 Yearbook of the National Council of Teachers of Mathematics) that this is the smallest dataset with nonzero numerators in which this reversal (Simpson's Paradox) happens. If we allow zeros, the smallest dataset is conjectured to be $9: \ 0/1 < 1/4$ and $2/3 < 1/1$, but $2/4 > 2/5$.

Prove these conjectures or find counterexamples.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

I wrote a small computer program that did an exhaustive search. It turned out that the first conjecture is wrong. The smallest value in the first case is 13 and not 19, and these are the solutions:

$$1/1 > 6/7, \quad 1/2 > 1/3, \quad 2/3 < 7/10$$

$$1/1 > 4/5, \quad 1/3 > 1/4, \quad 2/4 < 5/9$$

$$1/1 > 6/7, \quad 2/3 > 1/2, \quad 3/4 < 7/9$$

$$1/1 > 4/5, \quad 2/4 > 1/3, \quad 3/5 < 5/8$$

$$1/1 > 3/4, \quad 2/5 > 1/3, \quad 3/6 < 4/7$$

$$1/1 > 4/5, \quad 3/5 > 1/2, \quad 4/6 < 5/7$$

$$1/2 > 1/3, \quad 1/1 > 6/7, \quad 2/3 < 7/10$$

$$2/3 > 1/2, \quad 1/1 > 6/7, \quad 3/4 < 7/9$$

$$1/3 > 1/4, \quad 1/1 > 4/5, \quad 2/4 < 5/9$$

$$2/4 > 1/3, \quad 1/1 > 4/5, \quad 3/5 < 5/8$$

$$3/5 > 1/2, \quad 1/1 > 4/5, \quad 4/6 < 5/7$$

$$2/5 > 1/3, \quad 1/1 > 3/4, \quad 3/6 < 4/7$$

The smallest value in the second case is indeed 9 and these are the solutions:

$$1/1 > 3/4, \quad 1/3 > 0/1, \quad 2/4 < 3/5$$

$$1/1 > 2/3, \quad 1/4 > 0/1, \quad 2/5 < 2/4$$

$$1/3 > 0/1, \quad 1/1 > 3/4, \quad 2/4 < 3/5$$

$$1/4 > 0/1, \quad 1/1 > 2/3, \quad 2/5 < 2/4$$

Solution 2 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The conjecture is false for nonzero numerators since $4/5 < 1/1$ and $1/4 < 1/3$ but $5/9 > 2/4$, and the data set is $13 < 19$.

If zero numerator are allowed, then we will show that the smallest data set is indeed mine, that is if $\dfrac{a_2}{a_1} < \dfrac{A_2}{A_1}$ and $\dfrac{b_2}{b_1} < \dfrac{B_2}{B_1}$, then $\dfrac{a_2+b_2}{a_1+b_1} > \dfrac{A_2+B_2}{A_1+B_1}$ is impossible if $A_1 + B_1 + a_1 + b_1 \leq 8$ and $a_2 = 0$. To do so , we will maximize $\dfrac{a_2+b_2}{a_1+b_1}$ while minimizing $\dfrac{A_2+B_2}{A_1+B_1}$ Then $a_1 = 1$ and the maximum value of A_1 is 4.

If $A_1 = 4$, then maximizing $\dfrac{a_2+b_2}{a_1+b_1}$ and minimizing $\dfrac{A_2+B_2}{A_1+B_1}$ yields the following

$$\left.\begin{array}{l} 0/1 < 1/4 \\ 1/2 < 1/1 \end{array}\right\} \Longrightarrow 1/3 < 2/5.$$

Note that for other values of A_2, the fraction $\dfrac{A_2+B_2}{A_1+B_1} > \dfrac{2}{5}$ while $\dfrac{a_2+b_2}{a_1+b_1} = \dfrac{1}{3}$.

If $A_1 = 3$, then $b_1 + B_1 \leq 4$ implies b_1 is 2 or 3. If $b_1 = 2$, then maximizing $\dfrac{a_2+b_2}{a_1+b_1}$, one obtains

$$\left.\begin{array}{l} 0/1 < 1/3 \\ 1/2 < 1/1 \end{array}\right\} \Longrightarrow 1/3 < 2/4.$$

If $b_1 = 3$, then maximizing $\dfrac{a_2+b_2}{a+1+b+1}$ yields

$$\left.\begin{array}{l} 0/1 < 1/3 \\ 2/3 < 1/1 \end{array}\right\} \Longrightarrow 2/4 = 2/4.$$

If $A_1 = 2$, then $A_2 = 1$ and $b_1 + B_1 \leq 5$ implies b_1 is 2,3, or 4. If $b_1 = 2$, then $b_2 = 1$ and minimizing B_2/B_1 so that $b_2/b_1 < B_2/B_1$ implies $B_1 = 3$ and $B_2 = 2$. Thus,

$$\left.\begin{array}{l} 0/1 < 1/2 \\ 1/2 < 2/3 \end{array}\right\} \Longrightarrow 1/3 < 3/5.$$

If $b_1 = 3$ then $b_2 = 2$ and minimizing B_2/B_1 implies $B_1 = 1 = B_2$. Therefore,

$$\left.\begin{array}{l} 0/1 < 1/2 \\ 2/3 < 1/1 \end{array}\right\} \Longrightarrow 2/4 < 2/3.$$

If $b_1 = 4$ then $b_2 = 3$ and $B_1 = B_2 = 1$. Therefore,

$$\left.\begin{array}{l} 0/1 < 1/2 \\ 3/4 < 1/1 \end{array}\right\} \Longrightarrow 3/5 < 2/3.$$

If $A_1 = 1$ then $A_2 = 1$ and $b_1 + B_1 \leq 6$. Therefore $b_1$is 2,3, or 4. If $b_1 = 2$, then $b_2 = 1$ and minimizing B_2/B_1 implies $B_1 = 3$ and $B_2 = 2$. Therefore,

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 1/2 < 2/3 \end{array}\right\} \Longrightarrow 1/3 < 3/4.$$

If $b_1 = 3$, then $b_2 = 2$ and $b_2/b_1 < B_2/B_1$ implies $B_1 = B_2 = 1$ since $B_1 \leq 3$. Thus,

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 2/3 < 1/1 \end{array}\right\} \Longrightarrow 2/4 < 1/1.$$

Note if $b_1 = 3$ and $b_2 = 1$, then minimizing B_2/B_1 implies $B_1 = 2$ and $B_2 = 1$. Therefore,

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 1/3 < 1/2 \end{array}\right\} \Longrightarrow 1/4 < 2/3.$$

If $b_1 = 4$, then the only case when $\dfrac{A_2 + B_2}{A_1 + B_1} \neq 1$ is when $b_2 =$. Then $B_1 = 2$ and $B_2 = 1$. Then

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 1/4 < 1/2 \end{array}\right\} \Longrightarrow 1/5 < 2/3.$$

Hence, if zero numerators are allowed, then the smallest dataset in which Simpson's Paradox can happen is nine.

Comments by the **Michael N Fried of Kibbutz Revivim, Israel** and by **Lawrence Lesser,** the proposer.

Michael: The inequalities need not be strict, we have for example Bob 1/1, 2/10 and Al 1/2, 1/5. So Bob does better OR AS WELL as Al, while the total for Bob is 3/11, is worse than the total for Al, 2/7. Under this assumption, the total number of questions is 1+10+2+5=18<19.

Michael went on to say that these numbers can be represented as slopes of lines, i.e., the slopes of the lines from (0,0) to (1,1) and (10,2) are great than those from (0,0) to (2,1) and (5,1), while the slope of the line given by the vector sum of (1,1) and (10,2) is less than that given by the vector sum of (2,1) and (5,1).

Lawrence: By allowing equality we could actually get it all the way down to 9 (e.g, Bob 1/1, 2/4; Al 1/2, 1/2) but almost every formulation of the problem that I have seen maintains strict inequality.

Slopes of lines is one of many representations of problem that I complied in my chapter in the 2001 NCTM yearbook, <http://www.statlit.org/pdf/2001LesserNCTM. pdf>

- **5322:** *Proposed by D.M. Bătinetu-Girugiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "G.E. Palade", School, Buzău, Romania*

If $\lim\limits_{n\to\infty}\left(-\dfrac{3}{2}\sqrt[3]{n^2} + \displaystyle\sum_{k=1}^{n}\dfrac{1}{\sqrt[3]{k}}\right) = a > 0$, then compute $\lim\limits_{n\to\infty}\left(\dfrac{-\dfrac{3}{2}\sqrt[3]{n^2} + \displaystyle\sum_{k=1}^{n}\dfrac{1}{\sqrt[3]{k}}}{a}\right)^{\sqrt[3]{n}}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $[x]$ be the greatest integer not exceeding x. It is easy to prove by induction that for positive integers n,

$$\sum_{k=1}^{n} k^{-1/3} - \frac{3}{2}n^{2/3} = b + \frac{1}{2}n^{-1/3} + \frac{1}{3}\int_n^\infty \left(t - [t] - \frac{1}{2}\right) t^{-4/3} dt \qquad (1)$$

where $b = -\left(\frac{1}{2}+\frac{1}{3}\int_1^\infty (t-[t])\, t^{-4/3} dt\right)$. The constant b is finite since

$\left|\int_1^\infty (t-[t])\, t^{-4/3} dt\right| \le \int_1^\infty t^{-4/3} dt = 3$. Moreover it is negative by (1), $a = b$. For

$t \ge 0$, let $f(t) = \int_0^t \left(x-[x]-\frac{1}{2}\right) dx$. For any integer k, we have

$\int_k^{k+1} \left(x-[x]-\frac{1}{2}\right) dx = 0$, and so $f(t) = O(1)$. Integrating by parts, we see that the

integral in (1) equals $\frac{4}{3}\int_n^\infty f(t) t^{-7/3} dt = O\left(n^{-4/3}\right)$. Hence by (1), we have

$$\sqrt[3]{n} \lim_{n\to\infty} \left(\frac{-\frac{3}{2}\sqrt[3]{n^2}+\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}}{a}\right) = \sqrt[3]{n}\ln\left(1+\frac{1}{2a}n^{-1/3}+O\left(n^{-4/3}\right)\right) = \frac{1}{2a}+O\left(n^{-1/3}\right),$$

as $n \to \infty$. It follows that the limit of the problem equals $\mathrm{e}^{1/2a}$.

Solution 2 by Nicusor Zlota "Traian Vuia" Technical College, Focsani, Romania

We have the case of 1^∞.

Denoting $a_n = -\frac{3}{2}\sqrt[3]{n}^2 + \sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}$, we may write the limit as:

$$l = \lim_{n\to\infty}\left(1+\frac{a_n}{a}\right)^{\sqrt[3]{n}} = \lim_{n\to\infty}\left[\left(1+\frac{a_n-a}{a}\right)^{\frac{a}{a_n-a}}\right]^{\frac{a_n-a}{a}\sqrt[3]{n}} = e^{\lim_{n\to\infty}\frac{a_n-a}{a}\sqrt[3]{n}}$$

For $l_1 = \lim_{n\to\infty}\frac{a_n-a}{a}\sqrt[3]{n} = \frac{1}{a}\lim_{n\to\infty}\frac{a_n-a}{\frac{1}{\sqrt[3]{n}}}$, and by the Cesaro -Stolz lemma, we have successively:

$$l_1 = \frac{1}{a}\lim_{n\to\infty}\frac{a_{n+1}-a_n}{\frac{1}{\sqrt[3]{n+1}}-\frac{1}{\sqrt[3]{n}}} = \frac{1}{a}\lim_{n\to\infty}\frac{-\frac{3}{2}\sqrt[3]{(n+1)^2}+\frac{1}{\sqrt[3]{n+1}}=\frac{3}{2}\sqrt[3]{n}}{\frac{1}{\sqrt[3]{n+1}}-\frac{1}{\sqrt[3]{n}}}$$

$$l_1 = \frac{1}{2a}\lim_{n\to\infty}\frac{\left(3n+1-3\sqrt[3]{n^2(n+1)}\right)\sqrt[3]{n}}{\sqrt[3]{n+1}-\sqrt[3]{n}}$$

$$= \frac{1}{2a} \lim_{n\to\infty} \frac{(9n+1)\left(\sqrt[3]{n(n+1)^2} + \sqrt[3]{n^2(n+1)} + n\right)}{(3n+1)^2 + (9n+3)\sqrt[3]{n^2(n+1)} + 9n\sqrt[3]{n(n+1)^2}} = \frac{1}{2a}.$$

Therefore the limit is $l = 2^{1/2a}$.

Generalization:

If $\lim\limits_{n\to\infty}\left(-\frac{p}{p-1}\sqrt[p]{n^{p-1}} + \sum\limits_{k=1}^{n}\frac{1}{\sqrt[p]{k}}\right) = a > 0$, and we wish to compute

$$\lim_{n\to\infty}\left(\frac{-\frac{p}{p-1}\sqrt[p]{n^{p-1}} + \sum\limits_{k=1}^{n}\frac{1}{\sqrt[p]{k}}}{a}\right)^{\sqrt[p]{n}}, p \in N,\ p \geq 2$$

the answer is $e^{1/(p-1)a}$ and its proof is similar to the above.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania, and the proposers.

- **5323:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers greater than or equal to one. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{-2} + \left(\frac{1}{n^2}\prod_{k=1}^{n} a_k^{-2}\right)\left(\sum_{k=1}^{n}\left(a_k^2-1\right)^{1/2}\right)^2 \leq 1.$$

Solution 1 by Moti Levy, Rehovot, Israel

Let $p(x) = (x-1)\left(x - \prod\limits_{j=1}^{n} a_j^2\right)$. Then clearly $p(x) \leq 0$ for $1 \leq x \leq \prod\limits_{j=1}^{n} a_j^2$.

Every a_k^2, satisfies $1 \leq a_k^2 \leq \prod\limits_{j=1}^{n} a_j^2$, hence

$$p\left(a_k^2\right) = \left(a_k^2 - 1\right)\left(a_k^2 - \prod_{j=1}^{n} a_j^2\right) \leq 0, \quad 1 \leq k \leq n. \tag{1}$$

Rearranging the terms in (1), we obtain,

$$\frac{1}{a_k^2} + \left(\prod_{j=1}^{n} a_j^{-2}\right)\left(a_k^2 - 1\right) \leq 1, \quad 1 \leq k \leq n. \tag{2}$$

Taking average of both sides of (2), we get

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k^2} + \left(\prod_{j=1}^{n} a_j^{-2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left(a_k^2 - 1\right)\right) \leq 1. \tag{3}$$

The power mean $M_p(x_1, \ldots, x_n)$, is a mean of the form

$$M_p(x_1, \ldots, x_n) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k^p \right)^{\frac{1}{p}},$$
$$M_0(x_1, \ldots, x_n) = \left(\prod_{k=1}^{n} x_k \right)^{\frac{1}{n}}.$$

The monotonicity property of the power mean is

$$\text{if} \quad p < q, \quad \text{then} \quad M_p((x_1, \ldots, x_n)) \leq M_q((x_1, \ldots, x_n)). \tag{4}$$

By this property $M_{\frac{1}{2}} \leq M_1$, hence

$$\left(\frac{1}{n} \sum_{k=1}^{n} (a_k^2 - 1)^{\frac{1}{2}} \right)^2 \leq \frac{1}{n} \sum_{k=1}^{n} (a_k^2 - 1). \tag{5}$$

By (3) and (5),

$$\frac{1}{n} \sum_{k=1}^{n} \frac{1}{a_k^2} + \left(\prod_{j=1}^{n} a_j^{-2} \right) \left(\frac{1}{n} \sum_{k=1}^{n} (a_k^2 - 1)^{\frac{1}{2}} \right)^2 \leq 1. \tag{6}$$

Since the function $f(x) = \frac{1}{x^2}$ is convex for $x \geq 1$, then by Jensen's inequality

$$\frac{1}{n} \sum_{k=1}^{n} \frac{1}{a_k^2} \geq \frac{1}{\left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^2}. \tag{7}$$

It follows from (6) and (7) that

$$\frac{1}{\left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^2} + \left(\prod_{j=1}^{n} a_j^{-2} \right) \left(\frac{1}{n} \sum_{k=1}^{n} (a_k^2 - 1)^{\frac{1}{2}} \right)^2 \leq 1.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

For $k = 1, 2, \ldots, n$, let $a_k = \sec b_k$, where $0 \leq b_k < \dfrac{\pi}{2}$. Since the function $\sec x$ is convex for $0 < \dfrac{\pi}{2}$, so $\dfrac{1}{n} \displaystyle\sum_{k=1}^{n} a_k \geq \sec \left(\dfrac{\sum_{k=1}^{n} b_k}{n} \right)$. By the concavity of the function $\sin x$ for $0 \leq x < \dfrac{\pi}{2}$, we have

$$\left(\frac{1}{n} \prod_{k=1}^{n} a_k^{-1} \right) \left(\sum_{k=1}^{n} (a_k^2 - 1)^{1/2} \right) = \left(\frac{1}{n} \prod_{k=1}^{n} \cos b_k \right) \left(\sum_{k=1}^{n} \frac{\sin b_k}{\cos b_k} \right) \leq \frac{\sum_{k=1}^{n} \sin b_k}{n} \leq \sin \left(\frac{\sum_{k=1}^{n} b_k}{n} \right).$$

It follows that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{-2} + \left(\frac{1}{n^2}\prod_{k=1}^{n} a_k^{-2}\right)\left(\sum_{k=1}^{n}(a_k^2-1)^{1/2}\right)^2 \le \cos^2\left(\frac{\sum_{k=1}^{n} b_k}{n}\right) + \sin^2\left(\frac{\sum_{k=1}^{n} b_k}{n}\right) = 1,$$

as required.

Also solved by, Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

- **5324:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right) - 1 + \frac{1}{2n}\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\begin{aligned}
\sum_{n=1}^{N} n\ln\left(1+\frac{1}{n}\right) &= \sum_{n=1}^{N} n\ln(n+1) - \sum_{n=1}^{N} n\ln(n) = \sum_{n=1}^{N+1}(n-1)\ln(n) - \sum_{n=1}^{N} n\ln(n) \\
&= N\ln(N+1) - \sum_{n=1}^{N}\ln(n) = N\ln(N) + N\ln\left(1+\frac{1}{N}\right) - \ln(N!) \\
&= N\ln(N) + 1 + O\left(\frac{1}{N}\right) - \ln\left(\sqrt{2\pi N}\right) - N\ln(N) + N + o(1) \\
&= N + 1 - \frac{1}{2}\ln(N) - \frac{1}{2}\ln(2\pi) + o(1), \text{ as } N\to\infty,
\end{aligned}$$

where we have used Stirling's formula in the form $N! = \sqrt{2\pi N}N^N e^{-N+o(1)}$, as $N\to\infty$.

$$\sum_{n=1}^{N} 1 = N$$

$$\sum_{n=1}^{N}\frac{1}{2n} = \frac{1}{2}\ln(N) + \frac{\gamma}{2} + o\left(\frac{1}{N}\right), \text{as } N\to\infty.$$

Collecting results we find that

$$\sum_{n=1}^{N}\left(n\ln\left(1+\frac{1}{n}\right) - 1 + \frac{1}{2n}\right) = N + 1 - \frac{1}{2}\ln(N) - \frac{1}{2}\ln(2\pi) - N + \frac{1}{2}\ln(N) + \frac{\gamma}{2} + o(1)$$

$$= \quad 1-\frac{1}{2}\ln(2\pi)+\frac{\gamma}{2}+o(1), \text{ as } N\to\infty, \text{ and so}$$

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right) \quad = \quad 1-\frac{1}{2}\ln(2\pi)+\frac{\gamma}{2}.$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right)=\lim_{N\to\infty}\sum_{n=1}^{\infty}\left(n\ln\left(n+1\right)-n\ln n-1+\frac{1}{2n}\right)$$

$$= \quad \lim_{N\to\infty}\sum_{n=1}^{N}\left(\left(\left(n+1\right)\ln\left(n+1\right)\right)-n\ln n-\ln(n+1)-1+\frac{1}{2n}\right)$$

$$= \quad \lim_{N\to\infty}\sum_{n=1}^{N}(n+1)\ln(n+1)-n\ln n)-\lim_{N\to\infty}\sum_{n=1}^{N}\ln(n+1)-\lim_{N\to\infty}\sum_{n=1}^{N}1+\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{2n}$$

$$= \quad \lim_{N\to\infty}(N+1)\ln(N+1)-\lim_{N\to\infty}(\ln(N+1)!)-\lim_{N\to\infty}\ln N+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \quad \lim_{N\to\infty}\left(\ln(N+1)^{N}-\ln(N!)-\ln(e^{N})\right)+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \quad \lim_{N\to\infty}\ln\frac{(N+1)^{N}}{N!e^{N}}+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \quad \lim_{N\to\infty}\ln\frac{\dfrac{(N+1)^{N}}{N^{N}}N^{N}\sqrt{N}\dfrac{1}{\sqrt{N}}}{N!e^{N}}+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \quad \lim_{N\to\infty}\ln\frac{\dfrac{\left(1+\dfrac{1}{N}\right)^{N}}{N^{N}}N^{N}\sqrt{N}\dfrac{1}{\sqrt{N}}}{N!e^{N}}+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \quad \lim_{N\to\infty}\left(\ln\left(\left(1+\frac{1}{N}\right)^{N}\right)+\ln\frac{N^{N}\sqrt{N}}{N!e^{N}}+\ln\frac{1}{\sqrt{N}}\right)+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \quad \ln\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^{N}+\ln\lim_{N\to\infty}\frac{N^{N}\sqrt{N}}{N!e^{N}}+\frac{1}{2}\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln N\right)$$

$$= \ln e + \ln\left(\frac{1}{\sqrt{2\pi}}\right) + \frac{\gamma}{2}$$

$$= \frac{1}{2}\left(2 - \ln(2\pi) + \gamma\right),$$

where we have used the Stirling approximation for $N!$ and where γ is the Euler-Mascheroni constant.

Also solved by Ed Gray, Highland Beach, FL; G.E. Greubel, Newport News, VA; Moti Levy, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy, and the proposer.

Late Solutions, Comments, and an Announcement

A late solution to problem #5316 was received from **Raymon M. Melone of Waynesburg University, Waynesburg, PA.**

Comment by **Titu Zvonaru, Comănesti, Romania.** Solution 4 of problem #5317 is incorrect, because inequality (2) in the solution does not hold. For example: If

$$n = 3,\ b_1^{s+2} = 6,\ b_2^{s+2} = 9,\ b_3^{s+2} = 3,\ a_1 = \frac{1}{6},\ a_2 = \frac{1}{2} \text{ and } a_3 = \frac{1}{3}$$

then the LHS$= 36 + 18 + 9 + = 63$, while the RHS$= \frac{1}{3}\left(6 + 2 + 3\right)\left(6 + 9 + 3\right) = 66$.

The Chebyschev inequality maybe applied only if the sequences are both ascending or both descending. Of course, we may assume that one of the sequences is ascending but this assumption does not imply that the second sequence is also ascending:

$$b_1 \geq b_2 \geq \cdots \geq b_n \not\Longrightarrow a_1 \geq a_2 \geq \cdots \geq a_n.$$

For example, the inequality

$$\sum_{k=1}^{n} b_k^{s+2} \geq \frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{k=1}^{n} b_k^{s+1}\right)$$

is correct.

Announcement: Following is part of a letter that was received from **Don Allen of Brossard, Canada.** Don has agreed that I may distribute his pdf file and an accompanying article entitled “The verse problems of early American arithmetics” to anyone who is interested in receiving them. Please send your requests to me at <eisenbt@013.net>

—————————-

Dear Professor Eisenberg:

When we corresponded in late October, I related how SSM program 5314 had reminded me of the more challenging problems routinely posed in nineteenth-century school algebra and arithmetic texts, which I had searched through in a then-uncatalogued collection at the United States university when I was completing doctoral studies – Rutgers, in New Jersey. I copied hundreds of such early "word problems" (authors had been copying one another for decades), and used many of them as challenges for teachers and for abler students. When I was working in Canada's Eastern Arctic decades later, I assembled some of the more satisfying teacher columns that I had prepared for such problems and their suggested solutions, and shared them with able, interested students and their parents on an evening at the library/museum of an appropriate arctic community. I recently located the original of the 30-page handout, I would like to put them at your disposal. You may print any you wish, and use in SSM any you feel appropriate and desirable.

Cordially,

Don Allen
Brossard, Canada

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2015

- **5343:** *Proposed by Kenneth Korbin, New York, NY*

  Four different Pythagorean Triangles each have hypotenuse equal to $4p^4+1$ where p is prime.

  Express the sides of these triangles in terms of p.

- **5344:** *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan*

  Let $\triangle ABC$ be isosceles with $AB = AC$. Let D be a point on side BC. A line through point D intersects rays AB and AC at points E and F respectively. Prove that $ED \cdot DF \geq BD \cdot DC$.

- **5345:** *Proposed by Arkady Alt, San Jose, CA*

  Let $a, b > 0$. Prove that for any x, y the following inequality holds

  $$|a\cos x + b\cos y| \leq \sqrt{a^2+b^2+2ab\cos(x+y)},$$

  and find when equality occurs.

- **5346:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Show that in any triangle ABC, with the usual notations, the following hold,

  $$\frac{h_b+h_c}{h_a}r_a^2 + \frac{h_c+h_a}{h_b}r_b^2 + \frac{h_a+h_b}{h_c}r_c^2 \;\geq\; 2s^2,$$

  where r_a is the excircle tangent to side a of the triangle and s is the triangle's semiperimeter.

- **5347:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $0 < a < b$ be real numbers and let $f, g : [a,b] \to R_+^*$ be continuous functions. Prove that there exists $c \in (a,b)$ such that

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)\,dt}\right)\left(g(c) + \int_a^c f(t)\,dt\right) \geq 4$$

(R_+^* represents the set of non-negative real numbers.)

- **5348:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Prove that

$$\int_0^1 \ln^k(1-x)\ln x \,\mathrm{d}x = (-1)^{k+1}k!(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)),$$

where ζ denotes the Riemann zeta function.

Solutions

- **5325:** *Proposed by Kenneth Korbin, New York, NY*

Given the sequence $x = (1, 7, 41, 239, 1393, 8119, \dots)$, with $x_n = 6x_{n-1} - x_{n-2}$.

Let $y = \dfrac{x_{2n} + x_{2n-1}}{x_n}$. Find an explicit formula for y expressed in terms of n.

Solution by 1 D.M. Bătinetu-Giurgiu, National College "Matei Basarab," Bucharest, Romania

The recurrence sequence x_n has the equation $r^2 - 6r + 1 = 0$ with solutions

$$\begin{aligned} r_1 &= \left(\sqrt{2}+1\right)^2, \ r_2 = \left(\sqrt{2}-1\right)^2, \text{ so} \\ x_n &= ur_1^n + vr_2^n = \left(\sqrt{2}+1\right)^{2n} u + \left(\sqrt{2}-1\right)^{2n} v, \end{aligned}$$

and because $x_1 = 1,\ x_2 = 7$ yields that

$$(u,v) = \left(\frac{\sqrt{2}-1}{2}, -\frac{\sqrt{2}+1}{2}\right).$$

Therefore,

$$\begin{aligned} x_n &= \frac{\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}}{2}, \text{ and} \\ y_n &= \frac{x_{2n} + x_{2n-1}}{x_n} = 2\sqrt{2}\left(\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right), \end{aligned}$$

and we are done.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

In our solution to Problem 5308 (see Dec. 2014 issue of this column), we used the techniques for solving homogeneous linear difference equations to show that the closed form expression for x_n is

$$x_n = \frac{\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}}{2}$$

for all $n \geq 1$. It follows that for all $n \geq 1$,

$$\begin{aligned}
x_{2n} + x_{2n-1} &= \frac{\left(\sqrt{2}+1\right)^{4n-1} - \left(\sqrt{2}-1\right)^{4n-1}}{2} + \frac{\left(\sqrt{2}+1\right)^{4n-3} - \left(\sqrt{2}-1\right)^{4n-3}}{2} \\
&= \frac{\left(\sqrt{2}+1\right)^{4n-3}\left[\left(\sqrt{2}+1\right)^{2}+1\right] - \left(\sqrt{2}-1\right)^{4n-3}\left[\left(\sqrt{2}-1\right)^{2}+1\right]}{2} \\
&= \frac{\left(\sqrt{2}+1\right)^{4n-3}\left[2\left(2+\sqrt{2}\right)\right] - \left(\sqrt{2}-1\right)^{4n-3}\left[2\left(2-\sqrt{2}\right)\right]}{2} \\
&= \left(\sqrt{2}+1\right)^{4n-3}\left(2+\sqrt{2}\right) - \left(\sqrt{2}-1\right)^{4n-3}\left(2-\sqrt{2}\right) \\
&= \sqrt{2}\left[\left(\sqrt{2}+1\right)^{4n-2} - \left(\sqrt{2}-1\right)^{4n-2}\right] \\
&= \sqrt{2}\left[\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right]\left[\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= 2\sqrt{2}x_n\left[\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= \frac{x_{2n} + x_{2n-1}}{x_n} \\
&= 2\sqrt{2}\left[\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}\right]
\end{aligned}$$

for all $n \geq 1$.

Solution 3 by G. C. Greubel, Newport News, VA

First consider the difference equation

$$x_{n+2} = 6x_{n-1} - x_n \tag{1}$$

which has the general solution $x_n = Aa^{2n} + Bb^{2n}$ where $a = 1+\sqrt{2}$ and $b = 1-\sqrt{2}$. For the initial conditions $x_0 = 1$ and $x_1 = 7$ the sequence x_n has the solution $x_n = Q_{2n+1}/2$, where Q_n are the Pell-Lucas numbers with the recurrence relation $Q_{n+2} = 2Q_{n+1} + Q_n$. The element $x_{2n+1} + x_{2n-1}$ can be determined to be $4P_{4n}$, where P_n are the Pell numbers. This leads to the desired quantity being sought as

$$y_n = \frac{8P_{4n}}{Q_{2n+1}}. \tag{2}$$

Comment by Henry Ricardo, New York Math Circle, NY. The numbers x_n in the proposed problem are the **NSQ numbers** (named for Newman, Shanks, and Williams, authors of an influential 1980 group theory paper.) The *On-Line Encyclopedia of Integer Sequences* (OEIS) lists the sequence as entry $A002315$ and gives the formula (without proof)

$$x_n = \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n}{2} = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2},$$

for non-negative integers n. In addition to many comments on the sequence itself, the connection between this sequence and other OEIS entries are also pointed out.

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student at Taylor University), Upland IN; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Tsvetelina Karamfilova, Petko Rachov Slaveikov Secondary School, Kardzhali, Bulgaria; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis of the University of Tennessee at Martin, TN; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Angel Plaza, Universidad de Las Palmas, de Gran, Canaria, Spain; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Albert Stadler, Herrliberg, Switzerland,

- **5326:** *Proposed by Armend Sh. Shabani, University of Prishtina, Republic of Kosova*

Find all positive integer solutions to $m! + 2^{4k-1} = l^2$.

Solution 1 by Ed Gray, Highland Beach, FL

We note that $2^{(4k-1)}$ always seems to end with the integer 8 for all values of k. We prove this by induction. The statement is obviously true for $k = 1$. Assume that the statement is true for all positive integers up to and including k. I.e. $2^{(4k-1)}$ ends in with the integer 8. Does this $2^{(4k-1)}$ imply that $2^{(4(k+1)-1)}$ also ends with the integer 8?

$$2^{(4(k+1)-1)} = 2^{(4k+3)} = 2^4\left(2^{(4k-1)}\right) = 16\left(2^{(4k-1)}\right).$$

But by the induction hypothesis, $2^{(4k-1)}$ ends with integer 8 and so $16\left(2^{(4n-1)}\right)$ also ends with the integer 8.

Now we note that for all integers $m \geq 5$, the integer $m!$ ends with the integer 0. So, $m! + 2^{(4k-1)}$ ends in 8 for all integers $m \geq 5$. But there is no square number whose units digit is 8. So if there are any integer solutions to $m! + 2^{4k-1} = l^2$, the value of the positive integer m must be 1, 2, 3, or 4.

If $m = 4$, then $m! = 24$ ends in a 4 and so $m! + 2^{4k-1}$ ends with the unit's digit in 4+8, so l^2 must end in 2, but there is no integer whose square ends with a 2 So $m \neq 4$.
If $m = 3$, then $m! = 6$ and so $m! + 2^{4k-1}$ ends with a unit's digit of 6+8. That is, the units digit of l^2 must be 4, which implies that l must be even. Suppose that $l = 2r$. Then

$$6 + 2^{4k-1} = l^2$$

$$
\begin{aligned}
6 + 2^{4k-1} &= (2r)^2 \\
6 + 2^{4k-1} &= 4r^2
\end{aligned}
$$

But 4 divides the right hand side and 4 divides 2^{4k-1}, but 4 does not divide 6 so, $m \neq 3$.

If $m = 2$, then $m! = 4$ and so $m! + 2^{4k-1}$ ends with a unit's digit of 2, but there is no integer square has a units digit of 2. So,$m \neq 2$.

Finally, if $m = 1$ then $m! + 2^{4k-1}$ becomes

$$
\begin{aligned}
1 + 2^{4k-1} &= l^2 \\
2^{4k-1} &= l^2 - 1 \\
2^{4k-1} &= (l-1)(l+1).
\end{aligned}
$$

So, both factors $(l-1)$ and $(l+1)$ must be a power of 2.

Let $l - 1 = 2^a$ and $l + 1 = 2^b$. Subtracting gives $2 = 2^b - 2^a$ whose only solution is $b = 2$ and $a = 1$. So $l - 1 = 2^1 = 2$ and $l + 1 = 2^2$

Since

$$
\begin{aligned}
2^{4k-1} &= (l-1)(l+1) \\
2^{4k-1} &= (2)(4) \\
2^{4k-1} &= (2^3), \text{ so,} \\
k &= 1.
\end{aligned}
$$

The only solution to $m! + 2^{4k-1} = l^2$ is when $m = 1, k = 1$ and $l = 3$.

Solution 2 by Jerry Chu, (student at Saint George's School), Spokane, WA

We note that $2^{4k-1} \mod 3$ is 2. And $l^2 \mod 3$ is either 0 or 1. So, $m!$ must not be a multiple of 3. Therefore, $m = 1$ or 2.

When $m = 1$, $2^{4k-1} = l^2 - 1 = (l+1)(l-1)$.

Because $(l-1)$ and $(l+1)$ can only be powers of 2, l must equal 3. So
$m = 1, k = 1, l = 3$.

When $m = 2$, we take all terms mod 4 and see that $2 + 0 = 0 + 1$, which is impossible.

Therefore the only solution is $m = 1, k = 1, l = 3$.

Solution 3 by Adnan Ali (student in A.E.C.S-4), Mumbai, India

Assume that for $m \geq 3$, there exist solutions. Then putting the equation modulo 3, we see that

$$l^2 = m! + 2^{4k-1} \equiv (-1)^{4k-1} \equiv -1 \pmod 3$$

but -1 is not a quadratic residue modulo 3. So we conclude that $m \leq 2$. But now we may assume that there is a solution for $m = 2$, then we simply realize the fact that

$$l^2 = 2! + 2^{4k-1} \equiv 2 + 0 \pmod 4,$$

and since 2 is not a quadratic residue modulo 4, we are left with the only option $m = 1$. So, we have

$$2^{4k-1} + 1 = l^2 \Leftrightarrow (l+1)(l-1) = 2^{4k-1}$$

and so we must have both $l+1, l-1$ as powers of 2, so we let $l + 1 = 2^a > l - 1 = 2^b$ for integers a, b so that $a + b = 4k + 1$ and see that $2^a - 2^b = 2 = 2^b(2^{a-b} - 1)$ forcing $2^b = 2$ and $2^{a-b} - 1 = 1$ which has the only solution $(a, b) = (2, 1)$ and $2 + 1 = 3 = 4k - 1$ implies that $k = 1$.
So we conclude that the only possible solution is $(l, m, k) = (3, 1, 1)$.

Solution 4 by Henry Ricardo, New York Math Circle, NY

The triple $(m, k, l) = (1, 1, 3)$ is the only solution in positive integers.

To prove this assertion, we use the following easily established facts: (1) $2^{4k-1} \equiv 8 \pmod{10}$ for positive integers k; (2) If $l^2 \equiv r \pmod{10}$, then $r \in S = \{0, 1, 4, 5, 6, 9\}$.

First, if $m \geq 5$, then $m! \equiv 0 \pmod{10}$ so that $m! + 2^{4k-1} \equiv 8 \pmod{10}$. But $8 \notin S$. Thus $1 \leq m < 5$.

If $m = 2$, then $N = m! + 2^{4k-1} = 2(1 + 2^{4k-2})$, which can't be a perfect square since $1 + 2^{4k-2}$ is odd, implying that the prime divisor 2 does not appear with an even exponent in the prime power factorization of N. Similarly, if $m = 3$, then $m! + 2^{4k-1} = 2(3 + 2^{4k-2})$, which can't be a perfect square.

Finally, we eliminate $m = 4$ since $m! + 2^{4k-1} = 24 + 2^{4k-1} \equiv 4 + 8 \equiv 2 \pmod{10}$ and $2 \notin S$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Ethan Gegner (student, Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Bruno Salgueiro Fanego, Viveiro, Spain; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5327:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Show that in any triangle ABC, with the usual notations, that

$$\left(\frac{ab}{a+b}\right)^2 + \left(\frac{bc}{b+c}\right)^2 + \left(\frac{ca}{c+a}\right)^2 \geq 9r^2.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{ab}{a+b}\right)^2+\left(\frac{bc}{b+c}\right)^2+\left(\frac{ca}{c+a}\right)^2\geq\frac{1}{3}\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)^2.$$

Hence it suffices to show that

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\geq 3\sqrt{3}r. \qquad (1)$$

Let s be the semiperimeter and F the area of triangle ABC. It is well known that

$$F=rs=\frac{ab\sin C}{2}=\frac{bc\sin A}{2}=\frac{ca\sin B}{2}.$$ So (1) is equivalent to

$$(a+b+c)\left(\frac{1}{(a+b)\sin C}+\frac{1}{(b+c)\sin A}+\frac{1}{(c+a)\sin B}\right)\geq 3\sqrt{3}. \qquad (2)$$

Let $S_1=\dfrac{1}{\sin A}+\dfrac{1}{\sin B}+\dfrac{1}{\sin C}$ and $S_2=\dfrac{a}{\sin A(b+c)}+\dfrac{b}{\sin B(c+a)}+\dfrac{c}{\sin C(a+b)}$ so that the left side of (2) can be written as S_1+S_2. By the convexity of the function $\dfrac{1}{\sin x}$, for $0<x<\pi$, we have $S_1\geq 3\left(\dfrac{1}{\sin\left(\frac{A+B+C}{3}\right)}\right)=2\sqrt{3}$. By the sine formula, we have

$$\begin{aligned}
S_2 &= \frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}+\frac{1}{\sin A+\sin B}\\
&= \frac{1}{2}\left(\frac{1}{\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}+\frac{1}{\sin\left(\frac{C+A}{2}\right)\cos\left(\frac{C-A}{2}\right)}+\frac{1}{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\right)\\
&\geq \frac{1}{2}\left(\frac{1}{\sin\left(\frac{B+C}{2}\right)}+\frac{1}{\sin\left(\frac{C+A}{2}\right)}+\frac{1}{\sin\left(\frac{A+B}{2}\right)}\right)\\
&= \frac{1}{2}\left(\sec\left(\frac{A}{2}\right)+\sec\left(\frac{B}{2}\right)+\left(\frac{C}{2}\right)\right).
\end{aligned}$$

Hence by the convexity of the function $\sec x$ for $0<x<\dfrac{\pi}{2}$, we have

$$S_2\geq\frac{3}{2}\sec\left(\frac{A+B+C}{6}\right)=\sqrt{3}.$$

Thus (2) holds and this completes the solution.

Solution 2 by Perfetti Paolo, Department of Mathematics, University Tor Vergata, Rome, Italy

The Cauchy–Schwarz inequality yields

$$\left(\frac{ab}{a+b}\right)^2+\left(\frac{bc}{b+c}\right)^2+\left(\frac{ca}{c+a}\right)^2 \geq \frac{(ab+bc+ca)^2}{(a+b)^2+(b+c)^2+(c+a)^2} \geq 9r^2.$$

where $r=\sqrt{(s-a)(s-b)(s-c)/s}$, and $s=(a+b+c)/2$.

Letting $x=(b+c-a)/2$, using the symmetry in the statement of the problem and upon clearing the denominators we obtain

$$\frac{1}{A}\sum_{\text{sym}}\left(17x^3y^2+\frac{1}{2}x^5+7x^4y-9x^3yz-\frac{31}{2}x^2y^2z\right)\geq 0$$

and $A=(3(x^2+y^2+z^2)+5(xy+yz+zx))(x+y+z)>0$. Muirhead's theorem concludes the proof. Indeed

$$[3,2,0]\succ[2,2,1], \quad [5,0,0]\succ[3,1,1], \quad [4,1,0]\succ[3,1,1]$$

The underlying AGM's are

$$x^3y^2+x^3z^3\geq 2x^3yz, \quad 3x^5+y^5+z^5\geq 5x^3yz, \quad 9x^4y+y^4z+3z^4x\geq 13x^3yz$$

and symmetry.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Nikos Kalapodis (four solutions), Patras, Greece; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota (two solutions) "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5328:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the aid of a computer, find the positive solutions of the equation

$$2^{x+1}\left(1-\sqrt{1+x^2+2^x}\right)=\left(x^2+2^x\right)\left(1-\sqrt{1+2^{x+1}}\right).$$

Solution 1 by Junho Chang, Colegio Hispano-Inglés, and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Multiplying both terms of the given equation by $\left(1+\sqrt{1+x^2+2^x}\right)\left(1+\sqrt{1+2^{x+1}}\right)$ and simplifying we obtain

$$\begin{aligned}\sqrt{1+2^{x+1}} &= \sqrt{1+x^2+2^x}\\ 2^{x+1} &= x^2+2^x\\ 2^x &= x^2.\end{aligned}$$

Taking logarithms, the last equation may be written as $\dfrac{\ln x}{x}=\dfrac{\ln 2}{2}$. Let us consider the function $f(x)=\dfrac{\ln x}{x}$ defined for positive real numbers x. Since $f'(x)=\dfrac{1-\ln x}{x^2}$, $f(x)$ is increasing for $x \in (0, e)$ and it is decreasing for $x \in (e, +\infty)$. Since $\lim_{x \to 0^+} f(x) = -\infty$, and $f(e) = 1/e > \ln 2/2$ there is a unique root to the equation $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$ in $(0, e)$, which is $x = 2$. Also, since $\lim_{x \to +\infty} f(x) = 0$, there is a unique root to the equation $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$ in $(e, +\infty)$, which is $x = 4$. So, $x = 2$ and 4 are the only positive solutions to the problem.

Solution 2 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria

For convenience we set $a = 2^{x+1}, b = x^2 + 2^x$ then (1) is equivalent to

$$a\left(1 - \sqrt{1+b}\right) = b\left(1 - \sqrt{1+a}\right) \Leftrightarrow \frac{ab}{1+\sqrt{1+b}} = \frac{ab}{1+\sqrt{1+a}}.$$

Since $ab \neq 0$, we get

$$\begin{aligned} \frac{1}{1+\sqrt{1+b}} = \frac{1}{1+\sqrt{1+a}} &\Longrightarrow 1+\sqrt{1+b} = 1+\sqrt{1+a} \\ &\Longrightarrow a = b \end{aligned}$$

Which means that $2^{x+1} = 2^x + x^2$ then 2^x taking ln of both sides we get

$$\frac{\ln x}{x} = \frac{\ln 2}{2}, \quad x > 1$$

Let $f : (1, \infty) \mapsto R$ be defined by $f(x) = \frac{\ln x}{x} - \frac{\ln 2}{2}$, then $f'(x) = x^{-2}(1 - \ln x)$ then f cannot have more than two roots (since f increases on $(1, e)$ and decreases on $(e, +\infty)$) and since $2, 4$ are obvious roots we conclude that the only positive solutions to the equation (1) are $2, 4$.

Also solved by Adnan Ali (student in A.E.C.S-4), Mumbai, India; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu, (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5329:** *Proposed by Arkady Alt, San Jose, CA*

Find the smallest value of $\dfrac{x^3}{x^2+y^2} + \dfrac{y^3}{y^2+z^2} + \dfrac{z^3}{z^2+x^2}$ where real $x, y, z > 0$ and $xy + yz + zx = 1$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Since

$$
\begin{aligned}
& \frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} \\
= & \frac{1}{2}\left(\left(2x-y+\frac{y(x-y)^2}{x^2+y^2}\right)+\left(2y-z+\frac{z(y-z)^2}{y^2+z^2}\right)+\left(2z-x+\frac{x(z-x)^2}{z^2+x^2}\right)\right) \\
\geq & \frac{1}{2}\left((2x-y)+(2y-z)+(2z-x)\right) \\
= & \frac{x+y+z}{2} \\
= & \frac{1}{2\sqrt{2}}\sqrt{6(xy+yz+zx)+(x-y)^2+(y-z)^2+(z-x)^2} \\
= & \frac{1}{2\sqrt{2}}\sqrt{6} \\
= & \frac{\sqrt{3}}{2},
\end{aligned}
$$

and equality holds when $x=y=z=\dfrac{1}{\sqrt{3}}$, so the smallest value required is $\dfrac{\sqrt{3}}{2}$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since real $x,y,z>0$ and $xy+yz+zx=1$, there is an acute triangle ABC such that $\cot A=x, \cot B=y$ and $\cot C=z$ so

$$
\begin{aligned}
& \frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} \\
= & \cot A-\frac{\cot A\cot^2 B}{\cot^2 A+\cot^2 B}+\cot B-\frac{\cot B\cot^2 C}{\cot^2 B+\cot^2 C}+\cot C-\frac{\cot C\cot^2 A}{\cot^2 C+\cot^2 A} \\
\geq & \cot A+\cot B+\cot C-\frac{\cot A\cot^2 B}{2\cot A\cot B}-\frac{\cot B\cot^2 C}{2\cot B\cot C}-\frac{\cot C\cot^2 A}{2\cot C\cot A} \\
= & \frac{1}{2}\left(\cot A+\cot B+\cot C\right) \\
\geq & \frac{\sqrt{3}}{2}
\end{aligned}
$$

with equality iff $\cot A=\cot B=\cot C$ and $A=B=C=\pi/3$, that is iff $x=y=z=\dfrac{1}{\sqrt{3}}$, where we have used that $\cot A, \cot B > 0, (\cot A - \cot B)^2 \geq 0$ with equality iff $\cot A = \cot B$ and cyclically, and inequality 2.38 page 28, *Geometric Inequalities*, Bottema O., Djordjević, R.Ž., Janić, R.R., Mitrinović, D.S. Vasić, P.M., Wolters-Noordhoff, , Groningen, 1969.

Solution 3 by Henry Ricardo, New York Math Circle, NY

The AGM inequality gives us

$$\frac{x^3}{x^2+y^2} \;=\; x - \frac{xy^2}{x^2+y^2} \;\geq\; x - \frac{xy^2}{2xy} \;=\; x - \frac{y}{2}.$$

Similarly, we get

$$\frac{y^3}{y^2+z^2} \;\geq\; y - \frac{z}{2} \quad \text{and} \quad \frac{z^3}{z^2+x^2} \;\geq\; z - \frac{x}{2}.$$

Adding these three inequalities, we see that

$$f(x,y,z) \;=\; \frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \;\geq\; \frac{x+y+z}{2}. \qquad (A)$$

Now we have

$$(x+y+z)^2 \;=\; x^2+y^2+z^2+2(xy+yz+zx) \;=\; x^2+y^2+z^2+2,$$

so $x+y+z = \sqrt{x^2+y^2+z^2+2} \;\geq\; \sqrt{3}$, where we have used the well-known inequality $x^2+y^2+z^2 \;\geq\; xy+yz+zx$.

Thus $f(x,y,z) \;\geq\; \dfrac{\sqrt{3}}{2}$, with equality if and only if $x=y=z=1/\sqrt{3}$.

Editor's comment : The author also provided a second solution to the above problem. It starts off exactly as the one above up until statement A. Then:

$$\begin{aligned} f(x,y,z) \;=\; \frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \;&\geq\; \frac{x+y+z}{2} \;=\; \frac{3}{2}\left(\frac{x+y+z}{3}\right) \\ &\geq \frac{3}{2}\left(\frac{xy+yz+zx}{3}\right)^{1/2} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}. \end{aligned}$$

Thus $f(x,y,z) \;\geq\; \dfrac{\sqrt{3}}{2}$, with equality if and only if $x=y=z=1/\sqrt{3}$.

Solution 4 by Albert Stadler, Herrliberg, Switzerland

Suppose that $xy+y+zx=1$. We claim that

$$\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \quad \geq \frac{\sqrt{3}}{2}, \qquad (1)$$

with equality if and only if $x=y=z=\dfrac{1}{\sqrt{3}}$. By homogeneity, (1) is equivalent to the unconditional inequality

$$\frac{1}{\sqrt{xy+yz+zx}}\left(\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2} \geq \frac{\sqrt{3}}{2}\right). \qquad (2)$$

We first note that

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx \geq 3\,(xy+yz+zx)\,,$$

since by the Cauchy-Schwarz Inequality, $x^2+y^2+z^2 \geq xy+yz+zx$, with equality if and only if $x=y=z$.

So

$$\frac{1}{\sqrt{xy+yz+zx}}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right) \geq \frac{\sqrt{3}}{x+y+z}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right).$$

To prove (2) it is therefore enough to prove that

$$\frac{1}{x+y+z}\left(\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right) \geq \frac{1}{2} \tag{3}$$

with equality if and only if $x=y=z$.

Clearing denominators we see that (3) is equivalent to

$$\sum_{cycl} x^5y^2 + \sum_{cycl} x^2y^5 + \sum_{cycl} x^4y^3 \geq \sum_{cycl} x^3y^4 + \sum_{cycl} x^4y^2z + \sum_{cycl} x^4yz^2. \tag{4}$$

By the weighted AM-GM inequality,

$$\frac{1}{2}x^2y^2 + \frac{1}{2}x^4y^3 \geq x^3y^4,$$

$$\frac{3}{19}x^2y^5 + \frac{2}{19}y^3z^5 + \frac{14}{19}z^2x^5 \geq x^4yz^2,$$

$$\frac{1}{2}x^5y^2 + \frac{1}{2}x^3z^4 \geq x^4y^{z2},$$

$$\frac{10}{19}x^5y^5 + \frac{3}{76}y^5z^2 + \frac{7}{38}z^5x^2 + \frac{1}{4}x^4y^3 \geq x^4y^2z.$$

We conclude that

$$\frac{1}{2}\sum_{cycl} x^2y^5 + \frac{1}{2}\sum_{cycl} x^4y^3 \geq \sum_{cycl} x^3y^4, \tag{5}$$

$$\frac{1}{2}\sum_{cycl} x^2y^5 = \frac{1}{2}\left(\frac{3}{19}\sum_{cycl} x^2y^5 + \frac{2}{19}\sum_{cycl} y^2z^5 + \frac{14}{19}\sum_{cycl} z^2x^5\right) \geq \frac{1}{2}\sum_{cycl} x^4yz^2, \tag{6}$$

$$\frac{1}{4}\sum_{cycl} x^5y^2 + \frac{1}{4}\sum_{cycl} x^4y^3 = \frac{1}{4}\sum_{cycl} x^5y^2 + 14\sum_{cycl} x^3z^4 \geq \frac{1}{2}\sum_{cycl} x^4yz^3, \tag{7}$$

$$\frac{3}{4}\sum_{cycl} x^5y^2 + \frac{1}{4}\sum_{cycl} x^4y^3 = \frac{10}{19}\sum_{cycl} x^5y^2 + \frac{3}{76}\sum_{cycl} x^5z^2 + \frac{7}{38}\sum_{cycl} z^5x^2 + \frac{1}{4}\sum_{cycl} x^4y^3 \geq 4\sum_{cycl} x^4y^2z. \qquad (8)$$

Condition (4) follows by adding (5),(6),(7), and (8). Equality holds if and only if $x = y = z$. (This is the equality condition for weighted AM-GM inequalities.)

Also solved by Adnan Ali (student, in A.E.C.S-4), Mumbai, India; Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy; Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania; Nicusor Zlota (plus a generalization) "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5330:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $B(x) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$ and let $n \geq 2$ be an integer.

Calculate the matrix product

$$B(2)B(3)\cdots B(n).$$

Solution 1 by Neculai Stanciu, "George Emil Palade School," Buzău, Romania (jointly with) Titu Zvonaru, Comănesti, Romania

We denote A(n)=B(1)B(2)...B(n). We have

$$A(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A(2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

We assume that

$$A(n) = \begin{pmatrix} \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \\ \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \end{pmatrix}. \qquad (1)$$

Since

$$A(n) = \begin{pmatrix} \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \\ \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \end{pmatrix}\begin{pmatrix} n+1 & 1 \\ 1 & n+1 \end{pmatrix} = \begin{pmatrix} \frac{(n+2)!}{2} & \frac{(n+2)!}{2} \\ \frac{(n+2)!}{2} & \frac{(n+2)!}{2} \end{pmatrix},$$

we have shown, by mathematical induction that (1) holds for all integers $n \geq 2$.

Solution 2 by Moti Levy, Rehovot, Israel

Let $B(x) = xI + A$, where A is an involute matrix (i.e., $A^2 = I$).
Let $P_n = B\,(2)\,B\,(3)\cdots B\,(n)$.

Since A is an involute matrix then

$$P_n = \alpha_n I + \beta_n A,$$
$$P_2 = 2I + A.$$

$$P_{n+1} = P_n B(n+1) = (\alpha_n I + \beta_n A)\left((n+1)\, I + A\right).$$

A recurrence formula for α_n, β_n is

$$\begin{aligned} \alpha_{n+1} &= (n+1)\,\alpha_n + \beta_n \\ \beta_{n+1} &= (n+1)\,\beta_n + \alpha_n \\ \alpha_2 &= 2, \quad \beta_2 = 1. \end{aligned}$$

Let $x_n = \alpha_n - \beta_n$ and $y_n = \alpha_n + \beta_n$, then

$$\begin{aligned} x_{n+1} &= n x_n, \\ y_{n+1} &= (n+2)\, y_n, \\ x_2 &= 1, \quad y_2 = 3. \end{aligned}$$

The solution for x_n, y_n is

$$x_n = (n-1)!, \quad y_n = \frac{1}{2}\,(n+1)!.$$

Solving for α_n, β_n,

$$\begin{aligned} \alpha_n &= \frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)!, \\ \beta_n &= \frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)!. \end{aligned}$$

For any involutory matrix A,

$$P_n = \left(\frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)!\right) I + \left(\frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)!\right) A.$$

For the special case $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the solution is

$$B\,(2)\,B\,(3)\cdots B\,(n) = \begin{pmatrix} \frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)! & \frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)! \\ \frac{1}{4}\,(n+1)! - \frac{1}{2}\,(n-1)! & \frac{1}{4}\,(n+1)! + \frac{1}{2}\,(n-1)! \end{pmatrix}.$$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Jerry Chu (student at Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student at Taylor University), Upland IN; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Carl Libis of the University of Tennessee at Martin, TN; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Haroun

Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Aoxi Yao (student at Saint George's School), Spokane, WA; Ricky Wang, (student at Saint George's School), Spokane, WA, and the proposer.

Late Solutions

A late solution was received to problem #5319 and to 5321 by **Adnan Ali (student in A.E.C.S-4), Mumbai, India.**

Solutions to problems 5322, 5323 and 5324 were received from **Arkady Alt of San Jose, CA.** They were received on time but misfiled by me, and his name was inadvertently not listed as having solved these problems in the February 2015 issue of the column. Arkady, I am sorry; mea culpa (once again.)

Solutions to problems 5313, 5314, 5315, and 5318 were also received from **Carl Libis of the University of Tennessee at Martin, TN.** They too were received on time but misfiled by me–again, mea culpa.

Solutions 5320 and to 5322 were also submitted on time by **Albert Stadler of Herrliberg, Switzerland,** and inadvertently and misfiled by me.

And for the files submitted by **Moubinool Omarjee of Lyce Henri IV, Paris, France** my computer of its own accord, placed them into a "junk file." But he deserves credit for having solved problems 5257, 5269, 5275, 5276, and 5281.

To Albert and to Moubinool, and to all others for whom this might have also happened, mea culpa.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2015

- **5349:** *Proposed by Kenneth Korbin, New York, NY*

  Given angle A with $\sin A = \dfrac{5}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

- **5350:** *Proposed by Kenneth Korbin, New York, NY*

  The four roots of the equation

  $$x^4 - 96x^3 + 206x^2 - 96x + 1 = 0$$

  can be written in the form

  $$x_{1,2} = \left(\frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}}\right)^{\pm 1}$$

  $$x_{3,4} = \left(\frac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}}\right)^{\pm 1}$$

  where a, b, and c are positive integers.

  Find a, b, and c if $(a, b, c) = 1$.

- **5351:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Let x, y, z be positive real numbers. Show that

  $$\frac{xy}{x^3 + y^3 + xyz} + \frac{yz}{y^3 + z^3 + xyz} + \frac{zx}{z^3 + x^3 + xyz} \le \frac{3}{x + y + z}.$$

- **5352:** *Proposed by Arkady Alt, San Jose, CA*

  Evaluate $\sum\limits_{k=0}^{n} x^k - (x-1)\sum\limits_{k=0}^{n-1}(k+1)x^{n-1-k}$.

5353: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients. Prove that all its zeros lie in the disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < r\}$, where

$$r = \left\{ 1 + \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^3 \right)^{1/2} \right\}^{2/3}$$

- **5354:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c > 0$ be real numbers. Prove that the series

$$\sum_{n=1}^{\infty} \left[n \cdot \left(a^{\frac{1}{n}} - \frac{b^{\frac{1}{n}} + c^{\frac{1}{n}}}{2} \right) - \ln \frac{a}{\sqrt{bc}} \right],$$

converges if and only if $2\ln^2 a = \ln^2 b + \ln^2 c$.

Solutions

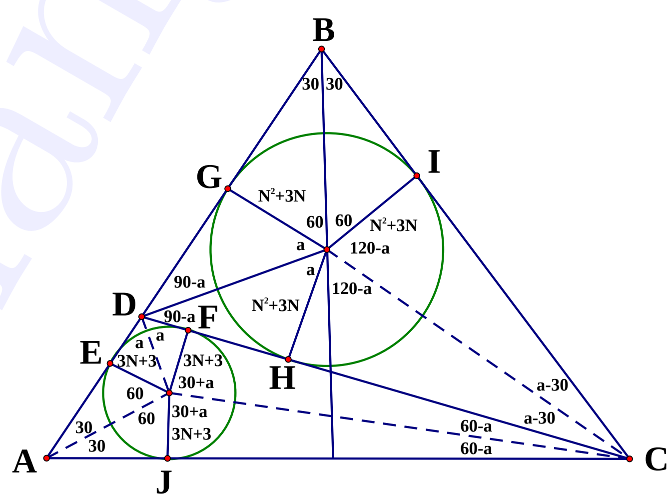
- **5331:** *Proposed by Kenneth Korbin, New York, NY*

Given equilateral $\triangle ABC$ with cevian $\overline{CD}$. Triangle ACD has inradius $3N + 3$ and $\triangle BCD$ has inradius $N^2 + 3N$ where N is a positive integer.

Find lengths $\overline{AD}$ and $\overline{BD}$.

Solution 1 by Ed Gray, Highland Beach, FL

Referring to the diagram, we can derive an equation which relates N and the angle a defined as the bisector of $\angle ADO$. (Points O and P are the centers of the incircles.



It is seen from the diagram that $IC = AG = AE + ED + DG$.

1) $\tan 30 = \dfrac{3N+3}{AE}$, so

2) $AE = \sqrt{3}(3N+3)$

3) $\tan a = \dfrac{3N+3}{ED}$, so

4) $ED = \dfrac{3N+3}{\tan a}$

5) $\tan a = \dfrac{DG}{N^2+3N}$

6) $DG = (N^2+3N)\tan a$

Adding 2), 4), and 6)

7) $AG = (3N+3)\sqrt{3} + \dfrac{3N+3}{\tan a} + \left(N^2+3N\right)\tan a$

To evaluate IC we note:

8) $\tan(a-30) = \dfrac{N^2+3N}{IC}$, or

9) $IC = \dfrac{N^2+3N}{\tan(a-30)}$

Equating 7) and 9) gives the basic equation:

10) $(3N+3)\left(\sqrt{3} + \dfrac{1}{\tan a}\right) + \left(N^2+3N\right)\tan a = \dfrac{N^2+3N}{\tan(a-30)}$.

We expand $\tan(a-30)$

11) $\tan(a-30) = \dfrac{\tan a - \tan 30}{1 + \tan a \tan 30} = \dfrac{\tan a - \sqrt{3}/3}{1 + \sqrt{3}/3\tan a} = \dfrac{3\tan a - \sqrt{3}}{3 + \sqrt{3}\tan a}$. So,

12) $(3N+3)\dfrac{1+\sqrt{3}\tan a}{\tan a} = \left(N^2+3N\right)\left(\dfrac{3+\sqrt{3}\tan a}{3\tan a - \sqrt{3}} - \tan a\right)$.

There is no way to eliminate all of these irrationals except to let:

13) $\tan a = r\sqrt{3}$, where r is, for now, unspecified. Making this substitution, eq-12) becomes:

14) $(3N+3)\dfrac{1+\sqrt{3}r\sqrt{3}}{r\sqrt{3}} = (N^2+3N)\left(\dfrac{3+3r}{3r\sqrt{3}-\sqrt{3}} - r\right)$.

Step 14) simplifies to

15) $\dfrac{(3N+3)(1+3r)}{r} = \dfrac{\left(N^2+3N\right)(9r^2-6r-3)}{(1-3r)}$ and dividing by 3

16) $\dfrac{(N+1)(1+3r)}{r} = (N^2+3N)\dfrac{(3r+1)(r-1)}{1-3r}$, and dividing by $3r+1$

17) $\dfrac{N+1}{r} = (N^2+3N)\dfrac{1-r}{3r-1}$, and simplifying gives

18) $(N+1)(3r-1) = (N^2+3N)(r-r^2)$.

Writing step 18) as a quadratic in r, we obtain,

19) $\left(N^2+3N\right) r^2+\left(3-N^2\right) r-(N+1)=0$, with solution

20) $2(N^2+3N)r=(N^2-3)+\sqrt{(N^2-3)^2+(4N+4)(N^2+3N)}$. The discriminat D^2 is:

21) $D^2=N^4+4N^3+10N^2+12N+9=\left(N^2+2N+3\right)^2$. So

22) $D=N^2+2N+3$, and equation 20) becomes

23) $2(N^2+3N)r=N^2-3+N^2+2N+3$

24) $2(N^2+3N)r=2N^2+2N$

25) $r=\dfrac{N^2+N}{N^2+3N}=\dfrac{N+1}{N+3}$

Then the value of $\tan a$ becomes

26) $\tan a=\dfrac{N+1}{N+3}\sqrt{3}$. So,

Finally, $AD=AE+ED$. So,

27) $AD=(3N+3)\sqrt{3}+(N+3)\sqrt{3}=2\sqrt{3}(2N+3)$, and

Similarly, $DB=DG+GB=2\sqrt{3}\left(N^2+2N\right)$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let I and J be respectively the inradius of $\triangle ACD$ and $\triangle BCD$, E and F be the tangent points of the incircles of $\triangle ACD$ and $\triangle BCD$ with $\overline{AB}$, respectively, $a=\overline{AB}=\overline{BC}=\overline{CA},\ d=\overline{CD},\ x=\overline{AD},\ e=\overline{AE}$ and $f=\overline{BF}$.

By the cosine theorem in $\triangle ACD$, $d^2=a^2+x^2-2ax\cos(\pi/3)$, d$=\sqrt{a^2-ax+x^2}$.

Since $\overline{AE}$ is a segment of the tangent from A to the incircle of $\triangle ACD$, whose semiperimeter is $\dfrac{a+x+d}{2}$, $e=\dfrac{a+x+d}{2}-d=\dfrac{a+x-d}{2}$ and analogously, $f=\dfrac{2a-x-d}{2}$; on the other hand, in $\triangle IAE$ we have that $\angle IAE=\angle(DAC/2)=\pi/6$, and $IE\perp AD$, so $e=(3N+3)\cot(\pi/6)=3\sqrt{3}(N+1)$ and analogously $f=\sqrt{3}N(N+3)$.

Thus, $a+x-\sqrt{x^2+a^2-xa}=6\sqrt{3}\,(N+1)$ and $2a-x-\sqrt{x^2+a^2-xa}=2\sqrt{3}N\,(N+3)$.

Subtracting the first equation from the second one, we obtain that $a=2x+2\sqrt{3}\left(N^2-3\right)$, and isolating the square root and squaring the first equation we obtain that

$$(a+x)^2-121\sqrt{3}(N+1)(a+x)+108(N+1)^2 \;=\; x^2+a-xa,\ \text{or equivalently}$$

$$ax-4\sqrt{3}(N+1)(a+x)+36(N+1)^2 \;=\; 0$$

Substituting here the obtained value of a as a function of x we deduce that $x^2+\sqrt{3}(N^2-6N-9)x-12N^2+6N^2+72N+54=0$, which is a quadratic equation with solutions

$$
\begin{aligned}
x &= \frac{1}{2}\left(-\sqrt{3}\left(N^2-6N-9\right) \pm \sqrt{3\left(N^2-6N-9\right)^2-4\left(-12N^2+6N^2+72N+54\right)}\right)\\
&= \frac{1}{2}\left(-\sqrt{3}\left(N^2-6N-9\right) \pm \sqrt{3\left(N^2+2N+3\right)^2}\right)\\
&= \frac{\sqrt{3}}{2}\left(-N^2+6N+9 \pm \left(N^2+2N+3\right)\right) \in \left\{-\sqrt{3}(N+1)(N-3),\ (2\sqrt{3}(2N+3)\right\},
\end{aligned}
$$

from where, being $a = 2x + 2\sqrt{3}(N^2-3)$, we deduce that $a \in \left\{4\sqrt{3}N,\ 2\sqrt{3}\left(N+1\right)\left(N+3\right)\right\}$, respectively, that is, $\overline{AD} = 2\sqrt{3}\left(2N+3\right)$ and $\overline{BD} = 2\sqrt{3}N\left(N+2\right)$.

Note that N is a positive integer, so the first case would be possible if $(N+1)(3-N)$ and $(N-1)(N+3)$ are positive, which is impossible, hence, $\overline{AD} = 2\sqrt{3}(2N+3)$ and $BD = 2\sqrt{3}N\left(N+2\right)$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\left(\overline{AD}, \overline{BD}\right) = \left(2\sqrt{3}(2N+3), 2\sqrt{3}N(N+2)\right)$.

Let $\overline{AD} = x$ and $\overline{BD} = y$ so that $\overline{AC} = \overline{BC} = x+y$. The area of $\triangle ACD = \dfrac{x(x+y)\sin 60^\circ}{2} = \dfrac{\sqrt{3}x(x+y)}{4}$ and the area of $\triangle BCD = \dfrac{\sqrt{3}y(x+y)}{4}$.

Applying the cosine formula to $\triangle ACD$, we obtain $\overline{CD} = \sqrt{x^2+xy+y^2}$.

Since the area of a triangle equals the product of its semiperimeter with its inradius, so

$$
\begin{aligned}
\frac{\sqrt{3}x(x+y)}{2\left(2x+y+\sqrt{x^2+xy+y^2}\right)} &= N+3, \qquad (1) \text{ and}\\
\frac{\sqrt{3}y(x+y)}{2\left(x+2y+\sqrt{x^2+xy+y^2}\right)} &= N^2+3N. \qquad (2)
\end{aligned}
$$

Since the left side of (1) equals $\dfrac{\sqrt{3}\left(2x+y-\sqrt{x^2+xy+y^2}\right)}{6}$ and the left side of (2) equals $\dfrac{\sqrt{3}\left(x+2y-\sqrt{x^2+xy+y^2}\right)}{6}$, so we obtain respectively from (1) and (2) that

$$
\begin{aligned}
\sqrt{x^2+xy+y^2} &= 2x+y-6\sqrt{3}(N+1) \qquad (3) \text{ and}\\
\sqrt{x^2+xy+y^2} &= x+2y-2\sqrt{3}N(N+3) \qquad (4)
\end{aligned}
$$

From (3) and (4), we obtain $y = x + 2\sqrt{3}\left(N^2 - 3\right)$. Substituting y back into (3), squaring and simplifying, we obtain,

$x^2 + \sqrt{3}\left(N^2 - 6N - 9\right) X - 12N^3 + 6N^2 + 72N + 54 = 0$. Hence either

$x = 2\sqrt{3}(2N+3),\ y = 2\sqrt{3}N(N+2)$ or $x = \sqrt{3}(3-N)(1+N),\ y = \sqrt{3}(N-1)(N+3)$.

Since only the former solution satisfies (3) and (4), so we obtain the claimed solution.

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5332:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Inspired by the prime number 1000000000000066600000000000001, known as *Belphegor's prime* where there are thirteen consecutive zeros to the left and right of 666, we consider the numbers $100\ldots0201500\ldots01$ where there are $k-$zeros left and right of 2015. For $k < 28$ only $k = 9$ and $k = 27$ yield prime numbers.

(a) Prove that the sequence $120151,\ 10201501,\ 1002015001, \ldots$ has an infinite subsequence of all composite numbers.

(b) Find the next prime in both the sequences $100\ldots066600\ldots01$ and $100\ldots0201500\ldots01$, after the ones noted above.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

(a) The sequence can be expressed as $a_k = 10^{2k+5} + 2015 \cdot 10^{k+1} + 1,\ k = 0, 1, \ldots,$ where k denotes the number of consecutive zeros to the right and to the left of 2015.

We note that $1000 \equiv -1 \pmod{13},\ 2015 \equiv 0 \pmod{13},\ 10^{13} \equiv 1 \pmod{53}$. So

$$a_{3n+2} = 10^{3(2n+3)} + 2015 \cdot 10^{3n+3} + 1 \equiv 1000^{2n+3} + 2015 \cdot 10^{3n+3} + 1 \equiv -1 + 0 + 1 = 0 \pmod{13},$$

$$a_{13n} = 10^{26n+5} + 2015 \cdot 10^{13n+1} + 1 \equiv 10^5 + 20150 + 1 \equiv 0 \pmod{53},$$

$$a_{13n+8} = 10^{26n+21} + 2015 \cdot 10^{13n+9} + 1 \equiv 10^8 + 2015 \cdot 10^9 + 1 \equiv 0 \pmod{53}.$$

So there are infinitely many indices k for which a_k is composite.

(b) Tom Moore is wrong in saying that

$$1\underbrace{0\ldots0}_{9\ zeros}2015\underbrace{0\ldots0}_{9\ zeros}1 \text{ and}$$

$$1\underbrace{0\ldots0}_{27\ zeros}2015\underbrace{0\ldots0}_{27\ zeros}1$$

are primes. The correct statement is that

$$1\underbrace{0\ldots0}_{7\ zeros}2015\underbrace{0\ldots0}_{7\ zeros}1 \text{ and}$$

$$1\underbrace{0\ldots0}_{25\ zeros}2015\underbrace{0\ldots0}_{25\ zeros}1$$

are primes.

Let $b_k = 10^{2k+4} + 666 \cdot 10^{k+1} + 1$. Then b_{13} is Belphegore's prime. Using the PrimeQ function of Mathematica we find that

- b_{42} is prime,
- b_k is composite for $14 \le k \le 41$,
- a_k is composite for$0 \le k \le 7000,$ except for $k = 7$ and $k = 25$.

I was not able to find a $k > 25$ for which a_k is prime.

Solution 2 by Pat Costello, Eastern Kentucky University, Richmond, KY

(a) The number 2015 is divisible by 13 and so starting with 1002015001, every third number in the sequence is divisible by 13 (the leading 1 is a $10^{3(2x+3)} \equiv -1 \pmod{13}$ which cancels with the final 1).

(b) The next primes in the sequence $100\ldots066600\ldots01$ are when then the number of zeroes is $k = 42$ and $k = 506$ (probably prime according to *Mathematica*).

In the sequence $100\ldots0201500\ldots-1$, I believe the k values that give primes should be $k = 7$ and $k = 25$ (not 9 and 27) and *Mathematica* did not find any more primes (or probably primes) in the sequence with $k < 2000$.

Solution 3 by Ashland University Undergraduate Problem Solving Group, Ashland, OH

a) We begin by noting $a_k = 10^{5+2k} + 2015(10^{k+1}) + 1$ is an explicit formula for the number with k-zeros to the left and right of 2015.

Suppose $k \equiv 2 \pmod 3$ so $k = 3n + 2$ for some integer n. Then $a_{3n+2} = 10^{6n+9} + 2015(10^{3n+3}) + 1$. Since $2015 \equiv 0 \pmod{13}$, we have $a_{3n+2} \equiv 10^{6n+9} + 1 \pmod{13}$. Thus $a_{3n} \equiv (10^3)^{2n+3} + 1 \pmod{13}$. Note $10^3 = 1000 \equiv -1 \pmod{13}$ and clearly $2n + 3$ is odd, so $a_{3n+2} \equiv (-1)^{2n+3} + 1 \equiv -1 + 1 \equiv 0 \pmod{13}$ and hence $13 \big| a_{3n+2}$ and a_{3n+2} is composite. Thus the subsequence $\{a_n\}$ where $k_n = 3n + 2$ for $n = 0, 1, 2, 3, \ldots$ is an infinite subsequence of all composite numbers.

b) For the sequence $10\ldots0666001,\ a_k = 10^{2k+4} + 666(10^{k+1}) + 1$ and we used MAPLE to find that the next prime occurs when $k = 42$, .i.e., there are 42 zeros to the left and right of 666. (The only additional primes in this sequence with $k \le 1000$ occur when $k = 506$ and $k = 608$).

For the sequence $10\ldots020150\ldots01,\ a_k = 10^{5+2k} + 2015(10^{k+1}) + 1$ and were unable to find the next prime in the sequence, using MAPLE to check all terms with $k \le 7000$ were composite.

Also solved by Brain D. Beasley, Presbyterian College, Clinton, SC; Ed

Gray, Highland Beach, FL; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene) Algeria, and the proposer.

- **5333:** *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy.*

Evaluate

$$\int_{-\pi/2}^{\pi/2} \frac{\left(\ln\left(1+\tan x+\tan^2 x\right)\right)^2}{1+\sin x\cos x}dx.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Denote the integral of the problem by I. We show that

$$I = \frac{2\sqrt{3}\pi\left(\pi^2+3\ln^2 3\right)}{9}. \qquad (1)$$

Let $J = \displaystyle\int_o^{\pi/2} \ln(\cos x)dx$ and $K = \displaystyle\int_0^{\pi/2} \ln^2(\cos x)dx$. It is known [1], p. 531, section 4.224, entries 6 and 8) that

$$J = \frac{-\pi\ln 2}{2} \qquad (2), \text{ and}$$

$$K = \frac{-\pi(\pi^2+12\ln^2 2)}{24}. \qquad (3)$$

By means of the substitution $\tan x = \dfrac{\sqrt{3}\tan y - 1}{2}$, we see that

$$I = \frac{2}{\sqrt{3}}\int_{-\pi/2}^{\pi/2} \ln^2\left(\frac{3\sec^2 y}{4}\right)dy = \frac{4}{\sqrt{3}}\int_0^{\pi/2} \ln^2\left(\frac{3\sec^2 y}{4}\right)dy.$$

Since $\ln^2\left(\dfrac{3\sec^2 y}{4}\right) = \ln^2\left(\dfrac{3}{4}\right) - 4\ln\left(\dfrac{3}{4}\right)\ln(\cos y) + 4\ln^2(\cos y)$, so

$$I = \frac{4}{\sqrt{3}}\left(\ln^2\left(\frac{3}{4}\right)\frac{\pi}{2} - 4\ln\left(\frac{3}{4}\right)J + 4K\right).$$

Using (2) and (3), we obtain (1).

Reference

1. I.S. Gradshteyn and I.M. Ryzhik: *Table of Integrals, Series, and Products,* seventh edition, Elsevier, Inc. 2007.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We claim that the integral equals $\dfrac{2\pi(\pi^2+3\ln^2 3)}{3\sqrt{3}}$.

We perform a change of variables and put $y = \tan x$. The $dy = \dfrac{1}{\cos^2 x}dx = (1+y^2)dx$ and

$$I = \int_{-\pi/2}^{\pi/2} \frac{\left(\ln\left(1+\tan x+\tan^2 x\right)\right)^2}{1+\sin x\cos x}dx = \int_{-\infty}^{\infty} \frac{\left(\ln\left(1+y+y^2\right)\right)^2}{1+\dfrac{y}{1+y^2}}\frac{dy}{1+y^2} = \int_{-\infty}^{\infty} \frac{\left(\ln\left(1+y+y^2\right)\right)^2}{1+y+y^2}dy =$$

$$= \int_{-\infty}^{\infty} \frac{\left(\ln\left(1+\left(y-\frac{1}{2}\right)+\left(y-\frac{1}{2}\right)^2\right)\right)^2}{1+\left(y-\frac{1}{2}\right)+\left(y-\frac{1}{2}\right)^2}dy = \int_{-\infty}^{\infty} \frac{\left(\ln\left(\frac{3}{4}+y^2\right)\right)^2}{\frac{3}{4}+y^2}dy = 2\int_0^{\infty} \frac{\left(\ln\left(\frac{3}{4}+y^2\right)\right)^2}{\frac{3}{4}+y^2}dy.$$

Put $f(s) = 2\displaystyle\int_0^{\infty} \frac{1}{\left(\frac{3}{4}+y^2\right)^s}dy$ for $\Re(s) > \dfrac{1}{2}$.

We evaluate $f(s)$ in terms of the beta function

$B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, by performing a change of variables in the defining integral of $f(s)$. Letting $z = \dfrac{1}{1+y^2}$, $y = \sqrt{\dfrac{1}{z}-1}$, $dy = \dfrac{-1}{2z^2\sqrt{\dfrac{1}{z}-1}}dz$ we obtain

$$f(s) = 2\int_0^{\infty} \frac{1}{\left(\frac{3}{4}+y^2\right)^s}dy = 2\sqrt{\frac{3}{4}}\int_0^{\infty} \frac{1}{\left(\frac{3}{4}+\frac{3}{4}y^2\right)^s}dy = 2\left(\frac{3}{4}\right)^{\frac{1}{2}-s}\int_0^{\infty} \frac{1}{\left(1+y^2\right)^s}dy =$$

$$= \left(\frac{3}{4}\right)^{\frac{1}{2}-s}\int_0^1 z^{s-\frac{3}{2}}\frac{1}{\sqrt{1-z}}dz =$$

$$= \left(\frac{3}{4}\right)^{\frac{1}{2}-s}\frac{\Gamma\left(s-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(s)} = \frac{\sqrt{3\pi}}{2}\left(\frac{4}{3}\right)^s\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)},$$

where we have used that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

We have $\dfrac{d^2}{ds^2}\dfrac{1}{\Gamma(s)} = \dfrac{d}{ds}\dfrac{-\Gamma'(s)}{\Gamma^2(s)} = -\dfrac{\Gamma''(s)}{\Gamma^2(s)} + 2\dfrac{\left(\Gamma'(s)\right)^2}{\Gamma^3(s)}$,

$$\frac{\frac{d^2}{ds^2}u(s)v(s)w(s)}{u(s)v(s)w(s)} = \frac{u''(s)}{u(s)} + \frac{v''(s)}{v(s)} + \frac{w''(s)}{w(s)} + 2\frac{u'(s)v'(s)}{u(s)v(s)} + 2\frac{v'(s)w'(s)}{v(s)w(s)} + 2\frac{w'(s)u'(s)}{w(s)u(s)}.$$

So

$$I = f''(1) = \frac{\sqrt{3\pi}}{2}\frac{4}{3}\Gamma\left(\frac{1}{2}\right)\ln^2\left(\frac{4}{3}\right) + \frac{\sqrt{3\pi}}{2}\frac{4}{3}\Gamma''\left(\frac{1}{2}\right) +$$

$$+\frac{\sqrt{3}\pi}{2}\frac{4}{3}\Gamma''\left(\frac{1}{2}\right)\left(-\Gamma''(1)+2\left(\Gamma'(1)\right)^2\right)+2\frac{\sqrt{3}\pi}{2}\frac{4}{3}\Gamma'\left(\frac{1}{2}\right)\ln\left(\frac{4}{3}\right)$$

$$+2\frac{\sqrt{3}\pi}{2}\frac{4}{3}\Gamma'\left(\frac{1}{2}\right)\left(-\Gamma'(1)\right)+2\frac{\sqrt{3}\pi}{2}\frac{4}{3}\ln\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{2}\right)\left(-\Gamma'(1)\right) \tag{1}$$

To evaluate $\Gamma'(1)$, $\Gamma''(1)$, $\Gamma'\left(\frac{1}{2}\right)$, $\Gamma''\left(\frac{1}{2}\right)$ we use the well known equations,

$$\frac{\Gamma'(z)}{\Gamma(z)}=\frac{1}{z}+\gamma+\sum_{n\geq 1}\left(\frac{1}{n+z}-\frac{1}{n}\right).$$

$$\frac{\Gamma''(z)}{\Gamma(z)}-\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)^2=\sum_{n\geq 0}\frac{1}{(n+z)^2},$$

from which we deduce

$$(i)\quad \Gamma(1) = -\gamma,$$

$$(ii)\quad \Gamma''(1) = \gamma^2+\sum_{n\geq 0}\frac{1}{(n+1)^2}=\gamma^2+\frac{\pi^2}{6},$$

$$\begin{aligned}(iii)\quad \Gamma'\left(\frac{1}{2}\right) &= -\Gamma\left(\frac{1}{2}\right)\left(2+\gamma+\sum_{n\geq 1}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n}\right)\right)=-\sqrt{\pi}\left(2+\gamma+2\sum_{n\geq 1}\left(\frac{1}{2n+1}-\frac{1}{2n}\right)\right)= \\ &= -\sqrt{\pi}\left(\gamma+2\ln 2\right),\end{aligned}$$

$$\begin{aligned}(iv)\quad \Gamma'\left(\frac{1}{2}\right) &= -\Gamma\left(\frac{1}{2}\right)\left(\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2+\sum_{n\geq 0}\frac{4}{(2n+1)^2}\right)= \\ &= \sqrt{\pi}\left((\gamma+2\ln 2)^2+4\left(\sum_{n\geq 0}\frac{1}{n^2}-\sum_{n\geq 0}\frac{1}{4n^2}\right)\right)=\sqrt{\pi}\left((\gamma+2\ln 2)^2+\frac{\pi^2}{2}\right).\end{aligned}$$

We plug $(i)-(iv)$ into (1) and get

$$I = \frac{2}{3}\sqrt{3}\pi\ln^2\left(\frac{4}{3}\right)+\frac{2}{3}\sqrt{3}\pi\left((\gamma+2\ln 2)^2+\frac{\pi^2}{2}\right)+\frac{2}{3}\sqrt{3}\pi\left(\gamma^2-\frac{\pi^2}{6}\right)+$$

$$- \quad \frac{4}{3}\sqrt{3}\pi\left(\gamma+2\ln 2\right)\ln\left(\frac{4}{3}\right)-\frac{4}{3}\sqrt{3}\pi\gamma\left(\gamma+2\ln 2\right)+\frac{4}{3}\sqrt{3}\pi\gamma\ln\left(\frac{4}{3}\right)=$$

$$= \quad \frac{2\pi\left(\pi^2+3\ln^2 3\right)}{3\sqrt{3}}.$$

Comment by editor. The numerical answer to this problem can be approximated to whatever degree of accuracy one wishes by piecing together various integrating techniques for power series expansions over specific domains and for estimating the area under the graph of a positively valued curve. This method of computing the value of the integral was employed by **Ed Gray of Highland Beach, FL** *in his 10 page solution that gave him a numerical answer that was correct to several decimal places. But as one can see from the above solutions, the problem was not as straight-forward as I had initially thought.*

This problem was also solved by its proposer.

- **5334:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $x_{ij},\ (1 \le i \le m, 1 \le j \le n)$ be nonnegative real numbers. Prove that

$$\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}\frac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}\frac{1}{1+\sqrt{x_{ij}}}\right)\ge 1.$$

Solution by Kee-Wai Lau, Hong Kong, China

For $1 \le i \le m,\ 1 \le j \le n$, let y_{ij} be real numbers satisfying $0 \le y_{ij} \le 1$. We prove by induction on $m+n$ that

$$\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}y_{ij}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}\left(1-y_{ij}\right)\right)\ge 1. \qquad (1)$$

For $m+n=2$, we have $m=n=1$, and (1) becomes an equality. So suppose that (1) holds for $m+n=k\ge 2$. We now consider $m+n=k+1$.

Denote the left side of (1) by $f\left(y_{mn}\right)$. Then

$$f\left(y_{mn}\right)\ge\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}y_{ij}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n-1}\left(1-y_{ij}\right)\right), \qquad (2)$$

and

$$f\left(y_{mn}\right)\ge\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m-1}y_{ij}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}\left(1-y_{ij}\right)\right). \qquad (3)$$

Here we assign the value 1 to any empty products. From (2), we obtain by the induction assumption that

$$f\left(0\right) \geq \prod_{j=1}^{n-1}\left(1-\prod_{i=1}^{m} y_{ij}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n-1}\left(1-y_{ij}\right)\right) \geq 1, \qquad (4)$$

and from (3), we obtain by the induction assumption that

$$f\left(1\right) \geq \prod_{j=1}^{n}\left(1-\prod_{i=1}^{m-1} y_{ij}\right)+\prod_{i=1}^{m-1}\left(1-\prod_{j=1}^{n}\left(1-y_{ij}\right)\right) \geq 1. \qquad (5)$$

Since $f(y_{mn})$ is a polynomial in y_{mn} with degree 0 or 1, so from (4) and (5), we see that $f(y_{mn}) \geq 1$, and (1) holds also for $m+n=k+1$. Hence (1) holds in general and the inequality of the problem follows by the substitution $y = \dfrac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student), Taylor University, Upland, IN; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5335:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any real $p>1$ and $x>1$ that

$$\frac{\ln x}{\ln(x+p)} \leq \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^{p}.$$

Solution 1 by Ethan Gegner (student), Taylor University, Upland, IN

The weighted AM-GM inequality, followed by Jensen's inequality applied to the concave function $\ln x$ yields

$$\begin{aligned}
(\ln x)^{1/p}\left(\ln(x+p)\right)^{\dfrac{p-1}{p}} &\leq \frac{1}{p}\ln x+\frac{p-1}{p}\ln(x+p)\\
&\leq \ln\left(\frac{1}{p}x+\frac{p-1}{p}\left(x+p\right)\right)\\
&= \ln(x+p-1).
\end{aligned}$$

Exponentiation by p and then rearranging yields the desired result.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

The inequality is true for any real $p \geq 1$ and $x>1$, because

$$\left(\frac{\ln\left(x+p-1\right)}{\ln\left(x+p\right)}\right)^{p}-\frac{\ln x}{\ln(x+p)} \quad \geq \quad 1+p\left(\frac{\ln(x+p-1)}{\ln(x+p)}-1\right)-\frac{\ln x}{\ln(x+p)}$$

$$
\begin{aligned}
&= \frac{p\ln\left(\dfrac{x+p-1}{x+p}\right) - \ln\left(\dfrac{x}{x+p}\right)}{\ln(x+p)} \\
&= \frac{qy\ln(1-y^{-1}) - \ln(1-q)}{\ln y} \\
&= \frac{q}{\ln y}\left(-\sum_{k=1}^{\infty} k^{-1}y^{1-k} + \sum_{k=1}^{\infty} k^{-1}q^{k-1}\right) \\
&= \frac{q}{\ln y}\sum_{k=1}^{\infty} k^{-1}\left(q^{k-1} - y^{1-k}\right) \geq 0,
\end{aligned}
$$

where we have used Bernoulli's inequality

$$(1+t)^p \geq 1 + pt \text{ for } t = \frac{\ln(x+p-1)}{\ln(x+p)} - 1 \geq -1.$$

Note that $p \geq 1, x > 1 \Rightarrow x+p-1 > 1, x+p > 1 \Rightarrow \ln(x+p-1), \ln(x+p) > 0$, the notation $y = x+p$ and $q = \dfrac{p}{y}$, the series expansion $\ln(1-u) = -\sum_{k=1}^{\infty} k^{-1}u^k$ for $u = y^{-1}$ and $u = q$ (observe that $0 < y^{-1}, q < 1$) and the fact that $q \geq y^{-1}$ with equality iff $p = 1 \Rightarrow q^{k-1} \geq \left(y^{-1}\right)^{k-1}$ for any integer $k \geq 1$.

Moreover, equality is attained iff it occurs in Bernouilli's inequality and in the inequality $q \geq y^{-1}$. Since there is equality in this last inequality iff $p = 1$ and in this case also in Bernoulli's inequality, we conclude that equality occurs iff $p = 1$.

Solution 3 by Paul M. Harms, North Newton, KS

All logarithms involved with the inequality are positive. Then the inequality is correct if the logarithm of the left side is less than the logarithm of the right side. Taking the natural logarithm of both sides and dividing by p the problem inequality is equivalent t o

$$\frac{\ln\ln x - \ln\ln(x+p)}{p} \leq \frac{\ln\ln(x+p-1) - \ln\ln(x+p)}{1},$$

Let $f(x) = \ln\ln x$ where $x > 1$. Multiplying both sides of the inequality by (-1) we can write the resulting inequality as

$$\frac{f(x+p) - f(x)}{(x+p) - x} \geq \frac{f(x+p) - f(x+p-1)}{(x+p) - (x+p-1)},$$

forms often associated with the Mean Value Theorem for derivatives.

Let the following letters and points be associated with each other:

$$A\,(x, f(x))\,,\ B\,((x+p), f(x+p))\,,\ C\,((x+p), f(x))\,,$$

$$E\left((x+p-1), f(x+p-1)\right),\ F\left((x+p), f(x+p-1)\right),$$

and let D be intersection of the line segment between A and B with the line segment between E and F.

Consider the right triangle $\triangle BEF$ and the similar right triangles $\triangle ABC$ and $\triangle DBF$. The left side of the last inequality is the ratio of the distances $\dfrac{BC}{AC} = \dfrac{BF}{DF}$ and the right side equals the ratio $\dfrac{BF}{EF}$.

Since $f^{'}(x) = \dfrac{1}{x \ln x} > 0$, and $f^{''}(x) = \dfrac{-1\,(1+\ln x)}{(x\ln x)^2} < 0$ for $x > 1$, the line segment from A to B is below the graph of $y = f(x)$. Point D then satisfies the distance inequality $DF < EF$ so we have $\dfrac{BF}{DF} \geq \dfrac{BF}{EF}$. The problem inequality is correct.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The inequality in the statement of the problem is equivalent to

$$\frac{\ln x}{\ln(x+p)} \leq \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p \iff \ln\left(\ln(x+p)\right)^{p-1} \leq \left(\ln(x+p-1)\right)^p. \qquad (*)$$

Knowing that $\ln x > 0$ and using the AM-GM inequality, we have:

$$\ln x\left(\ln(x+p)\right)^{p-1} \leq \left(\frac{\ln x + (p-1)\ln(x+p)}{p}\right)^p = \left(\ln \sqrt[p]{x(x+p)^{p-1}}\right)^p \leq (\ln(x+p-1))^p$$

for every $p > 1$ and $x > 1$. Using the fact that $\ln x$ is an increasing function, we deduce that $(*)$ is true and also the equivalent inequality in the statement of the problem.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5336:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Caculate:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Solution 1 by Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome, Italy

The first item we employ is

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + \gamma_n,\ \gamma_n = \gamma + o(1), \qquad n(\gamma_n - \gamma) \to 1/2.$$

The second item we use is the content of problem 1781 of Mathematics Magazine, vol.80–5, 2007.

Rearranging the sum up to n we get

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{n}{k}-\sum_{k=1}^{n-1}\frac{k}{k+1}-\sum_{k=1}^{n}\ln\left(k+\frac{1}{2}\right)-n\gamma=\\
&=n(\lg n+\gamma_n)-\sum_{k=1}^{n-1}\left(1-\frac{1}{k+1}\right)-\ln\prod_{k=1}^{n}\frac{2k+1}{2}-n\gamma=\\
&=n\ln n+n(\gamma_n-\gamma)-(n-1)+\ln n+\gamma_n-1-\ln\frac{(2n+1)!}{2^{2n}n!}
\end{aligned}$$

Stirling's formula $n!=(n/e)^n\sqrt{2\pi n(1+o(1))}$ and $\ln(1+x)\sim x$ for $x\to 0$ yields

$$\begin{aligned}
&n\ln n+n(\gamma_n-\gamma)-n+\ln n+\gamma_n-(2n+1)\ln(2n+1)+(2n+1)+\\
&-\frac{1}{2}\ln(2\pi(2n+1))+o(1)+2n\ln 2+n\ln n-n+\frac{1}{2}\ln(2\pi n)+o(1)
\end{aligned}$$

$(2n+1)\ln(2n+1)=(2n+1)(\ln 2+\ln n+o(\frac{1}{n})=2n\ln n+2n\ln 2+\ln n+\ln 2+o(1)$.

The sum becomes

$$\begin{aligned}
&n(\gamma_n-\gamma)+n\ln n(1-2+1)+n(-1-2\ln 2+2\ln 2+2-1)+\\
&+\ln n(1-1-\frac{1}{2}+\frac{1}{2})+(\gamma_n-\ln 2-\frac{1}{2}\ln(4\pi)+\frac{1}{2}\ln(2\pi)
\end{aligned}$$

and in the limit we obtain $\dfrac{1}{2}+\gamma-\dfrac{3}{2}\ln 2$.

Solution 2 by Anastasios Kotronis, Athens, Greece

Let

$$S_n:=\sum_{k=1}^{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Summing by parts we have

$$\begin{aligned}
S_n&=\sum_{k=1}^{n}(k+1-k)H_k-\ln\left(\prod_{k=1}^{n}\frac{2k+1}{2}\right)-n\gamma\\
&=kH_k\Big|_1^{n+1}-\sum_{k=1}^{n}(k+1)\left(H_{k+1}-H_k\right)-\ln\left(\prod_{k=1}^{n}\frac{2k(2k+1)}{2^2k}\right)-n\gamma\\
&=(n+1)\left(H_{n+1}-1\right)-\ln\left(\frac{(2n+1)!}{2^{2n}n!}\right)-n\gamma\\
&\to\frac{1}{2}+\gamma-\frac{3\ln 2}{2}
\end{aligned}$$

by Stirling's approximation.

Solution 3 by Haroun Meghaichi, (student, University of Science and Technology Houari Boumediene), Algiers, Algeria.

Let H_n be the n-th harmonic number then for any integer $n > 1$ we have

$$\begin{aligned} a_n = \sum_{k=1}^{n} H_k &= \sum_{k=1}^{n} (k+1)H_{k+1} - kH_k - 1 \\ &= (n+1)\left(H_{n+1} - 1\right) \\ &= n\left(\ln n + \gamma - 1\right) + \ln n + \gamma + \frac{1}{2} + o(1). \end{aligned}$$

And

$$\begin{aligned} b_n = \sum_{k=1}^{n} \ln\left(\frac{2k+1}{2}\right) &= \ln\left(\frac{(2n+1)!!}{2^n}\right) \\ &= \ln\left(\frac{(2n+1)!}{4^n(n!)}\right) = \ln(2n+1)! - \ln n! - 2n\ln 2 \\ &= n\left(\ln n - 1\right) + \ln n + \frac{3}{2}\ln 2 + o(1). \end{aligned}$$

The last line comes directly from Stirling approximation, then we have

$$\sum_{k=1}^{n}\left(H_k - \ln\left(k + \frac{1}{2}\right) - \gamma\right) = a_n - b_n - n\gamma = \gamma + \frac{1}{2} - \frac{3}{2}\ln 2 + o(1)$$

Hence, the answer is $\boxed{\gamma + \frac{1}{2} - \frac{3}{2}\ln 2 = \frac{1}{2}\ln\frac{e^{2\gamma+1}}{8}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee -Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Comments

Editor's note: The following comment was sent to me by **Henry Ricardo of the NY Math Circle.** In the March, 2015 solutions, Solution 1 of problem 5330 is incorrect. By throwing in the factor B(1), the solvers have replaced the original problem by one whose solution is almost trivial. The proposer (Ovidiu Furdui) no doubt specified that the matrix product start with B(2) to make it more challenging. The extra factor does not provide a generalization or extension but, rather, a simplification that is contrary to the spirit of the problem as proposed.

Solution 1 was solved by looking at a few examples, guessing a general form for the product and then proving the product held by induction. I thought it was a nice simple way to solve the problem. Henry disagreed. So I sent his comment on to Ovidiu Furdui, the proposer of the problem and asked him if the published solution were on a test, would he give full credit. Here is his response.

The reader is right, solution 2 is the correct one. On one hand, the problem asks for the calculation of the product starting from $B(2)$ up to $B(n)$, for $n \geq 2$ and in solution 1 basically the solvers have computed a product which simplifies very much the problem, so from a mathematical point of view the problem asks for one thing and the solvers give another. The product $A(n) = B(1)B(2) \cdots B(n)$ as they give it is correct but this is not what the problem asks for. (Me, Ted, speaking again; I don't see it this way– as I see it, they did answer the question. Now back to Ovidiu.)

On the other hand, to answer your question, if this problem would have been an exam problem and the student(s) would have solved the problem as in solution 1, then certainly I would give partial credit for this solution, but not full credit due to the fact that, strictly speaking the solution is not what the problem asks for. However, I would offer partial credit to the student for calculating the product A(n) (for observing its form and for proving that by induction) but not full credit.

Solution 2 is the correct solution of this problem.

(Editor again:) But still I wasn't satisfied that the solution was incorrect, and so I explained the solution to Michael Fried, and he agreed with Henry and with Ovidui. His reasoning was that the authors of the Solution 1 had changed the initial conditions of the sequence by saying that the sequence started with B(1) and not B(2). But I argued that the authors of Solution 1 stated in their argument, "we have shown, by mathematical induction that (1) holds for all integers $n \geq 2$," and again I felt that that they had shown that. To my way of thinking, we had the product of matrices $B(1)B(2) \cdots B(n)$. The authors of Solution 1 could obtain the correct answer by a simple translation. I also thought that they could obtain the answer by multiplying the product by the inverse of B(1), and therein I made a mistake. Matrix B(1) is not invertible. Anyway, at this point the score was two against me, nobody for me. I then sent the question (Was the published solution 1 incorrect?) to Albert Stadler, and he agreed with the others, and he pointed out my mistake that matrix B(1) was not invertible. The score was now 3-0, and I am now siding with the majority.

Solution 1 to 5332 misses the spirit of the intended problem; once again, mea culpa.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
October 15, 2015

- **5355:** *Proposed by Kenneth Korbin, New York, NY*

  Find the area of the convex cyclic pentagon with sides

$$(13,\ 13,\ 12\sqrt{3}+5,\ 20\sqrt{3},\ 12\sqrt{3}-5).$$

- **5356:** *Proposed by Kenneth Korbin, New York, NY*

  For every prime number p there is a circle with diameter $4p^4+1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area

$$(8p^3)(p+1)(p-1)(2p^2-1).$$

  Find the sides of the triangles if $p=2$ and if $p=3$.

- **5357:** *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

  Determine all triangles whose side-lengths are positive integers (of which at least one is a prime number), whose semiperimeter is a positive integer, and whose area is equal to its perimeter.

- **5358:** *Proposed by Arkady Alt, San Jose, CA*

  Prove the identity $\displaystyle\sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1} = (r+1)^m(mr-1)+1.$

5359: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1} \le \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

- **5360:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n \geq 1$ be an integer and let

  $$I_n = \int_0^\infty \frac{\arctan x}{(1+x^2)^n} dx.$$

  Prove that

  (a) $\displaystyle\sum_{n=1}^{\infty} \frac{I_n}{n} = \zeta(2);$

  (b) $\displaystyle\int_0^\infty \arctan x \ln\left(1+\frac{1}{x^2}\right) dx = \zeta(2).$

Solutions

- **5337:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex quadrilateral $ABCD$ with sides,

  $$\begin{aligned} \overline{AB} &= 1+3\sqrt{2} \\ \overline{BC} &= 6+4\sqrt{2} \text{ and} \\ \overline{CD} &= -14+12\sqrt{2}. \end{aligned}$$

  Find side $\overline{AD}$ so that the area of the quadrilateral is maximum.

  Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

  In the published solution to part (b) of problem 787 *Journal Crux Mathematicorum*, $1984, 10(2), 56-58$, it is proved that given three sides $\overline{AB}, \overline{BC}$, and $\overline{CD}$, the area of the quadrilateral $ABCD$ is maximum if, and only if, the length of the fourth side, $\overline{AD}$ is the diameter of the circle passing through B and C, and a root of the polynomial

  $$\begin{aligned} x^3 - \left(\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2\right) - 2\overline{AB}\cdot\overline{BC}\cdot\overline{CD} &= 0. \text{ That is} \\ x^3 - \left(571 - 282\sqrt{2}\right)x - 206 - 104\sqrt{2} &= 0, \end{aligned}$$

  whose only real positive root is $x = 7+5\sqrt{2}$; so $\overline{AD} = 7+5\sqrt{2}$.

  Solution 2 by Albert Stadler, Herrliberg, Switzerland

  The cyclic quadrilateral has the maximal area among all quadrilaterals having the same sequence of side lengths. This is a corollary to Bretschneider's formula (http://en.wikipedia.org/wiki/Bretschneider%27s_formula). It can also be proved using calculus (see([1]). The area of a cyclic quadrilateral with side a, b, c, d is given by Brahmagupta's formula

  $$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s = (a+b+c+d)/2.$$

So if $a = 1 + 3\sqrt{2},\ b = 6 + 4\sqrt{2}$, and $c = -14 + 12\sqrt{2}$ then

$$16A^2 = \left(d - 9 + 13\sqrt{2}\right)\left(d - 19 + 11\sqrt{2}\right)\left(d + 21 - 5\sqrt{2}\right)\left(-d - 7 + 19\sqrt{2}\right).$$

This is a polynomial of degree four whose extremal points are located at the zeros of its derivative. Brute force shows that the extremal points are

$$\begin{aligned} d_1 &= 7 + 5\sqrt{2} > 0, \\ d_2 &= \frac{-7 - 5\sqrt{2} + \sqrt{1987 - 1338\sqrt{2}}}{2} < 0, \\ d_3 &= \frac{-7 - 5\sqrt{2} - \sqrt{1987 - 1338\sqrt{2}}}{2} < 0. \end{aligned}$$

So $\overline{AD} = d_1 = 7 + 5\sqrt{2}$

References: (1) Thomas, Peter, "Maximizing the Area of a Quadrilateral," The College Mathematics Journal, Vol 34. No 4 (September 2003), pp. 315-316.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the area of the quadrilateral is maximum when $\overline{AD} = 7 + 5\sqrt{2}$.

Let $\overline{AD} = x, s$ be the semiperimeter and Δ the area of the quadrilateral. Since the length of any side of a quadrilateral must be less than the sum of the lengths of the other three sides, we have $19 - 112\sqrt{2} < x < -7 + 19\sqrt{2}$. It is well known that

$$\Delta \le \sqrt{\left(s - \overline{AB}\right)\left(s - \overline{BC}\right)\left(s - \overline{AB}\right)\left(s - \overline{AD}\right)},$$

so that $16\Delta^2 \le f(x)$, where

$$f(x) = -x^4 + 2\left(571 - 282\sqrt{2}\right)x^2 + 32(27 + 13\sqrt{2})x - 454337 + 314940\sqrt{2}.$$

It can be checked readily by differentiation that for $19 - 11\sqrt{2} < x < -7 + 19\sqrt{2},\ f(x)$ attains its unique maximum at $x = 7 + 5\sqrt{2}$. Hence

$$\Delta \le \frac{\sqrt{f(7 + 5\sqrt{2})}}{4} = 14\sqrt{-137 + 106\sqrt{2}}.$$

It can also be checked readily that the area of the quadrilateral with sides $\overline{AB} = 1 + 3\sqrt{2},\ \overline{BC} = 6 + 4\sqrt{2},\ \overline{CD} = -14 + 12\sqrt{2},\ \overline{AD} = 7 + 5\sqrt{2}$,

$$\overline{AC} = \sqrt{7\left(-55 + 58\sqrt{2}\right)} \text{ in fact equals } 14\sqrt{-137 + 106\sqrt{2}}.$$

This completes the solution.

Also solved by Arkardy Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Henry Ricardo, New York Math Circle, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5338:** *Proposed by Arkady Alt, San Jose, CA*

Determine the maximum value of

$$F(x,y,z) = \min\left\{\frac{|y-z|}{|x|}, \frac{|z-x|}{|y|}, \frac{|x-y|}{|z|}\right\},$$

where x, y, z are arbitrary nonzero real numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the maximum value of $F(x,y,z)$ is 1.

We first prove that

$$F(x,y,z) \le 1, \qquad (1)$$

by showing that at least one of the numbers $\dfrac{|y-z|}{|x|}, \dfrac{|z-x|}{|y|}, \dfrac{|x-y|}{|z|}$ is less than equal to 1.

Suppose, on the contrary, that all of them are greater than 1. From $\dfrac{|y-z|}{|x|} > 1$, we obtain

$$(y-z)^2 > x^2, \text{ or } (x+y-z)(x-y+z) < 0. \qquad (2)$$

Similarly from $\dfrac{|z-x|}{|y|} > 1$, and $\dfrac{|x-y|}{|x|} > 1$, we obtain respectively

$$(x-y-z)(x+y-z) > 0, \qquad (3)$$

and

$$(x-y-z)(x-y+z) > 0. \qquad (4)$$

Multiplying $(2), (3)$ and (4) together. we obtain

$$(x+y-z)^2(x-y+z)^2(x-y-z)^2 < 0,$$

which is false. Thus (1) holds. Since $F(2,-1,1) = 1$, we see that the maximum value of $F(x,y,z)$ is 1 indeed.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We claim that the maximum value equals 1.

Let $x > 0$. Then $F(x, x+1, -1) = \min\left\{\dfrac{x+2}{x}, \dfrac{x+1}{x+1}, \dfrac{1}{1}\right\} = 1.$

So the maximum value is ≥ 1.

Suppose the maximum value is > 1. Then there is a triple (x,y,z) with

$$|y-z| > |x|,\ |z-x| > |y|,\ |x-y| > |z|. \qquad (1)$$

By cyclic symmetry, we can assume that $x \le \min(y,z)$.

Assume first that $x \le y \le z$. Then (1) reads as

$z-y > |x|,\ z-x > |y|,\ y-x > |z|$. So $z-x = (z-y)+(y-x) > |x|+|z| \ge z-x$

which is a contradiction.

Assume next that $x \le z \le y$. Then (1) reads as

$y-z>|x|,\ z-x>|y|,\ y-x>|z|$. So $y-x=(y-z)+(z-x)>|x|+|y|\ge y-x$, which is a contradiction.

This concludes the proof.

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" Unversity, Rome Italy

Answer: 1

The symmetry of $F(x,y,z)$ allows us to set $z\le y\le x$. We have two cases:

$$\begin{aligned} &1)\ \ 0<z\le y\le x \text{ and}\\ &2)\ \ z<0,\ 0<y\le x. \end{aligned}$$

Moreover, by observing that $F(x,y,z)=F(-x,-y,-z)$, the case $z\le y<0$, $x>0$ is recovered by the case 2) simply changing sign to all the signs and the same happens if $z\le y\le x<0$.

Now we study the case 1)

$$\frac{|y-z|}{|x|}\le\frac{|x-z|}{|y|}\iff\frac{y-z}{x}\le\frac{x-z}{y}\iff z\le x+y$$

which evidently holds true. Moreover,

$$\frac{|y-z|}{|x|}\le\frac{|x-y|}{|z|}\iff\frac{y-z}{x}\le\frac{x-y}{z}\iff yx+yz\le x^2+z^2$$

This generates two subcases.

1.1) $0<z\le y\le x$ and $yx+yz\le x^2+z^2$. In this case we must find the maximum of the function $\dfrac{y-z}{x}$. We have

$$\frac{y-z}{x}\le\frac{y-z}{y}=1-\frac{z}{y}<1.$$

The value 1 is not attained because $z\ne 0$.

1.2) $0<z\le y\le x$ and $yx+yz>x^2+z^2$. In this case we must find the maximum of the function $\dfrac{x-y}{z}$. We have

$$\frac{x-y}{z}<\frac{y-z}{x}\le\frac{y-z}{y}=1-\frac{z}{y}<1.$$

Now we study case 2)

$$F(x,y,z)=\min\left\{\frac{y-z}{x},\frac{x-z}{y},\frac{x-y}{-z}\right\}$$

and

$$\frac{y-z}{x} \leq \frac{x-z}{y} \iff z \leq x+y$$

which evidently holds true.
Moreover,

$$\frac{y-z}{x} \leq \frac{z-y}{-z} \iff y \leq x+z.$$

This generates two subcases.

2.1) $z<0$, $0<y<x$, $y \leq x+z$. In this case we must find the maximum of

$$\frac{y-z}{x} \leq \frac{x}{x} = 1.$$

The maximum achieved.

2.2) $z<0$, $0<y<x$, $y>x+z$. In this case we must find the maximum of

$$\frac{x-y}{-z} \leq \frac{x-y}{x-y} = 1.$$

The maximum achieved.

Also solved by Jerry Chu, (student at Saint George's School), Spokane, WA; Ethan Gegner, (student, Taylor University), Upland, IN, and the proposer.

- **5339:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

Calculate: $\displaystyle\int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x} dx.$

Solution 1 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria

Consider the general case for $a, b > 0$:

$$I(a,b) = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{b\sin x + a\cos x}\, \mathrm{d}x,$$

Note that the derivative of the denominator (with respect to x) is $b\cos x - a\sin x$, and $\{b\sin x + a\cos x, b\cos x - a\sin x\}$ form a base on $R[\cos x, \sin x]$, then there are $\alpha, \beta \in R$ such that

$$a\sin x + b\cos x \;=\; \alpha\,(b\sin x + a\cos x) + \beta\,(b\cos x - a\sin x)\,, \quad \forall\, x \in R$$

$$\Leftrightarrow \; b - a\alpha - b\beta = a - b\alpha + a\beta = 0.$$

We can easily solve the system to get $(\alpha,\beta) = \left(\dfrac{2ab}{a^2+b^2}, \dfrac{b^2-a^2}{a^2+b^2}\right)$, then

$$I(a,b) \;=\; \frac{1}{a^2+b^2}\int_0^{\pi/2} 2ab + (b^2-a^2)\frac{b\cos x - a\sin x}{b\sin x + a\cos x}\, \mathrm{d}x$$

$$= \frac{1}{a^2+b^2}\Big[2abx + (b^2-a^2)\ln|a\cos x + b\sin x|\Big]_0^{\pi/2}$$

$$= \frac{1}{a^2+b^2}\left(ab\pi + (b^2-a^2)\ln\frac{b}{a}\right).$$

The proposed integral equals $I(4,3) = I(3,4) = \dfrac{1}{25}\left(12\pi + 7\ln\dfrac{4}{3}\right)$.

Solution 2 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker, Angelo State University, San Angelo, TX

We attack the problem by using the classical technique for converting a rational function of $\sin x$ and $\cos x$ into an ordinary rational function. If we set

$$u = \tan\left(\frac{x}{2}\right),$$

then the "half-angle" formulas imply that

$$u^2 = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{1-\cos x}{1+\cos x}$$

and hence,

$$\cos x = \frac{1-u^2}{1+u^2}. \tag{1}$$

Also, using (1) and the known identity

$$u = \tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x},$$

we get

$$\sin x = \frac{2u}{1+u^2}. \tag{2}$$

Finally,

$$du = \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}\,dx = \frac{1}{2}\left[1+\tan^2\left(\frac{x}{2}\right)\right]\,dx = \frac{1+u^2}{2}\,dx,$$

i. e.,

$$dx = \frac{2}{1+u^2}\,du. \tag{3}$$

Since $u=0$ when $x=0$ and $u=1$ when $x=\dfrac{\pi}{2}$, (1), (2), and (3) yield (upon simplification)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}\,dx &= 4\int_0^1 \frac{2u^2-3u-2}{(3u^2-8u-3)(1+u^2)}\,du\\ &= 4\int_0^1 \frac{2u^2-3u-2}{(3u+1)(u-3)(1+u^2)}\,du. \qquad (4)\end{aligned}$$

Then, (4) and the partial fraction expansion

$$\frac{2u^2-3u-2}{(3u+1)(u-3)(1+u^2)} = \frac{12}{25}\cdot\frac{1}{1+u^2} - \frac{7}{50}\cdot\frac{u}{1+u^2} + \frac{21}{100}\cdot\frac{1}{3u+1} + \frac{7}{100}\cdot\frac{1}{u-3}$$

imply that

$$\int_0^{\frac{\pi}{2}} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}\,dx = 4\int_0^1 \frac{2u^2-3u-2}{(3u+1)(u-3)(1+u^2)}\,du$$

$$= \frac{48}{25}\tan^{-1}u\Big]_0^1 - \frac{7}{25}\ln\left(1+u^2\right)\Big]_0^1 + \frac{7}{25}\ln|3u+1|\Big]_0^1$$

$$+ \frac{7}{25}\ln|u-3|\Big]_0^1$$

$$= \frac{12\pi}{25} - \frac{7}{25}\ln 2 + \frac{7}{25}\ln 4 + \frac{7}{25}\ln 2 - \frac{7}{25}\ln 3$$

$$= \frac{12\pi}{25} + \frac{7}{25}\ln\left(\frac{4}{3}\right)$$

Solution 3 by Ethan Gegner, (student, Taylor University), Upland, IN

The value of the integral is $\frac{1}{25}\left(12\pi + 7\log(4/3)\right)$.

Define

$$I = \int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}dx$$

$$A = \int_0^{\pi/2} \frac{\sin x}{3\cos x + 4\sin x}dx$$

$$B = \int_0^{\pi/2} \frac{\cos x}{3\cos x + 4\sin x}dx.$$

Then

$$I = 3A + 4B$$

$$I + A - B = \int_0^{\pi/2} \frac{3\cos x + 4\sin x}{3\cos x + 4\sin x} dx = \frac{\pi}{2}$$

$$I - 6A = \int_0^{\pi/2} \frac{-3\sin x + 4\cos x}{3\cos x + 4\sin x} dx = \int_3^4 \frac{1}{u} du = \log(4/3)$$

Solving this system yields $I = \dfrac{1}{25}\left(12\pi + 7\log(4/3)\right).$

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Since $\dfrac{d}{dx}\left(ax + b\ln(2\cos x + 4\sin x)\right) = \dfrac{(4a-3b)\sin x + (3a+4b)\cos x}{3\cos x + 4\sin x}$ when $3\cos x + 4\sin x > 0$ and $b \in \Re$, if we take $a, b, \in \Re$ such that $4a - 3b = 3$ and $3a + 4b = 4$, that is, $a = \dfrac{24}{25}$ and $b = \dfrac{7}{25}$, we obtain that $\dfrac{1}{25}\left(24x + 7\ln(3\cos x + 4\sin x)\right)$is a primitive of $\dfrac{3\sin x + 4\cos x}{3\cos x + 4\sin x}$ in $[0, \pi/2]$, so, by Barrow's rule,

$$\begin{aligned}
\int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x} dx &= \frac{1}{25}\left(24x + 7\ln(3 + 4()\right)\Big|_0^{\pi/2} \\
&= \frac{1}{25}\left(12x + 7\ln(3\cdot 0 + 4\cdot 1)\right) - \frac{1}{25}\left(24\cdot 0 + 7\ln(31 + 4\cdot 0\right) \\
&= \frac{12\pi}{25} + \frac{7}{25}\ln\left(\frac{4}{3}\right).
\end{aligned}$$

Solution 5 by Brian D. Beasely, Presbyterian College, Clinton, SC

We let $f(x) = 3\sin x + 4\cos x$ and $g(x) = 3\cos x + 4\sin x$. Since $g'(x) = -3\sin x + 4\cos x$, we seek constants A and B such that

$$\frac{f(x)}{g(x)} = A\left(\frac{g'(x)}{g(x)}\right) + B.$$

This produces $A = 7/25$ and $B = 24/25$, so

$$\begin{aligned}
\int_0^{\pi/2} \frac{f(x)}{g(x)} dx &= \int_0^{\pi/2} \left[A\left(\frac{g'(x)}{g(x)}\right) + B\right] dx \\
&= A\ln(g(x)) + Bx]_0^{\pi/2} \\
&= A\ln\left(\frac{4}{3}\right) + B\left(\frac{\pi}{2}\right) \\
&= \frac{7}{25}\ln\left(\frac{4}{3}\right) + \frac{12\pi}{25}.
\end{aligned}$$

Addendum. We may generalize the above technique to show that
$$\int_0^{\pi/2} \frac{m\sin x + n\cos x}{3\cos x + 4\sin x}dx = A\ln\left(\frac{4}{3}\right) + B\left(\frac{\pi}{2}\right),$$
where $A = (-3m + 4n)/25$ and $B = (4m + 3n)/25$.

We may further generalize to show that
$$\int_0^{\pi/2} \frac{m\sin x + n\cos x}{p\cos x + q\sin x}dx = A\ln\left|\frac{q}{p}\right| + B\left(\frac{\pi}{2}\right),$$
where $A = (-pm + qn)/(p^2 + q^2)$ and $B = (qm + pn)/(p^2 + q^2)$, provided we place appropriate restrictions on the values of p and q (to keep $p\cos x + q\sin x \neq 0$ for each x in $[0, \pi/2]$, to avoid $p = 0$ or $q = 0$, etc.).

Also solved by Arkady Alt, San Jose, CA; Andrea Fanchini, Gantú, Italy; Paul M. Harms, North Newton, KS; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kee-Wai Lau, - Hong Kong, China; Daniel López, Center for Mathematical Sciences, UNAM, Morelia, Mexico; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Henry Ricardo (two solutions), New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Vu Tran (student, Purdue University),West Lafayette, IN; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănesti, Romania, and the proposers.

- **5340:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

Let a, b and c be the side-lengths, and s the semi-perimeter of a triangle. Show that
$$\frac{a^2+b^2}{(s-c)^2} + \frac{b^2+c^2}{(s-a)^2} + \frac{c^2+a^2}{(s-b)^2} \geq 24.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Changing variables by letting $s - a = x$, $s - b = y$ and $s - c = z$ the proposed inequality is equivalent to the following one, for x, y and z positive real numbers:
$$\sum_{\text{cyclic}} \left(1 + \frac{y}{z}\right)^2 + \left(1 + \frac{x}{z}\right)^2 \geq 24.$$

The last inequality follows by the power-mean, arithmetic-mean, geometric-mean inequality:
$$\begin{aligned}
\sqrt{\frac{\sum\limits_{\text{cyclic}} \left(1+\frac{y}{z}\right)^2 + \left(1+\frac{x}{z}\right)^2}{6}} &\geq \frac{\sum\limits_{\text{cyclic}} \left(1+\frac{y}{z}\right) + \left(1+\frac{x}{z}\right)}{6} \\
&= 1 + \frac{\sum\limits_{\text{cyclic}} \left(\frac{y}{z} + \frac{x}{z}\right)}{6} \\
&\geq 1 + \sqrt[6]{\prod_{\text{cyclic}} \frac{y}{z}\cdot\frac{x}{z}} \\
&= 2
\end{aligned}$$

from where the result follows, with equality if and only if $x = y = z$, that is if $a = b = c$.

Solution 2 by Nikos Kalapodis, Patras, Greece

$a + b + c = 2s \Longrightarrow a^2 = (s - b + s - c)^2$.
Using the well-known inequality $(x + u)^2 \geq 4xy$ for $x = s - b$ and $y = s - c$ we have

$$(s - b + s - c)^2 \geq 4(s - b)(s - c), \text{ i.e.,}$$

$$a^2 \geq 4(s - b)(s - c) \qquad (1)$$

Similarly we obtain,

$$b^2 \geq 4(s - c)(s - a) \qquad (2)$$

$$c^2 \geq 4(s - a)(s - b). \qquad (3)$$

Applying the well known inequality $x^2 + y^2 \geq 2xy$, to (1), (2), and (3) we have

$$\frac{a^2 + b^2}{(s - c)^2} + \frac{b^2 + c^2}{(s - a)^2} + \frac{c^2 + a^2}{(s - b)^2} =$$

$$\left[\left(\frac{a}{s - b}\right)^2 + \left(\frac{a}{s - c}\right)^2\right] + \left[\left(\frac{b}{s - c}\right)^2 + \left(\frac{b}{s - a}\right)^2\right] + \left[\left(\frac{c}{(s - a)}^2 + \left(\frac{c}{(s - b)^2}\right)\right] \geq$$

$$\frac{2a^2}{(s - b)(s - c)} + \frac{2b^2}{(s - c)(s - a)} + \frac{2c^2}{(s - a)(s - b)} \geq 2(4 + 4 + 4) = 24.$$

Solution 3 by Arkady Alt, San Jose, CA

Note that $\sum_{cyc} \dfrac{a^2 + b^2}{(s - c)^2} \geq 24 \iff \displaystyle\sum_{cyc} \frac{a^2 + b^2}{(a + b - c)^2} \geq 6$.

Since $a^2 \geq a^2 - (b - c)^2 \iff \dfrac{a^2}{a + b - c} \geq c + a - b$

and

$$b^2 \geq b^2 - (c - a)^2 \iff \frac{b^2}{a + b - c} \geq b + c - a$$

then by AM-GM Inequality we have

$$\sum_{cyc} \frac{a^2}{(a + b - c)^2} \geq \sum_{cyc} \frac{c + a - b}{a + b - c} \geq 3\sqrt[3]{\frac{c + a - b}{a + b - c} \cdot \frac{a + b - c}{b + c - a} \cdot \frac{b + c - a}{c + a - b}} = 3$$

and

$$\sum_{cyc} \frac{b^2}{(a + b - c)^2} \geq \sum_{cyc} \frac{b + c - a}{a + b - c} \geq 3\sqrt[3]{\frac{b + c - a}{a + b - c} \cdot \frac{c + a - b}{b + c - a} \cdot \frac{a + b - c}{c + a - b}} = 3.$$

Thus, $\sum_{cyc} \dfrac{a^2 + b^2}{(a + b - c)^2} \geq 6$.

Solution 4 by D.M. Bătinetu-Giurgiu, Bucharest, Romania

We shall prove that

$$\frac{xa^m+yb^m}{(s-c)^m}+\frac{xb^m+yc^m}{(s-a)^m}+\frac{xc^m+ya^m}{(s-b)^m}\geq 3\sqrt{xy}\cdot 2^{m+1},\ \text{where } m,x,y>0.$$

Proof: We denote the area of the triangle by F, its circumradius by R and its inradius by r.

By the AM-GM inequality and taking into account that $F=sr=\sqrt{s(s-a)(s-b)(s-c)}$ we have that

$$\begin{aligned}
\sum_{cyclic}\frac{xa^m+yb^m}{(s-c)^m}\geq 2\sqrt{xy}\sum_{cyclic}\frac{\left(\sqrt{ab}\right)^m}{(s-c)^m} &\geq 2\sqrt{xy}\cdot 3\cdot\sqrt[3]{\prod_{cyclic}\frac{\left(\sqrt{ab}\right)^m}{(s-c)^m}}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{\left(\frac{abc}{(s-a)(s-b)(s-c)}\right)^m}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{\frac{(4RF)^m s^m}{(s(s-a)(s-b)(s-c))^m}}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{\frac{4^mR^mF^ms^m}{F^{2m}}}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{\frac{4^mR^ms^m}{F^m}}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{\frac{4^mR^ms^m}{s^mr^m}}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{4^m\left(\frac{R}{r}\right)^m}\\
&\geq {}^{Euler(R\geq 2r)}6\sqrt{xy}\cdot\sqrt[3]{4^m2^m}\\
&= 6\sqrt{xy}\cdot\sqrt[3]{2^{3m}}=6\sqrt{xy}\cdot\sqrt[3]{2^{3m}}=3\sqrt{xy}2^{m+1}
\end{aligned}$$

If we take $m=2$ we obtain a solution to problem 5340.

Solution 5 by Paul M. Harms, North Newton, KS

If $x > 0$, then using calculus we can show that the minimum value of both expressions

$$\begin{cases} x + \dfrac{1}{x} \\ x^2 + \dfrac{1}{x^2} \end{cases}$$

is 2 and occurs at $x = 1$. I will use several substitutions to get the left side of the problem inequality into a form easier to use.

First let $t > 0$ and $r > 0$ such that $a = rc$ and $b = tc$. Then $s = \dfrac{c}{2}(r + t + 1)$ and the left side of the problem inequality is

$$\frac{(r^2 + t^2)}{\left(\dfrac{t + r - 1}{2}\right)^2} + \frac{(t^2 + 1)}{\left(\dfrac{t - r + 1}{2}\right)^2} + \frac{(r^2 + 1)}{\left(\dfrac{r - t + 1}{2}\right)^2}.$$

Now let $\begin{cases} 2H = r + t - 1, \\ 2L = t - r + 1 \\ 2K = r - t + 1. \end{cases}$ Then $\begin{cases} r = H + K \\ t = H + L \\ L = 1 - K \end{cases}$ with H, L and K positive since $s - a,\ s - b$ and $s - c$ are positive.

The inequality in terms of the positive numbers H, K and L can be written as

$$\frac{(H + K)^2 + (H + L)^2}{H^2} + \frac{(H + L)^2 + 1}{L^2} + \frac{(H + K)^2 + 1}{K^2} \geq 24.$$

Working with the left side of the inequality we can obtain

$$\left(2 + 2\frac{K}{H} + \left(\frac{K}{H}\right)^2 + 2\frac{L}{H} + \left(\frac{L}{H}\right)^2\right) + \left(\left(\frac{H}{L}\right)^2 + 2\frac{L}{H} + 1 + \frac{1}{L^2}\right) + \left(\left(\frac{H}{K}\right)^2 + 2\frac{H}{K} + 1 + \frac{1}{K^2}\right)$$

$$= \quad 2\left(\frac{K}{H} + \frac{H}{K}\right) + 2\left(\frac{L}{H} + \frac{H}{L}\right) + 2\left(\left(\frac{H}{K}\right)^2 + \left(\frac{K}{H}\right)^2\right) + \left(\left(\frac{L}{H}\right)^2 + \left(\frac{H}{L}\right)^2\right) + 4 + \frac{1}{K^2} + \frac{1}{L^2}.$$

Each of the brackets in the last expression has the form $\left(x + \dfrac{1}{x}\right)$ or $\left(x^2 + \dfrac{1}{x^2}\right)$ so the minimum value of each bracket is 2. Then the left side of the original problem inequality is greater than or equal to $2(2) + 2(2) + 2 + 2 + 4 + \dfrac{1}{K^2} + \dfrac{1}{L^2}$. If we can show that this expression is greater than or equal 24, the original inequality is correct.

We must show that $\dfrac{1}{K^2} + \dfrac{1}{L^2}$ is at least 8. Since K and L are positive numbers such that $L = 1 - K$, the derivative of the two terms is $\dfrac{-2}{K^3} - \dfrac{2}{L^3}(-1)$. Letting the derivative equal to zero, we obtain $K = L = \dfrac{1}{2}$. The value of 8 is clearly a minimium for $\dfrac{1}{K^2} + \dfrac{1}{L^2}$. Thus the problem inequality is correct.

Solution 6 by Henry Ricardo, New York Math Circle, NY

It is a known consequence of the arithmetic-geometric mean inequality that the side-lengths of a triangle satisfy the inequality

$$(b+c-a)(c+a-b)(a+b-c) \ \leq \ abc.$$

Using this fact and the arithmetic-geometric mean inequality twice more, we have

$$\begin{aligned}
\frac{a^2+b^2}{(s-c)^2}+\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2} \ &\geq \ 3\left(\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(s-a)^2(s-b)^2(s-c)^2}\right)^{1/3} \\
&\geq \ 3\left(\frac{(2ab)(2bc)(2ac)}{[(b+c-a)(a+c-b)(a+b-c)]^2/64}\right)^{1/3} \\
&\geq \ 3\left(\frac{8a^2b^2c^2}{(abc)^2/64}\right)^{1/3} \ = \ 24.
\end{aligned}$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; D. M.Btinetu-Giurgiu, Bucharest, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Nikos Kalapodis (two additional solutions to #2 above), Patras, Greece; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru and Neculai Stanciu, Romania, and the proposer.

- **5341:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $z_1, z_2, \cdots, z_n$, and $w_1, w_2, \cdots, w_n$ be sequences of complex numbers. Prove that

$$Re\left(\sum_{k=1}^{n} z_k w_k\right) \leq \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2+\frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_k|^2.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We have

$$\begin{aligned}
Re\left(\sum_{k=1}^{n} z_k w_k\right) \leq \left|\sum_{k=1}^{n} z_k w_k\right| \ &\leq \ \sum_{k=1}^{n}|z_k|\,|w_k| \\
&= \ \sum_{k=1}^{n}\left|\frac{\sqrt{6}z_k}{\sqrt{(n+1)(n+2)}}\right|\left|\frac{\sqrt{(n+1)(n+2)}w_k}{\sqrt{6}}\right| \\
&\leq \ \frac{1}{2}\left(\sum_{k=1}^{n}\left(\left|\frac{\sqrt{6}z_k}{\sqrt{(n+1)(n+2)}}\right|^2+\left|\frac{\sqrt{(n+1)(n+2)}w_k}{\sqrt{6}}\right|^2\right)\right) \\
&= \ \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2+\frac{(n+1)(n+2)}{12}\sum_{k=1}^{n}|w_k|^2.
\end{aligned}$$

Since
$$\frac{(n+1)(n+2)}{12} = \frac{3n^2+6n+1}{20} - \frac{(n-1)(4n+7)}{60} \leq \frac{3n^2+6n+1}{20},$$
so the inequality of the problem holds.

Solution 2 by Ethan Gegner (student, Taylor University), Upland, IN

For $n \in N$, define
$$f(n) = \left(\frac{3}{(n+1)(n+2)}\right)\left(\frac{3n^2+6n+1}{20}\right)$$
and observe that f is an increasing function of n; thus, $f(n) \geq f(1) = 1/4$ for all $n \in N$. Applying AM-GM inequality and then Cauchy's inequality, we obtain

$$\begin{aligned}
\frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2 + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_k|^2 \quad &\geq \quad 2\sqrt{f(n)\left(\sum_{k=1}^{n}|z_k|^2\right)\left(\sum_{k=1}^{n}|w_k|\right)^2} \\
&\geq \quad \left(\sum_{k=1}^{n}|z_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|w_k|^2\right)^{1/2} \\
&\geq \quad \sum_{k=1}^{n}|z_k|\,|w_k| \\
&\geq \quad Re\left(\sum_{k=1}^{n} z_k w_k\right).
\end{aligned}$$

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

The AGM yields

$$\frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_x|^2 + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_x|^2 \geq 2\sqrt{\frac{3}{20}\frac{3n^2+6n+1}{n^2+3n+2}}\sqrt{\sum_{k=1}^{n}|z_x|^2 \cdot \cdot \sum_{r=1}^{n}|w_r|^2}.$$

Then we use Cauchy–Schwarz

$$\sqrt{\sum_{k=1}^{n}|z_x|^2 \cdot \sum_{r=1}^{n}|w_r|^2} \geq \sum_{k=1}^{n}|z_x| \cdot |w_k|$$

Moreover

$$Re\left(\sum_{k=1}^{n} z_k w_k\right) \leq \left|\sum_{k=1}^{n} z_k w_k\right| \leq \sum_{k=1}^{n} |z_k w_k|\,,$$

and the inequality amounts to show that

$$2\sqrt{\frac{3}{20}\frac{3n^2+6n+1}{n^2+3n+2}} \geq 1 \iff n \leq -\frac{7}{4},\ n \geq 1.$$

This completes the proof.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Let $z_k = x_k + iy_k$ and $w_k = a_k + ib_k$, for $0 \leq k \leq n$. We can assume that $x_k, y_k, a_k, b_k \geq 0$, because we can increase the left hand side of the statement of the problem by using absolute values.

We wish to prove the inequality:

$$\sum_{k=1}^{n}(a_k x_k - b_k y_k) \leq \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}\left(x_k^2 + y_k^2\right) + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}\left(a_k^2 + b_k^2\right).$$

Because of symmetry, we need only show that:

$$a_k x_k \leq \frac{3}{(n+1)(n+2)}x_k^2 + \frac{3n^2+6n+1}{20}a_k^2.$$

Considering this as a quadratic inequality for the variable x_k, we see that the discriminant is negative.

$$\Delta = a_k^2 - 4\frac{3}{(n+1)(n+2)}\frac{3n^2+6n+1}{20}a_k^2 = a_k^2\left(\frac{-4n^2+3n+7}{5(n+1)(n+2)}\right) < 0.$$

Hence, the problem is solved.

Also solved by Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL, and the proposer.

- **5342:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c, \alpha > 0$, be real numbers. Study the convergence of the integral

$$I(a,b,c,\alpha) = \int_1^{\infty}\left(a^{1/x} - \frac{b^{1/x}+c^{1/x}}{2}\right)^{\alpha} dx.$$

The problem is about studying the conditions which the four parameters, a, b, c, and α, should verify such that the improper integral would converge.

Solution 1 by Arkady Alt, San Jose, CA

Case 1. If $a = b = c$, then for any nonzero x, $a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} = 0$, so $I(a, b, c, \alpha) = 0$ for any real $\alpha > 0$.

Case 2. Suppose α isn't an integer. Then $a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}$ must be nonnegative for any x and in particular, it must be positive for $x = 1$, that is $a \geq \dfrac{b+c}{2}$.

Since $\begin{cases} 2a = b + c \\ b = c \end{cases} \iff a = b = c$ then, to avoid the trivial case 1, we will consider a, b, c such that

$a > \dfrac{b+c}{2}$ or $\begin{cases} 2a = b + c \\ b \neq c. \end{cases}$

Then, by the AM-PM inequality, for $x > 1$ we have

$$\frac{b+c}{2} > \left(\frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^x \iff \left(\frac{b+c}{2}\right)^{\frac{1}{x}} > \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2},$$

and we obtain $a^{\frac{1}{x}} > \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}$ for any $x > 1$ and that the integral is defined.

For any real $p > 0$ we have $\lim\limits_{t \to 0} \dfrac{p^t - 1}{t} = \ln p$. So, $\lim\limits_{x \to \infty} x \left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right) =$

$$\lim_{x \to \infty} x \left(a^{\frac{1}{x}} - 1\right) - \frac{1}{2}\left(\lim_{x \to \infty} x \left(b^{\frac{1}{x}} - 1\right) + \lim_{x \to \infty} x \left(c^{\frac{1}{x}} - 1\right)\right) = \ln a - \frac{\ln b + \ln c}{2} = \ln \frac{a}{\sqrt{bc}} > 0,$$

because $a > \sqrt{bc}$ if $b \neq c$ or if $a > \dfrac{b+c}{2}$.

Therefore, $\lim\limits_{x \to \infty} \dfrac{\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^{\alpha}}{\dfrac{1}{x^{\alpha}}} = \ln^{\alpha} \dfrac{a}{\sqrt{bc}} > 0$, and by the Limit Comparison Test, $I(a, b, c, \alpha)$ converges iff $\dfrac{1}{x^{\alpha}}$converges; that is, $I(a, b, c, \alpha)$ converges if $\alpha > 1$ and diverges if $\alpha \in (0, 1]$.

Case 3. Let α be a positive integer. Then the expression $\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^{\alpha}$ is defined for any positive a, b, c and since

$$\lim_{x \to \infty} \left(a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^{\alpha} = \ln^{\alpha} \frac{a}{\sqrt{bc}} > 0$$

is the limit of $I(a, b, c, \alpha)$ for $a > \sqrt{bc}$ and when $\alpha > 1$. So the situation of $a = \sqrt{bc}$ must be analyzed.

Then $\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^{\alpha} = \dfrac{(-1)^{\alpha}\left(b^{\frac{1}{2x}} - c^{\frac{1}{2x}}\right)^{2\alpha}}{2^{\alpha}}$.

Assume, without loss of generality, $b > c$. Since $\lim\limits_{x\to\infty} x\left(b^{\frac{1}{2x}} - a^{\frac{1}{2x}}\right) = \dfrac{1}{2}\ln\dfrac{b}{c} > 0$,
then $\lim\limits_{x\to\infty} \dfrac{\left(b^{\frac{1}{2x}} - a^{\frac{1}{2x}}\right)^{2\alpha}}{\frac{1}{x^{2\alpha}}} = \left(\dfrac{1}{2}\ln\dfrac{b}{c}\right)^{2\alpha} > 0,$ and by the Limit Comparison Test
$I(a,b,c,\alpha)$ is convergent iff $\dfrac{1}{x^{2\alpha}}$ convergent, that is $I(a,b,c,\alpha)$ convergent if $\alpha > 1/2$ and divergent if $\alpha \in (0, 1/2]$.

In summary,

- If $a = b = c$ then $I(a,b,c,\alpha) = 0$ is convergent for any real α;

- If $\alpha \in \Re_+/N$ and $a > \dfrac{b+c}{2}$ or $\begin{cases} 2a = b + c \\ b \neq c \end{cases}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha > 1$ and divergent for $\alpha \in (0, 1]$;

- If $\alpha \in \Re_+/N$ and $a > \sqrt{bc}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha > 1$ and divergent for $\alpha \in (0, 1]$;

- If $\alpha \in N$ and $a = \sqrt{bc}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha > 1/2$ and divergent for $\alpha \in (0, 1/2]$.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

To have the integral well defined, a necessary condition is $2a \geq b + c$.

The convergence occurs in one of the following cases:
1) if $a = b = c$ we have convergence for any value of α
2) if $\alpha > 1$ we have convergence regardless the values of a, b, c
3) if $1/2 < \alpha \leq 1$ and $a = \sqrt{bc}$ we have convergence.

Proof
If α is irrational or it is a rational p/q reduced to the lowest terms with q even, we must impose

$$2a^{1/x} - b^{1/x} - c^{1/x} \geq 0$$

but this doesn't seem to me easy to prove. A necessary condition is $2a \geq b + c$ corresponding to $x = 1$.

If $a = b = c$ the integrand is identically zero and then the integral converges regardless the value of α.

From now on, $a \neq b$ or $b \neq c$ or $a \neq c$.
We have $a^{1/x} = e^{\frac{\ln a}{x}} = 1 + \dfrac{\ln a}{x} + \dfrac{\ln^2 a}{2x^2} + \dfrac{\ln^3 a}{6x^3} + O(x^{-4})$ whence

$$\left[a^{1/x}-\frac{b^{1/x}+c^{1/x}}{2}\right]^{\alpha}=\left\{1+\frac{\ln a}{x}+\frac{\ln^2 a}{2x^2}+\frac{\ln^3 a}{6x^3}+\right.$$

$$\left.-\frac{1}{2}\Big(1+\frac{\ln b}{x}+\frac{\ln^2 b}{2x^2}+\frac{\ln^3 b}{6x^3}+1+\frac{\ln c}{x}+\frac{\ln^2 c}{2x^2}+\frac{\ln^3 c}{6x^3}+O(x^{-4})\Big)\right\}^{\alpha}=$$

$$=\frac{1}{x^{\alpha}}\Big(\ln\frac{a}{\sqrt{bc}}+\frac{\ln^2 a-\frac{\ln^2 b}{2}-\frac{\ln^2 c}{2}}{2x}+xA\Big)^{\alpha}$$

$$A=\frac{1}{6}\left(\frac{\ln^3 a}{x^3}-\frac{\ln^3 b}{2x^3}-\frac{\ln^3 c}{2x^3}\right)+O(x^{-4})$$

The positivity of $\ln\frac{a}{\sqrt{bc}}+\frac{\ln^2 a-\frac{\ln^2 b}{2}-\frac{\ln^2 c}{2}}{2x}+xA$ for x large enough, imposes $\ln\frac{a}{\sqrt{bc}}>0$ that is $a^2\geq bc$ which in turn follows by $2a\geq b+c$. Indeed

$$a^2\geq\frac{(b+c)^2}{4}=\frac{b^2+c^2+2bc}{4}\geq\frac{4bc}{4}=bc$$

Let $\alpha>1$. Since for any x large enough it is

$$\Big(\ln\frac{a}{\sqrt{bc}}+\frac{\ln^2 a-\frac{\ln^2 b}{2}-\frac{\ln^2 c}{2}}{2x}+xA\Big)^{\alpha}\leq C$$

if $\alpha>1$ the integral $\int_1^{\infty}\frac{1}{x^{\alpha}}\Big(\ln\frac{a}{\sqrt{bc}}+\frac{\ln^2 a-\frac{\ln^2 b}{2}-\frac{\ln^2 c}{2}}{2x}+xA\Big)^{\alpha}dx$ converges.

Let $1/2<\alpha\leq 1$ and $a=\sqrt{bc}$.

$$0\leq\left(a^{1/x}-\frac{b^{1/x}+c^{1/x}}{2}\right)^{\alpha}=\frac{1}{x^{2\alpha}}\Big(\frac{1}{4}(\ln b-\ln c)^2+x^2A\Big)^{\alpha}\leq\frac{C_1}{x^{2\alpha}}$$

whence convergence.

Let $0<\alpha\leq 1/2$, and $a=\sqrt{bc}$. To have convergence we need $\ln b=\ln c$ that is $b=c$, but this would yield $a=b=c$, a forbidden condition.

Also solved by the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2015

- **5361:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $ABCD$ has perimeter $P = 75 + 61\sqrt{15}$ and has $\angle B = \angle D = 90^\circ$. The lengths of the diagonals are 112 and 128. Find the lengths of the sides.

- **5362:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Two thousand forty seven death row prisoners were arranged from left to right with the numbers 1 through 2047 on their backs in this left to right order. Prisoner 1 was given a gun and shoots prisoner number 2 dead, and then gives the gun to prisoner number 3 who shoots prisoner number 4 and then gives the gun to number 5 and so on, so that every second originally numbered prisoner is shot dead.

  This process is then repeated from right to left, starting with the person (in this case number 2047) who last received the gun and then continues to proceed from right to left, and then the direction switches again, and then again until only one prisoner remains standing. What is the number of the prisoner who survives the left to right, right to left shootout? Note that if there had been 2048 prisoners, number 2047 would have no one to whom to hand the gun in the left to right direction after shooting number 2048, and so he would then start the gun in its opposite direction shooting the living prisoner to his immediate left i.e., number 2045. In this case, number 2047 gets to shoot two prisoners before he hands the gun off to another prisoner.

- **5363:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

  Let $x \in \Re$ and $A(x) = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 1 & x+1 & 1 & 1 \\ 1 & 1 & x+1 & 1 \\ 1 & 1 & 1 & x+1 \end{pmatrix}$.

  Compute $A(0) \cdot A(x) \cdot A(y) \cdot A(z), \forall x, y, z \in \Re$.

- **5364:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Prove that $\displaystyle\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}4^{-n}=1.$

- **5365:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $n\geq 3$ be a positive integer. Find all real solutions of the system

$$\left.\begin{array}{c} a_2^3(a_2^2+a_3^2+\ldots+a_{j+1}^2)=a_1^2 \\ a_3^3(a_3^2+a_4^2+\ldots+a_{j+2}^2)=a_2^2 \\ \cdots\cdots\cdots \\ a_n^3(a_n^2+a_1^2+\ldots+a_{j-1}^2)=a_{n-1}^2 \end{array}\right\}$$

  for $1<j<n$.

- **5366:** *Proposed by Ovidiu Furdui and Alina Sintămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all non-constant, differentiable functions $f:R\to R$ which verify the functional equation $f(x+y)-f(x-y)=2f'(x)f(y)$, for all $x,y\in R$.

Solutions

- **5343:** *Proposed by Kenneth Korbin, New York, NY*

  Four different Pythagorean Triangles each have hypotenuse equal to $4p^4+1$ where p is prime.

  Express the sides of these triangles in terms of p.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC

We designate the lengths of the legs of these triangles by a and b, so that $a^2+b^2=(4p^4+1)^2$. We then make use of the well-known identity

$$(w^2+x^2)(y^2+z^2)=(wy+xz)^2+(wz-xy)^2.$$

Since $4p^4+1=(2p^2)^2+(1)^2=(2p^2-1)^2+(2p)^2$, we make the appropriate substitutions into the above identity to obtain the following four expressions of $(4p^4+1)^2$ as the sum of two squares:

$$\begin{aligned}(4p^4+1)^2 &= (4p^4-1)^2+(4p^2)^2\\ &= (4p^4-8p^2+1)^2+(8p^3-4p)^2\\ &= (4p^4-2p^2+2p)^2+(4p^3-2p^2+1)^2\\ &= (4p^4-2p^2-2p)^2+(4p^3+2p^2-1)^2.\end{aligned}$$

Hence the four triangles have the following lengths for their legs:

$a=4p^4-1,\ b=4p^2;$

$a=4p^4-8p^2+1,\ b=8p^3-4p;$

$a=4p^4-2p^2+2p,\ b=4p^3-2p^2+1;$

$a = 4p^4 - 2p^2 - 2p,\ b = 4p^3 + 2p^2 - 1.$

Addendum. We note that for $p \geq 2$, these eight values of a and b are positive and distinct. We also observe that the condition that p be prime does not seem to be necessary.

Solution 2 by Trey Smith, Angelo State University, San Angelo, TX

It is well known that if $m > n$ are both positive integers then

$$(m^2 - n^2,\ 2mn,\ m^2 + n^2)$$

is a Pythagorean triple.

1. Letting $m_1 = 2p^2$ and $n_1 = 1$ yields the Pythagorean triple
$$(\ 4p^4 - 1,\ 4p^2,\ 4p^4 + 1\ ).$$
2. Letting $m_2 = 2p^2 - 1$ and $n_2 = 2p$ yields the Pythagorean triple
$$(\ 4p^4 - 8p^2 + 1,\ 8p^3 - 4p,\ 4p^4 + 1\ ).$$
3. $4p^4 + 1 = (2p^2 + 2p + 1)(2p^2 - 2p + 1)$, and
$2p^2 + 2p + 1 = p^2 + 2p + 1 + p^2 = (p + 1)^2 + p^2$. Letting $m_3 = p + 1$ and $n_3 = p$ yields the Pythagorean triple $(\ 2p + 1,\ 2p(p + 1),\ 2p^2 + 2p + 1\ )$. Multiplying each side of the associated Pythagorean triangle by $2p^2 - 2p + 1$ yields the triple
$$(\ (2p + 1)(2p^2 - 2p + 1),\ 2p(p + 1)(2p^2 - 2p + 1),\ 4p^4 + 1\ ).$$
4. Using a similar argument to 3 above, and letting $m_4 = p$ and $n_4 = p - 1$ then multiplying each side of the associated Pythagorean triangle by $2p^2 + 2p + 1$ yields the triple
$$(\ (2p - 1)(2p^2 + 2p + 1),\ 2p(p - 1)(2p^2 + 2p + 1),\ 4p^4 + 1\ ).$$

It is worth noting that the above computations produce the demonstrated four Pythagorean triangles for any given prime p. There are, however, cases where a particular choice of p yields more than four Pythagorean triangles. For example, when $p = 3$ we have the triples
$(36,\ 323,\ 325)$,
$(80,\ 315,\ 325)$,
$(91,\ 312,\ 325)$,
$(125,\ 300,\ 325)$,
$(165,\ 280,\ 325)$,
$(195,\ 260,\ 325)$,
$(204,\ 253,\ 325)$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We remove the restriction that p be prime, requiring only that p be an integer ≥ 2. It is very well known that very Pythagorean triangle (a, b, c) has the from

$$a \ = \ k(2mn)$$

$$\begin{aligned} b &= k\left(m^2 - n^2\right) \\ c &= k\left(m^2 + n^2\right), \end{aligned}$$

where $k \geq 1$, and m and n are relatively prime integers of opposite parity with $m > n$. Thus we need to write $4p^4 + 1$ in the form of $k\left(m^2 + n^2\right)$ in four different ways.

We have the obvious choices

$$4p^4 + 1 = 1 \cdot \left[\left(2p^2\right)^2 + 1^2\right] \quad \textit{and}\ 4p^4 + 1 = 4p^4 - 4p^2 + 1 + 4p^2 = 1 \cdot \left[\left(2p^2 - 1\right)^2 + (2p)^2\right].$$

A different factorization produces two more triangles:

$$\begin{aligned} 4p^4 + 1 &= \left(2p^2 - 2p + 1\right) \cdot \left(2p^2 + 2p + 1\right) \\ &= \left[p^2 + (p-1)^2\right]\left[(p+1)^2 + p^2\right] \\ &= \left(2p^2 - 2p + 1\right)\left[(p+1)^2 + p^2\right] \text{ and} \\ &= \left(2p^2 + 2p + 1\right)\left[p^2 + (p-1)^2\right]. \end{aligned}$$

We summarize the results in Table 1:

k	m	n	$a = k(2mn)$	$b = k(m^2 - n^2)$	$c = k(m^2 + n^2)$
1	$2p^2$	1	$4p^2$	$4p^4 - 1$	$4p^4 + 1$
1	$2p^2 - 1$	$2p$	$8p^3 - 4p$	$4p^4 - 8p^2 + 1$	$4p^4 + 1$
$2p^2 - 2p + 1$	$p + 1$	p	$4p^4 - 2p^2 + 2p$	$4p^3 - 2p^2 + 1$	$4p^4 + 1$
$2p^2 + 2p + 1$	p	$p - 1$	$4p^4 - 2p^2 - 2p$	$4p^3 - 2p^2 + 1$	$4p^4 + 1$

It appears that we have four triangles with the required hypotenuse, but we need to check they are really distinct. Since all of the "a legs" are even and the "b legs" odd, we only need to compare the values for a and show they are all distinct. This requires 6 comparisons.

For instance, if it were the case that the first two triangles were the same for some value of p, we would have $4p^2 = 8p^3 - 4p$.

then $0 = 8p^3 - 4p^2 - 4p = 4p(p-1)(2p+1)$, which is impossible.

The other comparisons also prove to be impossible.

Therefore, we do have four distinct Pythagorean triangles with hypotenuse $4p^2 + 1$.

An example with $p = 2$.

k	m	n	$a = k(2mn)$	$b = k(m^2 - n^2)$	$c = k(m^2 + n^2)$
1	8	1	16	63	65
1	7	4	56	33	65
5	3	2	60	250	65
13	2	1	52	39	65

Note that these four triples are all possible with triples with hypotenuse 65, so the result proved is, in general, the best possible.

The four triples produced for $p = 3$, so that $4 \cdot 3^4 + 1 = 325$:

k	m	n	a	b	c
1	18	1	36	323	325
1	17	6	204	253	325
13	4	3	312	91	325
25	3	2	300	125	325

A Deeper Look: There are many more such triangles having hypotenuse $4p^4 + 1$. Consider the following construction suggested the last row of our table.

The generating pair $m = 2, n = 1$ produces a Pythagorean triangle with hypotenuse 5. If we can find a value of p such that 5 divides $4p^4 + 1$, then we can let $k = \dfrac{4p^4+1}{5}$, $m = 2$ and $n = 1$ and produce the triangle.

$$a = k(2mn) = 4k;\ b = k\left(m^2 - n^2\right) = 3k;\ c = \left(m^2 + n^2\right) = \frac{4p^4+1}{5} \cdot 5 = 4p^4 + 1.$$

Are there any such p? Well,

$$5\big|(4p^4+1) \iff 4p^4 + 1 \equiv 0(\text{mod } 5) \iff -p^4 \equiv -1(\text{mod } 5) \iff p^4 \equiv (1\text{mod } 5).$$

By Fermat's Little Theorem, this last condition is true for all p relatively prime to 5. That is, for any p not divisible by 5, we have a Pythagorean triangle with hypotenuse $4p^4 + 1$.

For instance, with $p = 2$, $k = \dfrac{4 \cdot 2^4 + 1}{5} = \dfrac{65}{3} = 13$, and this construction re-creates the last row of our table.

Let's designate the triple found via this construction at $PT(2;\ 13, 2, 1)$.

In general, we designate by $PT(p; k, m, n)$ the triangle having hypotenuse $4 \cdot p^4 + 1$, generated by $k = \dfrac{4 \cdot p^4 + 1}{m^2 + n^2}$, m and n, where m and n are relatively prime integers of opposite parity with $m > n$.

With $p = 3$, $k = \dfrac{4 \cdot 3^4 + 1}{5} = \dfrac{325}{5} = 65$, and this construction yields a *new* triangle with hypotenuse 325; $(260, 195, 325)$ that is $PT(3;\ 65, 2, 1)$. Note that the four solutions given in Table 1 are $PT(p;\ 1, 2p^2, 1)$, $PT(p;\ 1, 2p^2 - 1, 2p)$, $PT(p;\ 2p^2 - 2p + 1, p + 1, p)$ and $PT(p;\ 2p^2 + 2p + 1, p, p - 1)$.

Continuing in this vein, the generating pair $m = 3, n = 2$ produces a Pythagorean triangle with hypotenuse 13. If we can find a value of p such that 13 divides $4p^4 + 1$, then we can let

$$k = \frac{4p^4+1}{13}, m = 3 \text{ and } n = 2 \text{ and produce the triangle}$$

$$a = k(2mn) = 12k,\ b = k\left(m^2 - n^2\right) = 5k,\ c = k\left(m^2 + n^2\right) = \frac{4p^4+1}{13} \cdot 13 = 4p^4 + 1.$$

Are there any such p? Well,

$$13 \big| 4p^4+1 \iff 4p^4+1 \equiv 0 (\text{mod } 13) \iff 4p^4 \equiv -1 (\text{mod } 13) \iff 4p^4 \equiv 12 (\text{mod } 13)$$
$$\iff p^4 \equiv 3 (\text{mod } 13).$$

It is easy to check that this last congruence is satisfied if and only if $p = 2, 3, 10$ or $11 (\text{mod } 13)$. Using any such p will produce a triangle generated by

$$k = \frac{4p^4+1}{13}, m = 3 \text{ and } n = 2 \text{ and of the form}$$

$$a = 12k,\ b = 5k,\ c = \frac{4p^4+1}{13} \cdot 13 = 4p^4+1.$$

This process can be used for any fundamental generating pair m and n.

<u>Theorem</u>: This construction produces all Pythagorean triples having the desired hypotenuse, $4p^4+1$.

First, some evidence. For instance, we re-examined the table for $p = 2$.

k	m	n	a	b	c	PT
1	8	1	16	63	65	PT(2;1,8,1)
1	7	4	56	33	65	PT(2;1,7,4)
5	3	2	60	25	65	PT(2;5,3,2)
13	2	1	52	39	65	PT(2;13,2,1)

For $p = 3$, we also look at all Pythagorean triples with hypotenuse $4 \cdot 3^4 + 1 = 325$, where the first four triples are those shown above, produced by our procedure shown in Table 1.

k	m	n	a	b	c	PT
1	18	1	36	323	325	PT(3;1,18,1)
1	17	6	204	253	325	PT(3;1,17,6)
13	4	3	312	91	325	PT(3;13,4,3)
25	3	2	300	125	325	PT(3;25,3,2)
			80	315	325	PT(3;5,8,1)
			280	165	325	PT(3;5,7,4)
			260	195	325	PT(3;13,2,1)

<u>Proof of the theorem</u>. Suppose we are given a Pythagorean triple (a, b, c) which has hypotenuse of the form $4p^2 + 1$. We can immediately computer p from
$c = 4p^4 + 1;\ \ p = \sqrt[4]{\frac{c-1}{4}}$.

We can also computer $k = \gcd(a, b)$.

This gives us a primitive Pythagorean triple $\left(\frac{a}{k}, \frac{b}{k}, \frac{c}{k}\right)$, in which we may choose $\frac{a}{k}$ to be the even leg.

That is, we must find appropriate m and n so that

$$\frac{a}{k} = 2mn,\quad \frac{b}{k} = m^2 - n^2,\quad \frac{c}{k} = \frac{4p^2+1}{k} = m^2 + n^2.$$

By solving the last two equations, we find that $m=\sqrt{\dfrac{b+c}{2k}}$ and $n=\sqrt{\dfrac{c-b}{2k}}$.

These must be coprime integers of opposite parity, because $\left(\dfrac{a}{k},\dfrac{b}{k},\dfrac{c}{k}\right)$ is a primitive Pythagorean triple.

Therefore, (a,b,c) is $PT(p;k,m,n)$.

Caveat: Producing triples by using this construction is rather random. Given an appropriate generating pair (m,n) we must find p (and thus k) by solving the congruence $4p^4+1\equiv 0 \bmod \left(m^2+n^2\right)$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University San Angelo, TX; Jerry Chu (Student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5344:** *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan*

Let $\triangle ABC$ be isosceles with $AB=AC$. Let D be a point on side BC. A line through point D intersects rays AB and AC at points E and F respectively. Prove that $ED\cdot DF\geq BD\cdot DC$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let Γ be the circle which passes through B,C and E and let J be the other point of intersection of the line DE with Γ. Since E and F are on the rays with origin A and with orientations $\overline{AB}$ and $\overline{AC}$ respectively, we have that $DF=DJ=JF\geq DJ$ with equality if, and only if $J=C=F$, that is, if, and only if the line EF=BC, so

$$ED\cdot DF\geq ED\cdot DJ \qquad (1)$$

with equality if, and only if the line through point D given in the statement of the problem is the line BC, and, by the intersecting chords theorem, the absolute value of the power of D with respect to Γ is ED$\cdot DJ$ and also $BD\cdot DC$ that is

$$ED\cdot DJ=BD\cdot DC. \qquad (2)$$

From (1) and (2) we deduce the inequality to be shown and that equality occurs if, and only if, the line through point D is the line BC.

Solution 2 by Titu Zvonaru, Comănesti, Romania

We denote by M the midpoint of $BC, a=MB,=MC, h=AM$ and $\tan(\angle FDC)=m$. Suppose that F lies between A and C. A parallel line to EF through M intersects AB and AC at points E' and F' respectively. By Similitude, we obtain:

$$\begin{aligned}\frac{DF}{MF'}&=\frac{DC}{MC}\iff DF=\frac{MF'\cdot DC}{MC},\\ \frac{DE}{ME'}&=\frac{DB}{MB}\iff DF=\frac{ME'\cdot DB}{MB}.\end{aligned}$$

(1)

Since

$$ED{\cdot}DF \geq BD{\cdot}DC \iff \frac{ME' \cdot DB}{MB} \cdot \frac{MF' \cdot DC}{BC} \geq BD{\cdot}DC \iff ME{\cdot}MF \geq MB{\cdot}MC,$$

we deduce that it suffices to prove the statement of the problem if D is the midpoint of BC. In the following we will assume that D is the midpoint of BC.

Lt T be the projection of F to BC. It results that

$$\frac{TC}{DC} = \frac{FT}{AD} \iff \frac{MC - DT}{DC} = \frac{DT \cdot m}{AD} \Rightarrow DT = \frac{ah}{h + am}.$$

By the Pythagorean Theorem, we obtain

$$DF = \sqrt{DT^2 + FT^2} = \sqrt{\frac{a^2h^2}{(h+am)^2} + \frac{a^2h^2}{(h+am)^2}m^2} = \frac{ah}{h+am}\sqrt{1+m^2},$$

and similarly, $DE = \dfrac{ah}{h - am}\sqrt{1+m^2}$.
It results that:

$$ED{\cdot}DF \geq BD{\cdot}DC \iff \frac{a^2h^2}{h^2 - a^2m^2}\left(1+m^2\right) \iff h^2 + h^2m^2 \geq h^2 - a^2m^2 \iff \left(a^2+h^2\right)m^2 \geq 0,$$

which is true. The equality holds if and only if $m = 0$, that is, the line through D is BC.

Solution 3 by Ed Gray, Highland Beach, FL

To be specific in the case you wish to draw a diagram, let the point D be on the left of middle of side BD so that point E is on side AB in the triangle closer to B than to A. The point F on the extension of AC and is external to the triangle ABC. We shall be interested in triangles EBE and DCF.

In $\triangle BED$, let $\alpha = \angle EBD$ and let $\beta = \angle EDB$. So $\angle DEB = 180 - \alpha - \beta$. Also $\angle BCA = \alpha$ because $\triangle ABC$ is isosceles.

In $\triangle$CDF, $\angle FDC = \beta; FCD = 180 - \alpha$, and although $\triangle EBD$ and $\triangle FCD$ are not similar to one another, the law of sines holds in each triangle.

In $\triangle BED$; $\dfrac{ED}{\sin\alpha} = \dfrac{BD}{\sin(180-\alpha-\beta)} = \dfrac{BD}{\sin(\alpha+\beta)}$. So, $ED = \dfrac{BD\sin a}{\sin(\alpha+\beta)}$.

In $\triangle DCF$; $\dfrac{DC}{\sin(\alpha-\beta)} = \dfrac{DF}{\sin(180-\alpha)} = \dfrac{DF}{\sin\alpha}$. So, $DF = \dfrac{DC\sin\alpha}{\sin(\alpha-\beta)}$.

To show $ED \cdot DF \geq BD \cdot DC$ we must show that

$$\frac{(BD\sin a)\cdot(DC\sin\alpha)}{\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)} \geq BD \cdot DC, \text{ or}$$

$$\frac{\sin^2\alpha}{\sin(\alpha+\beta)\sin(\alpha-\beta)} \geq 1, \quad \text{or}$$

$$\sin^2\alpha \quad \geq \quad \sin(\alpha+\beta)\sin(\alpha-\beta)$$

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta,\ \text{ or}\\
\sin^2\alpha &\geq (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta.
\end{aligned}
$$

Adding $\cos^2\alpha$ to both sides of the above inequality we obtain

$$
\begin{aligned}
1 &\geq \cos^2\alpha-\cos^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta=\cos^2\alpha(1-\sin^2\beta)+\sin^2\alpha\cos^2\beta\\
1 &\geq \cos^2\alpha\cos^2\beta+\sin^2\alpha\cos^2\beta=(\cos^2\beta)(\cos^2\alpha+\sin^2\alpha)\\
1 &\geq \cos^2\beta,\ \text{ and this proves the conjecture.}
\end{aligned}
$$

Also solved by Michael Brozinsky, Central Islip, NY; Jerry Chu (student, Saint George's School), Spokane, WA; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro GA, and the proposer.

- **5345:** *Proposed by Arkady Alt, San Jose, CA*

Let $a, b > 0$. Prove that for any x, y the following inequality holds

$$|a\cos x+b\cos y|\leq\sqrt{a^2+b^2+2ab\cos(x+y)},$$

and find when equality occurs.

Solution 1 by Michael Brozinsky, Central Islip, NY

Since $\sqrt{u^2}=|u|$, the left hand side of the given inequality can be written as

$$a^2\cos^2x+2ab\cos x\cos y+b^2\cos^2y,$$

and so using the identities $\sin^2u=1-\cos^2u$ and $\cos(x+y)=\cos x\cos y-\sin x\sin y$, it must be shown that

$$a^2\sin^2x+b^2\sin^2y\geq 2ab\sin x\sin y.$$

This is true from the AM-GM inequality, with equality if, and only if, $a\sin x=b\sin y$.

Solution 2 by Paul M. Harms, North Newton, KS

Since each side of the inequality is a nonnegative number, the inequality holds if the square of the left side is less than or equal to the square of the right side. We need to show that

$$(a\cos x+b\sin y)^2=a^2\cos^2x+2ab\cos x\cos y+b^2\cos^2y\leq a^2+b^2+2ab\cos(x+y).$$

The last inequality is equivalent to

$$
\begin{aligned}
0 &\le a^2\left(1-\cos^2 x\right)+b^2\left(1-\cos^2 y\right)+2ab\left(\cos(x+y)-\cos x\cos y\right)\\
&= a^2\sin^2 x+b^2\sin^2 y+2ab((\cos x\cos y-\sin x\sin y)-\cos x\cos y)\\
&= (a\sin x-b\sin y)^2.
\end{aligned}
$$

Clearly, $0\le (a\sin x-b\sin y)^2$ so the problem inequality holds. Equality will hold when $a\sin x=b\sin y$ or $\dfrac{a}{b}=\dfrac{\sin x}{\sin y}$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro GA; Vu Tran (student, Purdue University), West Lafayette, IN; Nicusor Zlota, "Traian Vula" Technical College, Focsani, Romania, and the proposer.

- **5346:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Show that in any triangle ABC, with the usual notations, the following hold,

$$\frac{h_b+h_c}{h_a}r_a^2+\frac{h_c+h_a}{h_b}r_b^2+\frac{h_a+h_b}{h_c}r_c^2 \;\ge\; 2s^2,$$

where r_a is the excircle tangent to side a of the triangle and s is the triangle's semiperimeter.

Solution 1 by Moti Levy, Rehovot, Israel

From geometry of the triangle:

$$h_a=\frac{2}{\dfrac{1}{r_b}+\dfrac{1}{r_c}},\ h_b=\frac{2}{\dfrac{1}{r_a}+\dfrac{1}{r_c}}\ ,\ h_c=\frac{2}{\dfrac{1}{r_b}+\dfrac{1}{r_a}}. \qquad (1)$$

Solving (1) for r_a, r_b and r_c, we get

$$r_a = \frac{h_ah_bh_c}{h_ah_b+h_ah_c-h_bh_c}$$

$$
\begin{aligned}
r_b &= \frac{h_c h_a h_b}{h_a h_b - h_a h_c + h_b h_c} \\
r_c &= \frac{h_a h_b h_c}{-h_a h_b + h_a h_c + h_b h_c} \qquad (2)
\end{aligned}
$$

Suppose $h_a \geq h_b \geq h_c$. It follows from (2) that $r_a \leq r_b \leq r_c$. It is also easy to see that $h_a \geq h_b \geq h_c$ implies $\dfrac{h_b + h_c}{h_a} \leq \dfrac{h_c + h_a}{h_b} \leq \dfrac{h_a + h_b}{h_c}$.

So now we can apply Chebyshev's sum inequality,

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2 \geq \frac{1}{3}\left(\frac{h_b + h_c}{h_a} + \frac{h_c + h_a}{h_b} + \frac{h_a + h_b}{h_c}\right)\left(r_a^2 + r_b^2 + r_c^2\right).$$

Since $x + \frac{1}{x} \geq 2$, for $x \geq 0$,

$$\frac{h_b + h_c}{h_a} + \frac{h_c + h_a}{h_b} + \frac{h_a + h_b}{h_c} = \frac{h_b}{h_a} + \frac{h_c}{h_a} + \frac{h_c}{h_b} + \frac{h_a}{h_b} + \frac{h_a}{h_c} + \frac{h_b}{h_c} \geq 6.$$

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2 \geq 2\left(r_a^2 + r_b^2 + r_c^2\right).$$

To complete the solution, we use the well known inequality

$$r_a^2 + r_b^2 + r_c^2 \geq s^2,$$

(which can be shown by proving that $\tan^2 \dfrac{\alpha}{2} + \tan^2 \dfrac{\beta}{2} + \tan^2 \dfrac{\gamma}{2} \geq 1$, and that $r_a = s \tan \dfrac{\alpha}{2}$, $r_b = s \tan \dfrac{\beta}{2}$, $r_c = s \tan \dfrac{\gamma}{2}$).

Reference: Bottemi O., et al. Geometric inequalities (Noordhoff, 1969), 2.35 p. 27, 5.34 p. 57.

Solution 2 by Nikos Kalapodis, Patras, Greece

Applying the Cauchy-Schwartz inequality,

$$\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) \geq \left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)^2$$

for $a_1 = \dfrac{r_a}{\sqrt{h_a}}$, $a_2 = \dfrac{r_b}{\sqrt{h_b}}$, $a_3 = \dfrac{r_c}{\sqrt{h_c}}$ and $b_1 = \sqrt{h_a}$, $b_2 = \sqrt{h_b}$, $b_3 = \sqrt{h_c}$ we have

$$\left(\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c}\right)(h_a + h_b + h_c) \geq (r_a + r_b + r_c)^2,$$

i.e.,

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_v} r_c^2 \geq 2\left(r_a r_b + r_b r_c + r_c r_a\right). \qquad (1)$$

Taking into account the well-known formulas $S^2 = s(s-a)(s-b)(s-c)$ and $S = r_a(s-a) = r_b(s-b) - r_c(s-c)$ for the area S of triangle ABC, we have

$$r_a r_b + r_b r_c + r_c r_a = \frac{S^2}{(s-a)(s-b)} + \frac{S^2}{(s-b)(s-c)} + \frac{S^2}{(s-c)(s-a)}$$

$$
\begin{aligned}
&= s(s-c)+s(s-a)+s(s-b) \\
&= s\left(3s-(a+b+c)\right) \\
&= s(3s-2s)=s^2 \qquad (2)
\end{aligned}
$$

Unsing (1)and (2) we obtain the required inequality.

Solution 3 by Titu Zvonaru, Comănesti, Romania

We suppose that $a \geq b \geq c$. Denoting by F the area of triangle ABC w have

$$
\begin{aligned}
a \geq b \geq c \quad &\Longleftrightarrow \quad \frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c} \Longleftrightarrow \frac{F}{a} \leq \frac{F}{b} \leq \frac{F}{c} \Longleftrightarrow h_a \leq h_b \leq h_c \\
&\Longleftrightarrow \quad \frac{h_a+h_b+h_c}{h_a} \geq \frac{h_a+h_b+h_c}{h_b a} \geq \frac{h_a+h_b+h_c}{h_c} \\
&\Longleftrightarrow \quad \frac{h_b+h_c}{h_a} \geq \frac{h_c+h_a}{h_b} \geq \frac{h_a+h_b}{h_c}.
\end{aligned}
$$

and

$$
a \geq b \geq c \Longleftrightarrow s-a \leq s-b \leq s-c \Longleftrightarrow \frac{F}{s-a} \geq \frac{F}{s-b} \geq \frac{F}{s-c} \Longleftrightarrow r_a \geq r_b \geq r_c.
$$

Applying the Chebyshev inequality and the well known inequality

$$
x^2+y^2+z^2 \geq xy+yz+zx,
$$

we obtain

$$
\begin{aligned}
&\frac{h_b+h_c}{h_a}r_a^2+\frac{h_c+h_a}{h_b}r_b^2+\frac{h_a+h_b}{h_c}r_c^2 \\
\geq\ & \frac{1}{3}\left(\frac{h_b+h_c}{h_a}+\frac{h_c+h_a}{h_b}+\frac{h_a+h_b}{h_c}\right)\left(r_a^2+r_b^2+r_c^2\right) \\
\geq\ & \frac{1}{3}\left(\frac{h_a}{h_b}+\frac{h_b}{h_a}+\frac{h_b}{h_c}+\frac{h_c}{h_b}+\frac{h_c}{h_a}+\frac{h_a}{h_c}\right)\left(r_a r_b+r_b r_c+r_c r_a\right) \\
\geq\ & \frac{1}{3}(2+2+2)\left(\frac{F^2}{(s-a)(s-b)}+\frac{F^2}{(s-b)(s-c)}+\frac{F^2}{(s-c)(s-a)}\right) \\
=\ & 2\cdot\frac{F^2(s-c+s-a+s-b)}{(s-c)(s-b)(s-c)}
\end{aligned}
$$

$$= \quad 2 \cdot \frac{s(s-a)(s-b)(s-c)s}{(s-a)(s-b)(s-c)} = 2s^2.$$

The equality holds if and only if $a = b = c$, that is, when triangle ABC is equilateral.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Since $h_a = b\sin C, h_b = c\sin A, h_c = a\sin B$, so by the sine formula we have

$$\begin{aligned}
\frac{h_b + h_c}{h_a} &= \frac{c\sin A + a\sin B}{b\sin C} \\
&= \frac{\sin A(\sin B + \sin C)}{\sin B \sin C} \\
&= \frac{\sin A}{\sin B + \sin C}\left(4 + \frac{(\sin B - \sin C)^2}{\sin B \sin C}\right) \\
&\geq \frac{4\sin A}{\sin B + \sin C} \\
&= \frac{4\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{B-C}{2}\right)} \\
&\geq 4\sin\left(\frac{A}{2}\right).
\end{aligned}$$

Similarly, $\dfrac{h_c + h_a}{h_b} \geq 4\sin\left(\dfrac{B}{2}\right)$ and $\dfrac{h_a + h_b}{h_c} \geq 4\sin\left(\dfrac{C}{2}\right)$. Hence using the well-known relations $r_a = s\tan\left(\dfrac{A}{2}\right)$, $r_b = s\tan\left(\dfrac{B}{2}\right)$, $r_c = s\tan\left(\dfrac{C}{2}\right)$, we see that

$$\frac{1}{s^2}\left(\frac{h_b + h_c}{h_a}r_a^2 + \frac{h_c + h_a}{h_b}r_b^2 + \frac{h_a + h_b}{h_c}r_c^2\right) \geq 4\left(f(A/2) + f(B/2) + f(C/2)\right),$$

where $f(x) = \sin x \tan^2 x$, for $0 < x < \dfrac{\pi}{2}$. Since

$$\frac{d^2 f(x)}{dx^2} = \sin x + \tan x \sec x + 4\tan x \sec^3 x + 2\tan^3 x \sec x > 0,$$

so, $f(A/2) + f(B/2) + f(C/2) \geq 3f\left(\dfrac{A+B+C}{6}\right) = \dfrac{1}{2}$, and therefore the inequality of the problem holds.

Also solved by Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vula" Technical College, Focsani, Romania, and the proposers.

- **5347:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $0 < a < b$ be real numbers and let $f, g : [a, b] \to R_+^*$ be continuous functions. Prove that there exists $c \in (a, b)$ such that

$$\left(\frac{1}{f(c)} + \frac{1}{\displaystyle\int_c^b g(t)\,dt}\right)\left(g(c) + \int_a^c f(t)\,dt\right) \geq 4$$

(R_+^* represents the set of non-negative real numbers.)

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

In order to avoid non-sense expressions, as zero denominators, we may assume that f, g are not identically null. The proposed inequality may be written as

$$\frac{g(c) + \displaystyle\int_a^c f(t)dt}{2} \geq \frac{2}{\dfrac{1}{f(c)} + \dfrac{1}{\displaystyle\int_c^b g(t)dt}}.$$

The right-hand side tends to zero for $c \to b$, because $\displaystyle\int_c^b g(t)dt \to 0$. On the other hand, g, and f are not identically null so the limit of the left-hand side is positive for $c \to b$, since at least $\displaystyle\int_a^b f(t)dt > 0$ and the conclusion follows.

Solution 2 by Henry Ricardo, New York Math Circle, NY

Define $F(x) = \displaystyle\int_a^x f(t)dt \cdot \int_x^b g(t)dt$. Since $F(a) = F(b) = 0$, Rolle's theorem tells us that there exists $c \in (a, b)$ such that $0 = F'(c) = f(c)\displaystyle\int_c^b g(t)dt - g(c)\int_a^c f(t)dt$, or

$$f(c)\int_c^b g(t)dt \;=\; g(c)\int_a^c f(t)dt. \qquad (1)$$

Since f and g are non-negative, the AM-GM inequality yields

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)dt}\right)\left(g(c) + \int_a^c f(t)dt\right) \geq \frac{2}{\sqrt{f(c)\int_c^b g(t)dt}} \cdot 2\sqrt{g(c)\int_a^c f(t)dt} = 4$$

by statement (1).

Comment by solver: We are tacitly assuming that $f(c) \neq 0$. It is better to alter the problem's hypothesis so that at least f is strictly positive on $[a, b]$.

Solution 3 by Michael Brozinsky, Central Islip, NY

Assume the contrary that no such c exists so that

$$\left(\frac{1}{f(c)}+\frac{1}{\int_c^b g(t)\,dt}\right)\left(g(c)+\int_a^c f(t)\,dt\right)\geq 4(*) \text{ for all } x \text{ on}(a,b).$$

Now $\displaystyle\int_x^b g(t)dt$ and $\displaystyle\int_a^x f(t)dt$ are continuous and positive functions of x for $a \leq x \leq b$ since $f(t)$ and $g(t)$ are nonnegative and continuous. Hence from $(*)$ we have $\dfrac{g(x)}{f(x)} < 4$ for all x on (a,b) $(**)$ and also $\displaystyle\int_a^x f(t)dt < 4\cdot\int_x^b g(t)dt$ $(***)$. From $(***)$, $(**)$ then implies that $\displaystyle\int_a^x f(t)dt < 4\cdot\int_x^b 4f(t)dt$ and so letting $x \longrightarrow b$ we have a contradiction that $\displaystyle\int_a^b f(t)dt \leq 0$. Hence there exists a c on (a,b) such that $F(c)>4$, in fact, there exists a c on (a,b) such that $F(c)>M$ where M is an arbitrary positive number as the above proof shows replacing the $4'$s by M throughout.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy

We argue by contradiction assuming that

$$\left(\frac{1}{f(c)}+\frac{1}{\int_c^b g(t)dt}\right)\left(g(c)+\int_a^c f(t)dt\right)<4$$

for any $c \in (a,b)$.

Cauchy Schwarz yields

$$4>\left(\frac{1}{f(c)}+\frac{1}{\int_c^b g(t)dt}\right)\left(g(c)+\int_a^c f(t)dt\right)\geq\left(\sqrt{\frac{\int_a^c f(t)dt}{f(c)}}+\sqrt{\frac{g(c)}{\int_c^b g(t)dt}}\right)^2$$

Now we prove the Lemma

Lemma *There exists $d \in (a,b)$ such that* $\dfrac{\displaystyle\int_a^d f(t)dt}{f(d)} \geq \dfrac{\displaystyle\int_d^b g(t)dt}{g(d)}$.

Proof
$\dfrac{\displaystyle\int_a^d f(t)dt}{f(d)} \geq \dfrac{\displaystyle\int_d^b g(t)dt}{g(d)}$ if and only if

$$g(d)\int_a^d f(x)dx \geq f(d)\int_d^b g(x)dx \qquad (1)$$

Now let $g(b)=g_0>0$. A value d can be chosen so close to b such that $|g(x)-g_0|\leq g_0/2$ for any $x\in(d,b]$. For the same reasons $|f(x)-f_0|\leq f_0/2$ for any $x\in(d,b]$ where $f_0=f(b)$. Moreover we can suppose $\displaystyle\int_a^d f(x)dx \geq \frac{1}{2}\int_a^b f(x)dx = I/2>0$. We can write

$$g(d)\int_a^d f(x)dx \ge \frac{1}{2}g_0\int_a^d f(x)dx \ge \frac{1}{2}g_0\frac{I}{2}$$

and

$$\frac{3}{2}f(b)\frac{3}{2}g_0(b-d) \ge f(d)\int_d^b g(x)dx.$$

To prove (1) it suffices

$$\frac{1}{2}g_0\frac{I}{2} \ge \frac{3}{2}f(b)\frac{3}{2}g_0(b-d) \iff I \ge 9f(b)(b-d)$$

and this clearly holds provided that d is very close to b. .

Thanks to the lemma, we can write

$$4 > \left(\sqrt{\frac{\int_a^c f(t)dt}{f(c)}} + \sqrt{\frac{g(c)}{\int_c^b g(t)dt}}\right)^2 \ge \left(\sqrt{\frac{\int_d^b g(t)dt}{g(d)}} + \sqrt{\frac{g(d)}{\int_d^b g(t)dt}}\right)^2 \ge 4$$

since $x + 1/x \ge 2$ for any $x > 0$, contradiction.

Also solved by Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the the proposer.

- **5348:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \ge 1$ be an integer. Prove that

$$\int_0^1 \ln^k(1-x)\ln x\, \mathrm{d}x = (-1)^{k+1}k!(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)),$$

where ζ denotes the Riemann zeta function.

Solution 1 by Moubinool Omarjee of Lycée Henri IV, Paris, France

We change the variable letting $u = -\ln(1-x)$.

$$\begin{aligned}
\int_0^1 \ln^k(1-x)\ln x dx &= (-1)^k\int_0^{+\infty} u^k \ln(1-e^{-u})e^{-u}du \\
&= (-1)^{k+1}\int_0^{+\infty}\sum_{n=1}^{\infty}\frac{1}{n}u^k e^{-u(n+1)}du \\
&= (-1)^{k+1}\sum_{n=2}^{\infty}\frac{1}{n-1}\int_0^{+\infty} u^k e^{-un}du \\
&= (-1)^{k+1}\sum_{n=2}^{\infty}\frac{1}{n-1}\frac{1}{n^{k+1}}\Gamma(k+1)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{k+1} \sum_{n=2}^{\infty} \frac{1}{n-1} \frac{1}{n^{k+1}} k! \\
&= (-1)^{k+1} k! \sum_{n=2}^{\infty} \frac{1}{n-1} \frac{1}{n^{k+1}} \\
&= (-1)^{k+1} k! \sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \frac{1}{n^2} - \frac{1}{n^3} - \ldots - \frac{1}{n^{k+1}} \right) \\
&= (-1)^{k+1} k! \sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \sum_{n=2}^{\infty} \frac{1}{n^2} - \sum_{n=2}^{\infty} \frac{1}{n^3} - \ldots - \sum_{n=2}^{\infty} \frac{1}{n^{k+1}} \right) \\
&= (-1)^{k+1} k! \, (k+1-\zeta(2)-\zeta(3)-\ldots-\zeta(k+1))
\end{aligned}
$$

Solution 2 by Anastasios Kotronis, Athens, Greece

It is straightforward to see that $\displaystyle\sum_{n\geq 1} \frac{x^n}{n} \ln^k x$ converges uniformly on $[0,1]$ and, integrating by parts, that for n, k non negative integers:

$$
\int x^n \ln^k x \, dx = x^{n+1} \left(\frac{\ln^k x}{n+1} - \frac{k \ln^{k-1} x}{(n+1)^2} + \frac{k(k-1)\ln^{k-2} x}{(n+1)^3} - \ldots + \frac{(-1)^k k!}{(n+1)^{k+1}} \right) + c.
$$

so we have

$$
\begin{aligned}
\int_0^1 \ln^k(1-x) \ln x \, dx &\stackrel{1-x=y}{=\!=\!=} \int_0^1 \ln(1-y) \ln^k y \, dy = -\int_0^1 \sum_{n\geq 1} \frac{y^n}{n} \ln^k y \, dy = -\sum_{n\geq 1} \frac{1}{n} \int_0^1 y^n \ln^k y \, dy \\
&= (-1)^{k+1} k! \sum_{n\geq 1} \frac{1}{n(n+1)^{k+1}} = (-1)^{k+1} k! \sum_{n\geq 2} \frac{1}{(n-1)n^{k+1}} \\
&= (-1)^{k+1} k! \sum_{n\geq 2} \frac{1-1+\frac{1}{n^{k+2}}}{1-\frac{1}{n}} = (-1)^{k+1} k! \sum_{n\geq 2} \left(\frac{n}{n-1} - \sum_{m=0}^{k+1} \frac{1}{n^m} \right) \\
&= (-1)^{k+1} k! \sum_{n\geq 2} \left(\frac{n}{n-1} - 1 - \frac{1}{n} - \sum_{m=2}^{k+1} \frac{1}{n^m} \right) \\
&= (-1)^{k+1} k! \left(\sum_{n\geq 2} \left(\frac{1}{n-1} - \frac{1}{n} \right) - \sum_{n\geq 2} \sum_{m=2}^{k+1} \frac{1}{n^m} \right) \\
&= (-1)^{k+1} k! \left(1 - \sum_{m=2}^{k+1} \sum_{n\geq 2} \frac{1}{n^m} \right) = (-1)^{k+1} k! \left(1 - \sum_{m=2}^{k+1} (\zeta(m)-1) \right) \\
&= (-1)^{k+1} k! \, (k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)) \, .
\end{aligned}
$$

Solution 3 by Moti Levy, Rehovot, Israel

Clearly,

$$\int_0^1 \ln^k(1-x)\ln x dx = \int_0^1 \ln(1-x)\ln^k x dx.$$

The Taylor series of $\ln(1-x)$ is $\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}, \qquad |x| < 1.$

$$\int_0^1 \ln(1-x)\ln^k x dx = -\int_0^1 \left(\sum_{m=1}^{\infty} \frac{x^m}{m}\right) \ln^k x dx.$$

The order of summation and integration can be interchanged (since $\int_0^1 \left(\sum_{m=1}^{\infty} \frac{x^m}{m}\right) \left|\ln^k x\right| dx < \int_0^1 |\ln(1-x)\ln x|\, dx = 2 - \frac{1}{6}\pi^2 < \infty$).
Hence,

$$\int_0^1 \ln(1-x)\ln^k x dx = -\sum_{m=1}^{\infty} \frac{1}{m} \int_0^1 x^m \ln^k x dx.$$

After integration by parts of $\int_0^1 x^m \ln^k x dx$, we get the recurrence,

$$\int_0^1 x^m \ln^k x dx = -\frac{k}{m+1} \int_0^1 x^m \ln^{k-1} x dx.$$

It follows from the recurrence relation that,

$$\int_0^1 x^m \ln^k x dx = (-1)^k \frac{k!}{(m+1)^{k+1}}.$$

$$\begin{aligned}
\int_0^1 \ln(1-x)\ln^k x dx &= -\sum_{m=1}^{\infty} \frac{1}{m} (-1)^k \frac{k!}{(m+1)^{k+1}} \\
&= (-1)^{k+1} k! \sum_{m=1}^{\infty} \frac{1}{m(m+1)^{k+1}} \\
&= (-1)^{k+1} k! \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1} - \frac{1}{(m+1)^2} - \frac{1}{(m+1)^3} - \cdots - \frac{1}{(m+1)^{k+1}}\right).
\end{aligned}$$

$$\sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1}\right) = 1, \quad \sum_{m=1}^{\infty} \frac{1}{(m+1)^l} = -1 + \sum_{m=1}^{\infty} \frac{1}{m^l} = -1 + \zeta(l).$$

$$\int_0^1 \ln(1-x)\ln^k x dx = (-1)^{k+1} k! \left(k + 1 - \sum_{l=2}^{k+1} \zeta(l)\right).$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The solution to 5340 of Paolo Perfetti of the Mathematics Department at Tor Verga University in Rome, Italy, was inadvertently omitted by the editor from the list of those who had solved the problem. But on the other hand, Paolo also solved 5322, but he inadvertently forgot to send it to the editor on time. Paolo Perfetti should be credited with having solved both 5322 and 5340.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2016

- **5367:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with integer length sides and integer area. The vertices have coordinates $A(0,0)$, $B(x,y)$ and $C(z,w)$ with $\sqrt{x^2+y^2}-\sqrt{z^2+w^2}=1$.

  Find positive integers x, y, z and w if the perimeter is 84.

- **5368:** *Proposed by Ed Gray, Highland Beach, FL*

  Let *abcd* be a four digit number in base 10, none of which are zero, such that the last four digits in the square of *abcd* are *abcd*, the number itself. Find the number *abcd*.

- **5369:** *Proposed by Chirita Marcel, Bucuresti, Romania*

  Let convex quadrilateral $ABCD$ have area S and side lengths $\overline{AB}=a, \overline{BC}=b, \overline{CD}=c, \overline{DA}=d$. Show that

  $$2\left(a+b+c+d\right)^2+a^2+b^2+c^2+d^2\geq 36\sqrt{\left(S^2+abcd\cos^2\frac{A+C}{2}\right)}.$$

- **5370:** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $f(x)$ and $g(x)$ be arbitrary functions defined for all $x\in\Re$. Prove that there is a function $h(x)$ such that

  $$\left(f(x)-h(x)\right)^{2015}\cdot\left(g(x)-h(x)\right)^{2015}$$

  is an odd function for all $x\in\Re$.

- **5371:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_1, a_2, \ldots, a_n$ be positive real numbers where $n\geq 4$. Prove that

  $$\left(\frac{a_1}{a_n+a_2}\right)^2+\left(\frac{a_2}{a_1+a_3}\right)^2+\ldots+\left(\frac{a_n}{a_{n-1}+a_1}\right)^2\geq\frac{4}{n}$$

- **5372:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  (a) Let $k \geq 2$ be an integer. Calculate

$$\int_0^\infty \frac{\ln(1+x)}{x\sqrt[k]{x}}dx.$$

  (b) Calculate

$$\int_0^\infty \frac{\ln(1-x+x^2)}{x\sqrt{x}}dx.$$

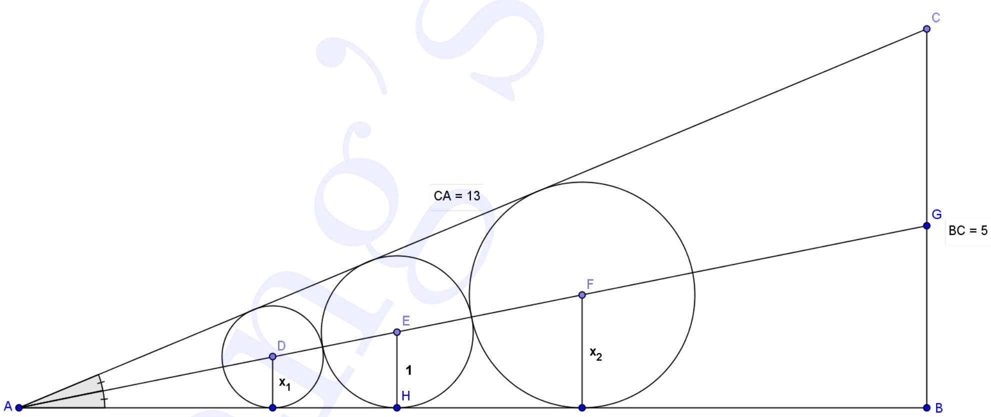
Solutions

- **5349:** *Proposed by Kenneth Korbin, New York, NY*

Given angle A with $\sin A = \dfrac{5}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

Solution by Andrea Fanchini, Cantú, Italy

I) angle A is acute.



With the notations of the figure we have

$$AB = \sqrt{13^2 - 5^2} = 12$$

the centers of the circles are on the bisector of A and we know that the bisector divides the opposite side as the ratio of the lengths of the adjacent sides, so

$$\frac{BG}{GC} = \frac{12}{13} \quad \Rightarrow \quad \frac{BG}{5 - BG} = \frac{12}{13} \quad \Rightarrow \quad BG = \frac{12}{5}$$

Now we have that

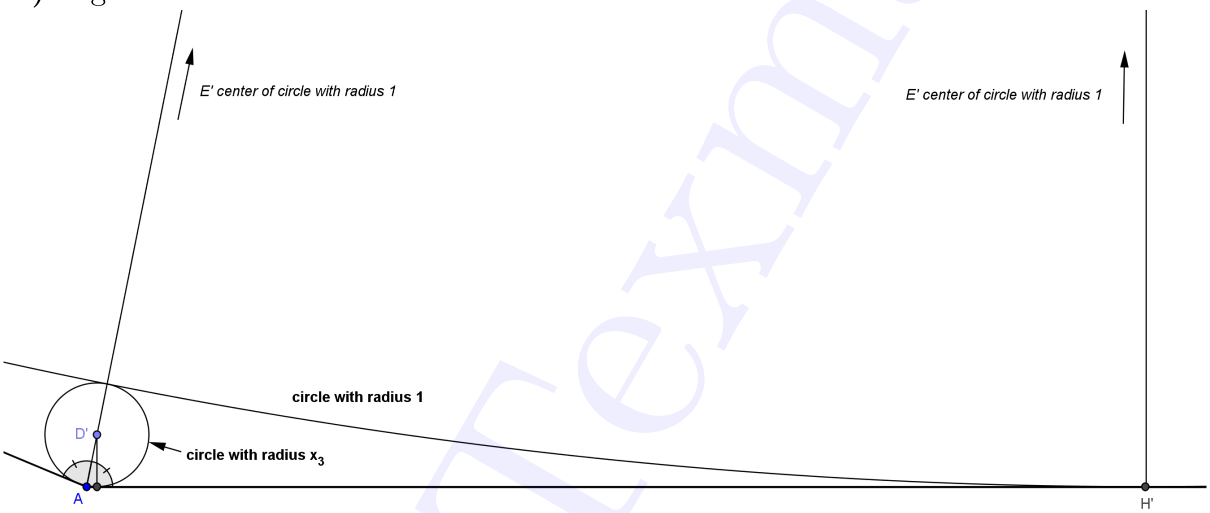
$$\tan\frac{A}{2} = \frac{\frac{12}{5}}{12} = \frac{1}{5} \quad \Rightarrow \quad AH = 5, \quad AE = \sqrt{5^2 + 1^2} = \sqrt{26}, \quad \sin\frac{A}{2} = \frac{1}{\sqrt{26}}$$

Finally, we obtain the two solutions

$$\sin\frac{A}{2}=\frac{x_1}{AD}\quad\Rightarrow\quad\frac{1}{\sqrt{26}}=\frac{x_1}{\sqrt{26}-1-x_1}\quad\Rightarrow\quad x_1=\frac{\sqrt{26}-1}{\sqrt{26}+1}$$

$$\sin\frac{A}{2}=\frac{x_2}{AF}\quad\Rightarrow\quad\frac{1}{\sqrt{26}}=\frac{x_2}{\sqrt{26}+1+x_2}\quad\Rightarrow\quad x_2=\frac{\sqrt{26}+1}{\sqrt{26}-1}$$

II) angle A is obtuse.



In this case $\angle E'AH'=90^\circ-\dfrac{A}{2}$, so with the notations of the figure we have

$$\tan\left(90^\circ-\frac{A}{2}\right)=\cot\frac{A}{2}=5\quad\Rightarrow\quad AH'=\frac{1}{5},\quad AE'=\sqrt{\left(\frac{1}{5}\right)^2+1^2}=\frac{\sqrt{26}}{5}$$

Finally, we obtain the other two solutions (where F' is the center of circle with radius x_4)

$$\sin\left(90^\circ-\frac{A}{2}\right)=\frac{x_3}{AD'}\quad\Rightarrow\quad\frac{5}{\sqrt{26}}=\frac{x_3}{\frac{\sqrt{26}}{5}-1-x_3}\quad\Rightarrow\quad x_3=\frac{\sqrt{26}-5}{\sqrt{26}+5}$$

$$\sin\left(90^\circ-\frac{A}{2}\right)=\frac{x_4}{AF'}\quad\Rightarrow\quad\frac{5}{\sqrt{26}}=\frac{x_4}{\frac{\sqrt{26}}{5}+1+x_4}\quad\Rightarrow\quad x_4=\frac{\sqrt{26}+5}{\sqrt{26}-5}$$

Solution 2 by Brain D. Beasley, Presbyterian College, Clinton, SC

Given such a circle of radius 1, there are two circles which are tangent to both sides of angle A and to the original circle; one is smaller than the original, and the other is larger. We denote the radius of the smallest of these three circles by x and the radius of the largest circle by X. We bisect angle A to create three similar right triangles, each with acute angle $A/2$ and with opposite sides of lengths x, 1, and X, respectively. Using the half-angle formula for sine, we have two cases:

If $\sin(A/2)=1/\sqrt{26}$, then the "middle" triangle (which has opposite side of length 1) has a hypotenuse of length $\sqrt{26}$. Thus the hypotenuse of the smallest triangle has length $\sqrt{26}x$, and since the smallest circle is tangent to the "middle" circle, this yields

$$\sqrt{26} = \sqrt{26}x + x + 1.$$

Hence $x = \dfrac{\sqrt{26}-1}{\sqrt{26}+1}$. Similarly, since the largest circle is tangent to the "middle" circle and has a hypotenuse of length $\sqrt{26}X$, we obtain

$$\sqrt{26}X = \sqrt{26} + 1 + X.$$

Hence $X = \dfrac{\sqrt{26}+1}{\sqrt{26}-1} = \dfrac{1}{x}$.

If $\sin(A/2) = 5/\sqrt{26}$, then the "middle" triangle (which has opposite side of length 1) has a hypotenuse of length $\sqrt{26}/5$. Thus the hypotenuse of the smallest triangle has length $\sqrt{26}x/5$, and since the smallest circle is tangent to the "middle" circle, this yields

$$\frac{\sqrt{26}}{5} = \frac{\sqrt{26}x}{5} + x + 1.$$

Hence $x = \dfrac{\sqrt{26}-5}{\sqrt{26}+5}$. Similarly, since the largest circle is tangent to the "middle" circle and has a hypotenuse of length $\sqrt{26}X/5$, we obtain

$$\frac{\sqrt{26}X}{5} = \frac{\sqrt{26}}{5} + 1 + X.$$

Hence $X = \dfrac{\sqrt{26}+5}{\sqrt{26}-5} = \dfrac{1}{x}$.

Comment: David Stone and John Hawkins of Georgia Southern University in Statesboro, GA extended the conjecture of the problem. They solved the problem and then applied the conditions of the problem again, showing that there is a third larger circle of radius $\left(\dfrac{\sqrt{26}+1}{\sqrt{26}-1}\right)^2$, or in the obtuse case, $\left(\dfrac{\sqrt{26}+5}{\sqrt{26}-5}\right)^2$, lying outside the second one. Continuing on in this manner they noted that there is an infinite sequence of circles, growing larger geometrically, lying inside angle A, each one tangent to the sides of A and to its predecessor.

And similarly they noted that there is a infinite sequence of circles *below* the circle of radius 1, growing smaller geometrically, lying inside angle A, with each one being tangent to the sides of A and to its predecessor.

Also solved by Michael Brozinsky, Central Islip, NY; Jerry Chu (Student at George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Michael Fried, Kibbutz Revivim, Israel; Ed Gray, Highland Beach, FL; Paul M.Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Nord, Saint George's School, Spokane, WA; Neculai Stanciu, "George Emil Palade" School, Buză, Romania and Titu Zvonaru,Comănesti, Romania; Cassidy Wyse, Becca Gerig and Josh Stimmel (jointly, students at Taylor University), Upland, IN; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5350:** *Proposed by Kenneth Korbin, New York, NY*

The four roots of the equation

$$x^4 - 96x^3 + 206x^2 - 96x + 1 = 0$$

can be written in the form

$$x_{1,2} = \left(\frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}}\right)^{\pm 1}$$

$$x_{3,4} = \left(\frac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}}\right)^{\pm 1}$$

where a, b, and c are positive integers.

Find a, b, and c if $(a, b, c) = 1$.

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The values a, b and c are $a = 10, b = 5$ and 21. One verifies that these values do yield the four roots of the polynomial equation. Also, note that
$(10, 5, 21) = ((10, 5), 21) = (5, 21) = 1$ as required.

Let $r = \dfrac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}}$ and $s = \dfrac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}}$. If $r, \dfrac{1}{r}, s$ and $\dfrac{1}{s}$ are the roots of the polynomial equation, then

$$(x - r)(x - \frac{1}{r})(x - s)(x - \frac{1}{s}) = x^4 - 96x^3 + 206x^2 - 96x + 1.$$

Expanding the left side of the equation and equating coefficients, one obtains

$$\begin{aligned} r + \frac{1}{r} + s + \frac{1}{s} &= 96 \\ \left(r + \frac{1}{r}\right)\left(s + \frac{1}{s}\right) &= 204. \end{aligned}$$

One calculates

$$r + \frac{1}{r} = \frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}} + \frac{\sqrt{a} - \sqrt{b + \sqrt{c}}}{\sqrt{a} + \sqrt{b + \sqrt{c}}} = \frac{2(a + b + \sqrt{c})}{a - b - \sqrt{c}},$$

$$s + \frac{1}{s} = \frac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}} + \frac{\sqrt{a} - \sqrt{b - \sqrt{c}}}{\sqrt{a} + \sqrt{b - \sqrt{c}}} = \frac{2(a + b - \sqrt{c})}{a - b + \sqrt{c}}.$$

Therefore,

$$r + \frac{1}{r} + s + \frac{1}{s} = \frac{a^2 - b^2 + c}{(a - b)^2 - c} = 24 \quad \mathbf{(1)}$$

$$\left(r+\frac{1}{r}\right)\left(s+\frac{1}{s}\right)=\frac{(a+b)^2-c}{(a-b)^2-c}=51 \quad (\mathbf{2})$$

Equation ($\mathbf{1}$) written $a^2-b^2+c=24(a-b)^2-c$ when expanded and simplified yields

$$23a^2+25b^2-48ab-25c=0. \qquad (\mathbf{3})$$

Rewrite equations ($\mathbf{1}$) and ($\mathbf{2}$) as follows:

$$\begin{aligned}(a-b)^2-c &= \frac{a^2=b^2+c}{24}\\[2ex] (a-b)^2-c &= \frac{(a+b)^2-c}{51}.\end{aligned}$$

Then $\dfrac{a^2-b^2+c}{24}=\dfrac{(a+b)^2-c}{51}$ or

$$9a^2-25b^2-16ab+25c=0. \qquad (\mathbf{4})$$

Adding equations ($\mathbf{3}$) and ($\mathbf{4}$) we get $32a^2-64ab=0$ or $a=2b$ since $a\neq 0$. Substituting $2b$ for a in equation ($\mathbf{4}$) one obtains $25c=2b^2$ or $c=\dfrac{21}{25}b^2$. Since b and c are positive integers, it follows that $b=5k$ for some integer k. Therefore, $c=21k^2$ and $a=2b=10k$. Hence, $b=5, a=10$, and $c=21$ since $(a,b,c)=1$.

Solution 2 by Jerry Chu (student, Saint George's School), Spokane, WA

Obviously, $x_1x_2=x_3x_4=1$. So we can factor the equation into $(x^2+kx+1)(x^2+lx+1)$; expanding this and equating its coefficients to those in the given equation we obtain $\begin{cases}k+l=-96\\ kl=204.\end{cases}$

Let $k=x_1+x_2=\dfrac{2(a+b+\sqrt{c}\,)}{a-b-\sqrt{c}}$, and similarly $l=x_3+x_4=\dfrac{2(a+b-\sqrt{c}\,)}{a-b+\sqrt{c}}$.

Subtracting, we get $k-l=\dfrac{8a\sqrt{c}}{(a-b)^2-c}$. And also from the above system of equations $k-l=\sqrt{(k+l)^2-4kl}=20\sqrt{21}$.

So $c=21$ because $a,b,c\in Z^+$ and $(a,b,c)=1$, therefore $5\left((a-b)^2-21\right)=2a$. Call this Equation $\mathbf{1}$.

On the other hand, $kl=\dfrac{4\left((a+b)^2-c\right)}{(a-b)^2-c}=204$. Therefore, $5\left((a+b)^2-21\right)=(51)(2a)$. Call this Equation ($\mathbf{2}$). Subtracting Equation $\mathbf{1}$ from Equation $\mathbf{2}$ gives us that

$$(a+b)^2-(a-b)^2 = \frac{50(2a)}{5}$$

$$\begin{aligned} 4ab &= 20a \\ b &= 5 \end{aligned}$$

Plugging $b = 5$ into equation 1 we obtain that $a = 10$. Therefore, $\begin{cases} a = 10 \\ b = 5 \\ c = 21. \end{cases}$

Solution 3 by Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria

Note that the equation in the statement of the problem is equivalent to

$$\left(x + \frac{1}{x}\right)^2 - 96\left(x + \frac{1}{x}\right) + 204 = 0.$$

If x is a solution to this equation, then x^{-1} is also a solution. Take

$$x = \frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}},$$

where $a, b, c \in N$ and c is not a perfect square and $(a, b, c) = 1$, which means that

$$\begin{aligned} x + \frac{1}{x} &= \frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}} + \frac{\sqrt{a} - \sqrt{b + \sqrt{c}}}{\sqrt{a} + \sqrt{b + \sqrt{c}}} \\ &= \frac{2\left(a + b + \sqrt{c}\right)}{a - b - \sqrt{c}}. \end{aligned}$$

with some basic algebraic manipulations we get

$$\left(x + \frac{1}{x}\right)^2 - 96\left(x + \frac{1}{x}\right) + 204 = \frac{16\left(a^2 + 25\left(b^2 - ab + c + \sqrt{c}\,(2b - a)\right)\right)}{(a - b - \sqrt{c})^2}.$$

therefore $2b = a$, the equation becomes $25c = 21b^2$. Since $(b, c) = 1$ then $b = 5k, c = 21n$ for some coprime positive integers k, n, and so $n = k^2$, but $(n, k) = 1$ so $n = k = 1$, and

$$(a, b, c) = (10, 5, 21).$$

The same technique works on $x_{3,4}$, so the solution to the problem is $(10, 5, 21)$.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Since $x^4 - 96x^3 + 206x^2 - 96x + 1 = (x^2 + ux + 1)(x^2 + vx + 1)$ with $u = -48 + 10\sqrt{21}$ and $v = -48 - 10\sqrt{21}$, the four roots are

$$x_{1,2} = 24 + 5\sqrt{21} \pm \sqrt{1100 + 240\sqrt{21}} = 24 + 5\sqrt{21} \pm (4\sqrt{35} + 6\sqrt{15})$$

and

$$x_{3,4} = 24 - 5\sqrt{21} \pm \sqrt{1100 - 240\sqrt{21}} = 24 - 5\sqrt{21} \pm (4\sqrt{35} - 6\sqrt{15}),$$

with $x_1 > x_2$ and $x_3 > x_4$. We also note that the roots in each pair are reciprocals, since $(24 + 5\sqrt{21})^2 - (4\sqrt{35} + 6\sqrt{15})^2 = 1$ and $(24 - 5\sqrt{21})^2 - (4\sqrt{35} - 6\sqrt{15})^2 = 1$.

To write the four roots in the desired form, we first set $d_1 = \sqrt{a} + \sqrt{b + \sqrt{c}}$, $d_2 = \sqrt{a} - \sqrt{b + \sqrt{c}}$, $d_3 = \sqrt{a} + \sqrt{b - \sqrt{c}}$, and $d_4 = \sqrt{a} - \sqrt{b - \sqrt{c}}$. Since

$d_1 > d_3 > d_4 > d_2$, this justifies our designating x_1 as the largest root above, with $x_1 > x_3 > x_4 > x_2$. As a result, we require $x_1 + x_2 = d_1/d_2 + d_2/d_1 = 48 + 10\sqrt{21}$ and $x_3 + x_4 = d_3/d_4 + d_4/d_3 = 48 - 10\sqrt{21}$. Then rationalizing produces

$$x_1 + x_2 = \frac{2(a^2 - b^2 + c + 2a\sqrt{c})}{(a-b)^2 - c} = 48 + 10\sqrt{21},$$

so we set $a^2 - b^2 + c = 24[(a-b)^2 - c]$ and $2a = 5[(a-b)^2 - c]$. Letting $c = 21$, we obtain $48ab - 23a^2 = 5(5b^2 - 105)$ and $2a + 10ab - 5a^2 = 5b^2 - 105$. Thus $10a + 2ab - 2a^2 = 0$, so $a - b = 5$, which yields $a = 10$ and $b = 5$. Similarly, we note that $(a, b, c) = (10, 5, 21)$ produces $x_3 + x_4 = 48 - 10\sqrt{21}$ as needed.

Finally, we observe that since there is a unique real number $x > 1$ with $x + 1/x = 48 + 10\sqrt{21}$, we may conclude

$$x_1 = 24 + 5\sqrt{21} + 4\sqrt{35} + 6\sqrt{15} = \frac{\sqrt{10} + \sqrt{5 + \sqrt{21}}}{\sqrt{10} - \sqrt{5 + \sqrt{21}}}.$$

Similarly, we have the corresponding results for x_2, x_3, and x_4.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Neculai Stanciu, "George Emil Palade" School, Buză, Romania and Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro GA; Vu Tran (student, Purdue University), West Lafayette, IN, and the proposer.

- **5351:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Let x, y, z be positive real numbers. Show that

$$\frac{xy}{x^3 + y^3 + xyz} + \frac{yz}{y^3 + z^3 + xyz} + \frac{zx}{z^3 + x^3 + xyz} \le \frac{3}{x + y + z}.$$

Solution 1 by Ed Gray, Highland Beach, FL

Divide the numerator and denominator of the first term on the left side of the inequality by xy, and the numerator and denominator of the second term by yz and similarly the third term by zx. Thus, the left hand side becomes

$$\frac{1}{\dfrac{x^3 + y^3}{xy} + z} + \frac{1}{\dfrac{y^3 + z^3}{yz} + x} + \frac{1}{\dfrac{z^3 + x^3}{zx} + y}.$$

$$\begin{aligned} \frac{x^3 + y^3}{xy} + z &= \frac{(x+y)(x^2 - xy + y^2)}{xy} + z \\ &= (x+y)\left(\frac{x^2}{xy} - 1 + \frac{y^2}{xy}\right) + z \end{aligned}$$

$$= \quad (x+y)\left(\frac{x}{y}-1+\frac{y}{x}\right)+z$$

But $\frac{x}{y}+\frac{y}{z}-1 \geq 1$, so $\frac{x^3+y^3}{xy}+z \geq (x+y+z)$, and $\frac{1}{\frac{x^3+y^3}{xy}+z} \leq \frac{1}{x+y+z}$.

Each of the other two terms are handled in precisely the same manner, so, to avoid repetition,

$$\frac{1}{\frac{x^3+y^3}{xy}+z}+\frac{1}{\frac{y^3+z^3}{yz}+x}+\frac{1}{\frac{z^3+x^3}{zx}+y} \leq \frac{1}{x+y+z}+\frac{1}{y+z+x}+\frac{1}{z+x+y}=\frac{3}{x+y+z}.$$

Note that equality holds if, and only if, $x=y=z$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We have

$$\begin{aligned}
& \frac{xy}{x^3+y^3+xyz}+\frac{yz}{y^3+z^3+xyz}+\frac{zx}{z^3+x^3+xyz} \\
= \quad & \frac{1}{x+y+z+\frac{(x+y)(x-y)^2}{xy}}+\frac{1}{x+y+z+\frac{(y+z)(y-z)^2}{yz}}+\frac{1}{x+y+z+\frac{(z+x)(z-x)^2}{zx}} \\
\leq \quad & \frac{1}{x+y+z}+\frac{1}{x+y+z}+\frac{1}{x+y+z} \\
= \quad & \frac{3}{x+y+z}, \text{ as required.}
\end{aligned}$$

Solution 3 by Arkady Alt, San Jose, CA

Since $x^3+y^3 \geq xy(x+y) \iff x^3+y^3-xy(x+y)=(x+y)(x-y)^2 \geq 0$ then

$$\sum_{cyc}\frac{xy}{x^3+y^3+xyz} \leq \sum_{cyc}\frac{xy}{xy(x+y)+xyz}=\sum_{cyc}\frac{1}{x+y+z}=\frac{3}{x+y+z}.$$

Also solved by Dionne T. Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Nikos Kalapodis, Patras, Greece; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria Spain; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănesti, Romania; and the proposers.

- **5352:** *Proposed by Arkady Alt, San Jose, CA*

  Evaluate $\sum\limits_{k=0}^{n} x^k - (x-1)\sum\limits_{k=0}^{n-1}(k+1)x^{n-1-k}$.

Solution 1 by G.C. Greubel, Newport News, VA

Consider the series $\displaystyle\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for which the series in question becomes

$$\begin{aligned}
S &= \sum_{k=0}^{n} x^k - (x-1)\sum_{k=0}^{n-1}(k+1)\,x^{n-k-1} \\
&= \frac{1-x^{n+1}}{1-x} + (1-x)\left[\sum_{k=0}^{n-1} x^{n-k-1} + \sum_{k=0}^{n} k\,x^{n-k-1}\right] \\
&= \frac{1-x^{n+1}}{1-x} + (1-x)\,x^{n-1}\left[\frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}} + x\,\partial_x\left(\frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}}\right)\right] \\
&= \frac{1-x^{n+1}}{1-x} + (1-x)\cdot\frac{1-x^n}{1-x} + (x-1)\,x^{n+2}\left[\frac{n(x-1)+1-x^n}{x^{n+2}}\right] \\
&= \frac{1-x^{n+1}}{1-x} + 1 - x^n + n - \frac{1-x^n}{1-x} \\
&= n+1.
\end{aligned}$$

From this it can be stated that

$$\sum_{k=0}^{n} x^k - (x-1)\sum_{k=0}^{n-1}(k+1)\,x^{n-k-1} = n+1.$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $\displaystyle\sum_{k=0}^{n-1}(k+1)x^{n-1-k} = \sum_{k=1}^{n} kx^{n-k}$, then $\displaystyle(x-1)\sum_{k=0}^{n-1}(k+1)x^{n-1-k} =$ $\displaystyle\sum_{k=1}^{n} kx^{n-k+1} - \sum_{k=2}^{n+1}(k-1)x^{n-k+1} = x^n + \sum_{k=2}^{n} x^{n-k+1} - n = -n + \sum_{k=1}^{n} x^k$, and therefore

$$\sum_{k=0}^{n} x^k - (x-1)\sum_{k=0}^{n-1}(k+1)x^{n-1-k} = 1+n.$$

Solution 3 by Henry Ricardo, New York Math Circle, NY

Denote the given expression as $F_n(x)$, where we assume that $n \geq 1$ and $x \neq 0$. Since $F_1(x) = 1 + x - (x-1)(0) = 2 = 1+1$ and $F_2(x) = (1+x+x^2) - (x-1)(x+2) = 3 = 2+1$, we conjecture that $F_n(x) = n+1$ for all nonzero values of x and prove this by induction.

Suppose that $F_N(x) = N+1$ for some integer $N \geq 3$ and all $x \neq 0$. Then

$$\begin{aligned}
F_{N+1}(x) &= \sum_{k=0}^{N+1} x^k - (x-1)\sum_{k=0}^{N}(k+1)x^{N-k} \\
&= x\sum_{k=0}^{N} x^k + 1 - (x-1)\left(\sum_{k=0}^{N-1}(k+1)x^{N-k} + N + 1\right) \\
&= x\sum_{k=0}^{N} x^k + 1 - (x-1)\left(x\sum_{k=0}^{N-1}(k+1)x^{N-k-1} + N + 1\right) \\
&= 1 + x\left(\sum_{k=0}^{N} x^k - (x-1)\sum_{k=0}^{N-1}(k+1)x^{N-k-1}\right) - (N+1)(x-1) \\
&= 1 + x(N+1) - (N+1)(x-1) = N+2 = (N+1)+1.
\end{aligned}$$

Also solved by Dionne T. Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Henry Ricardo (two additional solutions to his one above), New York Math Circle, New York; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University in Statesboro, GA, and the proposer.

5353: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients. Prove that all its zeros lie in the disk $\mathcal{D} = \{z \in C : |z| < r\}$, where

$$r = \left\{1 + \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^3\right)^{1/2}\right\}^{2/3}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

$A(z)$ is a polynomial of degree n. So $a_n \neq 0$. Let $|z| \geq r$. Then, by Hölder's inequality,

$$\frac{1}{|a_n|}|A(z)| \geq |z|^n - \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right||z|^k \geq |z|^n - \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^3\right)^{\frac{1}{3}}\left(\sum_{k=0}^{n-1}|z|^{\frac{3k}{2}}\right)^{\frac{2}{3}} = |z|^n - \left(r^{\frac{3}{2}}-1\right)^{\frac{2}{3}}\left(\frac{|z|^{\frac{3n}{2}}-1}{|z|^{\frac{3}{2}}-1}\right)^{\frac{2}{3}}$$

$$\geq |z|^n - \left(r^{\frac{3}{2}}-1\right)^{\frac{2}{3}}\left(\frac{|z|^{\frac{3n}{2}}-1}{|r|^{\frac{3}{2}}-1}\right)^{\frac{2}{3}}$$

$$= |z|^n - \left(|z|^{\frac{3n}{2}} - 1\right)^{\frac{2}{3}} > |z|^n - \left(|z|^{\frac{3n}{2}}\right)^{\frac{2}{3}} = 0.$$

So all zeros lie in the open disk $\mathcal{D}$

Solution 2 by Kee-Wai Lau, Hong Kong, China

According to Theorem (27.4) on p. 124 of [1], we have the following result:

For any p and q such that $p > 1, q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$, the polynomial $f(x) = a_0 + a_1 x + \cdots a_n z^n, a_n \neq 0$ has all of its zeros in the circle

$$|z| < \left\{ 1 + \left(\sum_{j=0}^{n-1} \left| \frac{a_j}{a_n} \right|^p \right)^{q/p} \right\}^{1/q} \leq \left(1 + n^{q/p} M^q\right)^{1/q},$$

where $M = \max \left| \dfrac{a_j}{a_n} \right|, j = 0, 1, \cdots, n-1.$

In particular, when $p = 3$, the result of the above problem follows.

Reference: 1. M. Marden: *Geometry of Polynomials,* Mathematical Surveys and Monographs Number 3, American Mathematical Society, (1966).

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5354:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c > 0$ be real numbers. Prove that the series

$$\sum_{n=1}^{\infty} \left[n \cdot \left(a^{\frac{1}{n}} - \frac{b^{\frac{1}{n}} + c^{\frac{1}{n}}}{2} \right) - \ln \frac{a}{\sqrt{bc}} \right],$$

converges if and only if $2 \ln^2 a = \ln^2 b + \ln^2 c$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let x be real. By Taylor's theorem there is a number $h = h(x),\ 0 \leq h \leq 1$, such that $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} e^{hx}$. We choose $x = \dfrac{\ln a}{n},\ x = \dfrac{\ln b}{n},\ x = \dfrac{\ln c}{n}$ and get

$$a^{\frac{1}{n}} = 1 + \frac{\ln a}{n} + \frac{\ln^2 a}{2n^2} + \frac{\ln^3 a}{6n^3} a^{\frac{h}{n}},\ 0 \leq h = h(a, n) \leq 1,$$

$$b^{\frac{1}{n}} = 1 + \frac{\ln b}{n} + \frac{\ln^2 b}{2n^2} + \frac{\ln^3 b}{6n^3} b^{\frac{h}{n}},\ 0 \leq h = h(b, n) \leq 1,$$

$$c^{\frac{1}{n}} = 1 + \frac{\ln c}{n} + \frac{\ln^2 c}{2n^2} + \frac{\ln^3 c}{6n^3} c^{\frac{h}{n}},\ 0 \leq h = h(c, n) \leq 1.$$

So

$$\sum_{n=1}^{\infty}\left[n\left(a^{\frac{1}{n}}-\frac{b^{\frac{1}{n}}+c^{\frac{1}{n}}}{2}\right)-\ln\frac{a}{\sqrt{bc}}\right]$$

$$=\sum_{n=1}^{\infty}\frac{1}{2n}\left(\ln^2 a-\frac{\ln^2 b+\ln^2 c}{2}\right)+\sum_{n=1}^{\infty}\frac{1}{6n^2}\left((\ln^3 a)a^{\frac{h(a,n)}{n}}-\frac{(\ln^3 b)b^{\frac{h(b,n)}{n}}+(\ln^3 c)b^{\frac{h(c,n)}{n}}}{2}\right).$$

The second sum is convergent. The first sum equals 0 if $\ln^2 a=\dfrac{\ln^2 b+\ln^2 c}{2}$ and it diverges if $\ln^2 a\neq\dfrac{\ln^2 b+\ln^2 c}{2}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

For $x>0$ real number it is

$$x^{\frac{1}{n}}=\exp\left(\frac{\ln x}{n}\right)=1+\frac{\ln x}{n}+\frac{\ln^2 x}{2n^2}+\mathcal{O}(n^{-3}). \tag{1}$$

Setting

$$A_n=n\cdot\left(a^{\frac{1}{n}}-\frac{b^{\frac{1}{n}}+c^{\frac{1}{n}}}{2}\right)-\ln\frac{a}{\sqrt{bc}}$$

and

$$A=\frac{\ln^2 a}{2}-\frac{\ln^2 b}{4}-\frac{\ln^2 c}{4},$$

so that $A=0\Longleftrightarrow 2\ln^2 a=\ln^2 b+\ln^2 c$, with a,b and c respectively in the place of x in (1) we get

$$A_n=\frac{A}{n}+\mathcal{O}(n^{-2}). \tag{2}$$

- If $A=0$, (2) gives that for some real $c>0$ and positive integer n_0,

  $$0\le|A_n|\le\frac{c}{n^2},\qquad n\ge n_0$$

  so $\sum_{n\ge n_0}A_n$ converges absolutely and hence the given series converges.

- If $A\neq 0$, (2) gives that for some real $c>0$ and positive integer n_0,

  $$-\frac{c}{n^2}+A\le A_n\le A+\frac{c}{n^2},\qquad n\ge n_0$$

  so $\sum_{n\ge n_0}A_n=\operatorname{sgn}(A)\cdot\infty$ and hence the given series diverges.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The general term of the series is

$$
\begin{aligned}
&n\Big(1+\frac{\ln a}{n}+\frac{\ln^2 a}{4n^2}+O\left(\frac{1}{n^3}\right)-\frac{1}{2}-\frac{\ln b}{2n}-\frac{\ln^2 a}{8n^2}+O\left(\frac{1}{n^3}\right)+\\
&-\frac{1}{2}-\frac{\ln c}{2n}-\frac{\ln^2 c}{8n^2}+O\left(\frac{1}{n^3}\right)-\ln\frac{a}{\sqrt{bc}}\Big)=\\
&=n(1-\frac{1}{2}-\frac{1}{2})+(\ln\frac{a}{\sqrt{bc}}-\ln\frac{a}{\sqrt{bc}})+n\frac{1}{8n}\left(2\ln^2 a-\ln^2 b-\ln^2 c\right)+O\left(\frac{1}{n^2}\right)=\\
&=O\left(\frac{1}{n^2}\right)
\end{aligned}
$$

whence the absolute convergence.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria, and the proposer.

Mea Culpa

Apologies to **Arkady Alt of San Jose, CA** for inadvertently not acknowledging his solutions to problems 5343, 5344 and 5346.

Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
February 15, 2016

- **5373:** *Proposed by Kenneth Korbin, New York, NY*

  Given the equation $\dfrac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}}=\sqrt{x+y\sqrt{5}}$.

  Find positive integers x and y.

- **5374:** *Proposed by Roger Izard, Dallas TX*

  In a certain triangle, three circles are tangent to the incircle, and all of these circles are tangent to two sides of the triangle. Derive a formula which gives the radius of the incircle in terms of the radii of these three circles.

- **5375*:** *Proposed by Kenneth Korbin, New York, NY*

  Prove or disprove the following conjecture. Let k be the product of N different prime numbers each congruent to $1(mod\ 4)$. Let a be any positive integer.

  Conjecture: The total number of different rectangles and trapezoids with integer length sides that can be inscribed in a circle with diameter k is exactly $\dfrac{5^N-3^N}{2}$.

  $Editor's\ comment$: The number for this problem carries with it an astrick. The astrick signifies that neither the proposer nor the editor are aware of a proof of this conjecture.

- **5376:** *Proposed by Arkady Alt , San Jose ,CA*

  Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be positive real numbers such that
  $b_1 < a_1 < b_2 < a_2 < ... < a_{n-1} < b_n < a_n$.
  Let
  $$F(x)=\frac{(x-b_1)(x-b_2)...(x-b_n)}{(x-a_1)(x-a_2)...(x-a_n)}.$$
  Prove that $F'(x) < 0$ for any $x \in Dom(F)$.

- **5377:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Show that if A, B, C are the measures of the angles of any triangle ABC and a, b, c the measures of the length of its sides, then holds

$$\prod_{cyclic} \sin^{1/3}(|A-B|) \le \sum_{cyclic} \frac{a^2+b^2}{3ab}\sin(|A-B|).$$

- **5378:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \ge 1$ be an integer. Calculate

$$\int_0^\infty \ln^k\left(\frac{e^x+1}{e^x-1}\right) dx.$$

Solutions

- **5355:** *Proposed by Kenneth Korbin, New York, NY*

Find the area of the convex cyclic pentagon with sides

$$(13,\ 13,\ 12\sqrt{3}+5,\ 20\sqrt{3},\ 12\sqrt{3}-5)$$

Solution by Kee-Wai Lau, Hong Kong, China

We show that the area of the pentagon equals $370\sqrt{3}$.

Let the pentagon be $ABCDE$ with
$\overline{AB}=\overline{BC}=13,\ \overline{CD}=12\sqrt{3}+5,\ \overline{DE}=20\sqrt{3},\ \overline{EA}=12\sqrt{3}-5$. Denote the center and radius of the circumcircle by O and R respectively.

We first consider the case when O lies inside the pentagon. We have
$\angle AOB+\angle BOC+\angle COD+\angle DOE+\angle EOA=2\pi$, so that

$$2\sin^{-1}\left(\frac{13}{2R}\right)+\sin^{-1}\left(\frac{12\sqrt{3}+5}{2R}\right)+\sin^{-1}\left(\frac{10\sqrt{3}}{R}\right)+\sin^{-1}\left(\frac{12\sqrt{3}-5}{2R}\right)=\pi. \tag{1}$$

The left side of (1) is a decreasing function of R for $R \ge 10\sqrt{3}$, so (1) has at most one real valued solution.

Using the addition formula for the inverse sine function, we have

$$\begin{aligned} 2\sin^{-1}\left(\frac{13}{37}\right) &= \sin^{-1}\left(\frac{520\sqrt{3}}{1369}\right), \\ \sin^{-1}\left(\frac{12\sqrt{3}+5}{37}\right)+\sin^{-1}\left(\frac{12\sqrt{3}-5}{37}\right) &= \sin^{-1}\left(\frac{20\sqrt{3}}{37}\right), \text{ and that} \end{aligned}$$

$$2\sin^{-1}\left(\frac{20\sqrt{3}}{37}\right) = \pi - \sin^{-1}\left(\frac{520\sqrt{3}}{1369}\right).$$

It follows that the unique solution to (1) is $R = \dfrac{37}{2}$.

Using Heron's formula, we obtain the area of the triangles
$OAB,\ OBC,\ OCD,\ ODE,\ OEA$ as $65\sqrt{3},\ 65\sqrt{3},\ \dfrac{175\sqrt{3}+39}{2},\ 65\sqrt{3},\ \dfrac{175\sqrt{3}-39}{2}$, and so the area of the pentagon equals $370\sqrt{3}$.

We next consider the case when O lies on or outside the pentagon.
In this case $\angle EOA + \angle AOB + \angle BOC + \angle COD = \angle DOE$, so that

$$\sin^{-1}\left(\frac{12\sqrt{3}-5}{2R}\right) + 2\sin^{-1}\left(\frac{13}{2R}\right) + \sin^{-1}\left(\frac{12\sqrt{3}+5}{2R}\right) - \sin^{-1}\left(\frac{10\sqrt{3}}{R}\right) = 0. \qquad (2)$$

For $R \geq 20$, let $f(R) = 4\sin^{-1}\left(\dfrac{13}{2R}\right) - \sin^{-1}\left(\dfrac{10\sqrt{3}}{R}\right)$. Since $f(20) > 0$, $\lim\limits_{R\to\infty} f(R) = 0$ and that f attains the maximum value of $0.29\cdots$ at $R = \dfrac{195\sqrt{470}}{188}$, so in fact $f(R) > 0$.
Thus the left side of (2) is always positive and so (2) has no solutions.

This completes the solution.

Comments by Editor

1. A sticky point with this problem was in showing that the center of the circle had to lie in the interior of the pentagon. **David Stone and John Hawkins of Georgia Southern University** argued it like this: Assume that the points E, A, B, C, and D are arranged on the circumference of the circumscribing circle such that
$\overline{EA} = 12\sqrt{3}-5, \overline{AB} = 13, \overline{BC} = 13, \overline{CD} = 12\sqrt{3}+5$, $\overline{DE} = 20\sqrt{3}$. And suppose that the center of the circumscribing circle lies in the exterior of the pentagon. If it lies on the longest side of the pentagon, then it lies on $\overline{DE}$ and this would make $\overline{DE}$ a diameter of the circumscribing circle, so the radius R of the circumscribing circle must be $10\sqrt{3}$ and the length of $arcDE = 1/2$ the circumference of the circle. I.e., $arcDE = \pi(10\sqrt{3})$.

The length of each arc of the circle is greater than the length of its corresponding chord. So,

$$\begin{aligned}
arc\ DE &= arc\ DC + arc\ CB + arc\ BA + arc\ AE,\\
arc\ DE &> \overline{DC} + \overline{CB} + \overline{BA} + \overline{AE},\\
\pi(10\sqrt{3}) &> \left(12\sqrt{3}+5\right) + 13 + 13 + \left(12\sqrt{3}-5\right),\\
\pi(10\sqrt{3}) &> 24\sqrt{3}+26;\ \ \pi(10\sqrt{3}) \approx 54.4,\ \text{and } 24\sqrt{3}+26 \approx 67.57;\ \text{So},\\
54.4 &> 67.57\ ?\ \text{No.}
\end{aligned}$$

Therefore the center of the circumscribing circle cannot lie on the longest side of the pentagon.

But as the center of the circle moves into the exterior of the pentagon, the radius of the circumcircle increases and $arcDE$ decreases. I.e., $arcDE \leq 10\pi\sqrt{3}$. So again we have

$$54.4 > arcDE = arcDE + arcDE + arcDE + arcDE = 67.57$$

Hence, the center of the circumscribing circle must be in interior of the pentagon.

2. Bruno Salgueiro Fanego of Viveiro, Spain mentioned in his solution that he was applying an algebraic approach that was developed in a paper by David P. Robbins (see: *Areas of Polygons Inscribed in a Circle,* **The American Mathematical Monthly,** 102(6)(June-July, 1995)). The background to this approach is that the area of a convex polygon with more than three sides is not uniquely determined by the length of its sides. But adding the restriction that the polygon must also be cyclic, circumvents this problem and allows us to extend Heron's formula for finding the area K of a triangle (with side lengths a, b, c and semiperimeter s to Brahmagupta's formula for finding the area of a quadrilateral, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Robbins' paper presents formulas for finding the areas of the cyclic pentagon and cyclic hexagon. He wrote: "We shall see that the calculations leading to the discovery of the pentagon formula are so complex that it would have been quite difficult to carry them out without the aid of a computer. In fact after some study of the problem I thought it likely that, even if I were to discover the formula, its complexity would make it of little interest to write down. However, it is possible to write the formulas for the areas of the cyclic pentagon and the cyclic hexagon in a compact form which is related to the formula of the discriminant of a cubic polynomial in one variable."

Using Robbins' method the formula for finding the area K of a triangle with sides a, b, and c is

$$16K^2 = 2a^{2b}2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4,$$

while the formula for finding the area K of a cyclic quadrilateral with sides a, b, c, and d is

$$16K^2 = 2a^2b^2 + \cdots + 2c^2d^2 - a^4 - b^4 - c^4 - d^4 + 8abcd.$$

The formulas for finding the areas of cyclic pentagons and hexagons are spelled out in Robbins' paper, and although they are formidable, his method works.

Also solved by Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray, Highland Beach, FL; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Southern Georgia University, and the proposer.

- **5356:** *Proposed by Kenneth Korbin, New York, NY*

For every prime number p there is a circle with diameter $4p^4 + 1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area

$$(8p^3)(p+1)(p-1)(2p^2-1).$$

Find the sides of the triangles if $p = 2$ and if $p = 3$.

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

We designate the side lengths of the triangle by a, b, and c. We also let A be the area of the triangle and r be the radius of the circle that circumscribes it. Then the formula for the circumradius and Heron's formula yield

$$abc = 4Ar = 16p^3(p+1)(p-1)(2p^2-1)(4p^4+1)$$

and

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16A^2 = 1024p^6(p+1)^2(p-1)^2(2p^2-1)^2.$$

Inspired by the factorization $4p^4+1 = (2p^2+2p+1)(2p^2-2p+1)$, we let $a = 4p(2p^2-1)$, $b = 2p(p-1)(2p^2+2p+1)$, and $c = 2p(p+1)(2p^2-2p+1)$. Then

$$abc = 16p^3(p+1)(p-1)(2p^2-1)(4p^4+1)$$

as needed, so to complete the argument, it suffices to verify the second formula above. Letting $P = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$, we calculate

$$\begin{aligned} P &= (2p)^4(4p^3+4p^2-2p-2)(4p^3-4p^2-2p+2)(4p^2)(4p^2-4) \\ &= 1024p^6(p+1)(2p^2-1)(p-1)(2p^2-1)(p+1)(p-1) \\ &= 1024p^6(p+1)^2(p-1)^2(2p^2-1)^2. \end{aligned}$$

Hence the result holds for any integer $p > 1$. In particular, when $p = 2$, the triangle side lengths are 56, 52, and 60; when $p = 3$, the triangle side lengths are 204, 300, and 312.

Addendum: The sides of every Heronian triangle have the form $d(m+n)(mn-k{-}^2)$, $dm(n^2+k^2)$, and $dn(m^2+k^2)$, where m, n and k are positive integers with $\gcd(m,n,k) = 1$ and where d is a proportionality factor; see [1] for more details. Given any integer $p > 1$, we may take $m = p^2, n = p^2-1, k = p(p-1)$, and $d = \dfrac{2}{k}$ to produce the values of a, b, and c given above.

[1] https://en.wikipedia.org/wiki/Heronian-triangle

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, NY; Toshihiro Shimizu, Kawasaki Japan; David Stone and John Hawkins, Southern Georgia University; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Bazău, Romania, and the proposer.

- **5357:** *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

Determine all triangles whose side-lengths are positive integers (of which at least one is a prime number), whose semiperimeter is a positive integer, and whose area is equal to its perimeter.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let a, b, c be the positive integer side-lengths of a triangle, $p = \dfrac{a+b+c}{2}$ its semiperimeter and let us suppose that the area of that triangle, given by Heron's formula $\sqrt{p(p-a)(p-b)(p-c)}$ is equal to its perimeter $2p$.

Let $x = p-b,\ y = p-c,\ z = p-a$; then $xyz = (p-a)(p-b)(p-c) = 4p = 4(x+y+z)$ so $x = \dfrac{4(x+y)}{xy-4}$. By the triangle inequalities, x, y, z are positive integers so $xy = 4$ must be a positive integer as well. Without loss of generality, suppose that $a \le b \le c$; since $a = x+y, b = y+z,\ c = z+x$, and this is equivalent to $y \le x \le z$, so

$x+y \le 2x \le 2z = \dfrac{8(x+y)}{xy-4}$, from where $xy-4 \le 8$; hence $y \le \dfrac{12}{x} \le \dfrac{12}{y}$ which implies $y \le 3$, that is $y \in \{1,2,3\}$.

If $y=1$, then $x \le \frac{1}{2}y = 12$ or equivalently $x \in \{1,2,3,4,5,6,7,8,9,10,11,12\}$. If $y=2$, then $2 = y \le 12y = 6$, or what is the same $x \in \{2,3,4,5,6\}$ and if $y=3$, then $3 = y \le x \le 12y = 4$, or equivalently, $x \in \{3,4\}$.

From these possibilities the only ones that give positive integers for $z = \dfrac{4(x+y)}{xy-4}$ are $(x,y) = \{(5,1),\ (6,1),\ (8,1),\ (9,1),\ (3,2),\ (4,2),\ (6,2)\}$, which give

$$(a,b,c) = (x+y,\ y+z,\ z+x\} \in \{6,25,29),\ (7,15,20),\ (9,10,17),\ (10,9,17),\ (5,12,13),\ (6,8,10),\ (8,6,10)\}.$$

Thus, the triples of positive integer side-lengths of triangles whose area is equal to its perimeter are $(6,25,29), (7,15,20), (9,10,17), (5,12,13), (6,8,10)$ and since at least one of a,b,c is a prime number, we exclude the triple $(6,8,10)$ and since in all the other four cases the semiperimeter $p = \dfrac{a+b+c}{2}$ is a positive integer, the triangles we are looking for are those whose side lengths are $(6,25,29), (7,15,20), (9,10,17)$, or $(5,12,13)$. (Note also that only the last of them corresponds to a right triangle.)

Solution 2 and Comment by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The $A=P$ problem has a long history. In [2], Markov tells us that Dickson [1] attributes the solution to Whitworth and Biddle in 1904, then lists the only triangles with Area = Perimeter:

$$\begin{gathered}(6,8,10)\\(5,12,13)\\(6,25,29)\\(7,15,20)\\(9,10,17).\end{gathered}$$

Because our problem requires that one side be a prime, we see that the only solutions to the stated problem are the last four triangles above (note that each has an integral semiperimeter).

(The above result can probably now be considered as "common knowledge": it even appeared recently online on answers Yahoo.com [3]).

1. L. Dickson, History of the Theory of Numbers, Vol II, Dover Publications, Inc, New York, 2005 (reprint from the 1923 edition), p. 199.

2. L. P. Markov, Pythagorean Triples and the Problem A = mP for Triangles, Mathematics Magazine 79(2006) 114−121

3. From Dan, answers.Yahoo.com/question/index?qid=2081130185149AAua2RD, 7 years ago.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Jerry Chu (student, Saint George's School), Spokane, WA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton,

KS; David E. Manes SUNY College at Oneonta, Oneonta, NY; Ken Korbin, NewYork, NY; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

- **5358:** *Proposed by Arkady Alt, San Jose, CA*

Prove the identity $\sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1} = (r+1)^m(mr-1)+1.$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$$\begin{aligned}
(r+1)^m(mr-1)+1 &= \sum_{k=0}^{m} m\binom{m}{k} r^{k+1} - \sum_{k=1}^{m}\binom{m}{k} r^k \\
&= \sum_{k=1}^{m} m\binom{m}{k} r^{k+1} - \sum_{k=1}^{m-1}\binom{m}{k+1} r^{k+1} \\
&= mr^{m+1} + \sum_{k=1}^{m-1}\left(m\binom{m}{k} - \binom{m}{k+1}\right) r^{k+1} \\
&= mr^{m+1} + \sum_{k=1}^{m-1} k\binom{m+1}{k+1} r^{k+1} \\
&= \sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1}
\end{aligned}$$

where we have used that $m\binom{m}{k} - \binom{m}{k+1} = k\binom{m+1}{k+1}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

We have

$$(1+r)^m = \sum_{k=0}^{m}\binom{m}{k} r^k \tag{1}$$

and differentiating

$$mr(1+r)^{m-1} = \sum_{k=0}^{m} k\binom{m}{k} r^k. \tag{2}$$

Now

$$\begin{aligned}
\sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1} &= \sum_{k=2}^{m+1}(k-1)\binom{m+1}{k} r^k = \sum_{k=2}^{m+1} k\binom{m+1}{k} r^k - \sum_{k=2}^{m+1}\binom{m+1}{k} r^k \\
&\overset{(2),(1)}{=\!=} (m+1)r(1+r)^m - (m+1)r - (1+r)^{m+1} + 1 + (m+1)r \\
&= (r+1)^m(mr-1)+1.
\end{aligned}$$

Solution 3 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy

Proof Induction. Let $m = 1$. We have

$$\binom{2}{2} r^2 = (r+1)(r-1) + 1$$

which clearly holds.

Let's suppose it is true for $2 \le m \le n-1$. For $m = n$ we have

$$\begin{aligned}
\sum_{k=1}^{m+1} k \binom{m+2}{k+1} r^{k+1} &= (m+1)r^{m+2} + \sum_{k=1}^{m} k \left[\binom{m+1}{k+1} + \binom{m+1}{k} \right] r^{k+1} = \\
&= (m+1)r^{m+2} + (r+1)^m (mr-1) + 1 \sum_{k=1}^{m} k \binom{m+1}{k} r^{k+1} \qquad (1)
\end{aligned}$$

$$\binom{m+2}{k+1} = \binom{m+1}{k+1} + \binom{m+1}{k}$$

and the induction hypothesis have been used. Moreover

$$\begin{aligned}
&\sum_{k=1}^{m} k \binom{m+1}{k} r^{k+1} \underbrace{=}_{k+1=p} r \sum_{p=0}^{m-1} (p+1) \binom{m+1}{p+1} r^{p+1} = \\
&= r \sum_{p=1}^{m-1} p \binom{m+1}{p+1} r^{p+1} + r \sum_{p=0}^{m-1} \binom{m+1}{p+1} r^{p+1} = \\
&= r \sum_{p=1}^{m} p \binom{m+1}{p+1} r^{p+1} - mr^{m+2} \underbrace{+}_{p+1=q} r \sum_{q=0}^{m+1} \binom{m+1}{q} r^{q} - r - r^{m+2}
\end{aligned}$$

The induction hypotheses and the Newton–binomial yield that it is equal to

$$r\left((r+1)^m (mr-1) + 1\right) - mr^{m+2} + r(1+r)^{m+1} - r - r^{m+2}.$$

By inserting in (1) we get

$$\begin{aligned}
&(m+1)r^{m+2} + \left((r+1)^m (mr-1) + 1\right)(r+1) - (m+1)r^{m+2} + r(1+r)^{m+1} - r = \\
&= (r+1)^{m+1}(mr-1) + (r+1) + r(1+r)^{m+1} - r = \\
&= (r+1)^{m+1}((m+1)r - 1) - r(1+r)^{m+1} + (r+1) + r(1+r)^{m+1} - r = \\
&= (r+1)^{m+1}((m+1)r - 1) + 1.
\end{aligned}$$

and the proof is complete.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Here we differentiate the given sum to get the Binomial Theorem, then integrate to get the desired sum.

Let $f(r) = \sum_{k=1}^{m} \binom{m+1}{k+1} r^{k+1} = \sum_{k=1}^{m} k \frac{(m+1)!}{(k+1)k(k-1)!(m+1-k-1)!} r^{k+1}$,

so,

$$\begin{aligned}
f'(r) &= \sum_{k=1}^{m} k \frac{(k+1)(m+1)!}{(k+1)k(k-1)!(m-k)!} r^k \\
&= \sum_{k=1}^{m} k \frac{(m+1)!}{(k-1)!(m-k)!} r^k \\
&= \sum_{k=0}^{m-1} mk \frac{(m+1)!}{(k-1)!(m-1-k)!} r^k, \text{by reindexing} \\
&= m(m+1) \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-1-k)!} r^k, \\
&= m(m+1)r \sum_{k=0}^{m-1} \binom{m-1}{k} r^k \\
&= m(m+1)r(r+1)^{m-1} \text{ by the Binomial Theorem.}
\end{aligned}$$

Now we can integrate by parts to fine $f(r)$:

$$\begin{aligned}
f(r) &= \int m(m+1)(r(r+1)^{m-1} dr \\
&= m(m+1) \int r(r+1)^{m-1} dr \\
&= m(m+1) \left[\frac{1}{m} r(r+1)^m - \int \frac{1}{m}(r+1) dr \right] \\
&= m(m+1) \left[\frac{1}{m} r(r+1)^m - \frac{1}{m} \frac{(r+1)^{m+1}}{m+1} \right] + C \\
&= m(m+1) \left\{ \frac{(r+1)^m}{m} \frac{(mr-1)}{m+1} \right\} + C \\
&= (r+1)^m (mr-1) + C
\end{aligned}$$

Using the initial condition $f(0) = 0$ we find $C = 1$, so $f(r) = (r+1)^m(mr-1)+1$, as desired.

Editor's note: David and John also submitted a second solution to this problem that was similar to Solution 2 above.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Eastern Connecticut State University, Willimantic, CT David E. Manes SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

5359: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1} \le \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since $15a^3b+1$ can be represented as $(2a)^3 \cdot \dfrac{15b+\dfrac{1}{a^3}}{8}$ then by AM-GM Inequality we obtain

$$\begin{aligned}
\sum_{cyc} \sqrt[4]{15a^3b+1} = \sum_{cyc} \sqrt[4]{(2a)^3 \cdot \frac{15b+\dfrac{1}{a^3}}{8}} \quad &\le \quad \sum_{cyc} \frac{3\cdot(2a)+\dfrac{15b+\dfrac{1}{a^3}}{8}}{4} \\
&= \quad \sum_{cyc} \frac{48a+15b+\dfrac{1}{a^3}}{32} \\
&\le \quad \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).
\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that

$$\sqrt[4]{11+15x^4} \le \frac{63}{32}x+\frac{1}{32x^3},\ x>0. \qquad (1)$$

Indeed,
$$\left(\frac{63}{32}x+\frac{1}{32x^3}\right)^4-\left(1+15x^4\right)=\frac{(x-1)^2\,(x+1)^2\left(x^2+1\right)^2\left(24321x^8+254x^4+1\right)}{2^{20}x^{12}}\geq 0$$
We replace x by $\sqrt[4]{a^3b}$ in (1) and use the AM$-$GM inequality to obtain

$$\sqrt[4]{1+15a^3b}\leq\frac{63}{32}\sqrt[4]{b^3c}+\frac{1}{32\sqrt[4]{a^9b^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot a+\frac{1}{4}\cdot b\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{a^3}+\frac{1}{4}\cdot\frac{1}{b^3}\right).\tag{2}$$

Similarly,

$$\sqrt[4]{1+15b^3c}\leq\frac{63}{32}\sqrt[4]{b^3c}+\frac{1}{32\sqrt[4]{b^9c^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot b+\frac{1}{4}\cdot c\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{b^3}+\frac{1}{4}\cdot\frac{1}{c^3}\right).\tag{3}$$

$$\sqrt[4]{1+15c^3a}\leq\frac{63}{32}\sqrt[4]{cb^3a}+\frac{1}{32\sqrt[4]{c^9a^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot c+\frac{1}{4}\cdot a\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{c^3}+\frac{1}{4}\cdot\frac{1}{a^3}\right).\tag{4}$$

We complete the proof by adding (2), (3), and (4).

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Let $f:(0,\infty)\longrightarrow\Re$ and $f(x)=\sqrt[4]{x}$, which is concave on $(0,\infty)$. Therefore,

$$\sqrt[4]{x}=f(x)\leq f(t)+f'(t)(x-t)=\sqrt[4]{t}+\frac{1}{4\sqrt[4]{t^3}}(x-t),\forall x,t>0.$$

Let $x=15a^3b+1$ and $t=16a^4$. Then we have:

$$\sqrt[4]{15a^3b+1}\leq 2a+\frac{1}{32a^3}\left(15a^3b+1-16a^4\right)=2a+\frac{1}{32}\left(15b+\frac{1}{a^3}-16a\right).$$

Summing the analogous upper bounds on the other two terms, gives

$$\begin{aligned}\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1} &\leq 2\sum a+\frac{15}{32}\sum a-\frac{1}{2}\sum a+\frac{1}{32}\sum\frac{1}{a^3}\\ &= \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).\end{aligned}$$

Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5360:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n\geq 1$ be an integer and let

$$I_n=\int_0^\infty\frac{\arctan x}{(1+x^2)^n}dx.$$

Prove that

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{I_n}{n}=\zeta(2);$

(b) $\displaystyle\int_0^{\infty}\arctan x\ln\left(1+\frac{1}{x^2}\right)dx=\zeta(2).$

Solution 1 by Anastasios Kotronis, Athens, Greece

- a) We have

$$I_n \overset{x=\tan y}{=\!=\!=} \int_0^{\frac{\pi}{2}} y\cos^{2n-2}y\,dy$$

and since the integrand doesn't change sign:

$$\begin{aligned}\sum_{n\geq 1}\frac{I_n}{n} &= \sum_{n\geq 1}\frac{1}{n}\int_0^{\frac{\pi}{2}} y\cos^{2n-2}y\,dy=\sum_{n\geq 1}\int_0^{\frac{\pi}{2}} y\sum_{n\geq 1}\frac{(\cos^2 y)^{n-1}}{n}\,dy=-\int_0^{\frac{\pi}{2}} y\frac{\ln(1-\cos^2 y)}{\cos^2 y}\,dy\\
&=-2\int_0^{\frac{\pi}{2}} y\frac{\ln(\sin y)}{\cos^2 y}\,dy=-2\int_0^{\frac{\pi}{2}}(y\tan y+\ln(\cos y))'\ln(\sin y)\,dy\\
&=-2\left(y\tan y+\ln(\cos y)\right)\ln(\sin y)\Big|_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}(y\tan y+\ln(\cos y))\cot y\,dy\\
&=\frac{\pi^2}{4}+2\int_0^{\frac{\pi}{2}}\cot y\ln(\cos y)\overset{\cos y=t}{=\!=\!=}\frac{\pi^2}{4}+2\int_0^1\frac{t\ln t}{1-t^2}\,dt=\frac{\pi^2}{4}+\int_0^1\frac{\ln t}{1-t}\,dt-\int_0^1\frac{\ln t}{1+t}\,dt\\
&=\frac{\pi^2}{4}+\int_0^1\sum_{n\geq 0}t^n\ln t-\int_0^1\sum_{n\geq 0}(-t)^n\ln t.\end{aligned}$$

From Dominated Convergence Theorem, the order of integration and summation can change, so

$$\sum_{n\geq 1}\frac{I_n}{n}=\frac{\pi^2}{4}+\sum_{n\geq 0}\int_0^1 t^n\ln t-\sum_{n\geq 0}\int_0^1(-t)^n\ln t=\frac{\pi^2}{4}-\sum_{n\geq 1}\frac{1}{n^2}+\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{6}=\zeta(2).$$

b)

$$\int_0^{\infty}\arctan x\ln\left(1+\frac{1}{x^2}\right)dx\overset{x=\tan y}{=\!=\!=}-2\int_0^{\frac{\pi}{2}} y\frac{\ln(\sin y)}{\cos^2 y}\,dy=\sum_{n\geq 1}\frac{I_n}{n},$$

from the first part of the problem, so the result is immediate.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

We first prove (b) and then (a).

b)

$$\int_0^{\infty}\arctan x\ln\left(1+\frac{1}{x^2}\right)dx=\left[\begin{array}{c}u=\arctan x\Longrightarrow du=\dfrac{1}{1+x^2}dx\\ dy=\ln\left(1+\dfrac{1}{x^2}\right)dx\Longrightarrow v=2\arctan x+x\ln\left(1+\frac{1}{x^2}\right)\end{array}\right]$$

$$
\begin{aligned}
&= \int_0^\infty u dv = uv]_0^\infty - \int_0^\infty v du \\
&= \arctan x \left(2 \arctan x + x \ln \left(1 + \frac{1}{x^2} \right) \right) \Bigg]_0^\infty - \int_0^\infty \frac{2 \arctan x + x \ln \left(1 + \dfrac{1}{x^2} \right)}{1+x^2} dx \\
&= \frac{\pi^2}{2} \left(2\frac{\pi}{2} + 0 \right) - 0 \left(2 \cdot 0 + 0 \right) - 2 \int_0^\infty \frac{\arctan x}{1+x^2} dx - \int_0^\infty \frac{x \ln \left(1a + \dfrac{1}{x^2} \right)}{1+x^2} dx \\
&\underset{1}{=} \frac{\pi^2}{2} - \left(\arctan^2 x \Big]_0^\infty \right) - \frac{\pi^2}{12} \\
&= \frac{\pi^2}{2} - \frac{\pi^2}{4} + 0 - \frac{\pi^2}{12} \;=\; \frac{\pi^2}{6} \;=\; \zeta(2).
\end{aligned}
$$

(a)

$$
\sum_{n=1}^\infty \frac{I_n}{n} = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty \frac{\arctan x}{(1+x^2)^n} dx = \int_0^\infty \arctan x \sum_{n=1}^\infty \frac{1}{n} \left(\frac{1}{1+x^2} \right)^n dx
$$

$$
= \int_0^\infty \arctan x \left(-\ln \left(1 - \frac{1}{1+x^2} \right) \right) dx = \int_0^\infty \arctan x \ln \left(1 + \frac{1}{x^2} \right) dx = \zeta(2), \text{ from part b.}
$$

(1) Table of Integrals, Series and Products, Gradshteyn, I.S. and Ryzhik, I.M., Seventh Edition Elsevier Inc., 2007, 4,298(16) page 564.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Part a)

$$
\begin{aligned}
I_n &= \frac{x \arctan x}{(1+x^2)^n} \Big|_0^\infty - \int_0^\infty \frac{x}{(1+x^2)^{n+1}} + 2n \int_0^\infty \frac{x^2 \arctan x}{(1+x^2)^{n+1}} \\
&= \frac{1}{2n} \frac{1}{(1+x^2)^n} \Big|_0^\infty + 2n \int_0^\infty \frac{\arctan x}{(1+x^2)^n} dx - 2n \int_0^\infty \frac{\arctan x}{(1+x^2)^{n+1}} dx \\
&= -\frac{1}{2n} + 2nI_n - 2nI_{n-1}.
\end{aligned}
$$

We have obtained the recursive sequence

$$I_{n+1} = I_n \left(1 - \frac{1}{2n}\right) - \frac{1}{4n^2} \iff \frac{1}{2n} I_n = I_n - I_{n+1} - \frac{1}{4n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = 2I_1 - \lim_{n\to\infty} I_n - \frac{\pi^2}{12}$$

$$I_1 = \int_0^{\infty} \frac{\arctan x}{1+x^2} dx \underbrace{=}_{x=\tan t} \int_0^{\pi/2} t dt = \frac{\pi^2}{8}$$

As for $\lim\limits_{n\to\infty} I_n$, we break I_n into two addends.

$$I_n = \int_0^1 \frac{\arctan x}{(1+x^2)^n} dx + \int_1^{\infty} \frac{\arctan x}{(1+x^2)^n} dx \doteq J_1 + J_2.$$

J_1 converges to zero for instance by the dominated convergence theorem of Lebesgue after observing that $\dfrac{\arctan x}{(1+x^2)^n} \to 0$.

As for J_2 we bound,

$$0 < \int_1^{\infty} \frac{\arctan x}{(1+x^2)^n} dx \le \frac{\pi}{2} \int_1^{\infty} \frac{1}{x^{2n}} dx = \frac{\pi}{2} \frac{1}{2n-1} \to 0.$$

We have obtained,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Part (b). Let's define I the integral. Integrating by parts,

$$I = x \arctan x \ln\left(1 + \frac{1}{x^2}\right)\Big|_0^{\infty} - \int_0^{\infty} \frac{x \ln\left(1 + \frac{1}{x^2}\right)}{1+x^2} dx + \int_0^{\infty} \frac{2 \arctan x}{1+x^2} dx \tag{1}$$

The first summand annihilates because

$$\lim_{x\to 0} x \arctan x \ln\left(1 + \frac{1}{x^2}\right) = \lim_{x\to 0} x \arctan x (\ln(1+x^2) - 2\ln x) = 0.$$

The third is equal to $(\arctan^2 x)\Big|_0^{\infty} = \dfrac{\pi^2}{4}$.

As for the second summand it is equal to

$$\lim_{a\to\infty} \int_0^a \frac{x\ln(1+x^2) - 2x\ln x}{1+x^2} dx = \lim_{a\to\infty} \frac{\ln^2(1+x^2)}{4}\Big|_0^a - \lim_{a\to\infty} 2\int_0^a \frac{x\ln x}{1+x^2}. \tag{2}$$

$$\lim_{a\to\infty} 2\int_0^a \frac{x\ln x}{1+x^2} = \lim_{a\to\infty} \ln x \ln(1+x^2)\Big|_0^a - \lim_{a\to\infty} \int_0^a \frac{\ln(1+x^2)}{x} dx\,;$$

$$\lim_{a\to\infty}\int_0^a \frac{\ln(1+x^2)}{x}\,dx = \int_0^1 \frac{\ln(1+x^2)}{x}\,dx + \lim_{a\to\infty}\int_1^a \frac{\ln(1+x^2)}{x}\,dx\,;$$

$$\int_0^1 \frac{\ln(1+x^2)}{x}\,dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{2k-1}dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k^2}$$

$$\begin{aligned}
\lim_{a\to\infty}\int_1^a \frac{\ln(1+x^2)}{x}\,dx \quad &\underbrace{=}_{x=1/y} \quad \lim_{a\to\infty}\int_{1/a}^1 \frac{\ln(1+y^2)-2\ln y}{y}dy \\
&= \int_0^1 \frac{\ln(1+y^2)}{y}dy - \lim_{a\to\infty}\ln^2 y\Big|_{1/a}^{1} \\
&= \int_0^1 \frac{\ln(1+y^2)}{y}dy + \lim_{a\to\infty}\ln^2 a
\end{aligned}$$

Plugging in (2) we get

$$\lim_{a\to\infty}\frac{\ln^2(1+a^2)}{4} - \ln a\ln(1+a^2) + \ln^2 a + 2\int_0^1 \frac{\ln(1+x^2)}{x}dx = 2\int_0^1 \frac{\ln(1+x^2)}{x}dx.$$

and

$$\int_0^1 2\frac{\ln(1+x^2)}{x}dx = \sum_{k=1}^{\infty}\frac{2(-1)^k}{k}\int_0^1 x^{2k-1}dx = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = \frac{\pi^2}{12},$$

and finally (1) is equal to

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Solution 4 by G.C. Greubel, Newport News, VA

Part a) Given the integral

$$I_n = \int_0^{\infty}\frac{\arctan x}{(1+x^2)^n}\,dx \tag{1}$$

make the change of variable $x = \tan t$ to obtain

$$I_n = \int_0^{\pi/2}\frac{t\,\sec^2 t}{(\sec^2 t)^n}\,dt = \int_0^{\pi/2} t\,\cos^{2n-2} t\,dt. \tag{2}$$

By considering the summation of I_n in the desired manner leads to

$$S = \sum_{n=1}^{\infty}\frac{I_n}{n} = -\int_0^{\pi/2}\frac{t\,\ln(\sin^2 t)}{\cos^2 t}\,dt. \tag{3}$$

The integral in (3) may be evaluated by use of the Dilogarithm function as seen by the following.

$$\begin{aligned}
J &= \int_0^{\pi/2} \frac{t\,\ln(\sin^2 t)}{\cos^2 t}\,dt \qquad (4)\\
&= \Bigg[-Li_2\left(-\tan^2\left(\frac{t}{2}\right)\right) - 2Li_2\left(\frac{1}{2}\sec^2\left(\frac{t}{2}\right)\right) - Li_2\left(\cos t\,\sec^2\left(\frac{t}{2}\right)\right)\\
&\quad -t^2 - \ln^2\left(\sec^2\left(\frac{t}{2}\right)\right) + 2\ln 2\,\ln\left(\sec^2\left(\frac{t}{2}\right)\right) + t\,\tan t\,\ln\left(\sin^2 t\right)\\
&\quad -\ln\left(\sin^2 t\right)\,\ln\left(\sec^2\left(\frac{t}{2}\right)\right) + \ln\left(\sin^2 t\right)\,\ln\left(\cos t\,\sec^2\left(\frac{t}{2}\right)\right)\\
&\quad -\ln\left(\tan^2\left(\frac{t}{2}\right)\right)\,\ln\left(\cos t\,\sec^2\left(\frac{t}{2}\right)\right)\Bigg]_0^{\pi/2}\\
&= -Li_2(-1) - Li_2(1) - \frac{\pi^2}{4} + \ln^2 2 + 2Li_2\left(\frac{1}{2}\right)\\
&= -Li_2(1) = -\zeta(2). \qquad (5)
\end{aligned}$$

By using the resulting integral value of (5) in (3) the desired result is obtained, namely,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = \zeta(2). \qquad (6)$$

Part b) The integral in question is given by

$$I = \int_0^{\infty} \tan^{-1} x\,\ln\left(1 + \frac{1}{x^2}\right)\,dx. \qquad (7)$$

Making the change of variable $x = \tan t$ leads to the integral

$$I = -\int_0^{\pi/2} \frac{t\,\ln(\sin^2 t)}{\cos^2 t}\,dt. \qquad (8)$$

This is the same integral defined as (4) and has the resulting value given by (5). By comparison of results the integral of this section is presented as

$$\int_0^{\infty} \tan^{-1} x\,\ln\left(1 + \frac{1}{x^2}\right)\,dx = \zeta(2) \qquad (9)$$

which is the desired result.

Also solved by Ed Gray, Highland Beach, FL; Kee Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2016

- **5379:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
$$\frac{(x+1)^4}{(x-1)^2} = 17x.$$

- **5380:** *Proposed by Arkady Alt, San Jose, CA*

  Let $\Delta(x,y,z) = 2(xy+yz+xz) - (x^2+y^2+z^2)$ and a,b,c be the side-lengths of a triangle ABC. Prove that
$$F^2 \geq \frac{3}{16} \cdot \frac{\Delta(a^3,b^3,c^3)}{\Delta(a,b,c)},$$
  where F is the area of $\triangle ABC$.

- **5381:** *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

  Prove: In any acute triangle ABC, with the usual notations, holds:
$$\sum_{cyclic} \left(\frac{\cos A \cos B}{\cos C}\right)^{m+1} \geq \frac{3}{2^{m+1}},$$
  where $m \geq 0$ is an integer number.

- **5382:** *Proposed by* Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

  Prove that if a,b,c are positive real numbers, then
$$\left(\sum_{cyclic} \frac{a}{b} + 8\sum_{cyclic} \frac{b}{a}\right)\left(\sum_{cyclic} \frac{b}{a} + 8\sum_{cyclic} \frac{a}{b}\right) \geq 9^3.$$

- **5383:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Find $\gcd(a_n, b_n)$, where a_n and b_n are the positive integers for which $(1-\sqrt{5})^n = a_n - b_n\sqrt{5}$.

- **5384:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all differentiable functions $f : \Re \to \Re$ which verify the functional equation

$$xf'(x) + f(-x) = x^2, \quad \text{for all} \quad x \in \Re.$$

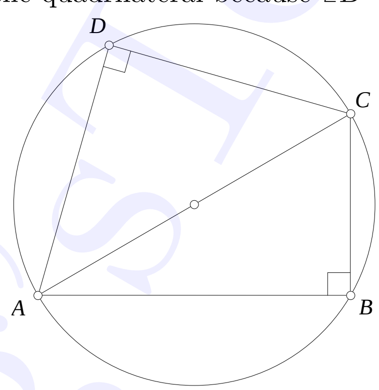
Solutions

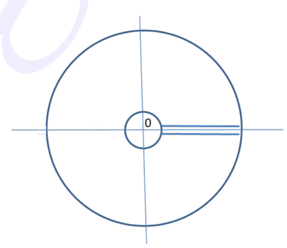
- **5361:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $ABCD$ has perimeter $P = 75 + 61\sqrt{15}$ and has $\angle B = \angle D = 90°$. The lengths of the diagonals are 112 and 128. Find the lengths of the sides.

  Solution by Ercole Suppa, Teramo, Italy

  Observe that $ABCD$ is a cyclic quadrilateral because $\angle B = \angle D = 90°$.





Denote $AB = a$, $BC = b$, $CD = c$, $DA = d$. By the Pythagorean theorem applied to triangles ABC, ACD and Ptolemy's theorem applied to the quadrilateral $ABCD$ we have

$$a^2+b^2=c^2+d^2=128^2 \tag{1}$$
$$ac+bd=112\cdot 128 \tag{2}$$

Taking into account of (1) and (2) we obtain

$$\begin{gathered}
(a+b+c+d)^2=\left(75+61\sqrt{15}\right)^2 \qquad \Leftrightarrow \\
a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)=61440+9150\sqrt{15} \qquad \Leftrightarrow \\
2\cdot 128^2+2(ab+ac+ad+bc+bd+cd)=61440+9150\sqrt{15} \qquad \Leftrightarrow \\
ac+bd+(a+c)(b+d)=14336+4575\sqrt{15} \qquad \Leftrightarrow \\
(a+c)(b+d)=4575\sqrt{15}
\end{gathered}$$

Putting $a+c=x$, $b+d=y$ we have

$$\begin{cases} x+y=75+61\sqrt{15} \\ xy=4575\sqrt{15} \end{cases}$$

from which, after some algebra, we find $(x,y)=\left(75,61\sqrt{15}\right)$ or $(x,y)=\left(61\sqrt{15},75\right)$. Finally, solving the system

$$\begin{cases} a+c=75 \\ b+d=61\sqrt{15} \\ a^2+b^2=128^2 \\ c^2+d^2=128^2 \end{cases}$$

we get $(a,b,c,d)=\left(7,33\sqrt{15},68,28\sqrt{15}\right),\left(33\sqrt{15},7,28\sqrt{15},68\right),\left(68,28\sqrt{15},7,33\sqrt{15},\right)$ or $\left(28\sqrt{15},68,33\sqrt{15},7\right)$, and the proof is completed.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Gail Nord, Gonzaga University, Spokane, WA; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Toshihiro Shimizu, Kawasaki, Japan; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

- **5362:** *Proposed by Michael Brozinsky, Central Islip, NY*

Two thousand forty seven death row prisoners were arranged from left to right with the numbers 1 through 2047 on their backs in this left to right order. Prisoner 1 was given a gun and shoots prisoner number 2 dead, and then gives the gun to prisoner number 3 who shoots prisoner number 4 and then gives the gun to number 5 and so on, so that every second originally numbered prisoner is shot dead.

This process is then repeated from right to left, starting with the person (in this case number 2047) who last received the gun and then continues to proceed from right to left, and then the direction switches again, and then again until only one prisoner remains standing. What is the number of the prisoner who survives the left to right, right to left shootout? Note that if there had been 2048 prisoners, number 2047 would have no one to whom to hand the gun in the left to right direction after shooting number 2048, and so he would then start the gun in its opposite direction shooting the living prisoner to his immediate left i.e.,number 2045. In this case, number 2047 gets to shoot two prisoners before he hands the gun off to another prisoner.

Solution 1 by Ashland University Undergraduate Problem Solving Group, Ashland, OH

Let $a(n)$ = the number of the prisoner who survives when n prisoners are in line. It is given in the problem that $a(2048) = a(2047)$, and from the explanation given, we can similarly conclude that $a(2k) = a(2k-1)$. We can also see that the prisoner left standing for $a(2k+1)$ is the $a(k+1)^{st}$ odd-numbered prisoner from the right end of the line since only odd numbers survived the first gun pass through the line. this gives the relation

$$a(2k+1) = 2k+1-2\left[a(k+1)-1\right] = 2k+3-2a(k+1).$$

From this we can see that

$$a(2^m) = a(2^m-1) = (2^m-1)-2\left[a(2^{m-1}-1+1)-1\right] = 2^m+1-2a(2^{m-1}).$$

We can then solve for an explicit formula using iteration.

$$\begin{aligned}
a(2^m) &= a(2^m-1) = 2^m+1-2a(2^{m-1}) \\
&= 2^m+1-2\left(2^{m-1}+1-2a(2^{m-2})\right) \\
&= 2^m+1-2\left(2^{m-1}+1-2\left[2^{m-2}+1-2a(2^{m-3})\right]\right) \\
&= 2^m+1-2\left(2^{m-1}+1-2\left[2^{m-2}+1-2\left(2^{m-3}+1-2a\left(2^{m-4}\right)\right)\right]\right).
\end{aligned}$$

So if we regroup these equations,

$$\begin{aligned}
a(2^m) &= (a^{2m}-1) = 2^m+1-2a(2^{m-1}) \\
&= (2^m-2^m)+(1-2)+2^2a(2^{m-2})a \\
&= (2^m-2^m+2^m)+(1-2+2^2)-2^3a(2^{m-3}) \\
&= (2^m-2^m+2^m-2\ ^m)+(1-2+2^2-2^3)+2^4a(2^{m-4}).
\end{aligned}$$

We can see that

$$\begin{aligned}
a(2^{2k}) &= 1\left(\frac{1-(-2)^{2k}}{1-(-2)}\right)+2^{2k}a(a^{2k-2k}) \\
&= \left(\frac{1-(-2)^{2k}}{3}\right)+2^{2k}a(1) \\
&= 2^{2k}+\left(\frac{1-2^{2k}}{3}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3(2^{2k}) + (1 - 2^{2k})}{3} \\
&= \frac{2^{2k} + 1}{3} \\
&= \frac{2^{2k+1} + 1}{3}
\end{aligned}
$$

And

$$
\begin{aligned}
a(2^{2k+1}) &= 2^{2k+1} + 1 + 1\left(\frac{1-(-2)^{2k+1}}{1-(-2)}\right) - 2^{2k+1} a\left(2^{[2k+1]-[2k+1]}\right) \\
&= 2^{2k+1} + \left(\frac{1+2^{2k+1}}{3}\right) - 2^{2k+1} a(1) \\
&= 2^{2k+1} - 2^{2k+1} + \frac{1+2^{2k+1}}{3} \\
&= \frac{2^{2k+1}+1}{3}
\end{aligned}
$$

So $a\left(2^{2k}\right) = a\left(2^{2k+1}\right) = \dfrac{2^{2k+1}+1}{3}$, and $a(2047) = a(2048) = a\left(2^{11}\right) = \dfrac{2^{11}+1}{3} = 683$.

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

We consider the case of 2^n prisoner. Let $f(n)$ be the index of the prisoner who remains alive. It's obvious that $f(0) = 1$. In the first left-to-right shootout, 2^{n-1} prisoners who were originally indexed as $1, 3, 5, \ldots, 2^n - 1$ are alive. Then, we reindex these prisoner as $2^{n-1}, 2^{n-1} - 1, \ldots, 2, 1$. So the prisoner with new-index $f(n-1)$ is alive. This prisoner is also the prisoner of original-index $f(n)$. Since the prisoner of new-index i corresponds to original index $2^n + 1 - 2i$, it follows that $f(n) = 2^n + 1 - 2f(n-1)$. This relation is equivalent to

$$
f(n) - 2^{n-1} - \frac{1}{3} = -2\left(f(n-1) - 2^{n-2} - \frac{1}{3}\right).
$$

Therefore, $f(n) - 2^{n-1} - 1/3 = (-2)^n(f(0) - 2^{-1} - 1/3) = (-2)^n/6$ or $f(n) = (-2)^n/6 + 2^{n-1} + 1/3$.

We consider the cases with 2047 prisoners and with 2048 prisoners. In the first left-to-right of the later case, the prisoner 2047 shoots 2048, while in the former case, the prisoner 2048 does not initially exist. Thus, in the both two cases, the original index of living prisoners are identical after first left-to-right movement. Thus, the prisoner who, in the end, remains alive, is also same. This prisoner is indexed $f(11) = 683$.

Solution 3 by David E. Manes, SUNY College at Oneonta, NY

At the end of the bloodbath, number 683 is the only prisoner standing and it takes ten stages to produce him.

Stage 1: Procedure goes from left to right. The odd numbered prisoners are alive and the even numbered ones are not. Therefore 1023 prisoners have been eliminated and 1024 prisoners are still alive.

Stage 2: Procedure goes from right to left starting with prisoner 2047. The prisoners' decreased are numbered $4k+1, 0 \le k \le 511$ while the prisoners alive are numbered $4k+3, 0 \le k \le 511$. There are now 512 prisoners alive.

Stage 3: Procedure goes from left to right starting with prisoner 3. Prisoners still alive after this stage are numbered $8k+7, 0 \le k \le 255$. There are now 256 prisoners alive.

Stage 4: Procedure goes from right to left starting with prisoner 2043. The prisoners dismissed after this stage have numbers $16k+3, 0 \le k \le 127$ and the prisoners still standing have numbers $16k+11, 0 \le k \le 127$. There are now 128 prisoners alive.

Stage 5 : Procedure goes from left to right starting with prisoner 11. After this stage the lifeless prisoners have numbers $32k+27, 0 \le k \le 63$ and the prisoners still alive have numbers $32k+11, 0 \le k \le 63$.

Stage 6: Procedure goes from right to left starting with prisoner 2027. Prisoners no longer playing are numbered $64k+11, 0 \le k \le 31$ and the prisoners still playing have numbers $64k+43, 0 \le k31$.

Stage 7: Procedure goes from left to right starting with prisoner 43. After this stage the prisoners asked to leave are numbered $128k+107, 0 \le k \le 15$ and the prisoners still living have numbers $128k+43, 0 \le k \le 15$.

Stage 8: Procedure goes from right to left starting with prisoner 1963. After this stage the prisoners not breathing have numbers $256k+171, 0 \le k \le 7$.

Stage 9: Procedure goes from left to right starting with prisoner 171. After this stage the extinct prisoners have numbers $512k+427, 0 \le k \le 3$ and the prisoners still alive have numbers $512k+171,\ 0k \le 171$. Prisoners no longer playing are numbered and the prisoners still playing have numbers $64k+43, 0 \le k \le 3$, that is, prisoners numbered 171, 683, 1195, and 1707,

Stage 10: Procedure goes from right to left starting with prisoner 1707. The deceased prisoners are numbered 1195 and 171. The only prisoners alive are 683 and 1707, but prisoner 683 has the loaded gun, hence the result.

Solution 4 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

Order of shootout	*Direction of shootout*	*Number of surviving inmates*	*Difference between numbers of two surviving inmates*	*Left − end, Right− end Surviving numbers*
1	$L \to R$	1024	2	$1 - 2047$
2	$L \leftarrow R$	512	4	$3 - 2047$
3	$L \to R$	256	8	$3 - 2043$
4	$L \leftarrow R$	128	16	$11 - 2043$
5	$L \to R$	64	32	$11 - 2027$
6	$L \leftarrow R$	32	64	$43 - 2027$
7	$L \to R$	16	128	$43 - 1963$
8	$L \leftarrow R$	8	256	$171 - 1963$
9	$L \to R$	4	512	$171 - 1707$
10	$L \leftarrow R$	2	1024	$683 - 1707$
11	$L \to R$	1		683

The last surviving inmate has the number 683.

Solution 5 by Carl Libis and Roland Depratti, Eastern Connecticut State University, Willimantic, CT

Let $f(x)$ =number of the prisoner that survives when there are x prisoners. Observe that

$$f(2^n+1) = \begin{cases} 1, \text{ if } n \ = 2,4,6,\cdots \\ 2^n+1, \text{ if } n, = 1,3,5,\cdots \end{cases}$$

$$f(2^n) \ \ = \begin{cases} \dfrac{2^{n+1}}{3}, \text{ if } n \ = 2,4,6,\cdots \\ \dfrac{2^n+1}{3}, \text{ if } n = 1,3,5,\cdots \end{cases}$$

$$f(2^n-1) = \begin{cases} \dfrac{2^{n+1}}{3}, \text{ if } n \ = 2,4,6,\cdots \\ \dfrac{2^n+1}{3}, \text{ if } n = 1,3,5,\cdots \end{cases}$$

Thus $f(2047) = f(2^{11}-1) = 683$, so when there are 2047 prisoners, then prisoner number 683 will survive.

Editor's comment: **Ulrich Abel of Technische Hochschule Mittelhessen in Freiberg, Germany,** wrote that "this problem is a variant of the famous Josephus Problem (see; e.g. http://en.wikpedia.org/wiki/Josephusproblem) or the book Concrete Mathematics by Graham, Knuth and Patashnik."

Also solved by Ed Gray, Highland Beach, FL; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5363:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

  Let $x \in \Re$ and $A(x) = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 1 & x+1 & 1 & 1 \\ 1 & 1 & x+1 & 1 \\ 1 & 1 & 1 & x+1 \end{pmatrix}$.

  Compute $A(0) \cdot A(x) \cdot A(y) \cdot A(z), \forall x, y, z \in \Re$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $I_4 \cdot M = M \cdot I_4 = M$ for all 4×4 matrices M. Also, for all $t \in R$, it is easily seen that

$$A(t) = A(0) + tI_4$$

and

$$[A(0)]^2 = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} = 4A(0).$$

As a result, we have

$$\begin{aligned} A(0) \cdot A(x) &= A(0) \cdot [A(0) + xI_4] \\ &= [A(0)]^2 + xA(0) \\ &= (x+4)A(0) \end{aligned}$$

and

$$\begin{aligned} A(y) \cdot A(z) &= [A(0) + yI_4] \cdot [A(0) + zI_4] \\ &= [A(0)]^2 + (y+z)A(0) + yzI_4 \\ &= (y+z+4)A(0) + yzI_4. \end{aligned}$$

Therefore,

$$\begin{aligned} A(0) \cdot A(x) \cdot A(y) \cdot A(z) &= (x+4)A(0) \cdot [(y+z+4)A(0) + yzI_4] \\ &= (x+4)[4(y+z+4)A(0) + yzA(0)] \\ &= (x+4)(yz+4y+4z+16)A(0) \\ &= (x+4)(y+4)(z+4)A(0). \end{aligned}$$

Solution 2 by Moti Levy, Rehovot, Israel

$$A(x) = xI_4 + E_4,$$

where $I_4 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $E_4 := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

$$\begin{aligned} A(0) \cdot A(x) \cdot A(y) \cdot A(z) &= E_4(E_4 + xI_4)(E_4 + yI_4)(E_4 + zI_4) \\ &= E_4^4 + (x+y+z)E_4^3 + (xy+xz+yz)E_4^2 + xyzE_4 \end{aligned}$$

$$\begin{aligned} E_4^2 &= 4E_4, \\ E_4^3 &= 16E_4 \\ E_4^3 &= 64E_4 \end{aligned}$$

$$\begin{aligned} A(0) \cdot A(x) \cdot A(y) \cdot A(z) &= E_4^4 + (x+y+z)E_4^3 + (xy+xz+yz)E_4^2 + xyzE_4 \\ &= (64 + 16(x+y+z) + 4(xy+xz+yz) + xyz)E_4 \\ &= (z+4)(y+4)(x+4)E_4. \end{aligned}$$

Solution 3 by Paul M. Harms, North Newton, KS

Computing $A(0), A(x)$, we obtain the value of $(x-4)$ for each element in the product. On the main diagonal of the product $A(x)A(z)$ we have
$(y+1)(z+1)+3 = yz+y+z+4$. The other elements have the value
$(y+1)+(z+1)+2 = y+z+4$. Then the product $A(0)\,[A(y)A(z)]$ has the value
$yz+y+z+4+3(y+z+4)$ for each element. This value is equal to
$yz+4y+4z+16 = (y+4)(z+4)$. The result of the computation requested in the problem is $(x+4)(y+4)(z+4)A(0)$ or a 4 by 4 matrix all of whose elements are $(x+4)(y+4)(z+4)$.

Solution 4 by David Stone and John Hawkins of Georgia Southern University in Statesboro, GA

Editor's comment : The authors of this solution generalized the problem as follows:

Let $A(x)$ be m instead of 4×4 and we shall compute
$A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_n)$, for $x_i \in \Re$.

Let A be the $m\times m$ matrix $A = A(0)\ A(x) = \begin{vmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 \end{vmatrix}$.

Then$A(x) = A + xI$ (where I is the $m\times m$ identity matrix).

Lemma 1: $A^k = m^{k-1}A,\ k\geq 1$.
Proof: Certainly $A^1 = m^{1-1}A$ and an easy computation shows that $A^2 = mA = m^{2-1}A$.
Upon the obvious induction hypothesis,
$A^{k+1} = AA^k = A\left(m^{k-1}A\right) = m^{k-1}A^2 = m^{k-1}\left(mA\right) = m^kA,$ as desired.

Lemma 2: For any real $x,\ A\cdot A(x) = (m+x)A$.
Proof:

$$\begin{aligned} A\cdot A(x) = A\cdot(A+xI) &= A^2 + xA \\ &= mA + xA, \text{ by Lemma 1} \\ &= (m+x)A. \end{aligned}$$

Theorem: For $x_1, x_2, x_3, \ldots, x_n, \in \Re$ we have
$A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)(x_n) = (m+x_1)\,(m+x_2)\,(m+x_3)\cdots(m+x_n)\,A$.

Proof: We proceed by induction on n.
For $n=1,\ A(0)\cdot A(x_1) = A\cdot A(x_1) = (m+x_1)A$ by Lemma 2.
Making the obvious induction hypothesis,

$$\begin{aligned} & A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_{n+1}) \\ &\quad = \{A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_n)\}\cdot A\left(x_{n+1}\right) \\ &\quad = \{(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)A\}\cdot A\left(x_{n+1}\right) \\ &\quad = \{(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)\}\cdot\{A\cdot A\left(x_{n+1}\right)\} \end{aligned}$$

$$= \{(m + x_1)(m + x_2)(m + x_3) \cdots (m + x_n)\} \cdot \{(m + x_{n+1})\, A\} \text{ by Lemma 2}$$

$$= (m + x_1)\,(m + x_2)(m + x_3) \cdots (m + x_n) \cdot (m + x_{n+1})\, A, \text{ as desired.}$$

That is, $A\,(0) \cdot A\,(x_1) \cdot A\,(x_2) \cdot A\,(x_3) \cdots A\,(x_n)$ equals the $m \times m$ matrix

$$(m + x_1)(m + x_2)(m + x_3) \cdots (m + x_n) \begin{vmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 \end{vmatrix}.$$

Norte. There are no concerns about non-commutativity in our algebra of matrices, because A commutes with powers of itself and with any scalar matrix c.

Note also that everything above remains true if we let all scalars come from an arbitrary ring with identity (instead of the reals).

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; David Diminnie and Michael Taylor, Texas Instruments Inc., Dallas, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Connor Greenhalgh (student, Eastern Kentucky University), Richmond, KY; G. C. Greubel, Newport News, VA; Carl Libis, Columbia Southern University, Orange Beach, AL; David E, Manes, SUNY College at Oneonta, NY; Gail Nord, Gonzaga University, Spokane, WA; Toshihiro Shimizu, Kawasaki, Japan; Morgan Wood (student, Eastern Kentucky University), Richmond, KY, and the proposer.

- **5364:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Prove that $\displaystyle\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} 4^{-n} = 1.$

Solution 1 by Henry Ricardo, New York Math Circle, NY

The generating function of the central binomial coefficient is well known:

$$f(x) \;=\; \frac{1}{\sqrt{1-4x}} \;=\; \sum_{k=0}^{\infty} \binom{2k}{k} x^k.$$

Applying a standard theorem on the Cauchy product of two power series,

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{j=0}^{\infty} b_j x^j\right) \;=\; \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k\right) x^n,$$

to $f^2(x)$ yields

$$\begin{aligned}
\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} &= \text{the coefficient of } x^n \text{ in } \left(\frac{1}{\sqrt{1-4x}}\right)^2 \\
&= \text{the coefficient of } x^n \text{ in } \frac{1}{1-4x} = 4^n,
\end{aligned}$$

which proves the given identity.

Comment: The identity in the problem has been known since at least the 1930s. In her article "Counting and Recounting: The Aftermath" (*The Mathematical Intelligencer*, Vol. 6, No. 2, 1984), Marta Sved provides some references and describes a number of purely combinatorial proofs of the identity, all based in some way on the count of lattice paths.

Solution 2 by Arkady Alt, San Jose ,CA

First note that

$$\begin{aligned}
\binom{-1/2}{n} &= \frac{-1/2\,(-1/2-1)\ldots.(-1/2-n+1)}{n!} \\
&= (-1)^n \cdot \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!} \\
&= (-1)^n \cdot \frac{(2n)!}{2^{2n}\,(n!)^2} \\
&= \frac{(-1)^n}{4^n}\binom{2n}{n} \text{ and therefore,} \\
\binom{2n}{n} &= (-4)^n \binom{-1/2}{n}.
\end{aligned}$$

Since,

$$\binom{2k}{k}\binom{2n-2k}{n-k} = (-4)^k \binom{-1/2}{k} (-4)^{n-k}\binom{-1/2}{n-k} = (-4)^n \binom{-1/2}{k}\binom{-1/2}{n-k},$$

we have

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}4^{-n} = 1 \iff \sum_{k=0}^{n}\binom{-1/2}{k}\binom{-1/2}{n-k} = (-1)^n.$$

Since $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum\limits_{n=0}^{\infty}\binom{-1/2}{n}x^n$ and $\sum\limits_{n=0}^{\infty}(-1)^n x^n = \dfrac{1}{1+x}$,

we obtain

$$\left(\frac{1}{\sqrt{1+x}}\right)^2 = \frac{1}{1+x} \iff \left(\sum_{n=0}^{\infty}\binom{-1/2}{n}x^n\right)^2 = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\Longleftrightarrow \quad \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = \sum_{n=0}^{\infty} n\,(-1)^n\, x^n.$$

Hence, $\sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = (-1)^n.$

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

We have $\dfrac{1}{\sqrt{1-x^2}} = \sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n}.$

On the other hand, we have $\dfrac{1}{1-x^2} = \sum_{n\geq 0} x^{2n}$. Squaring the first power series and comparing terms give us $\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} 2^{-2n} = 1$, q.e.d.

Editor′s comment : Several of those who solved this problem also commented on where variations and generalizations of it can be found. E.g., **Ulrich Abel of the Technische Hochschule Mittelhessen in Friedberg, Germany** cited the paper:

Chang, G., Xu, C., "Generalization and probabilistic proof of a combinatorial identity." **American Mathematical Monthly** 118, 175-177, (2011), and also a paper of his which was published in 2015 that further generalizes notions used in the Chang and Xu paper.

Ulrich Abel, Vijay Gupta, and Mircea Ivan, "A generalization of a combinatorial identity by Change and Xu," **Bulletin of Mathematical Sciences,** published by Springer, ISSN 1664-3607. This paper can also be seen at Springer's open line access site < SpringerLink.com > .

Another citation was given by **Moti Levy, of Rehovot Israel.** He mentioned that in Concrete Mathematics, by Graham, Knuth, and Patashnik (second edition) the problem is solved in Section 5.3, "Tricks of the trade," pages 186-187 . **And Carl Libis of Columbia Southern University, Orange Beach, AL** cited http://math.stackexchange.com/questions/687221/proving-sum-k-0n2k-choose-k2n-2k-choose-n-k-4n/688370688370

In addition, **Bruno Salgueirio Fanego of Viveiro, Spain** stated that a probabilistic interpretation of the problem can be found in <http://mathes.pugetsound.edu/∼mspivey/AltConvRepring.pdf>. He went on to say that: more generally, it can be demonstrated that, for any real l, $\sum_{k=0}^{n} \binom{2n-2k-l}{n-k}\binom{2k+l}{k} 4^{-n} = 1$ (see: http://arxiv.org/pdf/1307.6693.pdf) and that for any integer $m \geq 2$,

$$\sum_{k_1\cdot k_2\cdots k_m = n} \binom{2k_1}{k_1}\binom{2k_2}{k_2}\cdots\binom{2k_m}{k_m} 4^{-n} = \frac{\Gamma\left(\frac{m}{2}+n\right)}{n!\,\Gamma\left(\frac{m}{2}\right)},$$

as can be found in <http://129.81.170.14/~vhm/papers_html/prob-bin.pdf>.

Also solved by Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Gail Nord, Gonzaga University, Spokane, WA; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

5365: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $n \geq 3$ be a positive integer. Find all real solutions of the system

$$\left.\begin{array}{c} a_2^3(a_2^2 + a_3^2 + \ldots + a_{j+1}^2) = a_1^2 \\ a_3^3(a_3^2 + a_4^2 + \ldots + a_{j+2}^2) = a_2^2 \\ \cdots\cdots\cdots \\ a_n^3(a_n^2 + a_1^2 + \ldots + a_{j-1}^2) = a_{n-1}^2 \\ a_1^3(a_1^2 + a_2^2 + \ldots + a_j^2) = a_n^2 \end{array}\right\}$$

for $1 < j < n$.

Partial solution by the proposer

Since the RHS of all equations are nonnegative, then the system does not have solutions $(a_1, a_2, \ldots, a_n)$ with negative components. Moreover, $(0, 0, \ldots, 0)$ is a trivial solution. So, it remains to find the positive solutions of the system. To do it, let $m = \min\limits_{1 \leq k \leq n} \{a_k\} = a_p$ and $M = \max\limits_{1 \leq k \leq n} \{a_k\} = a_q$. Then, using the $(q-1)^{th}$ equation yields

$$jM^3m^2 \leq a_q^3(a_q^2 + a_{q+1}^2 + \ldots + a_{q+j-1}^2) = a_{q-1}^2 \leq M^2$$

and from the $(p-1)^{th}$ equation we get

$$jm^3M^2 \geq a_p^3(a_p^2 + a_{p+1}^2 + \ldots + a_{p+j-1}^2) = a_{p-1}^2 \geq m^2$$

Therefore,

$$jM^3m^2 \leq M^2 \Leftrightarrow M \leq \frac{1}{jm^2}$$

and

$$jm^3M^2 \geq m^2 \Leftrightarrow m \geq \frac{1}{jM^2}$$

Since $M \leq \dfrac{1}{jm^2}$, then $j^2m^4 \leq \dfrac{1}{M^2}$ and from $m \geq \dfrac{1}{jM^2}$ follows that

$$m \geq jm^4 \Rightarrow m \leq \sqrt[3]{1/j}$$

Likewise, from $M \leq \dfrac{1}{jm^2}$ and $m \geq \dfrac{1}{jM^2}$ immediately follows

$$M \leq jM^4 \Rightarrow M \geq \sqrt[3]{1/j}$$

So, $m = M = \sqrt[3]{1/j}$ and a positive solution of the given system is

$$\left(\sqrt[3]{1/j}, \sqrt[3]{1/j}, \ldots, \sqrt[3]{1/j}\right)$$

(*) It remains to prove if there exist or not other positive solutions.

Editor's comment: When the statement of this problem was published the last line in the system was not there. **Toshihiro Shimizu of Kawasaki, Japan** mentioned that for the sake of symmetry it would be advantageous to add this last line to the system, and the proposer agreed. But as we see, even with this additional condition, a definitive set of solutions was not received.

5366: *Proposed by Ovidiu Furdui and Alina Sintămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all nonconstant, differentiable functions $f : \Re \to \Re$ which verify the functional equation $f(x+y) - f(x-y) = 2f'(x)f(y)$, for all $x, y \in \Re$.

Solution 1 by Moti Levy, Rehovot, Israel

We will show that all the solutions of the *functional* equation (1) must satisfy the *differential* equation (2):

$$f(x+y) - f(x-y) = 2f^{'}(x)f(y), \quad \text{for all } x, y \in R, \tag{1}$$

$$f^{''}(x)f(x) - \left(f^{'}(x)\right)^2 + 1 = 0, \quad f(0) = 0, f^{'}(0) = 1. \tag{2}$$

We divide both sides of (1) by y and take the limit $y \to 0$.

$$\frac{f(x+y) - f(x-y)}{y} = 2f^{'}(x)\frac{f(y)}{y} \tag{3}$$

The left hand side approaches the derivative $f^{'}(x)$

$$\lim_{y\to 0} \frac{f(x+y) - f(x-y)}{2y} = f^{'}(x),$$

and the right hand side is equal to $f^{'}(x) \lim_{y\to 0} \frac{f(y)}{y}$.
It follows that

$$\lim_{y\to 0} \frac{f(y)}{y} = 1 \quad \Longrightarrow \quad f(0) = 0. \tag{4}$$

By Taylor's theorem,

$$f(y) = f(0) + f^{'}(\theta)\, y, \qquad 0 \le \theta \le y.$$

$$\lim_{y\to 0} \frac{f(y)}{y} = 1 = \lim_{y\to 0} \frac{f^{'}(\theta)\, y}{y} \Longrightarrow f^{'}(0) = 1.$$

Thus we have derived the initial conditions,

$$f(0) = 0, \; f^{'}(0) = 1. \tag{5}$$

Differentiation of (1) with respect to the variable y, gives

$$f^{'}(x+y) + f^{'}(x-y) = 2f^{'}(x)f^{'}(y). \tag{6}$$

Setting $x = y$ in (1) and in (6), we obtain

$$f(2x) = 2f^{'}(x)f(x), \tag{7}$$

$$f^{'}(2x) + 1 = 2\left(f^{'}(x)\right)^2. \tag{8}$$

Now, $f^{'}(x)=\frac{f(2x)}{2f(x)}$ from (7), implies that $f^{'}(x)$ it is differentiable function (for $f(x)\neq 0)$, (actually, by this argument $f(x)$ is infinitely differentiable). Differentiating (7) gives

$$f^{'}(2x)=f^{''}(x)f(x)+\left(f^{'}(x)\right)^2. \tag{9}$$

By equating $f^{'}(2x)$ in (8) and (9), we obtain the differential equation,

$$f^{''}(x)f(x)-\left(f^{'}(x)\right)^2+1=0 \tag{10}$$

Now we differentiate (10),

$$f^{(3)}(x)f(x)+f^{''}(x)f^{'}(x)-2f^{'}(x)f^{''}(x)=0$$

or

$$\frac{f^{(3)}(x)}{f^{''}(x)}=\frac{f^{'}(x)}{f(x)} \tag{11}$$

$$\ln f^{''}(x)=\ln f(x)+c$$
$$f^{''}(x)=k^2f(x)$$

$$f(x)=\alpha e^{kx}+\beta e^{-kx}$$

$$f(0)=0, \qquad \Longrightarrow \qquad \alpha+\beta=0$$
$$f^{'}(0)=1, \qquad \Longrightarrow \qquad k\alpha-k\beta=1$$

$$f(x)=\frac{e^{kx}-e^{-kx}}{2k}, \qquad k\in C.$$

Let $k=\sigma+i\tau$, $\sigma,\tau\in R$, then

$$\begin{aligned}
f(x) &= \frac{e^{\sigma x}(\cos\tau x+i\sin\tau x)-e^{-\sigma x}(\cos\tau x-i\sin\tau x)}{2(\sigma+i\tau)}\\
&= \frac{(\sigma-i\tau)\left(e^{\sigma x}(\cos\tau x+i\sin\tau x)-e^{-\sigma x}(\cos\tau x-i\sin\tau x)\right)}{2(\sigma^2+\tau^2)}\\
&= \frac{e^{\sigma x}(\sigma\cos(\tau x)+\tau\sin(\tau x))-e^{-\sigma x}(\sigma\cos(\tau x)-\tau\sin(\tau x))}{2(\sigma^2+\tau^2)}\\
&+i\frac{e^{\sigma x}(\sigma\sin\tau x-\tau\cos\tau x)-e^{-\sigma x}(\sigma\sin\tau x-\tau\cos\tau x)}{2(\sigma^2+\tau^2)}.
\end{aligned}$$

Since we are requested to find only the real functions, then σ must be equal to 0 or τ must be equal to 0.
When $\sigma=0$ then

$$f(x)=\frac{\sin(\tau x)}{\tau}, \qquad \tau\in R\backslash\{0\}.$$

When $\tau=0$ then

$$f(x)=\frac{\sin(i\sigma x)}{i\sigma}=\frac{\sinh(\sigma x)}{\sigma}, \quad \tau\in R\backslash\{0\}.$$

One can check that $f(x) = \lim_{\tau \to 0} \frac{\sin(\tau x)}{\tau} = x$ is also a solution of the differential equation (2).

It is easy to check that $f(x) = \begin{cases} \frac{\sin(\tau x)}{\tau}, & \tau \in R \backslash \{0\} \\ \frac{\sinh(\sigma x)}{\sigma}, & \sigma \in R \backslash \{0\} \\ x \end{cases}$, the family of solution of (2), are indeed solution of (1).

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

Let (F) be the functional equation in the problem statement.

There exists y_1 such that $f'(y_1) \neq 0$, otherwise f would be constant. Taking $x = y_1, y \neq 0$ to (F) we get

$$\frac{f(y_1 + y) - f(y_1 - y)}{2y} = f'(y_1)\frac{f(y)}{y}.$$

Taking the limit $y \to 0$, we get $f'(y_1) = f'(y_1)f'(0)$ or $f'(0) = 1$. Taking $x = 0$ to (F), we get $f(y) - f(-y) = 2f(y)$ or $f(-y) = -f(y)$. Especially, $f(0) = 0$.

Then, we get
$2f'(-x)f(y) = f(-x + y) - f(-x - y) = -f(y - x) + f(x + y) = 2f'(x)f(y)$. We take $y = y_0$ such that $f(y_0) \neq 0$, where such y_0 exists since f is not constant. Then, we get $f'(-x) = f'(x)$ for all $x \in R$.

We show that $f'$is differentiable. Taking $y = y_0$ to (F),
$f'(x) = \big(f(x + y_0) - f(x - y_0)\big)/\big(2f(y_0)\big)$ for all $x \in R$.

Thus, it follows that

$$\begin{aligned}
\frac{f'(x + h) - f'(x)}{h} &= \frac{f(x + h + y_0) - f(x + h - y_0) - f(x + y_0) + f(x - y_0)}{2f(y_0)h} \\
&= \frac{1}{2f(y_0)}\left(\frac{f(x + h + y_0) - f(x + y_0)}{h} - \frac{f(x + h - y_0) - f(x - y_0)}{h}\right) \\
&\to \frac{1}{2f(y_0)}\left(f'(x + y_0) - f'(x - y_0)\right) (h \to 0)
\end{aligned}$$

Thus f' is differentiable.

Differentiating with respect to x, we get

$$f'(x + y) - f'(x - y) = 2f''(x)f(y)$$

Exchanging x and y, (l.h.s) is not changed. Thus $f''(x)f(y) = f(x)f''(y)$ for any $x, y \in R$. Especially for $y = y_0$, we get the result that $f''(x) = cf(x)$ for some constant $c \in R$. It's known functional equation and we omit the detail.

If $c > 0$, we can write as $f(x) = C_1 \exp(Cx) + C_2 \exp(-Cx)$. From the fact that $f(0) = 0$ and $f'(0) = 1$, we get $C_1 + C_2 = 0$ and $C(C_1 - C_2) = 1$. Thus, we can write as

$f(x) = \big(\exp(Cx) - \exp(-Cx)\big)/(2C) = \sinh(Cx)/(2C)$. It is easy to check that this function satisfies (F).

If $c < 0$, we can write as $f(x) = C_1 \cos(Cx) + C_2 \sin(Cx)$. From the fact that $f(0) = 0$ and $f'(0) = 1$, we get $C_1 = 0$, $CC_2 = 1$. Thus, we can write as $f(x) = \sin(Cx)/C$. Again, it is easy to check that this function satisfies (F).

If $c = 0$, we can write as $f(x) = Cx + D$. From the fact that $f(0) = 0$ and $f'(0) = 1$. We get $f(x) = x$. It also satisfies (F).

Finally, we get $f(x) = \sinh(Cx)/(2C)$ or $f(x) = \sin(Cx)/C$ or $f(x) = x$ where $C \neq 0$ is constant.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $f(x) = x, \dfrac{e^{ax} - e^{-ax}}{2a}$ or $\dfrac{\sin(bx)}{b}$, where a and b are nonzero numbers.

By putting $y = 0$ into the into the given functional equation

$$f(x+y) - f(x-y) = 2f'(x)f(y) \qquad (1)$$

we obtain we obtain $f'(x)(0) = 0$. Since f is non-constant, so there exists $a \in \Re$ such that $f'(a) \neq 0$. Hence $f(0) = 0$. Differentiate (1) with respect to y, we obtain

$$f'(x+y) + f'(x-y) = 2f'(x)f'(y). \qquad (2)$$

By putting $y = 0$ and $x = a$ into (2), we obtain $f'(0) = 1$. By putting $x = 0$ into (1), we obtain $f(-y) = -f(y)$. Hence by interchanging x and y in (1), we obtain

$$f(x+y) + f(x-y) = 2f'(y)f(x), \qquad (3)$$

Adding up (1) and (3),) we obtain

$$f(x+y) = f'(x)f(y) + f'(y)f(x). \qquad (4)$$

Differentiating (4) with respect to x, we obtain

$$f'(x+y) = f''(x)f(y) + f'(y)f'(x). \qquad (5)$$

Differentiating (4) with respect to y, we obtain

$$f'(x+y) = f'(x)f'(y) + f''(y)f(x) \qquad (6)$$

From (5) and (6), we obtain $f''(x)f(y) = f''(y)f(x)$ for all $x, y \in \Re$.

It follows that $f''(x) = kf(x)$, where k is a constant.

If $k = 0$ then $f''(x) = 0$, so that f is a linear function. Since $f(0) = f'(0) - 1 = 0$, so $f(x)$. If $k = a^2$, then $f''(x) - a^2 f(x) = 0$.

By standard methods, we obtain $(x) = \dfrac{e^{ax} - e^{-ax}}{2a}$. If $k = -b^2$, then $f''(x) + b^2 f(x) = 0$.

By standard methods, we obtain $f(x) = \dfrac{\sin(bx)}{b}$.

This completes the solution.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC, and the proposers.

$Mea - Culpa$

The names of Bruno Salgueiro Fanego of Viveiro, Spain and **David E. Manes of SUNY College at Oneonta, NY** should have been listed as having solved problem 5358; their names were inadvertently omitted from the list.

Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2016

- **5385:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides and integer area has perimeter $P = 6^6$. Find the sides of the triangle when the area is minimum.

- **5386:** *Proposed by Michael Brozinsky, Central Islip, NY.*

  Determine whether or not there exit nonzero constants a and b such that the conic whose polar equation is
  $$r = \sqrt{\frac{a}{\sin(2\theta) - b\cos(2\theta)}}$$
  has a rational eccentricity.

- **5387:** *Proposed by Arkady Alt, San Jose, CA*

  Let $D := \{(x, y) \mid x, y \in R_+,\ x \neq y \text{ and } x^y = y^x\}$.(Obviously $x \neq 1$ and $y \neq 1$).
  Find $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1} + y^{-1}}{2}\right)^{-1}$

- **5388:** *Proposed by Jiglău Vasile, Arad, Romania*

  Let $ABCD$ be a cyclic quadrilateral, R and r its exradius and inradius respectively, and a, b, c, d its side lengths (where a and c are opposite sides.) Prove that
  $$\frac{R^2}{r^2} \ \geq \frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}.$$

- **5389:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let ABC be a scalene triangle with semi-perimeter s and area $\mathcal{A}$. Prove that
  $$\frac{3a + 2s}{a(a-b)(a-c)} + \frac{3b + 2s}{b(b-a)(b-c)} + \frac{3c + 2s}{c(c-a)(c-b)} < \frac{3\sqrt{3}}{4\mathcal{A}}.$$

- **5390:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $A \in \mathcal{M}_2(R)$ such that $AA^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a > b \geq 0$. Prove that $AA^T = A^T A$ if and only if $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ or $A = \begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$, where $\alpha = \dfrac{\pm\sqrt{a+b} \pm \sqrt{a-b}}{2}$ and $\beta = \dfrac{\pm\sqrt{a+b} \mp \sqrt{a-b}}{2}$. Here A^T denotes the *transpose* of A.

Solutions

- **5367:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with integer length sides and integer area. The vertices have coordinates $A(0,0)$, $B(x,y)$ and $C(z,w)$ with $\sqrt{x^2+y^2} - \sqrt{z^2+w^2} = 1$.

  Find positive integers x, y, z and w if the perimeter is 84.

  Solution by Ed Gray, Highland Beach, FL

  Let the sides of the triangle be a, b, c where $b = \sqrt{z^2+w^2}$ and $c = \sqrt{x^2+y^2}$. We are given that

$$\begin{aligned} c - b &= 1 \\ a + b + c &= 84. \text{ So, subtracting} \\ a + 2b &= 83, \text{ or, } a = 83 - 2b. \end{aligned}$$

  By Brahmagupta's formula, the area T is given by

$$\begin{aligned} T^2 &= s(s-a)(s-b)(s-c), \text{ where } s = \frac{1}{2}(a+b+c) = 42. \text{ Then,} \\ T^2 &= 42\left(42 - (83-2b)\right)(42-b)\left(42-(b+1)\right), \text{ or} \\ T^2 &= 42\,(2b-41)\,(42-b)(41-b) \Longrightarrow b = 34. \text{ So} \\ T^2 &= (42)(27)(8)(7) = (14)^2 \cdot 9^2 \cdot 2^2 = (252)^2 \Longrightarrow \\ T &= 252, \ b = 34, \ c = b+1 = 35, \text{ and } a = 15. \end{aligned}$$

  Since $b = \sqrt{z^2+w^2}$, $b^2 = 34^2 = 1156 = z^2 + w^2$ and we have $z = 30, w = 16$ since $900 + 256 = 1156$, or vice versa, $z = 16$ and $w = 30$. Similarly,

  $c = \sqrt{x^2+y^2}$, $c^2 = 35^2 = 1225 = x^2 + y^2$ and we have $x = 28, y = 21$ since $784 + 441 = 1225$, or vice versa, $x = 21$ and $y = 28$.

  In summary, $(x, y, z, w) \in \{(21, 28, 30, 16), (28, 21, 16, 30)\}$.

  Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo Sate University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania;

David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5368:** *Proposed by Ed Gray, Highland Beach, FL*

Let $abcd$ be a four digit number in base 10, none of which are zero, such that the last four digits in the square of $abcd$ are $abcd$, the number itself. Find the number $abcd$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $x = \left(a \times 10^3\right) + \left(b \times 10^2\right) + (c \times 10) + d$, with $a, b, c, d \in \{1, 2, \ldots, 9\}$,then

$$\begin{aligned} x^2 &= \left(a^2 \times 10^6\right) + \left(2ab \times 10^5\right) + \left[\left(b^2 + 2ac\right) \times 10^4\right] + \left[2\left(ad + bc\right) \times 10^3\right] \\ &\quad + \left[\left(c^2 + 2bd\right) \times 10^2\right] + (2cd \times 10) + d^2. \end{aligned}$$

In order for the units digit of x^2 to be d, we must have $d^2 \equiv d \pmod{10}$. Since $d \in \{1, 2, \ldots, 9\}$, this restricts our choices to $d = 1$, 5, or 6

Case 1. If $d = 1$, then $d^2 = 1$ and to obtain c as the tens digit of x^2, we need $2cd \equiv c \pmod{10}$. Since $d = 1$, this reduces to $c \equiv 0 \pmod{10}$, which is impossible when $c \in \{1, 2, \ldots, 9\}$. Therefore, this case fails.

Case 2. If $d = 5$, then $d^2 = 25$ and to get c as the tens digit of x^2, we require that $2cd + 2 \equiv c \pmod{10}$. With $d = 5$, this reduces to $c \equiv 2 \pmod{10}$ and hence, $c = 2$. When $c = 2$ and $d = 5$, we have $(2cd \times 10) + d^2 = 225$. To get b as the hundreds digit of x^2, we are forced to set

$$c^2 + 2bd + 2 \equiv b \pmod{10}.$$

This reduces to $b \equiv 6 \pmod{10}$ and thus, $b = 6$. When $d = 5$, $c = 2$, and $b = 6$, we have $\left(c^2 + 2bd\right) \times 10^2 + (2cd \times 10) + d^2 = 6625$. Finally, to obtain a as the thousands digit of x^2, we are left with

$$2\left(ad + bc\right) + 6 \equiv a \pmod{10},$$

which reduces to $a \equiv 0 \pmod{10}$. Since this is impossible when $a \in \{1, 2, \ldots, 9\}$, this case also fails.

Case 3. If $d = 6$, then $d^2 = 36$ and to get c as the tens digit of x^2, we must set $2cd + 3 \equiv c \pmod{10}$. This reduces to $c \equiv 7 \pmod{10}$ and hence, $c = 7$. When $d = 6$ and $c = 7$, $(2cd \times 10) + d^2 = 876$. To get b as the hundreds digit of x^2 now requires that $c^2 + 2bd + 8 \equiv b \pmod{10}$, i.e., $b \equiv 3 \pmod{10}$. Hence, $b = 3$ and $\left(c^2 + 2bd\right) \times 10^2 + (2cd \times 10) + d^2 = 9376$. Finally, in order for the thousands digit of x^2 to be a, we need $2\left(ad + bc\right) + 9 \equiv a \pmod{10}$ or $a \equiv 9 \pmod{10}$. This yields $a = 9$ and $x = 9376$. Since $(9376)^2 = 87909376$, our solution is complete.

Solution 2 by Bruno Salguerio Fanego, Viveiro, Spain

Note that $abcd$ can be expressed as $1000a + 100b + 10c + d$, whose square $(abcd)^2$ is

$$a^2 \cdot 10^6 + 2ab \cdot 10^5 + (2ac + b2) \cdot 10^4 + (2ad + 2bc) \cdot 1000 + (2bd + c^2) \cdot 100 + 2cd \cdot 10 + d^2.$$

Moreover, $1 \leq a, b, c, d \leq 9$. We distinguish several cases:

If $d \leq 3$, the last digit of $(abcd)^2$ is d^2, which, since its last four digits are $abcd$, must be equal to d, so $d = 1$, in which case, for $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, we obtain that the last two digits of $(abc1)^2 = \ldots + 2c \cdot 10 + 1$ are, respectively, $\{21, 41, 61, 81, 01, 21, 41, 61, 81\}$ and, on the other hand, since the last two digits of $(abc1)^2$ are equal to c1, they must be also equal to $\{11, 21, 31, 41, 51, 61, 71, 81, 91\}$. But none of the two possible ending digits for $(abcd)^2$ coincides with units digit of this last possible ending, and so we conclude that this case, that is, $d \leq 3$, is impossible, so $d \geq 4$. Since d^2 ends in $1, 4, 9, 6$ or $5, (abcd)^2$ ends in $1, 4, 9, 6$ or 5, so $d \in \{4, 5, 6, 9\}$ and, hence, $(abcd)^2$ ends in $6, 5, 6, 1$ respectively, so $d \in \{6, 5, 6, 1\}$ respectively, which implies that $d \in \{5, 6\}$.

When $d = 5, (abcd)^2 = \ldots + (2bd + c^2 + c) \cdot 10 + 25$ ends in 25, so $c = 2$. Then,

$$(abcd)^2 = \cdots + (2b \cdot 5 + 2 \cdot 2^2) \cdot 100 + (2 \cdot 2 \cdot 5 + 2^2) \cdot 10 + 25 = \ldots + 625,$$

which ends in 625, so $b = 6$. Hence,

$$(abcd)^2 = \ldots + (2 \cdot a \cdot 5 + 2 \cdot 6 \cdot 2) \cdot 1000 + (2 \cdot 6 \cdot 5 + 2^2) \cdot 100 + (2 \cdot 2 \cdot 5) \cdot 10 + 25,$$

which ends in 0625 and this contradicts the fact that $(abcd)^2$ must end in $abcd$ (because a cannot be equal to zero).

When $d = 6$, we obtain respectively that, for $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, (abcd $)^2$ ends in $56, 76, 96, 16, 36, 56, 76, 96, 16$. Thus, the only possible case is $c = 7$, being thus

$$(abcd)^2 = (ab76)^2 = (12a + 14b) \cdot 1000 + (12b + 49) \cdot 100 + 876.$$

Hence, we obtain that, when $b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, (abcd)^2$ ends in $976, 176, 376, 576, 776, 976, 176, 376, 576$, respectively, which implies that $b = 3$ is the only possibility.

Then, $(abcd)^2 = (a376)^2$, which ends $3376, 5376, 7376, 9376, 1376, 3376, 5376, 7376, 9376$ for a equal to $1, 2, 3, 4, 5, 6, 7, 8, 9$. This implies that $a = 9$ and since 9376^2 ends in 9376, we conclude that the only solution to the problem is the number 9376.

Solution 3 by Paul M. Harms, North Newton, KS

Let us look for the answer to the problem by checking one digit at a time. First consider a one-digit number whose square has the same units digit as the original number. The one-digit number will have to be 1, 5, or 6.

Let us now try two-digit numbers whose units digit is 1 and whose square has the same last two digits as the original number. It is easy to show that no two-digit number exists for this case.

Now consider the case where the units digit is 5. All numbers of this type have squares ending in 25. Thew number 25 is the only two-digit number whose square ends in 25.

We find 625 is the only three-digit number whose square ends in 625.

If a is any non-zero fourth digit, we find that $a625$ has a square that ends in 0625.Thus the number satisfying the problem cannot end in 5. We now consider the case where the units digit is 6. We see that $76^2 = 5776,\ 376^2 = 141376$, and $9376^2 = 87909376$. The number 9376 satisfies the problem.

Editor's comment: **Brian D. Beasley of Presbyterian College in Clinton SC, Kenneth Korbin of New York, NY, and the team of David Stone and John Hawkins of Georgia Southern University** each mentioned in their solution that such sequences are called “automorphic numbers” and start as $\{5, 25, 625, 0625, 90625, \ldots\}$ and $\{6, 76, 376, 9376, 09376, \ldots\}$. See: Weisstein, Eric W. “Automorphic Number” in MathWorld-A Wolfram Web Resource, <http://matheworld.wolfram.com/Automorphic Number.html>.

David Stone and John Hawkins constructed and proved the following theorem. For any $n \geq 1$, there are exactly four n-digit integers N such that the last n digits of N^2 are the digits of N. The four numbers are 0 and 1 (considered as n-digit integers), $2^{n\cdot 4\cdot 5^{n-1}}$ and $5^{n\cdot 2^{n-1}}$ (both being reduced mod 10^n).
They went on to say that they did not find the above theorem in the literature that they searched on automorphic numbers.

Also solved by Stephen Acampa (student at Eastern Kentucky University), Richmond, KY; Brian D. Beasley, Presbyterian College, Clinton SC; Kee-Wai Lau, Hong Kong, China; Kenneth Korbin, New York, NY; Carl Libis, Columbia Southern University, Orange Beach, AL; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Susan Popp (graduate student at Eastern Kentucky University), Richmond, KY; Erron Prickett (graduate student at Eastern Kentucky University), Richmond, KY; Neculai Stanciu, “George Emil Palade” School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA; Deven Turner (student at Eastern Kentucky University), Richmond, KY, and the proposer.

- **5369:** *Proposed by Chirita Marcel, Bucuresti, Romania*

A convex quadrilateral $ABCD$ has area S and side lengths $\overline{AB} = a, \overline{BC} = b, \overline{CD} = c, \overline{DA} = d$. Show that

$$2\left(a+b+c+d\right)^2 + a^2 + b^2 + c^2 + d^2 \geq 36\sqrt{\left(S^2 + abcd\cos^2\frac{A+C}{2}\right)}.$$

Solution by Nikos Kalapodis, Patras, Greece

Taking into account the Bretschneider’s formula (see [1]) for the area of a convex quadrilateral:
$S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\dfrac{A+C}{2}}$, where $s = \dfrac{a+b+c+d}{2}$,
we see that the given inequality is equivalent to

$$2(a+b+c+d)^2 + a^2 + b^2 + c^2 + d^2 \geq 36\sqrt{(s-a)(s-b)(s-c)(s-d)} \quad (*).$$

Now from the Cauchy-Schwartz inequality and the AM-GM inequality we have

$$\begin{aligned} 2(a+b+c+d)^2 + a^2 + b^2 + c^2 + d^2 &\geq 2(a+b+c+d)^2 + \frac{(a+b+c+d)^2}{4} \\ &= \frac{9}{4}(a+b+c+d)^2 \\ &= \frac{9}{4}[(s-a)+(s-b)+(s-c)+(s-d)]^2 \end{aligned}$$

$$\geq \frac{9}{4}\left[4\sqrt[4]{(s-a)(s-b)(s-c)(s-d)}\right]^2$$
$$= 36\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

We have thus proved $(*)$ and this completes the solution.

[1] https://en.wikipedia.org/wiki/Bretschneider

Also solved by Bruno Salguerio Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5370:** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let $f(x)$ and $g(x)$ be arbitrary functions defined for all $x \in \Re$. Prove that there is a function $h(x)$ such that

$$(f(x)-h(x))^{2015} \cdot (g(x)-h(x))^{2015}$$

is an odd function for all $x \in \Re$.

Solution by Moti Levy, Rehovot, Israel

If $f(x)$ is odd then $(f(x))^{2015}$ is odd, hence proving that there is a function $h(x)$ such that

$$(f(x)-h(x))(g(x)-h(x))$$

is an odd function for all $x \in R$ will suffice.
Let

$$h(x) = \frac{1}{2}\left(f(x)+f(-x)+g(x)-g(-x)\right).$$

$$(f(x)-h(x))\,(g(x)-h(x))$$
$$= \left(f(x)-\frac{1}{2}\left(f(x)+f(-x)+g(x)-g(-x)\right)\right)\left(g(x)-\frac{1}{2}\left(f(x)+f(-x)+g(x)-g(-x)\right)\right)$$
$$= \left(\frac{f(x)-f(-x)}{2}-\frac{g(x)-g(-x)}{2}\right)\left(\frac{g(x)+g(-x)}{2}-\frac{f(x)+f(-x)}{2}\right).$$

The first factor is odd function, while the second factor is even function, hence the product is an odd function, as required.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5371:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers where $n \geq 4$. Prove that

$$\left(\frac{a_1}{a_n+a_2}\right)^2+\left(\frac{a_2}{a_1+a_3}\right)^2+\ldots+\left(\frac{a_n}{a_{n-1}+a_1}\right)^2 \geq \frac{4}{n}$$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Define vector $\vec{u}$ and $\vec{v}$ in R^n such that

$$\vec{u} = (1, 1, 1, \ldots, 1) \text{ and } \vec{v} = \left(\frac{a_1}{a_n + a_2}, \frac{a_2}{a_1 + a_3}, \cdots, \frac{a_n}{a_{n-1} + a_1}\right).$$

Then the Cauchy-Schwarz inequality implies $||\vec{u}||||\vec{v}|| \geq \vec{u}\cdot\vec{v}$. Therefore,

$$\sqrt{n}\sqrt{\left(\frac{a_1}{a_n + a_2}\right)^2 + \left(\frac{a_2}{a_1 + a_3}\right)^2 + \cdots + \left(\frac{a_n}{a_{n-1} + a_1}\right)^2} \geq \frac{a_1}{a_n + a_2} + \frac{a_2}{a_1 + a_3} + \cdots + \frac{a_n}{a_{n-1} + a_1}.$$

Squaring the inequality, we obtain

$$\left(\frac{a_1}{a_n + a_2}\right)^2 + \left(\frac{a_2}{a_1 + a_3}\right)^2 + \cdots + \left(\frac{a_n}{a_{n-1} + a_1}\right)^2 \geq \frac{\left(\frac{a_1}{a_n + a_2}, \frac{a_2}{a_1 + a_3}, \cdots, \frac{a_n}{a_{n-1} + a_1}\right)^2}{n}.$$

The result now follows provided we can show that if $n \geq 4$, then

$$J_n = \frac{a_1}{a_n + a_2} + \frac{a_2}{a_1 + a_3} + \cdots + \frac{a_n}{a_{n-1} + a_1} \geq 2.$$

To this end, let $n = 4$. Then

$$J_4 = \frac{a_1}{a_4 + a_2} + \frac{a_2}{a_1 + a_3} + \frac{a_3}{a_2 + a_4} + \frac{a_4}{a_3 + a_1} = \frac{a_1 + a_3}{a_2 + a_4} + \frac{a_2 + a_4}{a_1 + a_3} \geq 2,$$

since $x + \dfrac{1}{x} \geq 2$ for all $x > 0$. Assume inductively tht k is a positive integer, $k \geq 4$, and $J_k \geq 2$. Consider $k + 1$ positive numbers $a_1, a_2, \ldots a_k, a_{k+1}$. Since J_{k+1} is symmetric with respect to these numbers, we can assume without loss of generality that $a_j \geq a_k + 1$ for $j = 1, 2, \ldots, k$. Then

$$J_{k+1} = \frac{a_1}{a_{k+1} + a_2} + \frac{a_2}{a_1 + a_3} + \cdots + \frac{a_k}{a_{k-1} + a_{k+1}} + \frac{a_{k+1}}{a_k + a_1}.$$

Observe that

$a_{k+1} \leq a_k$ implies $a_{k+1} + a_2 \leq a_k + a_2$ implies $\dfrac{a_1}{a_{k+1} + a_2} \geq \dfrac{a_1}{a_k + a_2}$, and similarly

$\dfrac{a_k}{a_{k-1} + a_{k+1}} \geq \dfrac{a_k}{a_{k-1} + a_1}$. Therefore,

$$J_{k+1} \geq J_k + \frac{a_{k+1}}{a_k + a_1} > J_k \geq 2$$

by the induction hypothesis. Hence, by induction $J_n \geq 2$ if $n \geq 4$.

Accordingly,

$$\left(\frac{a_1}{a_n + a_2}\right)^2 + \left(\frac{a_2}{a_1 + a_3}\right)^2 + \cdots + \left(\frac{a_n}{a_{n-1} + a_1}\right)^2 \geq \frac{(J_n)^2}{n} \geq \frac{4}{n}.$$

Solution 2 by Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania

Since $(a_n + a_2)^2 \leq 2\left(a_n^2 + a_2^2\right)$, it suffices to prove that

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq\frac{8}{n}, \text{ where } x_1=a_i^2.$$

We shall prove that

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq 2.$$

By Bergstrm's inequality we obtain

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq\frac{(x_1+x_2+\cdots+x_n)^2}{2\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right)},$$

so it suffices to show that

$$(x_1+x_2+\cdots+x_n)^2\geq 4\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right). \qquad (1)$$

The inequality (1) is cyclic; we can assume that $x_n=\min\{x_1,x_2,\ldots,x_{n-1},x_n\}$.

• For n odd we have

$$\begin{aligned}&(x_1+x_2+\cdots+x_n)^2-4\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right)\\ \geq\ &(x_1-x_2+\ldots-x_{n-1}+x_n)^2+4x_1x_{n-1}-4x_1x_n\geq 0.\end{aligned}$$

• For n even we have

$$\begin{aligned}&(x_1+x_2+\cdots+x_n)^2-4\left(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1\right)\\ \geq\ &(x_1-x_2+\ldots+x_{n-1}-x_n)^2.\end{aligned}$$

Remark. For $n\geq 8$ we have a simple solution, i.e.,

$$\frac{x_1}{x_n+x_2}+\frac{x_2}{x_1+x_3}+\cdots+\frac{x_n}{x_{n-1}+x_1}\geq\frac{x_1}{x_1+x_2+\cdots+x_n}+\cdots+\frac{x_n}{x_1+x_2+\cdots+x_n}=1\geq\frac{8}{n}.$$

Editor′s comment: **Paolo Perfetti** mentioned in his solution that $\dfrac{a_1}{a_n+a_2}+\dfrac{a_2}{a_1+a_3}+\cdots+\dfrac{a_n}{a_{n-1}+a_1}\geq 2$ is known as being one of the Shapiro inequalities, and that its proof by induction can be found in <http://olympiads.mccme.ru/1ktg/2010/5/5-1en.pdf>.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

• **5372:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $k\geq 2$ be an integer. Calculate

$$\int_0^\infty\frac{\ln(1+x)}{x\sqrt[k]{x}}dx.$$

(b) Calculate

$$\int_0^\infty \frac{\ln(1-x+x^2)}{x\sqrt{x}}dx.$$

Solution 1 by Moti Levy, Rehovot, Israel

Reference: Emil Artin, "*The Gamma Function*", Holt, Rinehart and Winston, 1964. Page 29.

(a)

The well known Euler's reflection formula for the Gamma function is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \qquad 0 < x < 1.$$

From the definition of the Beta function,

$$B(x, 1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} = \int_0^1 t^{x-1}(1-t)^{-x}\,dt.$$

Since $\Gamma(1) = 1$,

$$\int_0^1 t^{x-1}(1-t)^{-x}\,dt = \frac{\pi}{\sin \pi x}, \qquad 0 < x < 1.$$

Changing the variable of integration $u = \frac{t}{1-t}$, we get

$$\int_0^\infty \frac{u^{x-1}}{1+u}du = \frac{\pi}{\sin \pi x}, \qquad 0 < x < 1.$$

By integration by parts, we get

$$\int_0^\infty \frac{u^{x-1}}{1+u}du = (1-x)\int_0^\infty \frac{\ln(1+u)}{u^{2-x}}du$$

Now set $x = 1 - \frac{1}{k}$ to obtain,

$$\frac{1}{k}\int_0^\infty \frac{\ln(1+u)}{u^{1+\frac{1}{k}}}du = \frac{\pi}{\sin \pi\left(1-\frac{1}{k}\right)} = \frac{\pi}{\sin \frac{\pi}{k}}.$$

We conclude that

$$\int_0^\infty \frac{\ln(1+x)}{x\sqrt[k]{x}}dx = \frac{k\pi}{\sin \frac{\pi}{k}}, \quad k \geq 2.$$

(b)

$1-x+x^2=(x+\alpha)(x+\beta)$ with $\alpha\beta=1$ and $\alpha+\beta=-1$.

$$\int_0^\infty \frac{\ln\left(1-x+x^2\right)}{x\sqrt{x}}dx=\int_0^\infty \frac{\ln\left((x+\alpha)(x+\beta)\right)}{x\sqrt{x}}dx$$

$$=\int_0^\infty \frac{\ln(x+\alpha)}{x\sqrt{x}}dx+\int_0^\infty \frac{\ln(x+\beta)}{x\sqrt{x}}dx$$

$$=\int_0^\infty \frac{\ln\alpha+\ln\left(\frac{x}{\alpha}+1\right)}{x\sqrt{x}}dx+\int_0^\infty \frac{\ln\beta+\ln\left(\frac{x}{\beta}+1\right)}{x\sqrt{x}}dx$$

$$=\int_0^\infty \frac{\ln(\alpha\beta)}{x\sqrt{x}}dx+\frac{1}{\sqrt{\alpha}}\int_0^\infty \frac{\ln\left(\frac{x}{\alpha}+1\right)}{\frac{x}{\alpha}\sqrt{\frac{x}{\alpha}}}\frac{dx}{\alpha}+\frac{1}{\sqrt{\beta}}\int_0^\infty \frac{\ln\left(\frac{x}{\beta}+1\right)}{\frac{x}{\beta}\sqrt{\frac{x}{\beta}}}\frac{dx}{\beta}$$

Changing the variable of integration, we obtain

$$\int_0^\infty \frac{\ln\left(1-x+x^2\right)}{x\sqrt{x}}dx=\left(\frac{1}{\sqrt{\alpha}}+\frac{1}{\sqrt{\beta}}\right)\int_0^\infty \frac{\ln(u+1)}{u\sqrt{u}}du$$

$$\frac{1}{\sqrt{\alpha}}+\frac{1}{\sqrt{\beta}}=\frac{\sqrt{\alpha}+\sqrt{\beta}}{\sqrt{\alpha\beta}}=\sqrt{\alpha}+\sqrt{\beta}=\sqrt{\alpha+\beta+2\sqrt{\alpha\beta}}=\sqrt{-1+2}=1.$$

We conclude that

$$\int_0^\infty \frac{\ln\left(1-x+x^2\right)}{x\sqrt{x}}dx=\int_0^\infty \frac{\ln(u+1)}{u\sqrt{u}}du=\frac{2\pi}{\sin\frac{\pi}{2}}=2\pi.$$

Editor's comment: **Ulrich Abel of Technische Hochschule Mittelhessen in Freiberg, Germany,** wrote that "both integrals of Problem 5372 can be determined by using computer algebra. Mathematica V. 9" and he then stated:

(a) $\displaystyle\int_0^\infty \frac{\ln(1+x)}{x^a}dx=\pi\cdot\frac{\text{Cosec}(a\cdot\pi)}{1-a}$ for all constants a such that $1<Re[a]<2$. This is slightly more general than the proposed problem.

(b) $\displaystyle\int_0^\infty \frac{\ln(x^2-x+1)}{x^{3/2}}dx=2\pi$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

(a) Denote the integral by I. By substitution $x=y^k$, we obtain

$$I=k\int_0^\infty \frac{\ln\left(1+y^k\right)}{y^2}dy.\ \text{ Since }\lim_{y\to0^+}\left(\frac{\ln(1+y^k)}{y}\right)=\lim_{y\to\infty}\left(\frac{\ln(1+y^k)}{y}\right)=0,$$

so by integrating by parts, we obtain

$$I=-k\int_0^\infty \ln(1+y^k)d\left(\frac{1}{y}\right)=k^2\int_0^\infty \frac{y^{k-2}}{1+y^k}dy.$$

We next substitute $y = \dfrac{1}{z}$ to obtain $I = k^2 \displaystyle\int_0^\infty \frac{1}{1+z^k} dz$. It is known ([1], entry 34.24(2))
that $\displaystyle\int_0^\infty \frac{1}{1+z^k} dz = \frac{\pi}{k} \csc\left(\frac{\pi}{k}\right)$, and so $I = \pi k \csc\left(\dfrac{\pi}{k}\right)$.

(b) Denote the integral by J. By substitution $x = y^2$ we obtain

$$J = 2\int_0^\infty \frac{\ln(1-y^2+y^4)}{y^2} dy. \text{ Since}$$

$$\lim_{y\to 0^+} \left(\frac{\ln(1-y^2+y^4)}{y}\right) = \lim_{y\to\infty} \left(\frac{\ln(1-y^2+y^4)}{y}\right) = 0,$$

so by integrating by parts, we obtain

$$\begin{aligned} J &= 2\int_0^\infty \frac{\ln(1-y^2+y^4)}{y^2} dy = -2\int_0^\infty \ln(1-y^2+y^4) d\left(\frac{1}{y}\right) \\ &= 4\int_0^\infty \frac{2y^2-1}{1-y^2+y^4} dy = 8\int_0^\infty \frac{y^2}{1-y^2+y^4} dy - 4\int_0^\infty \frac{1}{1-y^2+y^4} dy. \end{aligned}$$

Substituting $y = \dfrac{1}{z}$, we obtain $\displaystyle\int_0^\infty \frac{y^2}{1-y^2+y^4} dy = \int_0^\infty \frac{1}{1-z^2+z^4} dz$, so that
$J = 4\displaystyle\int_0^\infty \frac{1}{1-y+y^4} dy$. It is known ([1], entry 3.242(1)) that $\displaystyle\int_0^\infty \frac{1}{1-y^2+y^4} dy = \frac{\pi}{2}$
and so
$J = 2\pi$.

Reference [1] I.S. Gradshteyn and I.M. Ryzhik: *Tables of Integrals, Series, and Products,* Seventh Edition, Elsevier, Inc., 2007.

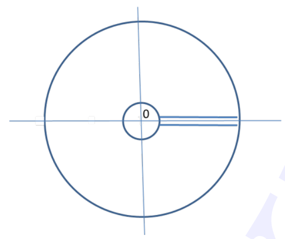
Solution 3 by Albert Stadler, Herrliberg, Switzerland

Both integrals can be evaluated by means of the following

Lemma

Let $0 < a < 1$. Let $0 < b < 2\pi$. Then $\displaystyle\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx = \frac{\pi e^{ia(\pi-b)}}{\sin(\pi a)}$.

Proof of the Lemma



Define a path C that consists of the following pieces:
$C_1 : Re^{it},\ 0 < t < 2\pi$, run through once in the positive direction,
$C_2 : t,\ \epsilon < t < R$, run through in the direction of decreasing real values,
$C_3 : \epsilon e^{it},\ 0 < t < 2\pi$ run through once in the negative direction,
$C_4 : t,\ \epsilon < t < R$, run through in the direction of increasing real values.,
Define the branch of z^{-a} such that $z^{-a} = \left(|z|\, e^{iArg(z)}\right)^{-a}$, where $0 < Arg(z) < 2\pi$.
Then, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz = Res\left(\frac{z^{-a}}{z-e^{ib}},\ z = e^{ib}\right) = e^{-abi}. \qquad (1)$$

The integral $\displaystyle\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz$ splits as follows:

$$\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz = \frac{1}{2\pi i}\int_{C_1} \frac{z^{-a}}{z-e^{ib}}dz + \frac{1}{2\pi i}\int_{C_2} \frac{z^{-a}}{z-e^{ib}}dz + \frac{1}{2\pi i}\int_{C_3} \frac{z^{-a}}{z-e^{ib}}dz + \frac{1}{2\pi i}\int_{C_4} \frac{z^{-a}}{z-e^{ib}}dz.$$

We treat each of these four integrals separately.

$$\left|\frac{1}{2\pi i}\int_{C_1} \frac{z^{-a}}{z-e^{ib}}dz\right| \le \frac{1}{2\pi}\frac{R^{-a}}{R-1}2\pi R = O\left(R^{-a}\right), as\ R \to \infty,$$

$$\left|\frac{1}{2\pi i}\int_{C_3} \frac{z^{-a}}{z-e^{ib}}dz\right| \le \frac{1}{2\pi}\frac{\epsilon^{-a}}{\epsilon-1}2\pi\epsilon = O\left(\epsilon^{1-a}\right), as\ \epsilon \to 0.$$

Therefore,

$$\frac{1}{2\pi i}\int_C \frac{z^{-a}}{z-e^{ib}}dz = \frac{1}{2\pi i}\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx - \frac{1}{2\pi i}\int_0^\infty \frac{\left(xe^{2\pi i}\right)^{-a}}{x-e^{ib}}dx = \frac{1}{2\pi i}\left(1-e^{-2\pi i a}\right)\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx. \qquad (2)$$

We combine (1) and (2) and get

$$\frac{1}{2\pi i}\left(1-e^{-2\pi i a}\right)\int_0^\infty \frac{x^{-a}}{x-e^{ib}}dx = e^{iab}$$

which is the claim of the lemma.

(a) Let $1 < a < 2$. Partial integration yields

$$\int_0^\infty \frac{\log(1+x)}{x^a}dx = \underbrace{\left.\frac{-x^{1-a}\log(1+x)}{a-1}\right|_0^\infty} + \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{1+x}dx = \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{1+x}dx,$$

because the first term evaluates to zero.

We set $b=\pi$ and apply the lemma to get

$$\int_0^\infty \frac{\log(1+x)}{x^a}dx = \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{1+x}dx = \frac{1}{a-1}\cdot\frac{\pi}{\sin(\pi(a-1))} = \frac{-1}{a-1}\cdot\frac{\pi}{\sin(\pi a)}.$$

(a) is the special case $a = 1+\dfrac{1}{k}$.

(b) Let $1 < a < 2$. Partial integration yields

$$\begin{aligned}\int_0^\infty \frac{\log(1-x+x^2)}{x^a}dx &= \underbrace{\left.\frac{-x^{1-a}\log(1-x+x^2)}{a-1}\right|_0^\infty} + \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}(2x-1)}{1-x+x^2}dx \\ &= \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{x-e^{\frac{\pi i}{3}}}dx + \frac{1}{a-1}\int_0^\infty \frac{x^{1-a}}{x-e^{\frac{5\pi i}{3}}},\end{aligned}$$

because the first time evaluates to zero.

We apply the lemma to get

$$\begin{aligned}\int_0^\infty \frac{\log(1-x+x^2)}{x^a}dx &= \frac{1}{a-1}\cdot\frac{\pi e^{i(a-1)(\pi-\frac{\pi}{3})}}{\sin(\pi(a-1))} + \frac{1}{a-1}\cdot\frac{\pi e^{i(a-1)(\pi-\frac{5\pi}{3})}}{\sin(\pi(a-1))} \\ &= \frac{2\pi}{a-1}\cdot\frac{\cos\left(\frac{2\pi}{3}(a-1)\right)}{\sin(\pi(a-1))} \\ &= \frac{-2\pi}{a-1}\cdot\frac{\cos\left(\frac{2\pi}{3}(a-1)\right)}{\sin(\pi(a))}.\end{aligned}$$

In particular, if $a=\dfrac{3}{2}$ then $\displaystyle\int_0^\infty \frac{\log(1-x+x^2)}{x\sqrt{x}}dx = -4\pi\cdot\frac{\cos\left(\frac{\pi}{3}\right)}{\sin(3\pi/2)} = 2\pi.$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2016

- **5391:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides $(49, b, b+1)$ has integer area. Find two possible values of b.

- **5392:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Prove that if $x, y, z > 0$, then

$$\frac{4\left(x^2+y^2+z^2\right)}{27\left(xy+yz+zx\right)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\geq\frac{13}{27}.$$

- **5393:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Through the midpoint of the diagonal BD in the convex quadrilateral $ABCD$ we draw a straight line parallel to the diagonal AC. This line intersects the side AD at the point E. Show that

$$\frac{1}{[ABC]}+\frac{1}{[AEC]}\geq\frac{4}{[CED]}.$$

  Here $[XYZ]$ represents the are of $\triangle XYZ$.

5394: *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let a, b and c be positive real numbers such that $ab+bc+ca=3$ and $n>1$. Prove that

$$\sqrt[n]{a+\frac{1}{abc}}+\sqrt[n]{b+\frac{1}{abc}}+\sqrt[n]{c+\frac{1}{abc}}\geq 3\sqrt[n]{2}.$$

- **5395:** *Proposed by Mohsen Soltanifar (Ph.D. student), Biostatistics Division, Dalla Lana School of Public Health, University of Toronto, Canada.*

  Given the sequence $\{\sigma_n^2\}_{n=1}^{\infty}$ of positive numbers and $X_1 \sim N(\mu, \sigma_1^2)$. Define recursively a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ via

$$X_{n+1}|X_n \sim N\left(X_n, \sigma_{n+1}^2\right) \qquad n=1,2,3,\ldots$$

Calculate the limit distribution X of $\{X_n\}_{n=1}^{\infty}$.

Reference: Rosenthal, J.S. (2007). A First Look at Rigorous Probability (2nd edition), World Scientific, p. 139.

Proposer's note concerning the problem:

This is a Bayesian Hierarchical Model of Human Heights from Adam & Eve to the end of time.

Consider a family with its children. We know that height has a normal distribution. We also know that height of children is due to genetic factors which are dependent on the height of their parents, but usually this distribution has the same mean as the mean height of their parents but may vary (some children are taller, some shorter, some are average- versus their parents). So, the height of children may be modeled as the normal distribution conditioned to the height of their parents with same mean but potentially different variance.

The first term in the sequence is the distribution of height of Adam & Eve. The second term is the conditional distribution of their children's height. This goes till the end of time consecutively when, according to some beliefs, the Messiah returns. Accordingly, the Messiah will return and a generation of humans will observe this return . But we do not know when this will occur. So, we may assume the Messiah will return as time approaches infinity, and that the distribution of the height of generations of humans that observe the return is "X". We are interested in knowing certain features of this distribution.

This problem is a mathematical modeling of the above belief.

- **5396:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all continuous functions $f : \Re \to \Re$ such that

$$f(-x) = x + \int_0^x e^{-t} f(x-t)dt, \quad \forall x \in \Re.$$

Solutions

- **5373:** *Proposed by Kenneth Korbin, New York, NY*

Given the equation $\dfrac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}} = \sqrt{x+y\sqrt{5}}$.

Find positive integers x and y.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\frac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}} = \frac{2\sqrt{2}}{\left(7-3\sqrt{5}\right)\sqrt{7-3\sqrt{5}}}.$$

$$\frac{2\sqrt{2}}{\left(7-3\sqrt{5}\right)\sqrt{76-3\sqrt{5}}} = \sqrt{x+y\sqrt{5}} \text{ implies}$$

$$x + y\sqrt{5} = \frac{8}{\left(7 - 3\sqrt{5}\right)^3} = \frac{8 \cdot \left(7 + 3\sqrt{5}\right)^3}{4^3} = 161 + 72\sqrt{5}.$$

So $(x, y) = (161, 72)$.

Solution 2 by Neculai Stanciu "George Emil Palade" School, Buzău and Titu Zvonaru, Comănesti, Romania

We have

$$\sqrt{343 - 147\sqrt{5}} = \sqrt{\frac{441}{2}} - \sqrt{\frac{245}{2}} = \frac{21\sqrt{2}}{2} - \frac{7\sqrt{10}}{2}.$$

$$\sqrt{315 - 135\sqrt{5}} = \sqrt{\frac{405}{2}} - \sqrt{\frac{225}{2}} = \frac{9\sqrt{10}}{2} - \frac{15\sqrt{2}}{2}.$$

$$\sqrt{343 - 147\sqrt{5}} - \sqrt{315 - 135\sqrt{5}} = \frac{21\sqrt{2}}{2} - \frac{7\sqrt{10}}{2} - \frac{9\sqrt{10}}{2} + \frac{15\sqrt{2}}{2} = 18\sqrt{2} - 8\sqrt{10}, \text{ and so}$$

$$\frac{2\sqrt{2}}{\sqrt{343 - 147\sqrt{5}} - \sqrt{315 - 135\sqrt{5}}} = \frac{1}{9 - 4\sqrt{5}} = 9 + 4\sqrt{5}.$$

Solving the equation $\sqrt{x + y\sqrt{5}} = 9 + 4\sqrt{5}$ gives $x = 161, y = 72$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Since $343 - 147\sqrt{5} = 49\left(7 - 3\sqrt{5}\right)$ and $315 - 135\sqrt{5} = 45\left(7 - 3\sqrt{5}\right)$, we obtain

$$\begin{aligned}\sqrt{343 - 147\sqrt{5}} - \sqrt{315 - 135\sqrt{5}} &= 7\sqrt{7 - 3\sqrt{5}} - \sqrt{45}\sqrt{7 - 3\sqrt{5}} \\ &= \left(7 - 3\sqrt{5}\right)\sqrt{7 - 3\sqrt{5}} \\ &= \sqrt{\left(7 - 3\sqrt{5}\right)^3} = \sqrt{8\left(161 - 72\sqrt{5}\right)} = 2\sqrt{2}\sqrt{161 - 72\sqrt{5}}.\end{aligned}$$

Hence,

$$\frac{2\sqrt{2}}{\sqrt{343 - 147\sqrt{5}} - \sqrt{315 - 135\sqrt{5}}} = \sqrt{\frac{1}{161 - 72\sqrt{5}}} = \sqrt{161 + 72\sqrt{5}}.$$

So, $x + y\sqrt{5} = 161 + 72\sqrt{5}$, that is $x - 161 = (72 - y)\sqrt{5}$. If $72 - y \neq 0$, then $\sqrt{5} = \dfrac{x - 161}{72 - y}$, which is impossible because, since $x - 161$ and $72 - y$ are integers, the left hand side is irrational and the right hand side is rational. Thus, $72 - y = 0$ and henceforth, $x - 161 = 0\sqrt{5}$. That is $(x, y) = (161, 72)$, is valid and the only solution, because of the stipulation in the problem that x and y be positive integers.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Squaring both sides of the given equation, we obtain

$$\begin{aligned}
x+y\sqrt{5} &= \frac{8}{658-282\sqrt{5}-2\sqrt{207270-92610\sqrt{5}}}\\
&= \frac{8}{658-282\sqrt{5}-2\left(147\sqrt{5}-315\right)}\\
&= \frac{1}{161-72\sqrt{5}}\\
&= 161+72\sqrt{5}.
\end{aligned}$$

Hence, $x=161$ and $y=72$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State Unversity, San Angelo TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the author.

- **5374:** *Proposed by Roger Izard, Dallas TX*

In a certain triangle, three circles are tangent to the incircle, and all of these circles are tangent to two sides of the triangle. Derive a formula which gives the radius of the incircle in terms of the radii of these three circles.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let ABC be the triangle, $a=\overline{BC}, b=\overline{CA}, c=\overline{AB}, A=\angle BAC, B=\angle CBA$, $C=\angle ACB, I$ the center of the incircle ABC, r its radius, O_a and r_a the respective center and radius of the circle tangent to that incircle and to the side AB and AC, O_b and r_b the center and radius of the circle tangent to that incircle and to the sides BA and BC, respectively, O_c and r_c the respective radius of the circle tangent to that incircle and to the sides CA and CB, T_a the point of tangency of the incircle of ABC and the circle with center O_a and radius $r_a, O_a^{'}$ the point of tangency of that circle with AB and $I^{'}$ the point of tangency of the incircle of ABC with AB. We shall prove that

$$r=\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}.$$

From triangles $AO_a'O_a$ and $AI'I$, since $\angle O_a'AO_a=\angle BAC/2=\angle I'AI$, we deduce that $\dfrac{r_a}{AO_a}=\sin\left(\dfrac{A}{2}\right)=\dfrac{r}{AI}$.

Since $AI = AO_a + O_aT_a + T_aI,\ r\csc\left(\frac{A}{2}\right) = r_a\csc\left(\frac{A}{2}\right) + r_a + r$ and , so,

$$\frac{r_a}{r} = \frac{1-\sin\left(\frac{A}{2}\right)}{1+\sin\left(\frac{A}{2}\right)} = \frac{1-2\sin\left(\frac{A}{4}\right)\cos\left(\frac{A}{2}\right)}{1+2\sin\left(\frac{A}{4}\right)\cos\left(\frac{A}{2}\right)} = \frac{1-2\dfrac{\tan\left(\frac{A}{4}\right)}{1+\tan^2\left(\frac{A}{4}\right)}}{1+2\dfrac{\tan\left(\frac{A}{4}\right)}{1+\tan^2\left(\frac{A}{4}\right)}} = \left(\frac{1-\tan\left(\frac{A}{4}\right)}{1-\tan\left(\frac{A}{4}\right)}\right)^2.$$

Analogously,

$$\frac{r_b}{r} = \left(\frac{1-\tan\left(\frac{B}{4}\right)}{1-\tan\left(\frac{B}{4}\right)}\right)^2 \text{ and } \frac{r_c}{r} = \left(\frac{1-\tan\left(\frac{C}{4}\right)}{1-\tan\left(\frac{C}{4}\right)}\right)^2.$$

Let us dnote $t_a = \tan\left(\frac{A}{4}\right)$, $t_b = \tan\left(\frac{B}{4}\right)$ and $t_c = \tan\left(\frac{C}{4}\right)$. Since $0 < \frac{A}{4} < \frac{\pi}{4},\ 0 < t_a < 1$, so since $1 - t_a > 0$, analogously we have $1 - t_b > 0$ and $1 - t_c > 0$. The equality to prove, $\sqrt{\frac{r_a}{r}}\sqrt{\frac{r_b}{r}} + \sqrt{\frac{r_b}{r}}\sqrt{\frac{r_c}{r}} + \sqrt{\frac{r_c}{r}}\sqrt{\frac{r_a}{r}} = 1$ is successively equivalent to showing that

$$\frac{1-t_a}{1+t_a}\cdot\frac{1-t_b}{1+t_b} + \frac{1-t_b}{1+t_b}\cdot\frac{1-t_c}{1+t_c} + \frac{1-t_c}{1+t_c}\cdot\frac{1-t_a}{1+t_a} = 1.$$

And this is equivalent to showing that

$$(1-t_a)(1-t_b)(1+t_c) + (1-t_b)(1-t_c)(1+t_a) + (1-t_a)(1-t_c)(1+t_b) = (1+t_a)(1+t_b)(1+t_c).$$

Expanding and simplifying we obtain:

$$t_at_b + t_bt_c + t_ct_a + t_a + t_b + t_c = t_at_bt_c + 1.$$

But this is true because

$$1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{A+B+C}{4}\right) = \tan\left(\frac{A}{4} + \frac{B}{4} + \frac{C}{4}\right) = \frac{t_a + t_b + t_c - t_at_bt_c}{1 - t_at_b - t_bt_c + t_ct_a}.$$

So, the formula $r = \sqrt{r_ar_b} + \sqrt{r_br_c} + \sqrt{r_cr_a}$ holds.

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

Let the triangle be $A_1A_2A_3$, the incenter and radius of the triangle be I,r, the centers of the three circles be I_1, I_2, I_3 and radius of them be r_1, r_2, r_3, respectively. Let the foot of perpendicular from I to A_2A_3, A_3A_1, A_1A_2 be H_1, H_2, H_3, respectively. Let $\alpha_i = \angle A_iIA_{i+1} = \angle A_iIA_{i+2}$ where the indices are considered to be the same $\pmod 3$. Now, we calculate $\tan\alpha_1$. Let the foot of perpendicular from I_1 to IH_2 be K. Then, $IK = r - r_1$, $II_1 = r + r_1$. Thus, $KI_1 = \sqrt{II_1^2 - IK^2} = 2\sqrt{rr_1}$. Therefore,

$$\tan\alpha_1 = \frac{2\sqrt{rr_1}}{r - r_1}.$$

Similarly,

$$\tan\alpha_2 = \frac{2\sqrt{rr_2}}{r-r_2}$$
$$\tan\alpha_3 = \frac{2\sqrt{rr_3}}{r-r_3}.$$

On the other hand, since $\alpha_1+\alpha_2+\alpha_3=\pi$, it follows that

$$\begin{aligned}\tan\alpha_1+\tan\alpha_2+\tan\alpha_3 &= \tan\alpha_1+\tan\alpha_2-\tan(\alpha_1+\alpha_2)\\ &= \tan\alpha_1+\tan\alpha_2-\frac{\tan\alpha_1+\tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}\\ &= \frac{-\tan\alpha_1\tan\alpha_2(\tan\alpha_1+\tan\alpha_2)}{1-\tan\alpha_1\tan\alpha_2}\\ &= -\tan\alpha_1\tan\alpha_2\tan(\alpha_1+\alpha_2)\\ &= \tan\alpha_1\tan\alpha_2\tan\alpha_3.\end{aligned}$$

Thus,

$$\frac{2\sqrt{rr_1}}{r-r_1}+\frac{2\sqrt{rr_2}}{r-r_2}+\frac{2\sqrt{rr_3}}{r-r_3} = \frac{2\sqrt{rr_1}}{r-r_1}\cdot\frac{2\sqrt{rr_2}}{r-r_2}\cdot\frac{2\sqrt{rr_3}}{r-r_3}$$
$$\sum_{cyc}\sqrt{r_1}(r-r_2)(r-r_3) = 4r\sqrt{r_1r_2r_3}$$

$$r^2\sum_{cyc}\sqrt{r_1} - r\left(\sum_{sym}\sqrt{r_1}r_2+4\sqrt{r_1r_2r_3}\right)+\sqrt{r_1r_2r_3}\sum_{cyc}\sqrt{r_2r_3}=0$$

We see it as a quadratic equation of r. Then the discriminant is

$$\begin{aligned}D &= \left(\sum_{sym}\sqrt{r_1}r_2+4\sqrt{r_1r_2r_3}\right)^2-4\sum_{cyc}\sqrt{r_1}\cdot\sqrt{r_1r_2r_3}\sum_{cyc}\sqrt{r_2r_3}\\ &= \left(\sum_{sym}\sqrt{r_1}r_2\right)^2+8\sqrt{r_1r_2r_3}\sum_{sym}\sqrt{r_1}r_2+16r_1r_2r_3-4\sqrt{r_1r_2r_3}\left(3\sqrt{r_1r_2r_3}+\sum_{sym}\sqrt{r_1}r_2\right)\\ &= \left(\sum_{sym}\sqrt{r_1}r_2\right)^2+4\sqrt{r_1r_2r_3}\sum_{sym}\sqrt{r_1}r_2+4r_1r_2r_3\\ &= \left(\sum_{sym}\sqrt{r_1}r_2+2\sqrt{r_1r_2r_3}\right)^2.\end{aligned}$$

Thus,

$$r=\frac{\sum_{sym}\sqrt{r_1}r_2+4\sqrt{r_1r_2r_3}\pm\left(\sum_{sym}\sqrt{r_1}r_2+2\sqrt{r_1r_2r_3}\right)}{2\sum_{cyc}\sqrt{r_1}}.$$

We first consider the minus sign of the case. In this case,

$$\begin{aligned} r &= \frac{\sqrt{r_1 r_2 r_3}}{\sum_{cyc}\sqrt{r_1}} \\ &\le \frac{\sqrt{r_1 r_2 r_3}}{3\sqrt[3]{\sqrt{r_1 r_2 r_3}}} \\ &= \frac{1}{3}(r_1 r_2 r_3)^{1/3} \\ &\le \frac{1}{3}\max\{r_1, r_2, r_3\} \end{aligned}$$

It contradicts with the fact that r is larger any of r_1, r_2, r_3.
Thus, the plus sign must be ocurred. Then,

$$\begin{aligned} r &= \frac{\sum_{sym}\sqrt{r_1}r_2 + 3\sqrt{r_1 r_2 r_3}}{\sum_{cyc}\sqrt{r_1}} \\ &= \frac{(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3})(\sqrt{r_1 r_2}+\sqrt{r_2 r_3}+\sqrt{r_3 r_1})}{\sum_{cyc}\sqrt{r_1}} \\ &= \sqrt{r_1 r_2}+\sqrt{r_2 r_3}+\sqrt{r_3 r_1} \end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer

- **5375*:** *Proposed by Kenneth Korbin, New York, NY*

Prove or disprove the following conjecture. Let k be the product of N different prime numbers each congruent to $1(mod\ 4)$. Let a be any positive integer.

Conjecture: The total number of different rectangles and trapezoids with integer length sides that can be inscribed in a circle with diameter k is exactly $\dfrac{5^N - 3^N}{2}$.

Editor′s comment: The number for this problem carries with it an asterisk. The asterisk signifies that neither the proposer nor the editor are aware of a proof of this conjecture. **Toshihiro Shimizu of Kawasaki, Japan** considered the case $k = 5 \cdot 17$, and stated:
"There are four rectangles satisfying the conditions of the problem:
$(a, b) = (13, 84), (36, 77), (40, 75), (51, 68)$, where a and b are the lengths of the sides of the rectangle."
and
"There are six trapezoids satisfying the conditions of the problem:

$$(a, b, c) = (13, 77, 40), (13, 77, 68), (36, 84, 40), (36, 84, 51), (43, 83, 34), (43, 83, 50),$$

where a, and b are the lengths of the two parallel sides of the trapezoid and c is the length of the other two sides." He went on to state that he came to these conclusions with the aid of a computer.

Editor′s update : No analytic solutions to the conjecture were received, so the problem will remain open. Ken Korbin, the author of 5375, sent a comment that we should also note that $\dfrac{5^N - 3^N}{2} = \displaystyle\sum_{j=1}^{N}\binom{N}{j}\left(2^{j-1}\right)\left(3^{N-j}\right).$

When a complete solution is received, it will be published.

- **5376:** *Proposed by Arkady Alt , San Jose ,CA*

Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be positive real numbers such that
$b_1 < a_1 < b_2 < a_2 < ... < a_{n-1} < b_n < a_n$.
Let

$$F(x) = \frac{(x-b_1)(x-b_2)...(x-b_n)}{(x-a_1)(x-a_2)...(x-a_n)}.$$

Prove that $F'(x) < 0$ for any $x \in Dom(F)$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that $F(x) = \prod_{m=1}^{n} \frac{x-b_m}{x-a_m}$ is a rational function with simple poles at $x = a_m$, $1 \leq m \leq n$.

The residue of $F(x)$ at $x = a_\mu$ equals $(a_\mu - b_\mu) \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m} > 0$, since $a_\mu > b_\mu$ and

$\frac{a_\mu - b_m}{a_\mu - a_m} > 0$, for $m \neq \mu$.

So $f(x) = F(x) - \sum_{\mu=1}^{n} \frac{a_\mu - b_\mu}{x - a_\mu} \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m}$ is a bounded entire function which implies that $f(x)$ is a constant. We conclude $f'(x) = 0$ which implies

$F'(x) = -\sum_{\mu=1}^{n} \frac{a_\mu - b_\mu}{(x-a_\mu)^2} \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m} < 0$ for any $x \in$Dom (F).

Solution 2 by Ethan Gegner (student), Taylor University, Upland, IN

For all $x \in Dom(F)$, we have

$$F'(x) = \frac{\left(\prod_{i=1}^n (x-a_i)\right)\left(\prod_{i=1}^n (x-b_i)\right)' - \left(\prod_{i=1}^n (x-b_i)\right)\left(\prod_{i=1}^n (x-a_i)\right)'}{\left(\prod_{i=1}^n (x-a_i)\right)^2} \qquad (1)$$

Suppose $x = b_j$ for some $1 \leq j \leq n$. Then

$$F'(x) = \frac{\prod_{i \neq j}(x-b_i)}{\prod_{i=1}^n (x-a_i)} = \frac{1}{(x-a_j)} \prod_{i \neq j} \frac{x-b_i}{x-a_i} < 0$$

since $x = b_j < a_j$ and $\frac{x-b_i}{x-a_i} > 0$ for all $i \neq j$.

Now suppose $x \notin \{b_1, \ldots, b_n\}$. Then $F(x) \neq 0$, so by equation (1) we have

$$\frac{F'(x)}{F(x)} = \frac{\left(\prod_{i=1}^n (x-b_i)\right)'}{\prod_{i=1}^n (x-b_i)} - \frac{\left(\prod_{i=1}^n (x-a_i)\right)'}{\prod_{i=1}^n (x-a_i)} = \sum_{i=1}^n \left(\frac{1}{x-b_i} - \frac{1}{x-a_i}\right)$$

$$= \sum_{i=1}^n \frac{b_i - a_i}{(x-b_i)(x-a_i)} \qquad (2)$$

If $x < b_1$ or $x > a_n$, then $F(x) > 0$, and $\frac{b_i - a_i}{(x-b_i)(x-a_i)} < 0$ for all $1 \leq i \leq n$, whence $F'(x) < 0$. Suppose there exists some $1 \leq j \leq n-1$ such that $a_j < x < b_{j+1}$. Then for every

$1 \le i \le n$, $x - b_i$ and $x - a_i$ have the same sign, whence $\dfrac{b_i - a_i}{(x-b_i)(x-a_i)} < 0$ and $F(x) = \prod_{i=1}^{n} \dfrac{x - b_i}{x - a_i} > 0$. Thus, equation (2) implies $F'(x) < 0$.

Finally, suppose that $b_j < x < a_j$ for some $1 \le j \le n$. Then

$$\frac{F'(x)}{F(x)} = \sum_{i=1}^{n} \left(\frac{1}{x-b_i} - \frac{1}{x-a_i} \right) = \frac{1}{x-b_1} - \frac{1}{x-a_n} + \sum_{i=1}^{n-1} \left(\frac{1}{x-b_{i+1}} - \frac{1}{x-a_i} \right) > 0$$

since every term on the right hand side is positive. Moreover, $F(x) = \dfrac{x-b_j}{x-a_j} \prod_{i \ne j} \dfrac{x-b_i}{x-a_i} < 0$, so again $F'(x) < 0$.

Solution 3 by the proposer

Lemma.

$F(x)$ can be represented in form

$$F(x) = 1 + \sum_{k=1}^{n} \frac{c_k}{x-a_k},$$

where $c_k, k = 1, 2, ..., n$ are some positive real numbers.

Proof.

Let $F_k(x) := \dfrac{(x-b_1)(x-b_2)\dots(x-b_k)}{(x-a_1)(x-a_3)\dots(x-a_k)}$, $k \le n$.

We will prove by Math Induction that for any $k \le n$ there are positive numbers $c_k(i), i = 1, ..., k$ such that $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$.

Let $d_k := a_k - b_k > 0, k = 1, 2, ..., n$.

Note that $F_1(x) = \dfrac{x-b_1}{x-a_1} = \dfrac{x-a_1+a_1-b_1}{x-a_1} = 1 + \dfrac{d_1}{x-a_1}$.

Since $\dfrac{x-b_{k+1}}{x-a_{k+1}} = 1 + \dfrac{d_{k+1}}{x-a_{k+1}}$ then in supposition $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$, where $c_k(i) > 0, i = 1, ..., k < n$ we obtain

$$\begin{aligned}
F_{k+1}(x) &= F_k(x) \cdot \frac{x-b_{k+1}}{x-a_{k+1}} = \left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i}\right)\left(1 + \frac{d_{k+1}}{x-a_{k+1}}\right) \\
&= 1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} + \sum_{i=1}^{k} \frac{d_{k+1}c_k(i)}{(x-a_i)(x-a_{k+1})} \\
&= 1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} - \sum_{i=1}^{k} \frac{d_{k+1}c_k(i)}{a_{k+1}-a_i}\left(\frac{1}{x-a_i} - \frac{1}{x-a_{k+1}}\right) \\
&= 1 + \frac{d_{k+1}}{x-a_{k+1}}\left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{a_{k+1}-a_i}\right) + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i}\left(1 - \frac{d_{k+1}}{a_{k+1}-a_i}\right)
\end{aligned}$$

$$= \quad 1+\frac{d_{k+1}F_k\left(a_{k+1}\right)}{x-a_{k+1}}+\sum_{i=1}^{k}\frac{c_k\left(i\right)}{x-a_i}\cdot\frac{b_{k+1}-a_i}{a_{k+1}-a_i}.$$

Since $F_k\left(a_{k+1}\right)>0$ and $b_{k+1}-a_i=\left(b_{k+1}-a_k\right)+\left(a_k-a_i\right)>0$ then

$c_{k+1}\left(k+1\right)=d_{k+1}F_k\left(a_{k+1}\right)>0,\ c_{k+1}\left(i\right):=\dfrac{\left(b_{k+1}-a_i\right)c_k\left(i\right)}{a_{k+1}-a_i}>0, i=1,2,...,k$

and $F_{k+1}\left(x\right)=1+\sum\limits_{i=1}^{k+1}\dfrac{c_{k+1}\left(i\right)}{x-a_i}$.

Therefore, since $F\left(x\right)=1+\sum\limits_{k=1}^{n}\dfrac{c_k}{x-a_k}$ and $c_k>0, k=1,2,...,n$ then
$F'\left(x\right)=-\sum\limits_{k=1}^{n}\dfrac{c_k}{\left(x-a_k\right)^2}<0$ for any $x\in Dom\left(F\right)=\left\{a_1,a_2,...,a_n\right\}$.

Solution 4 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

We find $F'(x)$,

$$\begin{aligned}F'(x) \quad = \quad & \frac{b_1-a_1}{(x-a_1)^2}\cdot\frac{x-b_2}{x-a_2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{x-b_n}{x-a_n}+\frac{x-b_1}{x-a_1}\cdot\frac{b_2-a_2}{(x-a_2)^2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{x-b_n}{x-a_n}+\ldots+\\ & \frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdots\frac{x-b_{j-1}}{x-a_{j-1}}\cdot\frac{b_j-a_j}{(x-a_j)^2}\cdot\frac{x-b_{j+1}}{x-a_{j+1}}\cdots\cdot\frac{x-b_n}{x-a_n}+\ldots+\\ & \frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{b_n-a_n}{(x-a_n)^2}. \qquad (1)\end{aligned}$$

We set

$$D_1(x)=\frac{b_1-a_1}{(x-a_1)^2}\cdot\frac{x-b_2}{(x-a_2)}\cdot\frac{x-b_3}{(x-a_3)}\cdots\frac{x-b_n}{x-a_n}$$

$$D_2(x)=\frac{x-b_1}{x-a_1}\cdot\frac{b_2-a_2}{(x-a_2)^2}\cdot\frac{x-b_3}{(x-a_3)}\cdots\frac{x-b_n}{x-a_n}$$

$$\vdots$$

$$D_j(x)=\frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdots\frac{x-b_{j-1}}{x-a_{j-1}}\cdot\frac{b_j-a_j}{(x-a_j)^2}\cdot\frac{x-b_{j+1}}{x-a_{j+1}}\cdots\frac{x-b_n}{x-a_n}$$

$$\vdots$$

$$D_n(x)=\frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{b_n-a_n}{(x-a_n)^2}.$$ Then

$$F'(x)=\sum_{k=1}^{n}D_k(x). \qquad (2)$$

We note that because

$$0<b_1<a_1<b_2<a_2<\cdots<a_{n-1}<b_n<a_n \qquad (3)$$

we have

$$\frac{b_j - a_j}{(x-a_j)^2} < 0, \text{for all } j \text{ with } 1 \le j \le n. \quad (4)$$

Let $x \in Dom(F)$, then we consider the following cases:

Case 1. Let $x = b_{j_0}$, for some $j_0 \in \{1, 2, \cdots, n\}$, then $D_j(b_{j_0}) = 0$, for all $j \neq j_0$, and because of (3), $\dfrac{b_{j_0} - b_j}{b_{j0} - a_j} > 0$, for all $j \neq j_0$ and with (4) we conclude that $F'(b_{j_0}) < 0$.

Case 2. Let $x < b_1$, then for all j with $1 \le j \le n$, and by using (3), we conclude that and that $\dfrac{x - b_j}{x - a_j} > 0. \quad (5)$

And then by (4) and (5) we get equation (6) that $D_j(x < b_1) < 0$, for all j with $1 \le j \le n$, and this implies that $F'(x < b_1) < 0$.

Case 3. Let $x \in (b_{j_0}, a_{j_0})$ for some $j_0 \in \{1, 2, \cdots, n\}$, we will show that $F(x)$ is decreasing on (b_{j_0}, a_{j_0}). We know that by (4) and (3),, each function $f_j(x) = \dfrac{x - b_j}{x - a_j}$ is decreasing and positive on (b_{j_0}, a_{j_0}), when $j \neq j_0$, then for all $s, t \in (b_{j_0}, a_{j_0})$ with $s < t$ we have

$$f_j(t) > f_j(s), \quad (7)$$

also $f_{j_0}(x) = \dfrac{x - b_{j_0}}{x - a_{j_0}}$ is decreasing but negative on (b_{j_0}, a_{j_0}) and

$$f_{j_0}(t) > f_{j_0}(s). \quad (8)$$

Now using (7) and (8), we have $\displaystyle\prod_{j=1}^{n} f_j(t) > \prod_{j=1}^{n} f_j(s)$, that is $F(t) > F(s)$, whenever $s, t \in (b_{j_0}, a_{j_0})$ with $s < t$, the means $F(x)$ is decreasing on (b_{j_0}, a_{j_0}) or $F(x) < 0$ on (b_{j_0}, a_{j0}).

Case 4. Let $x \in \left(a_{j_0}, b_{j0+1}\right)$, for some $j_0 \in \{1, 2, \cdots, n-1\}$, then $f_j(x) = \dfrac{x - b_j}{x - a_j} > 0$, on $\left(a_{j_0}, b_{j0+1}\right)$, for $j \in \{1, 2, \cdots, n\}$, and by (4) and (2), we conclude that $F'(x) < 0$, on $\left(a_n, b_{j0+1}\right)$.

Case 5. Let $x \in (b_n, \infty)$, then $f_j(x) = \dfrac{x - b_j}{x - a_j} > 0$, on (b_n, ∞) for all $j \in \{1, 2, \cdots, n\}$, and by (4) and (2), we conclude that $F'(x) < 0$, on (b_n, ∞).

Combining the results of Cases 1-5, we conclude that $F'(x) < 0$ for any $x \in Dom(F)$.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and Toshihiro Shimizu, Kawasaki, Japan.

- **5377:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Show that if A, B, C are the measures of the angles of any triangle ABC and a, b, c the measures of the length of its sides, then holds

$$\prod_{cyclic} \sin^{1/3}(|A-B|) \le \sum_{cyclic} \frac{a^2+b^2}{3ab} \sin(|A-B|).$$

Solution 1 by Andrea Fanchini Cantú, Italy

We know that

$$\sin A = \frac{2K}{bc}, \quad \sin B = \frac{2K}{ac}, \quad \sin C = \frac{2K}{ab}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2ca}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

where K is the area of the triangle.

So we have that

$$\sin(A-B) = \sin A \cos B - \cos A \sin B = \frac{2K(a^2-b^2)}{abc^2},$$

and cyclically,

$$\sin(B-C) = \frac{2K(b^2-c^2)}{a^2bc}, \quad \sin(C-A) = \frac{2K(c^2-a^2)}{ab^2c},$$

so we have

$$\prod_{cyc} \sin^{1/3}(A-B) = \frac{2K}{abc}\sqrt[3]{\frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{abc}}, \text{ and}$$

$$\sum_{cyc} \frac{a^2+b^2}{3ab}\sin(A-B) = \frac{2K}{3a^2b^2c^2}\left[(a^2+b^2)(a^2-b^2)+(b^2+c^2)(b^2-c^2)+(c^2+a^2)(c^2-a^2)\right].$$

Now if we assume $C > B > A$ then

$$\prod_{cyc} \sin^{1/3}|A-B| = \frac{2K}{abc}\sqrt[3]{\frac{(a+b)(b-a)(b+c)(c-b)(c+a)(c-a)}{abc}}, \text{ and}$$

$$\sum_{cyc} \frac{a^2+b^2}{3ab}\sin|A-B| = \frac{2K}{3a^2b^2c^2}\left[(a^2+b^2)(a+b)(b-a)+(b^2+c^2)(b+c)(c-b)+(c^2+a^2)(c+a)(c-a)\right].$$

Therefore we need to prove

$$\begin{aligned} & 3\sqrt[3]{a^2b^2c^2(a+b)(b-a)(b+c)(c-b)(c+a)(c-a)} \\ \leq \; & (a^2+b^2)(a+b)(b-a)+(b^2+c^2)(b+c)(c-b)+(c^2+a^2)(c+a)(c-a). \end{aligned}$$

But the AM-GM inequality gives us

$$\begin{aligned} & 3\sqrt[3]{a^2b^2c^2(a+b)(b-a)(b+c)(c-b)(c+a)(c-a)} \\ \leq \; & 3\sqrt[3]{(a^2+b^2)(a+b)(b-a)(b^2+c^2)(b+c)(c-b)(c^2+a^2)(c+a)(c-a)}. \end{aligned}$$

So it remains to show that

$$a^2b^2c^2 \leq (a^2+b^2)(b^2+c^2)(c^2+a^2).$$

But this follows immediately because this inequality is equivalent to
$0 \leq a^4(b^2+c^2)+b^4(a^2+c^2)+c^4(a^2+b^2)+a^2b^2c^2$ which is immediately evident. Therefore, the statement of the problem is true, with equality holding if and only if the triangle is isosceles.

Solution 2 Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will prove the slightly stronger inequality

$$\prod_{cyclic} \sin^{\frac{1}{3}}(|A-B|) \leq \sum_{cyclic} \frac{a^2+b^2}{6ab} \sin(|A-B|).$$

Since $0 < A, B, C < \pi$, it follows that

$$-\pi < A-B, B-C, C-A < \pi$$

and hence,

$$0 \leq |A-B|, |B-C|, |C-A| < \pi.$$

Then,

$$\sin(|A-B|), \sin(|B-C|), \sin(|C-A|) \geq 0$$

and the Arithmetic - Geometric Mean Inequality implies that

$$\begin{aligned} \prod_{cyclic} \sin^{\frac{1}{3}}(|A-B|) &= \sqrt[3]{\prod_{cyclic} \sin(|A-B|)} \\ &\leq \frac{1}{3} \sum_{cyclic} \sin(|A-B|). \end{aligned} \tag{1}$$

Further, The Arithmetic - Geometric Mean Inequality also yields

$$a^2+b^2 \geq 2ab, \quad \text{i.e.,} \quad \frac{a^2+b^2}{2ab} \geq 1.$$

Similar results hold for $\dfrac{b^2+c^2}{2bc}$ and $\dfrac{c^2+a^2}{2ca}$. If we combine these facts with condition (1), we get

$$\begin{aligned} \prod_{cyclic} \sin^{\frac{1}{3}}(|A-B|) &\leq \frac{1}{3} \sum_{cyclic} \sin(|A-B|) \\ &\leq \sum_{cyclic} \frac{a^2+b^2}{6ab} \sin(|A-B|). \end{aligned}$$

Solution 3 by Moti Levy, Rehovot, Israel

By AM-GM inequality and $x + \dfrac{1}{x} \geq 2$ for $x > 0$,

$$\begin{aligned} \prod_{cyclic} \sin^{\frac{1}{3}} |A-B| &\leq \frac{1}{3} \sum_{cyclic} \sin|A-B| \\ &\leq \frac{1}{3} \sum_{cyclic} 2\sin|A-B| \leq \frac{1}{3} \sum_{cyclic} \left(\frac{a}{b} + \frac{b}{a}\right) \sin|A-B| \\ &= \sum_{cyclic} \frac{a^2+b^2}{3ab} \sin|A-B|. \end{aligned}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, Albert Stadler, Herrliberg, Switzerland; Henry Ricardo, New York Math Circle, NY; Neculai Stanciu "George Emil Palade" General School, Buzău and Titu Zvonaru, Comănesti, Romania Toshihiro Shimizu, Kawasaki, Japan, and the proposer

- **5378:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Calculate

$$\int_0^\infty \ln^k\left(\frac{e^x+1}{e^x-1}\right)dx.$$

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

Let $y = \ln\left(\frac{e^x+1}{e^x-1}\right)$. Then $e^x = \frac{e^y+1}{e^y-1}$ or $x = \ln\left(\frac{e^y+1}{e^y-1}\right)$ and

$$\begin{aligned}\frac{dx}{dy} &= \frac{d}{dy}\left(\ln(e^y+1) - \ln(e^y-1)\right)\\ &= \frac{e^y}{e^y+1} - \frac{e^y}{e^y-1}\\ &= \frac{-1}{e^y+1} + \frac{-1}{e^y-1}\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^\infty \ln^k\left(\frac{e^x+1}{e^x-1}\right)dx &= \int_\infty^0 y^k\left(\frac{-1}{e^y+1} + \frac{-1}{e^y-1}\right)dy\\ &= \int_0^\infty \frac{y^k}{e^y+1}dy + \int_0^\infty \frac{y^k}{e^y-1}dy\\ &= \Gamma(k+1)\eta(k+1) + \Gamma(k+1)\zeta(k+1)\\ &= k!(1-2^{-k})\zeta(k+1) + k!\zeta(k+1)\\ &= k!(2-2^{-k})\zeta(k+1)\end{aligned}$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We calculate, for $k \geq 1$,

$$I(k) = \int_0^\infty \left(\log\frac{e^x+1}{e^x-1}\right)^k dx.$$

The change of variable

$$\begin{aligned} t &= \log\frac{e^x+1}{e^x-1}, \text{ or equivalently, } x = \log\frac{e^t+1}{e^t-1},\\ dx &= \left(\frac{e^t+1}{e^t-1}\right)^{-1}\frac{-2e^t}{(e^t-1)^2}dt = \frac{-2e^t}{e^{2t}-1}dt = \frac{-2e^{-t}}{1-e^{-2t}}dt\end{aligned}$$

yields

$$I(k) = \int_0^\infty t^k\frac{2e^{-t}}{1-e^{-2t}}dt.$$

Rewriting as a geometric series we have

$$I(k) = 2\sum_{j=0}^\infty \int_0^\infty t^k e^{-(2j+1)t}dt = 2\Gamma(k+1)\sum_{j=0}^\infty \frac{1}{(2j+1)^{k+1}},$$

since $\int_0^\infty t^k e^{-t} dt = \Gamma(k+1)$ and the interchange of integration and summation is justified by the Monotone Convergence Theorem. It is well known (and easy to verify) that

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)^{k+1}} = \left(1 - 2^{-(k+1)}\right) \zeta(k+1).$$

Hence,

$$I(k) = \left(2 - 2^{-k}\right) \Gamma(k+1) \zeta(k+1)$$

Remark: The above formula is valid even for complex k with $Re(k) > 0$.

Solution 3 by Moti Levy, Rehovot, Israel

Let $I_k := \int_0^\infty \ln^k \left(\frac{e^x+1}{e^x-1}\right) dx$. By change of variable $v = \ln\left(\frac{e^x+1}{e^x-1}\right)$,

$$I_k = 2\int_0^\infty v^k \frac{e^v}{e^{2v}-1} dv.$$

$$\frac{e^v}{e^{2v}-1} = \frac{1}{e^v-1} - \frac{1}{e^{2v}-1},$$

$$\begin{aligned} I_k &= 2\int_0^\infty \frac{v^k}{e^v-1} dv - 2\int_0^\infty \frac{v^k}{e^{2v}-1} dv \\ &= 2\int_0^\infty \frac{v^k}{e^v-1} dv - \left(\frac{1}{2}\right)^k \int_0^\infty \frac{v^k}{e^v-1} dv \\ &= \left(2 - \left(\frac{1}{2}\right)^k\right) \int_0^\infty \frac{v^k}{e^v-1} dv. \end{aligned}$$

An integral representation of the Zeta function is

$$\Gamma(s)\zeta(s) = \int_0^\infty v^{s-1} \frac{1}{e^v-1} dv, \quad \mathrm{Re}(s) > 1.$$

$$\int_0^\infty \ln^k \left(\frac{e^x+1}{e^x-1}\right) dx = \left(2 - \left(\frac{1}{2}\right)^k\right) k! \zeta(k+1).$$

Also solved by Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

Toshihiro Shimizu of Kawasaki, Japan should have been credited with having solved problems $5367, 5368, 5369, 5370, 5371$, and 5372. His name was inadvertently omitted from the listing. Also omitted from the list of having solved problems were**Charles McCracken, of Dayton, OH** for 5367 , **Paolo Perfetti of the Mathematics Department of Tor Vergata University in Rome, Italy** for 5372, **The Prishtina Math Gymnasium Problem Solving Group of the Republic of Kosova** for 5368 and 5370, and **Albert Stadler, Herrliberg, Switzerland for 5368. Bruno Salgueiro Fanego of Viveiro, Spain** noted that problem 5386 appeared in this column previously as problem 5304. For the above errors, duplications and omissions, mea culpa. Editor.

Editor′s addendum: Albert's proof to 5368 was very different from the others that were received. The problem (posed by Ed Gray of Highland Beach FL) was to find a four digit number *abcd* in base 10 such that the last four digits of the square of the number *abcd* was again, *abcd*. Most solvers considered various cases for the digits *abcd*, starting with the digit $d \in \{1, 5, 6\}$, and then, employing the conditions of the problem, eliminated various values. Following is Albert's solution to 5368.

5368: Solution by Albert Stadler, Herrliberg, Switzerland

Let x be the four digit number is base 10. By assumption, $x^2 \equiv x \mod (10^4)$, which implies that $x(x-1)$ is divisible by 10^4. x and $x-1$ are relatively prime. So either 2^4 divides x and 5^4 divides $x-1$ or 5^4 divides x and 2^4 divides $x-1$.

We now invoke the Chinese remainder theorem. The first alternative implies $x = 9376$, while the second implies 625. However, 625 is not a four digit number, so 9376 is the only solution.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2016

- **5397:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation $\sqrt[3]{x+9} = \sqrt{3} + \sqrt[3]{x-9}$ with $x > 9$.

- **5398:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  If $(2n-1)!! = 1 \cdot 3 \cdot 5 \ldots (2n-1)$, then evaluate

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n}\right).$$

- **5399:** *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

  Let a, b, c be positive real numbers. Prove that

$$\sum_{cyclic} \frac{2a+2b}{\sqrt{6a^2+4ab+6b^2}} \leq 3.$$

5400: *Proposed by Arkady Alt, San Jose, CA*

Prove the inequality

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \leq 12\,(2R-3r),$$

where a, b, c and m_a, m_b, m_c are respectively sides and medians of $\triangle ABC$, with circumradius R and inradius r.

- **5401:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let a, b, c be three positive real numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2} + \frac{c^{-1}}{(4\sqrt{b}+3\sqrt{c})^2} + \frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2} \geq \frac{3}{49}.$$

- **5402:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
$$\int_0^{\infty} \left(\frac{\cos(ax) - \cos(bx)}{x}\right)^2 \mathrm{d}x,$$
  where a and b are real numbers.

Solutions

- **5379:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
$$\frac{(x+1)^4}{(x-1)^2} = 17x.$$

Solution 1 by Ed Gray, Highland Beach, FL

Cross-multiplying and simplifying gives $x^4 - 13x^3 + 40x^2 - 13x + 1 = 0$. Obviously $x \neq 0$, so dividing the polynomial by $40x^2$ gives
$$\begin{aligned}
\frac{x^2}{40} - \frac{13}{40}x + 1 - \frac{13}{40}\cdot\frac{1}{x} + \frac{1}{40}\cdot\frac{1}{x^2} &= 0, \\
\frac{1}{40}\left(\left(x^2 + \frac{1}{x^2}\right) - 13\left(x + \frac{1}{x}\right)\right) + 1 &= 0.
\end{aligned}$$

Letting $t = x + \dfrac{1}{x}$, squaring $t^2 = x^2 + \dfrac{1}{x^2} + 2$ and then substituting into the above gives
$$\begin{aligned}
\frac{1}{40}\left((t^2 - 2) - 13t\right) + 1 &= 0 \\
t^2 - 13t + 38 &= 0, \text{ so} \\
t_1 &= \frac{1}{2}\left(13 + \sqrt{17}\right) \\
t_2 &= \frac{1}{2}\left(13 - \sqrt{17}\right).
\end{aligned}$$

Since $t = x + \dfrac{1}{x}$, we have $x^2 - tx + 1 = 0$, and solving for x gives
$$\begin{aligned}
x_1 &= \frac{1}{2}\left(t_1 + \sqrt{t_1^2 - 4}\right) \quad x_2 = \frac{1}{2}\left(t_1 - \sqrt{t_1^2 - 4}\right) \\
x_3 &= \frac{1}{2}\left(t_2 + \sqrt{t_2^2 - 4}\right) \quad x_4 = \frac{1}{2}\left(t_2 - \sqrt{t_2^2 - 4}\right).
\end{aligned}$$

Substituting in the respective values of t and simplifying gives:
$$x_1 = \frac{13}{4} + \frac{\sqrt{17}}{4} + \frac{1}{2}\sqrt{\frac{85}{2} + \frac{13\sqrt{17}}{2}}$$

$$
\begin{aligned}
x_2 &= \frac{13}{4} + \frac{\sqrt{17}}{4} - \frac{1}{2}\sqrt{\frac{85}{2} + \frac{13\sqrt{17}}{2}} \\
x_3 &= \frac{13}{4} - \frac{\sqrt{17}}{4} + \frac{1}{2}\sqrt{\frac{85}{2} - \frac{13\sqrt{17}}{2}} \\
x_4 &= \frac{13}{4} - \frac{\sqrt{17}}{4} - \frac{1}{2}\sqrt{\frac{85}{2} - \frac{13\sqrt{17}}{2}}.
\end{aligned}
$$

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

Given a real number k, we seek all real solutions of

$$\frac{(x+1)^4}{(x-1)^2} = kx.$$

We require

$$x^4 + (4-k)x^3 + (6+2k)x^2 + (4-k)x + 1 = (x^2+ax+1)(x^2+bx+1) = 0,$$

where $a = (4-k+\sqrt{k^2-16k})/2$ and $b = (4-k-\sqrt{k^2-16k})/2$. Hence there are no real solutions unless $k \in (-\infty, 0] \cup [16, \infty)$. Solving for x, we obtain $x = (-a \pm \sqrt{a^2-4})/2$ or $x = (-b \pm \sqrt{b^2-4})/2$. We note that if $k = 0$, then there is one real solution; if $k < 0$ or $k = 16$, then there are two real solutions; and if $k > 16$, then there are four real solutions.

For the given equation with $k = 17$, we have four real solutions:

Letting $a = (-13+\sqrt{17})/2$ and $b = (-13-\sqrt{17})/2$, we obtain

$$x = (-a \pm \sqrt{a^2-4})/2 \approx 4.200 \text{ or } 0.238;$$

$$x = (-b \pm \sqrt{b^2-4})/2 \approx 8.443 \text{ or } 0.118.$$

Comments: **Arkady Alt of San Jose, CA** noted in his solution that the 17 in the statement of the problem could be replaced with any of the three numbers $15, 16$, or 18 to obtain a more elegant answer. For example, the equation

$$\frac{(x+1)^4}{(x-1)^3} = 18x \text{ gives the solutions}$$

$$
\begin{aligned}
x &= 5 \pm 2\sqrt{6} = \left(\sqrt{3} \pm \sqrt{2}\right)^2 \\
x &= 2 \pm \sqrt{3} = \left(\frac{\sqrt{6} \pm \sqrt{2}}{2}\right)^2.
\end{aligned}
$$

Kenneth Korbin, proposer of the problem, stated: If $b > 2$, then the equation $\dfrac{(x+1)^4}{(x-1)^3} = \left(4b^2\right)x$ gives the solutions

$$x_1 = \frac{\sqrt{a} + \sqrt{b+\sqrt{c}}}{\sqrt{a} - \sqrt{b+\sqrt{c}}} = \frac{1}{x_2},$$

$$x_3 = \frac{\sqrt{a}+\sqrt{b-\sqrt{c}}}{\sqrt{a}-\sqrt{b-\sqrt{c}}} = \frac{1}{x_4},$$

with $a = 2b$ and with $c = b^2 - 4$.
In the given equation $4b^2 = 17$. Then

$$b^2 = \frac{17}{4}, \quad b = \frac{\sqrt{17}}{2} > 2, \ a = 2b = \sqrt{17}, \ c = b^2 - 4 = \frac{1}{4}.$$

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; Henry Ricardo, New York Math Circle, NY. Toshihiro Shimizu, Kawasaki, Japan; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; (David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5380:** *Proposed by Arkady Alt, San Jose, CA*

Let $\Delta(x,y,z) = 2(xy+yz+xz) - (x^2+y^2+z^2)$ and a,b,c be the side-lengths of a triangle ABC. Prove that

$$F^2 \geq \frac{3}{16} \cdot \frac{\Delta(a^3,b^3,c^3)}{\Delta(a,b,c)},$$

where F is the area of $\triangle ABC$.

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

From the Heron's formula,

$$\begin{aligned} F^2 &= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{16} \\ &= \frac{\Delta(a^2,b^2,c^2)}{16} \end{aligned}$$

Thus, it suffices to show that $\Delta(a^2,b^2,c^2)\Delta(a,b,c) - 3\Delta(a^3,b^3,c^3) \geq 0 \quad (\heartsuit)$. The (l.h.s) can be written as

$$\sum_{cyc}(a-b)(a-c)q(a,b,c),$$

where $q(a,b,c) = 4a^4 + 2a^3(b+c) + a^2(b-c)^2 \geq 0$. Moreover, since

$$q(a,b,c) - q(b,c,a) = (a-b)(bc^2 + ac^2 + 2b^2c + 2a^2c + 4b^3 + 6ab^2 + 6a^2b + 4a^3),$$

the relation, which is larger $q(a,b,c)$ or $q(b,c,a)$, depends on the value of a or b. Without loss of generality, we assume $a \geq b \geq c$. Then, $q(a,b,c) \geq q(b,c,a) \geq q(c,a,b)$. Thus,

$$\begin{aligned}\sum_{cyc}(a-b)(a-c)q(a,b,c) &= (a-b)\big((a-c)q(a,b,c)-(b-c)q(b,c,a)\big)+q(c,a,b)(a-c)(b-c)\\ &\geq 0.\end{aligned}$$

Therefore, ($\heartsuit$) is true.
Note: It is similar to the proof of Schur's inequality. It seems that ($\heartsuit$) is valid for any a,b,c, even if the constraint that a,b,c are the side-lengths of a triangle is not satisfied.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Denote by s the semiperimeter of the triangle put $s_a = s-a, s_b = s-b, s_c = s-c$. By the triangle inequality, $s_a \geq 0, s_b \geq 0, s_c \geq 0$. Also $a = s_b + s_c,\ b = s_c + s_a,\ c = s_a + s_b$. Furthermore, we note that

$$\Delta(a,b,c) = \Delta\left(s_b+s_c, s_c+s_a, s_a+s_b\right) = 4\left(s_a s_b + s_b s_c + s_c s_a\right) \geq 0.$$

By Heron's formula $F^2 = s \cdot s_a \cdot s_b \cdot s_c = \left(s_a+s_b+s_c\right) s_a \cdot s_b \cdot s_c$.
Therefore we need to prove that

$$64\left(s_a+s_b+s_c\right) \cdot s_a \cdot s_b \cdot s_c \left(s_a s_b + s_b s_c + s_c s_a\right) \geq 3\Delta\left(\left(s_b+s_c\right)^3,\ \left(s_c+s_a\right)^3,\ \left(s_a+s_b\right)^3\right)$$

which is equivalent to

$$27\sum_{symm} s_a^4 s_b^2 + 21\sum_{symm} s_a^3 s_b^3 + 5\sum_{symm} s_a^2 s_b^2 s_c^2 \geq 27\sum_{symm} s_a^4 s_b s_c + 26\sum_{symm} s_a^3 s_b^2 s_c \qquad (1)$$

(as is seen by simply multiplying out).

By Schur's inequality

$$\sum_{cycl} s_a s_b \left(s_a s_b - s_b s_c\right)\left(s_a s_b - s_c s_a\right) \geq 0$$

which is equivalent to

$$\sum_{symm} s_a^3 s_b^3 + \sum_{symm} s_a^2 s_2^2 s_c^2 \geq 2\sum_{symm} s_a^3 s_b^2 s_c \qquad (2)$$

(as is seen again by multiplying out).

We have the following inequalities

$$\begin{aligned}5\sum_{symm} s_a^3 s_b^3 + 5\sum_{symm} s_a^2 s_b^2 s_c^2 \quad &\geq \quad 10\sum_{symm} s_a^3 s_b^2 s_c, \quad (\text{by}(2)),\\ 27\sum_{symm} s_a^4 s_b^2 \quad &\geq \quad 27\sum_{symm} s_a^4 s_b s_c,\ \text{by Muirhead's inequality},\\ 16\sum_{symm} s_a^3 s_b^3 \quad &\geq \quad 16\sum_{symm} s_a^3 s_b^2 s_c,\ \text{by Muirhead's inequality}.\end{aligned}$$

(1) follows by adding these three inequalities.

Solution 3 by proposer

Let $s := \dfrac{t_1+t_2+t_3}{2}$. Since $t_i < s, i = 1,2,3$ (triangle inequalities) then our problem is:
Find $\max s$ for which there are positive integer numbers
t_1, t_2, t_3 satisfying $t_i \le \min\{a_i, s-1\}, i = 1,2,3,\ t_1+t_2+t_3 = 2s$.
First note that $s \ge 3, t_i \ge 2, i = 1,2,3$. Indeed, since $t_i \le s-1$, then
$1 \le s - t_i, i = 1,2,3$ and,
therefore, $t_1 = 2s - t_2 - t_3 = (s-t_2) + (s-t_3) \ge 2$. Cyclic we obtain $t_2, t_3 \ge 2$. Hence, $2s \ge 6 \iff s \ge 3$.

Since $t_3 = 2s - t_1 - t_2,\ 2 \le t_3 \le \min\{a_3, s-1\}$
then $1 \le 2s - t_1 - t_2 \le \min\{a_3, s-1\} \iff$
$\max\{2s - t_1 - a_3, s+1-t_1\} \le t_2 \le 2s - 1 - t_1$ and, therefore, for t_2 we obtain the inequality

(1) $\max\{2s - t_1 - a_3, s+1-t_1, 2\} \le t_2 \le \min\{2s-1-t_1, a_2, s-1\}$
with conditions of solvability :

(2) $$\begin{cases} 2s - t_1 - a_3 \le s-1 \\ 2s - t_1 - a_3 \le a_2 \\ s+1-t_1 \le a_2 \\ 2 \le 2s-1-t_1 \end{cases} \iff \begin{cases} s+1-a_3 \le t_1 \\ 2s - a_2 - a_3 \le t_1 \\ s+1-a_2 \le t_1 \\ t_1 \le 2s-3 \end{cases}.$$

Since $s - 1 \le 2s - 3$ then **(2)** together with $2 \le t_1 \le \min\{a_1, s-1\}$ it gives us bounds for t_1 :

(3) $\max\{s+1-a_3, 2s-a_2-a_3, s+1-a_2, 2\} \le t_1 \le \min\{a_1, s-1\}$.

Since $2 \le a_i, i = 2,3$ then $s+1-a_2 \le s-1, s+1-a_3 \le s-1$ and solvability condition for **(3)** becomes

$s+1-a_3 \le a_1 \iff s \le a_1+a_3-1, 2s-a_2-a_3 \le a_1 \iff s \le \left\lfloor \dfrac{a_1+a_2+a_3}{2} \right\rfloor$,
$s+1-a_2 \le a_1 \iff s \le a_1+a_2-1, 2s-a_2-a_3 \le s-1 \iff s \le a_2+a_3-1$.

Thus, $s^* = \min\left\{ \left\lfloor \dfrac{a_1+a_2+a_3}{2} \right\rfloor, a_1+a_2-1, a_2+a_3-1, a_3+a_1-1 \right\}$ is the largest value of integer semiperimeter.

Solution 4 by Andrea Fanchini, Cantú, Italy

We know that
$$F^2 = s(s-a)(s-b)(s-c)$$
where s is the semiperimeter of $\triangle ABC$.
Now making the substitutions and clearing the denominators we have to prove

$$16s(s-a)(s-b)(s-c)\left[2(ab+bc+ca) - (a^2+b^2+c^2)\right] \ge 3\left[2(a^3b^3+b^3c^3+c^3a^3) - (a^6+b^6+c^6)\right]$$

now we make the following substitutions (with $x, y, z > 0$)

$$a = y+z, \quad b = z+x, \quad c = x+y$$

and expanding out into symmetric sums the given inequality yields
LHS:

$$27(x^4y^2 + x^4z^2 + y^4z^2 + x^2y^4 + x^2z^4 + y^2z^4) + 42(x^3y^3 + y^3z^3 + x^3z^3)+$$
$$+6(x^3yz^2 + x^3y^2z + x^2y^3z + xy^3z^2 + x^2yz^3 + xy^2z^3) + 78x^2y^2z^2$$

RHS:

$$38(x^4yz + xy^4z + xyz^4)$$

so it remains to prove that

$$27(x^4y^2 + x^4z^2 + y^4z^2 + x^2y^4 + x^2z^4 + y^2z^4) \geq 38(x^4yz + xy^4z + xyz^4)$$

or

$$27[4, 2, 0] \geq 19[4, 1, 1]$$

which is true because

$$19[4, 2, 0] \succ 19[4, 1, 1]$$

it follows from Muirhead's Theorem, q.e.d.

Solution 5 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

As well known $F^2 = s(s-a)(s-b)(s-c)$ and $s = (a+b+c)/2$. Upon setting $a = y + z$, $b = x + z$, $c = x + y$,
the inequality becomes

$$\sum_{\text{sym}} \left(27x^4y^2 + 21(xy)^3 + 5(xyz)^2\right) \geq \sum_{\text{sym}} \left(27x^4yz + 26x^3y^2z\right).$$

The third degree Schür inequality is

$$(a^3 + b^3 + c^3) + 3abc \geq \sum_{\text{sym}} a^2b,$$

which applied to the triple $(xy), (yz), (zx)$, yields

$$5\sum_{\text{sym}}(xy)^3 + 5\sum_{\text{sym}}(xyz)^2 \geq 10\sum_{\text{sym}} x^3y^2z.$$

The inequality becomes

$$\sum_{\text{sym}} \left(27x^4y^2 + 16(xy)^3\right) \geq \sum_{\text{sym}} \left(27x^4yz + 16x^3y^2z\right),$$

and the proof is complete upon observing that by the AGM we have

$$x^4y^2 + x^4z^2 \geq 2x^4yz, \qquad (xy)^3 + (xy)^3 + (xz)^3 \geq 3x^3y^2z.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel, and Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

- **5381:** *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

Prove: In any acute triangle ABC, with the usual notations, holds:

$$\sum_{cyclic}\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}\geq\frac{3}{2^{m+1}},$$

where $m\geq 0$ is an integer number.

Solution 1 by Nikos Kalapodis, Patras, Greece

We first recall Barrow's Inequality:
If x,y,z are positive real numbers and $A+B+C=\pi$ then

$$\frac{yz}{2x}+\frac{zx}{2y}+\frac{xy}{2z}\geq x\cos A+y\cos B+z\cos C \qquad (1)$$

(This inequality first appeared in [1]. For a solution see [2] or [3] (inequality 2.20, pp. 23-24)).
Applying inequality (1) for $x=\cos A$, $y=\cos B$, and $z=\cos C$ (note that $\cos A,\cos B,\cos C>0$, since ABC is an acute triangle) we obtain

$$\sum_{cyclic}\frac{\cos A\cos B}{\cos C}\geq 2(\cos^2 A+\cos^2 B+\cos^2 C) \qquad (2)$$

By the following well-known trigonometric identities
$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$ and
Euler's inequality ($2r\leq R$) we obtain that $\cos A+\cos B+\cos C=1+\frac{r}{R}\leq\frac{3}{2}$ (3)
Using the AM-GM inequality and inequality (3) we have
$\cos A\cos B\cos C\leq\left(\frac{\cos A+\cos B+\cos C}{3}\right)^3\leq\left(\frac{1}{3}\cdot\frac{3}{2}\right)^3=\frac{1}{8}$ (4)
Furthermore, by the identity $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$ and inequality (4) we obtain

$$\cos^2 A+\cos^2 B+\cos^2 C\geq\frac{3}{4} \qquad (5)$$

By (2) and (5), we have

$$\sum_{cyclic}\frac{\cos A\cos B}{\cos C}\geq\frac{3}{2} \qquad (6)$$

Finally, applying Radon's Inequality and using inequality (6) we have that

$$\sum_{cyclic}\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}=$$

$$\frac{\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}}{1^m}+\frac{\left(\frac{\cos B\cos C}{\cos A}\right)^{m+1}}{1^m}+\frac{\left(\frac{\cos C\cos A}{\cos B}\right)^{m+1}}{1^m}\geq$$

$$\frac{\left(\sum_{cyclic}\frac{\cos A\cos B}{\cos C}\right)^{m+1}}{(1+1+1)^m} \geq \frac{\left(\frac{3}{2}\right)^{m+1}}{3^m} = \frac{3}{2^{m+1}}.$$

References:
[1] L. J. Mordell and D. F. Barrow, Solution 3740, *The American Mathematical Monthly* Vol. 44, No. 4 (Apr., 1937) pp. 252-254
(http://www.jstor.org/stable/2300713)
[2] R. R. Janic, *On A Geometric Inequality Of D. F. Barrow*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-196 (1967), pp.73-74
(http://pefmath2.etf.bg.ac.rs/files/71/194.pdf)
[3] O. Bottema, R. Z. Djordjevic, R. R. Janic, D. S. Mitrinovic, and P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.
Remark: Inequalities (3), (4), and (5) also appear respectively as inequalities 2.16, 2.23 and 2.21 in reference [3].

Solution 2 byÁngel Plaza, University of Las Palmas de Gran Canaria, Spain

By the RMS-AM inequality it is enough to prove that

$$\sum_{cyclic}\frac{\cos A\cos B}{\cos C} \geq \frac{3}{2}.$$

Taking into account that $A+B+C=\pi$, then
$\cos C = \cos\left(\frac{\pi}{2}-A+\frac{\pi}{2}-B\right) = \sin A\sin B - \cos A\cos B$, so the inequality to be proved may be written with cotangents as

$$\sum_{cyclic}\frac{\cot A\cot B}{1-\cot A\cot B} \geq \frac{3}{2}, \text{ or } \sum_{cyclic}\frac{1}{1-\cot A\cot B} \geq \frac{9}{2}.$$

It is well known that if $\alpha=\cot A$, $\beta=\cot B$, and $\gamma=\cot C$, then $\alpha\beta+\beta\gamma+\gamma\alpha=1$. Therefore, taking $x=\alpha\beta$, $y=\beta\gamma$, and $z=\gamma\alpha$ we have to prove that $\displaystyle\sum_{cyclic}\frac{1}{1-x} \geq \frac{9}{2}$ which follows by Jensen's inequality, since function $f(x)=\dfrac{1}{1-x}$ is convex for $x\in(0,1)$.

Solution 3 by Henry Ricardo, New York Math Circle, NY

Elementary calculations show that for $A,B,C\in(0,\pi/2)$

$$\frac{\cos A\cos B}{\cos C} = \frac{\tan C}{\tan A+\tan B}. \tag{1}$$

Furthermore, we have

$$\sum_{cyclic}\frac{\tan C}{\tan A+\tan B} \geq \frac{3}{2}. \tag{2}$$

by Nesbitt's inequality.

Finally, $(1), (2)$, and the power mean inequality give us

$$\begin{aligned}
\sum_{cyclic}\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1} &= \sum_{cyclic}\left(\frac{\tan C}{\tan A+\tan B}\right)^{m+1}\\
&\geq 3\left(\frac{1}{3}\sum_{cyclic}\frac{\tan C}{\tan A+\tan B}\right)^{m+1}\\
&\geq 3\left(\frac{1}{2}\right)^{m+1} = \frac{3}{2^{m+1}}.
\end{aligned}$$

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Using the inequality $a^{m+1}+b^{m+1}+c^{m+1}\geq \dfrac{1}{3^m}(a+b+c)^{m+1}$ $(*)$ we have

$$\sum\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1} = \sum\left(\frac{\tan C}{\tan A+\tan B}\right)^{m+1} \underset{\text{by }(*)}{\geq} \frac{1}{3^m}\left(\sum\frac{\tan C}{\tan A+\tan B}\right)^{m+1}. \quad (**)$$

Setting $\tan A = x,\ \tan B = y,\ \tan C = z$, and using Nesbitt's inequality, we have

$$\sum\frac{\tan C}{\tan A+B} = \sum\frac{z}{x+y} \underset{\text{by Nesbitt)}}{\geq} \frac{3}{2}, \qquad (***)$$

The statement of the problem follows from $(**)$ and $(***)$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5382:** *Proposed by* Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Prove that if a, b, c are positive real numbers, then

$$\left(\sum_{cyclic}\frac{a}{b}+8\sum_{cyclic}\frac{b}{a}\right)\left(\sum_{cyclic}\frac{b}{a}+8\sum_{cyclic}\frac{a}{b}\right)\geq 9^3.$$

Solution 1 by Henry Ricardo, New York Math Circle, NY

By the arithmetic-geometric mean inequality, each of the sums $\displaystyle\sum_{cyclic}\frac{a}{b}$, $\displaystyle\sum_{cyclic}\frac{b}{a}$ is greater than or equal to 3. Thus

$$\left(\sum_{cyclic}\frac{a}{b}+8\sum_{cyclic}\frac{b}{a}\right)\left(\sum_{cyclic}\frac{b}{a}+8\sum_{cyclic}\frac{a}{b}\right)\geq(3+8\cdot 3)(3+8\cdot 3)=27^2=9^3.$$

Solution 2 by Ed Gray, Highland Beach, FL

Clearly, if $a=b=c$, the above product becomes

$$(1+1+1+8(1+1+1))\,(1+1+1+8(1+1+1)) = (3+24)(3+24) = 27^2 = 729 = 9^3.$$

Therefore, if we show that the product is minimum when all variables are equal, then the conjecture would be true. It is sufficient to calculate the product in two different ways.

First, suppose that $a = b$ and $c = 0.99a$. Second, suppose $a = b$ and $c = 1.01a$. If both of these products exceed 729, that would show that if all variables are not equal, we do not have a minimum.

Case 1: $a = b, c = 0.99a$. The product becomes

$$\left(1 + \frac{1}{0.99} + 0.99 + 8(1 + 0.99 + \frac{1}{0.99})\right)\left(1 + 0.99 + \frac{1}{0.99} + 8(1 + \frac{1}{0.99} + 0.99)\right) = 729.049$$

Case 2: $a = b, c = 1.01a$. The product becomes

$$\left(1 + \frac{1}{0.99} + 0.99 + 8(1 + 1.01 + \frac{1}{1.01})\right)^2 = 729.048$$

QED

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We begin by applying the extension of the Arithmetic - Geometric Mean Inequality which states that if $\alpha, \beta, x, y > 0$ and $\alpha + \beta = 1$, then

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta},$$

with equality if and only if $x = y$. It follows that

$$\begin{aligned}
\sum_{cyclic} \frac{a}{b} + 8 \sum_{cyclic} \frac{b}{a} &= \sum_{cyclic} \left(\frac{a}{b} + 8\frac{b}{a}\right) \\
&= 9 \sum_{cyclic} \left(\frac{1}{9}\frac{a}{b} + \frac{8}{9}\frac{b}{a}\right) \\
&\geq 9 \sum_{cyclic} \left(\frac{a}{b}\right)^{\frac{1}{9}} \left(\frac{b}{a}\right)^{\frac{8}{9}} \\
&= 9 \sum_{cyclic} \left(\frac{b}{a}\right)^{\frac{7}{9}},
\end{aligned}$$

with equality if and only if $\dfrac{a}{b} = \dfrac{b}{a}$, $\dfrac{b}{c} = \dfrac{c}{b}$, and $\dfrac{c}{a} = \dfrac{a}{c}$, i.e., if and only if $a = b = c$.

Next, apply the standard version of the Arithmetic - Geometric Mean Inequality to get

$$\begin{aligned}
\sum_{cyclic} \frac{a}{b} + 8 \sum_{cyclic} \frac{b}{a} &\geq 9 \sum_{cyclic} \left(\frac{b}{a}\right)^{\frac{7}{9}} \\
&\geq 27 \sqrt[3]{\prod_{cyclic} \left(\frac{b}{a}\right)^{\frac{7}{9}}} \\
&= 27, \qquad (1)
\end{aligned}$$

with equality if and only if $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{a}{c}$, i.e., if and only if $a = b = c$.

A similar set of steps yields

$$\sum_{cyclic} \frac{b}{a} + 8 \sum_{cyclic} \frac{a}{b} \geq 27, \tag{2}$$

with equality if and only if $a = b = c$.

Therefore, by (1) and (2),

$$\left(\sum_{cyclic} \frac{a}{b} + 8 \sum_{cyclic} \frac{b}{a} \right) \left(\sum_{cyclic} \frac{b}{a} + 8 \sum_{cyclic} \frac{a}{b} \right) \geq 27^2 = 9^3,$$

with equality if and only if $a = b = c$.

Solution 4 by Andrea Fanchini, Cantú, Italy

Clearing the denominators and making the multiplications we have

$$8(a^4b^2 + a^4c^2 + a^2b^4 + b^4c^2 + b^2c^4 + a^2c^4) + 65(a^4bc + ab^4c + abc^4) + 65(a^3b^3 + b^3c^3 + a^3c^3) +$$

$$+16(a^3b^2c + a^3bc^2 + a^2b^3c + ab^3c^2 + a^2bc^3 + ab^2c^3) \geq 534a^2b^2c^2$$

or

$$16[4,2,0] + 65[4,1,1] + 65[3,3,0] + 32[3,2,1] \geq 178[2,2,2]$$

which is true because

$$16[4,2,0] \succ 16[2,2,2]$$

$$65[4,1,1] \succ 65[2,2,2]$$

$$65[3,3,0] \succ 65[2,2,2]$$

$$32[3,2,1] \succ 32[2,2,2]$$

each of which follows from Muirhead's Theorem, q.e.d.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michael Brozinsky (3 solutions), Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel; Nikos Kalapodis, Patras, Greece; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; Neculai Stanciu,"George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Helliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5383: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Find $\gcd(a_n, b_n)$, where a_n and b_n are the positive integers for which $(1-\sqrt{5})^n = a_n - b_n\sqrt{5}$.

Solution 1 by Ethan Gegner (Undergraduate student, Taylor University), Upland, IN

The equation

$$a_{n+1} - b_{n+1}\sqrt{5} = (a_n - b_n\sqrt{5})(1-\sqrt{5}) = a_n + 5b_n - (a_n + b_n)\sqrt{5}$$

yields the recurrence relations

$$\begin{aligned} a_{n+1} &= a_n + 5b_n \\ b_{n+1} &= a_n + b_n \end{aligned}$$

Thus,

$$\begin{aligned} \gcd(a_{n+1}, b_{n+1}) = \gcd(a_n + 5b_n, a_n + b_n) &= \gcd(6a_{n-1} + 10b_{n-1}, 2a_{n-1} + 6b_{n-1}) \\ &= \gcd(16a_{n-2} + 40b_{n-2}, 8a_{n-2} + 16b_{n-2}) \\ &= 8\gcd(2a_{n-2} + 5b_{n-2}, a_{n-2} + 2b_{n-2}) \\ &= 8\gcd(b_{n-2}, a_{n-2} + 2b_{n-2}) \\ &= 8\gcd(b_{n-2}, a_{n-2}) \end{aligned}$$

Since $a_1 = b_1 = 1$, $a_2 = 6, b_2 = 2$, $a_3 = 16, b_3 = 8$, we have $\gcd(a_1, b_1) = 2^0, \gcd(a_2, b_2) = 2^1, \gcd(a_3, b_3) = 2^3$. It follows inductively that

$$\gcd(a_n, b_n) = \begin{cases} 2^n & : 3|n \\ 2^{n-1} & : \text{otherwise} \end{cases}$$

Solution 2 Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

For $n = 1$, $a_1 = b_1 = 1$. For $n > 1$,

$$\begin{aligned} a_n - b_n\sqrt{5} &= \left(a_{n-1} - b_{n-1}\sqrt{5}\right)\left(1+\sqrt{5}\right) \\ &= a_{n-1} + 5b_{n-1} - \sqrt{5}\,(a_{n-1} + b_{n-1}) \end{aligned}$$

so $a_n = a_{n-1} + 5b_{n-1}$, and $b_n = a_{n-1} + b_{n-1}$, or in matrix form

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \Rightarrow \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, $a_n = 2^{n-1} \cdot L_n$ and $b_n = 2^n \cdot F_n$, where L_n and F_{n+1} respectively are the nth Lucas and the nth Fibonacci numbers. Since $L_n = F_{n-1} + F_{n+1}$, then $gcd(L_n, F_{n+1}) = 1$ and hence $gcd(a_n, b_n) = 2^{n-1}$, if L_n is odd, while $gcd(a_n, b_n) = 2^n$ if L_n is even, that is when n is a multiple of 3.

Solution 3 by Carl Libis, Columbia Southern University, Orange Beach, AL

Let $\left(1-\sqrt{5}\right)^n = a_n - b_n\sqrt{5}$. Then

$$a_{n+1} - b_{n+1}\sqrt{5} = \left(a_n - b_n\sqrt{5}\right)\left(1-\sqrt{5}\right) = (a_n + 5b_n) - (a_n + b_n)\sqrt{5}.$$

Thus,

$$(i)\ a_{n+1} = a_n + 5b_n,$$

$$
\begin{aligned}
(ii)\ b_{n+1} &= a_n + b_n, \text{ and using (i), and (ii) we can show that}\\
(iii)\ a_{n+1} &= 2a_n + 4a_{n-1},\\
(iv)\ b_{n+1} &= 2b_n + ba_{n-1}. \text{ By observation we note from the first few terms}\\
(v)\ a_n &= 2^{n-1} l_n,\\
(vi)\ b_n &= 2^{n-1} f_n,
\end{aligned}
$$

where l_n and f_n are Lucas and Fibonacci numbers. We can verify (v) and (vi) by substituting them into (iii) and (iv).

It is well known that $\gcd(f_n, l_n) = \begin{cases} 2, & \text{if } 3|\, n \\ 1, & \text{otherwise.} \end{cases}$.
See<http://mathhelpforum.com/discrete-math/40492-proof-about-fibonacci-lucas-numbers-gcd.html> or
<https://cms.math.ca/crux/v3/n4/page232-236.pdf>.

Therefore,
$\gcd(a_n, b_n) = \gcd\left(2^{n-1} f_n, 2^{n-1} l_n\right) = 2^{n-1} \begin{cases} 2, & \text{if } 3|\, n \\ 1, & \text{otherwise.} \end{cases} = \begin{cases} 2^n, & \text{if } 3|\, n \\ 2^{n-1}, & \text{otherwise.} \end{cases}$

Comment by Editor: **Kenneth Korbin of NewYork, NY** observed a connection between this problem and the solution to problem 5373, that required us to find positive integers x and y such that $\dfrac{2\sqrt{2}}{\sqrt{343 - 147\sqrt{5}} - \sqrt{315 - 135\sqrt{5}}} = \sqrt{x + y\sqrt{5}}$, the unique answer of which was $(x, y) = 161 + 72\sqrt{5}$. He continued on as follows.

Observe that:

$\left(1 - \sqrt{5}\right)^{12} = (4096)(161 - 72\sqrt{5}) = 2^{12}(161 - 72\sqrt{5})$, and also $(161)^2 - (72\sqrt{5})^2 = 1$. So,

$$\left(161 - 72\sqrt{5}\right)\left(161 + 72\sqrt{5}\right)\left(1 - \sqrt{5}\right)^{12} = 2^{12}(161 - 72\sqrt{5})$$

$$161 + 72\sqrt{5} = \frac{2^{12}}{\left(1 - \sqrt{5}\right)^{12}}$$

$$\sqrt{161 + 72\sqrt{5}} = \frac{2^6}{\left(1 - \sqrt{5}\right)^{6}}.$$

And additionally: $\dfrac{2^6}{\left(1 - \sqrt{5}\right)^{6}} = \dfrac{2\sqrt{2}}{\sqrt{343 - 147\sqrt{5}} - \sqrt{315 - 135\sqrt{5}}}$.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China;

Moti Levi, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5384: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : \Re \to \Re$ which verify the functional equation

$$xf'(x) + f(-x) = x^2, \quad \text{for all} \quad x \in \Re.$$

Solution 1 by Michael Brozinsky, Central Islip, NY

We have at once from the given equation $xf'(x) + f(-x) = x^2$ that $f(0) = 0$, and since $x^2 = (-x)^2$ that

$$xf'(x) + f(x) = -xf'(-x) + f(x) \text{ so that}$$
$$x\left(f'(x) + f'(-x)\right) = f(x) - f(-x)$$

which can be cast as $xG'(x) = G(x)$, which we label as equation (1) and in which $G(x) = f(x) - f(-x)$.

From (1) we have $G(x) = cx$ for some constant c and thus $G''(x) = 0$, and so $f''(x) = f''(-x)$ and we label this as equation (2), where we have used the chain rule. Now, by differentiating the given equation twice we have

$$x \cdot f'''(x) + f''(x) + f''(x) + f''(-x) = 2$$

and so from (2) we have

$$f''(0) = \frac{2}{3} \ \ \textit{and}\ xf'''(x) + 3f''(x) = 2. \qquad (3)$$

Letting $v = f''(x)$ in (3) we have the linear differential equation $x \cdot \left(\dfrac{dv}{dx}\right) + 3v = 2$, and using the integrating factor x^3 we obtain

$$\begin{aligned} x^3dv + 3x^2vdx &= 2x^2dx \text{ so that} \\ x^3v = \frac{2x^3}{3} + A \quad \textit{and} \quad f''(x) &= v = \frac{2}{3} + \frac{A}{x^3} \qquad (4) \end{aligned}$$

where the constant $A = 0$ since $f''(0) = \dfrac{2}{3}$. Integrating (4) twice we obtain

$f(x) = \dfrac{x^2}{3} + Bx + C$ where B and C are constants and since $f(0) = 0$, we have $C = 0$.

Hence, the general solution is $f(x) = \dfrac{x^2}{3} + Bx$, where B is an *arbitrary* constant.

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

Let $P(x)$ be the given equation. From $P(x) + P(-x)$, we get

$$x\frac{d}{dx}\left(f(x) + f(-x)\right) + \left(f(x) + f(-x)\right) = 2x^2. \qquad (1)$$

From $P(x)-P(-x)$, we get

$$x\frac{d}{dx}\left(f(x)-f(-x)\right)-\left(f(x)-f(-x)\right)=0. \tag{2}$$

First, we solve (1). Let $g(x)=f(x)+f(-x)-\frac{2}{3}x^2$.Then, (1) can be rewritten as

$$x\frac{dg}{dx}=-g(x)$$

The root of this differential equation is $g(x)=C/x$ for constant $C\in R$.
Next, we solve (2). Let $h(x)=f(x)-f(-x)$. Then, (2) can be rewritten as

$$x\frac{dh}{dx}=h(x)$$

The root of this differential equation is $h(x)=Dx$ for constant $D\in R$.
Thus, $f(x)=(g(x)+2/3x^2+h(x))/2=C/x+Dx+x^2/3$ for some constant $C,D\in R$.
Since, $f(x)$ should be defined for all $x\in R$, C must be 0. Therefore, $f(x)=Dx+x^2/3$, where $D\in R$ is a constant and this satisfies $P(x)$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$xf'(x)+f(-x)=x^2,\forall x\in\Re \implies -xf'(-x)+f\left(-(-x)\right)-(-x)^2,\forall x\in\Re$, that is
$-xf'(-x)+f(x)=x^2,\forall x\in\Re \implies xf'(x)+f(-x)=x^2=-xf'\left(-x\right)+f(x)$,
$\forall x\in\Re \implies x\left(f'(x)-f'(-x)\right)+f(x)+f(-x)=2x^2,\forall\in\Re$, or equivalently,
$xg'(x)+g(x)=2x^2,\forall x\in\Re$, where $g:\Re\to\Re$ is the function defined by
$g(x)=f(x)+f(-x),\forall x\in\Re$, that is $h'(x)=2x^2$, with $h:\Re\to\Re,\forall x\in\Re$.
$h(x)=xg(x),\forall x\in\Re \implies h(x)=\dfrac{2x^2}{3}+C$, for some $C\in\Re,\forall x\in\Re$
implies $f(x)+f(-x)=g(x)=\dfrac{h(x)}{x}=\dfrac{2x^2}{3}+\dfrac{C}{x},\ \forall x-\{0\}$.
f differentiable implies f differentiable at $x=0 \implies f$ continuous at $x=0$.
This fact and the equality $f(x)+f(-x)=\dfrac{2x^2}{3}+\dfrac{C}{x}$ imply that $C=0$.
Hence, $f(-x)=\dfrac{2x^2}{3}-f(x)$ and thus $xf'(x)+\dfrac{2x^2}{3}-f(x)=xf'(x)+f(-x)=x^2$.
$\forall x\in\Re-\{0\} \implies xf'(x)-f(x)=\dfrac{x^2}{3}\forall x\in\Re-0\} \implies \dfrac{f'(x)}{x}-\dfrac{f(x)}{x^2}=\dfrac{1}{3},\forall x\in\Re-\{0\}$.
$\implies k'(x)\dfrac{1}{3}$, where $k:\Re\to\Re$ is the function defined by $k(x)=\dfrac{f(x)}{x},\forall x-\{0\}$
$\implies k(x)=\dfrac{x}{3}+D$ with $D\in\Re,\forall x\in\Re-\{0\} \implies f(x)=\dfrac{x^2}{3}+Dx,\forall x\in\Re-\{0\}$. Since
$\left(f'(x)-f'^{(-x)}\right)+f(x)+f(-x)=2x^2,\forall x\in\Re \implies$
$2f(0)=0\left(f'(0)-f'(-0)\right)+f(0)+f(-0)=2\cdot 0^2=0$, so $f(0)=0$, we conclude that
$f(x)=\dfrac{x^2}{3}+Dx,\forall x$, where D is any real constant.

Solution 4 by Moti Levy, Rehovot, Israel

The derivative of $f:R\to R$ satisfies the functional equation

$$f^{'}(x) = \frac{x^2 - f(-x)}{x}, \tag{1}$$

hence it is also differentiable function (maybe except for $x = 0$).

Differentiation of the functional equation gives,

$$xf^{''}(x) + f^{'}(x) - f^{'}(-x) = 2x. \tag{2}$$

Substitution of (1) into (2) gives,

$$xf^{''}(x) + f^{'}(x) + \frac{x^2 - f(x)}{x} = 2x,$$

or

$$x^2 f^{''}(x) + xf^{'}(x) - f(x) = x^2. \tag{3}$$

All the differentiable functions which satisfy the functional equation $xf^{'}(x) + f(-x) = x^2$, must satisfy (3).
The solutions of the differential equation (3) are

$$f(x) = \frac{1}{3}x^2 + \alpha\left(x + \frac{1}{x}\right) + \beta\left(x - \frac{1}{x}\right) \tag{4}$$

Now we substitute (4) in the left side of the original functional equation:

$$x\frac{d\left(\frac{1}{3}x^2 + \alpha\left(x + \frac{1}{x}\right) + \beta\left(x - \frac{1}{x}\right)\right)}{dx} + \frac{1}{3}x^2 - \alpha\left(x + \frac{1}{x}\right) + \beta\left(\frac{1}{x} - x\right)$$
$$= \frac{1}{x}\left(x^3 - 2\alpha + 2\beta\right) = x^2 - \frac{2}{x}(\alpha - \beta).$$

It follows that α must be equal to β for (4) to be a solution.
All the differentiable functions $f : R \to R$, which satisfy the functional equation $xf^{'}(x) + f(-x) = x^2$, for all $x \in R$ are

$$f(x) = \frac{1}{3}x^2 + cx, \quad c \in R.$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

Denote the given functional equation by (1). We show that

$$f(x) = \frac{x^2}{3} + kx, \tag{2}$$

where k is an arbitrary constant.

Replacing x bt $-x$ in (1), we obtain

$$-xf'(-x) + f(x) = x^2. \tag{3}$$

Subtracting (3) fromf (1), we obtain

$$x\left(f'(x) + f'(-x)\right) - \left(f(x) - f(-x)\right) = 0. \tag{4}$$

Integrating (4), we obtain $f(x) - f(-x) = ax$, where a is an arbitrary constant. By substituting $f(-x) = f(x) - ax$ back into (1). we obtain

$$xf'(x) + f(x) = x^2 + ax. \qquad (5)$$

Integrating (5), we obtain $xf(x) = \dfrac{x^3}{3} + \dfrac{ax^2}{2} + b$, where b is a constant. By putting $x = 0$ we see that $b = 0$. Thus (2) hold for $x \neq 0$. By putting $x = 0$ into (1), we obtain $f(0) = 0$ and so (2) hold for $x = 0$ as well.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Henry Ricardo, New York Math Circle, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

Editor's Notes

The conjecture in **5375*** has been revised by its author **Kenneth Korbin of NY, NY** to the following:

5375* (revised): Prove or disprove the following conjecture. Let k be the product of N different prime numbers each congruent to 1(mod 4).

The total number of different rectangles and trapezoids with integer length sides and diagonals that can be inscribed in a circle with diameter k is exactly $\dfrac{5^N - 3^N}{2}$.

Toshihiro Shimizu of Kawasaki, Japan provided a counter example to the original statement of the problem that did not require the diagonals to also be integers. He let $k = 5 \cdot 17 = 85$ and then developed the trapezoids $(34, 43, 34, 83)$ and $(50, 43, 50, 83)$. The diagonals of these two trapezoids are not of integral length. Ken commented on Toshihiro's examples by saying that: "It never occurred to me that a trapezoid with integer length sides inscribed in a circle with diameter k could have non-integer length diagonals." So with the revision, 5375* remains an open problem.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
October15, 2016

- **5403:** *Proposed by Kenneth Korbin, New York, NY*

  Let $\phi = \dfrac{1+\sqrt{5}}{2}$. Solve the equation $\sqrt[3]{x+\phi} = \sqrt[3]{\phi} + \sqrt[3]{x-\phi}$ with $x > \phi$.

- **5404:** *Proposed Arkady Alt, San Jose, CA*

  For any given positive integer $n \geq 3$, find the smallest value of the product of $x_1 x_2 \ldots x_n$, where $x_1, x_2, x_3, \ldots x_n > 0$ and $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1$.

- **5405:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu,"George Emil Palade" School, Buzău, Romania*

  If $a, b \in \Re$ such that $a+b=1$, $e_n = \left(1+\dfrac{1}{n}\right)^n$ and $c_n = -\ln n + \displaystyle\sum_{k=1}^{n} \frac{1}{k}$, then compute

  $$\lim_{n\to\infty} \left((n+1)^a \sqrt[n+1]{\left((n+1)!c_n\right)^b} - n^a \sqrt[n]{(n!e_n)^b} \right).$$

- **5406:** *Proposed by Cornel Ioan Vălean, Timis, Romania*

  Calculate:

  $$\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} \right),$$

  where $H_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$ denotes the harmonic number.

- **5407:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all triples (a,b,c) of positive reals such that

$$\begin{aligned} a+b+c &= 1, \\ \frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} &= \frac{243}{16}. \end{aligned}$$

- **5408:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate:
$$\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}dx.$$

Solutions

- **5385:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides and integer area has perimeter $P=6^6$. Find the sides of the triangle when the area is minimum.

Solution by Toshihiro Shimizu, Kawasaki, Japan

Let $s=P/2=23328$. Let the sides of the triangle be a,b,c. The square of area of the triangle can be written as $s(s-a)(s-b)(s-c)$. Thus, $(s-a)(s-b)(s-c)$ must be minimized and this value must be twice the square of an integer. Let $\alpha=s-a$, $\beta=s-b$, $\gamma=s-c$ and $T=(s-a)(s-b)(s-c)=\alpha\beta\gamma$. Then, $\alpha+\beta+\gamma=s$ and without loss of generality, we assume $\alpha\geq\beta\geq\gamma>0$. When $(\alpha,\beta,\gamma)=(23276,44,8)$, we have $T=2\cdot 2024^2$. We show that this case is the unique smallest case. In this case it follows that $(a,b,c)=(52,23284,23320)$ and $Area=437,184$.

First, we assume that if $\beta\gamma=t$ for some positive integer t. Then, it follows that $\alpha=s-\beta-\gamma\geq s-t-1$ and

$$T=\alpha\beta\gamma\geq(s-t-1)\cdot t$$

Thus, we need to find the case that $(s-t-1)\cdot t<2\cdot 2024^2$ or $t^2-23327t+2\cdot 2024^2>0$ or $t<356.6$.
Therefore, we only need to consider the case that $\beta\gamma\leq 356$ and the range of γ is $\gamma\leq\lfloor\sqrt{356}\rfloor=18$.
We consider the case $\gamma=1$. The range of β is $1\leq\beta\leq 356$.For case $(\beta,\gamma)=(1,1)$, $\alpha=s-\beta-\gamma=23326$ and $T=23326, T/2=11663$, It's not a square of an integer.
For case $(\beta,\gamma)=(2,1)$, $\alpha=s-\beta-\gamma=23325$ and $T=46650, T/2=23325$, It's not a square of an integer.
For case $(\beta,\gamma)=(3,1)$, $\alpha=s-\beta-\gamma=23324$ and $T=69972, T/2=34986$, It's not a square of an integer.

$$\vdots$$

Editor's interlude : The solution continues on in the above manner, and after 49 pages, with each line similar to the output listed above, the proof by exhaustion ends with the final entries listed as:

$$\vdots$$

We consider the case $\gamma = 17$. The range of β is $17 \leq \beta \leq 20$. For case $(\beta, \gamma) = (17, 17)$, $\alpha = s - \beta - \gamma = 23294$ and $T = 6731966, T/2 = 3365983$, It's not a square of an integer.
For case $(\beta, \gamma) = (18, 17)$, $\alpha = s - \beta - \gamma = 23293$ and $T = 7127658, T/2 = 3563829$, It's not a square of an integer.
For case $(\beta, \gamma) = (19, 17)$, $\alpha = s - \beta - \gamma = 23292$ and $T = 7523316, T/2 = 3761658$, It's not a square of an integer.
For case $(\beta, \gamma) = (20, 17)$, $\alpha = s - \beta - \gamma = 23291$ and $T = 7918940, T/2 = 3959470$, It's not a square of an integer.
We consider the case $\gamma = 18$. The range of β is $18 \leq \beta \leq 19$.For case $(\beta, \gamma) = (18, 18)$, $\alpha = s - \beta - \gamma = 23292$ and $T = 7546608, T/2 = 3773304$, It's not a square of an integer.
For case $(\beta, \gamma) = (19, 18)$, $\alpha = s - \beta - \gamma = 23291$ and $T = 7965522, T/2 = 3982761$, It's not a square of an integer.

Editor again: Each of the complete solutions submitted used Hero's formula on an expression connecting the perimeter of the triangle with its area, and then used a computer in proving that they had the minimal area. But sometimes computers get it wrong. **David Stone and John Hawkins of Southern Georgia University in Statesboro, GA** stated that the area of the triangle with integer length sides of $(1, 23327, 23328)$ is *essentially* zero, which of course they quickly dismissed. They then listed the areas of the following three Heronian triangles each having perimeter $6^6 = 46,656$.

$$\begin{pmatrix} a & b & c & s & s-a & s-b & s-c & area \\ 52 & 23284 & 23320 & 23328 & 23276 & 44 & 8 & 437184 \\ 72 & 23290 & 23294 & 23328 & 23256 & 38 & 343 & 837218 \\ 153 & 23225 & 23278 & 23328 & 23175 & 103 & 50 & 1,668,60 \end{pmatrix}$$

Ed Gray of Highland Beach, FL showed that the Heronian Triangle with side lengths of $\{1928, 21402, 23326\}$ has an area of 1386720, and **Kenneth Korbin,** proposer of the problem, showed that a triangle with side lengths $\{2600, 2073, 23319\}$ has an area of 3,357,936. **Kee-Wai Lau of Hong Kong, China** also showed that the triangle with integer side lengths of $\{52, 23284, 23320\}$ has a perimeter of 6^6 and produces the triangle with the minimal integral area.

- **5386:** *Proposed by Michael Brozinsky, Central Islip, NY.*

Determine whether or not there exit nonzero constants a and b such that the conic whose polar equation is

$$r = \sqrt{\frac{a}{\sin(2\theta) - b\cos(2\theta)}}$$

has a rational eccentricity.

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, the given polar equation can be written in x and y as follows:

$$by^2 + 2xy - bx^2 \quad = \quad a. \tag{1}$$

Noting that (1) has the form $Dx^2 + Exy + Fy^2 = a$, the angle of rotation is found to be

$$\tan(2\theta) \quad = \quad \frac{E}{D-F} = -\frac{1}{b}. \tag{2}$$

With some perseverance and the standard rotation formulas with $x = u\,cos(\theta) - v\,\sin(\theta)$ and $y = u\,\sin(\theta) + v\,\cos(\theta)$, (1) can be written as

$$\big(\sin(2\theta) - b\,\cos(2\theta)\big)u^2 + \big(b\,\cos(2\theta) - \sin(2\theta)\big)v^2 \quad = \quad a. \tag{3}$$

Thus, using (2), $\sin(2\theta) = \dfrac{1}{\sqrt{b^2+1}}$ and $\cos(2\theta) = -\dfrac{b}{\sqrt{b^2+1}}$. (3) can now be simplified and displayed in standard form of a conic as

$$\sqrt{b^2+1}\,u^2 - \sqrt{b^2+1}\,v^2 = a$$

$$\frac{u^2}{\dfrac{a}{\sqrt{b^2+1}}} - \frac{v^2}{\dfrac{a}{\sqrt{b^2+1}}} = 1. \tag{4}$$

If we consider A to be the distance from the center of the hyperbola to a vertex, B to be the distance from the center to an end of the conjugate axis, and C to be the distance from the center to a focus, then from (4), $A^2 = \dfrac{a}{\sqrt{b^2+1}}$, $B^2 = \dfrac{a}{\sqrt{b^2+1}}$, and

$$C^2 \quad = \quad A^2 + B^2 = \frac{2a}{\sqrt{b^2+1}}. \tag{5}$$

Using (5), eccentricity is defined to be $e = \dfrac{C}{A} = \sqrt{2}$. Thus, there do not exist nonzero constants a and b to yield a rational eccentricity.

Editor's comment: This problem appeared before in this column as problem 5304; mea culpa, once again.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray; Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5387:** *Proposed by Arkady Alt, San Jose, CA*

Let $D := \{(x,y) \mid x,y \in R_+,\ x \neq y \text{ and } x^y = y^x\}$.(Obviously $x \neq 1$ and $y \neq 1$).
Find $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$

Solution 1 by Henry Ricardo, New York Math Circle, NY

The power mean inequality gives us

$$M_{-1}(x,y) \;=\; \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;\le\; M_0(x,y) \;=\; \sqrt{xy},$$

so that

$$\sup_{(x,y)\in D}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;\le\; \sup_{(x,y)\in D}\sqrt{xy}.$$

Now it is well known that the general solution of the equation $x^y = y^x$ in the first quadrant is given parametrically by

$$x \;=\; \left(1+\frac{1}{u}\right)^{u}, \quad y \;=\; \left(1+\frac{1}{u}\right)^{u+1}, \; u>0,$$

a form attributed to Christian Goldbach. This gives us

$$x\cdot y \;=\; \left(1+\frac{1}{u}\right)^{u}\cdot\left(1+\frac{1}{u}\right)^{u+1},$$

implying that

$$\sup_{(x,y)\in D}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;=\; \lim_{u\to\infty}\sqrt{xy} \;=\; \sqrt{e\cdot e} \;=\; e.$$

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

It is well-known that for any positive integer n,

$$(x,y) = \left(\left(1+\frac{1}{n}\right)^{n}, \left(1+\frac{1}{n}\right)^{n+1}\right)$$

satisfies the equation $x^y = y^x$ and $x \neq y$. Letting $n\to\infty$, both x and y converges to e.
Thus, the value $((x^{-1}+y^{-1})/2)^{-1}$ also converges to e.
Next, we show that for any real number satisfying $x^y = y^x, x\neq y$, the equation $((x^{-1}+y^{-1})/2)^{-1} \le e$ holds. $x^y = y^x$ is equivalent to $\log x/x = \log y/y$. Since $\log x/x$ is negative and monotone decreasing for $x<1$, and it's positive and monotone increasing for $1\le x\le e$ and also it's positive and monotone decreasing on $e\le x$, it is obvious that $1<x,y$ and without loss of generality , we assume $y<e<x$. We write $x=1/s$, $y=1/t$.
Then, $s<1/e<t$ and $s\log s = t\log t$. The inequality $((x^{-1}+y^{-1})/2)^{-1}\le e$ is equivalent to $1/e \le (s+t)/2$.
Let $f(x) = x\log x$. Then, $f'(x) = 1+\log x, f''(x) = 1/x, f'''(x) = -x^{-2} < 0$ for $x>0$.
Thus, $f'(x)$ is concave and it follows that

$$\frac{f'(z)+f'(\frac{2}{e}-z)}{2} \le f'(\frac{z+\frac{2}{e}-z}{2}) = f'(\frac{1}{e}) = 0$$

for any $z > 0$. Integrating from $z = s$ to $z = 1/e$, we get

$$\frac{f(1/e) - f(s) + f(\frac{2}{e} - s) - f(1/e)}{2} \le 0,$$

or $f(2/e - s) \le f(s) = f(t)$. Since, $f(z)$ is monotone increasing on $1/e \le z$, it follows that $2/e - s \le t$ or $1/e \le (s+t)/2$. Therefore we have shown that $((x^{-1} + y^{-1})/2)^{-1} \le e$ for any $(x, y) \in D$.
Finally we conclude that the supremum value is e.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

In is known that $D \cap \left\{(x,y)\middle| x \ne 1, y \ne 1\right\}$ can be parametrized by
$(0,1) \cup (1, +\infty) \ni t \to \big(x(t), y(t)\ \big) = \left(t^{\frac{1}{t-1}}, t^{\frac{t}{t-1}}\right).$

(Note that $t = \dfrac{y(t)}{x(t)}$ is the slope of the line from $(0,0)$ to $\big(x(t), y(t)\big)$; moreover,

$$y(t)^{x(t)} = \left(t^{\frac{t}{t-1}}\right)^{t^{\frac{1}{t-1}}} = t^{\frac{t}{t-1}\cdot t^{\frac{1}{t-1}}} = t^{\frac{t\cdot t^{\frac{1}{t-1}}}{t-1}} = t^{\frac{t^{1+\frac{1}{t-1}}}{t-1}} = t^{\frac{t^{\frac{t}{t-1}}}{t-1}} = t^{\frac{1}{t-1}\cdot t^{\frac{1}{t-1}}} = \left(t^{\frac{1}{t-1}}\right)^{t^{\frac{t}{t-1}}} = x(t)^{y(t)}.$$

Hence,

$$\left(\frac{x(t)^{-1} + y(t)^{-1}}{2}\right)^{-1} = \frac{2x(t)y(t)}{x(t) + y(t)} = \frac{2t^{\frac{1}{t-1}} \cdot t^{\frac{t}{t-1}}}{t^{\frac{1}{t-1}} + t^{\frac{t}{t-1}}} = \frac{2t^{\frac{1+t}{t-1}}}{t^{\frac{1}{t-1}} \cdot (1+t)} = \frac{2t^{\frac{t}{t-1}}}{t+1}.$$

Let us define $(0,1) \cup (1, \infty) \ni \mu \to f(u) = \dfrac{2u^{\frac{u}{u-1}}}{u+1.}$

Then $f'(u) = \dfrac{2u^{\frac{u}{u-1}}\,(2u - 2 - (u+1)\ln u)}{(u^2-1)^2}$ so $f'(u) > 0$ for $u \in (0,1)$ and $f'(u) < 0$ for $u \in (1, +\infty)$, with implies that f is strictly increasing in $(0,1)$ and strictly decreasing in $(1, +\infty)$, which implies that

$$\begin{aligned}
\sup_{u \in (0,1)\cup(1,+\infty)} f(u) &= \lim_{u\to 1} f(u) = \lim_{n\to 1} \frac{2}{u+1} \cdot \lim_{u \to 1} u^{\frac{u}{u-1}} = \lim_{n\to 1} u^{\frac{u}{u-1}} = e^{\ln \lim\limits_{u\to 1} u^{\frac{u}{u-1}}} \\
&= e^{\lim\limits_{u\to 1} \ln u^{\frac{u}{u-1}}} = e^{\lim\limits_{u\to 1} \frac{u}{u-1}\ln u} = e^{\lim\limits_{u\to 1} \frac{u}{u-1}\left(-\sum\limits_{n=1}^{\infty} \frac{1-u^n}{n}\right)} = e^{\lim\limits_{u\to 1} u \sum\limits_{n=1}^{\infty} \frac{(1-u)^{n-1}}{n}} \\
&= e^{\lim\limits_{u\to 1} u + \sum\limits_{n=2}^{\infty} \frac{u(1-u)^{n-1}}{n}} = e^{1+0} = e.
\end{aligned}$$

Thus, $\displaystyle\sup_{(x,y)\in D} \left(\frac{x^{-1} + y^{-1}}{2}\right)^{-1} = \sup_{t \in (0,1)\cup(1,+\infty)} \left(\frac{x(t)^{-1} + y(t)^{-1}}{2}\right)^{-1} = \sup_{t\in(0,1)\cup(1,+\infty)} f(t) = e.$

Solutions 4 and 5 by Michael Brozinsky, Central Islip, NY

For simplicity, we shall use $\dfrac{2xy}{x+y}$, which equals the given expression.
We shall also use the Lambert function $W(x)$ which is the inverse of $f(x) = x \cdot e^x$ (with the domain of $f(x)$ being $\{-1, \infty\,)$ so that $W(x)$ has domain $\left[-\dfrac{1}{e}, \infty\right)$ and

$$\begin{aligned} W\left(x \cdot e^x\right) &= x \; if \; x \geq -1, \text{and} \\ x &= W(x) \cdot e^{W(x)}, \; if \; x \geq \frac{1}{e} \qquad (*) \end{aligned}$$

From $y^x = x^y$ we have $\dfrac{\ln(x)}{x} = \dfrac{\ln(y)}{y}$ (Δ), and since $F(t) = \dfrac{\ln(t)}{t}$ is one to one and negative on $(0,1)$, one to one and positive on $(1,e)$ and one to one and positive on (e,∞) and since $x \neq y$, we can assume that $1 < y < e$ and $x > e$ so that in particular $\ln(y) > -1$ and from $(*)$, $W\left(-\ln(y) \cdot e^{-\ln(y)}\right) = -\ln(y)$ which we will encounter later when we obtain $(**)$ below.

From $y^x = x^y$ we have by raising both sides to the $\dfrac{1}{xy}$ power that $y^{\frac{1}{y}} = x^{\frac{1}{x}}$. The left hand side can be written as $\left(e^{\ln(y)}\right)^{\frac{1}{y}} = \left(e^{\ln(y)}\right)^{e^{-\ln(y)}} = e^{\ln(y) \cdot e^{-\ln(y)}}$ and so we have $e^{\ln(y) \cdot e^{-\ln(y)}} = x^{\frac{1}{x}}$. If we take natural logs of both sides of this equation and multiply both sides by -1 we have

$$-\ln(y) \cdot e^{-\ln(y)} = \frac{-\ln(x)}{x} \qquad (1).$$

Now $\dfrac{-\ln(x)}{x} > -\dfrac{1}{e}$ (since $\dfrac{\ln(x)}{x}$ has it s maximum of $\dfrac{1}{e}$) when $x = e$ and thus $W\left(-\dfrac{\ln(x)}{x}\right) > -1$ and so $1 + W\left(-\dfrac{\ln(x)}{x}\right) > 0$. (Note $W(u) \geq -1$ with equality only if $u = -\dfrac{1}{e}$).

Taking W of both sides of (1) and using $(*)$ we have from (1) that
$-\ln(y) = W\left(-\dfrac{\ln(x)}{x}\right)$ $\quad(**)$ and so

$$y = \frac{1}{e^{-\ln(y)}} = \frac{1}{e^{W\left(-\frac{\ln(x)}{x}\right)}} \;=\; using(*) \; \frac{W\left(-\frac{\ln(x)}{x}\right)}{-\frac{\ln(x)}{x}} = -\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)$$

The expression whose supremum we wish to find is thus

$$\frac{2xy}{x+y} = \frac{2x\left(-\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)\right)}{x + \left(-\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)\right)} - \frac{2x^2 W\left(-\frac{\ln(x)}{x}\right)}{\ln(x) \cdot \left(x - \frac{xW\left(-\frac{\ln(x)}{x}\right)}{\ln(x)}\right)} \qquad (***)$$

Now differentiating the second equation in $(*)$ shows $W'(x) = \dfrac{1}{e^{W(x)} \cdot (W(x)+1)}$ and so differentiating $(***)$ gives, after simplification

$$\frac{2W\left(-\dfrac{\ln(x)}{x}\right)^2\left(\ln(x) - W\left(-\dfrac{\ln(x)}{x}\right) - 2\right)}{\left(\ln(x) - W\left(-\dfrac{\ln(x)}{x}\right)\right)^2\left(1 + W\left(-\dfrac{\ln(x)}{x}\right)\right)} = -\frac{2\ln(y)^2\left(\ln(x) + \ln(y) - 2\right)}{\left(\ln(x) + \ln(y)\right)^2\left(1 - \ln(y)\right)} \; using \; (**) \;\; (1).$$

Recall $1 - \ln(y) = 1 + W\left(\dfrac{\ln(x)}{x}\right) > 0$. The expression in (1) thus is positive when $\ln(x) + \ln(y) - 2 < 0$ and negative when $\ln(x) + \ln(y) - 2 > 0$. This last expression in $(***)$ increases if $xy < e^2$ and decreases when $xy > e^2$ and thus has maximum of e when $xy = e^2$ and so e is the desired supremum.

Solution 5

For simplicity, we shall use $\dfrac{2xy}{x+y}$, which equals the given expression. From $y^x = x^y$ we have $\dfrac{\ln(x)}{x} = \dfrac{\ln(y)}{y}$ (Δ), and since $F1(t) = \dfrac{\ln(t)}{t}$ is one to one and negative on $(0,1)$, one to one and positive on $(1,e)$ and one to one and positive on (e,∞) and since $x \neq y$, we can assume that $1 < x < e$ and $y > e$

Now since $y \cdot \ln(x) = x \cdot \ln(y)$, we have that $y \cdot \ln(x) - x = x \cdot (\ln(y) - 1) > 0 \quad (*)$. Since $\dfrac{d}{dx}\left(u(x)^{v(x)}\right) = u(x)^{v(x)} \cdot \left(\dfrac{v(x)}{u(x)}u'(x) + \ln\left(u(x)\right) \cdot v'(x)\right)$ we readily have from $y^x = x^y$ by implicit differentiation that $y' = \dfrac{y \cdot \ln(y) - \dfrac{y^2}{x}}{y \cdot \ln(x) - x}$ and since $\dfrac{d}{dx}\left(\dfrac{2xy}{x+y}\right) = \dfrac{2\left(x^2y' + y^2\right)}{(x+y)^2}$ we have by substitution that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2xy}{x+y}\right) &= \frac{2y\left(\ln(y)x^2 + \ln(x)y^2 - 2xy\right)}{\left(y\ln(x) - x\right)(x+y)} \text{ and factoring out } xy \\
&= \frac{2xy^2\left(\dfrac{\ln(y)}{y}x + \dfrac{\ln(x)}{x}y - 2\right)}{(y\ln(x) - x)(x+y)^2}, \text{ and since } x^y = y^x, \\
&= \frac{2xy^2\left(\dfrac{\ln(x^y)}{y} + \dfrac{\ln(y^x)}{x} - 2\right)}{(y\ln(x) - x)(x+y)^2} \\
&= \frac{2xy^2\left(\ln(x) + \ln(y) - 2\right.}{\left(y\ln(x) - x\right)(x+y)^2}.
\end{aligned}$$

The expression is thus positive (recall $y\ln(x) - x > 0$) when $\ln(x) + \ln(y) - 2 < 0$ and negative when $\ln(x) + \ln(y) - 2 > 0$. Thus $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$ increases if $xy < e^2$ and decreases when $xy > e^2$ and so e is the desired supremum.

Editor′s comment: Michael Brozinsky also submitted two more solutions to this problem, each in the spirit of solutions the above.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5388: *Proposed by Jiglău Vasile, Arad, Romania*

Let $ABCD$ be a cyclic quadrilateral, R and r its exradius and inradius respectively, and a, b, c, d its side lengths (where a and c are opposite sides.) Prove that

$$\frac{R^2}{r^2} \geq \frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}.$$

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

Remark: We assume that $ABCD$ is inscribable (and thus $ABCD$ is bicentric) and excircle is circumcircle.

Let the circumcircle and incircle of $ABCD$ be Γ(with center O), Γ'(with center I), respectively. We fix Γ,Γ' and move A such that $ABCD$ has circumcircle Γ and incircle Γ'. The existence of such quadrilateral is assured by the Poncelet's closure theorem (see also https://en.wikipedia.org/wiki/Poncelet%27s_closure_theorem).

If Γ and Γ' are concentric, the quadrilateral is square and we can easy to check that $R = \sqrt{2}r$ and $\frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2} = 2$. Thus the equality holds. We assume that Γ and Γ' are not concentric.

As A vary, we only show the case when (r.h.s), that is $\frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}$, is maximum. The value is maximum when $\frac{ac}{bd}$ is maximum. We calculate the maximum value.

Let P be the intersection of AC and BD. Let W, X, Y, Z be the tangency point of Γ' with AB, BC, CD, DA, respectively.

Then, we show the following lemma. The point P is a fixed point as A varies. Let E be the intersection of AB and CD. Let F be the intersection of BC and DA. Since the quadrilateral $ABCD$ is inscribable, AC, BD, ZX, WY are all concurrent at point P. (it can be shown by Brianchon's theorem and we omit) Then, ZX is the polar line of F with respect to Γ' and WY is the polar line of E with respect to Γ'. Thus, FE is the polar line of P (intersection of ZX and WY) with respect to Γ'. Moreover, E, P is on the polar line of F with respect to Γ and F, P is on the polar line of E with respect to Γ. (This fact is well known and I saw it in my Japanese book.) Therefore, EF and P are polar line and pole with respect to both Γ and Γ'. We will show that this situation only occurs when P is one of the particular two points. More precisely, since EF is polar line of P with respect to both

Γ and Γ', both PO, PO' are perpendicular to EF. Thus, P must be on OO'. We calculate the position of P (see Figure 1) Let $x = IP$ and $d = IO$. From the point P, draw a line perpendicular to OO' and let S, S' be one of the intersection with Γ, Γ', respectively. Let Q be the intersection of tangent line of Γ at S and OO' and Q' be the intersection of tangent line of Γ' at S' and OO'. We find the condition that $Q = Q'$. This situation is equivalent to the above since $\triangle QSO$ and $\triangle SPO$ is similar right triangle,
$OQ = OS \cdot OS/OP = R^2/(x + d)$. Similarly, $IQ' = IS' \cdot IS'/IP = r^2/x$. Thus,

$$\frac{R^2}{x+d} = d + \frac{r^2}{x}$$

must be hold. Since this equation is quadratic equation, there are at most two valid value of x. As A varies continuously, P moves continuously and can't jump to another point. Thus, P must be fixed point as A varies. Therefore, lemma has been shown.

Now we have fixed point P and line EF are fixed as A varies. We show that EI and FI are perpendicular. Since $WY \perp EI$ and $ZX \perp FI$, it suffices to show that $ZX \perp WY$. Since $\angle ZAP = \angle DAC = \angle DBC = \angle PBX$ and $\angle AZP = \angle FZX = \angle FXZ = \angle BXP$, we have $\triangle ZPA \sim \triangle XPB$. Thus, $\angle ZPA = \angle XPB$. Similarly, $\angle APW = \angle DPY$, $\angle WPB = \angle YPC$, $\angle XPC = \angle ZPD$. Since $\angle APW = \angle YPC$ and $\angle XPC = \angle ZPA$,
$\angle ZPA + \angle WPA = 360°/4 = 90°$. Thus, $ZX \perp WY$.

Let $\theta = \angle IEF$, $\angle DEA = 2\alpha$, $\angle DFC = 2\beta$. The distance between I and EF be $p(> r)$, this value is constant as θ vary. Then, since $EI = p/\sin\theta$ and $FI = p/\cos\theta$,
$\sin\alpha = r/EI = (r\sin\theta)/p$ and $\sin\beta = r/FI = (r\cos\theta)/p$. Thus,
$\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2(r^2\sin^2\theta)/p^2$ and $\cos 2\beta = 1 - 2\sin^2\beta = 1 - 2(r^2\cos^2\theta)/p^2$.

Then, from the Law of Sines, it follows that

$$\begin{aligned} a &= AB \\ &= FB \cdot \frac{\sin\angle BFA}{\sin\angle FAB} \\ &= EF \cdot \frac{\sin\angle FEB}{\sin\angle EBF} \cdot \frac{\sin\angle BFA}{\sin\angle FAB} \\ &= EF \cdot \frac{\sin(\theta-\alpha)\sin 2\beta}{\cos(\beta-\alpha)\cos(\alpha+\beta)} \\ c &= CD \\ &= CF \cdot \frac{\sin\angle DFC}{\sin\angle CDF} \\ &= EF \cdot \frac{\sin\angle CEF}{\sin\angle ECF} \cdot \frac{\sin\angle DFC}{\sin\angle CDF} \\ &= EF \cdot \frac{\sin(\theta+\alpha)\sin 2\beta}{\cos(\alpha+\beta)\cos(\beta-\alpha)} \end{aligned}$$

b, d are calculated by replacing θ by $\pi/2 - \theta$ and swapping α and β from a, c respectively.

Then, since both denominators are unchanged under these replacement, we get

$$\begin{aligned}
\frac{ac}{bd} &= \frac{\sin(\theta-\alpha)\sin(\theta+\alpha)}{\sin(\pi/2-\theta-\beta)\sin(\pi/2-\theta+\beta)}\cdot\frac{\sin^2 2\beta}{\sin^2 2\alpha}\\
&= \frac{\cos 2\theta-\cos 2\alpha}{\cos(\pi-2\theta)-\cos 2\beta}\cdot\frac{1-\cos^2 2\beta}{1-\cos^2 2\alpha}\\
&= \frac{\cos 2\theta-\cos 2\alpha}{\cos(\pi-2\theta)-\cos 2\beta}\cdot\frac{1-\cos 2\beta}{1-\cos 2\alpha}\cdot\frac{1+\cos 2\beta}{1+\cos 2\alpha}\\
&= \frac{1-2\sin^2\theta-\left(1-2(r^2\sin^2\theta)/p^2\right)}{1-2\cos^2\theta-\left(1-2(r^2\cos^2\theta)/p^2\right)}\cdot\frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{2-2(r^2\cos^2\theta)/p^2}{2-2(r^2\sin^2\theta)/p^2}\\
&= \frac{p^2-r^2\cos^2\theta}{p^2-r^2\sin^2\theta}.
\end{aligned}$$

Since $\sin^2\theta+\cos^2\theta=1$, this value takes maximum when $\sin\theta=0$ and the maximum value is $\frac{p^2-r^2}{p^2}$. Similarly, the minimal value is $\frac{p^2}{p^2-r^2}$ when $\sin\theta=1$. Therefore, the maximal value of $\frac{a^2c^2}{b^2d^2}+\frac{b^2d^2}{a^2c^2}$ is $\left(\frac{p^2-r^2}{p^2}\right)^2+\left(\frac{p^2}{p^2-r^2}\right)^2$. Thus we only need to show that

$$\frac{R^2}{r^2}\geq\left(\frac{p^2-r^2}{p^2}\right)^2+\left(\frac{p^2}{p^2-r^2}\right)^2$$

Now we derive relation between p,r,R. Let K, L be the intersection of line QI and Γ', where K is closer to Q than L. Let the tangent line at K meet Γ at N and the tangent line at L meet Γ at N'. We can see that Q,N,N' are collinear and this line is a tangent line of Γ (see figure 2). Let the tangency point be M. Then, since $\triangle QKN$ and $\triangle QMI$ are similar, $KN=MI\cdot QK/QM=r(p-r)/\sqrt{p^2-r^2}$. Thus, $KO=\sqrt{R^2-\left(\frac{r(p-r)}{\sqrt{p^2-r^2}}\right)^2}$. Similarly, since $\triangle QLN'$ and $\triangle QMI$ are similar, $LN'=MI\cdot QL/QM=r(p+r)/\sqrt{p^2-r^2}$. Thus, $LO=\sqrt{R^2-\left(\frac{r(p+r)}{\sqrt{p^2-r^2}}\right)^2}$. Therefore, since $2r=KO+LO$, it follows that

$$\begin{aligned}
2r &= \sqrt{R^2-\left(\frac{r(p-r)}{\sqrt{p^2-r^2}}\right)^2}+\sqrt{R^2-\left(\frac{r(p+r)}{\sqrt{p^2-r^2}}\right)^2}\\
2r &= \sqrt{R^2-\frac{p+r}{p-r}\cdot r^2}+\sqrt{R^2-\frac{p-r}{p+r}\cdot r^2}
\end{aligned}$$

Squaring, we get

$$4r^2=2R^2-\left(\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)r^2+2\sqrt{R^2-\frac{p+r}{p-r}\cdot r^2}\sqrt{R^2-\frac{p-r}{p+r}\cdot r^2}$$

$$\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)r^2-2R^2=2\sqrt{R^2-\frac{p+r}{p-r}\cdot r^2}\sqrt{R^2-\frac{p-r}{p+r}\cdot r^2}$$

Squaring again, we get

$$\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)^2 r^4-4R^2r^2\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)+4R^4=4R^4-4R^2r^2\left(\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)+4r^4$$

$$\left(\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)^2-4\right)r^2=16R^2$$

Therefore,

$$\begin{aligned}
\frac{R^2}{r^2}&=\frac{1}{16}\left(\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)^2-4\right)\\
&=\frac{1}{16}\left(14+\left(\frac{p-r}{p+r}\right)^2+\left(\frac{p+r}{p-r}\right)^2+8\left(\frac{p-r}{p+r}+\frac{p+r}{p-r}\right)\right)\\
&=\frac{14\left(p^2-r^2\right)^2+(p-r)^4+(p+r)^4+8\left(p-r\right)\left(p+r\right)\left((p-r)^2+(p+r)^2\right)}{16\left(p-r\right)^2\left(p+r\right)^2}\\
&=\frac{2p^4-p^2r^2}{\left(p^2-r^2\right)^2}
\end{aligned}$$

Therefore we need to show that

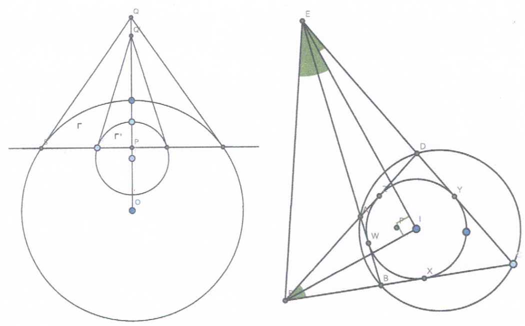
$$\frac{2p^4-p^2r^2}{\left(p^2-r^2\right)^2}\geq\left(\frac{p^2-r^2}{p^2}\right)^2+\left(\frac{p^2}{p^2-r^2}\right)^2$$

It is equivalent to

$$\begin{aligned}
\frac{p^4-p^2r^2}{\left(p^2-r^2\right)^2}&\geq\left(\frac{p^2-r^2}{p^2}\right)^2\\
\frac{p^2}{p^2-r^2}&\geq\left(\frac{p^2-r^2}{p^2}\right)^2\\
p^6&\geq\left(p^2-r^2\right)^3.
\end{aligned}$$

The last inequality is obvious.

Following are the diagrams for Lemma 1.



Solution 2 by Kee-Wai Lau, Hong Kong, China

It is well known that $R = \dfrac{1}{4}\sqrt{\dfrac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}$ and $r = \dfrac{\sqrt{abcd}}{a+c} = \dfrac{\sqrt{abcd}}{b+d}$ for the bicentric quadrilateral $ABCD$. Hence the inequality of the problem is equivalent to

$$(ab+cd)(ac+bd)(ad+bc)((a+c)(b+d) - 16\left(a^4c^4+b^4d^4\right) \geq 0 \qquad (1)$$

By homogeneity, we may assume without loss of generality that

$$c = 1-a \qquad (2)$$

and

$$d = 1-b \qquad (3)$$

It can be checked readily, using (2) and (3), that the left hand side of (1) equals

$$(1+4a\,(1-a))\,a^2\,(a-1)^2\,(2a-1)^2+(1+4b(1-b))\,b^2(b-1)^2(2b-1)^2$$
$$+ab(1-a)(1-b)((2a-1)^2+(2b-1)^2).$$

Since $0<a<1$ amnd $0<b<1$, so the last expression is nonnegative. Thus (1) holds and this completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer

5389: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let ABC be a scalene triangle with semi-perimeter s and area $\mathcal{A}$. Prove that

$$\frac{3a+2s}{a(a-b)(a-c)}+\frac{3b+2s}{b(b-a)(b-c)}+\frac{3c+2s}{c(c-a)(c-b)}<\frac{3\sqrt{3}}{4\mathcal{A}}.$$

Solution 1 by Neculai Stanciu, "George Emil Palade" School Buzău, Romania and Titu Zvonaru, Comaesti, Romania

$$\frac{bc^3-b^3c+a^3c-ac^3+ab^3-a^3b}{abc\,(a-b)(b-c)(c-a)}<\frac{3\sqrt{3}}{4A} \iff \frac{(a-b)(b-c)(c-a)(a+b+c)}{abc(a-b)(b-c)(c-a)}<\frac{3\sqrt{3}}{4A}$$

$$\iff \frac{2s}{4AR}<\frac{3\sqrt{3}}{4A}$$

$$\iff 2s<3R\sqrt{3},$$

which is the well-known Mitrinović's inequality (see, e.g., item 5.3 in Geometric Inequalities by O.Bottema et. al., Groningen, 1969.)

Solution 2 by Ed Gray, Highland Beach, FL

Let the statement of the problem be labeled as (1).

Outline of solution: We will show that the left hand side of (1), l.h.s. (1) =l.h.s. (9)
Statement (12) below is derived from a well known identity.
Statement (13) and onward shows that the l.h.s (9)=l.h.s. (12). So in summary,
l.h.s (1)=l.h.s. (9) =l.h.s.(12) $\leq \dfrac{3\sqrt{3}}{4A}$.

Collecting the terms on the l.h.s.(1) gives us

(2) $\dfrac{bc(c-b)(4a+b+c)+ac(a-c)(a+4b+c)+ab(bb-a)(a+b+4c)}{abc(a-b)(b-c)(c-a)}$ or

(3) $\dfrac{bc(c-b)4a+(bc)(c^2-b^2)+(ac)(a-c)4b+ac(a^2-c^2)+ab(b-a)4c+ab(b^2-a^2)}{abc(a-b)(b-c)(c-a)}$

(4) $\dfrac{4abc(c-b)+4abc(a-c)+4abc)(b-a)+bc(c^2-b^2)+ab(a^2-c^2)+ab(b^2-a^2)}{abc(a-b)(b-c)(c-a)}$ or

(5) $\dfrac{4abc(c-b+a-c+b)+bc(c-b)(c+b)+ac(a-c)(a+c)+ab(b-a)(b+a)}{abc(a-b)(b-c)(c-a)}=$

(6) $\dfrac{bc(c-b)(c+b)+a-c+b)}{abc(a-b)(b-c)(c-a)}+\dfrac{ac(a-c)(a+c)+a-c+b)}{abc(a-b)(b-c)(c-a)}+\dfrac{ab(b-a)(b+a)}{abc(a-b)(b-c)(c-a)}=$

(7) $\dfrac{c+b}{a(a-b)(a-c)}+\dfrac{a+c}{b(c-b)(a-b)}+\dfrac{a+b}{c(c-b)(c-a)}=$

(8) $\dfrac{bc(c^2-b^2)+ac(a^2-c^2)+ab(b^2-a^2)}{abc(a-b)(b-c)(c-a)}$.

Slightly re-arranging (1) becomes

(9) $\dfrac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{bc(a^3)(c-b)+ac(b^3)(a-c)+ab(c^3)(b-a)}<\dfrac{3\sqrt{3}}{4A}$.

A well known identity (GOOGLE) is:

(10) $\dfrac{9abc}{a+b+c}\geq 4A\sqrt{3}$, or inverting

(11) $\dfrac{a+b+c}{9abc}\leq\dfrac{1}{4A\sqrt{3}}=\dfrac{\sqrt{3}}{12A}=\dfrac{3\sqrt{3}}{36A}$

Multiplying by 9

(12) $\dfrac{a+b+c}{abc}\leq\dfrac{3\sqrt{3}}{4A}$

Hence, it is sufficient to show:

(13) $\dfrac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{bc(a^3)(c-b)+ac(b^3)(a-c)+ab(c^3)(b-a)}\leq\dfrac{a+b+c}{abc}$ or

(14)

$a^3(c-b)+b^3(a-c)+c^3(b-a)\leq(a+b+c)\left(\dfrac{bc(a^3)(c-b)+ac(b^3)(a-c)+ab(c^3)(b-a)}{abc}\right)$

or

(15)

$a^3(c-b)+b^3(a-c)+c^3(b-a)\leq(c-b)\left(a^3+a^2b+a^2c\right)+(a-c)\left(ab^2+b^3+cb^2\right)+(b-a)\left(ac^2+bc^2+c^3\right))$

Transposing from left to right

(16)

$$0\leq\ (c-b)(a^2)(b+c)+(a-c)(b^2)(a+c)+(b-a)(c^2)(b+a)$$
$$(a^2)(c^2-b^2)+(b^2)(a^2-c^2)+(c^2)(b^2-a^2)\text{ or}$$

(17)

$$0\leq a^2c^2-a^2b^2+b^2a^2-b^2c^2+b^2c^2-a^2c^2=0.\text{ Q.E.D.}$$

Solution 3 by Moti Levy, Rehovot, Israel

The left hand side of the inequality can be simplified,

$$\frac{3a+2s}{a\,(a-b)\,(a-c)}+\frac{3b+2s}{b\,(b-c)\,(b-a)}+\frac{3c+2s}{c\,(c-a)\,(c-b)}=2sabc.$$

Hence the original inequality is equivalent to

$$\frac{2s}{abc}<\frac{3\sqrt{3}}{4\mathcal{A}} \quad \text{or to} \quad 4\mathcal{A}\sqrt{3}<\frac{9abc}{a+b+c}. \qquad (1)$$

It is well known that in any triangle, $\sin A+\sin B+\sin C\leq\dfrac{3\sqrt{3}}{2}$. Hence

$$\begin{aligned}\sin A+\sin B+\sin C &= \frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\leq\frac{3\sqrt{3}}{2} \quad \text{or}\\ a+b+c &\leq 3R\sqrt{3}. \qquad (2)\end{aligned}$$

It is well known that $R=\dfrac{abc}{4\mathcal{A}}$. Labeling this equation as (3), it follows from (2) and (3) that $a+b+c\leq 3\dfrac{abc}{4\mathcal{A}}\sqrt{3}$, which implies (1).

Remark: Inequality (1) was proposed by T. R. Curry in the "American Mathematical Monthly", Vol. 73 (1966) as elementary problem number 1861.
The solution by Leon Bankoff (who served as the editor of the Problem Department of PME magazine for several years) was selected.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Toshihiro Shimizu, Kawasaki, Japan; Kee-Wai Lau, Hong Kong, China; Nicusor Zlota "Trian Vuia" Technical College, Focsani, Romania, and the proposer.

5390: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A\in\mathcal{M}_2\,(R)$ such that $AA^T=\begin{pmatrix} a & b\\ b & a\end{pmatrix}$, where $a>b\geq 0$. Prove that $AA^T=A^TA$ if and only if $A=\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}$ or $A=\begin{pmatrix}\beta & \alpha\\ \alpha & \beta\end{pmatrix}$, where $\alpha=\dfrac{\pm\sqrt{a+b}\pm\sqrt{a-b}}{2}$ and $\beta=\dfrac{\pm\sqrt{a+b}\mp\sqrt{a-b}}{2}$. Here A^T denotes the *transpose* of A.

Solution 1 by Toshihiro Shimizu of Kawasaki, Japan

Remark: I assume that $a > b > 0$.
Let
$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$
Then, from the $AA^T = A^T A$, it follows that $x^2 + y^2 = x^2 + z^2 = y^2 + w^2 = z^2 + w^2 = a$, $xz + yw = xy + zw = b$. Thus, $y^2 = z^2, x^2 + z^2 = y^2 + w^2$ and $(x - w)(y - z) = 0$. Thus, it follows that $x = w$ or $y = z$ and $y = \pm z$.
If $y \neq z$, $y = -z \neq 0$ and $x = w$ must be satisfied. Then, we can write
$$A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$
Then, $a = x^2 + y^2, b = 0$, a contradiction.
Thus $y = z$, then $x = \pm w$. Since $xz + yw = b > 0$, the plus sign must be occured. Thus, we can write
$$A = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$
and $x^2 + y^2 = a,\ 2xy = b$. Then, $(x + y)^2 = a + b$ implies $x + y = \pm\sqrt{a + b}$. Thus, x, y is a two root of the equation $t^2 \mp \sqrt{a + b}t + b/2 = 0$. Thus,
$\{x, y\} = \left\{ \frac{\pm\sqrt{a+b}+\sqrt{a+b-2b}}{2}, \frac{\pm\sqrt{a+b}-\sqrt{a+b-2b}}{2} \right\}$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

If $a > b = 0$, then the matrix $A_0 = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$ satisfying $A_0 A_0^T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $A_0 A_0^T = A_0^T A_0$, is neither of the form $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ nor of the form $\begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$.
Hence in what follows we always assume that $a > b > 0$.

Let $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ so that $AA^T = \begin{pmatrix} w^2 + x^2 & wy + xz \\ wy + xz & y^2 + z^2 \end{pmatrix}$ and $AA^T = \begin{pmatrix} w^2 + y^2 & wx + yz \\ wx + yz & x^2 + z^2 \end{pmatrix}$.
Hence if $AA^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, then
$$w^2 + x^2 = y^2 + z^2 = a, \qquad (1)$$
and
$$wy + xz = b. \qquad (2)$$
Suppose that $AA^T = A^T A$, then
$$x^2 = y^2, \qquad (3)$$
and
$$wy + xz = wx + yz. \qquad (4)$$
From (4) we obtain $(x - y)(z - w) = 0$. We first suppose that $x = y$. Then by (1), we have $w^2 = z^2$ and by (2) we have $x(w + z) = b$. Since $b > 0$, so $w = z$ and we have

$$\begin{aligned} w^2+x^2 &= a \\ 2wx &= b \qquad (5) \end{aligned}$$

Solving (5), we obtain $(w, x) =$

$$\left(\frac{\sqrt{a+b}+\sqrt{a-b}}{2}, \frac{\sqrt{a+b}-\sqrt{a-b}}{2}\right), \ \left(-\frac{\sqrt{a+b}+\sqrt{a-b}}{2}, \frac{\sqrt{a-b}-\sqrt{a+b}}{2}\right),$$
$$\left(\frac{\sqrt{a+b}-\sqrt{a-b}}{2}, \frac{\sqrt{a+b}+\sqrt{a-b}}{2}\right), \left(\frac{\sqrt{a-b}-\sqrt{a+b}}{2}, -\frac{\sqrt{a+b}+\sqrt{a-b}}{2}\right),$$

with corresponding matrices

$$A_1 = \begin{pmatrix} \dfrac{\sqrt{a+b}+\sqrt{a-b}}{2} & \dfrac{\sqrt{a+b}-\sqrt{a-b}}{2} \\ \\ \dfrac{\sqrt{a+b}-\sqrt{a-b}}{2} & \dfrac{\sqrt{a+b}+\sqrt{a-b}}{2} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\dfrac{\sqrt{a+b}+\sqrt{a-b}}{2} & \dfrac{\sqrt{a-b}-\sqrt{a+b}}{2} \\ \\ \dfrac{\sqrt{a-b}-\sqrt{a+b}}{2}, & -\dfrac{\sqrt{a+b}+\sqrt{a-b}}{2} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \dfrac{\sqrt{a+b}-\sqrt{a-b}}{2} & \dfrac{\sqrt{a+b}+\sqrt{a-b}}{2} \\ \\ \dfrac{\sqrt{a+b}+\sqrt{a+b}}{2}, & \dfrac{\sqrt{a+b}-\sqrt{a-b}}{2} \end{pmatrix}, \text{and}$$

$$A_4 = \begin{pmatrix} \dfrac{\sqrt{a-b}-\sqrt{a-b}}{2} & -\dfrac{\sqrt{a+b}+\sqrt{a-b}}{2} \\ \\ -\dfrac{\sqrt{a+b}+\sqrt{a-b}}{2}, & \dfrac{\sqrt{a-b}-\sqrt{a+b}}{2} \end{pmatrix}.$$

It is easy to check that A_k satisfies $A_k A_k^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and that $\mathrm{A}_k A_k^T = A_k^T A_k$ for $k = 1, 2, 3, 4$.

Next we suppose that $w = z$. Then by (2), we have $w(x+y) = b$. Since $b > 0$, so by (3), we have $x = y$, and we arrive at (5) again. This completes the solution.

Solution 3 by the Proposer

One implication is easy to prove. If $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ or $A = \begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$, with $\alpha = \dfrac{\pm\sqrt{a+b} \pm \sqrt{a-b}}{2}$ and $\beta = \dfrac{\pm\sqrt{a+b} \mp \sqrt{a-b}}{2}$, then

$$AA^T = A^T A = \begin{pmatrix} \alpha^2+\beta^2 & 2\alpha\beta \\ 2\alpha\beta & \alpha^2+\beta^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Now we prove the other implication. First we note, since $\det(AA^T) = \det^2 A = a^2 - b^2 > 0$, that A is invertible. The equation $AA^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ implies that $A^T = A^{-1}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = A^{-1}(aI_2 + bJ)$, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The equation $AA^T = A^T A$ implies that $AA^T = aI_2 + bJ = (aA^{-1} + bA^{-1}J)A = A^T A$, and this in turn implies $bA^{-1}JA = bJ$ and, since $b \neq 0$, we get that $JA = AJ$. Let $A = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$. Since $JA = AJ$ we get that $u = y$ and $v = x$, so $A = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$. We have

$$AA^T = \begin{pmatrix} x^2+y^2 & 2xy \\ 2xy & x^2+y^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

and this implies that $x^2 + y^2 = a$ and $2xy = b$. Since we have a symmetric system it is clear that the values of x and y could be interchanged. Adding and subtracting these equations we get that $(x+y)^2 = a+b$ and $(x-y)^2 = a-b$, and we have $x + y = \pm\sqrt{a+b}$ and $x - y = \pm\sqrt{a-b}$. Thus, $x = \dfrac{\pm\sqrt{a+b} \pm \sqrt{a-b}}{2}$, $y = \dfrac{\pm\sqrt{a+b} \mp \sqrt{a-b}}{2}$ and the problem is solved.

Also solved by Boris Rays, Brooklyn, NY; Dexter Harrell (Undergraduate Student), Auburn University Montgomery, AL; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

Problem 5375* once again

Toshihiro Shimizu of Kawasaki, Japan has the solved 5375*. (We can now remove the asterisk from its label.) Following is a restatement of Kenneth Korbin's problem and Toshihiro's solution to it.

5375 (revised): Prove or disprove the following conjecture. Let k be the product of N different prime numbers each congruent to 1(mod 4).

The total number of different rectangles and trapezoids with integer length sides and diagonals that can be inscribed in a circle with diameter k is exactly $\dfrac{5^N - 3^N}{2}$.

Solution

Let a, b be the length of longer and shorter sides of the trapezoid (or rectangle, in this case let $a = b$), c be the length of other sides and d be the length of the diagonal. Let $\alpha, \beta, \gamma, \delta$ be the central angle with respect to the circumcircle of the segment (side or diagonal) with length a, b, c, d, respectively. We can see that

$$\sin\frac{\alpha}{2} = \frac{a}{k}$$
$$\sin\frac{\beta}{2} = \frac{b}{k}$$
$$\sin\frac{\gamma}{2} = \frac{c}{k}$$
$$\sin\frac{\delta}{2} = \frac{d}{k}.$$

Moreover, $\alpha = \delta - \gamma$, $\beta = \delta + \gamma$. Thus, it follows that

$$\frac{a}{k} = \sin\frac{\alpha}{2} = \sin\left(\frac{\delta}{2} - \frac{\gamma}{2}\right) = \frac{d\sqrt{k^2 - c^2} - c\sqrt{k^2 - d^2}}{k^2}$$
$$\frac{b}{k} = \sin\frac{\beta}{2} = \sin\left(\frac{\delta}{2} + \frac{\gamma}{2}\right) = \frac{d\sqrt{k^2 - c^2} + c\sqrt{k^2 - d^2}}{k^2}.$$

Thus, $k^2 - c^2$ and $k^2 - d^2$ must be perfect square. (♡)Let these perfect squares be c'^2, d'^2, respectively. Then, $ak = dc' - cd'$, $bk = dc' + cd'$. Thus, both dc', cd' must be divisible by k. Since $dc' - cd' > 0$, it must follow that $d > c$.

Conversely, if we are given (c, d) with these condition, we can get a, b and the trapezoid (or rectangle) is determined. Thus, we calculate the number of (c, d).

It follows that

$$k^2 = c^2 + c'^2$$
$$k^2 = d^2 + d'^2$$

Let $k_1 = \gcd(c, k)$ and $k_2 = k/k_1$. Then, $\gcd(c', k) = k_1$ and d is divisible by k_2.

Let k_1 be the product of M prime numbers. We calculate the number of (c, d) with the fixed k_1. Since, the case that $c = d$ is impossible we ignore the condition $d > c$ and divide the result by 2.

The number of c with simply $k_1 \mid c$ is $3^{N-M} - 1$(see Note 2), since the condition is $(k/k_1)^2 = (c/k_1)^2 + (c'/k_1)^2$. But this value over-counts the case that $k_1 p \mid c$, where p is a prime divisor of k but not of k_1. Thus, we need to subtract $3^{N-M-1} - 1$. We also undercounted the case that cpq, where p, q is a prime divisor of k but not k_1, and so on. Thus the number of c is calculated, by Inclusion-exclusion principle, that

$$\begin{aligned}\sum_{t=0}^{N-M} \binom{N-M}{t} \cdot (-1)^t \left(3^{N-M-t} - 1\right) &= (3-1)^{N-M} - (1-1)^{N-M} \\ &= 2^{N-M}\end{aligned}$$

The number of d can be simply calculated as $3^M - 1$. Thus, summing up about M, the total number of $\{c, d\}$ is

$$\begin{aligned}\sum_{M=0}^{N} \binom{N}{M} 2^{N-M} \cdot (3^M - 1) &= (2+3)^N - (2+1)^N \\ &= 5^N - 3^N\end{aligned}$$

Thus, the total number of (c, d) is $\dfrac{5^N - 3^N}{2}$.

Note 1: about (♡): precisely, I think we can show that if $a\sqrt{x} + b\sqrt{y}$ is rational, where $a, b \in Q^+$ and x, y are non-negative integer, then both x, y must be perfect square.

Note 2: From *Jacobi's two square theorem* (http://web.maths.unsw.edu.au/˜mikeh/webpapers/paper21.pdf), the number of integer (x, y) with $k^2 = x^2 + y^2$ is

$$4 \sum_{2 \nmid d | n} (-1)^{\frac{d-1}{2}} = 4 \sum_{d|n} 1 = 4 \cdot 3^N.$$

Among these integer roots, there are four with at least one of them is zero $(\pm k, 0), (0, \pm k)$. Other $4 \cdot 3^N - 4$ of them are classified to $(\pm x, \pm y)$ with $x, y > 0$. Thus, the number of positive integer roots can be written as $3^N - 1$.

$$Mea - Culpa$$

Mistakes happen. **Arkady Alt of San Jose, CA** should have been credited with having solved 5381, and **G. C. Greubel of Newport News, VA** should have been listed for having solved, in two different ways, 5384. I am sorry for these mistakes.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2016

- **5409:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles trapezoid $ABCD$ with $\overline{AB} < \overline{CD}$, and with diagonal $\overline{AC} = \overline{AB} + \overline{CD}$. Find the perimeter of the trapezoid if $\triangle ABC$ has inradius 12 and if $\triangle ACD$ has inradius 35.

- **5410:** *Proposed by Arkady Alt, San Jose, CA*

  For the given integers $a_1, a_2, a_3 \geq 2$ find the largest value of the integer semiperimeter of a triangle with integer side lengths t_1, t_2, t_3 satisfying the inequalities $t_i \leq a_i,\ i = 1, 2, 3$.

- **5411:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be real valued positive sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a \in R_+^*$

  If $\lim\limits_{n\to\infty} (n(a_n - a)) = b \in R$ and $\lim\limits_{n\to\infty} (n(b_n - a)) = c \in R$ compute

  $$\lim_{n\to\infty} \left(a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!} \right).$$

  Note: R_+^* means the positive real numbers without zero.

- **5412:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

  Given positive integer M. Find a continuous, non-constant function $f : R \to R$ such that $f(f(x)) = f([x])$, for all real x, and for which the maximum value of $f(x)$ is M. Note: $[x]$ is the greatest integer function.

- **5413:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Compute

  $$\lim_{n\to\infty} \frac{1}{n} \sum_{1\leq i\leq j\leq n} \frac{1}{\sqrt{(n^2 + (i+j)n + ij)}}.$$

- **5414:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $A, B \in M_2(C)$ be such that $2015AB - 2016BA = 2017I_2$. Prove that
  $$(AB - BA)^2 = O_2.$$
  Here, C is the set of complex numbers.

Solutions

- **5391:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides $(49, b, b+1)$ has integer area. Find two possible values of b.

  Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

  If we let s and A denote the triangle's semi-perimeter and area, respectively, then
  $$s = \frac{49 + b + (b+1)}{2} = b + 25$$
  and Heron's Formula yields
  $$\begin{aligned} A^2 &= s\,(s-49)\,(s-b)\,(s-(b+1)) \\ &= (b+25)\,(b-24)\,(25)\,(24) \\ &= 600\,(b+25)\,(b-24)\,. \end{aligned}$$
  Since A is a positive integer, we must have
  $$(b+25)\,(b-24) = 6k^2 \tag{1}$$
  for some positive integer k. If we expand the left side of (1) and complete the square, we ultimately obtain
  $$(2b+1)^2 - 24k^2 = 2401 = 49^2.$$
  One way to find acceptable values for b, k, and A is to solve the Pell Equation
  $$x^2 - 24y^2 = 1 \tag{2}$$
  and then set $2b + 1 = 49x$, $k = 49y$, and $A = 60k$. Since the solution of (2) with the smallest value of x is $x = 5$, $y = 1$, we get all solutions (x_n, y_n) of (2) by setting
  $$x_n + y_n\sqrt{24} = \left(5 + \sqrt{24}\right)^n.$$
  For each solution, we let $b = \dfrac{49x_n - 1}{2}$, $k = 49y_n$, and $A = 60k$. The first five solutions of (2) and the corresponding values of b, k, and A are listed in the following table.

x	y	b	k	A
5	1	122	49	2,940
49	10	1,200	490	29,400
485	99	11,882	4,851	291,060
4,801	980	117,624	48,020	2,881,200
47,525	9,701	1,164,362	475,349	28,520,940

Editor's note: Some readers extended the above table a bit further. **David Stone and John Hawkins of Georgia Southern University** added the following to the above table.

x	y	b	k	A
47,525	9,701	1,164,362	475,349	28,520,940
		11,526,000	28,232,820	282,328,200
		114,095,642	279,476,106	2,794,761,060
		1,129,430,424	2,766,528,240	27,665,282,400
		11,180,208,602	27,385,806,294	273,858,062,940

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Brian Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Carl Libis, Columbia Southern University, Orange Beach, AL; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Toshihiro Shimizu, Kawasaki Japan; Neculai Stanciu "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

Editor's note: The problem solving column is sometimes used in a problem solving course offered at **Taylor University in Upland, IN,** where the students in the course often work in small groups. Each of the following students at Taylor University should also be credited with having solved 5391.

Group 1: Madison Massot, Julia Noonan and Benjamin Thayer
Group 2: Amish Mishra, Raquel Helton, and Allie Ternet
Group 3: Matt Garringer, Sarah Glett, and Erin Song
Group 4: Caleb Belmont, Caleb Holleman, and Nick Iorio

A comment and a question about 5391 from the proposer, Kenneth Korbin.

It can be seen that there are infinitely many Primitive Heronian Triangles that have a side with length 49.

Question: Is there another positive integer less than 1500 that can also be the length of a side of infinitely many Primitive Heronian Triangles?

- **5392:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Prove that if $x, y, z > 0$, then

$$\frac{4\left(x^2+y^2+z^2\right)}{27\left(xy+yz+zx\right)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\geq\frac{13}{27}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By homogeneity, we assume without loss of generality that $x+y+z=1$.

Let $t = x^2 + y^2 + z^2$ so that $xy + yz + zx = \dfrac{(x+y+z)^2 - (x^2+y^2+z^2)}{2} = \dfrac{1-t}{2}$ and $\dfrac{4(x^2+y^2+z^2)}{27(xy+yz+zx)} = \dfrac{8t}{27(1-t)}$. Since the function $\dfrac{1}{1+6s}$ is convex for $s > 0$, so by Jensen's inequality, we have

$$\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z} = \frac{x}{1+6x}+\frac{y}{1+6y}+\frac{z}{1+6z} \geq \frac{1}{1+6(x^2+y^2+z^2)} = \frac{1}{1+6t}.$$

Hence the left side of the inequality of the problem is greater than or equal to

$$\frac{8t}{27(1-t)} + \frac{1}{1+6t} = \frac{14(3t-1)^2}{27(1-t)(1+6t)} + \frac{13}{27} \geq \frac{13}{27},$$

as desired.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy

By Cauchy–Schwarz reversed, we get

$$\sum_{\text{cyc}} \frac{x}{7x+y+z} = \sum_{\text{cyc}} \frac{x^2}{x(7x+y+z)} \geq \frac{(x+y+z)^2}{7(x^2+y^2+z^2)+2(xy+yz+zx)}.$$

The inequality is implied by

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)} + \frac{x^2+y^2+z^2+2(xy+yz+zx)}{7(x^2+y^2+z^2)+2(xy+yz+zx)} - \frac{13}{27} \geq 0 \qquad (1)$$

Since the l.h.s. of (1) is equal to $756(x^2+y^2+z^2-xy-yz-zx)^2 \geq 0$
the original inequality is proven.

Solution 3 by Nicusor Zlota, "Traian Vuia"Technical College, Focsani, Romania

$$\begin{aligned}
\frac{1}{3} - \sum_{\textit{cyclic}} \frac{x}{7x+y+z} = \sum_{\textit{cyclic}} \left(\frac{1}{9} - \frac{x}{7x+y+z}\right) &= \sum_{\textit{cyclic}} \frac{y+z-2x}{9(7x+y+z)} \\
&= \sum_{\textit{cyclic}} \frac{y-x}{9(7x+y+z)} + \sum_{\textit{cyclic}} \frac{z-x}{9(7x+y+z)} \\
&= \sum \frac{y-x}{9(7x+y+z)} + \sum \frac{x-y}{9(x+7y+z)} \\
&= \sum \frac{(x-y)^2}{3(7x+y+z)(x+7y+z)},
\end{aligned}$$

and

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)} - \frac{4}{27} = \frac{2\sum(x-y)^2}{27(xy+yz+zx)}.$$

So, the inequality becomes

$$\sum \frac{(x-y)^2}{3(7x+y+z)(x+7y+z)} \le \frac{2\sum(x-y)^2}{27(xy+yz+zx)}.$$

To show this it suffices to show that

$$\frac{(x-y)^2}{3(7x+y+z)(x+7y+z)} \le \frac{2(x-y)^2}{27(xy+yz+zx)} \iff 2(7x+y+z)(x+7y+z) \ge 9(xy+yz+zx).$$

I.e.,

$$14x^2+14y^2+z^2+91xy+7yz+7zx \ge 0,$$

which is obviously true.

Equality holds if $x=y=z$.

Solution 4 by Moti Levy, Rehovot, Israel

This inequality deserves brute force attack by Muirhead's inequality.

$$\frac{4\left(x^2+y^2+z^2\right)}{xy+yz+zx}+\frac{27x}{7x+y+z}+\frac{27y}{x+7y+z}+\frac{27z}{x+y+7z}-13 \ge 0.$$

After some tedious manipulations, our inequality is equivalent to:

$$28\sum_{cyc} x^5+164\sum_{cyc} x^4y+164\sum_{cyc} xy^4+728\sum_{cyc} x^3yz \ge 80\sum_{cyc} x^3z^2+80\sum_{cyc} x^2z^3+924\sum_{cyc} x^2y^2z,$$

or to

$$14\sum_{sym} x^5+164\sum_{sym} x^4y+364\sum_{sym} x^3yz \ge 80\sum_{sym} x^3z^2+462\sum_{sym} x^2y^2z.$$

Now we prepare the inequality for application of Muirhead's inequality by splitting some terms in left and right hand sides:

$$14\sum_{sym} x^5+66\sum_{sym} x^4y+98\sum_{sym} x^4y+364\sum_{sym} x^3yz \ge 14\sum_{sym} x^3z^2+66\sum_{sym} x^3z^2+98\sum_{sym} x^2y^2z+364\sum_{sym} x^2y^2z.$$

We use the following majorization relations:

$$\begin{aligned}(5,0,0) &\succ (3,2,0)\,,\\ (4,1,0) &\succ (3,2,0)\,,\\ (4,1,0) &\succ (2,2,1)\,,\\ (3,1,1) &\succ (2,2,1)\end{aligned}$$

to show that

$$\begin{aligned}\sum_{sym} x^5 &\ge \sum_{sym} x^3z^2,\\ \sum_{sym} x^4y &\ge \sum_{sym} x^3z^2,\\ \sum_{sym} x^4y &\ge \sum_{sym} x^2y^2z,\\ \sum_{sym} x^3yz &\ge \sum_{sym} x^2y^2z.\end{aligned}$$

A nice tutorial on the application of Muirhead's inequality can be found at:
https://kheavan.files.wordpress.com/2010/06/muirhead-69859.pdf

Also solved by Arkady Alt, San Jose, CA; Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Toshihiro Shimizu, Kawasaki Japan; Albert Stadler, Herrliberg, Switzerland, and the proposer.

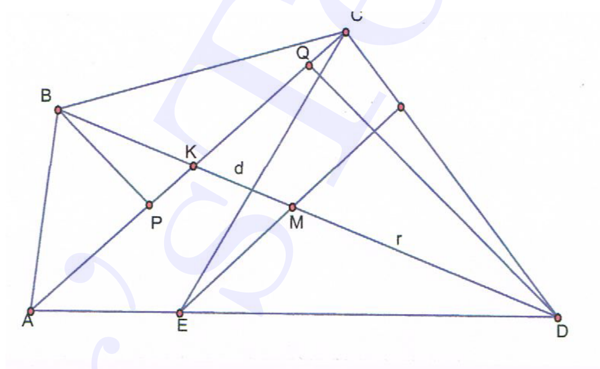
- **5393:** *Proposed by José Luis Díaz-Barrero, Barcelona, Tech, Barcelona, Spain*

Through the midpoint of the diagonal BD in the convex quadrilateral $ABCD$ we draw a straight line parallel to the diagonal AC. This line intersects the side AD at the point E. Show that

$$\frac{1}{[ABC]} + \frac{1}{[AEC]} \geq \frac{4}{[CED]}.$$

Here $[XYZ]$ represents the are of $\triangle XYZ$.

Solution 1 by Arkady Alt, San Jose, CA



We assume that midpoint M of diagonal BD does not coincide with K (the point of intersection of AC and BD) because otherwise $[AEC] = 0$.

Also, w.l.o.g. assume that $KD > BK$.

Let $r = BM = MD$ and $d := KM$. Since $ME \parallel AC$ then $\dfrac{AE}{ED} = \dfrac{KM}{MD} = \dfrac{d}{r}$ and, therefore,
$[AEC] = [ACD] \cdot \dfrac{AE}{AD} = [ACD] \cdot \dfrac{d}{r+d}$ and $[CED] = [ACD] \cdot \dfrac{ED}{AD} = [ACD] \cdot \dfrac{r}{r+d}$.

Let $BP, DQ \perp AC$.Then $\triangle KPB \approx \triangle KQD \implies \dfrac{BP}{DQ} = \dfrac{BK}{DK}$ and since
$BK = r - d, KD = r + d$ we have $\dfrac{[ABC]}{[ADC]} = \dfrac{AC \cdot BP}{AC \cdot DQ} = \dfrac{r-d}{r+d}$.

Thus, $\dfrac{1}{[ABC]} + \dfrac{1}{[AEC]} \geq \dfrac{4}{[CED]} \iff \dfrac{1}{[ABC]} + \dfrac{r+d}{d \cdot [ACD]} \geq \dfrac{4(r+d)}{r \cdot [ACD]} \iff$

$\dfrac{[ACD]}{[ABC]} + \dfrac{r+d}{d} \geq \dfrac{4(r+d)}{r} \iff \dfrac{r+d}{r-d} + \dfrac{r+d}{d} \geq \dfrac{4(r+d)}{r} \iff \dfrac{1}{r-d} + \dfrac{1}{d} \geq \dfrac{4}{r}$
and we have $\dfrac{1}{r-d} + \dfrac{1}{d} - \dfrac{4}{r} = \dfrac{(r-2d)^2}{dr(r-d)} \geq 0$.

Solution 2 by Toshihiro Shimizu, Kawasaki Japan

Let M be the midpoint of BD, F be the intersection of AC and BD and S be the area of the quadrilateral $ABCD$. Let $x = FB/DB$. Then, $[ABC] = xS$, $[ADC] = (1-x)S$ and $DM/DF = 1/(2(1-x))$. Thus, $[AEC] = [ADC]\,|AE/AD| = [ADC]\,|MF/DF| = (1-x)S\,((1/2-x)/(1-x)) = (1/2-x)\,S$. Similarly,
$[CED] = [ADC]\,|ED/AD| = [ADC]\,|MD/DF| = (1-x)S\,|(1/2)/(1-x)| = 1/2S$.
Therefore we need to show that

$$\frac{1}{x} + \frac{1}{\frac{1}{2} - x} \geq \frac{4}{1/2} = 8.$$

From Cauchy-Schwarz's inequality,

$$\left(\frac{1}{x} + \frac{1}{\frac{1}{2} - x}\right)\left(x + \frac{1}{2} - x\right) \geq 2^2$$

.It's equivalent to the desired inequality.

Soltuion 3 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, let $\overline{BD} = 2$. Let
$O = (0,0)$, $A = (x_A, y_A)$, $B = (0,1), C = (x_C, y_C)$, $D(0,-1)$, where $x_A > 0$ and $x_C < 0$.
Suppose that AC and BD interset at $F = (0, f)$. Since $OE || AC$ and E lies on AD, so $f > 0$. Since quadrilateral $ABCD$ is convex, so $f < 1$. Suppose that slope of $AC =$ slope of $OE = m$. We readily obtain $y_A = mx_A + f$, $y_C = mx_C + f$ and
$E = \left(\dfrac{x_A}{1+f}, \dfrac{mx_A}{1+f}\right)$. By the standard formula, we obtain $[ABC] = \dfrac{(1-f)(x_A - x_C)}{2}$.
$[AEC] = \dfrac{f(x_A - x_C)}{2}$ and $[CED] = \dfrac{x_A - x_C}{2}$. Hence the inequality of the problem is

equivalent to $\dfrac{1}{1-f}+\dfrac{1}{f}\geq 4$. But this follows from the fact that

$$\frac{1}{1-f}+\frac{1}{f}-4=\frac{(1-2f)^2}{(1-f)f}\geq 0.$$

This completes the solution.

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Let M be the midpoint of the diagonal BD and let us draw a straight line parallel to the diagonal AC through B.

NB is parallel to AC, triangles ABC and ANC have the same basis AC and the same altitude, so $[ABC]=[ANC]$.

By the Arithmetic Mean -Harmonic Mean Inequality applied to the positive numbers $[ANC]$ and $[AEC]$

$$\frac{1}{[ABC]}+\frac{1}{[AEC]}=\frac{1}{[ANC]}+\frac{1}{[AEC]}\geq\frac{4}{[ANC]+[AEC]}=\frac{4}{[CNE]},$$

with equality iff $[ANC]=[AEC]$, that is, iff $AN=AE$, or equivalently, $NE=2AE$; that $DE=2AE$, or what is the same $AD=3AE$, i.e., $DP=3MP$, where P denotes the intersection of the diagonals AC and BD. Since M is the midpoint of BD, this is equivalent to $DP=3BP$.

ME and $BN$are parallel lines (because they are both parallel to the diagonal AC) and M is the midpoint of BD, so E is the midpoint of DN and, hence triangles CNE and CED have the same area (because they have the same base-lengths $NE=DE$, and the same altitude (distance (C,AD)).

Thus, $\dfrac{1}{[ABC]}+\dfrac{1}{[AEC]}\geq\dfrac{4}{[DEC]}$, and equality occurs iff the diagonal AC divides the diagonal BD in the ratio $3:1$.

Also solved by Boris Rays, Brooklyn, NY; Yaqub Aliyev, Qafqaz University, Baku, Azerbaijan, and the proposer.

5394: *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let a, b and c be positive real numbers such that $ab+bc+ca=3$ and $n>1$. Prove that

$$\sqrt[n]{a+\frac{1}{abc}}+\sqrt[n]{b+\frac{1}{abc}}+\sqrt[n]{c+\frac{1}{abc}}\geq 3\sqrt[n]{2}.$$

Solutions 1 and 2 by Henry Ricardo, New York Math Circle, NY.

We have, using the AM-GM inequality several times,

$$\begin{aligned}\sum_{cyclic}\sqrt[n]{a+\frac{1}{abc}} &\geq 3\left[\left(a+\frac{1}{abc}\right)\left(b+\frac{1}{abc}\right)\left(c+\frac{1}{abc}\right)\right]^{1/3n}\\ &\geq 3\left[2\sqrt{1/bc}\cdot 2\sqrt{1/ac}\cdot 2\sqrt{1/ab}\right]^{1/3n}\\ &= 3\left(\frac{8}{abc}\right)^{1/3n} = \frac{3\sqrt[n]{2}}{(abc)^{1/3n}} \geq 3\sqrt[n]{2}\end{aligned}$$

since $3 = ab + bc + ca \geq 3\sqrt[3]{(abc)^2}$ implies that $1/abc \geq 1$.

Equality holds if and only if $a = b = c = 1$.

Solution 2:

We have, using the AM-GM inequality several times,

$$\begin{aligned}
\sum_{cyclic} \sqrt[n]{a + \frac{1}{abc}} &\geq \sqrt[n]{2} \cdot \sum_{cyclic} \frac{1}{(bc)^{1/2n}} \\
&\geq \sqrt[n]{2}\left(3\sqrt[3]{\left(\frac{1}{bc}\right)^{1/2n}\left(\frac{1}{ca}\right)^{1/2n}\left(\frac{1}{ab}\right)^{1/2n}}\right) \\
&= 3\sqrt[n]{2}\left(\frac{1}{(abc)^{1/3n}}\right) \geq 3\sqrt[n]{2}
\end{aligned}$$

since $3 = ab + bc + ca \geq 3\sqrt[3]{(abc)^2}$ implies that $1/abc \geq 1$.

Equality holds if and only if $a = b = c = 1$.

Solution 3 by Nikos Kalapodis, Patras, Greece

By the AM-GM inequality we have $3 = ab+bc+ca \geq 3\sqrt[3]{(abc)^2}$. It follows that $\frac{1}{abc} \geq 1$.
Using again AM-GM inequality properly, we have

$$\sqrt[n]{a + \frac{1}{abc}} + \sqrt[n]{b + \frac{1}{abc}} + \sqrt[n]{c + \frac{1}{abc}} \geq \sqrt[n]{\frac{2}{\sqrt{bc}}} + \sqrt[n]{\frac{2}{\sqrt{ca}}} + \sqrt[n]{\frac{2}{\sqrt{ab}}} \geq 3\sqrt[3]{\sqrt[n]{\frac{2^3}{abc}}} =$$
$$3\sqrt[3n]{\frac{2^3}{abc}} \geq 3\sqrt[3n]{2^3} = 3\sqrt[n]{2}.$$

Solution 4 by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru Comănesti, Romania

By the AM-GM inerquality we have

$$\sqrt[2n]{bc} = \sqrt[2n]{1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1(bc)} \leq \frac{2n-1+bc}{2n},$$ and the other two inequalities analogously.

Hence, by the AM-GM inequality and Bergström's inequality we obtain

$$\begin{aligned}
\sqrt[n]{a + \frac{1}{abc}} + \sqrt[n]{b + \frac{1}{abc}}\sqrt[n]{c + \frac{1}{abc}} &\geq \sqrt[n]{2}\left(\frac{1}{\sqrt[2n]{bc}} + \frac{1}{\sqrt[2n]{ca}} + \frac{1}{\sqrt[2n]{ab}}\right) \\
&\overset{Bergstr\ddot{o}m}{\geq} \sqrt[n]{2} \cdot \frac{9}{\sqrt[2n]{ab} + \sqrt[2n]{bc} + \sqrt[2n]{ca}} \\
&\geq \sqrt[n]{2} \cdot \frac{9}{\frac{2n-1+ab+2n-1+bc+2n-1+ca}{2n}}
\end{aligned}$$

$$= \qquad \sqrt[n]{2} \cdot \frac{18n}{6n} = 3\sqrt[n]{2}, \text{ and we are done.}$$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Toshihiro Shimizu, Kawasaki Japan; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5395: *Proposed by Mohsen Soltanifar (Ph.D. student), Biostatistics Division, Dalla Lana School of Public Health, University of Toronto, Canada.*

Given the sequence $\{\sigma_n^2\}_{n=1}^{\infty}$ of positive numbers $X_1 \sim N(\mu, \sigma_1^2)$. Define recursively a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ via

$$X_{n+1}|X_n \sim N\left(X_n, \sigma_{n+1}^2\right) \qquad n = 1, 2, 3, \ldots$$

Calculate the limit distribution X of $\{X_n\}_{n=1}^{\infty}$.

Reference: Rosenthal, J.S. (2007). A First Look at Rigorous Probability (2nd edition), World Scientific, p. 139.

Proposer's note concerning the problem:

This is a Bayesian Hierarchical Model of Human Heights from Adam & Eve to the end of time.

Consider a family with its children. We know that height has a normal distribution. We also know that height of children is due to genetic factors which are dependent on the height of their parents, but usually this distribution has the same mean as the mean height of their parents but may vary (some children are taller, some shorter, some are average- versus their parents). So, the height of children may be modeled as the normal distribution conditioned to the height of their parents with same mean but potentially different variance.

The first term in the sequence is the distribution of height of Adam & Eve. The second term is the conditional distribution of their children's height. This goes till the end of time consecutively when, according to some beliefs, the Messiah returns. Accordingly, the Messiah will return and a generation of humans will observe this return . But we do not know when this will occur. So, we may assume the Messiah will return as time approaches infinity, and that the distribution of the height of generations of humans that observe the return is "X". We are interested in knowing certain features of this distribution.

This problem is a mathematical modeling of the above belief.

Solutions 1 and 2 by Moti Levy, Rehovot, Israel

The random variables $\{X_n\}_{n=1}^{\infty}$ have normal distribution $X_n \sim \mathcal{N}\left(\mu_n, s_n^2\right)$.

The probability density function of X_n is $f_{X_n}(t) = \frac{1}{\sqrt{2\pi}s_n} e^{-\frac{(t-\mu_n)^2}{2s_n^2}}$.The probability density function of $X_{n+1}|X_n$ is $f_{X_{n+1}|X_n}(t) = \frac{1}{\sqrt{2\pi}\sigma_{n+1}} e^{-\frac{(t-X_n)^2}{2\sigma_{n+1}^2}}$.
The probability density function of X_{n+1} is given by the following integral

$$\begin{aligned}
f_{X_{n+1}}(t) &= \int_{-\infty}^{\infty} f_{X_{n+1}|X_n}(\xi) f_{X_n}(\xi)\, d\xi = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}\sigma_{n+1}} e^{-\frac{(t-\xi)^2}{2\sigma_{n+1}^2}} \right) \left(\frac{1}{\sqrt{2\pi}s_n} e^{-\frac{(\xi-\mu_n)^2}{2s_n^2}} \right) d\xi \\
&= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_{n+1}^2 + s_n^2}} e^{-\frac{(t-\mu_n)^2}{2\left(\sigma_{n+1}^2 + s_n^2\right)}} = \mathcal{N}\left(\mu_n, \sigma_{n+1}^2 + s_n^2\right).
\end{aligned}$$

By induction argument, the mean of $X_{n+1} = \mu$ and the variance of $X_{n+1} = \sum_{k=1}^{n+1} \sigma_k^2$.
The limit distribution of $\{X_n\}_{n=1}^{\infty}$ is $\mathcal{N}\left(\mu, \sum_{k=1}^{\infty} \sigma_k^2\right)$.

Solution 2 Remark: Alternative method for finding the probability density function of X_{n+1} is by employing characteristic functions defined as follows:

$$\varphi_X(t) := E\left[e^{itX}\right].$$

The characteristic function of normal random variable $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is
$\varphi_X(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$.

$$\begin{aligned}
\varphi_{X_{n+1}}(t) &= E\left[E\left[e^{itX_{n+1}}|X_n\right]\right] = E\left[e^{iX_n t - \frac{1}{2}\sigma_{n+1}^2 t^2}\right] \\
&= e^{-\frac{1}{2}\sigma_{n+1}^2 t^2} E\left[e^{iX_n t}\right] = e^{-\frac{1}{2}\sigma_{n+1}^2 t^2} \varphi_{X_n}(t) \\
&= e^{-\frac{1}{2}\sigma_{n+1}^2 t^2} e^{i\mu_n t - \frac{1}{2}s_n^2 t^2} = e^{i\mu_n t - \frac{1}{2}\left(\sigma_{n+1}^2 + s_n^2\right)t^2}
\end{aligned}$$

It follows that $X_{n+1} \sim \mathcal{N}\left(\mu_n, \sigma_{n+1}^2 + s_n^2\right)$.

Solution 3 by Toshihiro Shimizu, Kawasaki Japan

Let $f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ be the density function of the normal distribution. Let $g_n(x)$ be the density function of the distribution X_n. We show that $g_n(x) = f\left(x \mid \mu, \sum_{i=1}^{n} \sigma_i^2\right)$. $n = 1$ is obvious. We asume that the statement is true for some n. Then,

$$\begin{aligned}
g_{n+1}(x) &= \int_{-\infty}^{\infty} f(x \mid t, \sigma_{n+1}^2) \cdot g_n(t) dt \\
&= \int_{-\infty}^{\infty} f(x - t \mid 0, \sigma_{n+1}^2) \cdot f\left(t - \mu \mid 0, \sum_{i=1}^{n} \sigma_i^2\right) dt \\
&= f\left(x - \mu \mid 0, \sum_{i=1}^{n+1} \sigma_i^2\right) \\
&= f\left(x \mid \mu, \sum_{i=1}^{n+1} \sigma_i^2\right).
\end{aligned}$$

Thus, the statement is true for $n + 1$.

If the value $\sum_{i=1}^{\infty} \sigma_i^2$ is bounded, it converges to some value σ^2. Then, the limit distribution X is $N(\mu, \sigma^2)$. If the value $\sum_{i=1}^{\infty} \sigma_i^2$ is not bounded. There is no limit distribution.

Also solved by the proposer.

5396: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all continuous functions $f: \Re \to \Re$ such that

$$f(-x) = x + \int_0^x e^{-t} f(x-t)dt, \quad \forall x \in \Re.$$

Solution 1 by Michael Brozinsky, Central Islip,NY

If we rewrite the given equation as $f(-x) = x + e^{-x}\left(\int_0^x e^{x-t} f(t)dt\right)$ and use the change of variable $u = x - t$, the given equation can be written as

$f(-x) = x + e^{-x}\left(\int_0^x e^u f(u)du\right)$. (1)

Note that the right hand side is a differentiable function of x follows from the fundamental theorem of integral calculus and thus so is $f(x)$ or equivalently,

$(f(-x) - x) \cdot e^x \left(\int_0^x e^u f(u)du\right)$. (2)

If we differentiate both sides of (2) with respect to x and use the chain rule and the fundamental theorem of integral calculus we obtain

$(f(-x) - x) \cdot e^x + e^x \cdot (-f'(-x) - 1) = e^x \cdot f(x)$ and so dividing by e^x gives

$(f(-x) - x) + (f'(-x) - 1) = f(x)$ which can be written as

$f'(x) = f(-x) - x - 1 - f(x)$ (3)

The right hand side of (3) is differentiable and using (3) (or its equivalent form in which all $x's$ are replaced by $-x$) we have by differentiation

$$\begin{aligned} f''(-x) \cdot (-1) &= f'(-x) \cdot (-1) - 1 - f'(x) \\ &= (f(-x) - x - 1 - f(x)) \cdot (-1) - 1 - (f(x) + x - 1 - f(-x)) \\ &= 1. \qquad (4) \end{aligned}$$

If we subtract two times equation (3) from equation (4) we obtain

$$\begin{aligned} f''(-x) \cdot (-1) - 2 \cdot f'(-x) &= 3 - 2 \cdot f(-x) + 2x + 2 \cdot f(x) \\ &\overset{using(3)}{=} 3 - 2 \cdot f(-x) + 2x + 2 \cdot (f(-x) - x - 1 - f'(-x)) \\ &= -2 \cdot f'(-x) + 1, \end{aligned}$$

so that $f(x)$ satisfies the differential equation.

$f''(x) = -1$ where the initial conditions $f(0) = 0$ and $f'(0) = 1$ and $f''(0) = -1$ follow from the given equation and from (3) and (4) respectively.

Hence, $f(x) = a + bx + cx^2$ and the initial conditions readily give $a = 0,\ b = -1$ and $c = -\dfrac{1}{2}$, so that $f(x) = -x - \dfrac{1}{2}x^2$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We perform a change of variables and get

$$f(-x) = x + \int_0^x e^{-t} f(x-t)dt = x + e^{-x}\int_0^x d^t f(t)dt. \qquad (1)$$

The right hand side is a differentiable function, since f is continuous . The f is differentiable. Suppose f is n-times differentiable. Then $x + e^{-x}\displaystyle\int_0^x e^t f(t)dt$ is differentiable $n+1$ times, and so f is $n+1$ times differentiable. So f is differentiable infinitely many times. Thus,

$$\begin{aligned}
(f(x)+x)e^{-x} &= \int_0^{-x} e^t f(t)dt,\\
\frac{d}{dx}\left((f(x)+x)e^{-x}\right) &= -e^{-x}f(-x) = -e^{-x}\left(x + e^{-x}\int_0^x e^t f(t)dt\right),\\
-f(x)e^{-x} + f'(x)e^{-x} + e^{-x} - xe^{-x} &= -xe^{-x} - e^{-2x}\int_0^x e^t f(t)dt,\\
-f(x)e^{x} + f'(x)e^{x} + e^{x} &= -\int_0^x e^t f(t)dt, \qquad (2)\\
\frac{d}{dx}\left(-f(x)e^x + f'(x)e^x\right) + e^x &= {}^-e^x f(x),\\
-f(x)e^x + f''(x) + e^x &= -f(x)e^x,\\
f''(x) &= -1
\end{aligned}$$

We deduce from (1) and (2) that $f(0) = 0$ and $f'(0) = 1$. So

$$f(x) = -\frac{1}{2}x^2 - x = -\frac{1}{2}x(x+2).$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Toshihiro Shimizu, Kawasaki Japan, and the proposer.

Late Solutions Received

The name of Paul M. Harms of North Newton, KS should be added to the list of those who solved 5390. He also noted in his solution that the case of $b = 0$, which was allowed in the statement of the problem, led to a counter example of the statement, and like the others who solved this problem, he showed that the problem was only true for $b > 0$.

Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2017

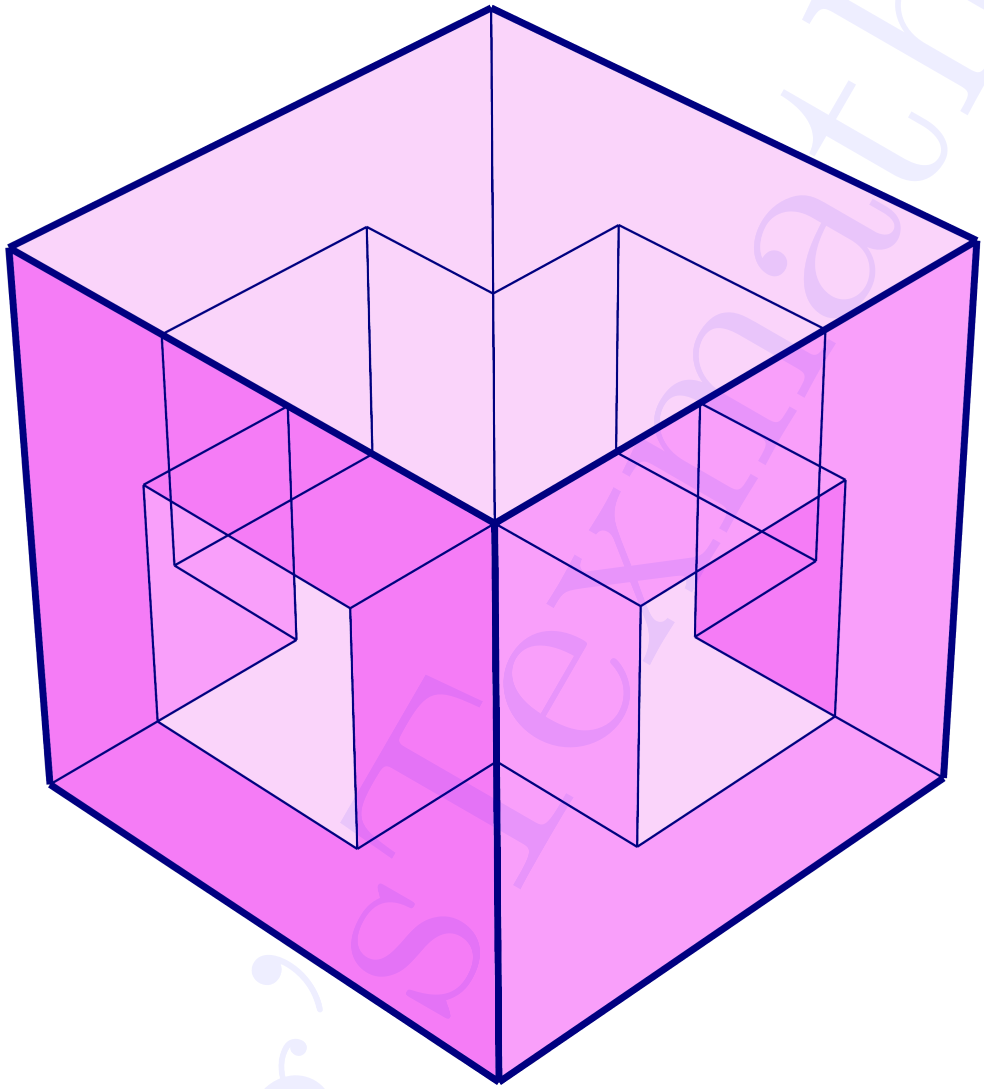
- **5415:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with inradius r and with cevian $\overline{CD}$. Triangle ACD has inradius x and triangle BCD has inradius y, where x, y and r are positive integers with $(x, y, r) = 1$.
  Part 1: Find x, y, and r if $x + y - r = 100$

  Part 2: Find x, y, and r if $x + y - r = 101$.

- **5416:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Two congruent intersecting holes, each with a square cross-section were drilled through a cube. Each of the holes goes through the opposite faces of the cube. Moreover, the edges of each hole are parallel to the appropriate edges of the original cube, and the center of each hole is at the center of the original cube. Letting the length of the original cube be a, find the length of the square cross-section of each hole that will yield the largest surface area of the solid with two intersecting holes. What is the largest surface area of the solid with two intersecting holes?



- **5417:** *Proposed by Arkady Alt, San Jose, CA*

  Prove that for any positive real number x, and for any natural number $n \geq 2$,

$$\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\cdots+x^{n-1}}{n}}.$$

- **5418:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Let ABC be an acute triangle with circumradius R and inradius r. If $m \geq 0$, then prove that
$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

- **5419:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that
$$\prod_{k=1}^{n} \left(\sum_{k=1}^{n} a_k^{t_k} \right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}} \right)^n$$
  where for all $k \geq 1$, t_k is the k^{th} tetrahedral number defined by $t_k = \dfrac{k(k+1)(k+2)}{6}$.

- **5420:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$. Calculate
$$\lim_{n\to\infty} \frac{1}{n} \left(I_2 + \frac{A^n}{n} \right)^n.$$

Solutions

- **5397:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation $\sqrt[3]{x+9} = \sqrt{3} + \sqrt[3]{x-9}$ with $x > 9$.

  Solution 1 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

  Cube both sides of the given equation and rearrange to obtain
$$(x-9)^{2/3} + \sqrt{3}\,(x-9)^{1/3} + \left(1 - 2\sqrt{3}\right) = 0.$$
  This is a quadratic equation with respect to $u = \sqrt[3]{x-9}$ with solutions
$$u = \frac{-\sqrt{3} \pm \sqrt{8\sqrt{3}-1}}{2}.$$
  When $x > 9$, we have $u > 0$ and
$$x = 9 + \left(\frac{-\sqrt{3} + \sqrt{8\sqrt{3}-1}}{2} \right)^3$$

$$
\begin{aligned}
&= \left(1+\sqrt{3}\right)\left(8\sqrt{3}-1\right)^{1/2}\\
&= \sqrt{44+30\sqrt{3}}.
\end{aligned}
$$

Solution 2 by Brain D. Beasely, Presbyterian College, Clinton, SC

Rewriting the given equation and cubing both sides yields

$$(x+9)-3\sqrt[3]{(x+9)^2(x-9)}+3\sqrt[3]{(x+9)(x-9)^2}-(x-9)=3\sqrt{3},$$

or $3\sqrt[3]{x^2-81}(\sqrt[3]{x-9}-\sqrt[3]{x+9})=3\sqrt{3}-18$. Then $-3\sqrt{3}\sqrt[3]{x^2-81}=3\sqrt{3}-18$, so cubing once more produces

$$-81\sqrt{3}(x^2-81)=2997\sqrt{3}-7290.$$

Hence $x^2=30\sqrt{3}+44$, so requiring $x>9$ yields $x=\sqrt{30\sqrt{3}+44}\approx 9.795995$.

Solution 3 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well known that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$. (1)

From the equation we have $\sqrt[3]{x+9}-\sqrt{3}-\sqrt[3]{x-9}=0$, and with the help of (1) we get

$$
\begin{aligned}
x+9-3\sqrt{3}-(x-9) &= 3\sqrt{3}\cdot\sqrt[3]{x^2-81}\\
18-3\sqrt{3} &= 3\sqrt{3}\cdot\sqrt[3]{x^{-}81}, \text{ and dividing both sides by } 3\sqrt{3}, \text{gives}\\
2\sqrt{3}-1 &= \sqrt[3]{x^2-81}. \qquad (2)
\end{aligned}
$$

From (2) we have $(2\sqrt{3}-1)^3=x^2-81$, which yields $x=\pm\sqrt{81+(2\sqrt{3}-1)^3}$ and since $x>9$, the only solution is $x=\sqrt{81+(2\sqrt{3}-1)^3}=\sqrt{30\sqrt{3}+44}$.

Solution 4 by Kee-Wai Lau, Hong Kong, China

By the substitution $x=y^3+9$, we obtain $\sqrt[3]{y^3+18}=\sqrt{3}+y$. Cubing both sides and simplifying, we have $\mathrm{y}^2+\sqrt{3}y+1-2\sqrt{3}=0$, so that the only positive solution is $y=\dfrac{\sqrt{8\sqrt{3}-1}-\sqrt{3}}{2}$. Hence the solution to the equation of the problem is $x=(1+\sqrt{3})\left(\sqrt{8\sqrt{3}-1}\right)=9.79\cdots$.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Karl Havlak, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, State University of New York at Oneonta, Oneonta, NY; Boris Rays, Brooklyn, NY; Toshihiro Shimizu, Kawaskaki, Japan; Albert, Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota "Traian Vuia Technical College, Focsani, Romania and the proposer.

Students from Taylor University in Upland, IN.
Group 1: Ben Byrd, Maddi Guillaume, and Makayla Schultz (jointly)
Group 2: Rebekah Couch, Hannah Keyser, and Nolan Willoughby (jointly)
Group 3: Michelle Franch, Caleb Knuth, and Savannah Porter (jointly)
Group 4: Lauren Moreland, Anna Souzis, and Boni Hernandez (jointly).

- **5398:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $(2n-1)!! = 1\cdot 3\cdot 5\ldots(2n-1)$, then evaluate

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{n!(2n-1)!!}}{n}\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

By Stirling's asymptotic formula,

$$n! = \left(\sqrt{2\pi n}\right) n^n e^{-n+O\left(\frac{1}{n}\right)}, \text{ as } n\to\infty.$$

So

$$\begin{aligned}
\frac{\sqrt[n]{n!\,(2n-1)!!}}{n} = \frac{1}{n}\sqrt[n]{\frac{(2n)!}{2^n n!}} = \frac{1}{2n}\sqrt[n]{(2n)!} &= \frac{1}{2n}\sqrt[2n]{4\pi n}\,(2n)^2\,e^{-2+O\left(\frac{1}{n^2}\right)}\\
&= \frac{2n}{e^2}e^{\frac{\ln(4\pi n)}{2n}+O\left(\frac{1}{n^2}\right)}\\
&= \frac{2n}{e^2}\left(1+\frac{(\ln 4\pi n)}{2n}+O\left(\frac{\ln^2 n}{n^2}\right)\right)\\
&= \frac{2n}{e^2}+\frac{\ln(4\pi n)}{e^2}+O\left(\frac{\ln^2 n}{n}\right).
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{(n!)!(2n-1)!!}}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{2(n+1)}{e^2}+\frac{\ln(4\pi(n+1))}{e^2}-\frac{2n}{e^2}-\frac{\ln(4\pi n)}{e^2}+O\left(\frac{\ln^2 n}{n}\right)\right)\\
&= \frac{2}{e^2}+\lim_{n\to\infty}\left(\frac{\ln\left(\frac{n+1}{n}\right)}{e^2}+O\left(\frac{\ln^2 n}{n}\right)\right)=\frac{2}{e^2}.
\end{aligned}$$

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

For each positive integer n, we let

$$a_n = \frac{\sqrt[n]{n!(2n-1)!!}}{n} = \frac{1}{n}\sqrt[n]{\frac{n!(2n)!}{2^n \cdot n!}} = \frac{1}{2n}\sqrt[n]{(2n)!}.$$

Next, we apply a version of Stirling's formula due to Robbins [1], namely $n! = \sqrt{2\pi n}(n/e)^n e^{r_n}$, where $1/(12n+1) < r_n < 1/(12n)$. This yields

$$a_n = \frac{(4\pi n)^{1/(2n)}(2n/e)^2 e^{r_{2n}/n}}{2n} = \frac{2n}{e^2}\left(e^{r_{2n}}\sqrt{4\pi n}\right)^{1/n}.$$

Hence

$$\begin{aligned} a_{n+1} - a_n &= \frac{2n+2}{e^2}\left(e^{r_{2n+2}}\sqrt{4\pi n + 4\pi}\right)^{1/(n+1)} - \frac{2n}{e^2}\left(e^{r_{2n}}\sqrt{4\pi n}\right)^{1/n} \\ &= \frac{2n}{e^2}\left[\left(e^{r_{2n+2}}\sqrt{4\pi n + 4\pi}\right)^{1/(n+1)} - \left(e^{r_{2n}}\sqrt{4\pi n}\right)^{1/n}\right] + \frac{2}{e^2}\left(e^{r_{2n+2}}\sqrt{4\pi n + 4\pi}\right)^{1/(n+1)}, \end{aligned}$$

so $\lim_{n\to\infty}(a_{n+1} - a_n) = 0 + \frac{2}{e^2} = \frac{2}{e^2}$.

[1] H. Robbins, A remark on Stirling's formula, *The American Mathematical Monthly* 62(1), Jan. 1955, 26-29.

Solution 3 by Adnan Ali (student), A.E.C.S-4, Mumbai, India

Lemma. [1] *If the positive sequence* (p_n) *is such that*

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = p > 0,$$

then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \frac{p}{e}.$$

Taking $p_n = \dfrac{n!(2n-1)!!}{n^n}$, we have

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = \lim_{n\to\infty}\frac{n^{n-1}(2n+1)}{(n+1)^n} = \lim_{n\to\infty}\frac{2n+1}{n+1}\left(1 - \frac{1}{n+1}\right)^{n-1} = 2e^{-1},$$

and so from our Lemma, the required limit evaluates to $2/e^2$.

REFERENCES

[1] Gh. Toader, Lalescu sequences, Publikacije-Elektrotehnickog Fakulteta Univerzitet U Beogradu Serija Matematika, 9 (1998), 1928.

Editor's comment : The authors of this problem, **D. M. Bătinetu-Giurgiu,** and **Neculai Stanciu** proved in their solution the following generalization:

If $t \in R_+^*$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ are positive real sequences such that $\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a \in R_+^*$ and $\lim_{n\to\infty}\frac{b_{n+1}}{n^t b_n} = b \in R_+^*$ then

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_nb_n}}{n^t}\right)=\frac{ab}{e^{t+1}}.$$

Letting $t=1$, $a_n=n!$ and $b_n=(2n-1)!!$, then

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=\lim_{n\to\infty}\frac{(n+1)!}{n\cdot n!}=\lim_{n\to\infty}\frac{n+1}{n}=1 \text{ and } \lim_{n\to\infty}\frac{(2n+1)!!}{n\cdot(2n-1)!!}=\lim_{n\to\infty}\frac{2n+1}{n}=2,$$

I.e., $a=1$ and $b=2$. So

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{n!(2n-1)!!}}{n}\right)=\frac{ab}{e^{t+1}}=\frac{1\cdot 2}{e^{1+1}}=\frac{2}{e^2}.$$

Also solved by Arkady Alt, San Jose, CA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawaskaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

- **5399:** *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let a,b,c be positive real numbers. Prove that

$$\sum_{cyclic}\frac{2a+2b}{\sqrt{6a^2+4ab+6b^2}}\leq 3.$$

Solution by Ed Gray, Highland Beach, FL

By symmetry it is sufficient to show that when x and y are positive, real numbers then

$$f(x,y)=\frac{2x+2y}{\sqrt{6x^2+4xy+6y^2}}\leq 1.$$

Squaring both sides, is

$$(2x+2y)^2 \leq 6x^2+4xy+6y^2? \text{ Or equivalently, is}$$

$$0 \leq 2x^2-4xy+2y^2=(2)(x-y)^2? \text{ But this is obviously true.}$$

Therefore the statement of the problem is true.

Editor's comment : **D.M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania** with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania** generalized the problem as follows:

If $a,b,c,m,n,p\in R_+^*$, then $\displaystyle\sum_{cyclic}\frac{m(a+b)}{\sqrt{(n+2p)a^2+2nab+(n+2p)b^2}}\leq\frac{3m}{\sqrt{n+p}}$.

After proving the generalization, they let $m=n=p=2$, obtaining the statement of the problem.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Nikos Kalapodis, Patras, Greece; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawaskaki, Japan; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota "Traian Vuia" Technical College, Focansi,Romania, and the proposer

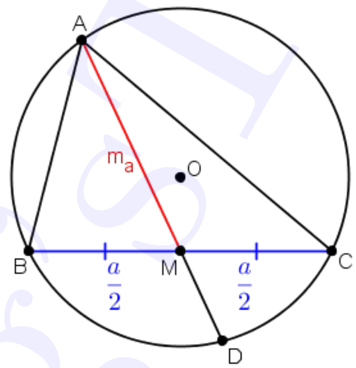
5400: *Proposed by Arkady Alt, San Jose, CA*

Prove the inequality

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \leq 12\,(2R - 3r),$$

where a, b, c and m_a, m_b, m_c are respectively sides and medians of $\triangle ABC$, with circumradius R and inradius r.

Solution 1 by Nikos Kalapodis, Patras, Greece



Let the median $AM = m_a$ intersects the circumcircle of triangle ABC at D.

Then by the intersecting chords theorem we have
$AM \cdot MD = MB \cdot MC$ or $AM \cdot (AD - AM) = MB \cdot MC$.

It follows that $m_a \cdot AD - m_a^2 = \dfrac{a^2}{4}$ i.e. $\dfrac{a^2}{m_a} = 4AD - 4m_a$.

By the obvious inequality $AD \leq 2R$ we obtain that $\dfrac{a^2}{m_a} \leq 8R - 4m_a$ (1).

Taking into account the other two similar inequalities we have

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \leq 24R - 4(m_a + m_b + m_c) \quad (2).$$

Inequality (1) can be rewritten as $m_a \geq \dfrac{a^2 + 4m_a^2}{8R}$. Adding the other two similar inequalities and using the following well-known identities

$m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$, $bc = 2Rh_a$, $\dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c} = \dfrac{1}{r}$ we get that

$$\begin{aligned} m_a+m_b+m_c \geq \frac{a^2+b^2+c^2+4(m_a^2+m_b^2+m_c^2)}{8R} &= \frac{a^2+b^2+c^2}{2R} \geq \frac{bc+ca+ab}{2R} \\ &= h_a+h_b+h_c \geq \frac{9}{\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}} \\ &= \frac{9}{\dfrac{1}{r}} = 9r, \quad \text{i.e.} \quad m_a+m_b+m_c \geq 9r \quad (3). \end{aligned}$$

Combining (2) and (3) we have

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq 24R-4(m_a+m_b+m_c) \leq 24R-4\cdot 9r = 12(2R-3r).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let A', B', C' and $A", B", C"$ be respectively the midpoints of the sides BC, CA, AB and the intersections of the medians AA', BB', CC' with the circumcircle of $\triangle ABC$ and let us denote h_a, h_b, h_c the heights and $n_a = A'A'', n_b = B'B'', n_c = C'C''$

Taking into account that the absolute value of the power of A' with respect to the circumcircle of $\triangle ABC$ is $A'B \cdot A'C$ and also $A'A \cdot A'A''$, that is $\frac{a}{2}\cdot\frac{a}{2} = m_a \cdot n_a$ or equivalently $\frac{a^2}{m_a} = 4n_a$.

Since $m_a + n_a \leq 2R$ ($AA^{'}$ is a chord of the circumcircle whose diameter is $2R$) and $h_a \leq m_a$ (the height is the minimum distance from the vertex to its opposite side), we conclude that $na_a \leq 2R - m_a \leq 2R - h_a$.

Thus $\frac{a^2}{m_a} \leq 4(2R-h_a)$ and analogously $\frac{b^2}{m_b} \leq 4(2R-h_b)$ and $\frac{c^2}{m_c} \leq 4(2R-h_c)$ so

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq 12\left(2R-\frac{1}{3}(h_a+h_b+h_c)\right).$$

The result follows from $h_a+h_b+h_c \geq 9r$, with equality iff $\triangle ABC$ is equilateral which is equality 6.8 from page 61 in the book *Geometric inequalities* by O. Bottema, R. Ž. Djordjević, R.R. Janić, D.S. Mitrinović and P.M Vasić, Wolters Noordhoff, Groningen, 1969.

Equality is attained iff $m_a+n_a = 2R,\ h_a = m_a$ and $\mathrm{h}_a + h_b + h_c = 9r$ and cyclically, that is, iff $\triangle ABC$ is an equilateral triangle.

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Using the inequality $m_a \geq \frac{b^2+c^2}{4R}$, we obtain

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \quad \leq \quad \sum\frac{4Ra^2}{b^2+c^2}$$

$$\frac{2(2R-3r)}{R}-\sum\frac{a^2}{b^2+c^2} \geq 0 \quad \Longleftrightarrow \quad \frac{3(2R-3r)}{R}-\frac{3}{2} \geq 0 \Longrightarrow R \geq 2r, \text{ which is true.}$$

(3)

* Where, using Nesbitt's inequality, we have $\sum \dfrac{a^2}{b^2+c^2} \geq \dfrac{3}{2}$.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawaskaki, Japan, and the proposer.

5401: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c be three positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2}+\frac{c^{-1}}{(4\sqrt{b}+3\sqrt{c})^2}+\frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2} \geq \frac{3}{49}.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The proposed inequality may be written as

$$\frac{1}{\left(4\sqrt{ab}+3b\right)^2}+\frac{1}{\left(4\sqrt{bc}+3c\right)^2}+\frac{1}{(4\sqrt{ca}+3a)^2} \geq \frac{3}{49}.$$

Now, by the Cauchy-Shwartz inequality in Engel form, the left-hand side is

$$\begin{aligned} LHS &\geq \frac{3^2}{\left(4\sqrt{ab}+3b\right)^2+\left(4\sqrt{bc}+3c\right)^2+(4\sqrt{ca}+3a)^2} \\ &= \frac{3^2}{16\,(ab+bc+ca)+9\,(a^2+b^2+c^2)+24\left(b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca}\right)}. \end{aligned}$$

By the rearrangement inequality, $ab+bc+ca \leq a^2+b^2+c^2$ and $b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca} \leq a^2+b^2+c^2$, so

$$LHS \geq \frac{3^2}{(16+9+24)\,(a^2+b^2+c^2)} = \frac{3^2}{(16+9+24)\,3} = \frac{3}{49}$$

with equality if and only if $a=b=c=1$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $x, y > 0$, then two forms of the Arithmetic - Geometric Mean Inequality state that

$$2\sqrt{xy} \leq x+y \quad \text{and} \quad 2xy \leq x^2+y^2.$$

In both cases, equality is attained if and only if $x = y$. As a result, we have

$$\begin{aligned}
y\left(4\sqrt{x}+3\sqrt{y}\right)^2 &= y\left(16x+24\sqrt{xy}+9y\right)\\
&\le y\left[16x+12\left(x+y\right)+9y\right]\\
&= 7y\left(4x+3y\right)\\
&= 7\left(4xy+3y^2\right)\\
&\le 7\left[2\left(x^2+y^2\right)+3y^2\right]\\
&= 7\left(2x^2+5y^2\right), \qquad (1)
\end{aligned}$$

with equality if and only if $x = y$.

We will also need the known result that if $X, Y, Z > 0$, then

$$(X+Y+Z)\left(\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}\right) \ge 9. \qquad (2)$$

(This is a direct result of applying the Cauchy - Schwarz Inequality to the vectors $\overrightarrow{V} = \left(\sqrt{X}, \sqrt{Y}, \sqrt{Z}\right)$ and $\overrightarrow{W} = \left(\frac{1}{\sqrt{X}}, \frac{1}{\sqrt{Y}}, \frac{1}{\sqrt{Z}}\right)$.)

By (1), (2), and the constraint equation $a^2+b^2+c^2 = 3$,

$$\begin{aligned}
&\frac{b^{-1}}{\left(4\sqrt{a}+3\sqrt{b}\right)^2}+\frac{c^{-1}}{\left(4\sqrt{b}+3\sqrt{c}\right)^2}+\frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2}\\
&\ge \frac{1}{7}\left[\frac{1}{2a^2+5b^2}+\frac{1}{2b^2+5c^2}+\frac{1}{2c^2+5a^2}\right]\\
&= \frac{1}{147}\cdot 21\cdot\left[\frac{1}{2a^2+5b^2}+\frac{1}{2b^2+5c^2}+\frac{1}{2c^2+5a^2}\right]\\
&= \frac{1}{147}\left[\left(2a^2+5b^2\right)+\left(2b^2+5c^2\right)+\left(2c^2+5a^2\right)\right]\left[\frac{1}{2a^2+5b^2}+\frac{1}{2b^2+5c^2}+\frac{1}{2c^2+5a^2}\right]\\
&\ge \frac{9}{147}\\
&= \frac{3}{49},
\end{aligned}$$

with equality if and only if $a = b = c = 1$.

Solution 3 by Henry Ricardo, New York Math Circle, NY

The arithmetic-geometric mean (AM-GM) inequality gives us

$$(4\sqrt{a}+3\sqrt{b})^2 = 16a+24\sqrt{ab}+9b \le 16a+24\left(\frac{a+b}{2}\right)+9b = 28a+21b.$$

Then, using the Cauchy-Schwarz inequality and the AM-GM inequality, we see that

$$\begin{aligned}
\sum_{cyclic}\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2} &\ge \sum_{cyclic}\frac{b^{-1}}{28a+21b} = \sum_{cyclic}\frac{1}{28ab+21b^2}\\
&\ge \frac{(1+1+1)^2}{\sum_{cyclic}(28ab+21b^2)} = \frac{9}{28(ab+bc+ca)+21(a^2+b^2+c^2)}\\
&\ge \frac{9}{28(a^2+b^2+c^2)+21(a^2+b^2+c^2)} = \frac{9}{49(3)} = \frac{3}{49}.
\end{aligned}$$

Equality holds if and only if $a = b = c = 1$.

Solution 4 by Toshihiro Shimizu, Kawaskaki, Japan

From Cauchy-Schwartz's inequality,

$$\begin{aligned}(a^2+b^2+c^2)\left(\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2}+\frac{c^{-1}}{(4\sqrt{b}+3\sqrt{c})^2}+\frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2}\right) &\geq \left(\sum_{cyclic}\frac{a}{\sqrt{b}(4\sqrt{a}+3\sqrt{b})}\right)^2\\ &= \left(\sum_{cyclic}\frac{1}{4\sqrt{\frac{b}{a}}+3\cdot\frac{b}{a}}\right)^2\end{aligned}$$

Let $x=\log\left(\sqrt{\frac{b}{a}}\right)$, $y=\log\left(\sqrt{\frac{c}{b}}\right)$, $z=\log\left(\sqrt{\frac{a}{c}}\right)$.Then, $x+y+z=0$. The (r.h.s) of the above inequality is equal to

$$\left(\sum_{cyclic}\frac{1}{4e^x+3e^{2x}}\right)^2$$

Let $f(x)=1/\left(4e^x+3e^{2x}\right)$. Since $f''(x)=4e^{-x}\left(9e^x+9e^{2x}+4\right)/\left(3e^x+4\right)^3>0$, f is convex. Thus, from Jensen's inequality, it follows that

$$\begin{aligned}f(x)+f(y)+f(z) &\geq 3f\left(\frac{x+y+z}{3}\right)\\ &= 3f(0)\\ &= \frac{3}{7}\end{aligned}$$

Solution 5 by David E. Manes, SUNY Oneonta, Oneonta, NY

Let

$$L=\sum_{cyclic}\frac{b^{-1}}{(4\sqrt{2}+3\sqrt{b})^2}=\sum_{cyclic}\frac{1}{b(4\sqrt{a}+3\sqrt{b})^2}.$$

Define vectors $\vec{u}$ and $\vec{v}$ such that

$$\begin{aligned}\vec{u} &= \left(\frac{1}{\sqrt{b}(4\sqrt{a}+3\sqrt{b})},\frac{1}{\sqrt{c}(4\sqrt{b}+3\sqrt{c})},\frac{1}{\sqrt{a}(4\sqrt{c}+3\sqrt{a})}\right).\\ \vec{v} &= \left(\sqrt{b}(4\sqrt{a}+3\sqrt{b}),\sqrt{c}(4\sqrt{b}+3\sqrt{c}),\sqrt{a}(4\sqrt{c}+3\sqrt{a})\right).\end{aligned}$$

Then the dot product of $\vec{u}$ and $\vec{v}$ is less than or equal to the product of the norms of $\vec{u}$ and $\vec{v}$ by the Cauchy-Schwarz inequality. Therefore,

$$1+1+1\leq\sqrt{\sum_{cyclic}\frac{1}{b(4\sqrt{a}+3\sqrt{b})^2}}\sqrt{\sum_{cyclic}b(4\sqrt{a}+3\sqrt{b})^2}$$

or

$$L \geq \frac{9}{\sum_{cyclic} b(4\sqrt{a}+3\sqrt{b})^2}.$$

Expanding the denominator, one obtains

$$\sum_{cyclic} b(4\sqrt{a}+3\sqrt{v})^2 = 16\left(\sum_{cyclic} ab\right) + 24\left(\sum_{cyclic} \sqrt{ab^3}\right) + 9(a^2+b^2+c^2).$$

The Rearrangement inequality implies

$$\sum_{cyclic} ab + \sum_{cyclic} \sqrt{ab^3} \leq (a^2+b^2+c^2) + \left(\sqrt{a^4}+\sqrt{b^4}+\sqrt{c^4}\right)$$

with equality if and only if $a=b=c$. Therefore,

$$\frac{1}{\sum_{cyclic} b(4\sqrt{a}+3\sqrt{b})^2} \geq \frac{1}{16\sum_{cyclic} a^2 + 24\sum_{cyclic} a^2 + 9\sum_{cyclic} a^2}.$$

Since $a^2+b^2+c^2=3$, it follows that

$$\frac{1}{\sum_{cyclic} b(4\sqrt{a}+3\sqrt{b})^2} \geq \frac{1}{16(3)+24(3)+9(3)} = \frac{1}{3(49)}.$$

Hence,

$$L = \sum_{cyclic} \frac{b^{-1}}{(4\sqrt{2}+3\sqrt{b})^2} \geq \frac{9}{3(49)} = \frac{3}{49}$$

with equality if and only if $a=b=c=1$.

Editor's comment : **D.M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania** with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania** generalized the problem as follows:

If $a,b,c,m,n \in R_+^*$, then $\sum_{cyclic} \frac{b^{-1}}{(m\sqrt{a}+n\sqrt{b})^2} \geq \frac{3}{(n+p)^2}$.

They did this by showing that

$$\sum_{cyclic} \frac{b^{-1}}{(m\sqrt{a}+n\sqrt{b})^2} \geq \frac{27}{(m+n)^2(a+b+c)^2}. \qquad (2)$$

Then they used the hypothesis concluding that

$$3 = a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3} \iff 9 \geq (a+b+c)^2. \qquad (3)$$

By (2) and (3) they obtained

$$\sum_{cyclic} \frac{b^{-1}}{(m\sqrt{a}+n\sqrt{b})^2} \geq \frac{27}{(m+n)^2(a+b+c)^2} = \frac{3}{(m+n)^2}.$$

Letting $m = 1$and $n = 3$ they obtained the statement of the problem.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Nikos Kalapodis, Patras, Greece; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department of Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5402: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 \mathrm{d}x,$$

where a and b are real numbers.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Obviously we may assume $a \neq b$, since otherwise the integral is null. Let us suppose that $a > b > 0$. Using parity, write the integral as

$$I = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 dx,$$

and then deform the contour to be the line C slightly below the real axis. Next express cosines in terms of exponentials. Then we obtain I equal to

$$\frac{1}{8}\left(\int_C \frac{-2\left(e^{-(a+b)xi}+e^{(a-b)xi}+e^{(b-a)xi}+e^{(a+b)xi}\right)+e^{-2axi}+e^{2axi}+e^{-2bxi}+e^{2bxi}+4}{x^2}dx\right).$$

For $a > b > 0$, in the integrals containing terms of the form e^{-kxi}, with $k > 0$, the contour can be closed in the lower half plane (by Jordan lemma) and therefore these integrals vanish (as there are no singularities inside).

The integrals containing terms of the form e^{kxi}, with $k \geq 0$, can only be closed in the upper half plane and are therefore given by the residues at $x = 0$. Therefore

$$\begin{aligned} I &= \frac{\pi i}{4}\, Res_{x=0}\left(\frac{-2e^{(a-b)xi}-2e^{(a+b)xi}+e^{2axi}+e^{2bxi}+4}{x^2}\right) \\ &= \frac{\pi i}{4}\left(-2i(a-b)-2i(a+b)+2ia+2ib\right) \\ &= \frac{\pi i 2(-a+b)i}{4} = \frac{\pi(a-b)}{2}. \end{aligned}$$

Solution 2 by Toshihiro Shimizu, Kawaskaki, Japan

For real number $a \neq 0$, we have

$$
\begin{aligned}
\int_0^\infty \frac{\sin^2 ax}{x^2} dx &= \frac{1}{2} \int_{-\infty}^\infty \frac{\sin^2 ax}{x^2} dx \\
&= \frac{a^2}{2} \int_{-\infty}^\infty \frac{\sin^2 y}{y^2} \frac{dy}{|a|} \, (\text{where } y = |a|\, x) \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \frac{\sin^2 y}{y^2} dy \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \sin^2 y \left(-\frac{1}{y}\right)' dx \\
&= \frac{|a|}{2} \left[\sin^2 y \left(-\frac{1}{y}\right)\right]_{-\infty}^\infty - \frac{a}{2} \int_{-\infty}^\infty 2 \sin y \cos y \left(-\frac{1}{y}\right) dy \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \frac{\sin 2y}{y} dy \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \frac{\sin y}{y} dy \\
&= \frac{|a|\, \pi}{2}.
\end{aligned}
$$

For $a = 0$, the value of l.h.s is 0 and r.h.s is also 0. Thus, this result is true for any real number a. Then, since

$$
\begin{aligned}
(\cos ax - \cos bx)^2 &= \cos^2 ax + \cos^2 bx - 2 \cos ax \cos bx \\
&= 1 - \sin^2 ax + 1 - \sin^2 bx - \cos(ax + bx) - \cos(ax - bx) \\
&= 2 - \sin^2 ax - \sin^2 bx \\
&\quad - \left(1 - 2\sin^2\left(\frac{ax+bx}{2}\right)\right) - \left(1 - 2\sin^2\left(\frac{ax-bx}{2}\right)\right) \\
&= -\sin^2 ax - \sin^2 bx + 2\sin^2\left(\frac{ax+bx}{2}\right) + 2\sin^2\left(\frac{ax-bx}{2}\right),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\int_0^\infty \left(\frac{\cos ax - \cos bx}{x}\right)^2 dx &= -\int_0^\infty \frac{\sin^2 ax}{x^2} dx - \int_0^\infty \frac{\sin^2 bx}{x^2} dx \\
&\quad + 2 \int_0^\infty \frac{\sin^2\left(\frac{ax+bx}{2}\right)}{x^2} dx + 2 \int_0^\infty \frac{\sin^2\left(\frac{ax-bx}{2}\right)}{x^2} dx \\
&= -\frac{|a|\, \pi}{2} - \frac{|b|\, \pi}{2} + 2 \cdot \frac{|a+b|}{4}\pi + 2 \cdot \frac{|a-b|}{4}\pi \\
&= \frac{1}{2}\left(-|a| - |b| + |a+b| + |a-b|\right).
\end{aligned}
$$

If a, b are the same sign or 0, we have $-|a| - |b| + |a+b| = 0$ and the answer is $\frac{|a-b|}{2}$, if a, b are the opposite sign, $-|a| - |b| + |a-b| = 0$ and the answer is $\frac{|a+b|}{2}$.
Also, we can write this answer as

$$
\min\left\{\frac{|a-b|}{2}, \frac{|a+b|}{2}\right\}.
$$

Solution 3 by Ed Gray, Highland Beach, FL

In order to calculate: $\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 \mathrm{d}x$, where a and b are real numbers we first expand the numerator so that the integral becomes

$$\int_0^\infty \frac{\cos^2(ax)-2\cos(ax)\cos(bx)+\cos^2(bx)}{x^2}\mathrm{d}x. \qquad (1)$$

But the expression $2\cos(ax)\cos(bx)=\cos(ax+bx)+\cos(ax-bx)$, so equation (1) becomes

$$\int_0^\infty \frac{\cos^2(ax)}{x^2} - \int_0^\infty \frac{\cos(ax+bx)}{x^2} - \int_0^\infty \frac{\cos(ax-bx)}{x^2} + \int_0^\infty \frac{\cos^2(bx)}{x^2}$$

We evaluate each of these four integrals.

We may use "integration by parts" and other standard procedures to obtain the following:

$$\begin{aligned}
\int_0^\infty \frac{\cos^2(ax)}{x^2} &= \frac{-a\pi}{2}\\
\int_0^\infty \frac{-\cos(ax+bx)}{x^2} &= \frac{(a+b)\pi}{2}\\
\int_0^\infty \frac{-\cos(ax-bx)}{x^2} &= \frac{(a-b)\pi}{2} \text{ if } a>b; \quad = \frac{(b-a)\pi}{2} \text{ if } b>a.\\
\int_0^\infty \frac{\cos^2(bx)}{x^2} &= \frac{-b\pi}{2}
\end{aligned}$$

Summing the four integrals above we see that

$$\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 \mathrm{d}x = \begin{cases} \dfrac{(a-b)\pi}{2}, & \text{if } b<a \\[2ex] \dfrac{(b-a)\pi}{2}, & \text{if } a<b. \end{cases}$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland

We claim that $f(a,b)=\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 dx = \frac{\pi}{2}\,||b|-|a||$.

Obviously, $f(a,b)=f(b,a)=-f(-a,b)=f(a,-b)$. $\quad$ (1)

Let $r>0$ and let L be the "indented" line: $-\infty<t\le -r,\ re^{i\varphi},\ \pi\ge\varphi\ge 0,\ r\le t<\infty$, run through "from left to right". Let a be a real number. Then $\int_L \frac{e^{iaz}}{z^2}dz=\pi\,(a\,|a|)$.

Indeed, By Cahuchy's theorem, the integral does not end on r. Assume that $a\ge 0$. Then

$$\left|\int_L \frac{e^{iaz}}{z^2}dz\right| \le 2\int_r^\infty \frac{1}{t^2}dt+\frac{\pi r}{r^2}\max_{0\le\varphi\le\pi}\left|e^{iare^{i\varphi}}\right| = \frac{1}{r}(2+\pi)\to 0,\ as\ r\to\infty.$$

So $\displaystyle\int_L \frac{e^{iaz}}{z^2} dz = 0$, if $a \geq 0$, where $\overline{L}$ is the complex conjugate of L, i.e., the line L reflected at the abscissa

By the residue theorem,

$$\int_{\overline{L}} \frac{e^{iaz}}{z^2} dz - \int_L \frac{e^{iaz}}{z^2} dz = \int_{|z|=r} \frac{e^{iaz}}{z^2} dz = 2\pi i Res\left(\frac{e^{iaz}}{z^2},\ z = 0\right) = -2\pi a.$$

So $\displaystyle\int_L \frac{e^{iaz}}{z^2} dz = \int_{\overline{L}} \frac{e^{iaz}}{z^2} dz - 2\pi i Res\left(\frac{e^{iaz}}{z^2},\ z = 0\right) = 2\pi a, \text{ if } a < \mathit{0}.$

To sum up:

$$\int_L \frac{e^{iaz}}{z^2} dz = \begin{cases} 0, & a \geq 0 \\ 2\pi a, & a < 0. \end{cases} = \pi\left(a - |a|\right), \text{as claimed.}$$

We conclude that

$$f(ab) = \int_0^\infty \left(\frac{\cos(ax) - \cos(bx)}{x}\right)^2 dx = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{\cos(ax) - \cos(bx)}{x}\right)^2 dx = \frac{1}{2}\int_L \left(\frac{\cos(az) - \cos(bz)}{z}\right)^2 dz =$$

$$= \frac{1}{2}\int_L \frac{\cos^2(az) + -2\cos(az)\cos(bz) + \cos^2(bz)}{z^2} dz$$

$$= \frac{1}{2}\int_L \frac{\left(e^{iaz} + e^{-iaz}\right)^2 - 2\left(e^{iaz} + e^{-iaz}\right)\left(e^{iaz} + e^{-iaz}\right) + \left(e^{iaz} + e^{-iaz}\right)^2}{4z^2} dz$$

$$= \frac{1}{2}\int_L \frac{e^{2iaz} + e^{2ibz} + e^{-2iaz} + e^{-2ibz} + 4 - 2e^{i(a+b)z} - 2e^{i(a-b)z} - 2e^{i(-a+b)z} - 2e^{i(-a-b)z}}{4z^2} dz$$

$$\begin{aligned} &= \frac{\pi}{8}\Big(2a - |2a| + 2b - |2b| - 2a - |2a| - 2b - |2b| - 2(a+b) \\ &+2|a+b| - 2(a-b) + 2|a-b| - 2(-a+b) + 2|-a+b| - 2(-a-b) + 2|-a-b|\Big) \\ &= \frac{\pi}{4}\Big(-|a| - |b| - |a| - |b| + |a+b| + |a-b| + |-a+b| + |-a-b|\Big) \\ &= \frac{\pi}{2}\Big(-|a| - |b| + |a+b| + |a-b|\Big). \end{aligned}$$

By (1) we can assume that $0 \leq a \leq b$. Then

$$\begin{aligned} f(a,b) &= \frac{\pi}{2}\left(-|a| - |b| + |a+b| + |a-b|\right) \\ &= \frac{\pi}{2}\left(-a - b + a + b + b - a\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{2}(b-a) \\
&= \frac{\pi}{2}\big||b|-|a|\big|, \text{ as claimed.}
\end{aligned}
$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

Denote the given integral by I. We show that

$$I = \frac{(|a+b|+|a-b|-|a|-|b|)\,\pi}{2} \qquad (1)$$

4pt It is well known that for any real number r, we have

$$\int_0^\infty \frac{\sin(rx)}{x}dx = \begin{cases} \pi/2 & r>0 \\ 0 & r=0. \\ -\pi/2 & r<0. \end{cases} \qquad (2)$$

Since $\lim\limits_{x\to 0} \dfrac{(\cos(ax)-\cos(bx))^2}{x} = 0$, so intergrating by parts , we obtain

$$
\begin{aligned}
I &= \int_0^\infty \frac{f(a,b,x)}{x}dx, \text{ where} \\
& \quad f(a,b,x) \\
&= 2\left(\cos(ax)-\cos(bx)\right)\left(b\sin(bx)-a\sin(ax)\right) \\
&= 2b\sin(bx)\cos(ax)+2a\sin(ax)\cos(bx)-a\sin(2ax)-b\sin(2bx) \\
&= (a+b)\sin\left((a+b)x\right)+(a-b)\sin\left((a-b)x\right)-a\sin(2ax)-b\sin(2bx).
\end{aligned}
$$

Using (2), we now obtain (1). This completes the proof.

Solution 6 by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

We prove that the value of the proposed integral is $(a-b)\dfrac{\pi}{2}$. It is trivial when $a=b$, so we assume that $a \neq b$. We make repeated use of the following integral (proof of which is provided at the end, for the sake of completion)

$$\int_0^\infty e^{-\alpha x}\cos(\beta x)dx = \frac{\alpha}{\alpha^2+\beta^2}$$

We have the identity (easily verified) $\dfrac{1}{x^2} = \displaystyle\int_0^\infty te^{-xt}dt$. Using this, the proposed integral becomes

$$\int_0^\infty \int_0^\infty te^{-xt}(\cos(ax)-\cos(bx))^2 dtdx.$$

Since everything is positive, by Tonelli's Theorem, we can reverse the order of integration so that the integral now becomes

$$\int_0^\infty \int_0^\infty te^{-xt}(\cos(ax) - \cos(bx))^2 dxdt.$$

From the trigonometric identities $\dfrac{\cos(2x)+1}{2} = \cos^2(x)$ and $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$, we easily obtain (using (1))

$$\int_0^\infty e^{-xt}(\cos^2(ax) + \cos^2(bx))dx = \frac{1}{t} + \frac{1}{2}\left(\frac{t}{t^2+(2a)^2} + \frac{t}{t^2+(2b)^2}\right)$$

and

$$\int_0^\infty e^{-xt}(2\cos(ax)\cos(bx))dx = \frac{t}{t^2+(a+b)^2} + \frac{t}{t^2+(a-b)^2}.$$

Thus, (2) becomes

$$\int_0^\infty \int_0^\infty te^{-xt}(\cos(ax) - \cos(bx))^2 dxdt$$

$$= \int_0^\infty t\left(\frac{1}{t} + \frac{1}{2}\left(\frac{t}{t^2+(2a)^2} + \frac{t}{t^2+(2b)^2}\right) - \frac{t}{t^2+(a+b)^2} - \frac{t}{t^2+(a-b)^2}\right)dt$$

$$= \left[(a-b)\arctan\frac{t}{a-b} + (a+b)\arctan\frac{t}{a+b} - a\arctan\frac{t}{2a} - b\arctan\frac{t}{2b}\right]_{t=0}^{t=\infty} = (a-b)\frac{\pi}{2}.$$

Proof of (1):

Let $I = \int_0^\infty e^{-\alpha x}\cos(\beta x)dx$. Then $\cos(\beta x) = \dfrac{e^{i\beta x} + e^{-i\beta x}}{2}$ gives

$$I = \frac{1}{2}\int_0^\infty \left(e^{-x(\alpha - i\beta)} + e^{-x(\alpha+i\beta)}\right)dx = \frac{1}{2}\left(\frac{1}{\alpha - i\beta} + \frac{1}{\alpha + i\beta}\right) = \frac{\alpha}{\alpha^2+\beta^2}.$$

Alternatively, one can do integration by parts to get the same result.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

Mea Culpa

Paolo Perfetti of the Mathematics Department of Tor Vergata University in Rome, Italy should have been credited with having solved 5394.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2017

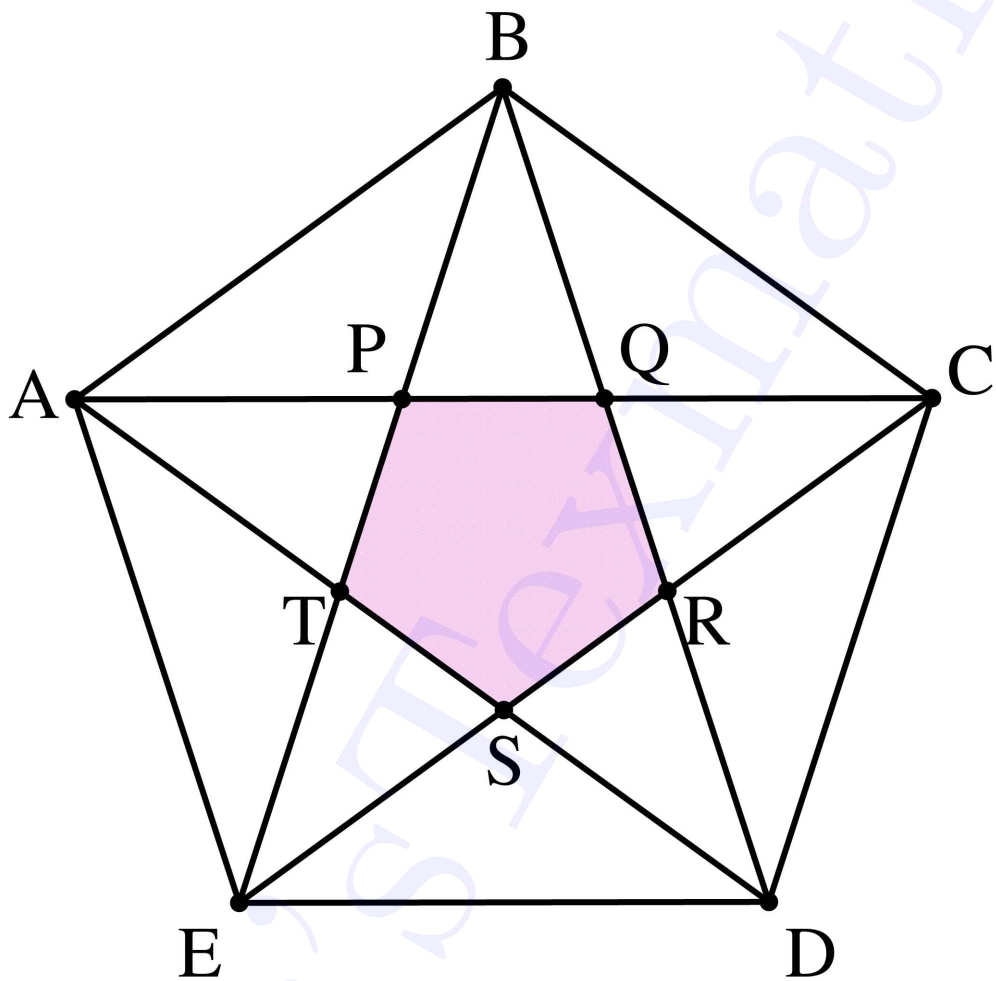
- **5421:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $1-d$ bisects two of its sides.

- **5422:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Polygon $ABCDE$ is a regular pentagon. Pentagon $PQRST$ is bounded by diagonals of pentagon $ABCDE$ as shown. Find the following:

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE}.$$



- **5423:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

  Let a, b, c be the side-lengths, r_a, r_b, r_c be the radii of the ex-circles and R, r the radii of the circumcircle and incircle respectively, and s the semiperimeter of $\triangle ABC$. Show that

  $$\frac{(r_a - r)^2 + r_b r_c}{(s-b)(s-c)} + \frac{(r_b - r)^2 + r_c r_a}{(s-c)(s-a)} + \frac{(r_c - r)^2 + r_a r_b}{(s-a)(s-b)} \geq 13.$$

- **5424:** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

  Let a, b, c and d be positive real numbers such that $abc + bcd + cda + dab = 4$. Prove that $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) \geq 256$.

- **5425:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$. If n is an odd positive integer then show that $1 + \det(A)$ is the product of two consecutive Fibonacci numbers, where

$$A = \begin{pmatrix} F_1^2 - 1 & F_1F_2 & F_1F_3 & \dots & F_1F_n \\ F_2F_1 & F_2^2 - 1 & F_2F_3 & \cdots & F_2F_n \\ F_3F_1 & F_3F_2 & F_3^2 - 1 & \cdots & F_3F_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_nF_1 & F_nF_2 & F_nF_3 & \cdots & F_n^2 - 1 \end{pmatrix}$$

5426: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of natural numbers. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]} \text{ converges.}$$

Here $[x, y]$ denotes the least common multiple of the natural numbers x and y.

Solutions

- **5403:** *Proposed by Kenneth Korbin, New York, NY*

Let $\phi = \dfrac{1+\sqrt{5}}{2}$. Solve the equation $\sqrt[3]{x+\phi} = \sqrt[3]{\phi} + \sqrt[3]{x-\phi}$ with $x > \phi$.

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak, Angelo State University, San Angelo, TX

Let $a = \sqrt[3]{x+\phi}$ and $b = \sqrt[3]{x-\phi}$. We may write

$$\begin{aligned} a - b &= \sqrt[3]{\phi} \\ (a-b)^3 &= \phi \\ a^3 - 3a^2b + 3ab^2 - b^3 &= \phi \\ a^3 - b^3 - 3ab(a-b) &= \phi \\ x + \phi - (x - \phi) - 3\sqrt[3]{x^2 - \phi^2}\sqrt[3]{\phi} &= \phi. \end{aligned}$$

Simplifying this last equation we obtain $\sqrt[3]{x^2-\phi^2} = \dfrac{\phi^{2/3}}{3}$. Under the condition $x > \phi$, the solution to this equation is $x = \sqrt{\dfrac{\phi^2}{27} + \phi^2} = \dfrac{2\sqrt{21}}{9}\phi$.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC.

Given any real number $a > 0$, we solve the equation $\sqrt[3]{x+a} = \sqrt[3]{a} + \sqrt[3]{x-a}$ with $x > a$. (Similarly, given any real number $a < 0$, we may solve the equation $\sqrt[3]{x+a} = \sqrt[3]{a} + \sqrt[3]{x-a}$ with $x < a$.)

Rewriting the given equation and cubing both sides yields

$$(x+a) - 3\sqrt[3]{(x+a)^2(x-a)} + 3\sqrt[3]{(x+a)(x-a)^2} - (x-a) = a,$$

or $3\sqrt[3]{x^2-a^2}(\sqrt[3]{x-a} - \sqrt[3]{x+a}) = -a$. Then $-3\sqrt[3]{a}\sqrt[3]{x^2-a^2} = -a$, so cubing once more produces

$$-27a(x^2-a^2) = -a^3.$$

Hence $x^2 = \dfrac{28}{27}a^2$, so requiring $x > a$ yields $x = \dfrac{2\sqrt{21}}{9}a$. In particular, when $a = \phi$, we obtain the solution $x = \dfrac{2\sqrt{21}}{9}\phi = \dfrac{\sqrt{21}+\sqrt{105}}{9}$.

Solution 3 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The value of $x > \phi$ that satisfies the equation is

$$x = \phi\Big[\Big(\frac{-3+\sqrt{21}}{6}\Big)^3 + 1\Big] \approx 1.48363835038.$$

One notes that $x > \phi$ and does satisfy the equation.
Let $v = \sqrt[3]{x-\phi}$. Then $v^3 = x - \phi$ so that $x = v^3 + \phi$. Since we want the solution $x > \phi$, it follows that x must be positive. The original equation in terms of v is

$$\sqrt[3]{x+2\phi} = \sqrt[3]{\phi} + v.$$

Cubing both sides of this equation, we get

$$3\sqrt[3]{\phi}\cdot v^2 + 3(\sqrt[3]{\phi})^2 v - \phi = 0.$$

Dividing by $3\sqrt[3]{\phi}$ reduces this equation to the monic quadratic equation

$$v^2 + \sqrt[3]{\phi}\cdot v - \frac{1}{3}(\sqrt[3]{\phi})^2 = 0$$

with roots

$$v = \frac{-\sqrt[3]{\phi} \pm \sqrt[3]{\phi}\cdot\sqrt{\frac{7}{3}}}{2}.$$

Rejecting the negative root yields

$$v = \frac{-\sqrt[3]{\phi} + \sqrt[3]{\phi}\cdot\sqrt{\frac{7}{3}}}{2} = \sqrt[3]{\phi}\Big(\frac{-3+\sqrt{21}}{6}\Big).$$

Hence,

$$x = v^3 + \phi = \phi\Big[\Big(\frac{-3+\sqrt{21}}{6}\Big)^3 + 1\Big] = \frac{2\sqrt{21}}{9}\phi.$$

Editor's comment : **D. M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania** with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania** generalized the problem as follows:

Let $a, b, c, > 0$, with $a + b = 2c$ then it can be shown that the unique real-valued solution to the equation $\sqrt[3]{x+a} = \sqrt[3]{x-b} + \sqrt[3]{c}$, where $x > c$ is $x = \dfrac{3\sqrt{3}\,(b-a) + 4c\sqrt{7}}{6\sqrt{3}}$.

If $a = b = \phi$, then $= \phi$ and the equation $\sqrt[3]{x+\phi} = \sqrt[3]{x-\phi} + \sqrt[3]{\phi}$ with $x > \phi$, has the solution

$$x = \frac{3\sqrt{3}\,(\phi - \phi) + 4\phi\sqrt{7}}{6\sqrt{3}} = \frac{2\sqrt{21}}{9}\phi.$$

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Ashland University Undergraduate Problem Solving Group, Ashland, OH; D. M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; the proposer, and Students from Taylor University (see below);**

Students at Taylor University, Upland, IN.
Group 1. Ben Byrd, Maddi Guillaume, and Makayla Schultz.
Group 2. Caleb Knuth, Michelle Franch and Savannah Porter.
Group 3. Lauren Moreland, Anna Souzis, and Boni Hermandez

- **5404:** *Proposed Arkady Alt, San Jose, CA*

For any given positive integer $n \geq 3$, find the smallest value of the product of $x_1 x_2 \ldots x_n$, where $x_1, x_2, x_3, \ldots x_n > 0$ and $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1$.

Solution 1 by Ed Gray, Highland Beach, FL

Suppose each term had the value of $\dfrac{1}{n}$. Since there are n terms, the sum is equal to 1, satisfying the problem restriction.

In the event for each $k, 1 \leq k \leq n$

1. $\dfrac{1}{1+x_k} = \dfrac{1}{n}$, so $x_k = n-1$, and the value of the product is:
2. $(n-1)^n$.

If this is not the smallest product, at least one value of x_k must be less than $n-1$. Suppose $x_k = n-1-e$ where $e > 0$.

Then the series contains the therm $\dfrac{1}{1+x_k} = \dfrac{1}{n-e}$. We must increase the value of another term so that the sum maintains the value of 1. We must have:

3. $\dfrac{1}{n-e} + \dfrac{1}{1+x_m} = \dfrac{2}{n}$
4. $\dfrac{1}{1+x_m} - \dfrac{2}{n} - \dfrac{1}{n-e} = \dfrac{2(n-e-n)}{n(n-e)} = \dfrac{2n-2e-n}{n(n-e)}$
5. $\dfrac{1}{1+x_m} = \dfrac{n-2e}{n(n-e)}$
6. $(1+x_m)(n-2e) = n(n-e)$
7. $1 + x_m = \dfrac{n(n-e)}{n-2e}$

8. $x_m = \dfrac{n(n-e)}{n-2e} - 1 = \dfrac{n(n-e)-n-2e)}{n-2e} = \dfrac{n^2-ne-n+2e}{n-2e}$

9. The new product is: $\Big((n-1)^{n-2}\Big)\, x_k x_m$. If the new product is to be smaller, we must have:

10. $\dfrac{(n-1)^{n-2}(n-1-e)(n^2-n-e(n-2)}{n-2e} < (n-1)^n$, or dividing by $(n-1)^{n-2}$

11. $(n-1-e)(n^2-n-en+2e) < (n-2e)(n-1)^2$,

12. $(n-1-e)(n^2-n-en+=2e) < (n-2e)(n^2-2n+1)$, which simplifies to:

13. $2en^2+ne2 < 2e^2$. Dividing by e^2,

14. $\dfrac{2n^2}{e}+n<2$, which is a contraction. Therefore, we did not decrease the product, but increased it.

So $(n-1)^n$ is the minimum product.

Solution 2 by Ramya Dutta (student), Chennai Mathematical Institute) India

Consider the polynomial $P(x) = \displaystyle\prod_{j=1}^{n}(x+x_j)$, then $\dfrac{P'(x)}{P(x)} = \displaystyle\sum_{j=1}^{n}\frac{1}{x+x_j}$, i.e., $P'(1)=P(1)$.

Denoting the j-th symmetric polynomial by, $\sigma_j = \displaystyle\sum_{1\le k_1<k_2<\cdots<k_j\le n} x_{k_1}x_{k_2}\cdots x_{k_j}$ for $j\ge 1$ and $\sigma_0 = 1$,

$P(x) = \displaystyle\sum_{j=0}^{n}\sigma_j x^{n-j}$ and $P'(x) = \displaystyle\sum_{j=0}^{n-1}(n-j)\sigma_j x^{n-j-1}$

Therefore, the condition $P(1) = P'(1)$ is equivalent to,

$$\sigma_n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j$$

Using, AM-GM inequality: $\sigma_j \ge \dbinom{n}{j}\sigma_n^{j/n}$ for $j\ge 1$.

I.e., writing $\sigma_n^{1/n} = \alpha$, we have,

$$\begin{aligned}
\alpha^n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j &\ge \sum_{j=0}^{n-1}(n-j-1)\binom{n}{j}\alpha^j \\
&= (n-1)\sum_{j=0}^{n-1}\binom{n}{j}\alpha^j - n\sum_{j=1}^{n-1}\binom{n-1}{j-1}\alpha^j \\
&= (n-1)\left((1+\alpha)^n-\alpha^n\right) - n\alpha\left((1+\alpha)^{n-1}-\alpha^{n-1}\right) \\
&= \alpha^n-(1+\alpha)^n+n(1+\alpha)^{n-1}
\end{aligned}$$

that is, $(1+\alpha)^n \ge n(1+\alpha)^{n-1} \implies \alpha\ge n-1$ (since, $\alpha>0$)
So, the minimum value of $x_1x_2\cdots x_n$ is $(n-1)^n$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We shall use the Method of Lagrange Multipliers to show that the smallest value of the product is $(n-1)^n$, achieved when each $x_i = n-1$.

First suppose that all but one of the x_i are equal: let $x_i = b$ for $1 \le i \le n-1$ and choose x_n so that the constraint $\displaystyle\sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1$ is satisfied:

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = (n-1)\frac{1}{1+b} + \frac{1}{1+x_n} = 1, \Longrightarrow x_n = \frac{n-1}{b-(n-2)}, \text{ where}$$

$b > n-2$ to make $x_n > 0$.

Then the product $f(x_1, x_2, \ldots, x_)n) = \displaystyle\prod_{i=1}^{n} x_i = b^{n-1}\frac{n-1}{b-(n-2)}$.

We note that as b becomes unbounded positive, the product of the $x_i's$ becomes unbounded positive, and as b approaches $n-2$ from above, the product of the $x_i's$ also becomes unbounded positive. Thus if the product has an absolute extremum subject to the given constraint, it must be a minimum since the product is unbounded above.

For $b = n-1$, we see that $\mathrm{x}_n = n-1$, so every $x_i = n-1$ and the product is equal to $(n-1)^n$,

We consider this as a Lagrange Multiplier problem where we minimize the product $f(x_1, x_2, \ldots, x_n) = \displaystyle\prod_{i=1}^{n} x_i$ subject to the constraint

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1.$$

That is, subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1.$$

By the Method of Lagrange Multipliers, we'll find the minimum of f where $\dfrac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_n) = \lambda \dfrac{\partial}{\partial x_i} g(x_1, x_2, \ldots, x_n)$ for $1 \le k \le n$.

We see that: $\dfrac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_n) = \displaystyle\prod_{\substack{i=1 \\ i=k}}^{n} x_i$ and $\dfrac{\partial}{\partial x_i} g(x_1, x_2, \ldots, x_n) = \dfrac{1}{(1+x_i)^2}$ for $1 \le k \le n$.

Thus we want to solve the system, $\displaystyle\prod_{\substack{i=1 \\ i=k}}^{n} x_i = \frac{\lambda}{(1+x_k)^2}$, for $1 \le k \le n$.

Solving each equation for λ gives $\lambda = -(1+x_k)^2 \displaystyle\prod_{\substack{i=1 \\ i\neq k}}^{n} x_i$ for $1 \le k \le n$.

Hence, for any $1 \le j,\ k \le n$ we must have $\lambda = -(1+x_i)^2 \displaystyle\prod_{\substack{i=1 \\ i\neq k}}^{n} x_i = -(1+x_j)^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} x_i$

Algebra gives $\dfrac{x_j}{(1+x_j)^2} = \dfrac{x_k}{(1+x_k)^2}, \quad 1 \le j,\ k \le n.$

We claim this forces $x_i = x_k$. Suppose that $x_k \ne x_i$ for some $k \ne j$.

Now consider the function $h(x) = \dfrac{x}{(1+x)^2}$ for $x > 0$.

Note that $h(x_i) = h(x_k)$ for $1 \le j,\ k \le n$

By calculus, $h(x)$ is strictly increasing for $0 < x < 1$ to a maximum (of $1/4$) at $x = 1$, and is then strictly decreasing for $x > 1$. That is, h except for the peak at $x = 1$ is two- to- one function (for $x > 0$).

Moreover, $h(x)$ has the reflective property $h\left(\dfrac{1}{x}\right) = h(x)$. Hence, for

$1 \le j \ne k \le n,\ h(x_j) = h(x_k)$ and $x_j \ne x_k \Longrightarrow x_j = \dfrac{1}{x_k}$. Then jour constraint becomes

$$\begin{aligned}
1 &= \frac{1}{1+x_k} + \frac{1}{1+x_j} + (\text{other positive terms}) \\
&= \frac{1}{1+x_k} + \frac{1}{1+\dfrac{1}{x_k}} + (\text{other positive terms}) \\
&= \frac{1}{1+x_k} + \frac{x_k}{1+x_k} + (\text{other positive terms}) \\
&= 1 + (\text{other positive terms})
\end{aligned}$$

which is impossible. Therefore, $x_k = x_j$.

Hence, to achieve the extreme value, which must be a minimum, all of the x_i are equal and must equal $n-1$, forcing the minimum value of the product to be $(n-1)^n$.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Denote by $\dfrac{1}{1+x_i} = y_i \Longrightarrow x_i = \dfrac{1-y_i}{y_i},\ y_i > 0, i = 1, 2, , n$

By the AM-GM, we get

$$x_1x_2\ldots x_n = \prod_{i=1}^{n} \frac{1-y_i}{y_i} = \frac{y_2+y_3+\ldots+y_n}{y_1} \ldots \frac{y_1+y_2+\ldots+y_{n-1}}{y_n} \ge \frac{(n-1)^n \sqrt[n-1]{(y_1y_2\ldots y_n)^{n-1}}}{y_1y_2\cdots y_n} = (n-1)^n.$$

So, $x_1x_2\ldots x_n \ge (n-1)^n$. Equality occurs for $x_1 = x_2 = \ldots = x_n = n-1$.

Editor's comment : In addition to a general solution to this problem, the problem's author, **Arkady Alt of San Jose, CA,** also provided 4 different solutions for the cases $n = 2 = 3$.

Solution A.

Let $n = 3$. We have $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \dfrac{1}{1+x_3} = 1 \Longleftrightarrow$
$3 + 2(x_1+x_2+x_3) + x_1x_2 + x_2x_3 + x_3x_1 = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_2x_3 +$
$x_3x_1 + x_1x_2x_3 \iff 2 + x_1 + x_2 + x_3 = x_1x_2x_3$. Since $x_1 + x_2 + x_3 \ge 3\sqrt[3]{x_1x_2x_3}$

then $x_1x_2x_3 \geq 2 + 3\sqrt[3]{x_1x_2x_3} \iff \left(\sqrt[3]{x_1x_2x_3} - 2\right)\left(\sqrt[3]{x_1x_2x_3} + 1\right)^2 \geq 0 \iff$ $\sqrt[3]{x_1x_2x_3} - 2 \geq 0 \iff x_1x_2x_3 \geq 2^3$.

Solution B.

Since $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \dfrac{1}{1+x_3} = 1 \iff \dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} = \dfrac{x_3}{1+x_3} \iff$ $\dfrac{1+x_3}{1+x_1} + \dfrac{1+x_3}{1+x_2} = x_3 \implies x_3 \geq 2\left(1+x_3\right)\sqrt{\dfrac{1}{1+x_1}.\dfrac{1}{1+x_2}} = \dfrac{2\left(1+x_3\right)}{\sqrt{\left(1+x_1\right)\left(1+x_2\right)}}$.

Similarly we obtain $x_2 \geq \dfrac{2\left(1+x_2\right)}{\sqrt{\left(1+x_3\right)\left(1+x_1\right)}}, x_1 \geq \dfrac{2\left(1+x_1\right)}{\sqrt{\left(1+x_2\right)\left(1+x_3\right)}}$.

Hence, $x_1x_2x_3 \geq \dfrac{2^3\left(1+x_1\right)\left(1+x_2\right)\left(1+x_3\right)}{\sqrt{\left(1+x_2\right)\left(1+x_3\right)} \cdot \sqrt{\left(1+x_3\right)\left(1+x_1\right)} \cdot \sqrt{\left(1+x_1\right)\left(1+x_2\right)}} = 2^3$.

Solution C.

Let $a := \dfrac{1}{1+x_1}, b := \dfrac{1}{1+x_2}, c := \dfrac{1}{1+x_3}$ then $a, b, c \in (0,1), a+b+c = 1$ and $x_1 = \dfrac{1-a}{a} = \dfrac{b+c}{a} \geq \dfrac{2\sqrt{bc}}{a}, x_2 = \dfrac{1-b}{b} = \dfrac{c+a}{b} \geq \dfrac{2\sqrt{ca}}{b}, x_3 = \dfrac{1-c}{c} = \dfrac{a+b}{c} \geq \dfrac{2\sqrt{ab}}{c}$.

Therefore, $x_1x_2x_3 \geq \dfrac{2\sqrt{bc}}{a} \cdot \dfrac{2\sqrt{ca}}{b} \cdot \dfrac{2\sqrt{ab}}{c} = 8$.

Solution D.

First note that at least one of the products x_1x_2, x_2x_3, x_3x_1 must be greater then 1.

Indeed,assume that $x_1x_2, x_2x_3, x_3x_1 \leq 1$. Then since $2 + x_1 + x_2 + x_3 = x_1x_2x_3 \iff$ $1 = \dfrac{2}{x_1x_2x_3} + \dfrac{1}{x_1x_2} + \dfrac{1}{x_2x_3} + \dfrac{1}{x_3x_1}$ and $x_1x_2x_3 = \sqrt{x_1x_2 \cdot x_2x_3 \cdot x_3x_1} \leq 1$

we obtain a contradiction $1 = \dfrac{2}{x_1x_2x_3} + \dfrac{1}{x_1x_2} + \dfrac{1}{x_2x_3} + \dfrac{1}{x_3x_1} \geq 2 + 1 + 1 + 1 \geq 5$.

Let it be $x_1x_2 > 1$ and let $t := \sqrt{x_1x_2}, r := x_1x_2x_3$.

Then $2 + x_1 + x_2 + x_3 = x_1x_2x_3$ becomes

$+\dfrac{r}{t^2} = r$ and, since $x_1 + x_2 \geq 2\sqrt{x_1x_2} = 2t,\ t > 1$, we obtain

$r - \dfrac{r}{t^2} = 2 + x_1 + x_2 \geq 2 + 2t \iff \dfrac{r\left(t^2-1\right)}{t^2} \geq 2\left(t+1\right) \iff r \geq \dfrac{2t^2}{t-1} = 2\left(\dfrac{t^2-1+1}{t-1}\right) =$ $2\left(\left(t-1+\dfrac{1}{t-1}\right)+2\right) \geq 2\left(2+2\right) = 8$, because $t - 1 + \dfrac{1}{t-1} \geq 2$.

Comment by Editor: **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru of Comănesti, Romania,** stated that there is a paper in the Romanian Mathematical Gazette, (Volume CXX, number 11, 2015) pp. 489-498 by **Eugen Păltănea** that presents five solutions and extensions for the following proposition: Let $x_1, x_2, \ldots, x_n > 0, n \geq 2$. If $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1$, then $\sqrt[n]{x_1x_2\ldots x_n} \geq n-1$. They presented a new solution to this proposition and then applied it to problem 5404.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg,

Switzerland; and the authors.

- **5405:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a, b \in \Re$ such that $a+b=1$, $e_n = \left(1+\frac{1}{n}\right)^n$ and $c_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$, then compute

$$\lim_{n\to\infty}\left((n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{e_n}\right)^b.$$

Solution 1 by Ramya Dutta (student, Chennai Mathematical Institute) India

Using $\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$, for $-1 < x < 1$ and the Stirling Approximation:

$\log n! = \left(n+\frac{1}{2}\right)\log n - n + \frac{1}{2}\log 2\pi + O\left(\frac{1}{n}\right)$

For $n > 2$,

$$\begin{aligned}
(n!e_n)^{b/n} &= \exp\left(\frac{b\log n!}{n}\right)\left(1+\frac{1}{n}\right)^b \\
&= \exp\left(b\log n + \frac{b\log n}{2n} - b + \frac{b\log 2\pi}{2n} + O\left(\frac{1}{n^2}\right)\right)\left(1+\frac{1}{n}\right)^b \\
&= e^{-b}n^b\exp\left(\frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + O\left(\frac{1}{n^2}\right)\right)\left(1+\frac{1}{n}\right)^b \\
&= e^{-b}n^b\left(1+\frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + O\left(\frac{\log^2 n}{n^2}\right)\right)\left(1+\frac{b}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= e^{-b}n^b\left(1+\frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right)
\end{aligned}$$

Again, $c_n = H_n - \log n = \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$

Therefore,

$$\begin{aligned}
c_n^{b/(n+1)} &= \exp\left(\frac{b\log c_n}{n+1}\right) \\
&= \exp\left(\frac{b\log\gamma}{n+1} + \frac{b}{n+1}\log\left(1+\frac{1}{2\gamma n} + O\left(\frac{1}{n^2}\right)\right)\right) \\
&= \exp\left(\frac{b\log\gamma}{n} + O\left(\frac{1}{n^2}\right)\right)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&((n+1)!c_n)^{b/(n+1)} \\
&= e^{-b}(n+1)^b \exp\left(\frac{b\log(n+1)}{2(n+1)} + \frac{b\log 2\pi}{2(n+1)} + O\left(\frac{1}{n^2}\right)\right) c_n^{b/(n+1)} \\
&= e^{-b}(n+1)^b \exp\left(\frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + O\left(\frac{1}{n^2}\right)\right) \exp\left(\frac{b\log\gamma}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= e^{-b}(n+1)^b \left(1 + \frac{b\log n}{2n} + \frac{b\log(2\pi\gamma^2)}{2n} + O\left(\frac{\log^2 n}{n^2}\right)\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{n\to\infty} (n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{(n!e_n)^b} \\
&= \lim_{n\to\infty} e^{-b}(n+1)\left(1 + \frac{b\log n}{2n} + \frac{b\log(2\pi\gamma^2)}{2n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&\quad - e^{-b}n\left(1 + \frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&= \lim_{n\to\infty} e^{-b}\left(1 + O\left(\frac{\log n}{n}\right)\right) + e^{-b}n\left(\frac{b\log\gamma}{n} - \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&= \lim_{n\to\infty} e^{-b}(1 + b\log\gamma - b) + O\left(\frac{\log n}{n}\right) = e^{-b}(a + b\log\gamma)
\end{aligned}
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The n^{th} harmonic number admits the asymptotic expansion $\sum_{k=1}^{n}\frac{1}{k} = \ln n + \gamma + O\left(\frac{1}{n}\right)$, as $n\to\infty$. (See for instance *https://en/wikipedia.org/wiki/Harmonic_number.*)

Stirling's formula states that $n! = \sqrt{2\pi n}n^n e^{-n}\left(1 + O\left(\frac{1}{n}\right)\right)$, as $n\to\infty$. (See for instance *https://en/wikipedia.org/wiki/Stirling %27s approximation*).

So

$$
\begin{aligned}
&(n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} = \\
= \quad & (n+1)^{a+b}(2\pi)^{\frac{b}{2(n+1)}}(n+1)^{\frac{b}{2(n+1)}} e^{-b}\left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{b}{n+1}}\left(\gamma + O\left(\frac{1}{n}\right)\right)^{\frac{b}{n+1}} \\
= \quad & (n+1)e^{-b}\left(1 + \frac{b}{2(n+1)}\log(2\pi) + \frac{b}{2(n+1)}\log(n+1) + \frac{b}{(n+1)}\log\gamma + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
= \quad & e^{-b}\left(n + 1 + \frac{b}{2}\log(2\pi) + \frac{b}{2}\log(n+1) + b\log\gamma + O\left(\frac{\log^2 n}{n}\right)\right),
\end{aligned}
$$

$$n^n \sqrt[n]{(n!e_n)^b} = n^{a+b}(2\pi)^{\frac{b}{2n}} n^{\frac{b}{2n}} e^{-b} \left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{b}{n}} \left(1 + \frac{1}{n}\right)^b$$

$$= \quad ne^{-b}\left(1 + \frac{b}{2n}\log(2\pi) + \frac{b}{2n}\log(n) + \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right),$$

$$= \quad e^{-b}\left(n + \frac{b}{2}\log(2\pi) + \frac{b}{2}\log(n) + b + O\left(\frac{\log^2 n}{n}\right)\right).$$

Thus

$$\lim_{n\to\infty}\left((n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{(n!e_n)^b}\right)$$

$$= \lim_{n\to\infty}\left(e^{-b}\left(n + 1 + \frac{b}{2}\log(2\pi) + \frac{b}{2}\log(n+1) + b\log(\gamma) - n - \frac{b}{2}\log(2\pi) - \frac{b}{2}\log(n) - b + O\left(\frac{\log^2 n}{n}\right)\right)\right)$$

$$= \quad e^{-b}\left(1 + b\log\gamma - b\right) = e^{-b}\left(a + b\log\gamma\right).$$

Also solved by Arkady Alt, San Jose, CA; Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the proposers.

- **5406:** *Proposed by Cornel Ioan Vălean, Timis, Romania*

Calculate:

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the harmonic number.

Solutions 1 and 2 by Ramya Dutta (student), Chennai Mathematical Institute India

Solution (1):

Changing the order of summation in ($\star$) and using $\sum_{n=1}^{k} \frac{H_n}{n} = \frac{H_k^2 + H_k^{(2)}}{2}$, we have:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(3) - \sum_{k=1}^{n}\frac{1}{k^3}\right) &= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{H_n}{nk^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^4} \qquad (\star)\\ &= \sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{k}\frac{H_n}{n} - \sum_{n=1}^{\infty}\frac{H_n}{n^4}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k^2 + H_k^{(2)}}{k^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^4}\end{aligned}$$

Lemma: $\displaystyle\sum_{k=1}^{\infty}\frac{H_k}{k(n+k)}=\frac{1}{n}\left(\frac{1}{2}H_n^2+\frac{1}{2}H_n^{(2)}+\zeta(2)-\frac{H_n}{n}\right)$

Proof:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{H_k}{k(n+k)} &= \sum_{k=1}^{\infty}\sum_{j=1}^{k}\frac{1}{jk(n+k)}=\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\frac{1}{jk(n+k)} && (1)\\
&= \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\frac{1}{jk(n+k)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (2)\\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(k+j)(n+k+j)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (3)\\
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{jk(n+k+j)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (4)\\
&= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{n+k}}{k(n+k)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (5)\\
&= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{n+k}}{k(n+k)}+\frac{1}{n}\left(\zeta(2)-\frac{H_n}{n}\right) && (6)
\end{aligned}$$

Justifications: (1) Interchanged order of summation, (3) made the change in variable $k\mapsto k+j$, (4) used the symmetry of the summation w.r.t. k and j,

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(k+j)(n+k+j)}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(k+j)(n+k+j)}\\
&=\frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(k+j)(n+k+j)}+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(k+j)(n+k+j)}\right)\\
&=\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{jk(n+k+j)},
\end{aligned}$$

(5) used the identity, $\displaystyle\frac{H_m}{m}=\frac{1}{m}\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{m+j}\right)=\sum_{j=1}^{\infty}\frac{1}{j(m+j)}$ and

(6) used partial fraction, $\displaystyle\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)}=\sum_{j=1}^{\infty}\left(\frac{1}{nj^2}-\frac{1}{nj(n+j)}\right)=\frac{1}{n}\left(\zeta(2)-\frac{H_n}{n}\right)$.

Again,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{H_{n+k}}{k(n+k)} &= \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{H_k}{k} - \frac{H_{n+k}}{n+k} \right) + \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{H_{n+k} - H_k}{k} \right) && (7) \\
&= \frac{1}{n} \sum_{k=1}^{n} \frac{H_k}{k} + \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{j=1}^{n} \frac{1}{k+j} \right) && (8) \\
&= \frac{1}{n} \sum_{k=1}^{n} \frac{H_k}{k} + \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+j} \right) && (9) \\
&= \frac{2}{n} \sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2 + H_n^{(2)}}{n} && (10)
\end{aligned}$$

Thus, combining lines (6) and (10),

$$\sum_{k=1}^{\infty} \frac{H_k}{k(n+k)} = \frac{1}{n} \left(\frac{1}{2} H_n^2 + \frac{1}{2} H_n^{(2)} + \zeta(2) - \frac{H_n}{n} \right) \qquad \square$$

Now, dividing both sides of the identity with n^2 and summing over $n \geq 1$,

$$\begin{aligned}
\frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^3} + \zeta(2)\zeta(3) - \sum_{n=1}^{\infty} \frac{H_n}{n^4} &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_k}{kn^2(n+k)} \\
&= \sum_{k=1}^{\infty} \frac{H_k}{k^2} \left(\zeta(2) - \frac{H_k}{k} \right)
\end{aligned}$$

where, we used partial fraction decomposition from line (6) earlier. That is,

$$\frac{3}{2} \sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^3} = \zeta(2) \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \zeta(2)\zeta(3) + \sum_{n=1}^{\infty} \frac{H_n}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} \qquad (\mathbf{I})$$

Now we provide an evaluation of the Euler sum: $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3}$.

Consider the partial fraction decomposition,

$$\begin{aligned}
\sum_{k=1}^{n-1} \left(\frac{1}{k(n-k)} \right)^2 &= \frac{1}{n^2} \sum_{k=1}^{n-1} \left(\frac{1}{k} + \frac{1}{n-k} \right)^2 \\
&= \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{(n-k)^2} + \frac{2}{n} \left(\frac{1}{k} + \frac{1}{n-k} \right) \\
&= \frac{2}{n^2} \left(H_n^{(2)} + \frac{2H_n}{n} - \frac{3}{n^2} \right)
\end{aligned}$$

Dividing both sides by n and summing over $n \geq 1$,

$$
\begin{aligned}
2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(H_n^{(2)}+\frac{2H_n}{n}-\frac{3}{n^2}\right) &= \sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{1}{nk^2(n-k)^2} \qquad \text{(change of variable } n=m+k\text{)}\\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2m^2(k+m)}\\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{k(m+k)-k^2}{k^3m^3(k+m)}\\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2m^3}-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{km^3(k+m)}\\
&= \zeta(2)\zeta(3)-\sum_{m=1}^{\infty}\frac{H_m}{m^4}
\end{aligned}
$$

i.e.,

$$
\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=\frac{1}{2}\zeta(2)\zeta(3)-\frac{5}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+3\zeta(5) \qquad \textbf{(II)}
$$

Thus, combining the results from (**I**) and (**II**),

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right) &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n^3}-\sum_{n=1}^{\infty}\frac{H_n}{n^4}\\
&= \frac{1}{3}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\frac{1}{3}\zeta(2)\zeta(3)-\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}\\
&= \frac{1}{3}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\frac{1}{6}\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\zeta(5)
\end{aligned}
$$

Using Euler's summation formula:

$$
\sum_{n=1}^{\infty}\frac{H_n}{n^q}=\left(1+\frac{q}{2}\right)\zeta(q+1)-\frac{1}{2}\sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j), \qquad \text{for } q\geq 2
$$

we have the particular cases, $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3)$ and $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^4}=3\zeta(5)-\zeta(2)\zeta(3)$,

i.e.,

$$
\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right)=2\zeta(2)\zeta(3)-\frac{7}{2}\zeta(5)
$$

Solution (2):

We start with evaluating the integral for $a > 0$,

$$\begin{aligned}
\int_0^1 x^{a-1} \log^2(1-x)\, dx &= \lim_{b \to 1} \frac{\partial^2}{\partial b^2} \int_0^1 x^{a-1}(1-x)^{b-1}\, dx \\
&= \lim_{b \to 1} \frac{\partial^2}{\partial b^2} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\
&= \frac{1}{a}\left((\gamma + \psi(a+1))^2 + \zeta(2) - \psi^{(1)}(a+1) \right)
\end{aligned}$$

Thus, $\displaystyle\int_0^1 x^{n-1} \log^2(1-x)\, dx = \frac{H_n^2 + H_n^{(2)}}{n}$

So,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3}\right) &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^3} - \sum_{n=1}^{\infty} \frac{H_n}{n^4} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x^{n-1} \log^2(1-x)\, dx - \sum_{n=1}^{\infty} \frac{H_n}{n^4} \\
&= \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2(x) \log^2(1-x)}{x}\, dx - \sum_{n=1}^{\infty} \frac{H_n}{n^4}
\end{aligned}$$

Using the reflection formula for Dilogarithm,
$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log x \log(1-x)$
we may rewrite the integral as,

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{Li}_2(x) \log^2(1-x)}{x}\, dx \\
&= \zeta(2) \underbrace{\int_0^1 \frac{\log^2(1-x)}{x}\, dx}_{\textbf{(I)}} - \underbrace{\int_0^1 \frac{\log x \log^3(1-x)}{x}\, dx}_{\textbf{(II)}} - \underbrace{\int_0^1 \frac{\mathrm{Li}_2(1-x) \log^2(1-x)}{x}\, dx}_{\textbf{(III)}}
\end{aligned}$$

The first integral **(I)**:

$$\begin{aligned}
\int_0^1 \frac{\log^2(1-x)}{x}\, dx &= \int_0^1 \frac{\log^2 x}{1-x}\, dx \\
&= \sum_{n=1}^{\infty} \int_0^1 x^{n-1} \log^2 x\, dx \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3)
\end{aligned}$$

The second integral **(II)**: Using $\displaystyle\frac{\log(1-x)}{1-x} = -\sum_{n=1}^{\infty} H_n x^n$,

$$\begin{aligned}
\int_0^1 \frac{\log x \log^3(1-x)}{x}\, dx &= \int_0^1 \frac{\log^3 x \log(1-x)}{1-x}\, dx \\
&= -\sum_{n=1}^{\infty} \int_0^1 H_n x^n \log^3 x\, dx \\
&= 6\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4} = 6\sum_{n=1}^{\infty} \frac{H_n}{n^4} - 6\zeta(5)
\end{aligned}$$

The third integral **(III)**:

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(1-x)\log^2(1-x)}{x}\,dx &= \int_0^1 \frac{\mathrm{Li}_2(x)\log^2 x}{1-x}\,dx \\ &= \sum_{n=1}^{\infty}\int_0^1 H_n^{(2)}x^n\log^2 x\,dx \\ &= 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3} = 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} - 2\zeta(5)\end{aligned}$$

Combining the results,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right) &= \zeta(2)\zeta(3) - 4\sum_{n=1}^{\infty}\frac{H_n}{n^4} + 4\zeta(5) - \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} \\ &= \frac{1}{2}\zeta(2)\zeta(3) - \frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4} + \zeta(5) \\ &= 2\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5)\end{aligned}$$

Editor's comment : **Albert Stadler of Herrliberg, Switzerland** mentioned in his solution that the expression $\displaystyle\sum_{k=1}^{\infty}\frac{H_k}{k^4} = -\frac{\pi^2}{6}\zeta(3) + 3\zeta(5)$ is due to Euler and that Euler went on to generalize this formula as follows:

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^m} = m + 2\zeta(m+1) - \sum_{n=1}^{m-2}\zeta(m-n)\zeta(n+1), m = 2,3,\ldots$$

The reference he gave for this is: L.Euler, Meditationes circa singulare serierum genus, Novi Comm. Acad. Sci. Petropolitanae 20 (1775), 140-186. Reprinted in Opera Omnia, ser. I, vol. 15, B.G. Teubner, Berlin, 1927, pp 217-267.

Solution 3 by Moti Levy, Rehovot, Israel

We calculate the sum by expressing it as a sum of definite integrals (involving polylogarithmic function) and then make use of results by Prof. Pedro Freitas [1].
The tail of $\zeta(3)$ is

$$\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} = \sum_{k=1}^{\infty}\frac{1}{(n+k)^3}. \tag{11}$$

The following definite integral is known [2]:

$$\int_0^1 x^n \ln^2 x dx = \frac{2}{(n+1)^3}. \tag{12}$$

Substituting (11) in (12) and changing the order of summation and integration give,

$$\begin{aligned}\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} &= \frac{1}{2}\sum_{k=1}^{\infty}\int_0^1 x^{n+k-1}\ln^2 x dx \\ &= \frac{1}{2}\int_0^1 x^n \ln^2 x\sum_{k=1}^{\infty}x^{k-1}dx = \frac{1}{2}\int_0^1 \frac{x^n}{1-x}\ln^2 x dx.\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)=\sum_{n=1}^{\infty}\frac{H_n}{n}\frac{1}{2}\int_0^1\frac{x^n}{1-x}\ln^2 x dx=\frac{1}{2}\int_0^1\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\frac{\ln^2 x}{1-x}dx \tag{13}$$

Let $F(x) := \sum_{n=1}^{\infty}\frac{H_n}{n}x^n$, then $\frac{dF}{dx}=\frac{1}{x}\sum_{n=0}^{\infty}H_n x^n$. The generating function of the sequence $(H_n)_{n\geq 0}$ is well known [3]

$$\sum_{n=0}^{\infty}H_n x^n = -\frac{\ln(1-x)}{1-x}.$$

It follows that $\frac{dF}{dx}=-\frac{\ln(1-x)}{x(1-x)}$. To find $F(x)$ we integrate,

$$F(x)=-\int_0^x\frac{\ln(1-t)}{t(1-t)}dt=-\int_0^x\frac{\ln(1-t)}{1-t}dt-\int_0^x\frac{\ln(1-t)}{t}dt=\frac{1}{2}\ln^2(1-x)+\mathrm{Li}_2(x) \tag{14}$$

Now we substitute (14) in (13) and obtain the required sum as a sum of two definite integrals,

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)=\frac{1}{4}\int_0^1\frac{\ln^2 x\ln^2(1-x)}{1-x}dx+\frac{1}{2}\int_0^1\frac{\ln^2 x}{1-x}\mathrm{Li}_2(x)dx.$$

These definite integrals appear in [1] as entries in Table 6:

$$\int_0^1\frac{\ln^2 x\ln^2(1-x)}{1-x}dx=-4\zeta(2)\zeta(3)+8\zeta(5).$$

$$\int_0^1\frac{\ln^2 x}{1-x}\mathrm{Li}_2(x)dx=6\zeta(2)\zeta(3)-11\zeta(5).$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)&=\frac{1}{2}(6\zeta(2)\zeta(3)-11\zeta(5))+\frac{1}{4}(-4\zeta(2)\zeta(3)+8\zeta(5))\\&=2\zeta(2)\zeta(3)-\frac{7}{2}\zeta(5)=\frac{\pi^2}{3}\zeta(3)-\frac{7}{2}\zeta(5)\cong 0.325\,36.\end{aligned}$$

References:

[1] Freitas Pedro, "*Integrals of Polylogarithmic functions, recurrence relations, and associated Euler sums*", arXiv:math/0406401v1 [math.CA] 21 Jun 2004.
[2] Gradshteyn and Ryzhik, "*Table of Integrals, Series and Products*" (7Ed, Elsevier, 2007), Entry **2.723-2**.
[3] Ronald L. Graham, Donald E. Knuth, Oren Patashnik "*Concrete Mathematics, A Foundation for Computer Science*", 2nd Edition 1994, page 352, (7.43).

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the sum of the problem, denoted by S, equals $\dfrac{4\zeta(2)\zeta(3)-7\zeta(5)}{2}$.

We need the facts that

$$\begin{aligned}\frac{H_n}{n} &= -\int_0^1 x^{n-1}\ln(1-x)dx, \quad \text{(see p. 206, of [2])},\\ \frac{1}{(n+m)^3} &= \frac{1}{2}\int_0^1 x^{m+n-1}\ln^2 x dx, \quad \text{(see formula 2.723 of [3]), and}\end{aligned}$$

$$\gamma(3)-\sum_{m=1}^{n}\frac{1}{m^3} \quad = \quad \sum_{m=1}^{\infty}\frac{1}{(n+m)^3}.$$

For $-1\leq 1$ let $\text{Li}_2(x)=\sum_{m=1}^{\infty}\frac{x^n}{n^2}$. By following closely the method of solution of problem 3.62 in $[2,\ p.\ 211-213]$, we obtain,

$$\begin{aligned} S=-\frac{1}{2}\int_0^1\int_0^1\frac{y\ln^2 y\ln(1-x)}{(1-y)(1-xy)}dxdy \quad &= \quad -\frac{1}{2}\int_0^1\frac{y\ln^2 y}{1-y}\left(\frac{-\frac{1}{2}\ln^2(1-y)-Li_2(y)}{y}\right)dy \\ &= \quad \frac{1}{4}I+\frac{1}{2}J, \end{aligned}$$

where $I=\displaystyle\int_0^1\frac{\ln^2 y\ln^2(1-y)}{1-y}dy$ and $J=\displaystyle\int_0^1\frac{\ln^2 y_2(1-y)}{1-y}dy$. It is known [1, p.1436, Table 6] that $I=8\zeta(5)-4\zeta(2)\zeta(3)$ and $J=6\zeta(2)\zeta(3)-11\zeta(5)$.

Hence the claimed result for the sum of the problem.

References:

1. Freitas P.: Integrals of polylogarithmic functions, recurrence relations and associated Euler sums, Mathematics of Computation, vol. 74, number 251, 1425-1440 (2005).

2. Furdui O.: Limits, Series, and Fractional Part Integrals, Springer, New Hork, (2013)

3. Gradshteyn, I.S. and Ryzhik, I.M.: Tables of Intgerals, Series, and Products, Seventh Edition, Elsevier (2007).

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5407:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all triples (a,b,c) of positive reals such that

$$\begin{aligned} a+b+c &= 1, \\ \frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} &= \frac{243}{16}. \end{aligned}$$

Solution 1 by Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru of Comănesti, Romania

Since $a+b+c=1$ then $a+bc=a\cdot 1+bc=a(a+b+c)+bc=(a+b)(a+c)$. We denote $a+b=x, b+c=y$ and $c+a=z$ then $x+y+z=2$. Using well-known inequalities we have

$$\frac{243}{16} \quad = \quad \frac{1}{x^2y^2}+\frac{1}{y^2z^2}+\frac{1}{z^2x^2}$$

$$\geq \frac{1}{xy}\cdot\frac{1}{yz}+\frac{1}{yz}\cdot\frac{1}{zx}+\frac{1}{zx}\cdot\frac{1}{xy}$$

$$= \frac{1}{xyz}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq \frac{1}{\frac{x+y+z}{3}}\cdot\frac{9}{x+y+z}$$

$$= \frac{27}{8}\cdot\frac{9}{2}=\frac{243}{16}.$$

Hence, $x=y=z \Longrightarrow a=b=c=\dfrac{1}{3}$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Assume that $a,b,c>0$ and $a+b+c=1$. Then, by the Arithmetic - Geometric Mean Inequality,

$$a+bc \leq a+\frac{(b+c)^2}{4}=a+\frac{(1-a)^2}{4}=\frac{(a+1)^2}{4},$$

with equality if and only if $b=c$. Since $a,b,c>0$, it follows that

$$\frac{1}{(a+bc)^2} \geq \frac{16}{(a+1)^4}, \tag{1}$$

with equality if and only if $b=c$. Similar steps show that

$$\frac{1}{(b+ca)^2} \geq \frac{16}{(b+1)^4}, \tag{2}$$

with equality if and only if $c=a$, and

$$\frac{1}{(c+ab)^2} \geq \frac{16}{(c+1)^4}, \tag{3}$$

with equality if and only if $a=b$. By (1), (2), and (3), we have

$$\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} \geq 16\left[\frac{1}{(a+1)^4}+\frac{1}{(b+1)^4}+\frac{1}{(c+1)^4}\right], \tag{4}$$

with equality if and only if $a=b=c=\dfrac{1}{3}$.

Further, if $f(x)=\dfrac{1}{x^4}$, then $f''(x)=\dfrac{20}{x^6}>0$ on $(0,\infty)$, and hence, $f(x)$ is strictly convex on $(0,\infty)$. If we use Jensen's Theorem, we obtain

$$\begin{aligned}\frac{1}{(a+1)^4}+\frac{1}{(b+1)^4}+\frac{1}{(c+1)^4} &= f(a+1)+f(b+1)+f(c+1)\\ &\geq 3f\left[\frac{(a+1)+(b+1)+(c+1)}{3}\right]\\ &= 3f\left(\frac{4}{3}\right)\\ &= \frac{243}{256}, \end{aligned} \tag{5}$$

with equality if and only if $(a+1)=(b+1)=(c+1)$, i.e., if and only if $a=b=c=\dfrac{1}{3}$.

By combining (4) and (5), we see that the conditions $a,b,c>0$ and $a+b+c=1$ imply that

$$\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2}\geq 16\left(\frac{243}{256}\right)=\frac{243}{16},$$

with equality if and only if $a=b=c=\dfrac{1}{3}$. Therefore, the unique solution for our system must be $a=b=c=\dfrac{1}{3}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$(1-a)^2-4bc=(b+c)^2-4bc=(b-c)^2\geq 0$ with equality iff $b=c$

$\Longrightarrow \dfrac{(1+a)^2}{4}=a+\dfrac{(1-a)^2}{4}\geq a+bc>0$ with equality iff $b=c\Longrightarrow \dfrac{1}{(a+bc)^2}\geq\dfrac{16}{(1+a)^4}$ with equality iff $b=c$, and cyclically, so

$$\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2}\geq 16\left(\frac{1}{(1+a)^4}+\frac{1}{(1+b)^4}+\frac{1}{(1+c)^4}\right)$$

with equality iff $a=b=c=\dfrac{1}{3}$. By the arithmetic mean$-$geometric mean inequality,

$$\begin{aligned}\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} &\geq 16\cdot 3\sqrt[3]{\frac{1}{(1+a)^4(1+b)^4(1+c)^4}}\\ &= \frac{48}{\left(\sqrt[3]{(1+a)(1+b)(1+c)}\right)^4}\\ &\geq \frac{48}{\left(\dfrac{1+a+1+b+1+c}{3}\right)^4}=\frac{48}{\left(\dfrac{4}{3}\right)^4}=\frac{243}{16}\end{aligned}$$

with equality iff $a=b=c=\dfrac{1}{3}$, so from this and the second of the given equations we conclude that $a=b=c=\dfrac{1}{3}$.

Editor′s comment: **D.M. Bătinetu-Giurgiu, of "Matei Basarab" National College, Bucharest, Romanina and Neculai Stanciu of "George Emil Palade" School Buzău, Romania** generalized the problem as follows:

If $a,b\geq 0, a+b,c,d,m>0,x,y,z>0$, such that $x+y+z=s>0$ and

$\dfrac{(as+bx)^{m+1}}{(cx+dyz)^m}+\dfrac{(as+bt)^{m+1}}{(cy+dzx)^m}+\dfrac{(as+bz)^{m+1}}{(cz+dxy)^m}=\dfrac{3^m(3a+b)^{m+1}s}{(3c+ds)^m}$, then find all triples$(x,y,z)$.

They found the solution that since $x+y+z=s$, then $s^2\geq 3(xy+yz+zx)$ with equality iff $x=y=z=\dfrac{s}{3}$.

If $s=1, m=2, a=1, b=0, c=1, d=1$ we obtain $x+y+z=1$ and $\sum_{cyc} \dfrac{1}{(x+yz)^2} = \dfrac{243}{16}$, i.e., problem 5407.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Ramya Dutta (student), Chennai Mathematical Institute, India; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro,GA, and the proposer.

- **5408:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate:

$$\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)} dx.$$

Solution 1 by Albert Stadler, Herriliberg, Switzerland

$$\begin{aligned}
\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}\, dx &= \int_0^1 \left(\frac{1}{x} - \frac{1}{1-x}\right) \ln(x) \ln(1-x) dx \\
&= 2\int_0^1 \frac{(\ln x)(\ln(1-x)}{x} dx \\
&= -2\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln x dx \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3).
\end{aligned}$$

Solution 2 by Moti Levy, Rehovot, Israel

Since $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$ and $\int_0^1 \frac{\ln x \ln(1-x)}{x} dx = \int_0^1 \frac{\ln x \ln(1-x)}{(1-x)} dx$ then

$$I := \int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)} dx = 2\int_0^1 \frac{\ln x \ln(1-x)}{x} dx.$$

Using the Taylor series of $\ln(1-x)$ for $0 < x < 1$, and changing the order of summation and integration,

$$I = -2\int_0^1 \frac{\ln x}{x} \left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right) dx = -2\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^{k-1} \ln x dx.$$

Gradshteyn and Ryzhik, entry **2.723-1,**

$$\int x^n \ln x dx = x^{n+1} \left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right).$$

$$I = -2\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^{k-1} \ln x dx = 2\sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{k^2} = 2\zeta(3).$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the integral of the problem , denoted by I equals $2\sum_{n=1}^{\infty}\frac{1}{n^3}$.

It is well known that for non-negative integers n.

$$\int x^n \ln x dx = x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right) + constant.$$

Hence for $0 < a < 1$, we have

$$\begin{aligned}
\int_0^a \frac{\ln x\ (1-x)}{x}dx &= -\int_0^a \ln x \sum_{n=0}^{\infty}\frac{x^n}{n+1}dx = -\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^a x^n \ln x dx \\
&= -\ln a \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)^2} + \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)^3}, \text{ so that} \\
\int_0^1 \frac{\ln x\ (1-x)}{x}dx &= \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)^3}.
\end{aligned}$$

Since $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$, so

$$I = \int_0^1 \frac{\ln x \ln(1-x)}{x}dx + \int_0^1 \frac{\ln x \ln(1-x)}{1-x}dx = 2\int_0^1 \frac{\ln x \ln(1-x)}{x}dx = 2\sum_{n=0}^{\infty}\frac{1}{(n+1)^3},$$

as asserted.

Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA

A generating function for the Harmonic numbers is

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}.$$

The radius of convergence for this series is 1, so the order of summation and integration can be reversed to yield

$$\begin{aligned}
\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}dx &= -\int_0^1 \frac{\ln x}{x}\left(\sum_{n=1}^{\infty} H_n x^n\right)dx \\
&= -\sum_{n=1}^{\infty} H_n \int_0^1 x^{n-1}\ln x dx \\
&= \sum_{n=1}^{\infty}\frac{H_n}{n^2}
\end{aligned}$$

$$= \quad 2\zeta(3).$$

Solution 5 by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

Let I denote the above integral and let $f(x) = \ln x \ln(1-x)$ and $g'(x) = \dfrac{1}{x(1-x)} = \dfrac{1}{x} + \dfrac{1}{1-x}$. Then $f'(x) = \dfrac{\ln(1-x)}{x} - \dfrac{\ln x}{1-x}$ and $g(x) = \ln x - \ln(1-x)$. Evaluating I by parts we have

$$\begin{aligned}
I &= [f(x)g(x)]_0^1 - \int_0^1 f'(x)g(x)dx \\
&= [\ln x \ln(1-x)(\ln x - \ln(1-x))]_0^1 - \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln x}{1-x}\right)(\ln x - \ln(1-x))dx \\
&= \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln x}{1-x}\right)(\ln(1-x) - \ln x)dx \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}dx + \int_0^1 \frac{\ln^2 x}{1-x}dx - \int_0^1 \frac{\ln x \ln(1-x)}{1-x}dx - \int_0^1 \frac{\ln x \ln(1-x)}{x}dx
\end{aligned}$$

Let $I_1 = \int_0^1 \dfrac{\ln^2(1-x)}{x}dx$, then $\int_0^1 \dfrac{\ln^2 x}{1-x}dx = I_1$ (with the substitution $y = 1-x$). Similarly let $I_2 = \int_0^1 \dfrac{\ln x \ln(1-x)}{x}dx$, then $\int_0^1 \dfrac{\ln x \ln(1-x)}{1-x}dx = I_2$ (with the substitution $y = 1-x$). So, $I = 2(I_1 - I_2)$. But we also notice that integration of I_2 by parts yields (taking $1/x$ as second function)

$$\begin{aligned}
I_2 = \int_0^1 \frac{\ln x \ln(1-x)}{x}dx &= \left[\ln^2 x \ln(1-x)\right]_0^1 - \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln x}{1-x}\right)\ln x dx \\
&= \int_0^1 \frac{\ln^2 x}{1-x}dx - \int_0^1 \frac{\ln x \ln(1-x)}{x}dx = I_1 - I_2.
\end{aligned}$$

Thus $I_2 = \frac{1}{2}I_1$ and so $I = 2(I_1 - I_2) = I_1$. Now to calculate I_1, we notice that

$$I_1 = \int_0^1 \frac{\ln^2 x}{1-x}dx = \sum_{n=0}^{\infty}\int_0^1 x^n \ln^2 x dx \tag{15}$$

Now from integration by parts we have (by taking x^n as the second function)

$$\begin{aligned}
\int_0^1 x^n \ln^2 x dx &= \left[(\ln^2 x)\frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{2\ln x}{x}\cdot\frac{x^{n+1}}{n+1}dx = -\frac{2}{n+1}\int_0^1 x^n \ln x dx \\
&= -\frac{2}{n+1}\left[\left[(\ln x)\frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}dx\right] = \frac{2}{n+1}\int_0^1 \frac{x^n}{n+1}dx = \frac{2}{(n+1)^3}.
\end{aligned}$$

Substituting the result obtained above in (1), we get $I_1 = \sum_{n=0}^{\infty} \dfrac{2}{(n+1)^3} = 2\zeta(3)$. Thus, $I = I_1 = 2\zeta(3)$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Pat Costello, Eastern Kentucky University, Richmond, KY; Ramya Dutta (student Chennai Mathematical Institute), India; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Late Acknowledgment

Henry Ricardo of the New York Math Circle should have been credited for having solved problem 5397.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2017

- **5427:** *Proposed by Kenneth Korbin, New York, NY*

  Rationalize and simplify the fraction

  $$\frac{(x+1)^4}{x(2016x^2-2x+2016)} \text{ if } x=\frac{2017+\sqrt{2017-\sqrt{2017}}}{2017-\sqrt{2017-\sqrt{2017}}}.$$

- **5428:** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

  If $x>0$, then $\dfrac{[x]}{\sqrt[4]{[x]^4+([x]+2\{x\})^4}}+\dfrac{\{x\}}{\sqrt[4]{\{x\}^4+([x]+2\{x\})^4}}\geq 1-\dfrac{1}{\sqrt[4]{2}}$, where $[.]$ and $\{.\}$ respectively denote the integer part and the fractional part of x.

- **5429:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Prove that there are infinitely many positive integers a, b such that
  $18a^2-b^2-6a-b=0$.

- **5430:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

  Let a, b, c be the side-lengths, α, β, γ the angles, and R, r the radii respectively of the circumcircle and incircle of a triangle. Show that

  $$\frac{a^3\cdot\cos(\beta-\gamma)+b^3\cdot\cos(\gamma-\alpha)+c^3\cdot\cos(\alpha-\beta)}{(b+c)\cos\alpha+(c+a)\cos\beta+(a+b)\cos\gamma}=6Rr.$$

- **5431:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let F_n be the n^{th} Fibonacci number defined by $F_1=1, F_2=1$ and for all $n\geq 3$,
  $\mathrm{F}_n=F_{n-1}+F_{n-2}$. Prove that

  $$\sum_{n=1}^{\infty}\left(\frac{1}{11}\right)^{F_nF_{n+1}}$$

  is an irrational number and determine it (*).

The asterisk $(*)$ indicates that neither the author of the problem nor the editor are aware of a closed form for the irrational number.

- **5432:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all differentiable functions $f:(0,\infty)\to(0,\infty)$, with $f(1)=\sqrt{2}$, such that

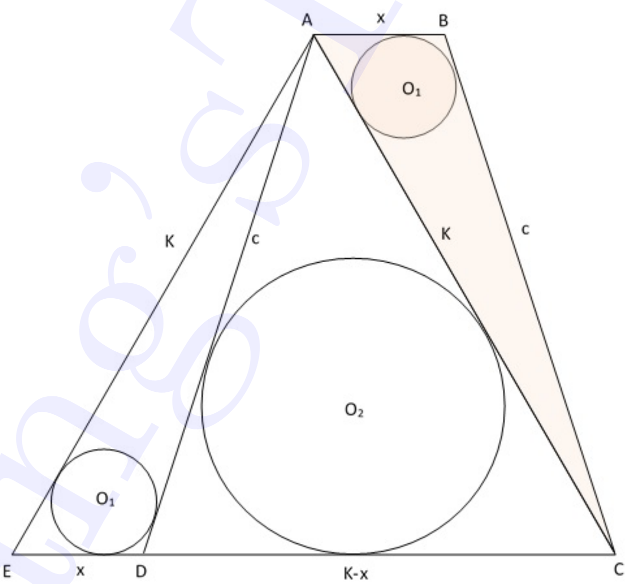
$$f'\left(\frac{1}{x}\right)=\frac{1}{f(x)},\ \forall x>0.$$

Solutions

- **5409:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles trapezoid $ABCD$ with $\overline{AB}<\overline{CD}$, and with diagonal $\overline{AC}=\overline{AB}+\overline{CD}$. Find the perimeter of the trapezoid if $\triangle ABC$ has inradius 12 and if $\triangle ACD$ has inradius 35.

  Solution 1 by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel



Let $|AB|=x, |AC|=|AD|=c, |AC|=K$ so that, since $|AC|=|AB|+|CD|$, we can write $|DC|=K-x$.
The key observation is that *if the triangle ABC is reflected and transposed so that BC coincides with AD, the resulting figure $AEDC$ is an equilateral triangle.* This is so because:
1) The trapezoid is isosceles, so that $EDA=\pi-ADC$, and, therefore, EDC is a straight line
2) By the given, $|AB|+|DC|=|ED|+|DC|=x+K-x=K$, and, therefore, $|EA|=|AC|=|CE|=K$.

With the geometry of the situation in mind, one can now easily see that since the diameter of O_1 is 24 and the diameter of O_2 is 70, the length of the side of the equilateral triangle (i.e. the diagonal of the original trapezoid) cannot be less than 94 units. This will be important later.
Now, since the twice the area of a triangle is the product of its inradius and its perimeter, we find that twice the area of the triangle AED is $12(c+K+x)$ and twice the area of the triangle ADC is $35(c+K+K-x)$. On the other hand, since we have observed that EAC is equilateral, twice the areas of these triangles are also, respectively, $\frac{\sqrt{3}}{2}Kx$ and $\frac{\sqrt{3}}{2}K(K-x)$. Thus, we can write the following two equations:

$$c+K+x=\frac{\sqrt{3}}{24}Kx \tag{1}$$

$$c+2K-x=\frac{\sqrt{3}}{70}K(K-x) \tag{2}$$

Using the law of cosines in the triangle AED and the fact that angle $< AED = \frac{\pi}{3}$, we have a third equation:

$$c^2=K^2+x^2-Kx \tag{3}$$

Thus, we have three quadratic equations in three unknowns, c, K, and x. We will show that this can be reduced to a single quadratic equation in K, from which we will be able to find x and c.
To make the algebra easier to write out, let us use the following notations:
$q=\frac{\sqrt{3}}{24}$
$p=\frac{\sqrt{3}}{70}$
$Q=qK-1$
$P=pK-1$
The reason for the latter two will become clear in a moment.
Eliminating c from equations 1 and 2, we find that $x=\frac{K(pK-1)}{K(p+q)-2}$ or using our notation above:

$$x=\frac{KP}{P+Q} \tag{4}$$

Note, 1 can be written as $K+c=(qK-1)x=Qx$ (similarly, 2 can be written $K+c=P(K-x)$), so substituting 4 into 1 in this form, we find, after some easy manipulations:

$$c=\frac{K}{P+Q}(PQ-(P+Q)) \tag{5}$$

On the other hand, substituting 4 into 3 and rearranging terms we obtain:

$$c^2=\left(\frac{K}{P+Q}\right)^2((P+Q)^2+P^2-P(P+Q)) \tag{6}$$

Squaring 5, equating it with 6, and canceling $\left(\frac{K}{P+Q}\right)^2$, we obtain:

$$(PQ-(P+Q))^2=(P+Q)^2+P^2-P(P+Q)$$

And after some simplification, we have the equation:

$$PQ(PQ-2(P+Q)+1)=0$$

But since $Qx = K + c > 0$ and $P(K - x) = K + c > 0$, neither P nor Q can be 0, so, we have:

$$PQ - 2(P + Q) + 1 = 0$$

At this point, we can substitute $pK - 1 = P$ and $qK - 1 = Q$, to obtain the following quadratic equation in K:

$$pqK^2 - 3(p + q)K + 6 = 0 \tag{7}$$

Substituting $p = \frac{\sqrt{3}}{70}$ and $q = \frac{\sqrt{3}}{24}$ and solving, we obtain two solutions:
$K = 80\sqrt{3} \approx 138.564$ or $K = 14\sqrt{3} \approx 24.245$. As we noted above, K cannot be less than 94, so we have only $K = 80\sqrt{3}$. Using 4 and 5 to find x and c, we find then that the perimeter of ABCD$= 2c + K = 226\sqrt{3} \approx 391.443$

Solution 2 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let $x = \overline{AB}$, $y = \overline{CD}$ and $z = \overline{AD} = \overline{BC}$. Then the perimeter of the trapezoid $ABCD$ is $x + y + 2z = 226\sqrt{3}$ when $x = 17\sqrt{3}$, $y = 63\sqrt{3}$ and $z = 73\sqrt{3}$.
Denote the area of polygon X by $[X]$. Then, by Ptolemy's theorem, $\overline{AC} = \sqrt{xy + z^2}$. Therefore, x + y = $\sqrt{xy + z^2}$. Solving for z^2, we get $z^2 = x^2 + xy + y^2$. The height h of the trapezoid, according to the Pythagorean theorem, is given by

$$h = \sqrt{z^2 - (\frac{y - x}{2})^2} = \frac{\sqrt{3}}{2}(x + y).$$

Therefore,

$$[ABC] = \frac{1}{2} \cdot x \cdot \frac{\sqrt{3}}{2}(x + y)$$

and

$$[ACD] = \frac{1}{2} \cdot y \cdot \frac{\sqrt{3}}{2}(x + y).$$

Let r denote the inradius of triangle T. Then $r \cdot s = [T]$ where s is the semiperimeter of T. For each of the triangles ACD and ABC, this formula reduces to

$$\frac{35}{2}(x + 2y + z) = \frac{\sqrt{3}}{4}y(x + y),$$
$$6(2x + y + z) = \frac{\sqrt{3}}{4}x(x + y),$$

respectively. Multiplying the first equation by x, the second by y and then subtracting the second equation from the first yields the following upon simplification:

$$z(\frac{35}{2}x - 6y) = 6y^2 - 23xy - \frac{35}{2}x^2.$$

Since $z = \sqrt{x^2 + xy + y^2}$, we have

$$\sqrt{x^2 + xy + y^2}(\frac{35}{2}x - 6y) = 6y^2 - 23xy - \frac{35}{2}x^2.$$

Squaring both sides of this equation and then simplifying it, one obtains the equation

$$136y^2 - 249xy - 945x^2 = 0.$$

Regarding this equation as a quadratic in y, one obtains the following roots

$$\begin{aligned} y &= \frac{249x \pm \sqrt{(249x)^2 + 4(136)(945x^2)}}{272} \\ &= \frac{249x \pm 759x}{272}. \end{aligned}$$

Since $y > 0$ we disregard the negative root so that

$$y = \frac{1008}{272}x = \frac{63}{17}x.$$

Moreover,

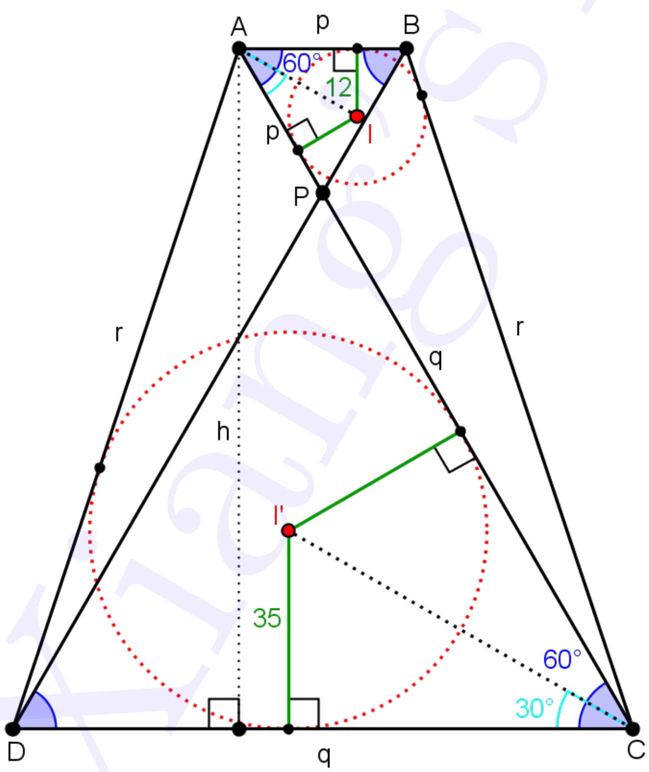
$$z = \sqrt{x^2 + xy + y^2} = \sqrt{x^2 + \frac{63}{17}x^2 + \left(\frac{63}{17}x\right)^2} = \frac{73}{17}x.$$

Thus, our solutions are parametrized by x and the problem now is to find the value(s) of x that satisfy the two equations for the inradius. To that end suppose

$$6\left(2x + \frac{63}{17}x + \frac{73}{17}x\right) = \frac{\sqrt{3}}{4}x\left(x + \frac{63}{17}x\right).$$

Then $x = 17\sqrt{3}$. Similarly, the equation $\frac{35}{2}(x + 2y + z) = \frac{\sqrt{3}}{4}y(x + y)$ yields $x = 17\sqrt{3}$. Hence, $y = \frac{63}{17}x = 63\sqrt{3}$ and $z = \frac{73}{17}x = 73\sqrt{3}$ so that the perimeter of the trapezoid $ABCD$ is $x + y + 2z = 226\sqrt{3}$.

Solution 3 by Nikos Kalapodis, Patras, Greece



Let P be the intersection of diagonals AC and BD. Since trapezoid $ABCD$ is isosceles, the triangles ABC and BAD, as well as, the triangles ACD and BDC are congruent, (SAS criterion).

It follows that the triangles PAB and PCD are isosceles with $PA = PB$ and $PC = PD$ (1).
Furthermore, since they are similar (congruent angles) we have
$\dfrac{PA}{AB} = \dfrac{PC}{CD} = \dfrac{PA + PC}{AB + CD} = \dfrac{AC}{AB + CD} = 1.$ Thus, $PA = AB$ and $PC = CD$ (2).
From (1) and (2) we conclude that triangles PAB and PCD are equilateral.
Let $p = AB = PA = PB$, $q = CD = PC = PD$, $r = BC = AD$, $t = p + q + r$ and h the height of the trapezoid. Then we have
$\dfrac{p}{q} = \dfrac{ph}{qh} = \dfrac{2[ABC]}{2[ACD]} = \dfrac{12(2p + q + r)}{35(p + 2q + r)} = \dfrac{12(p + t)}{35(q + t)}$ or $23pq = t(12q - 35p)$ (3)
Since the trapezoid is isosceles, it is cyclic, so by Ptolemy's Theorem we have
$pq + r^2 = (p + q)^2$ (4) or $pq = t(p + q - r)$ (5)
By (3) and (5) we obtain $58p + 11q = 23r$ (6)
Finally, applying the well-known formula $r = (s - a)\tan\dfrac{A}{2}$ in triangles ACD and BAC we have
$23 = 35 - 12 = \left(\dfrac{p + 2q + r}{2} - r\right)\dfrac{\sqrt{3}}{3} - \left(\dfrac{2p + q + r}{2} - r\right)\dfrac{\sqrt{3}}{3} = \dfrac{q - p}{2}\cdot\dfrac{\sqrt{3}}{3}$, i.e.
$q - p = 46\sqrt{3}$ (7).
Solving the system of equations (4), (6) and (7) we find $p = 17\sqrt{3}$, $q = 63\sqrt{3}$, and $r = 73\sqrt{3}$.
Therefore the perimeter of trapezoid is $p + q + 2r = 226\sqrt{3}$.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Malik Sheykhov (student at the France-Azerbaijan University in Azerbaijan) and Talman Residli (student at Azerbaijan Medical University in Baku, Azerbaijan); David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer

- **5410:** *Proposed by Arkady Alt, San Jose, CA*

For the given integers $a_1, a_2, a_3 \geq 2$ find the largest value of the integer semiperimeter of a triangle with integer side lengths t_1, t_2, t_3 satisfying the inequalities $t_i \leq a_i,\ i = 1, 2, 3$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, we assume that $a_1 \geq a_2 \geq a_3$. Let

$$T_1 = \{2, 3, \ldots, a_1\},\ T_2 = \{2, 3, \ldots, a_2\},\ T_3 = \{2, 3, \ldots, a_3\}$$

$$T = \{(t_1, t_2, t_3) : t_1 \in T_1,\ t_2 \in T_2,\ t_3 \in T_3\} \text{ and}$$

$$S = T \cap \{(t_1, t_2 t_3) : t_1, t_2, t_3 \text{ are the side lengths of a triangle}\}.$$

Let $L = \underset{(\mathrm{t_1, t_2, t_3}) \in \mathrm{S}}{\mathrm{Maximum}} \dfrac{t_1 + t_2 + t_3}{2}$. We show that $L = \begin{cases} \dfrac{a_1 + a_2 + a_3}{2}, & \text{if } a_2 + a_3 > a_1 \\ a_2 + a_3 - \dfrac{1}{2}, & \text{if } a_2 + a_3 \leq a_1. \end{cases}$

Case 1: $\mathrm{a}_2 + a_3 > a_1$

We have $(a_1, a_2, a_3) \in S$ and clearly $L = \dfrac{a_1 + a_2 + a_3}{2}$.

Case 2: $a_2 + a_3 \le a_1$

We have $(a_2 + a_3 - 1, a_2, a_3) \in S$ so that $L \ge a_2 + a_3 - \dfrac{1}{2}$. If $(t_1, t_2, t_3) \in T$ and $t_1 > a_2 + a_3 - 1$, then $(t_1, t_2, t_3) \notin S$. If $(t_1, t_2, t_3) \in T$ then $t_1 < a_2 + a_3 - 1$, then $\dfrac{t_1 + t_2 + t_3}{2} < \dfrac{(a_2 + a_3 - 1) + a_2 + a_3}{2} = a_2 + a_3 - \dfrac{1}{2}$. Hence, $L = a_2 + a_3 - \dfrac{1}{2}$ in this case.

This completes the solution.

Solution 2 by proposer

Let $s = \dfrac{t_1 + t_2 + t_3}{2}$. Since $t_i < s, i = 1, 2, 3$ then by the triangle inequality our problem becomes the following: Find the maximum of s for which there are positive integer numbers t_1, t_2, t_3 satisfying $\mathrm{t}_i \le \min\{a_i, s - 1\}, i = 1, 2, 3, t_1 + t_2 + t_3 = 2s$.

First note that $s \ge 3, t_i \ge 2, i = 1, 2, 3$. Indeed, since $t_i \le s - 1$ then $1 \le s - t_i, i = 1, 2, 3$ and therefore $t_1 = 2s - t_2 - t_3 = (s - t_2) + (s - t_3) \ge 2$. Cyclicly we obtian $t_2, t_3 \ge 2$ Hence, $2s \ge 6 \iff s \ge 3$.

Since $t_3 = 2s - t_1 - t_2,\ 2 \le t_3 \le \min\{a_3, s - 1\}$, then
$1 \le 2s - t_1 - t_2 \le \min\{a_3, s - 1\} \iff \max\{2s - t_1 - a_3, s + 1 - t_1\} \le t_2 \le 2s - 1 - t_1$, and therefore, we obtain the inequality for t_2, namely that

$$(\mathbf{1}) \quad \max\{2s - t_1 - a_3, s + 1 - t_1, 2\} \le t_2 \le \min\{2s - 1 - t_1, a_2, s - 1\}$$

with the conditions of solvability being:

$$(\mathbf{2}) \begin{cases} 2s - t_1 - a_3 & \le & s - 1 \\ 2s - t_1 - a_3 & \le & a_2 \\ s + 1 - t_1 & \le & a_2 \\ 2 & \le & 2s - 1 - t_1 \end{cases} \iff \begin{cases} s + 1 - a_3 & \le & t_1 \\ 2s - a_2 - a_3 & \le & t_1 \\ s + 1 - a_2 & \le & t_1 \\ t_1 & \le & 2s - 3 \end{cases}$$

Since $s - 1 \le 2s - 3$, then $(\mathbf{2})$ together with $2 \le t_1 \le \min\{a_1, s - 1\}$ gives us the bounds for t_1

$(\mathbf{3}) \max\{s + 1 - a_3, 2s - a_2 - a_3, s + 1 - a_2, 2\} \le t_1 \le= \min\{a_1, s - 1\}$.

Since $2 \le a_i, i = 1, 2, 3$ then $s + 1 - a_2 \le s - 1, s + 1 - a_3 \le s - 1$ and the solvability condition for $(\mathbf{3})$ becomes

$$s + 1 - a_3 \le a_1 \iff s \le a_1 + a_3 - 1, 2s - a_2 - a_3 \le a_1 \iff s \le \left\lfloor \frac{a_1 + a_2 + a_3}{2} \right\rfloor,$$

$$s + 1 - a_2 \le a_1 \iff s \le a_1 + a_2 - 1, 2s - a_2 - a_3 \le s - 1 \iff s \le +a_2 + a_3 - 1.$$

Thus, $s^* = \min\left\{\left\lfloor \dfrac{a_1 + a_2 + a_3}{2} \right\rfloor, a_1 + a_2 - 1, a_2 + a_3 - 1, a_3 + a_1 - 1\right\}$ is the largest integer value of the semiperimeter.

Solution 3 by Ed Gray, Highland Beach, FL

We consider several special cases:

a) If $a_1 = a_2 = a_3 = 2k$, we can equate $t_i = a_i$ for each i. The perimeter is then $6k$ and the semiperimeter is $3k$.

b) Suppose $a_1 = a_2 = a_3 = 2k+1$. We note that $a_1 + a_2 = 4k+2$ and $\mathrm{a}_3 - 1 = 2k$. We define $t_1 = a_1,\ t_2 = a_2$ and $t_3 = a_3 - 1$.

c) Suppose that $a_1 = a_2$ and a_3 is larger than either one. In this case we set $t_1 = a_1$ and $t_2 = a_2$. It doesn't matter if a_1, a_2 are both even or both odd, $t_1 + t_2$ is even. We now have to avoid a potential problem. It must be true that $t_1 + t_2 \geq t_3$. Therefore, since if a_3 is large, we need to define $t_3 = a_3 - x$, where x is the integer which is the smallest such that $a_3 - x$ is even and $t_1 + t_2 > t_3$. Since $t_1 + t_2 + t_3$ is even, the semiperimeter is integral.

d) Suppose that $a_1 = a_2$ and a_3 is smaller than either one, in this case set $t_1 = a_1, t_2 = a_2$, so that $t_1 + t_2$ is even. If a_3=2, we let $t_3 = 2$. If $a_3 > 2$, but odd, we we set $t_3 = a_3 - 1$. Then $t_1 + t_2 + t_3$ equals the perimeter which is even and with an integer semiperimeter, and the triangle inequalities hold.

e) Finally, we have the general case: $a_1 < a_2 < a_3$. We set $t_1 = a_1, t_2 = a_2$. If $t_1 + t_2$ is even we need t_3 to be even. If a_3 is very far so that $a_1 + a_2 < a_3$, we let $t_3 = a_3 - x$, where x is the smallest integer which simultaneously makes $t_1 + t_2 + t_3$ even and $t_1 + t_2 > t_3$. If $t_1 + t_2$ is odd, we employ a similar calculation.

Solution 4 by Paul M. Harms, North Newton, KS

Suppose $a_1 \leq a_2 \leq a_3$. The largest perimeter would be $a_1 + a_2 + a_3$ where $t_i = a_i,\ i = 1, 2, 3$ provided that we have a triangle, i.e., $a_1 + a_2 > a_3$.

If $a_1 + a_2 > a_3$, and the perimeter is an even integer, then the largest value of an integer semiperimeter is $\dfrac{a_1 +_2 + a_3}{2}$.

If the perimeter is an odd integer, then a_3 must be at least 3 and we could use sides $t_1 = a_1, t_2 = a_2$ and $t_3 = a_3 - 1$. The largest integer semiperimeter for this case is $\dfrac{a_1 + a_2 + a_3 - 1}{2}$.

Now consider the case where $a_1 + a_2 \leq a_3$. A triangle with a maximum perimeter is when $t_1 = a_1,\ a_2 = t_2$, and $t_3 = a_1 + a_2 - 1$. Here $t_3 > a_1, a_2$ and the perimeter is the odd integer $2a_1 + 2a_2 - 1$. To get the largest integer semiperimeter we could use $t_1 = a_1,\ t_2 = a_2$ and $t_3 = a_1 + a_2 - 2$ which has $a_1 + a_2 - 1$ as the largest integer semiperimeter.

Also solved by Jeremiah Bartz and Timothy Prescott, University of North Dakota, Grand Forks, ND; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

- **5411:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be real valued positive sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a \in R_+^*$

If $\lim\limits_{n\to\infty} (n(a_n - a)) = b \in R$ and $\lim\limits_{n\to\infty} (n(b_n - a)) = c \in R$ compute

$$\lim_{n\to\infty} \left(a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!}\right).$$

Note: R_+^* means the positive real numbers without zero.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

By Stirling's approximation,

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n},$$

so

$$\sqrt[n]{n!} \sim \frac{n}{e} \quad \text{and} \quad \sqrt[n+1]{(n+1)!} \sim \frac{n+1}{e}.$$

It then follows that

$$\begin{aligned} a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!} &\sim \frac{(n+1)a_{n+1}}{e} - \frac{nb_n}{e} \\ &= \frac{1}{e}\left[(n+1)(a_{n+1}-a) - n(b_n-a) + a\right] \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty}\left(a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!}\right) &= \lim_{n\to\infty} \frac{1}{e}\left[(n+1)(a_{n+1}-a) - n(b_n-a) + a\right] \\ &= \frac{1}{e}(b-c+a). \end{aligned}$$

Solution 2: by Moti Levy, Rehovot, Israel.

$$\begin{aligned} &\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} a_{n+1} - (n!)^{\frac{1}{n}} b_n\right) \\ &= \lim_{n\to\infty}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\left((n+1)\left(a_{n+1}-a\right)\right) + ((n+1)!)^{\frac{1}{n+1}} a - \frac{(n!)^{\frac{1}{n}}}{n}\left(n\left(b_n-a\right)\right) - (n!)^{\frac{1}{n}} a\right) \\ &= \lim_{n\to\infty} \frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\left((n+1)\left(a_{n+1}-a\right)\right) - \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\left(n\left(b_n-a\right)\right) + a\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) \\ &= \lim_{n\to\infty} \frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\lim_{n\to\infty}\left((n+1)\left(a_{n+1}-a\right)\right) \\ &- \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\lim_{n\to\infty}\left(n\left(b_n-a\right)\right) + a\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right). \end{aligned}$$

So we are challenged with two limits: $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ and $\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right)$. We will show that both limits equal to $\frac{1}{e}$.
We begin by stating the well-known asymptotic expansion of the Gamma function:

$$\frac{e^x}{x^x\sqrt{2\pi x}}\Gamma(x+1) \sim 1 + \frac{1}{12x}, \qquad x\to\infty.$$

For positive integer n,

$$\left(\frac{e}{n}\right)^n \frac{n!}{\sqrt{2\pi n}} \sim 1 + \frac{1}{12n}, \qquad n\to\infty.$$

Using $\left(1+\frac{1}{12n}\right)^{\frac{1}{n}} \sim 1 + \frac{1}{12n^2}$ and $\left(\sqrt{2\pi n}\right)^{\frac{1}{n}} \sim 1$, we get

$$\frac{e}{n}(n!)^{\frac{1}{n}} \sim 1 + \frac{1}{12n^2}, \qquad n\to\infty,$$

or

$$\frac{(n!)^{\frac{1}{n}}}{n} \sim \frac{1}{e}\left(1+\frac{1}{12n^2}\right), \quad n \to \infty,$$

which implies

$$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}.$$

$$\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) = \lim_{n\to\infty}(n!)^{\frac{1}{n}}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} - 1\right)$$

$$\frac{(n!)^{\frac{1}{n}}}{n} \sim \frac{1}{e}\left(1+\frac{1}{12n^2}\right); \quad \frac{((n+1)!)^{\frac{1}{n+1}}}{n+1} \sim \frac{1}{e}\left(1+\frac{1}{12n^2}\right)$$

$$\frac{\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}}{\frac{(n!)^{\frac{1}{n}}}{n}} \sim 1 \quad \Rightarrow \quad \frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} \sim \frac{n+1}{n} = 1 + \frac{1}{n}$$

$$\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} - 1 \sim \frac{1}{n}$$

$$(n!)^{\frac{1}{n}}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} - 1\right) \sim \frac{n}{e}\left(1+\frac{1}{12n^2}\right)\frac{1}{n} \sim \frac{1}{e}.$$

We conclude that

$$\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) = \frac{1}{e}.$$

Now back to the original limit

$$\begin{aligned}
&\lim_{n\to\infty}\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\lim_{n\to\infty}\left((n+1)\left(a_{n+1}-a\right)\right) \\
&-\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\lim_{n\to\infty}\left(n\left(b_n-a\right)\right) + a\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) \\
&= \frac{1}{e}b - \frac{1}{e}c + a\frac{1}{e} = \frac{a+b-c}{e}.
\end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Paul M. Harms, North Newton, KS; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy, and the proposers.

- **5412:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

Given positive integer M. Find a continuous, non-constant function $f : R \to R$ such that $f(f(x)) = f([x])$, for all real x, and for which the maximum value of $f(x)$ is M. Note: $[x]$ is the greatest integer function.

Solution 1 by Tommy Dreyfus, Tel Aviv University, Israel

Let $f(x) = 0$ except for $M < x < M+1$, where $f(x) = M - 2M\left|x - \left(M + \frac{1}{2}\right)\right|$.

Then f is continuous, attains its maximum at $f\left(M+\frac{1}{2}\right)=M$, and $f(f(x))=f([x])=0$ for all x.

Solution 2 by Albert Stadler, Herrliberg Switzerland

Let $f(x)=\begin{cases} M\sin^2(\pi x), & \text{if} \quad x<0 \text{ or } x>M \\ 0, & \text{if} \quad 0\le x\le M. \end{cases}$

$f(x)$ is continuous and non-constant. In addition $f(n)=0$ for all integers n.
$0\le f(x)\le M$ and the maximum M is assumed.

$f([x])=0$ for all real x since $[x]$ is an integer. $f(f(x))=0$ for all real x, since $0\le(x)\le M$ and $f(y)=0$ for $0\le y\le M$.

Solution 3 by Moti Levy, Rehovot, Israel
Let $f: R\to R$ be defined as follows (M is positive integer):

$$f(x)=\begin{cases} M\left[\frac{x}{M+1}\right]\sin^2(\pi x), & for \quad M+2\ge x\ge M+1 \\ 0, & otherwise. \end{cases}$$

The function $f(x)$ is continuous and its maximum value over R is M.
Clearly (since $\sin^2(\pi[x])=0$) ,

$$f([x])=0.$$

By its definition $0\le f(x)\le M$. Hence, $f(f(x))=0$, since $\left[\frac{f(x)}{M+1}\right]=0$.

$$f(f(x))=\begin{cases} M\left[\frac{f(x)}{M+1}\right]\sin^2(\pi f(x))=0, & for \quad M+2\ge x\ge M+1 \\ 0, & otherwise. \end{cases}$$

We conclude that $f(x)$ is continuous and non-constant function with maximum value M, which satisfies $f(f(x))=f([x])=0$.

Solution 4 by The Ashland University Undergraduate Problem Solving Group, Ashland, OH

The following function satisfies the given conditions;

$$f(x)=\begin{cases} -x+2M-2 & \text{if } M-2\le x\le M-3/2 \\ x+1 & \text{if } M-3/2<x\le M-1 \\ M & \text{otherwise.} \end{cases}$$

We can easliy check that f is continuous by noting that:

$$f(M-2)=M,\ f\left(M-\frac{3}{2}\right)=M-\frac{1}{2},\ \text{and } f(M-1)=M.$$

We now show f satisfies $f(f(x))=f([x])$.
When $x\le M-2,\ [x]\le M-2$ and $f(f(x))=f(M)=M=f([x])$.
When $x\ge M-1,\ [x]\ge M-1$ and $f(f(x))=f(M)=M=f([x]$.
Finally, when $M-2<x<M-1,\ [x]=M-2$ and $M-\frac{1}{2}\le f(x)<M$.
Thus, $f(f(x))=M=f([x])$.

Also solved by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel, and the proposer.

- **5413:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Compute

$$\lim_{n\to\infty}\frac{1}{n}\sum_{1\le i\le j\le n}\frac{1}{\sqrt{(n^2+(i+j)n+ij)}}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{1\le i\le j\le n}\frac{1}{\sqrt{n^2+(i+j)n+ij}} &= \lim_{n\to\infty}\sum_{1\le i\le j\le n}\frac{1}{\sqrt{(1+i/n)(1+j/n)}}\cdot\frac{1}{n^2}\\
&= \int_0^1\int_0^x\frac{1}{\sqrt{(1+x)(1+y)}}\,dy\,dx\\
&= \int_0^1\frac{2}{\sqrt{1+x}}\cdot\sqrt{1+y}\Big|_0^x\\
&= \int_0^1\left(2-\frac{2}{\sqrt{1+x}}\right)\,dx\\
&= \left(2x-4\sqrt{1+x}\right)\Big|_0^1\\
&= 6-4\sqrt{2}.
\end{aligned}$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

Since $n^2+(i+j)n+ij=(n+i)(n+j)$, it is easy to check that

$$\sum_{1\le i\le j\le n}\frac{1}{\sqrt{n^2+(i+j)n+ij}}=\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{\sqrt{n+i}}\right)^2+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{n+i}\qquad(1)$$

Now $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{n+i}}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n\sqrt{n+\frac{1}{n}}}=\int_0^1\frac{dx}{\sqrt{1+x}}=2\left(\sqrt{2}-1\right)$, and from

$\displaystyle 0<\sum_{i=1}^{n}\frac{1}{n+i}\le\frac{n}{n+1}$, we have $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{n+i}=0$, so by (1), we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{1\le i\le j\le n}\frac{1}{\sqrt{n^2+(i+j)n+ij}}=2(3-2\sqrt{2}).$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5414:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A, B \in M_2(C)$ be such that $2015AB - 2016BA = 2017I_2$. Prove that

$$(AB - BA)^2 = O_2.$$

Here, C is the set of complex numbers.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Recall that the characteristic polynomial of a 2x2 matrix M is $p_M(x) = \det(M - xI_2)$. An easy calculation shows that $p_M(x) = x^2 - \mathrm{tr}(M)x + \det(M)$ where $\det(M)$ is the determinate of M and $\mathrm{tr}(M)$ is its trace. By the Cayley-Hamilton Theorem we have $p_M(M) = 0_2$.

We first note that AB and BA have the same characteristic polynomial $p(x)$ because $\det(AB) = \det(BA)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

We given $2015AB - 2016BA = 2017I_2$. Adding AB to both sides of this equation yields

$$2016(AB - BA) = AB + 2017I_2.$$

Taking the determinant of this we find

$$2016^2 \det(AB - BA) = \det(AB + 2017I_2) = p(-2017).$$

Similarly adding BA to both sides of the original equation and taking the determinant yields

$$2015^2 \det(AB - BA) = 2015^2 \det(AB - BA) = p(-2017).$$

Thus

$$2016^2 \det(AB - BA) = 2015^2 \det(AB - BA)$$

and so $\det(AB - BA) = 0$.

Since $\mathrm{tr}(AB - BA)$=$\mathrm{tr}(AB)$−$\mathrm{tr}(BA) = 0$ we see that the characteristic polynomial of $AB - BA$ is x^2. Thus, $(AB - BA)^2 = 0_2$.

Essentially the same argument would establish the following mild generalization: Let $A, B \in M_2(K)$ were K is a field. Let $s, t \in K$ with $s \neq \pm 1$ and t$\neq 0$. Then $AB - sBA = tI_2$ implies $(AB - BA)^2 = 0_2$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

AB and BA have the same eigenvalues, since $\det(xI_2 - AB) = \det(xI_2 - BA)$. Indeed, when A is nonsingular this result follows from the fact that AB and BA are similar: $BA = A^{-1}(AB)A$.

For the case where both A and B are singular, one may remark that the desired identity is an equality between polynomials in x and the coefficients of the matrices. Thus, to prove this equality, it suffices to prove that it is veified on a non-empty open subset (for the usual topology, or, more generally, for the Zariski topology) of the space of all the coefficients. As the non-singular matrices form such an open subset of the space of all matrices, this proves the result.

Let x be an eigenvalue of AB. Then

$$\begin{aligned}
0 = \det(xI_2 - AB) &= \det\left(xI_2 - \frac{2016}{2015}BA - \frac{2017}{2015}I_2\right) \\
&= \frac{2016^2}{2015^2}\det\left(\frac{x-\frac{2017}{2015}}{\frac{2016}{2015}}I_2 - BA\right) \\
&= \frac{2016^2}{2015^2}\det\left(\frac{x-\frac{2017}{2015}}{\frac{2016}{2015}}I_2 - AB\right) \\
&= \frac{2016^2}{2015^2}\det\left(\frac{2015x-2017}{2016}I_2 - AB\right). \qquad (1)
\end{aligned}$$

$\det(xI_2 - AB)$ is a quadratic polynomial in x, let's say $\det(xI_2 - AB) = ax^2 + bx + c$. (1) then implies that

$ax^2 + bx + c = \dfrac{2016^2}{2015^2}\left(a\left(\dfrac{2015x-2017}{2016}\right)^2 + b\left(\dfrac{2015-2017}{2016}\right) + c\right)$. We compare the coefficients of the polynomials and see that $b = 4034a,\ c = 2017^2a$.

So the quadratic polynomial reads as $ax^2 + 4034ax + 2017^2a = a(x+2017)^2$ which shows that the characteristic polynomial of AB and BA has -2017 as a double zero, x is an eigenvector of both AB and BA corresponding to the eigenvalue -2017. Therefore there are numbers u and v such that AB is similar to $\begin{pmatrix} -2017 & u \\ 0 & -2017 \end{pmatrix}$

and BA is similar to $\begin{pmatrix} -2017 & v \\ 0 & -2017 \end{pmatrix}$. Therefore, $(AB - BA)^2$ is thus similar to $\begin{pmatrix} 0 & u-v \\ 0 & 0 \end{pmatrix} = 0_2$, which implies that $(AB - BA)^2 = 0_2$.

Solution 3 by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel

Let us write $[A, B]$ for $AB - BA$. Since $traceAB = traceBA$, we have, as is well-known, $trace[A, B] = 0$. Thus, keeping in mind that $[A, B]$ is a 2×2 matrix, the characteristic polynomial of $[A, B]$ is $x^2 + det[A, B] = 0$, so that if its eigenvalues are λ_1 and λ_2, we have $\lambda = \lambda_1 = -\lambda_2$ and $\lambda^2 = -det[A, B]$. Moreover, since every matrix satisfies its own characteristic polynomial,

$$[A, B]^2 = -det[A, B]I$$

Therefore, $[A, B]^2 = 0$, which is what we want to show, if and only if $\lambda^2 = -det[A, B] = 0$, that is, if and only if $\lambda = 0$. We will show that, indeed, $\lambda = 0$ Consider the given equation $pAB - (p+1)BA = (p+2)I$. By adding BA or AB to both sides, we obtain, respectively:

$$p[A, B] = (p+2)I + BA \qquad (8)$$

$$(p+1)[A, B] = (p+2)I + AB \qquad (9)$$

Let λ be an eigenvalue for $[A, B]$ and v the corresponding eigemvector. Thus, we have by (8):

$$p[A, B]v = p\lambda v = ((p+2)I + BA)v$$

Thus,

$$BAv = (p\lambda - (p+2))v$$

so that, $p\lambda - (p+2)$ is an eigenvalue for BA. Since $-\lambda$ is the other eigenvalue of $[A, B]$, we find that $-(p\lambda + (p+2))$ is the second eigenvalue of BA.
In the same way, using equation (9), we find the eigenvalues of AB to be
$(p+1)\lambda - (p+2)$ and $-((p+1)\lambda + (p+2))$
The determinant of any matrix is of course equal to the product of the eigenvalues. Moreover, $detAB = detBA$. Hence:

$$-(p\lambda + (p+2))(p\lambda - (p+2)) = -((p+1)\lambda + (p+2))((p+1)\lambda - (p+2))$$

From which we have:

$$((p+1)^2 - p^2)\lambda^2 = 0$$

So that $\lambda = 0$, which is what we wished to prove.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Assume $A, B \in M_2(C)$ with $nAB - (n+1)BA = (n+2)I_2$ for some positive integer n. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

Then $(AB - BA)^2 = kI_2$, where

$$k = (bg - cf)^2 + (af + bh - be - df)(ce + dg - ag - ch).$$

By hypothesis, we have:

$$\begin{array}{ll} n(ae+bg) - (n+1)(ae+cf) = n+2 & \quad n(af+bh) - (n+1)(be+df) = 0 \\ n(cf+dh) - (n+1)(bg+dh) = n+2 & \quad n(ce+dg) - (n+1)(ag+ch) = 0. \end{array}$$

Thus $n(af + bh - be - df) = be + df$ and $n(ce + dg - ag - ch) = ag + ch$. Also, $(n+1)/n = (af + bh)/(be + df) = (ce + dg)/(ag + ch)$, and $ae - dh = (2n+1)(bg - cf)$.

Substituting yields

$$k = \frac{(ae - dh)(bg - cf)}{2n+1} + \frac{(be + df)(ag + ch)}{n^2}.$$

Then

$$\begin{aligned}
(2n+1)k &= abeg - acef - bdgh + cdfh + \frac{2n+1}{n^2}(abeg + bceh + adfg + cdfh) \\
&= \left(\frac{n+1}{n}\right)^2 (abeg + cdfh) - (acef + bdgh) + \frac{2n+1}{n^2}(adfg + bceh) \\
&= \left(\frac{n+1}{n}\right)^2 [(ag+ch)(be+df) - adfg - bceh] - (acef + bdgh) + \frac{2n+1}{n^2}(adfg + bceh) \\
&= \left(\frac{n+1}{n}\right)^2 (ag+ch)(be+df) - (adfg + bceh) - (acef + bdgh) \\
&= \left(\frac{af+bh}{be+df}\right)\left(\frac{ce+dg}{ag+ch}\right)(ag+ch)(be+df) - (adfg + bceh) - (acef + bdgh) \\
&= (af+bh)(ce+dg) - (adfg + bceh) - (acef + bdgh) \\
&= 0.
\end{aligned}$$

Hence $k = 0$ as needed.

Also solved by Moti Levy, Rehovot, Israel, and the proposer.

Mea Culpa

Paul M. Harms of North Newton, KS and Jeremiah Bartz of University of North Dakota, Grand Forks, ND should have each been credited with having solved problem 5403.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2017

- **5433:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation: $\sqrt[4]{x+x^2} = \sqrt[4]{x} + \sqrt[4]{x-x^2}$, with $x > 0$.

- **5434:** *Proposed by Titu Zvonaru, Comnesti, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Calculate, without using a calculator or log tables, the number of digits in the base 10 expansion of 2^{96}.

- **5435:** *Proposed by Valcho Milchev, Petko Rachov Slaveikov Seconday School, Bulgaria*

  Find all positive integers a and b for which $\dfrac{a^4+3a^2+1}{ab-1}$ is a positive integer.

- **5436:** *Proposed by Arkady Alt, San Jose, CA*

  Find all values of the parameter t for which the system of inequalities

$$\mathbf{A} = \begin{cases} \sqrt[4]{x+t} \geq 2y \\ \sqrt[4]{y+t} \geq 2z \\ \sqrt[4]{z+t} \geq 2x \end{cases}$$

  a) has solutions;
  b) has a unique solution.

- **5437:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $f : C - \{2\} \to C$ be the function defined by $f(z) = \dfrac{2-3z}{z-2}$. If $f^n(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{n})(z)$, then compute $f^n(z)$ and $\lim_{n\to+\infty} f^n(z)$.

- **5438:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 0$ be an integer and let $\alpha > 0$ be a real number. Prove that

$$\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} + \frac{z^{2k}}{(1-z^2)^\alpha} \geq \frac{x^k y^k}{(1-xy)^\alpha} + \frac{y^k z^k}{(1-yz)^\alpha} + \frac{x^k z^k}{(1-xz)^\alpha},$$

for $x, y, z \in (-1, 1)$.

Solutions

- **5415:** *Proposed by Kenneth Korbin, New York, NY*

Given equilateral triangle ABC with inradius r and with cevian $\overline{CD}$. Triangle ACD has inradius x and triangle BCD has inradius y, where x, y and r are positive integers with $(x, y, r) = 1$.
Part 1: Find x, y, and r if $x + y - r = 100$

Part 2: Find x, y, and r if $x + y - r = 101$.

Solution by Ed Gray, Highland Beach, FL

Editor's comment: Ed's solution to this problem was 18 pages in length. Listed below is my greatly abbreviated outline of his solution method. All formulas listed below were proved and/or referenced in Ed's complete solution. He started his solution with the second part of the problem and then applied the methods constructed there to the first part of the problem. The reason for this will soon become apparent. Following is Ed's solution.

The following equations will be used in the solution.

$$\begin{aligned} x &= \frac{6r^2 - 3rp}{4r - p + \sqrt{4r^2 + p^2 - 2rp}} && (1) \\ y &= \frac{3rp}{p + 2r + \sqrt{4r^2 + p^2 - 2pr}} && (2) \\ k &= p(2r - p) && (3) \\ x + y - r &= \frac{k}{2r + \sqrt{4r^2 - k}} && (4) \end{aligned}$$

Solution to Part 2. $x + y - r = 101$.

Substituting the value $x + y - r = 101$ into (4) and solving for k we see that

$$\frac{k}{2r + \sqrt{4r^2 - k}} = 101 \Longrightarrow k = 404r - 10201,$$

and substituting this into (3) above we see that

$$404r - 10201 = 2pr - p^2 \Longrightarrow p = r - \sqrt{r^2 - 404r + 10201}.$$

Letting D equal the value under the square root we have

$$D^2 = r^2 - 404r + 10201 \Longrightarrow r^2 - 404r + 10201 - D^2 = 0.$$

Solving for r gives $r = 202 \pm \sqrt{30603 + D^2}$

Letting $b^2 = 30603 + D^2$ we have $(b-D)(b+D) = 30603 = 3^1 \cdot 101^2$.

This implies that there are three possible factorizations:

$$\begin{array}{rl} \text{Case I}: & 1 \times 30603 \\ \text{Case II}: & 3 \times 10201 \\ \text{Case III}: & 101 \times 303 \end{array}$$

Case I: $\begin{cases} b - D &= 1 \\ b + D &= 30603 \end{cases} \Longrightarrow \begin{cases} b &= 15302 \\ D &= 15301 \end{cases}.$

So,

$$\begin{aligned} r_1 &= 202 + b = 202 + 15302 = 15504 \\ r_2 &= 202 - b = 202 - 15302 < 0 \\ p &= r - D = 155404 - 15301 = 203. \end{aligned}$$

Therefore, $r = 15504$ and $p = 203$.

For these values of r and p, we evaluate x and y by using formulas (1) and (2) above.

$$\begin{aligned} x &= \frac{6r^2 - 3rp}{4r - p + \sqrt{4r^2 + p^2 - 2rp}} \\ & \frac{6(155040)^2 - 3(15504)(203)}{4(15504) - 203 + \sqrt{4(15504)^2 + (203)^2 - 2(15504)(203)}} \\ &= 15453. \end{aligned}$$

$$\begin{aligned} y &= \frac{3pr}{p + 2r + \sqrt{4r^2 + p^2 - 2pr}} \\ &= 152. \end{aligned}$$

So for Case I, $r = 15504,\ x = 15453,\ y = 152$, and $x + y - r = 101$. Since x, y, r have no common factor, they represent a solution.

Case II: $\begin{cases} b - D &= 3 \\ b + D &= 10201 \end{cases} \Longrightarrow \begin{cases} b &= 5102 \\ D &= 5099 \end{cases}.$ So, $\begin{cases} r_1 &= 202 + 5102 = 5304 \\ r_2 &= 202 - 5102 < 0 \\ p &= r - D = 205. \end{cases}.$

Following the path in Case I, we find that $\begin{cases} r &= 5304 \\ x &= 5252 \\ y &= 153, \text{ and} \\ x + y - r &= 101. \end{cases}$

These terms have no common factor and so represent a solution.

Case III: $\begin{cases} b-D &= 101 \\ b+D &= 303 \end{cases} \Longrightarrow \begin{cases} b &= 202 \\ D &= 101 \end{cases}$. So, $\begin{cases} r_1 &= 202+202=4043 \\ r_2 &= 202-202=0, \text{ not viable} \\ p &= r-D=4043-101=303. \end{cases}$.

Given $r=404, p=303$, and calculating as before, we have for Case III,
$r=404,\ x=303,\ y=202,\ x+y-r=101$. However 101 divides all three terms, violating $(x,y,r)=1$, so we do not have a solution.

In summary, and taking into account the interchangeability of x and y, there are four solutions for Part 2 of the problem:
$\begin{pmatrix} x \\ y \\ r \end{pmatrix} = \begin{pmatrix} 15453 \\ 152 \\ 15504 \end{pmatrix}, \begin{pmatrix} 5252 \\ 153 \\ 5304 \end{pmatrix}, \begin{pmatrix} 152 \\ 15453 \\ 15504 \end{pmatrix}, \begin{pmatrix} 153 \\ 5252 \\ 5304 \end{pmatrix}.$

Solution to Part 1. $x+y-r=100$. In solving Part 1 of the problem we employ the same techniques that were used in Part 2. We start off by finding that if $\dfrac{k}{2r+\sqrt{4r^2-k}}=100$ then $k=400r-10,000$. Substituting this into Equation (3), gives us $400r-10,000=2pr-p^2$ and solving for p gives us $p=r-\sqrt{r^2-4004r+10,000}$. The discriminant, D is given by $D^2=r^2-400r+10000$. Writing this as a quadratic in r and solving for r gives us

$$r^2-400r+10,000-D^2=0$$

$$r=200\pm\sqrt{30,000+D^2}.$$

And as before, letting $b^2=30,000+D^2$ we obtain

$$(b-D+)(b+D)=30,000=2^4\cdot 3^1\cdot 5^4.$$

Hence there are $5\times 2\times 5=50$ factors which need to be written as the product of two factors. Since $2b$ must equal the sum of the two factors, they cannot be of opposite parity. Following is a table listing all factorizations. We eliminate those factorizations that have an odd factor by placing an asterisk in front of them.

$*1\times 30,000$	$2\times 15,000$	$*3\times 10,000$
4×7500	$*5\times 6,000$	8×3750
10×3000	12×2500	$*15\times 2000$
$*16\times 1875$	20×1500	24×1250
$*25\times 1200$	30×1000	40×750
$*48\times 625$	50×600	60×500
$*75\times 400$	$*80\times 375$	100×300
120×250	$*125\times 240$	150×200

The remaining factorizations represent potential solutions. We will do the first one in detail but the others we will only check to see if $(x,y,r)=1$.

$\begin{cases} b-D &= 2 \\ b+D &= 15000 \end{cases} \Longrightarrow \begin{cases} b &= 7501 \\ D &= 7499. \end{cases}$. So, $\begin{cases} r_1 &= 200=7501=7701 \\ r_2 &= 200-7501<0 \\ p &= r-D=7701-7499=202. \end{cases}$

Given $p=r-D=7701-7499=202$. For $r=7701, p=202$ we calculate x and y using the standard formulas.

$$x=\frac{6r^2-3rp}{4r-p+\sqrt{4r^2+p^2-2rp}} \Longrightarrow x=7650$$

$$y = \frac{3rp}{p + 2r + \sqrt{4r^2 + p^2 - 2pr}} \Longrightarrow y = 151.$$

So $x = 7650,\ y = 152,\ r = 7701$. These have no common factor and so represent a solution.

We now move to the next case. $\begin{cases} b - D = 4 \\ b + D = 7500 \end{cases} \Longrightarrow x = 3900,\ y = 152,\ r = 3952$. Since $(x, y, z) \neq 1$, this is not a solution.

And the next case. $\begin{cases} b - D = 6 \\ b + D = 5000 \end{cases} \Longrightarrow x = 2650,\ y = 153,\ r = 2703$. Since $x + y - r = 100$ and $(x, y, r) = 1$ this is a solution.

And the next. $\begin{cases} b - D = 8 \\ b + D = 3750 \end{cases} \Longrightarrow x = 2025,\ y = 154,\ r = 2079$. Since $x + y - r = 100$ and $(x, y, r) = 1$ this is a solution 10pt Working our way through the table of potential solutions we find that $\begin{cases} b - D = 24 \\ b + D = 1250 \end{cases} \Longrightarrow x = 775,\ y = 162,\ r = 837$ and since $x + y - r = 100$ and $(x, y, r) = 1$ this is a solution

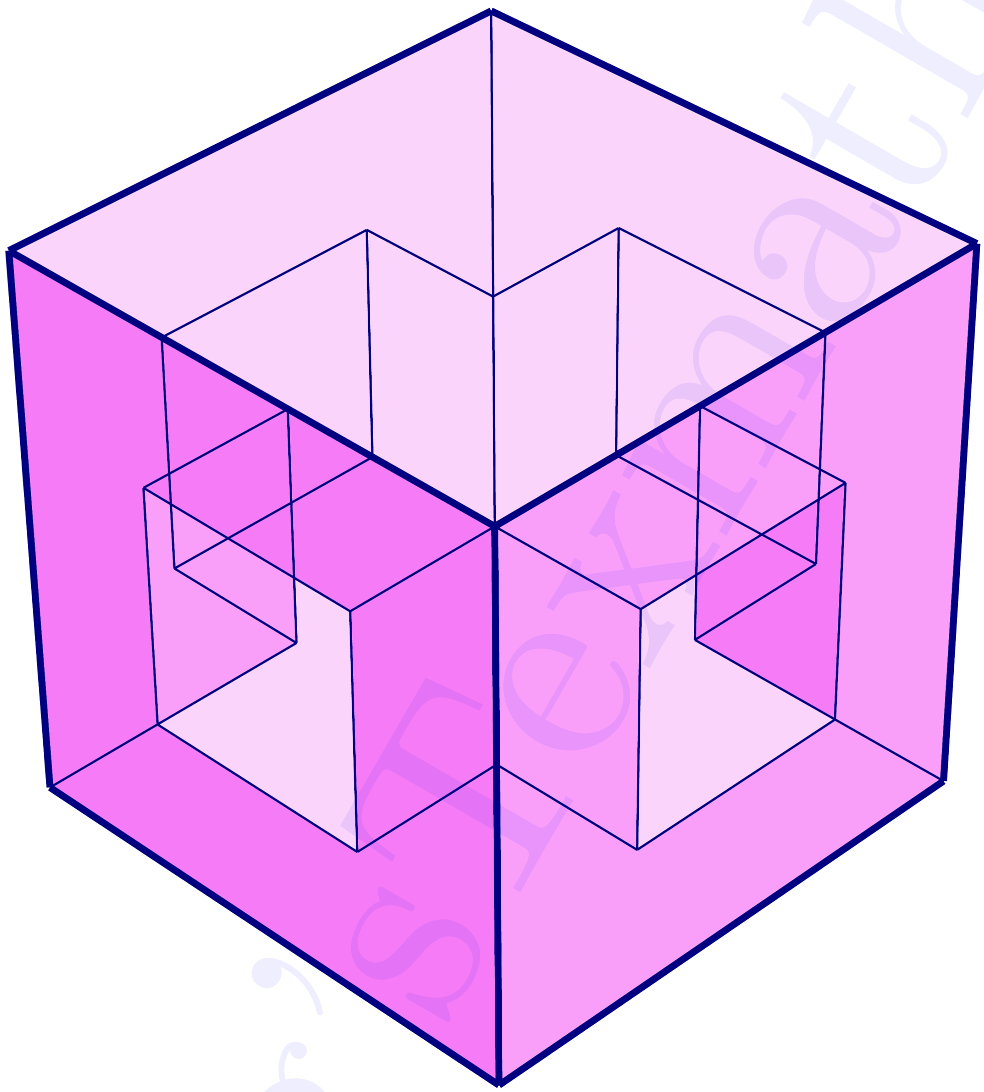
Systemically working our way the table we see that many values did not result in an answer to the problem. Summarizing Part 1 of the problem, and taking into account the interchangeability of x and y, we see that there are exactly eight solutions.

$$\begin{array}{l} 1)\ x = 7650,\ y = \ 151,\ r = 7701 \\ 2)\ x = 2650,\ y = \ 153,\ r = 2703 \\ 3)\ x = 2025,\ y = \ 154,\ r = 2079 \\ 4)\ x = \ 775,\ y = \ 162,\ r = \ 837 \\ 5)\ x = \ 151,\ y = 7650,\ r = 7701 \\ 6)\ x = \ 153,\ y = 2650,\ r = 2703 \\ 7)\ x = \ 154,\ y = 2025,\ r = 2079 \\ 8)\ x = \ 162,\ y = 775,\ r = \ 837. \end{array}$$

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5416:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Two congruent intersecting holes, each with a square cross-section were drilled through a cube. Each of the holes goes through the opposite faces of the cube. Moreover, the edges of each hole are parallel to the appropriate edges of the original cube, and the center of each hole is at the center of the original cube. Letting the length of the original cube be a, find the length of the square cross-section of each hole that will yield the largest surface area of the solid with two intersecting holes. What is the largest surface area of the solid with two intersecting holes?



Solution by Paul M. Harms, North Newton, KS

Let the side lengths of the drilled squares be x at the surface of the original cube. The surface area of the one side of the original cube, with a square hole cut out of it, is $a^2 - x^2$. There are four of these sides on the original cube.

On a side of the original cube the shortest distance between an edge of the original cube and a parallel side of the drilled square hole is $\dfrac{a-x}{2}$.

Now consider the surface area "inside" the cube made by the part of the drilled square that starts at a side of the original cube and ends when the drilled square meets the other drilled square originating from an adjacent side of the cube. This surface area looking at one side of the cube includes four rectangles with one side length of x and "depth" length of $\dfrac{a-x}{2}$, so this surface area is $\dfrac{4x(a-x)}{2} = 2(a-x)$. There are four of these around the original cube. The surface area of each of the two sides of the original cube which have no holes is a.

In the middle of the original cube at the intersection of the two drilled square holes, there are two squares of side length x with are parallel to the sides of the original cube with no holes . The area of each square is x^2.

The total surface area of the problem is

$$4(a^2-x^2)+4\left(2x(a-x)\right)+2a^2+2x^2 = 6a^2+8ax-10x^2.$$

The maximum surface area occurs when $8a-20x=0$ or $x=\dfrac{2a}{5}$. The maximum surface area is $\dfrac{38a^2}{5}$ when a side of the drilled square holes as a length of $\dfrac{2a}{5}$.

Editor′s comment: **David Stone and John Hawkins, both from Georgia Southern University, Statesboro, GA** accompanied their solution by placing the statement of the problem into a story setting. They wrote:

"An interpretation: in the ancient Martian civilization, the rulers favorite meditational spot was a levitating cube having a cubical inner sanctum formed by two horizontal square tunnels, meeting at the center of the cube, from which he could see out in all four directions. The designers were charged to construct the ship with a maximum amount of wall space for inscriptions and carved likenesses of His Highness. There are four short hallways leading from the inner room to the outside walls." They let x be the side length of the square tunnels that are drilled through the original cube and noted that each tunnel has an $x \times x$ cross section and has length a. The inner most cubical room is $x \times x \times x$. They then mentioned that "by drilling the tunnels and opening up an interior chamber, the surface area has increased from $6a^2$ to $\dfrac{38}{5}a^2$, an increase of $\dfrac{8}{5}a^2$ or 27%. So the King has his private getaway and more space for pictures and wall hangings."

Also solved by Jeremiah Bartz, University of North Dakota, Grand Forks, ND and Nicholas Newman, Francis Marion University, Florence SC; Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel; David A. Huckaby, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5417:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any positive real number x, and for any natural number $n \geq 2$,

$$\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\cdots+x^{n-1}}{n}}.$$

Solution 1 by Henry Ricardo, New York Math Circle, NY

Let $\alpha_n = (1 + x + \cdots + x^n)/(n+1)$ and define

$$F(x) = \frac{(1 + x + x^2 + \cdots + x^{n-1})^n}{(1 + x + x^2 + \cdots + x^n)^{n-1}}.$$

Then, for $x > 0$ and $n \geq 2$, we see that

$$\sqrt[n-1]{\alpha_{n-1}} \leq \sqrt[n]{\alpha_n} \;\Leftrightarrow\; \alpha_{n-1}^n \leq \alpha_n^{n-1} \;\Leftrightarrow\; F(x) \leq \frac{n^n}{(n+1)^{n-1}} = F(1).$$

Now we show that $F(x)$ attains its absolute maximum value at $x = 1$.

For $x \neq 1$, we have

$$\begin{aligned} F'(x) &= \frac{(x^n - 1)^{n-1}(x^{n+1} - 1)^{-n}\left(-x^{2n+1} + n^2x^{n+2} + 2(1 - n^2)x^{n+1} + n^2x^n - x\right)}{x(x-1)^2} \\ &= \overbrace{\frac{(x^n - 1)^{n-1}}{(x^{n+1} - 1)^n (x-1)^2}}^{G(x)} \cdot \overbrace{\left(-x^{2n} + n^2x^{n+1} + 2(1 - n^2)x^n + n^2x^{n-1} - 1\right)}^{H(x)}. \end{aligned}$$

Noting that $G(x)$ is negative for $0 < x < 1$ and positive for $x > 1$, we examine the factor $H(x)$ to see that

$$\begin{aligned} H(x) &= -(x^n - 1)^2 + n^2x^{n-1}(x-1)^2 \\ &= -n^2(x-1)^2\left[\frac{(x^{n-1} + x^{n-2} + \cdots + x + 1)^2}{n^2} - x^{n-1}\right] \\ &= -n^2(x-1)^2\left[\left(\frac{x^{n-1} + x^{n-2} + \cdots + x + 1}{n}\right)^2 - \left(\sqrt[n]{x^{n-1} \cdot x^{n-2} \cdots x \cdot 1}\right)^2\right] \end{aligned}$$

is negative for all $x > 0$ by the AM-GM inequality.

Thus $F'(x) > 0$ for $0 < x < 1$ and $F'(x) < 0$ for $x > 1$, implying that $F(x)$ has an absolute maximum value at $x = 1$—that is, $F(x) \leq F(1)$ on $(0, \infty)$, which proves the proposed inequality.

COMMENT: This was proposed by Walther Janous as problem 1763 (1992, p. 206) in *Crux Mathematicorum*. My solution is based on the published solution of Chris Wildhagen.

Solution 2: by Moti Levy, Rehovot, Israel

If $x = 1$ then the inequality holds, since

$$\sqrt[n]{\frac{1 + x + \cdots + x^n}{n+1}} = \sqrt[n-1]{\frac{1 + x + \cdots + x^{n-1}}{n}} = 1.$$

We assume that $x > 1$.
Let us define the continuous functions $g(t)$, and $f(t)$, $t \in R$, $t > 1$, as follows,

$$g(t) := \frac{x^{t+1} - 1}{x - 1}\frac{1}{t+1}, \quad f(t) := (g(t))^{\frac{1}{t}}.$$

Clearly, $\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} = \sqrt[n]{\frac{1}{n+1}\frac{x^{n+1}-1}{x-1}} = f(n)$. The original inequality (in terms of the function f) is

$$f(n) \geq f(n-1), \quad \text{for} \quad n \geq 2.$$

For $n = 2$, $\sqrt{\frac{1+x+x^2}{3}} \geq \frac{1+x}{2}$ follows from $\frac{1+x+x^2}{3} - \left(\frac{1+x}{2}\right)^2 = \frac{1}{12}(x-1)^2 \geq 0$.
Therefore, it suffices to prove that $f(t)$ is monotone increasing function for $t \geq 1$.
We will show this by proving that the derivative of $\ln f(t)$ is postive for $t \geq 1$.
The derivative is given by

$$t^2 \frac{d}{dt}(\ln f) = -\ln g + t\frac{\frac{dg}{dt}}{g}.$$

The first step is showing $-\ln g + t\frac{\frac{dg}{dt}}{g} > 0$ for $t = 1$.

$$-\ln g + t\frac{\frac{dg}{dt}}{g}\Bigg|_{t=1} = -\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}.$$

To show that $-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)} > 0$ for $x > 0$, we see that $\lim_{x\to 0}\left(-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}\right) = \ln 4 > 0$.
Now we show that the derivative of $-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}$ is positive:

$$\frac{d\left(-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}\right)}{dx} = \frac{1}{x^2-1} - \frac{2x\ln x}{(x^2-1)^2}.$$

We use the well known inequality: $\ln x \leq \frac{x^2-1}{2x}$ for $x > 0$ to show that

$$\frac{1}{x^2-1} - \frac{2x\ln x}{(x^2-1)^2} \geq 0.$$

The second step is showing that the derivative of $-\ln g + t\frac{\frac{dg}{dt}}{g}$ is positive for $t > 0$,

$$\frac{d\left(-\ln g + t\frac{\frac{dg}{dt}}{g}\right)}{dt} = -\frac{\frac{dg}{dt}}{g} + \frac{\frac{dg}{dt}}{g} + \frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right) = \frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right).$$

After some tedious calculation we arrive at,

$$\frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right) = \frac{\left(x^{t+1}-1\right)^2 - x^{t+1}\ln^2 x^{t+1}}{\left(x^{t+1}-1\right)^2(t+1)^2}.$$

To show that $\left(x^{t+1}-1\right)^2 \geq x^{t+1}\ln^2 x^{t+1}$, or that $\ln x^{t+1} \leq \frac{1}{\sqrt{x^{t+1}}}\left(x^{t+1}-1\right)$, we use again the inequality $\ln y \leq \frac{y^2-1}{2y}$ for $y > 0$,

$$\ln y \leq \frac{y-1}{\sqrt{y}}\frac{y+1}{2\sqrt{y}}.$$

But $\frac{y+1}{2\sqrt{y}} \geq 1$; hence,

$$\ln y \leq \frac{y-1}{\sqrt{y}}, \quad y > 0.$$

Now set $y = x^{t+1}$ to finish the proof.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the inequality of the problem by $(*)$. It is easy to see that if $(*)$ holds for $x = t$ then it also holds for $x = \dfrac{1}{t}$. Hence it suffices to prove $(*)$ for $0 < x \leq 1$.

Let $f(x) = (n-1)\ln\left(\displaystyle\sum_{k=0}^{n} x^k\right) - n\ln\left(\displaystyle\sum_{k=0}^{n-1} x^k\right) + \ln\left(\dfrac{n^n}{(n+1)^{n-1}}\right)$, where $0 < x \leq 1$.

By taking logarithms, we see that $(*)$ is equivalent to $f(x) \geq 0$.

We have $f(1) = 0$ and for $0 < x < 1$,

$$f(x) = (n-1)\ln(1 - x^{n+1}) - n\ln(1 - x^n) + \ln(1 - x) + \ln\left(\frac{n^n}{(n+1)^{n-1}}\right).$$

Hence to prove $(*)$, we need only prove that $f'(x) < 0$ for $0 < x < 1$.

Since $f'(x) = \dfrac{g(x)}{(x-1)(x^n-1)(x^{n+1}-1)}$, where

$g(x) = x^{2n} - n^2x^{n+1} + 2(n-1)(n+1)x^n - n^2x^{n-1} + 1$, it suffices to show

$g(x) > 0$, for $0 < x < 1$. Now

$$\begin{aligned}
g'(x) &= 2nx^{2n-1} - (n+1)n^2x^n + 2n(n-1)(n+1)x^{n-1} - (n-1)n^2x^{n-2}, \\
g''(x) &= 2n(2n-1)x^{2n-2} - (n+1)n^3x^{n-1} + 2n(n+1)(n-1)^2x^{n-2} - (n-1)(n-2)n^2x^{n-3}, \text{and} \\
g'''(x) &= 4n(n-1)(2n-1)x^{2n-3} - (n-1)(*n+1)n^3xn - 2+ \\
&\quad 2n(n-2)(n+1)(n-1)^2x^{n-3} - (n-1)(n-2)(n-3)n^2x^{n-4}.
\end{aligned}$$

Thus $g(1) = g'(1) = g''(1) = g'''(x) = 0$ so that 1 is a root of multiplicity 4 of the equation $g(x) = 0$. By Descartes' rule of signs, the equation $g(x) = 0$ has no other positive roots. Since $g(0) = 1 > 0$, so $g(x) > 0$ for $0 < x < 1$.

This completes the proof.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $f(t) = 1/x$. The inequality goes unchanged because

$$\sqrt[n]{\frac{1 + \frac{1}{t} + \ldots + \frac{1}{t^n}}{t^n(n+1)}} \geq \sqrt[n-1]{\frac{1 + \frac{1}{t} + \ldots + \frac{1}{t^{n-1}}}{t^{n-1}n}}$$
$$\iff \sqrt[n]{\frac{1 + t + \ldots + t^n}{n+1}} \geq \sqrt[n-1]{\frac{1 + t + \ldots + t^{n-1}}{n}}.$$

This means that we may assume $x \geq 1$.

Let $x = 1$. The inequality becomes

$$1 = \sqrt[n]{\frac{1}{n+1}(\underbrace{1+1+\ldots+1}_{n+1\ times})} \geq \sqrt[n]{\frac{1}{n}(\underbrace{1+1+\ldots+1}_{n\ times})} = 1.$$

Let $x > 1$. The inequality is also

$$\sqrt[n]{\frac{1}{n+1}\frac{1-x^{n+1}}{1-x}} \geq \sqrt[n-1]{\frac{1}{n}\frac{1-x^n}{1-x}},$$

that is

$$\sqrt[n]{\frac{1}{x-1}\int_1^x t^n} \geq \sqrt[n-1]{\frac{1}{x-1}\int_1^x t^{n-1}}.$$

This is the Power–Means inequality for integrals.

Also solved by Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5418:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

Let ABC be an acute triangle with circumradius R and inradius r. If $m \geq 0$, then prove that

$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 1 by Nikos Kalapodis, Patras, Greece

Applying Radon's Inequality and taking into account that $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$ and $\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C} \geq \frac{3}{2}$ (see Solution 1 of Problem 5381, SSMA, April 2016) we have

$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} = \sum_{cyclic} \frac{\left(\dfrac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \geq \frac{\left(\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\displaystyle\sum_{cyclic} \cos A\right)^m} \geq$$

$$\frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 2 by Arkady Alt, San Jose, CA

Firstly, we will prove that in any acute triangle the inequality

(1) $\displaystyle\sum_{cyc} \frac{\cos A \cos B}{\cos C} \geq \frac{3}{2}$, holds.

Let $\alpha := \pi - 2A, \beta := \pi - 2B, \gamma := \pi - 2C$. Then $\alpha, \beta, \gamma > 0$ (since $A, B, C < \pi/2$), $\alpha + \beta + \gamma = \pi$ and **(1)** $\Longleftrightarrow \sum_{cyc} \frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\gamma}{2}} \geq \frac{3}{2}$.

Let a, b, c be sidelenghts of a triangle with angles $\alpha, \beta, \gamma,$ respectively, and s be semiperimeter of this triangle.

Then $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ and,

similarly, $\sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$. Hence,

$$\sum_{cyc} \frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\gamma}{2}} = \sum_{cyc} \frac{\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-c)(s-a)}{ca}}}{\sqrt{\frac{(s-a)(s-b)}{ab}}} = \sum_{cyc} \frac{s-c}{c} = \sum_{cyc} \frac{s}{c} - 3 =$$

$$\frac{1}{2}(a+b+c) \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 \geq \frac{1}{2} \cdot 9 - 3 = \frac{3}{2}.$$

Noting that $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ and using a combination of the Weighted Power Mean-Arithmetic Inequality with weights $\cos A, \cos B, \cos C > 0$ and inequality **(1)** we obtain:

$$\sum_{cyc} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} = \sum_{cyc} \cos A \left(\frac{\cos B}{\cos C}\right)^{m+1} = \sum_{cyc} \cos A \cdot \left(\frac{\sum_{cyc} \cos A \left(\frac{\cos B}{\cos C}\right)^{m+1}}{\sum_{cyc} \cos A}\right) \geq$$

$$\sum_{cyc} \cos A \cdot \left(\frac{\sum_{cyc} \cos A \left(\frac{\cos B}{\cos C}\right)}{\sum_{cyc} \cos A}\right)^{m+1} = \sum_{cyc} \cos A \cdot \frac{\left(\sum_{cyc} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\sum_{cyc} \cos A\right)^{m+1}} =$$

$$\frac{\left(\sum_{cyc} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\sum_{cyc} \cos A\right)^{m}} \geq \frac{\left(\frac{3}{2}\right)^{m+1}}{\left(1 + \frac{r}{R}\right)^{m}} = \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The inequality is equivalent to and Radon's inequality, and applying it we obtain

$$\sum \frac{\cos A \cos^{m+1} B}{cos^{m+1} C} = \sum \frac{\left(\frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \underset{Radon}{\geq} \frac{\left(\sum \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\sum \cos A)^m} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m},$$

where $\sum \cos A = 1 + \frac{r}{R}$ and $\sum \frac{\cos A \cos B}{\cos C} = \sum \frac{\tan C}{\tan A + \tan B}$.

Denote $\tan A = x, \ \tan B = y, \ \tan C = z$. Using Nesbitt's inequality, we have

$\sum \frac{\tan C}{\tan A+\tan B} = \sum \frac{z}{x+y} \underset{Nesbitt}{\geq} \frac{3}{2}$.

Solution 4 by Henry Ricardo, New York Math Circle, NY.

We will use the following known results: (1) Radon's inequality: If $x_k, a_k > 0\ \forall k, p > 0$, then $\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \left(\sum_{k=1}^{n} x_k\right)^{p+1} / \left(\sum_{k=1}^{n} a_k\right)^p$; (2) $\sum_{cyclic} \frac{\cos A \cos B}{\cos C} \geq 3/2$; (3) $\sum_{cyclic} \cos A = (R+r)/R$.

Now we have

$$\begin{aligned}
\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} &= \sum_{cyclic} \frac{\left(\frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \\
&\overset{(1)}{\geq} \frac{\left(\sum_{cyclic} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\sum_{cyclic} \cos A\right)^m} \\
&\overset{(2),(3)}{\geq} \frac{(3/2)^{m+1}}{((R+r)/R)^m} = \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.
\end{aligned}$$

Comments: (a) Inequality (2) appeared as problem 4053, proposed by Šefket Arslanagić, in *Crux Mathematicorum* and reappeared in several solutions to problem 5381 in this Journal; (b) Inequality (3) appeared in Solution 1 to problem 5381 in this Journal. It is also Lemma 2.5.1 in *Inequalities: A Mathematical Olympiad Approach* by R. Manfrino et. al.; (c) The related inequality $\sum_{cyclic} \left(\frac{\cos A \cos B}{\cos C}\right)^{m+1} \geq 3/2^{m+1}$ appeared as problem 5381 by the current proposers.

Editor's comment: **Moti Levy of Rehovot Israel** stated in his solution that: "A nice article on Radon's inequality is *A generalization of Radon's Inequality* by D. M. Bătineţu-Giurgiu and Ovidiu T. Pop, in CREATIVE MATH. & INF. 19 (2010), No. 2, 116 - 121."

Also solved by Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5419: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n} \left(\sum_{k=1}^{n} a_k^{t_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}\right)^n$$

where for all $k \geq 1,\ t_k$ is the k^{th} tetrahedral number defined by $t_k = \frac{k(k+1)(k+2)}{6}$.

Counter example by Moti Levy, Rehovot, Israel

The index k appears twice in the left hand side. This seems odd. The proposer has been asked and here is his response:
"*Here, index k is used in both sum and product.*
But indices in sums and product are dummy variables and they do not need to be distinct.
Surely, it is convenient but not necessary."

Following the proposer's argument that the index k is a dummy variable, we change the first index designation from the letter k to the letter j.

Now the proposed inequality becomes:

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{n} a_k^{t_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}\right)^n.$$

But

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{n} a_k^{t_k}\right) = \left(\sum_{k=1}^{n} a_k^{t_k}\right)^n,$$

hence the proposed inequality implies

$$\sum_{k=1}^{n} a_k^{t_k} \geq \sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}.$$

Let us check this inequality for the special case $n = 2$, for example:

$$\sum_{k=1}^{2} a_k^{t_k} = a_1^{t_1} + a_2^{t_2} = a_1 + a_2^4$$

$$\sum_{k=1}^{2} a_k^{\frac{t_3}{4}} = a_1^{\frac{5}{2}} + a_2^{\frac{5}{2}}$$

Now take $a_1 = 4$ and $a_2 = 1$. Since

$$4 + 1 \leq 4^{\frac{5}{2}} + 1,$$

the inequality is not true.

Editor's note : The impossibility of this problem as it originally appeared was also noted by **Albert Stadler of Herrliberg, Switzerland.** I, as editor, should have noticed this mistake, but didn't; mea culpa.

In correspondence with the proposer of the problem, José Luis Díaz-Barrero, it was acknowledged that the problem should have read as follows:

Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{t_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}\right)^n$$

where for all $k \geq 1$, t_k is the k^{th} tetrahedral number defined by
$t_k = \dfrac{k(k+1)(k+2)}{6}$.

However, by changing the index in this manner, as Moti Levy mentioned, "changes the meaning of the problem." Below is a proof of the problem as it was intended to be in the first place.

Solution by the proposer. We consider the function $f(x) = \ln(a_1^x + a_2^x + \cdots + a_n^x)$ that is convex in R, as can be easily proven. Applying Jensen's inequality to $f(x)$, we obtain

$$\sum_{k=1}^{n} p_k \ln\left(a_1^{x_k} + \cdots + a_n^{x_k}\right) \geq \ln\left(a_1^{\sum_{k=1}^{n} p_k x_k} + \cdots + a_n^{\sum_{k=1}^{n} p_k x_k}\right)$$

where p_k are positive numbers of sum one and $x_1, x_2, \cdots, x_n \in R$. Taking into account that $f(x) = \ln(x)$ is injective, then the preceding expression becomes

$$\ln\left(\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{x_k}\right)^{p_k}\right) \geq \ln\left(a_1^{\sum_{k=1}^{n} p_k x_k} + \cdots + a_n^{\sum_{k=1}^{n} p_k x_k}\right)$$

or equivalently,

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{x_k}\right)^{p_k} \geq \left(a_1^{\sum_{k=1}^{n} p_k x_k} + \cdots + a_n^{\sum_{k=1}^{n} p_k x_k}\right)$$

Setting $p_k = \dfrac{1}{n}$, $1 \leq k \leq n$ and $x_k = t_k$, $1 \leq k \leq n$, and taking into account that $\displaystyle\sum_{k=1}^{n} t_k = \frac{n}{4}\, t_{n+1}$, as can be easily proven for instance by induction, then we have

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^{t_k}\right)^{1/n} \geq \sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}}$$

from which the statement follows. Equality holds when $n = 1$, and we are done.

Comment: On account of the preceding for the particular case $n = 2$, we have

$$\prod_{k=1}^{2}\left(\sum_{j=1}^{2} a_j^{t_k}\right) \geq \left(\sum_{k=1}^{2} a_k^{\frac{t_{n+1}}{4}}\right)^2$$

or

$$(a_1^{t_1} + a_2^{t_1})(a_1^{t_2} + a_2^{t_2}) \geq (a_1^{t_3/4} + a_2^{t_3/4})^2$$

Letting $a_1 = 4, a_2 = 1, t_1 = 1, t_2 = 4, t_3 = 10$ in the last expression, we obtain

$$(4^1 + 1)(4^4 + 1) \geq (4^{5/2} + 1)^2 \iff 1285 \geq 1089$$

5420: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$. Calculate

$$\lim_{n\to\infty} \frac{1}{n}\left(I_2 + \frac{A^n}{n}\right)^n.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Let

$$A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $\lambda^2 - 2\lambda + 1$, so $\lambda = 1$ is an eigenvalue of A with algebraic multiplicity 2. The vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

forms a basis for the eigenspace of A corresponding to $\lambda = 1$. One solution of the equation $A - I = \mathbf{v}$ is the vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The matrix A can therefore be written in the form

$$A = T \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} T^{-1},$$

where

$$T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}.$$

A straightforward induction argument establishes that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

so that

$$A^n = T \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} T^{-1} = \begin{bmatrix} 2n+1 & n \\ -4n & -2n+1 \end{bmatrix}.$$

Thus,

$$\frac{A^n}{n} = \begin{bmatrix} 2 + \dfrac{1}{n} & 1 \\ -4 & -2 + \dfrac{1}{n} \end{bmatrix},$$

and

$$I_2 + \frac{A^n}{n} = \begin{bmatrix} 3 + \dfrac{1}{n} & 1 \\ -4 & -1 + \dfrac{1}{n} \end{bmatrix} = T \begin{bmatrix} 1 + \dfrac{1}{n} & 1 \\ 0 & 1 + \dfrac{1}{n} \end{bmatrix} T^{-1}.$$

Another straightforward induction argument establishes that

$$\begin{bmatrix} 1 + \dfrac{1}{n} & 1 \\ 0 & 1 + \dfrac{1}{n} \end{bmatrix}^n = \begin{bmatrix} \left(1 + \dfrac{1}{n}\right)^n & n\left(1 + \dfrac{1}{n}\right)^{n-1} \\ 0 & \left(1 + \dfrac{1}{n}\right)^n \end{bmatrix},$$

so that

$$\left(I_2+\frac{A^n}{n}\right)^n = T\left[\begin{array}{cc}\left(1+\frac{1}{n}\right)^n & n\left(1+\frac{1}{n}\right)^{n-1}\\ 0 & \left(1+\frac{1}{n}\right)^n\end{array}\right]T^{-1}$$

$$= \left[\begin{array}{cc}2n\left(1+\frac{1}{n}\right)^{n-1}+\left(1+\frac{1}{n}\right)^n & n\left(1+\frac{1}{n}\right)^{n-1}\\ -4n\left(1+\frac{1}{n}\right)^{n-1} & -2n\left(1+\frac{1}{n}\right)^{n-1}+\left(1+\frac{1}{n}\right)^n\end{array}\right].$$

Finally,

$$\lim_{n\to\infty}\frac{1}{n}\left(I_2+\frac{A^n}{n}\right)^n=\left[\begin{array}{cc}2e & e\\ -4e & -2e\end{array}\right].$$

Solution 2 by Henry Ricardo, New York Math Circle, NY.

To simplify the solution, we invoke a known result $(*)$ that is a consequence of the Cayley-Hamilton theorem: If $A\in M_2(C)$ and the eigenvalues λ_1,λ_2 of A are equal, then for all $n\geq 1$ we have $A^n=\lambda_1^nB+n\lambda_1^{n-1}C$, where $B=I_2$ and $C=A-\lambda_1I_2$. (See, for example, Theorem 2.25(b) in *Essential Linear Algebra with Applications* by T. Andreescu, Birkhäuser, 2014.)

The eigenvalues of the given matrix A are both equal to 1, so we apply $(*)$ to get $A^n=nA-(n-1)I_2$. Now we use the last expression to see that $M=I_2+A^n/n=A+I_2/n$; and, since M's eigenvalues are both equal to $1+1/n$, we apply $(*)$ again to determine that

$$\begin{aligned}\frac{1}{n}\left(I_2+\frac{A^n}{n}\right)^n &= \frac{1}{n}M^n\\ &= \frac{1}{n}\left[\left(1+\frac{1}{n}\right)^nI_2+n\left(1+\frac{1}{n}\right)^{n-1}\left(M-\left(1+\frac{1}{n}\right)I_2\right)\right]\\ &= \frac{1}{n}\left[n\left(1+\frac{1}{n}\right)^{n-1}M+\left(1+\frac{1}{n}\right)^n(1-n)I_2\right]\\ &= \frac{1}{n}\left[n\left(1+\frac{1}{n}\right)^{n-1}\left(A+\frac{I_2}{n}\right)+\left(1+\frac{1}{n}\right)^n(1-n)I_2\right]\\ &= \left(1+\frac{1}{n}\right)^n\cdot\frac{n^2A-(n^2-n-1)I_2}{n(n+1)}\\ \to e(A-I_2) &= \begin{pmatrix}2e & e\\ -4e & -2e\end{pmatrix}.\end{aligned}$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Put $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $J=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $S=\begin{pmatrix}3&-2\\-6&7\end{pmatrix}$.

Then
$AS=SJ,\ S^{-1}=\frac{1}{9}\begin{pmatrix}7&2\\6&3\end{pmatrix},\ A=SJS^{-1}, A^n=\left(SJS^{-1}\right)^n=SJ^nS^{-1},\ J^n=\begin{pmatrix}1&n\\0&1\end{pmatrix},$

$$\frac{1}{n}\left(I+\frac{A^n}{n}\right)^n = \frac{1}{n}\left(I+\frac{(SJS^{-1})^n}{n}\right)^n = \frac{1}{n}\left(I+S\frac{J^n}{n}S^{-1}\right)^n$$

$$= \frac{1}{n}\left(S\left(I+\frac{J^n}{n}\right)S^{-1}\right)^n$$

$$= \frac{1}{n}S\left(I+\frac{J^n}{n}\right)^n S^{-1}$$

$$= \frac{1}{n}S\begin{pmatrix}1+\frac{1}{n} & 1\\ 0 & 1+\frac{1}{n}\end{pmatrix}^n S^{-1}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{n}S\begin{pmatrix}1 & \frac{1}{1+\frac{1}{n}}\\ 0 & 1\end{pmatrix}^n S^{-1}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{n}S\begin{pmatrix}1 & \frac{n}{1+\frac{1}{n}}\\ 0 & 1\end{pmatrix}^n S^{-1} \longrightarrow$$

$$= eS\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}S^{-1}$$

$$= \frac{e}{9}S\begin{pmatrix}6 & 3\\ 0 & 0\end{pmatrix}$$

$$= e\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}, \text{ as } n \longrightarrow \infty.$$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Solution. Let $B_n = I_2 + (1/n)A^n$. It is straightforward to show by induction that $B_n = A + (1/n)I_2$. Using the characteristic polynomial of B_n, we have $B_n^2 = 2(1+1/n)B_n - (1+1/n)^2 I_2$. It then follows by induction on k that for each positive integer k,

$$B_n^k = k\left(1+\tfrac{1}{n}\right)^{k-1}B_n - (k-1)\left(1+\tfrac{1}{n}\right)^k I_2.$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}B_n^n &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n-1}B_n-\left(\frac{n-1}{n}\right)\left(1+\frac{1}{n}\right)^{n}I_2\right]\\ &= eA-eI_2\\ &= e(A-I_2)\\ &= e\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}.\end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David R. Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2017

- **5439:** *Proposed by Kenneth Korbin, New York, NY*

  Express the roots of the equation $\dfrac{(x+1)^4}{(x-1)^2} = 20x$ in closed form.
  "Closed form" means that the roots cannot be expressed in their approximate decimal equivalents.

- **5440:** *Proposed by Roger Izard, Dallas,TX*

  The vertices of rectangle ABCD are labeled in clockwise order, and point F lines on line segment AB. Prove that $AD + AC > DF + FC$.

- **5441:** *Proposed by Larry G. Meyer, Fremont, OH*

  In triangle ABC draw a line through the ex-center corresponding to side c so that it is parallel to side c. Extend the angle bisectors of A and B to meet the constructed lines at points A' and B' respectively. Find the length of $\overline{A'B'}$ if given either

  (1) Angles A, B, C and the circumradius R

  (2) Sides a, b, c

  (3) The semiperimeter s, the inradius r and the exradius r_c

  (4) Semiperimeter s and side c.

- **5442:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

  Let L_n be the n^{th} Lucas number defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Prove that for all $n \geq 0$,

$$\frac{1}{2}\begin{vmatrix} (L_n + 2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n + L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{vmatrix}$$

  is the cube of a positive integer and determine its value.

- **5443:** *Proposed by D.M. Bătinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "Geroge Emil Palade" General School, Buzău, Romania*

  Compute $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x} dx$.

- **5444:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Solve in $\Re$ the equation $\{(x+1)^2\} = 2x^2$, where $\{a\}$ denotes the fractional part of a.

Solutions

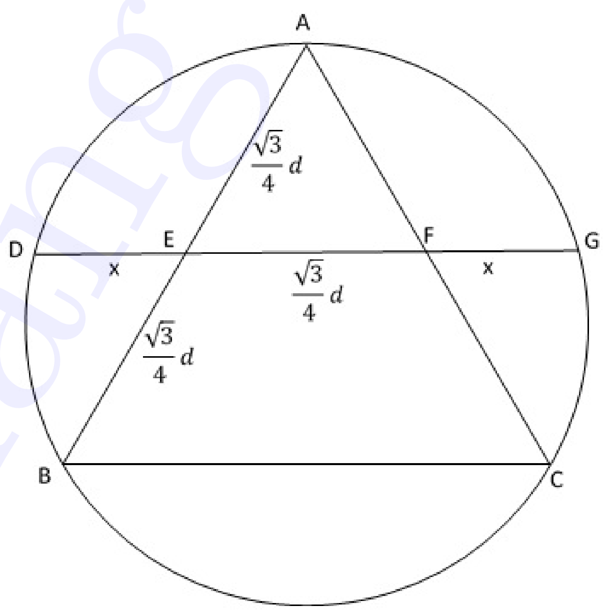
- **5421:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $1-d$ bisects two of its sides.

  Solution by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel

  Let the triangle be ABC and the chord, $DEFG$. Since the diameter is d, the side of the triangle is $\frac{\sqrt{3}d}{2}$ so that $AE = EB = EF = \frac{\sqrt{3}d}{4}$. Let $DE = FG = x$. So that by the theorem on intersecting chords, we have:

$$x\left(x + \frac{\sqrt{3}d}{4}\right) = \frac{3}{16}d^2$$



On the other hand, since $2x + \frac{\sqrt{3}d}{4} = 1 - d$, we have, $x = \frac{1}{2}\left(1 - d - \frac{\sqrt{3}d}{4}\right)$.
Substituting into the equation from the intersecting chords theorem and simplifying, we obtain a quadratic equation for d:

$$d^2 - 32d + 16 = 0$$

whose solutions are $d = 16 \pm 4\sqrt{15}$. But since $1 - d$ is the length of the chord DG, $d < 1$, so that we have the single solution $d = 16 - 4\sqrt{15}$.

Thus the perimeter of the triangle is:

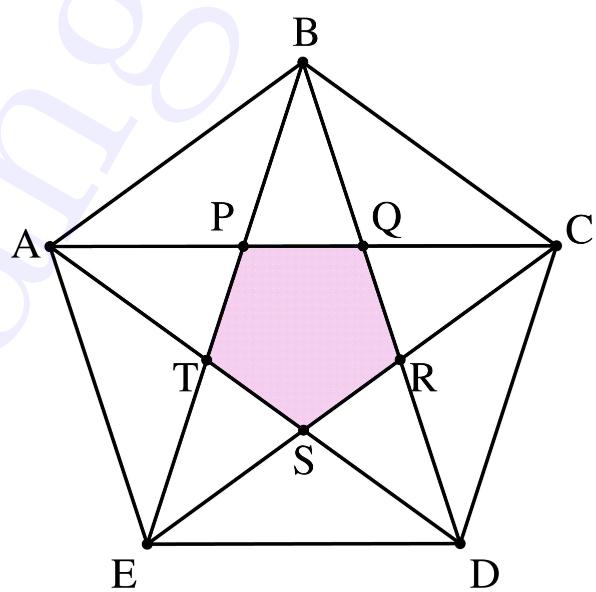
$$3\frac{\sqrt{3}d}{2} = 3\left(\frac{\sqrt{3}}{2}\right)\left(16 - 4\sqrt{15}\right) = 6\left(4\sqrt{3} - 3\sqrt{5}\right)$$

Also solved by Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David A. Huckaby, Angelo State University, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comămesto, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania, and the proposer.

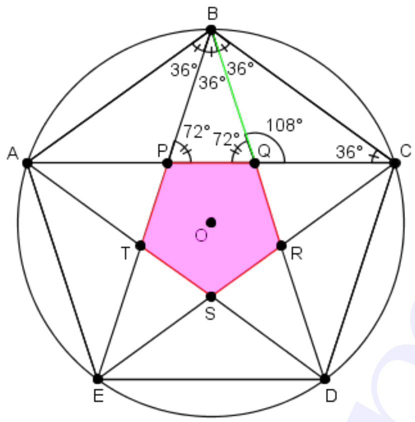
- **5422:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygon $ABCDE$ is a regular pentagon. Pentagon $PQRST$ is bounded by diagonals of pentagon $ABCDE$ as shown. Find the following:

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE}.$$



Solution 1 by Nikos Kalapodis, Patras, Greece



It can be easily checked that pentagon $PQRST$ is regular (since it is equiangular and equilateral). Therefore it is similar to pentagon $ABCDE$. Since the ratio of the areas of two similar polygons is equal to the square of the ratio λ of the corresponding sides, it follows that

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \lambda^2.$$

By the law of sines in triangles BPQ and QBC we have

$$\lambda = \frac{PQ}{BC} = \frac{\dfrac{PQ}{BQ}}{\dfrac{BC}{BQ}} = \frac{\dfrac{\sin 36^\circ}{\sin 72^\circ}}{\dfrac{\sin 108^\circ}{\sin 36^\circ}} = \frac{\sin^2 36^\circ}{\sin 72^\circ \sin 108^\circ} = \left(\frac{\sin 36^\circ}{\sin 72^\circ}\right)^2 = \left(\frac{\sin 36^\circ}{2\sin 36^\circ \cos 36^\circ}\right)^2$$

$$= \frac{1}{4\cos^2 36^\circ} = \frac{1}{4\left(\dfrac{\sqrt{5}+1}{4}\right)^2} = \frac{4}{6+2\sqrt{5}} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}.$$

Therefore $\lambda^2 = \left(\dfrac{3-\sqrt{5}}{2}\right)^2 = \dfrac{14-6\sqrt{5}}{4} = \dfrac{7-3\sqrt{5}}{2}$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $a = \overline{AB}$ and $d = \overline{AC}$ be the lengths of the side and the diagonal of the regular pentagon $ABCDE$.

It is a known result that $d = \varphi a$, where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. (This can be shown, for example by taking into account that triangle ACS and DES have their sides respectively parallel, so they are similar, from where $\dfrac{\overline{AC}}{\overline{DE}} = \dfrac{\overline{CS}}{\overline{ES}}$. Since $ABCS$ is a rhombus, itis a parallelogram and, thus $\overline{CS} = \overline{AB}$ and since $\overline{ES} = \overline{CE} - \overline{CS} = \overline{CE} - \overline{AB}$, we conclude that $\dfrac{\overline{AC}}{\overline{DE}} = \dfrac{\overline{AB}}{\overline{CE} - \overline{AB}}$, or equivalently, $\dfrac{d}{a} = \dfrac{a}{d-a}$, which impies $d^2 - ad - a^2 = 0$ and, hence, $d = \dfrac{a \pm \sqrt{a^2 + 4a^2}}{2}$, so $d = \phi a$ because $a > 0$ and $d > 0$.)

We have that $\overline{ES} = \overline{CE} - \overline{AB} = d - a = (\varphi - 1)a$ and

$$\overline{SR} = \overline{ER} - \overline{ES} = \overline{AB} - \overline{ES} = a - (\varphi - 1)a = (2 - \varphi)a$$

and since the ratio of the areas of $PQRST$ and $ABCDE$ equals the square of their similarity ratio $\dfrac{\overline{SR}}{\overline{AB}} = \dfrac{(2-\varphi)a}{a} = 2 - \varphi$, we conclude that

$$\frac{\text{area}(PQRST)}{\text{area}(ABCDE)} = (2 - \varphi)^2 \approx 0.145898$$

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA

The area of a regular pentagon is proportional to the square of its side length, so

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \left(\frac{\overline{PQ}}{\overline{DE}}\right)^2. \qquad (1)$$

Because triangle BPQ is similar to triangle BED,

$$\left(\frac{\overline{PQ}}{\overline{DE}}\right)^2 = \left(\frac{\overline{BQ}}{\overline{BD}}\right)^2. \qquad (2)$$

Without loss of generality, suppose that pentagon $ABCDE$ has sides of length 1. By the Law of Cosines,

$$\overline{BD}^2 = 2 - 2\cos 108^\circ = 4\sin^2 54^\circ. \qquad (3)$$

Moreover, triangle BQC is isosceles with $\overline{BQ} = \overline{QC}$; thus, by the Law of Cosines,

$$1 = \overline{BQ}^2(2 - 2\cos 108^\circ) = 4\overline{BQ}^2 \sin^2 54^\circ,$$

so that

$$\overline{BQ}^2 = \frac{1}{4\sin^2 54^\circ}. \qquad (4)$$

Combining equations (1) – (4), it follows that

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \frac{1}{16\sin^4 54^\circ}.$$

Now,

$$\sin 54^\circ = \frac{1 + \sqrt{5}}{4} = \frac{1}{2}\varphi,$$

where φ denotes the Golden Ratio, so

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \frac{1}{\varphi^4}.$$

Solution 4 by David E. Manes, SUNY at Oneonta, Oneonta, NY

Let [X] denote the area of polygon X. Then

$$\frac{[PQRST]}{[ABCDE]} = \frac{7-3\sqrt{5}}{2} \approx \frac{3}{20},$$

where $ABCDE$ is a regular pentagon.

Assume that $ABCDE$ is inscribed in the unit circle $x^2+y^2=1$. Then the vertices of the pentagon can be chosen as follows:
$B=(0,1), C=(s_1,c_1), D=(s_2,-c_2), E=(-s_2,-c_2)$ and $A=(-s_1,c_1)$, where

$$\begin{aligned}
c_1 &= \cos\left(\frac{2\pi}{5}\right) = \frac{1}{4}(\sqrt{5}-1),\\
c_2 &= \cos\left(\frac{\pi}{5}\right) = \frac{1}{4}(\sqrt{5}+1),\\
s_1 &= \sin\left(\frac{2\pi}{5}\right) = \frac{1}{4}\sqrt{10+2\sqrt{5}},\\
s_2 &= \sin\left(\frac{4\pi}{5}\right) = \frac{1}{4}\sqrt{10-2\sqrt{5}}.
\end{aligned}$$

Furthermore, the pentagon is symmetric with respect to the y-axis and the pentagon $PQRST$ is also regular since its sides are the bases of five congruent isosceles triangles. If t is the side length of a regular pentagon T, then its area is given by
$[T] = \frac{1}{4}\sqrt{25+10\sqrt{5}}\cdot t^2$.

Let a and b be the side lengths of pentagons $ABCDE$ and $PQRST$, respectively. Then

$$\begin{aligned}
a = BC = \sqrt{s_1^2+(1-c_1)^2} &= \sqrt{\frac{1}{16}(10+2\sqrt{5})+(1-\frac{1}{4}(\sqrt{5}-1)^2}\\
&= \frac{\sqrt{10-2\sqrt{5}}}{2}.
\end{aligned}$$

Therefore,

$$[ABCDE] = \frac{1}{4}\sqrt{25+10\sqrt{5}}\cdot a^2 = \frac{1}{4}\sqrt{25+10\sqrt{5}}\cdot\frac{1}{4}(10-2\sqrt{5}).$$

To find b, note that the equation of the line containing B and E is

$$y-1 = \Big(\frac{5+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\Big)x.$$

If $y=c_1$, then $x=\frac{(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}$ so that the coordinates for point P are

$$P = \Big(\frac{(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}, c_1\Big).$$

By symmetry,

$$Q = \Big(\frac{-(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}, c_1\Big).$$

Therefore,

$$b = PQ = \frac{-2(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}$$

so that

$$[PQRST] = \frac{\sqrt{25+10\sqrt{5}}(c_1-1)^2(10-2\sqrt{5})}{(5+\sqrt{5})^2}$$
$$= \frac{\frac{1}{16}\sqrt{25+10\sqrt{5}}(\sqrt{5}-5)^2(10-2\sqrt{5})}{(5+\sqrt{5})^2)}.$$

Hence,

$$\frac{[PQRST]}{[ABCDE]} = \frac{\left(\frac{\sqrt{25+10\sqrt{5}}(\sqrt{5}-5)^2)(10-2\sqrt{5})}{16(5+\sqrt{5})^2}\right)}{\left(\frac{\sqrt{25+10\sqrt{5}}(10-2\sqrt{5})}{16}\right)}$$
$$= \frac{(\sqrt{5}-5)^2}{(5+\sqrt{5})^2}$$
$$= \frac{7-3\sqrt{5}}{2} \approx 0.145\,898\,033\,75 \approx \frac{3}{20}.$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland

PQRST is similar to pentagon ABCDE. Therefore,

$$\frac{\text{the area of Pentagon PQRST}}{\text{the area of Pentagon ABCDE}} = \left(\frac{SR}{CD}\right)^2 = \left(\frac{SR}{CS}\right)^2 = \left(\frac{CS-CR}{CS}\right)^2$$

$$= \left(1-\frac{CR}{CS}\right)^2 = \left(1-\frac{\sqrt{5}-1}{2}\right)^2\left(\frac{3-\sqrt{5}}{2}\right)^2 = \left(\frac{6-2\sqrt{5}}{4}\right)^2 = \left(\frac{1-\sqrt{5}}{2}\right)^4,$$

where we have used the fact that in a regular pentagon diagonals are cut in sections whose proportions follow the golden ratio (https://en.wikipedia.org/wiki/Pentagon).

Editor′s comment: At first glance it appears that different answers were obtained for this problem. But letting φ equal the golden ratio, and using the equation $\varphi^2 - \varphi + 1 = 0$ it can be shown that the answers are equivalent to one another.

Scott Brown of Auburn University at Montgomery noted that: The material regarding the area of both pentagons can be found on pp. 308-315 in Tom Koshy's book "Fibonacci and Lucas Numbers with Applications". He went on to state that "evidently the problem is not new," to which I add, but it is still very interesting.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jeremiah Bartz, University of North Dakota Grand Forks, ND; Michael N. Fried, Ben-Gurion Univesity, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS: David Huckaby, Angelo State University, San Angelo, TX; Ken Korbin (two solutions), NewYork, NY; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5423:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

Let a, b, c be the side-lengths, r_a, r_b, r_c be the radii of the ex-circles and R, r the radii of the circumcircle and incircle respectively of $\triangle ABC$. Show that

$$\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)}+\frac{(r_b-r)^2+r_cr_a}{(s-c)(s-a)}+\frac{(r_c-r)^2+r_ar_b}{(s-a)(s-b)}\geq 13.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

It is well known (https://en.wikipedia.org/wiki/Incircle-and-excircles-of-a-triangle) that

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)},\ r=\frac{\Delta}{s},\ r_a=\frac{\Delta}{s-a},\ r_b=\frac{\Delta}{s-b},\ r_c=\frac{\Delta}{s-c}.$$

The stated inequality is therefore equivalent to

$$\sum_{cycl}\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)}=\sum_{cycl}s(s-a)\left(\frac{1}{s-a}-\frac{1}{s}\right)^2+\sum_{cycl}\frac{s(s-a)}{(s-b)(s-c)}\geq 13. \qquad (1)$$

Put $u:=s-a,\ v:=s-b,\ w:=s-c$. Then $s=u+v+w$. By the triangle inequality, $u\geq 0,\ v\geq 0,\ w\geq 0$. So (1) is equivalent to

$$\begin{aligned}
&\sum_{cycl}(u+v+w)u\left(\frac{1}{u}-\frac{1}{u+v+w}\right)^2+\sum_{cycl}\frac{(u+v+w)}{vw}\\
=&\ -6+\sum_{cycl}\frac{u+v+w}{u}+\sum_{cycl}\frac{u}{u+v+w}+\sum_{cycl}\frac{u^2}{vw}+\sum_{cycl}\frac{u}{w}+\sum_{cycl}\frac{u}{v}\\
=&\ -2+\sum_{cycl}\frac{v}{u}+\sum_{cycl}\frac{w}{u}+\sum_{cycl}\frac{u^2}{vw}+\sum_{cycl}\frac{u}{w}+\sum_{cycl}\frac{u}{v}\\
=&\ -2+2\sum_{cycl}\frac{v}{u}+2\sum_{cycl}\frac{w}{u}+\sum_{cycl}\frac{u^2}{vu}\geq 13. \qquad (2)
\end{aligned}$$

By the AM-GM inequality,

$$\sum_{cycl}\frac{v}{u}\geq 3\sqrt[3]{\frac{v}{u}\cdot\frac{w}{v}\cdot\frac{u}{w}}=3,\ \sum_{cycl}\frac{w}{u}\geq 3\sqrt[3]{\frac{w}{u}\cdot\frac{u}{v}\cdot\frac{v}{w}}=3,\ \sum_{cycl}\frac{u^2}{vw}\geq 3\sqrt[3]{\frac{u^2}{vw}\cdot\frac{v^2}{wu}\cdot\frac{w^2}{uv}}=3.$$

So (2) holds true.

Solution 2 by Arkady Alt, San Jose, CA

Let F be area of the triangle. Since $r_a=\dfrac{F}{s-a},\ r_b=\dfrac{F}{s-b},\ r_c=\dfrac{F}{s-c},\ r=\dfrac{F}{s}$ then

$$\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)}\ =\ \frac{\left(\dfrac{F}{s-a}-\dfrac{F}{s}\right)^2+\dfrac{F}{s-b}\cdot\dfrac{F}{s-c}}{(s-b)(s-c)}$$

$$
\begin{aligned}
&= \frac{F^2\left(\dfrac{a^2}{s^2\left(s-a\right)^2}+\dfrac{1}{\left(s-b\right)\left(s-c\right)}\right)}{(s-b)(s-c)} \\[2em]
&= \frac{F^2\left(a^2\left(s-b\right)\left(s-c\right)+s^2\left(s-a\right)^2\right)}{s^2\left(s-a\right)^2\left(s-b\right)^2\left(s-c\right)^2} \\[1em]
&= \frac{a^2\left(s-b\right)\left(s-c\right)+s^2\left(s-a\right)^2}{F^2} \\[1em]
&= \frac{4a^2\left(a+c-b\right)\left(a+b-c\right)+\left(a+b+c\right)^2\left(b+c-a\right)^2}{16F^2} \\
&= \frac{\left(4\left(bc^3+b^3c\right)-6\left(a^2b^2+6a^2c^2-b^2c^2\right)+5a^4+b^4+c^4+4a^2bc\right)}{16F^2} \text{ and, therefore,}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{cyc}\frac{\left(r_a-r\right)^2+r_br_c}{(s-b)(s-c)} &= \frac{1}{F^2}\sum_{cyc}\left(4bc\left(a^2+b^2+c^2\right)-6\left(a^2b^2+6a^2c^2-b^2c^2\right)+5a^4+b^4+c^4\right) \\[1em]
&= \frac{4\left(a^2+b^2+c^2\right)\left(ab+bc+ca\right)-6\left(a^2b^2+b^2c^2+a^2c^2\right)+7\left(a^4+b^4+c^4\right)}{16F^2} \\[1em]
&= \frac{4\left(a^2+b^2+c^2\right)\left(ab+bc+ca\right)-20\left(a^2b^2+b^2c^2+a^2c^2\right)+7\left(a^2+b^2+c^2\right)^2}{16F^2}.
\end{aligned}
$$

Let $x := s-a,\ y := s-b,\ z := s-c,\ p := xy+yz+zx,\ q := xyz$. Due to the homogeneity of the original inequality we can assume that $s=1$. Then $a=1-x, b=1-y, c=1-z$,

$$x,y,z>0,\ x+y+z=1,\ a+b+c=2,\ abc=p-q,\ F=\sqrt{xyz}=\sqrt{q},$$

$$ab+bc+ca=1+p, a^2+b^2+c^2=2\left(1-p\right),$$

$$a^2b^2+b^2c^2+a^2c^2=\left(ab+bc+ca\right)^2-2abc\left(a+b+c\right)$$

$$=\ \left(1+p\right)^2-4\left(p-q\right)=\left(1-p\right)^2+4q, \text{ and original inequality becomes}$$

$$\frac{8\left(1-p^2\right)-20\left(\left(1-p\right)^2+4q\right)+28\left(1-p\right)^2}{16q}\geq 13 \iff \frac{1-p-5q}{q}\geq 13 \iff 1-p\geq 18q.$$

Since $q = xyz \le \frac{x+y+z}{3} \cdot \frac{xy+yz+zx}{3} = \frac{p}{9}$ and

$p = xy + yz + zx \le \frac{(x+y+z)^2}{3} = \frac{1}{3}$, then

$1 - p - 18q \ge 1 - p - 18 \cdot \frac{p}{9} = 1 - 3p \ge 0.$

Solution 3 and 4 by Nikos Kalapodis, Patras, Greece

Using the well-known formulas $S = sr$, $S = r_a(s-a)$ and $S = \sqrt{s(s-a)(s-b)(s-c)}$ we have

$(r_a - r)^2 = \left(\frac{S}{s-a} - \frac{S}{s}\right)^2 = \frac{a^2S^2}{s^2(s-a)^2} = \frac{a^2(s-b)(s-c)}{s(s-a)}$ and

$r_b rc = \frac{S}{s-b} \cdot \frac{S}{s-c} = \frac{S^2}{(s-b)(s-c)} = s(s-a).$

It follows that $\frac{(r_a-r)^2}{(s-b)(s-c)} = \frac{a^2}{s(s-a)}$, and $\frac{r_b rc}{(s-b)(s-c)} = \frac{s(s-a)}{(s-b)(s-c)} = \frac{(s-a)^2}{r^2}$.

By Cauchy-Schwarz inequality and the well-known inequality $s \ge 3\sqrt{3}r$ we have

$$\sum_{cyc} \frac{(r_a-r)^2 + r_br_c}{(s-b)(s-c)} = \sum_{cyc} \frac{(r_a-r)^2}{(s-b)(s-c)} + \sum_{cyc} \frac{r_br_c}{(s-b)(s-c)} = \sum_{cyc} \frac{a^2}{s(s-a)} + \sum_{cyc} \frac{(s-a)^2}{r^2} \ge$$

$$\frac{(a+b+c)^2}{s(s-a+s-b+s-c)} + \frac{(s-a+s-b+s-c)^2}{3r^2} = \frac{4s^2}{s^2} + \frac{s^2}{3r^2} \ge 4 + \frac{1}{3} \cdot 27 = 4 + 9 = 13.$$

Solution 4

Using the well-known formulas $(s-b)(s-c) = rr_a$, $r_a + r_b + r_c = r + 4R$,
$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$ and Euler's inequality $(R \ge 2r)$ we have

$$\sum_{cyc} \frac{(r_a-r)^2}{(s-b)(s-c)} = \sum_{cyc} \frac{r_a^2 + r^2 - 2rr_a}{rr_a} = \sum_{cyc} \left(\frac{r_a}{r} + \frac{r}{r_a} - 2\right) = \frac{1}{r}\sum_{cyc} r_a + r\sum_{cyc} \frac{1}{r_a} - 6$$

$= \frac{1}{r}(r+4R) + r \cdot \frac{1}{r} - 6 = \frac{4R}{r} - 4 \ge 4$, and

$$\sum_{cyc} \frac{r_br_c}{(s-b)(s-c)} = \sum_{cyc} \frac{r_br_c}{rr_a} = \frac{r_ar_br_c}{r}\sum_{cyc} \frac{1}{r_a^2} = \frac{r_ar_br_c}{r}\sum_{cyc} \left(\frac{1}{r_a}\right)^2 \ge \frac{r_ar_br_c}{r}\sum_{cyc} \frac{1}{r_ar_b}$$

$= \frac{1}{r}\sum_{cyc} r_a = \frac{1}{r}(r+4R) = 1 + \frac{4R}{r} \ge 9.$

Therefore $\sum_{cyc} \frac{(r_a-r)^2 + r_br_c}{(s-b)(s-c)} = \sum_{cyc} \frac{(r_a-r)^2}{(s-b)(s-c)} + \sum_{cyc} \frac{r_br_c}{(s-b)(s-c)} \ge 4 + 9 = 13.$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Albert Stadler Herrliberg, Switzerland; Titu Zvonaru,Comănesti, Romania and Neculai Stanciu "Geroge Emil Palade" School, Buză, Romania, and the proposer.

- **5424:** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Forcsani, Romania*

Let a, b, c and d be positive real numbers such that $abc + bcd + cda + dab = 4$. Prove that $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) \geq 256$.

Solution 1 by Ed Gray, Highland Beach, FL

Like most of these inequality problems, I find there is generally a solution evident by inspection that actually makes the inequality a strict equality. Then, by choosing numbers with the correct orientation, one can show that the numbers seen by inspection actually provide an extremum. Let's put labels on the givens in the statement of the problem.

(1) $abc + bcd + cda + dab = 4$. We wish to prove:
(2) $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) \geq 256$.
Clearly, if $a = b = c = d = 1$, the following relations hold:
(3) $abc = 1$,
(4) $bcd = 1$,
(5) $cda = 1$,
(6) $dab = 1$, and
(7) $abc + bcd + cda + dab = 4$. Also,
(8) $a^8 - a^4 = 0$,
(9) $b^7 - b^3 = 0$,
(10) $c^6 - c^2 = 0$,
(11) $d^5 - d = 0$, so that the product in (2) becomes $4^4 = 256$.
Therefore, if we show that choices for a, b, c, d with at least one < 1 all makes the product of $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) > 256$, the conjecture would be true.

It would be sufficient to consider three cases:
(A) $a < 1,\ b = 1,\ c = 1,\ d > 1$,
(B) $a < 1,\ b < 1,\ c = 1,\ d > 1$,
(C) $a < 1,\ b < 1,\ c < 1,\ d > 1$,
In each case, we choose a, b, c as necessary and compute d by using (1).
To be explicit, we choose the following numbers:
(A) $a = .99,\ b = 1,\ c = 1$, calculated value of d is 1.010067114
Evaluating Eq.(2) using these numbers, the product is 256.1952096
(B) $a = .99,\ b = .99,\ c = 1$, calculated value of d is 1.02020202
Evaluating Eq.(2) using these numbers, the product is 256.489
(C) $a = .99, b = .99, c = .99$, calculated value of d is 1.030405401
Evaluating Eq.(2) using these numbers, the product is 256.9590651

Solution 2 by Moti Levy, Rehovot, Israel

The inequality can be simplified by applying

$$\begin{aligned} a^8 - a^4 &\geq a^4 - 1, \\ b^7 - b^3 &\geq b^4 - 1, \\ c^6 - c^2 &\geq c^4 - 1, \\ d^5 - d &\geq d^4 - 1. \end{aligned}$$

Hence it suffices to prove that

$$(a^4+3)(b^4+3)(c^4+3)(d^4+3) \geq 256.$$

We rewrite the left hand side:

$$\begin{aligned} &(a^4+3)(b^4+3)(c^4+3)(d^4+3) \\ &= (a^4+1+1+1)(1+b^4+1+1)(1+1+c^4+1)(1+1+1+d^4). \end{aligned}$$

By Holder's inequality,

$$(a^4+1+1+1)(1+b^4+1+1)(1+1+c^4+1)(1+1+1+d^4) \geq (a+b+c+d)^4 .$$

Thus, what is left is to show that $abc+bcd+cda+dab = 4$ implies that $a+b+c+d \geq 4$.

To this end, we employ elementary symmetric polynomials notation:

$$\begin{aligned} p_1 &= e_1 = a+b+c+d, \\ p_2 &= a^2+b^2+c^2+d^2, \\ p_3 &= a^3+b^3+c^3+d^3, \\ e_3 &= abc+bcd+cda+dab. \end{aligned}$$

It is well known (from Newton's identities) that

$$p_1^3 - 3p_1p_2 + 2p_3 = 6e_3.$$

We also have from the power mean inequality $\sqrt{\frac{p_2}{4}} \geq \frac{p_1}{4}$,

$$p_2 \geq \frac{p_1^2}{4},$$

and from Chebyshev's inequality $a^3+b^3+c^3+d^3 \geq abc+bcd+cda+dab$,

$$p_3 \geq e_3.$$

It follows that,

$$p_1^3 - 3p_1p_2 = 6e_3 - 2p_3 \geq 4e_3.$$

$p_2 \geq \frac{p_1^2}{4}$ implies that

$$p_1^3 - 3p_1\frac{p_1^2}{4} \geq p_1^3 - 3p_1p_2,$$

or that

$$\frac{p_1^3}{4} \geq 4e_3 = 16.$$

We conclude that $p_1 \geq 4$ and this fact completes the solution.

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5425:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$. If n is an odd positive integer the show that $1 + det(A)$ is the product of two consecutive Fibonacci numbers, where

$$A = \begin{pmatrix} F_1^2 - 1 & F_1F_2 & F_1F_3 & \dots & F_1F_n \\ F_2F_1 & F_2^2 - 1 & F_2F_3 & \cdots & F_2F_n \\ F_3F_1 & F_3F_2 & F_3^2 - 1 & \cdots & F_3F_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_nF_1 & F_nF_2 & F_nF_3 & \cdots & F_n^2 - 1 \end{pmatrix}$$

Brian Bradie, Christopher Newport University, Newport News, VA

We will establish the more general result that for any positive integer n the quantity $1 + (-1)^{n-1}\det(A)$ is the product of two consecutive Fibonacci numbers. Toward this end, let

$$B = A + I = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{bmatrix} \begin{bmatrix} F_1 & F_2 & F_3 & \cdots & F_n \end{bmatrix}.$$

The matrix B is a rank 1 matrix with eigenvalue

$$\sum_{j=1}^{n} F_j^2 = F_nF_{n+1}$$

of algebraic multiplicity 1 and eigenvalue 0 of algebraic multiplicity $n - 1$. It then follows that the matrix A has eigenvalue $F_nF_{n+1} - 1$ of algebraic multiplicity 1 and eigenvalue -1 of algebraic multiplicity $n - 1$. Thus,

$$\det(A) = (-1)^{n-1}\left(F_nF_{n+1} - 1\right),$$

and

$$1 + (-1)^{n-1}\det(A) = 1 + (F_nF_{n+1} - 1) = F_nF_{n+1}.$$

Also solved by Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5426:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of natural numbers. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]} \text{ converges.}$$

Here $[x, y]$ denotes the least common multiple of the natural numbers x and y.

Solution 1 by Moti Levy, Rehovot, Israel

It is known that
$$\operatorname{lcm}(a,b)\gcd(a,b)=ab.$$
Clearly, $a=A\gcd(a,b)$ and $b=B\gcd(a,b)$. If $a>b$ then $A>B$ and $a-b=(A-B)\gcd(a,b)>\gcd(a,b)$.
$$a-b>\gcd(a,b)=\frac{ab}{\operatorname{lcm}(a,b)},$$
or
$$\frac{1}{\operatorname{lcm}(a,b)}<\frac{a-b}{ab}=\frac{1}{b}-\frac{1}{a},\qquad a>b. \tag{1}$$
It follows from (1) that
$$\frac{\sqrt{a_n}}{\operatorname{lcm}(a_n,a_{n+1})}<\sqrt{a_n}\left(\frac{1}{a_n}-\frac{1}{a_{n+1}}\right)=\frac{1}{\sqrt{a_n}}-\frac{1}{\sqrt{a_{n+1}}}\sqrt{\frac{a_n}{a_{n+1}}},$$
so that
$$\sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{\operatorname{lcm}(a_n,a_{n+1})}<\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{a_n}}-\frac{1}{\sqrt{a_{n+1}}}\sqrt{\frac{a_n}{a_{n+1}}}\right). \tag{2}$$
Let us define a sequence of positive real numbers $(b_n)_{n\geq 1}$ as follows:
$$b_{2k-1} = \frac{1}{\sqrt{a_k}}, \tag{3}$$
$$b_{2k} = \frac{1}{\sqrt{a_{k+1}}}\sqrt{\frac{a_{k+1}}{a_{k+2}}} \tag{4}$$

By definition (4), $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{a_n}}-\frac{1}{\sqrt{a_{n+1}}}\sqrt{\frac{a_n}{a_{n+1}}}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}b_n$.

The terms of the sequence $(b_n)_{n\geq 1}$ satisfy: $b_n>b_{n+1}>0$ and $\lim_{n\to\infty}b_n=0$.

The series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges by the Alternating Series Test (called also Leibniz Criterion).
Inequality (2) implies that the series $\sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{\operatorname{lcm}(a_n,a_{n+1})}$ converges as well.

Remark: The idea for this solution came from the enjoyable short paper by D. Borwein, who solved a conjecture of P. Erdös.

Reference: D. Borwein, "A Sum of Reciprocals of Least Common Multiples", Canadian Mathematical Bulletin, Volume 20 (1), 1978, pp. 117-118.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote by (x,y) the greatest common divisor of the natural numbers x and y.
It is well known that $(x,y)[x,y]=xy$. Hence for any natural number $M\geq 2$, we have
$$\sum_{n=1}^{M}\frac{\sqrt{a_n}}{[a_n,a_{n+1}]} = \sum_{n=1}^{M}\frac{(a_n,a_{n+1})}{\sqrt{a_n}a_{n+1}}$$

$$
\begin{aligned}
&= \sum_{n=1}^{M} \frac{(a_n, a_{n+1} - a_n)}{\sqrt{a_n}a_{n+1}} \\
&\leq \sum_{n=1}^{M} \frac{a_{n+1} - a_n}{\sqrt{a_n}a_{n+1}} \\
&= \sum_{n=1}^{M} \frac{1}{\sqrt{a_n}} - \sum_{n=2}^{M+1} \frac{\sqrt{a_{n-1}}}{a_n} \\
&= \frac{1}{\sqrt{a_1}} - \frac{\sqrt{a_M}}{a_{M+1}} + \sum_{n=2}^{M} \frac{\sqrt{a_n} - \sqrt{a_n - 1}}{a_n} \\
&\leq \frac{1}{\sqrt{a_1}} + \sum_{n=2}^{M} \frac{\sqrt{a_n} - \sqrt{a_n - 1}}{\sqrt{a_n}\sqrt{a_{n-1}}} \\
&= \frac{1}{\sqrt{a_1}} + \sum_{n=2}^{M} \frac{1}{\sqrt{a_{n-1}}} - \sum_{n=2}^{M} \frac{1}{\sqrt{a_n}} \\
&= \frac{2}{\sqrt{a_1}} - \frac{1}{\sqrt{a_M}} \\
&\leq \frac{2}{\sqrt{a_1}}.
\end{aligned}
$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]}$ converges.

Also solved by Ed Gray, Highland Beach, FL and the author.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

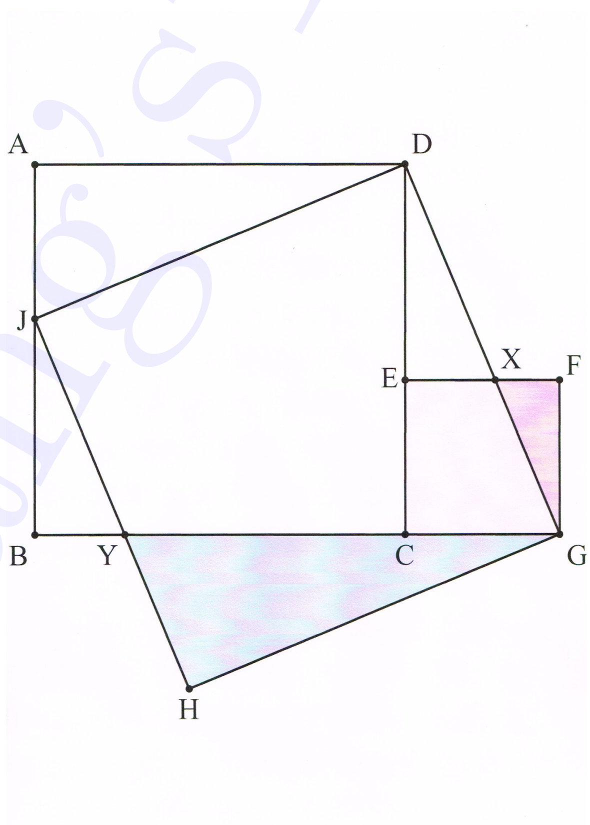
Solutions to the problems stated in this issue should be posted before June 15, 2017

- **5445:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of a triangle with exradii $(3, 4, 5)$.

- **5446:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Polygons $ABCD, CEFG$, and $DGHJ$ are squares. Moreover, point E is on side $DC, X = DG \cap EF$, and $Y = BC \cap JH$. If GX splits square $CEFG$ in regions whose areas are in the ratio 5:19. What part of square $DGHJ$ is shaded? (Shaded region in $DGHJ$ is composed of the areas of triangle YHG and trapezoid $EXGC$.)



- **5447:** *Proposed by Iuliana Trască, Scornicesti, Romanai*

  Show that if x, y, and z is each a positive real number, then

$$\frac{x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3}{x^2 \cdot y^2 \cdot z^2} \geq \frac{x^3 + y^3 + z^3 + 3x \cdot y \cdot z}{2}.$$

- **5448:** *Proposed by Yubal Barrios and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

  Evaluate: $\lim\limits_{n \to \infty} \sqrt[n]{\sum\limits_{\substack{0 \leq i,j \leq n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j}}.$

- **5449:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the use of a computer, find the real roots of the equation

$$x^6 - 26x^3 + 55x^2 - 39x + 10 = (3x-2)\sqrt{3x-2}.$$

- **5450:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let k be a positive integer. Calculate

$$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dxdy,$$

  where $\lfloor a \rfloor$ denotes the floor (the integer part) of a.

Solutions

5427: *Proposed by Kenneth Korbin, New York, NY*

Rationalize and simplify the fraction

$$\frac{(x+1)^4}{x(2016x^2 - 2x + 2016)} \quad \text{if} \quad x = \frac{2017 + \sqrt{2017 - \sqrt{2017}}}{2017 - \sqrt{2017 - \sqrt{2017}}}.$$

Solution 1 by David E. Manes, SUNY at Oneonta, Oneonta, NY

Let $F = (x+1)^4/(x(2016x^2 - 2x + 2016))$ and let $y = \sqrt{2017 - \sqrt{2017}}$. Then $y^2 = 2017 - \sqrt{2017}$ and $y^4 = 2017(2018 - 2\sqrt{2017})$. Moreover,

$$x = \frac{2017 + y}{2017 - y}, \quad \frac{1}{x} = \frac{2017 - y}{2017 + y}, \quad x + 1 = \frac{2(2017)}{2017 - y} \quad \text{and}$$

$$
\begin{aligned}
x^2+1 &= \Big(\frac{2017+y}{2017-y}\Big)^2+1 = \frac{(2017+y)^2+(2017-y)^2}{(2017-y)^2}\\
&= \frac{2(2017^2+y^2)}{(2017-y)^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2016(x^2+1)-2x &= 2\Big[\frac{2016(2017^2+y^2)}{(2017-y)^2}-\frac{2017+y}{2017-y}\Big]\\
&= 2\Big[\frac{2016(2017^2+y^2)-(2017^2-y^2)}{(2017-y)^2}\Big]\\
&= 2\Big[\frac{2015\cdot 2017^2+2017y^2}{(2017-y)^2}\Big]\\
&= 2(2017)\Big[\frac{2015(2017)+y^2}{(2017-y)^2}\Big]
\end{aligned}
$$

Substituting these values into the fraction F and simplifying, we obtain

$$
\begin{aligned}
F &= \frac{\left(\frac{2(2017)}{2017-y}\right)^4(2017-y)}{(2017+y)(2(2017)\left(\frac{2015(2017)+y^2}{(2017-y)^2}\right)}\\
&= \frac{(2(2017))^3}{(2017^2-y^2)(2015\cdot 2017+y^2)}\\
&= \frac{8(2017)^3}{2015\cdot 2017^3+2\cdot 2017(2017-\sqrt{2017})-2017(2018-2\sqrt{2017})}\\
&= \frac{8(2017)^2}{2015\cdot 2017^2+2016}\\
&= \frac{32546312}{8197604351}\\
&\approx 0.003\,970\,222\,349.
\end{aligned}
$$

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

For notational convenience we set $d=2017-\sqrt{2017}$, $y=2017+\sqrt{d}$, and $z=2017-\sqrt{d}$. Thus our x is y/z. We have

$$
\begin{aligned}
\frac{(x+1)^4}{x(2016x^2-2x+2016)} &= \frac{\left(\dfrac{y}{z}+1\right)^4}{\left(\dfrac{y}{z}\right)\left(2016\left(\dfrac{y}{z}\right)^2-2\left(\dfrac{y}{z}\right)+2016\right)}\cdot\frac{z^4}{z^4}\\
&= \frac{(y+z)^4}{yz(2016y^2-2yz+2016z^2)}
\end{aligned}
$$

Now

$$
y+z=2\cdot 2017,
$$

$$
\begin{aligned}
yz &= 2017^2 - d \\
&= 2017^2 - 2017 + \sqrt{2017} \\
&= 2017 \cdot 2016 + \sqrt{2017},
\end{aligned}
$$

and

$$
\begin{aligned}
2016y^2 - 2yz + 2016z^2 &= 2016(y^2 + z^2) - 2yz \\
&= 2016((y+z)^2 - 2yz) - 2yz \\
&= 2016(y+z)^2 - 2 \cdot 2017yz \\
&= 2016(2 \cdot 2017)^2 - 2 \cdot 2017(2017 \cdot 2016 + \sqrt{2017}) \\
&= 2 \cdot 2017(2017 \cdot 2016 - \sqrt{2017}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{(y+z)^4}{yz(2016y^2 - 2yz + 2016z^2)} &= \frac{2^4 \cdot 2017^4}{2 \cdot 2017(2017^2 \cdot 2016^2 - 2017)} \\
&= \frac{2^3 \cdot 2017^2}{2017 \cdot 2016^2 - 1} \\
&= \frac{32546312}{8197604351}.
\end{aligned}
$$

Solution 3 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Let $y = 2017$ and $w = \sqrt{y - \sqrt{y}}$. Observe

$$
\begin{aligned}
x &= \frac{y+w}{y-w} \\
x + 1 &= \frac{2y}{y-w} \\
w^2 &= y - \sqrt{y} \\
w^4 &= y^2 + y - 2y\sqrt{y}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{(x+1)^4}{x(2016x^2 - 2x + 2016)} &= \frac{2^4y^4}{(y-w)^4} \cdot \frac{y-w}{y+w} \cdot \frac{1}{2016\left(\frac{y+w}{y-w}\right)^2 - 2\left(\frac{y+w}{y-w}\right) + 2016} \\
&= \frac{2^4y^4}{2016(y+w)^3(y-w) - 2(y+w)^2(y-w)^2 + 2016(y+w)(y-w)^3} \\
&= \frac{2^4y^4}{2(2015y^4 + 2y^2w^2 - 2017w^4)} \\
&= \frac{2^3y^3}{2015y^3 + 2yw^2 - w^4} \qquad \text{using } y = 2017 \\
&= \frac{8y^3}{2015y^3 + 2y(y - \sqrt{y}) - (y^2 + y - 2y\sqrt{y})} \\
&= \frac{8y^3}{2015y^3 + y^2 - y} \\
&= \frac{8y^2}{2015y^2 + y - 1}
\end{aligned}
$$

so that

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{8(2017)^2}{2015(2017)^2+2016} = \frac{32546312}{8197604351}.$$

Solution 4 by Arkady Alt, San Jose, CA

Let $x = \dfrac{a+\sqrt{a-\sqrt{a}}}{a-\sqrt{a-\sqrt{a}}}$. Then, $x+\dfrac{1}{x} = \dfrac{a+\sqrt{a-\sqrt{a}}}{a-\sqrt{a-\sqrt{a}}} + \dfrac{a-\sqrt{a-\sqrt{a}}}{a+\sqrt{a-\sqrt{a}}} =$
$\dfrac{\left(a+\sqrt{a-\sqrt{a}}\right)^2+\left(a-\sqrt{a-\sqrt{a}}\right)^2}{a^2-a+\sqrt{a}} = \dfrac{2\left(a^2+a-\sqrt{a}\right)}{a^2-a+\sqrt{a}} = \dfrac{2\left(-a^2+a-\sqrt{a}+2a^2\right)}{a^2-a+\sqrt{a}} =$
$-2+\dfrac{4a^2}{a^2-a+\sqrt{a}} \iff x+\dfrac{1}{x}+2 = \dfrac{4a^2}{a^2-a+\sqrt{a}}$ and, therefore,
$\dfrac{(x+1)^4}{x((a-1)\,x^2-2x+(a-1))} = \dfrac{(x+1)^4}{x^2((a-1)\left(x+\dfrac{1}{x}+2\right)-2a)} =$

$\dfrac{\left(x+\dfrac{1}{x}+2\right)^2}{(a-1)\left(x+\dfrac{1}{x}+2\right)-2a} = \dfrac{\left(\dfrac{4a^2}{a^2-a+\sqrt{a}}\right)^2}{(a-1)\cdot\dfrac{4a^2}{a^2-a+\sqrt{a}}-2a} =$

$\dfrac{16a^4}{((a-1)\cdot 4a^2-2a\left(a^2-a+\sqrt{a}\right))\left(a^2-a+\sqrt{a}\right)} = \dfrac{16a^4}{2a\left(a^2-a-\sqrt{a}\right)\left(a^2-a+\sqrt{a}\right)} =$
$\dfrac{8a^3}{\left(a^2-a\right)^2-a} = \dfrac{8a^2}{a\,(a-1)^2-1}.$

For $a=2017$ we get $\dfrac{(x+1)^4}{x(2016x^2-2x+2016)} = \dfrac{8\cdot 2017^2}{2017\cdot 2016^2-1}.$

Solution 5 by Kee-Wai Lau, Hong Kong, China

We show that

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{32546312}{8197604351} \qquad (1)$$

Firstly we have

$$\begin{aligned}
x+\frac{1}{x} &= \frac{2017+\sqrt{2017-\sqrt{2017}}}{2017-\sqrt{2017-\sqrt{2017}}} + \frac{2017-\sqrt{2017-\sqrt{2017}}}{2017+\sqrt{2017-\sqrt{2017}}} \\
&= \frac{\left(2017+\sqrt{2017-\sqrt{2017}}\right)^2+\left(2017-\sqrt{2017-\sqrt{2017}}\right)^2}{\left(2017-\sqrt{2017-\sqrt{2017}}\right)^2+\left(2017+\sqrt{2017-\sqrt{2017}}\right)^2} \\
&= \frac{2(4070306-\sqrt{2017})}{4066272+\sqrt{2017}} \\
&= \frac{2(4070306-\sqrt{2017})(4066272-\sqrt{2017})}{(4066272+\sqrt{2017})(4066272-\sqrt{2017})}
\end{aligned}$$

$$= \frac{2(8205736897 - 4034\sqrt{2017})}{8197604351}.$$

Next, we have

$$\left(x + \frac{1}{x} + 2\right)^2 = \frac{131291822608(8197604353 - 4032\sqrt{2017})}{67200717095534131201}$$

and

$$2016\left(x + \frac{1}{x}\right) - 2 = \frac{4034(8197604353 - 4032\sqrt{2017})}{8197604351}.$$

Since $\dfrac{(x+1)^4}{x(2016x^2 - 2x + 2016)} = \dfrac{\left(x + \dfrac{1}{x} + 2\right)^2}{2016\left(x + \dfrac{1}{x}\right) - 2}$, so (1) follows.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Telman Rashidov, Azerbaijan Medical University, Baku Azerbaijan; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5428: *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

If $x > 0$, then $\dfrac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \dfrac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} \geq 1 - \dfrac{1}{\sqrt[4]{2}}$, where $[.]$ and $\{.\}$ respectively denote the integer part and the fractional part of x.

Solution 1 by Soumava Chakraborty, Kolkata, India

Case 1: $0 < x < 1 \quad [x] = 0$. Therefore,

$$LHS = \frac{\{x\}}{\sqrt[4]{17\{x\}^4}} = \frac{1}{\sqrt[4]{17}} > 1 - \frac{1}{\sqrt[4]{2}}.$$

Case 2: $[x] \geq 1$ and $\{x\} = 0$. Therefore,

$$LHS = \frac{[x]}{\sqrt[4]{2[x]^4}} = \frac{1}{\sqrt[4]{2}} > 1 - \frac{1}{\sqrt[4]{2}}.$$

Case 3: $[x] \geq 1$ and $0 < \{x\} < 1$. Therefore,

$$\begin{aligned}
\{x\} < 1 \leq [x] \quad &\Rightarrow \quad \{x\} < [x]\Big(2\{x\} + [x]\Big)^4 + [x]^4 < 82[x]^4 \\
&\Rightarrow \quad \frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} > \frac{1}{\sqrt[4]{82}}, \text{and } \frac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} > 0, \text{ and therefore} \\
LHS \quad &> \quad \frac{1}{\sqrt[4]{82}} > 1 - \frac{1}{\sqrt[4]{2}}.
\end{aligned}$$

Combining the 3 cases, the LHS is always $> \dfrac{1}{\sqrt[4]{82}}$ which is $> 1 - \dfrac{1}{\sqrt[4]{2}}$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since $x = [x] + \{x\}$ and $[x] \le x < [x] + 1$, we have that $[x] + 2\{x\} = x + \{x\}$ and $\{x\} = x - [x] < 1$, so $[x] + 2\{x\} = x + \{x\} \le x + x = 2x$ and, thus, since $x > 0$, $(x + \{x\})^4 < (2x)^4$; hence, $[x]^4 + (x + \{x\})^4 < x^4 + 16x^4$ and $\{x\}^4 + (x + \{x\})^4 < x^4 + 16x^4$.

It follows that $0 < \sqrt[4]{[x]^4 + (x + \{x\})^4} < \sqrt[4]{17x^4}$ and $0 < \sqrt[4]{\{x\}^4 + (x + \{x\})^4} < \sqrt[4]{17x^4}$ so

$0 < \dfrac{1}{\sqrt[4]{[x]^4 + (x + \{x\})^4}} \le \dfrac{1}{\sqrt[4]{17}x}$ and $0 < \dfrac{1}{\sqrt[4]{\{x\}^4 + (x + \{x\})^4}} \le \dfrac{1}{\sqrt[4]{17}x}$ and hence,

$\dfrac{[x]}{\sqrt[4]{\{x]\}^4 + (x + \{x\})^4}} \le \dfrac{[x]}{\sqrt[4]{17}x}$ with equality iff $[x] = 0$ and

$0 < \dfrac{\{x\}}{\sqrt[4]{\{x]\}^4 + (x + \{x\})^4}} \le \dfrac{\{x\}}{\sqrt[4]{17}x}$ with equality iff $\{x\} = 0$, so

$$\frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \frac{\{x\}}{\sqrt[4]{[\{x\}^4 + ([x] + 2\{x\})^4}} = \frac{[x]}{\sqrt[4]{[x]^4 + (x + \{x\})^4}} + \frac{\{x\}}{\sqrt[4]{[\{x\}^4 + (x + \{x\})^4}}$$

$$\ge \quad \frac{[x]}{\sqrt[4]{17}x} + \frac{\{x\}}{\sqrt[4]{17}x} = \frac{[x] + \{x\}}{\sqrt[4]{17}x} = \frac{x}{\sqrt[4]{17}x} = \frac{1}{\sqrt[4]{17}}$$

with equality iff $[x] = 0$ and $\{x\} = 0$, that is, iff $0 < x < 1$ and $x \in N$, with is impossible.

Hence, we have proved the more general and strict inequality

$$\frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \frac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} > \frac{1}{\sqrt[4]{17}}$$

(which implies, because $\dfrac{1}{\sqrt[4]{17}} + \dfrac{1}{\sqrt[4]{2}} = 1.33338\cdots > 1$, the initial result.)

Also solved by Moti Levy, Rehovot, Israel; Nirapada Pal-India, and the proposer.

5429: *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Prove that there are infinitely many positive integers a, b such that $18a^2 - b^2 - 6a - b = 0$.

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Define

$$g(a, b) = 18a^2 - 6a - b^2 - b$$

and

$$f(a, b) = (577a + 136b - 28, 2448a + 577b - 120).$$

By direct computation we see that $g(f(a,b)) = g(a,b)$. If $g(a_0, b_0) = 0$ with $a_0, b_0 \in N$ then the iterates $(a_n, b_n) = f(a_{n-1}, b_{n-1})$ are in $N \times N$ and satisfy $g(a_n, b_n) = 0$, for all $n \in N$.

Since $g(1,3) = 0$, starting with $(a_0, b_0) = (1,3)$ we obtain the infinite sequence of solutions

$$(1,3)\,,\ \ (957, 4059)\,,\ \ (1104185, 4684659)\,,\ \ (1274228341, 5406093003)\,,$$
$$(1470458401137, 6238626641379)\,, \ldots$$

Since $g(5,20) = 0$, starting with $(a_0, b_0) = (5,20)$ we obtain another infinite sequence of solutions:

$$(5,20)\,,\ \ (5577, 23660)\,,\ \ (6435661, 27304196)\,,\ \ (7426747025, 31509019100)\,,$$
$$(8570459630997, 36361380737780)\,, \ldots$$

Solution 2 by Trey Smith, Angelo State University, San Angelo, TX

Solution by Trey Smith, Angelo State University, San Angelo, TX 76909

We start by observing that

$$18a^2 - b^2 - 6a - b = 0 \quad \Rightarrow \quad (2b+1)^2 - 2(6a-1)^2 = -1$$

which is suspiciously close to being Pell's Equation. Our particular equation is of the form

$$x^2 - 2y^2 = -1.$$

Notice that $(7,5)$ ($x = 7$ and $y = 5$) is a solution to $x^2 - 2y^2 = -1$. We will now create a sequence of solutions starting with $(c_0, d_0) = (7,5)$ in the following recursive manner. For $n \geq 0$, let

$$c_{n+1} = c_n^3 + 6c_n d_n^2, \quad d_{n+1} = 3c_n^2 d_n + 2d_n^3.$$

We prove the following facts regarding this sequence.

Fact 1. For all n, (c_n, d_n) is a solution to $x^2 - 2y^2 = -1$.

Proof: We use induction to prove this. In the ground case, it is clear that $(c_0, d_0) = (7,5)$ is a solution to $x^2 - 2y^2 = -1$.

Assume that (c_n, d_n) is a solution.

$$\begin{aligned}
& c_{n+1}^2 - 2d_{n+1}^2 \\
= \ & (c_n^3 + 6c_n d_n^2)^2 - 2(3c_n^2 d_n + 2d_n^3)^2 \\
= \ & c_n^6 + 12c_n^4 d_n^2 + 36c_n^2 d_n^4 - 2(9c_n^4 d_n^2 + 12c_n^2 d_n^4 + 4d_n^6) \\
= \ & c_n^6 + 12c_n^4 d_n^2 + 36c_n^2 d_n^4 - 18c_n^4 d_n^2 - 24c_n^2 d_n^4 - 8d_n^6 \\
= \ & c_n^6 - 6c_n^4 d_n^2 + 12c_n^2 d_n^4 - 8d_n^6 \\
= \ & (c_n^2 - 2d_n^2)^3 \\
= \ & -1.
\end{aligned}$$

For the next two facts, we use the notation $q \equiv_m t$ to represent the statement $q \equiv t \pmod{m}$.

Fact 2. For all n, $c_n \equiv_3 1$ and $c_n \equiv_2 1$.

Proof: We proceed by induction noting, first, that $c_0 \equiv_3 1$ and $c_0 \equiv_2 1$. Then assuming that $c_n \equiv_3 1$ we have that

$$c_{n+1} = c_n^3 + 6c_n d_n^2 \equiv_3 1^3 + 0 = 1.$$

Also, assuming that $c_n \equiv_2 1$, we have

$$c_{n+1} = c_n^3 + 6c_n d_n^2 \equiv_2 1^3 + 0 = 1.$$

Fact 3. For all n, $d_n \equiv_2 1$.

Proof: Clearly $d_0 \equiv_2 1$. Assuming that $d_n \equiv_2 1$, we have

$$d_{n+1} = 3c_n^2 d_n + 2d_n^3 \equiv_2 3 \cdot 1^2 \cdot 1 + 0 = 3 \equiv_2 1.$$

Fact 4. For all n, $d_{2n} \equiv_3 2$.

Proof: Certainly $d_0 \equiv_3 2$. Assume that for n, $d_{2n} \equiv_3 2$. Then

$$d_{2n+1} = 3c_{2n}^2 d_{2n} + 2d_{2n}^3 \equiv_3 0 + 2 \cdot 2^3 \equiv_3 1,$$

so that

$$d_{2(n+1)} = d_{2n+2} = 3c_{2n+1}^2 d_{2n+1} + 2d_{2n+1}^3 \equiv_3 0 + 2 \cdot 1^3 \equiv_3 2.$$

Using the facts above, we show that there are infinitely many pairs (a, b) that satisfy $(2b+1)^2 - 2(6a-1)^2 = -1$. Fix an even number m. Then (c_m, d_m) satisfies $x^2 - 2y^2 = -1$. Since $c_m \equiv_2 1$ we have that $c_m - 1$ is even (and greater than 0) so that

$$b = \frac{c_m - 1}{2}$$

is an integer. Also, $d_m \equiv_3 2$ which tells us that $d_m + 1$ is divisible by 3, and since $d_m \equiv_2 1$, $d_m + 1$ is divisible by 2. Hence $d_m + 1$ is divisible by 6. Then

$$a = \frac{d_m + 1}{6}$$

is an integer. Thus, the pair (a, b) is a solution to $18a^2 - b^2 - 6a - b = 0$.

Solution 3 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Observe two such solutions (a,b) are given by $(1,3)$ and $(5,20)$. We claim that if (a_i,b_i) is a solution in positive integers, then so is (a_{i+1},b_{i+1}) where

$$\begin{aligned} a_{i+1} &= 577a_i+136b_i-28 \\ b_{i+1} &= 2448a_i+577b_i-120. \end{aligned}$$

To see this, note that (a_{i+1},b_{i+1}) are clearly positive integers and

$$\begin{aligned} 18a_{i+1}^2-b_{i+1}^2-6a_{i+1}-b_{i+1} &= 18(577a_i+136b_i-28)^2-(2448a_i+577b_i-120)^2 \\ &\quad -6(577a_i+136b_i-28)-(2448a_i+577b_i-120) \\ &= 18a_i^2-b_i^2-6a_i-b_i \\ &= 0. \end{aligned}$$

The solutions $(1,3)$ and $(5,20)$ are seeds which produce two infinite families of solutions. The first four solutions in each family is given below.

i	a_i	b_i	a_i	b_i
1	1	3	5	20
2	957	4059	5577	23660
3	1104185	4684659	6435661	27304196
4	1274228341	5406093003	7426747025	31509019100

Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The proposed equation may be written as follows:

$$\begin{aligned} 18a^2-b^2-6a-b &= 0 \\ 18\left(a-\frac{1}{6}\right)^2-\frac{1}{2}-\left(b+\frac{1}{2}\right)^2+\frac{1}{4} &= 0 \\ 18\left(a-\frac{1}{6}\right)^2-\left(b+\frac{1}{2}\right)^2 &= \frac{1}{4} \\ 72\left(a-\frac{1}{6}\right)^2-4\left(b+\frac{1}{2}\right)^2 &= 1 \\ (2b+1)^2-2\,(6a-1)^2 &= -1. \end{aligned}$$

The last equation is a Pell-type equation $x^2-2y^2=-1$, by doing $x=2b+1$ and $y=6a-1$. The smallest solution of $x^2-2y^2=-1$ is $(1,1)$ and therefore all its solutions are given by $x_i+y_i\sqrt{2}=\left(1+\sqrt{2}\right)^{2i+1}$. Note that x_i and y_i are allways odd so b is an integer. Also $6a=1+\sum_{k\ge0}\binom{2i+1}{2k+1}$. Since the expression $1+\sum_{k\ge0}\binom{2i+1}{2k+1}$ is even and multiple of 3 for i of the form $i=6m-1$, for m integer, the proposed equation has infinitely many positive integral solutions.

Solution 5 by David E. Manes, SUNY at Oneonta, NY

Solution. Writing the equation as a quadratic in b, one obtains $b^2+b-6a(3a-1)=0$ and, since we want positive integer solutions,

$$b=\frac{-1+\sqrt{1+72a^2-24a}}{2}.$$

Note that the above fraction is a positive integer provided that $72a^2 - 24a + 1 = c^2$ for some integer c. This last equation is equivalent to a negative Pell equation $c^2 - 2d^2 = -1$, where $d = 6a - 1$. This equation is solvable and the positive integer solutions are given by the odd powers of $1 + \sqrt{2}$. More precisely, if n is a positive integer and (c_n, d_n) is a solution of $c^2 - 2d^2 = -1$, then $c_n + d_n\sqrt{2} = (1 + \sqrt{2})^{2n-1}$. The problem is that not all the solutions for d_n yield solutions for a_n.

Observe: 1) if $n \equiv 0 \pmod 4$, then $c_n \equiv 5 \pmod 6$ and $d_n \equiv 1 \pmod 6$, 2) if $n \equiv 1 \pmod 4$, then $c_n \equiv d_n \equiv 1 \pmod 6$, 3) if $n \equiv 2 \pmod 4$, then $c_n \equiv 1 \pmod 6$ and $d_n \equiv 5 \pmod 6$, 4) if $n \equiv 3 \pmod 4$, then $c_n \equiv d_n \equiv 5 \pmod 6$.

The above observations provide straightforward inductive arguments for the following consequences. If $n \equiv 0$ or $1 \pmod 4$, then there are no solutions since $d_n \equiv 1 \pmod 6$ implies no integer solution for a_n. On the other hand, if $n \equiv 2$ or $3 \pmod 4$, then $a_n = \dfrac{d_n + 1}{6}$ is a positive integer and $b_n = (-1 + \sqrt{72a_n^2 - 24a_n + 1})/2$. Since there are infinitely many positive integers congruent to 2 or 3 modulo 4, the result follows.

Some of the infinitely many solutions are: if $n = 2$, then $c_2 = 7, d_2 = 5$ and $(a_2, b_2) = (1, 3)$; if $n = 3$, then $c_3 = 41, d_3 = 29$ and $(a_3, b_3) = (5, 20)$; if $n = 6$, then $c_6 = 8119, d_6 = 5741$ and $(a_6, b_6) = (957, 4059)$; if $n = 7$, then $c_7 = 47321, d_7 = 33461$ and $(a_7, b_7) = (5577, 23660)$; if $n = 10$, then $c_{10} = 9369319, d_{10} = 6625109$ and $(a_{10}, b_{10}) = (1104185, 4684659)$; if $n = 11$, then $c_{11} = 54608393, d_{11} = 38613965$ and $(a_{11}, b_{11}) = (6435661, 27304196)$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Anthony J. Bevelacqua, University of North Dakota, ND; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Kenneth Korbin, NY, NY; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5430: *Proposed by Oleh Faynshteyn, Leipzig, Germany*

Let a, b, c be the side-lengths, α, β, γ the angles, and R, r the radii respectively of the circumcircle and incircle of a triangle. Show that

$$\frac{a^3 \cdot \cos(\beta - \gamma) + b^3 \cdot \cos(\gamma - \alpha) + c^3 \cdot \cos(\alpha - \beta)}{(b + c)\cos\alpha + (c + a)\cos\beta + (a + b)\cos\gamma} = 6Rr.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

By the Law of Cosines,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

and hence,

$$(b + c)\cos\alpha = \frac{(b + c)\left(b^2 + c^2 - a^2\right)}{2bc} = \frac{a\,(b + c)\left(b^2 + c^2 - a^2\right)}{2abc}.$$

Similarly,

$$(c+a)\cos\beta = \frac{b\,(c+a)\left(c^2+a^2-b^2\right)}{2abc}$$

and

$$(a+b)\cos\gamma = \frac{c\,(a+b)\left(a^2+b^2-c^2\right)}{2abc}.$$

Therefore,

$$\begin{aligned}
&(b+c)\cos\alpha + (c+a)\cos\beta + (a+b)\cos\gamma \\
&= \frac{a\,(b+c)\left(b^2+c^2-a^2\right) + b\,(c+a)\left(c^2+a^2-b^2\right) + c\,(a+b)\left(a^2+b^2-c^2\right)}{2abc} \\
&= \frac{2a^2bc + 2ab^2c + 2abc^2}{2abc} \\
&= a+b+c. \qquad (1)
\end{aligned}$$

If K is the area of the given triangle, then

$$K = \frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta$$

and we have

$$\sin\alpha = \frac{2K}{bc}, \quad \sin\beta = \frac{2K}{ca}, \quad \text{and} \quad \sin\gamma = \frac{2K}{ab}.$$

Thus,

$$\begin{aligned}
a^3\cos(\beta-\gamma) &= a^3\left[\cos\beta\cos\gamma + \sin\beta\sin\gamma\right] \\
&= a^3\left[\frac{\left(c^2+a^2-b^2\right)}{2ca}\cdot\frac{\left(a^2+b^2-c^2\right)}{2ab} + \frac{4K^2}{(ca)(ab)}\right] \\
&= a\left[\frac{a^4-\left(b^2-c^2\right)^2+16K^2}{4bc}\right] \\
&= \frac{a^2}{4abc}\left[a^4-\left(b^2-c^2\right)^2+16K^2\right].
\end{aligned}$$

By Heron's Formula,

$$\begin{aligned}
16K^2 &= (a+b+c)(a+b-c)(b+c-a)(c+a-b) \\
&= \left[(a+b)^2-c^2\right]\left[c^2-(a-b)^2\right] \\
&= 2\left(a^2b^2+b^2c^2+c^2a^2\right) - \left(a^4+b^4+c^4\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
a^3\cos(\beta-\gamma) &= \frac{a^2}{4abc}\left[a^4-\left(b^2-c^2\right)^2+2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right] \\
&= \frac{a^2}{4abc}\left[-2b^4-2c^4+2\left(a^2b^2+2b^2c^2+c^2a^2\right)\right] \\
&= \frac{a^2}{2abc}\left(-b^4-c^4+a^2b^2+2b^2c^2+c^2a^2\right).
\end{aligned}$$

Similarly,

$$b^3\cos(\gamma-\alpha) = \frac{b^2}{2abc}\left(-c^4-a^4+a^2b^2+b^2c^2+2c^2a^2\right)$$

and

$$c^3\cos(\alpha-\beta)=\frac{c^2}{2abc}\left(-a^4-b^4+2a^2b^2+b^2c^2+c^2a^2\right).$$

As a result,

$$\begin{aligned}
&a^3\cos(\beta-\gamma)+b^3\cos(\gamma-\alpha)+c^3\cos(\alpha-\beta)\\
&=\frac{a^2}{2abc}\left(-b^4-c^4+a^2b^2+2b^2c^2+c^2a^2\right)+\frac{b^2}{2abc}\left(-c^4-a^4+a^2b^2+b^2c^2+2c^2a^2\right)\\
&+\frac{c^2}{2abc}\left(-a^4-b^4+2a^2b^2+b^2c^2+c^2a^2\right)\\
&=\frac{1}{2abc}\cdot 6a^2b^2c^2\\
&=3abc. \qquad (2)
\end{aligned}$$

By (1) and (2),

$$\frac{a^3\cos(\beta-\gamma)+b^3\cos(\gamma-\alpha)+c^3\cos(\alpha-\beta)}{(b+c)\cos\alpha+(c+a)\cos\beta+(a+b)\cos\gamma}=\frac{3abc}{a+b+c}. \qquad (3)$$

Finally, if $s=\dfrac{a+b+c}{2}$, then

$$R=\frac{abc}{4K}\quad\text{and}\quad K=rs$$

and we get

$$\begin{aligned}
6Rr&=6\left(\frac{abc}{4K}\right)\left(\frac{K}{s}\right)\\
&=\frac{3abc}{2s}\\
&=\frac{3abc}{a+b+c}. \qquad (4)
\end{aligned}$$

Conditions (3) and (4) yield the desired result.

Solution 2 by Moti Levy, Rehovot, Israel

After substituting $Rr=\frac{abc}{2(a+b+c)}$ in the right hand side of the original inequality, it becomes

$$\frac{\sum_{cyc}a^3\cos(\beta-\gamma)}{\sum_{cyc}(b+c)\cos\alpha}=\frac{3abc}{a+b+c}.$$

Thus, we actually need to prove two identities (which appeared many times before in the literature):

$$\sum_{cyc}(b+c)\cos\alpha=a+b+c, \qquad (1)$$

$$\sum_{cyc}a^3\cos(\beta-\gamma)=3abc. \qquad (2)$$

Dropping a perpendicular from C to side c, it divides the triangle into two right triangles, and c into two pieces $c = a\cos\beta + b\cos\alpha$, and similarly for all sides:

$$\begin{aligned} c &= a\cos\beta + b\cos\alpha, \\ a &= b\cos\gamma + c\cos\beta, \\ b &= c\cos\alpha + a\cos\gamma. \end{aligned}$$

To prove (1), we add the three equations, and get immediately:

$$a+b+c = a\cos\beta + b\cos\alpha + b\cos\gamma + c\cos\beta + c\cos\alpha + a\cos\gamma = \sum_{cyc}(b+c)\cos\alpha.$$

To prove (2), we use the following trigonometric identity

$$\cos(x-y) = \frac{\sin x\cos x + \sin y\cos y}{\sin(x+y)},$$

and the triangle identity

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

$$\begin{aligned} a^3\cos(\beta-\gamma) &= a^3\frac{\sin\beta\cos\beta + \sin\gamma\cos\gamma}{\sin(\beta+\gamma)} \\ &= a^3\frac{\sin\beta\cos\beta + \sin\gamma\cos\gamma}{\sin\alpha} \\ &= a^3\frac{b\cos\beta + c\cos\gamma}{a} = a^2b\cos\beta + a^2c\cos\gamma \end{aligned}$$

$$\begin{aligned} \sum_{cyc} a^3\cos(\beta-\gamma) &= \sum_{cyc}\left(a^2b\cos\beta + a^2c\cos\gamma\right) \\ &= ab(a\cos\beta + b\cos\alpha) + ac(c\cos\alpha + a\cos\gamma) + bc(b\cos\gamma + c\cos\beta) \\ &= 3abc. \end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Kevin Soto Palacios, Huarmey, Peru; Neculai Stanciu, "Geroge Emil Palade" School Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5431: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let F_n be the n^{th} Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all $n \geq 3$, $\mathrm{F}_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{n=1}^{\infty}\left(\frac{1}{11}\right)^{F_nF_{n+1}}$$

is an irrational number and determine it (*).

The asterisk $(*)$ indicates that neither the author of the problem nor the editor are aware of a closed form for the irrational number.

Solution 1 by Moti Levy, Rehovot, Israel

It is well known that

$$F_n F_{n+1} = \sum_{k=1}^{n} F_k^2, \qquad (1)$$

hence $x := \sum_{n=1}^{\infty} \left(\frac{1}{11}\right)^{F_n F_{n+1}}$ can be expressed as

$$x = \frac{1}{11^{F_1^2}} + \frac{1}{\left(11^{F_1^2}\right)\left(11^{F_2^2}\right)} + \frac{1}{\left(11^{F_1^2}\right)\left(11^{F_2^2}\right)\left(11^{F_3^2}\right)} + \cdots,$$

or

$$x = \sum_{k=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_k}, \qquad a_k = 11^{F_k^2}. \qquad (2)$$

The series (2) is the *Engel expansion* of the positive real number x. See [1] for definition of Engel expansion.

In 1913, Engel established the following result (See [2] page 303):
Every real number x has a unique representation $c + \sum_{k=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_k}$, where c is an integer and $2 \le a_1 \le a_2 \le a_3 \le \cdots$ is a sequence of integers. Conversely, every such sequence is convergent and its sum is irrational if and only if $\lim_{k\to\infty} a_k = \infty$.
Therefore, by Engel's result, $\sum_{n=1}^{\infty} \frac{1}{11^{F_n F_{n+1}}}$ is irrational, since $\lim_{k\to\infty} 11^{F_k^2} = \infty$.

I do not know how to express x in closed form. However, it can be shown that it is *transcendental*. To this end, I rely on a result given in [2] (on page 315):

Let $(f(n))_{n\ge1}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = \mu > 2$. *Then for every integer $d \ge 2$, the number $x = \sum_{n=1}^{\infty} \frac{1}{d^{f(n)}}$ is transcendental.*

In our case, $d = 11$ and $f(n) = F_n F_{n+1}$. We check that

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = \lim_{n\to\infty} \frac{F_{n+1}F_{n+2}}{F_n F_{n+1}} = \lim_{n\to\infty} \frac{F_{n+2}}{F_n} = \lim_{n\to\infty} \frac{F_{n+1}+F_n}{F_n}$$
$$= 1 + \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{3+\sqrt{5}}{2} \cong 2.618 > 2.$$

Then $x = \sum_{n=1}^{\infty} \frac{1}{11^{F_n F_{n+1}}}$ is transcendental.

References:

[**1**] Wikipedia "Engel expansion".

[**2**] Ribenboim Paulo, "*My Numbers, My Friends: Popular Lectures on Number Theory*", Springer 2000.

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

Let p be a prime. For the sake of brevity put $c_k = F_k F_{k+1}$. We prove that the number

$$s = \sum_{k=1}^{\infty} \left(\frac{1}{p}\right)^{c_k}$$

is transcendental, in particular irrational.

The partial sum

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{p}\right)^{c_k} = \frac{a_n}{b_n}$$

with positive integers a_n and $b_n \le p^{c_n}$ satisfies

$$\begin{aligned} 0 \quad < \quad s - s_n &= \sum_{k=n+1}^{\infty} \left(\frac{1}{p}\right)^{c_k} \le \left(\frac{1}{p}\right)^{c_{n+1}} \sum_{k=0}^{\infty} \left(\frac{1}{p}\right)^{k} \\ &= \frac{1}{p-1}\left(\frac{1}{p}\right)^{c_{n+1}-1} \le \frac{1}{(p^{c_n})^{\frac{c_{n+1}-1}{c_n}}}, \end{aligned}$$

because $c_{k+1} - c_k = F_{k+1}F_{k+2} - F_kF_{k+1} = F_{k+1}^2 \ge 1$. Since

$$\lim_{n\to\infty} \frac{c_{n+1}-1}{c_n} = \lim_{n\to\infty} \frac{F_{n+1}F_{n+2}-1}{F_nF_{n+1}} = \lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\cdot\frac{F_{n+2}}{F_{n+1}}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} > 2$$

By the theorem of Thue, Siegel and Roth, for any (fixed) algebraic number x and $\varepsilon > 0$, the inequality

$$0 < \left|x - \frac{a}{b}\right| < \frac{1}{b^{2+\varepsilon}}$$

is satisfied only by a finite number of integers a and b. Hence, s is transcendental.

Also solved by the Kee-Wai Lau, Hong Kong, China (first part of the problem), and the proposer, (first part of the problem)

5432: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : (0,\infty) \to (0,\infty)$, with $f(1) = \sqrt{2}$, such that

$$f'\left(\frac{1}{x}\right) = \frac{1}{f(x)},\ \forall x > 0.$$

Solution 1 by Arkady Alt, San Jose, CA

First note that $f'\left(\dfrac{1}{x}\right) = \dfrac{1}{f(x)},\ \forall x > 0 \iff f'(x) = \dfrac{1}{f\left(\dfrac{1}{x}\right)},\ \forall x > 0.$

Then, since $f''(x) = \left(\dfrac{1}{f\left(\dfrac{1}{x}\right)}\right)' = -\dfrac{f'\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)}{f^2\left(\dfrac{1}{x}\right)}$ and

$\dfrac{1}{f^2\left(\dfrac{1}{x}\right)} = (f'(x))^2,\ f'\left(\dfrac{1}{x}\right) = \dfrac{1}{f(x)},$

we obtain $f''(x) = \dfrac{1}{x^2}(f'(x))^2\dfrac{1}{f(x)} \iff \dfrac{f(x)f''(x)}{(f'(x))^2} = \dfrac{1}{x^2} \iff$

$\dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} - 1 = -\dfrac{1}{x^2} \iff$

$$\left(\frac{f(x)}{f'(x)}\right)' = 1 - \frac{1}{x^2} \iff \frac{f(x)}{f'(x)} = x + \frac{1}{x} + c \iff \frac{f'(x)}{f(x)} = \frac{x}{x^2+cx+1}.$$

Since $f'(1) = \frac{1}{f(1)} = \frac{1}{\sqrt{2}}$ then $\frac{f(1)}{f'(1)} = 1 + \frac{1}{1} + c \iff 2 = 2 + c \iff c = 0.$

Therefore, $\frac{f(x)}{f'(x)} = x + \frac{1}{x} \iff \frac{f'(x)}{f(x)} = \frac{x}{x^2+1} \iff \ln f(x) = \frac{1}{2}\ln\left(x^2+1\right) + d$ and, using $f(1) = \sqrt{2}$

again, we obtain $\ln f(1) = \frac{1}{2}\ln\left(1^2+1\right) + d \iff \ln\sqrt{2} = \frac{1}{2}\ln 2 + d \iff d = 0.$

Thus, $f(x) = \sqrt{x^2+1}$.

Solution 2 by Albert Stadler, Hirrliberg, Switzerland

The differential equation $f'(x) = \dfrac{1}{f\left(\dfrac{1}{x}\right)}$ shows that f is differentiable infinitely often in

$x > 0$. We differentiate the equation $f'(x)f\left(\dfrac{1}{x}\right) = 1$ and get

$f''(x)f\left(\dfrac{1}{x}\right) - f'(x)f'\left(\dfrac{1}{x}\right)\dfrac{1}{x^2} = \dfrac{f''(x)}{f'(x)} - \dfrac{f'(x)}{f(x)}\dfrac{1}{x^2} = 0,$

or equivalently

$$\frac{f''(x)f(x)}{(f'(x))^2} = \frac{1}{x^2}. \qquad (1)$$

By assumption $f(1) = \sqrt{2}$ and thus $f'(1) = \dfrac{1}{f(1)} = \dfrac{\sqrt{2}}{2}$.

We integrate (1) and apply partial integration to get

$$\begin{aligned}
1 - \frac{1}{x} = \int_1^x \frac{dt}{t^2} &= \int_1^x \frac{f''(t)f(t)}{(f'(t))^2}dt \\
&= \int_1^x \frac{d}{dt}\left(\frac{-1}{f'(t)}\right)f(t)dt \\
&= -\frac{f(t)}{f'(t)}\bigg|_1^x + \int_1^x \frac{f'(t)}{f'(t)}dt \\
&= \frac{f(1)}{f'(1)} - \frac{f(x)}{f'(x)} + x - 1 \\
&= 1 - \frac{f(x)}{f'(x)} + x.
\end{aligned}$$

So $\dfrac{f(x)}{f'(x)} = \dfrac{1}{x} + x$ or equivalently $\dfrac{f'(x)}{f(x)} = \dfrac{x}{1+x^2}$.

We integrate again and get

$$\ln f(x) - \ln f(1) = \int_1^x \frac{f'(t)}{f(t)} dt = \int_1^x \frac{t}{1+t^2} dt = \frac{1}{2}\ln(1+x^2) - = \frac{1}{2}\ln 2.$$

Therefore $f(x) = \sqrt{1+x^2}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $f:(0,+\infty)\to(0,+\infty)$ be a differentiable function that satisfies the hypothesis of the problem and let $g:(0,+\infty)\to(0,+\infty)$ be the differentiable function defined by $g(x)=\dfrac{1}{x}$. Since f is differentiable, and by the hypothesis $f'(x)=\dfrac{1}{(f\circ g)(x)}, \forall x>0$, we conclude that f' is also differentiable and, differentiating both side of the equality $f'(x)f\left(\dfrac{1}{x}\right)=1$, we obtain that $f''(x)f\left(\dfrac{1}{x}\right)+f'(x)f'\left(\dfrac{1}{x}\right)\dfrac{-1}{x^2}=0$, and since $f\left(\dfrac{1}{x}\right)=\dfrac{1}{x^2}$, or equivalently, $\dfrac{(f'(x))^2-f''(x)f(x)}{(f'(x))^2}=1-\dfrac{1}{x^2}$, or what is the same, $\left(\dfrac{f}{f'}\right)'(x)=1-\dfrac{1}{x^2},\ \forall x>0.$

Integrating both sides, we conclude that $\dfrac{f(x)}{f'(x)}=x+\dfrac{1}{x}+C,\ \forall x>0$, for some $C\in\Re$. If we take $x=1$ at the start of the inequality, and since $f(1)=\sqrt{2}$, we obtain that $f'(1)=\dfrac{1}{\sqrt{2}}$ and $\dfrac{f(1)}{f'(1)}=2+C$, from where $C=0$, which implies, because $f(x)>0\ \forall x>0$ by hypothesis and $\dfrac{f(x)}{f'(x)}=x+\dfrac{1}{x}+0$ and $\dfrac{f'(x)}{f(x)}=\dfrac{x}{x^2+1},\ \forall x>0.$ Integrating both sides of this last equality, we conclude that $\ln\left(f(x)\right)=\log\left(\sqrt{x^2+1}\right)+D,\ \forall x>0$ for some $D\in\Re$. Taking $x=1$ in this equality and using the fact that $f(1)=\sqrt{2}$, we find that $D=0$ and therefore $f(x)=\sqrt{x^2+1},\ \forall x>0.$

Since the function $f:(0,+\infty)\to(0,+\infty)$ defined by $f(x)=\sqrt{x^2+1}, \forall x>0$, is differentiable with $f'(x)=\dfrac{x}{\sqrt{x^2+1}}$ and satisfies that $f(1)=\sqrt{2}$, and that

$$f\left(\frac{1}{x}\right)=\frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}+1}}=\frac{1}{f(x)},\ \forall x>0,$$ we conclude that the only differentiable function

that satisfies the conditions of the problem is the function $f(x)=\sqrt{x^2+1},\ \forall x>0.$

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

We have $f'\left(\frac{1}{x}\right)f(x)=1$. Letting x to $\frac{1}{x}$ we also have $f'(x)f\left(\frac{1}{x}\right)=1\ \ (*)$. Thus,

$$\begin{aligned}\frac{d}{dx}\left(f(x)f\left(\frac{1}{x}\right)\right)&=f'(x)f\left(\frac{1}{x}\right)+\left(-x^{-2}\right)f(x)f'\left(\frac{1}{x}\right)\\&=1-x^{-2}.\end{aligned}$$

Integrating it, we have

$$f(x) f\left(\frac{1}{x}\right) = x + \frac{1}{x} + C$$

Letting $x = 1$, we have $2 = 2 + C$ or $C = 0$. Therefore $f(x) f\left(\frac{1}{x}\right) = x + \frac{1}{x}$. Multiplying $f(x)$ to $(*)$, we have

$$\left(x + \frac{1}{x}\right) f'(x) = f(x)$$
$$\frac{f'(x)}{f(x)} = \frac{1}{x + \frac{1}{x}}$$

Integrating again, we have

$$\begin{aligned}\log f(x) &= \int \frac{dx}{x + \frac{1}{x}} \\ &= \int \frac{x}{x^2 + 1} dx \\ &= \frac{1}{2} \int \frac{\left(x^2 + 1\right)'}{x^2 + 1} dx \\ &= \frac{1}{2} \log\left(x^2 + 1\right) + D\end{aligned}$$

Thus, we can write $f(x) = D\sqrt{x^2 + 1}$ where D is some constant. Letting $x = 1$, we have $D = 1$. Therefore, we have $f(x) = \sqrt{x^2 + 1}$, this function actually satisfies the condition.

Also solved by Abdallah El Farsi, Bechar, Algeria; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel; Moti Levy, Rehovot, Israel; Ravi Prakash, New Delhi, India, and the proposers.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
September 15, 2017

- **5451:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with sides $a = 8, b = 19$ and $c = 22$. The triangle has an interior point P where $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ each have positive integer length. Find $\overline{AP}$ and $\overline{BP}$, if $\overline{CP} = 4$.

- **5452:** *Proposed by Roger Izard, Dallas, TX*

  Let point O be the orthocenter of a given triangle ABC. In triangle ABC let the altitude from B intersect line segment AC at E, and the altitude from C intersect line segment AB at D. If AC and AB are unequal, derive a formula which gives the square of BC in terms of AC, AB, EO, and OD.

- **5453:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  If $a, b, c \in (0, 1)$ or $a, b, c \in (1, \infty)$ and m, n are positive real numbers, then prove that

  $$\frac{\log_a b + \log_b c}{m + n \log_a c} + \frac{\log_b c + \log_c a}{m + n \log_b a} + \frac{\log_c a + \log_a b}{m + n \log_c b} \geq \frac{6}{m+n}$$

- **5454:** *Proposed by Arkady Alt, San Jose, CA*

  Prove that for integers k and l, and for any $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, the following inequality holds:

  $$k^2 \tan\alpha + l^2 \tan\beta \geq \frac{2kl}{\sin(\alpha+\beta)} - \left(k^2 + l^2\right)\cot(\alpha+\beta).$$

- **5455:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Find all real solutions to the following system of equations:

  $$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{abc} \\ a + b + c &= abc + \frac{8}{27}(a+b+c)^3 \end{aligned}$$

- **5456:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let k be a positive integer. Calculate

$$\lim_{x\to\infty} \mathrm{e}^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left(\mathrm{e}^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right).$$

Solutions

- **5433:** *Proposed by Kenneth Korbin, New York, NY*

Solve the equation: $\sqrt[4]{x+x^2} = \sqrt[4]{x} + \sqrt[4]{x-x^2}$, with $x > 0$.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Let $f(x) = \sqrt[4]{x} + \sqrt[4]{x-x^2} - \sqrt[4]{x+x^2}$. Then $f(x)$ is continuous on $[0,1]$. We have $f(1/2) > 0$ and $f(1) < 0$. By the Intermediate Value Theorem our original equation has at least one solution with $x > 0$.

Now consider

$$\begin{aligned}
\sqrt[4]{x+x^2} = \sqrt[4]{x} + \sqrt[4]{x-x^2} \quad &\Longrightarrow \quad \sqrt[4]{1+x} = 1 + \sqrt[4]{1-x} \\
&\Longrightarrow \quad \sqrt[4]{1+x} - \sqrt[4]{1-x} = 1 \\
&\Longrightarrow \quad \sqrt{1+x} - 2\sqrt[4]{1-x^2} + \sqrt{1-x} = 1 \\
&\Longrightarrow \quad \sqrt{1+x} + \sqrt{1-x} = 1 + 2\sqrt[4]{1-x^2} \\
&\Longrightarrow \quad 1 + x + 2\sqrt{1-x^2} + 1 - x = 1 + 4\sqrt[4]{1-x^2} + 4\sqrt{1-x^2} \\
&\Longrightarrow \quad 1 - 2\sqrt{1-x^2} = 4\sqrt[4]{1-x^2} \\
&\Longrightarrow \quad 1 - 4\sqrt{1-x^2} + 4(1-x^2) = 16\sqrt{1-x^2} \\
&\Longrightarrow \quad 5 - 4x^2 = 20\sqrt{1-x^2} \\
&\Longrightarrow \quad 25 - 40x^2 + 16x^4 = 400(1-x^2) \\
&\Longrightarrow \quad 16x^4 + 360x^2 - 375 = 0
\end{aligned}$$

As a quadratic in x^2 the roots of this polynomial are

$$x^2 = \frac{-360 \pm 160\sqrt{6}}{32} = \frac{-45 \pm 20\sqrt{6}}{4}$$

and so

$$x = \pm\frac{\sqrt{-45 \pm 20\sqrt{6}}}{2}$$

This is a positive real number only if we choose both signs positive. Thus our original equation has at most one positive real solution.

Our last two paragraphs show that

$$x = \frac{\sqrt{20\sqrt{6} - 45}}{2}.$$

is the unique positive real solution to our original equation.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since $x > 0$, we lose no solutions if we divide by $\sqrt[4]{x}$ to obtain

$$\sqrt[4]{1+x} = 1 + \sqrt[4]{1-x}.$$

If we let $X = \sqrt[4]{1+x}$ and $Y = \sqrt[4]{1-x}$, then $X^4 + Y^4 = 2$ and we can solve for XY in the following steps:

$$\begin{aligned}
X - Y &= 1 \\
(X-Y)^4 &= 1 \\
X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4 &= 1 \\
X^4 + Y^4 - 2XY\left(2X^2 - 3XY + 2Y^2\right) &= 1 \\
-2XY\left[2(X-Y)^2 + XY\right] &= -1 \\
2XY(XY+2) &= 1 \\
2X^2Y^2 + 4XY - 1 &= 0 \\
XY &= \frac{-2 \pm \sqrt{6}}{2}.
\end{aligned}$$

The condition $XY = \sqrt[4]{1-x^2} \geq 0$ implies that

$$\begin{aligned}
\sqrt[4]{1-x^2} &= \frac{\sqrt{6}-2}{2} \\
1 - x^2 &= \left(\frac{\sqrt{6}-2}{2}\right)^4 = \frac{49 - 20\sqrt{6}}{4} \\
x^2 &= 1 - \frac{49 - 20\sqrt{6}}{4} = \frac{20\sqrt{6} - 45}{4}.
\end{aligned}$$

Because $x > 0$, our solution is

$$x = \frac{\sqrt{20\sqrt{6} - 45}}{2}.$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Solution. Since $x > 0$, we may divide the given equation by $\sqrt[4]{x}$ to produce

$$\sqrt[4]{1+x} = 1 + \sqrt[4]{1-x}.$$

Squaring both sides then yields $\sqrt{1+x} = 1 + 2\sqrt[4]{1-x} + \sqrt{1-x}$, or $\sqrt{1+x} - \sqrt{1-x} - 1 = 2\sqrt[4]{1-x}$. Squaring yet again produces

$$(1+x) + (1-x) + 1 - 2\sqrt{1+x} + 2\sqrt{1-x} - 2\sqrt{1-x^2} = 4\sqrt{1-x},$$

or $3 - 2\sqrt{1-x^2} = 2\sqrt{1+x} + 2\sqrt{1-x}$. We square once more to obtain

$$9 - 12\sqrt{1-x^2} + 4(1-x^2) = 4(1+x) + 4(1-x) + 8\sqrt{1-x^2}$$

and thus $5-4x^2 = 20\sqrt{1-x^2}$. Squaring for the last time yields $25-40x^2+16x^4 = 400(1-x^2)$ and hence $16x^4+360x^2-375=0$. Finally, the only real positive solution of this equation is

$$x = \sqrt{-\frac{45}{4}+5\sqrt{6}} = \frac{\sqrt{-45+20\sqrt{6}}}{2}.$$

Addendum. It is interesting to note that this solution is approximately 0.99872354, very close to 1. In particular, this implies that $49/4$ is a good rational approximation of $5\sqrt{6}$, which also means that $7/2$ is a good rational approximation of $\sqrt[4]{150}$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Aykut Ismailov, Shumen, Bulgaria; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Boris Rays, Brooklyn, NY; Brandon Richardson (student), Auburn University at Montgomery, AL; Toshihiro Shimizu, Kawasaki, Japan; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

- **5434:** *Proposed by Titu Zvonaru, Comnesti, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Calculate, without using a calculator or log tables, the number of digits in the base 10 expansion of 2^{96}.

Solution 1 by Ed Gray, Highland Beach, FL

$$\left(2^{12}\right)^8 = 2^{96} > (4\cdot 10^3)^8 = 4^8\cdot 10^{24} > 6\cdot 10^4\cdot 10^{24} = 6\cdot 10^{28}.$$

Also

$$\left(2^8\right)^{12} = 2^{96} < \left(3\cdot 10^2\right)^{12} = 3^{12}\cdot 10^{24} < \left(6\cdot 10^5\right)\cdot 10^{24} = 6\cdot 10^{29}.$$

Therefore, $6\cdot 10^{28} < 2^{96} < 6\cdot 10^{29}$. So $n=29$.

Solution 2 by Paul M. Harms, North Newton, KS

We see that

$$4(10^3) < 2^{12} = 4096 < 4.1(10^3).$$

Then

$$16(10^6) < 2^{24} < 16.81(10^6) < 17(10^6).$$

Taking the fourth power of the appropriate terms we obtain,

$$16^4(10^{24}) = 65536(10^{24}) = 0.65536(10^{29}) < 2^{96} < 17^4(10^{24}) = 83521(10^{24}) = 0.83521(10^{29}).$$

Since 2^{96} is bounded by integers who have 29 digits in the base 10 expansion, the integer 2^{96} must also have 29 digits in its base 10 expansion.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

The required number of digits is 29 because, as we shall show, $10^{28} \leq 2^{96} < 10^{29}$. More exactly, we shall prove that $1 < \dfrac{2^{96}}{10^{28}} < 10$. Since

$$\frac{2^{96}}{10^{28}} = \left(\frac{2^{24}}{10^7}\right)^4 = \left(\frac{\left(2^{12}\right)^2}{10^7}\right)^4 = \left(\frac{4096^2}{10^7}\right)^4 = \left(\frac{1,6777216 \cdot 10^7}{10^7}\right)^4 = (1,6777216)^4,$$

we obtain that

$1^4 < \dfrac{2^{96}}{10^{28}} < 1,68)^4$, that is $1 < \dfrac{2^{96}}{10^{28}} < (2.8224)^2$ and, hence, $1 < \dfrac{2^{96}}{10^{28}} < 3^2 < 10$.

Note: another way to show that $10^{28} < 2^{96}$ is, for example:

$$\left.\begin{matrix} 5^2 < 2^5 \\ 5 < 2^3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 5^2 < 2^5 \\ 5^3 < 2^9 \end{matrix}\right\} \Rightarrow 5^5 < 2^5 \cdot 5^3 < 2^{12} \Rightarrow \left.\begin{matrix} 5^5 < 2^{12} \\ 5^2 < 2^5 \end{matrix}\right\} \quad \Rightarrow 5^7 < 2^5 \cdot 5^5 < 2^{17} \Rightarrow$$

$$\Rightarrow 2^7 \cdot 5^7 < 2^{24} \Rightarrow$$

$$\Rightarrow \left(10^7\right)^4 < \left(2^{24}\right)^4 \Rightarrow$$

$$\Rightarrow 10^{28} < 2^{96}.$$

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

Since $10^3 < 2^{10} = 1024 < 1.03 \times 10^3$ and $2^{96} = \left(2^{10}\right)^9 \times 2^6 = \left(2^{10}\right)^9 \times 10 \times 6.4$ we have

$$6.4 \times 10 \times 10^{3\times 9} < 2^{96} < 6.4 \times 10 \times 10^{3\times 9} \times (1.03)^9.$$

We evaluate 1.03^9. We have $1.03 \times 1.03 \times 1.03 = 1.0609 \times 1.03 = 1.092727 < 1.1$ and $1.1 \times 1.1 \times 1.1 = 1.331 < 1.4$ (I never use calculator.) Therefore, we have

$$10^{28} < 6.4 \times 10^{28} < 2^{96} < 6.4 \times 1.4 \times 10^{28} = 8.96 \times 10^{28} < 10^{29}.$$

Therefore, the number of digits in 2^{96} is 29.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

- **5435:** *Proposed by Valcho Milchev, Petko Rachov Slaveikov Seconday School, Bulgaria*

Find all positive integers a and b for which $\dfrac{a^4 + 3a^2 + 1}{ab - 1}$ is a positive integer.

Solution 1 by Moti Levy, Rehovot, Israel

This solution is based on similar problem and solution which appeared in [1].

$\dfrac{a^4+3a^2+1}{ab-1}$ may be replaced by equivalent expression with *symmetric* polynomial in the numerator.

Indeed,

$$\frac{a^4+3a^2+1}{ab-1}=\frac{a^2\left(a^2+b^2+3\right)-(ab-1)\,(ab+1)}{ab-1}.$$

Now, a and $ab-1$ satisfy the equation $b*a+(-1)*(ab-1)=1$, which implies that a and $ab-1$ are relatively prime and clearly a^2 and $ab-1$ are also relatively prime.
Thus, $\dfrac{a^4+3a^2+1}{ab-1}$ is a positive integer if and only if $\dfrac{a^2+b^2+3}{ab-1}$ is a positive integer.
We call the ordered pair (a,b) a *solution* if

$$\frac{a^2+b^2+3}{ab-1}=m, \tag{1}$$

where m is a positive integer. The set of solutions is not empty since $(1,2)$ is a solution.
We exclude (a,a) from the set of solutions since $\dfrac{2a^2+3}{a^2-1}=2+\dfrac{5}{a^2-1}\notin N$ for all $a>0$.
Equation (1) is re-written as follows

$$a^2-mab+b^2=-\left(m+3\right). \tag{2}$$

It is easily verified (see (3)) that if (a,b) is a solution then $(ma-b,a)$ is a solution as well.

$$\left(ma-b\right)^2-m\left(ma-b\right)a+a^2=a^2-mab+b^2, \tag{3}$$

Let (a_0,b_0) be the “smallest” solution in the sense that $a_0+b_0\le a+b$, where (a,b) is any solution.

$$a_0+b_0\le\left(ma_0-b_0\right)+a_0,$$

or

$$\frac{2b_0}{a_0}\le m. \tag{4}$$

$$\frac{2b_0}{a_0}\le\frac{a_0^2+b_0^2+3}{a_0b_0-1}$$

$$0\le-2a_0b_0^2+2b_0+a_0^3+3a_0 \tag{5}$$

Let (a_0,a_0+k) be a solution. Then substituting in (5) gives,

$$\begin{aligned}0&\le-2a_0\left(a_0+k\right)^2+2\left(a_0+k\right)+a_0^3+3a_0\\&=-2k^2a_0-4ka_0^2+2k-a_0^3+5a_0.\end{aligned}$$

Solving $-2k^2a_0-4ka_0^2+2k-a_0^3+5a_0\ge0$, we get

$$\frac{1}{2a_0}\left(1-2a_0^2-\sqrt{6a_0^2+2a_0^4+1}\right)\le k\le\frac{1}{2a_0}\left(1-2a_0^2+\sqrt{6a_0^2+2a_0^4+1}\right),$$

hence, k will have positive values only if

$$\sqrt{6a_0^2+2a_0^4+1}+1\ge2a_0^2.$$

This inequality holds for $a_0=1$ and $a_0=2$. For $a_0=1$, possible values for k are 1 or 2; for $a_0=2$, possible value for k is 1.

Thus we have to check the following set of potential solutions: $\{(1,2),(1,3),(2,1)\}$. Clearly $(1,2)$ and $(2,1)$ are solutions, but $(1,3)$ is not.

For $(1,2)$ and $(2,1)$ the value of m is 8. We conclude that the sole value of m is 8.

It follows from (3) that the pairs (a_n, b_n) (and by symmetry (b_n, a_n)), which satisfy condition (1) are expressed by the recurrence formulas

$$\begin{aligned} a_{n+1} &= 8a_n - b_n, \\ b_{n+1} &= a_n, \end{aligned}$$

which are equivalent to the recurrence formulas

$$\begin{aligned} a_{n+2} &= 8a_{n+1} - a_n, \\ b_{n+2} &= 8b_{n+1} - b_n. \end{aligned} \tag{6}$$

We have two sets of initial conditions:

1) $a_0 = 1$, $a_1 = 6, b_0 = 2, b_1 = 1$; the pairs resulting from these initial conditions are $(1,2)$, $(6,1)$, $(47,6)$, $(370,47)$,. . . .

$$\begin{aligned} a_n &= \left(\frac{1}{2} - \frac{1}{\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(\frac{1}{2} + \frac{1}{\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n, \\ b_n &= \left(1 + \frac{7}{2\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(1 - \frac{7}{2\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n. \end{aligned}$$

2) $a_0 = 2$, $a_1 = 15, b_0 = 1, b_1 = 2$; the pairs resulting from these initial conditions are $(2,1)$, $(15,2)$, $(118,15)$, $(929,118)$, ...

$$\begin{aligned} a_n &= \left(1 - \frac{7}{2\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(1 + \frac{7}{2\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n, \\ b_n &= \left(\frac{1}{2} + \frac{1}{\sqrt{15}}\right)\left(4-\sqrt{15}\right)^n + \left(\frac{1}{2} - \frac{1}{\sqrt{15}}\right)\left(4+\sqrt{15}\right)^n. \end{aligned}$$

Reference:

[1] La Gaceta de la RSME, Vol. 18 (2015), No. 1, "*Solution to Problem 241, by Roberto de la Cruz Moreno*".

Solution 2 by Anthony Bevelacqua, University of North Dakota, Grand Forks, ND

1) There are no solutions to our problem with $a = b$. We have $a^4 + 3a^2 + 1 \equiv 5 \bmod (a^2 - 1)$. Assume there is a solution with $a = b$. Then $a^2 - 1$ divides $a^4 + 3a^2 + 1$ so $a^4 + 3a^2 + 1 \equiv 0 \bmod (a^2 - 1)$. Thus $5 \equiv 0 \bmod (a^2 - 1)$ and so $a^2 - 1$ divides 5. But then $a^2 = 2$ or $a^2 = 6$, a contradiction in either case.

2) The only solutions with $a \leq 4$ are $(a,b) = (1,2)$, $(2,1)$, $(1,6)$ and $(2,15)$. Suppose (a,b) is a solution to our problem. If $a = 1$ then $b - 1$ divides 5 so $b - 1 = 1$ or $b - 1 = 5$. Both $(1,2)$ and $(1,6)$ are solutions. If $a = 2$ then $2b - 1$ divides 29 so $2b - 1 = 1$ or $2b - 1 = 29$. Both $(2,1)$ and $(2,15)$ are solutions. If $a = 3$ then $3b - 1$ divides 109 so $3b - 1 = 1$ or $3b - 1 = 109$, a contradiction. If $a = 4$ then $4b - 1$ divides $305 = 5 \cdot 61$ so $4b - 1 \in \{1, 5, 61, 305\}$, a contradiction.

3) $ab-1$ divides a^4+3a^2+1 if and only if $ab-1$ divides a^2+b^2+3.
We have

$$\begin{aligned}(ab-1)(a^3b+3ab+a^2+3) &= a^4b^2+3a^2b^2+a^3b+3ab-a^3b-3ab-a^2-3\\ &= a^4b^2+3a^2b^2-a^2-3\end{aligned}$$

and so

$$b^2(a^4+3a^2+1)-(ab-1)(a^3b+3ab+a^2+3)=a^2+b^2+3.$$

Thus if $ab-1$ divides a^4+3a^2+1 then $ab-1$ divides a^2+b^2+3. Conversely suppose $ab-1$ divides a^2+b^2+3. Then $ab-1$ divides $b^2(a^4+3a^2+1)$. Since $ab-1$ and b^2 are relatively prime we have that $ab-1$ divides a^4+3a^2+1.

Now if $k>0$ and (a,b) is a solution to $a^2+b^2+3=k(ab-1)$ then b is a root of the polynomial $a^2+x^2+3=k(ax-1)$ which can be rewritten as $x^2-kax+(a^2+3+k)=0$. Thus if b' is the other root we have, by Vieta's formulas, $b+b'=ka$ and $bb'=a^2+3+k$. The first shows that b' is an integer and the second shows that $b'>0$. Thus (a,b') is another solution to $a^2+b^2+3=k(ab-1)$.

4) If $ab-1$ divides a^2+b^2+3 then $a^2+b^2+3=8(ab-1)$. Suppose there are positive integers a,b,k such that $a^2+b^2+3=k(ab-1)$. For this fixed k let S be the set of all positive integer pairs (a,b) such that $a^2+b^2+3=k(ab-1)$. Choose an $(a,b)\in S$ such that $a+b$ is minimal. Without loss of generality we have $a\le b$. Since $a\ne b$ by 1) we have $a<b$. Now (a,b') is another solution. Since $a+b$ is minimal we have $a+b\le a+b'$ and hence $b\le b'$. Thus

$$b^2\le bb'=a^2+3+k \implies k\ge b^2-a^2-3$$

and so

$$\begin{aligned}a^2+b^2+3 &= k(ab-1)\\ &\ge (b^2-a^2-3)(ab-1)\\ &= ab^3-b^2-a^3b+a^2-3ab+3.\end{aligned}$$

Hence

$$3ab+2b^2\ge ab^3-a^3b \implies 3a+2b\ge ab^2-a^3.$$

Since $a<b$ we have $3a+2b<5b$ and $ab^2-a^3=a(b+a)(b-a)>ab$. Thus $5b>ab$ and so $a<5$. By 2) the only possible (a,b) are then $(1,2)$, $(1,6)$, and $(2,15)$. Each of these gives $k=8$.

Thus 3) and 4) show that our original problem is equivalent to finding all positive integers a and b such that $a^2+b^2+3=8(ab-1)$. We could rewrite this as $(a-4b)^2-15b^2=-11$ and apply the theory of equations of the form $x^2-Dy^2=N$ as found in, say, section 58 of Nagell's *Number Theory*. Instead we will determine the solutions by "Vieta jumping" as in the proof of (4).

Let S be the set of all positive integers pairs (a,b) such that $a^2+b^2+3=8(ab-1)$. Clearly if $(a,b)\in S$ then $(b,a)\in S$, and, by 1) there are no $(a,b)\in S$ with $a=b$. Recall that if $(a,b)\in S$ then $(a,b')\in S$ where $b+b'=8a$ and $bb'=a^2+11$.

5) For any $(a,b)\in S$ define $\rho(a,b)=(b',a)$ and $\lambda(a,b)=(b,8b-a)$. Then $\rho(a,b)\in S$, $\lambda(a,b)\in S$, and $\lambda(\rho(a,b))=(a,b)$.

Let $(a,b) \in S$. We have $(a,b') \in S$ and hence $\rho(a,b) = (b',a) \in S$. Now

$$\begin{aligned} b^2 + (8b-a)^2 + 3 &= 64b^2 - 16ab + (a^2+b^2+3) \\ &= 64b^2 - 16ab + 8(ab-1) \\ &= 64b^2 - 8ab - 8 \\ &= 8(b(8b-a)-1) \end{aligned}$$

so $\lambda(a,b) = (b, 8b-a) \in S$. Finally,

$$\lambda(\rho(a,b)) = \lambda(b',a) = (a, 8a-b')$$

where

$$8a - b' = 8a - \frac{a^2+11}{b} = \frac{8ab - a^2 - 11}{b} = \frac{b^2}{b} = b.$$

6) The only $(a,b) \in S$ such that $a < b \leq 10$ are $(a,b) = (1,2)$ and $(1,6)$.

Since $a^2 + b^2 + 3 \equiv 0 \bmod 8$ we see that a and b must have opposite parity and neither can be divisible by 4. Moreover the only such solutions with a or b less than 4 are $(1,2)$ and $(1,6)$ by 2). This leaves only

$$(a,b) = (5,6), (6,7), (6,9), (5,10), (7,10), (9,10)$$

and none of these satisfy $a^2 + b^2 + 3 = 8(ab-1)$.

7) Let $(a,b) \in S$ such that $b \geq 11$. If $a < b$ then $b' < a$

Suppose first that $b' \leq 10$. Assume $a \leq b'$. Since $(a,b') \in S$ we have $a \neq b'$. Thus $a < b' \leq 10$. So, by 6), we must have $a = 1$. But if $a = 1$ we have $b = 1$ or $b = 6$, a contradiction with $b \geq 11$. Hence $b' < a$.

Suppose now that $b' \geq 11$. Again assume $a \leq b'$. Then, as in the last paragraph, $a < b'$. We have

$$bb' = a^2 + 11 < (b')^2 + 11 \implies b < b' + \frac{11}{b'} \leq b' + 1$$

and so $b \leq b'$. Now swapping b and b' we have

$$bb' = a^2 + 11 < b^2 + 11 \implies b' < b + \frac{11}{b} \leq b + 1$$

and so $b' \leq b$. Thus $b = b'$. Since $8a = b + b' = 2b$ we have $b = 4a$. But then

$$a^2 + 16a^2 + 3 = 8(4a^2 - 1) \implies 11 = 15a^2,$$

a contradiction. Hence $b' < a$.

Finally,

8) $(a,b) \in S$ if and only if $\{a,b\} = \{s_n, s_{n+1}\}$ or $\{a,b\} = \{t_n, t_{n+1}\}$ for $n \geq 0$ where

$$s_0 = 1,\ s_1 = 2,\ \text{and } s_n = 8s_{n-1} - s_{n-2} \text{ for } n \geq 2$$

and

$$t_0 = 1,\ t_1 = 6,\ \text{and } t_n = 8t_{n-1} - t_{n-2} \text{ for } n \geq 2.$$

Note that $\lambda^n(1,2) = (s_n, s_{n+1})$ and $\lambda^n(1,6) = (t_n, t_{n+1})$ for all $n \geq 0$.

Since $(1,2)$ and $(1,6) \in S$ we see that $(a,b) \in S$ for any $\{a,b\} = \{s_n, s_{n+1}\}$ or $\{a,b\} = \{t_n, t_{n+1}\}$ and $n \geq 0$ by (5).

Now suppose $(a,b) \in S$. Since $(b,a) \in S$ as well, we can suppose without loss of generality that $a < b$. By 5) and 7) there exists an integer $d \geq 0$ such that $\rho^d(a,b) = (a^*, b^*)$ with $a^* < b^* \leq 10$. By (6) we must have $\rho^d(a,b) = (1,2)$ or $\rho^d(a,b) = (1,6)$. Since $(a,b) = \lambda^d(\rho^d(a,b))$ we have $(a,b) = \lambda^d(1,2)$ or $(a,b) = \lambda^d(1,6)$.

Thus $ab-1$ divides a^4+3a^2+1 if and only if a and b are consecutive elements of either of the sequences s_n or t_n given above. Since the first few terms of s_n are $1, 2, 15, 118, 929, 7314, 57583, \ldots$ and the first few terms of t_n are $1, 6, 47, 370, 2913, 22934, 180559, \ldots$ the first few solutions to our problem (with $a \leq b$) are

$$(a,b) = (1,2), (2,15), (15,118), (118,929), (929,7314), (7314,57583), \ldots$$

and

$$(a,b) = (1,6), (6,47), (47,370), (370,2913), (2913,22934), (22934,180559), \ldots$$

Also solved by Ed Gray, Highland Beach, FL; Kenneth Korbin, NewYork, NY; Toshihiro Shimizu, Kawasaki, Japan; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

- **5436:** *Proposed by Arkady Alt, San Jose, CA*

Find all values of the parameter t for which the system of inequalities

$$\mathbf{A} = \begin{cases} \sqrt[4]{x+t} \geq 2y \\ \sqrt[4]{y+t} \geq 2z \\ \sqrt[4]{z+t} \geq 2x \end{cases}$$

a) has solutions;

b) has a unique solution.

Solution by the Proposer

a) Note that $(\mathbf{A}) \iff \begin{cases} t \geq 16y^4 - x \\ t \geq 16z^4 - y \\ t \geq 16x^4 - z \end{cases} \implies 3t \geq 16y^4 - x + 16z^4 - y + 16x^4 - z =$

$\left(16x^4 - x\right) + \left(16y^4 - y\right) + \left(16z^4 - z\right) \geq 3 \min\limits_x \left(16x^4 - x\right) \implies t \geq \min\limits_x \left(16x^4 - x\right).$

For $x \in \left(0, \dfrac{1}{16}\right)$, using the AM-GM Inequality, we obtain

$x - 16x^4 = x\left(1 - 16x^3\right) = \sqrt[3]{x^3\left(1-16x^3\right)^3} = \sqrt[3]{\dfrac{\left(48x^3\right)\left(1-16x^3\right)^3}{48}} \leq$

$\sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{48x^3 + 3 - 3 \cdot 16x^3}{4}\right)^4} = \sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{3}{4}\right)^4} = \dfrac{3}{16}$. And since $x - 16x^4 \leq 0$ for

$x \notin \left(0, \frac{1}{16}\right)$, then for all x the inequality $x - 16x^4 \le \frac{3}{16}$ holds. Since the upper bound is $\frac{3}{16}$ for values

$x - 16x^4$ is attainable when $x = \frac{1}{4}$, then $\max\left(x - 16x^4\right) = \frac{3}{16} \iff$

$\min\limits_x \left(16x^4 - x\right) = -\frac{3}{16}$.

Thus $t \ge -\frac{3}{16}$ is a necessary condition for the solvability of system **(A).**

Let's prove sufficiency.

Let $t \ge -\frac{3}{16}$. Since function $h(x)$ is continuous in R and $\min\limits_x \left(16x^4 - x\right) = -\frac{3}{16}$, then $\left[-\frac{3}{16}, \infty\right)$ is the range of $h(x)$. This means that for any $t \ge -\frac{3}{16}$ the equation

$16x^4 - x = t$

has solution in R and since for any u which is a solution of the equation $16x^4 - x = t$ the triple $(x, y, z) = (u, u, u,)$ is a solution of the system **(A)** then for such t system **(A)** solvable as well.

Remark.

Actually the latest reasoning about the solvability of system **(A)** if $t \ge -\frac{3}{16}$ is redundant for **(a)** because suffices to note that for such t the triple $(x, y, z) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ satisfies to **(A)**.

But for **(b)** criteria of solvability of equation $16x^4 - x = t$ in form of inequality

$t \ge -\frac{3}{16}$ is

important.

b) Note that system **(A)** always have more the one solution if $t > -\frac{3}{16}$.

Indeed, let for any $t_1, t_2 \in \left(-\frac{3}{16}, t\right)$ such that $t_1 \ne t_2$ equation $16u^4 - u = t_i$ has solution $u_i, i = 1, 2$.

Then $u_1 \ne u_2$ and two distinct triples $(u_1, u_1, u_1), (u_2, u_2, u_2)$ satisfy to the system **(A).**

Let $t = -\frac{3}{16}$.Then $-\frac{3}{16} \ge 16y^4 - x \implies -\frac{3}{16} + x - y \ge 16y^4 - y \ge -\frac{3}{16}$.

Hereof $x - y \ge 0 \iff x \ge y$. Similarly $-\frac{3}{16} \ge 16z^4 - y$ and $-\frac{3}{16} \ge 16x^4 - z$ implies $y \ge z$ and $z \ge x$, respectively. Thus in that case $x = y = z$ and all solutions of the system **(A)** are represented by solutions of one equation $16x^4 - x = -\frac{3}{16} \iff$

$16x^4 - x + \frac{3}{16} = 0 \iff 256x^4 - 16x + 3 = 0$ which has only root $\frac{1}{4}$ because

$256x^4 - 16x + 3 = (4x - 1)^2 \left(16x^2 + 8x + 3\right)$.

Thus, system **(A)** has unique solution iff $t = \frac{1}{4}$.

Also solved by Ed Gray, Highland Beach,FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David Stone and John Hawkins, Georgia

Southern University, Statesboro, GA, and Toshihiro Shimizu, Kawasaki, Japan.

- **5437:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f : C - \{2\} \to C$ be the function defined by $f(z) = \dfrac{2-3z}{z-2}$. If $f^n(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{n})(z)$, then compute $f^n(z)$ and $\lim\limits_{n\to+\infty} f^n(z)$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Assume first that $z \neq 2$ and $f^n(z)$ exists for all $n \geq 1$. Then, direct computation yields

$$f^2(z) = \frac{10-11z}{5z-6} \quad \text{and} \quad f^3(z) = \frac{42-43z}{21z-22}. \tag{1}$$

When these are combined with the formula for $f(z)$, it appears that there is a sequence $\{x_n\}$ of positive integers such that

$$f^n(z) = \frac{2x_n - (2x_n+1)z}{x_n z - (x_n+1)} \tag{2}$$

for all $n \geq 1$. Since $f(z) = \dfrac{2-3z}{z-2}$, we have $x_1 = 1$. Further, if (2) holds for some $n \geq 1$, then

$$\begin{aligned}
f^{n+1}(z) &= f\left(f^n(z)\right) \\
&= \frac{2-3f^n(z)}{f^n(z)-2} \\
&= \frac{2-3\left[\dfrac{2x_n-(2x_n+1)z}{x_nz-(x_n+1)}\right]}{\left[\dfrac{2x_n-(2x_n+1)z}{x_nz-(x_n+1)}\right]-2} \\
&= \frac{2\left[x_nz-(x_n+1)\right]-3\left[2x_n-(2x_n+1)z\right]}{\left[2x_n-(2x_n+1)z\right]-2\left[x_nz-(x_n+1)\right]} \\
&= \frac{(8x_n+2)-(8x_n+3)z}{(4x_n+1)z-(4x_n+2)}.
\end{aligned}$$

This suggests that $x_{n+1} = 4x_n + 1$ for $n \geq 1$. These conditions on $\{x_n\}$ are consistent with the formula for $f(z)$ and property (2). Note finally that

$$x_1 = 1 = \frac{3}{3} = \frac{4-1}{3}, \quad x_2 = 5 = \frac{15}{3} = \frac{4^2-1}{3}, \quad \text{and} \quad x_3 = 21 = \frac{63}{3} = \frac{4^3-1}{3}.$$

This leads us to conjecture that $x_n = \dfrac{4^n - 1}{3}$ and hence,

$$f^n(z) = \frac{2\left(\dfrac{4^n-1}{3}\right) - \left[2\left(\dfrac{4^n-1}{3}\right)+1\right]z}{\left(\dfrac{4^n-1}{3}\right)z - \left[\left(\dfrac{4^n-1}{3}\right)+1\right]} = \frac{2\left(4^n-1\right) - \left(2\cdot 4^n+1\right)z}{\left(4^n-1\right)z - \left(4^n+2\right)}$$

for all $n \geq 1$.

If $f^n(z)$ exists for all $n \geq 1$, let $P(n)$ be the statement

$$f^n(z) = \frac{2\left(4^n-1\right) - \left(2\cdot 4^n+1\right)z}{\left(4^n-1\right)z - \left(4^n+2\right)}. \tag{3}$$

If $n = 1$,

$$\begin{aligned}\frac{2(4-1)-(2\cdot 4+1)z}{(4-1)z-(4+2)} &= \frac{6-9z}{3z-6} \\ &= \frac{2-3z}{z-2}\end{aligned}$$

and thus, $P(1)$ is true. Assume that $P(n)$ is true, i.e.,

$$f^n(z) = \frac{2\left(4^n-1\right) - \left(2\cdot 4^n+1\right)z}{\left(4^n-1\right)z - \left(4^n+2\right)}$$

for some $n \geq 1$. Then,

$$\begin{aligned}f^{n+1}(z) &= f\left(f^n(z)\right) \\ &= \frac{2 - 3\left[\dfrac{2\left(4^n-1\right)-\left(2\cdot 4^n+1\right)z}{\left(4^n-1\right)z-\left(4^n+2\right)}\right]}{\left[\dfrac{2\left(4^n-1\right)-\left(2\cdot 4^n+1\right)z}{\left(4^n-1\right)z-\left(4^n+2\right)}\right]-2} \\ &= \frac{2\left[\left(4^n-1\right)z-\left(4^n+2\right)\right]-3\left[2\left(4^n-1\right)-\left(2\cdot 4^n+1\right)z\right]}{\left[2\left(4^n-1\right)-\left(2\cdot 4^n+1\right)z\right]-2\left[\left(4^n-1\right)z-\left(4^n+2\right)\right]} \\ &= \frac{\left[2\left(4^n-1\right)+3\left(2\cdot 4^n+1\right)\right]z-\left[2\left(4^n+2\right)+6\left(4^n-1\right)\right]}{\left[2\left(4^n-1\right)+2\left(4^n+2\right)\right]-\left[2\cdot 4^n+1+2\left(4^n-1\right)\right]z} \\ &= \frac{\left(2\cdot 4^{n+1}+1\right)z-2\left(4^{n+1}-1\right)}{\left(4^{n+1}+2\right)-\left(4^{n+1}-1\right)z} \\ &= \frac{2\left(4^{n+1}-1\right)-\left(2\cdot 4^{n+1}+1\right)z}{\left(4^{n+1}-1\right)z-\left(4^{n+1}+2\right)}\end{aligned}$$

and therefore, $P(n+1)$ is also true. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

Because formula (3) required the assumption that $f^n(z)$ exists for all $n \geq 1$, we need to determine if there are points $z \in C \setminus \{2\}$ for which there is a positive integer m such that

$f^n(z)$ does not exist for $n > m$. The existence of $f^n(z)$ requires that z, $f(z)$, …, $f^{n-1}(z) \neq 2$. Therefore, we have to find all points z for which $f^m(z) = 2$ for some $m \geq 1$. One way to do this is to consider the inverse function

$$f^{-1}(z) = \frac{2z+2}{z+3}$$

and describe

$$f^{-m}(z) = \Big(\underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{m}\Big)(z)$$

in a manner similar to that used to find formula (3). If we do so, we see that for $z \neq -3$,

$$f^{-m}(z) = \frac{(4^m+2)z + 2(4^m-1)}{(4^m-1)z + 2\cdot 4^m + 1}.$$

In particular,

$$f^{-m}(2) = \frac{(4^m+2)\cdot 2 + 2(4^m-1)}{(4^m-1)\cdot 2 + 2\cdot 4^m + 1} = \frac{4^{m+1}+2}{4^{m+1}-1}.$$

If $z_m = \dfrac{4^{m+1}+2}{4^{m+1}-1}$ for some $m \geq 1$, then it follows that $f^m(z_m) = 2$ and hence, $f^n(z_m)$ is undefined for $n > m$. Therefore, $\lim\limits_{n\to+\infty} f^n(z_m)$ does not exist for these points.

Let

$$S = \{2\} \cup \left\{\frac{4^{m+1}+2}{4^{m+1}-1} : m \in N\right\}.$$

For $z \notin S$, $f^n(z)$ exists for all $n \geq 1$. If $z = 1$, then $z \notin S$ and (3) implies that

$$\begin{aligned} f^n(1) &= \frac{2(4^n-1) - (2\cdot 4^n+1)}{(4^n-1)-(4^n+2)} \\ &= \frac{-3}{-3} \\ &= 1 \end{aligned}$$

for all $n \geq 1$. Hence, $\lim\limits_{n\to+\infty} f^n(1) = 1$. For all other values of $z \notin S$,

$$\begin{aligned} \lim_{n\to+\infty} f^n(z) &= \lim_{n\to+\infty} \frac{2(4^n-1) - (2\cdot 4^n+1)z}{(4^n-1)z - (4^n+2)} \\ &= \lim_{n\to+\infty} \frac{2(1-4^{-n}) - (2+4^{-n})z}{(1-4^{-n})z - (1+2\cdot 4^{-n})} \\ &= \frac{2-2z}{z-1} = -2. \end{aligned}$$

Therefore, for $z \notin S$,

$$\lim_{n\to+\infty} f^n(z) = \begin{cases} 1 & \text{if } z = 1 \\ -2 & \text{otherwise} \end{cases}$$

Solution 2 by Henry Ricardo, Westchester Math Circle, NY

We take advantage of the well-known homomorphism between 2×2 matrices and Möbius transformations: $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow f(z)=\dfrac{az+b}{cz+d}$. In this relation, the n-fold composition $f^n(z)$ corresponds to the nth power of A. Here we are dealing with powers of the matrix $A=\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$.

Now we invoke a known result that is a consequence of the Cayley-Hamilton theorem: If $A\in M_2(C)$ and the eigenvalues λ_1,λ_2 of A are not equal, then for all $n\geq 1$ we have

$$A^n=\lambda_1^n B+\lambda_2^n C, \text{ where } B=\frac{1}{\lambda_1-\lambda_2}\left(A-\lambda_2 I_2\right) \text{ and } C=\frac{1}{\lambda_2-\lambda_1}\left(A-\lambda_1 I_2\right). \ (*)$$

(See, for example, Theorem 2.25(a) in *Essential Linear Algebra with Applications* by T. Andreescu, Birkhäuser, 2014.)

The eigenvalues of the given matrix A are -1 and -4, so we apply $(*)$ to get

$$\begin{aligned} A^n &= \frac{(-1)^n}{3}\left(A+4I_2\right) - \frac{(-4)^n}{3}\left(A+I_2\right) \\ &= \left(\frac{(-1)^n-(-4)^n}{3}\right)A + \left(\frac{4\cdot(-1)^n-(-4)^n}{3}\right)I_2 \\ &= \begin{pmatrix} \frac{1}{3}(-1)^n(1+2\cdot 4^n) & \frac{2}{3}(-1)^n + \frac{2}{3}(-1)^{n+1}4^n \\ \frac{1}{3}(-1)^n+\frac{1}{3}(-1)^{n+1}4^n & \frac{1}{3}(-1)^n(2+4^n) \end{pmatrix}. \end{aligned}$$

After some simplification, we see that

$$f^n(z) = \frac{(2\cdot 4^n+1)z - 2(4^n-1)}{(1-4^n)z + (4^n+2)}.$$

Finally, we note that $f^n(1)=3/3=1$; and, for $z\neq 1$, we have

$$\lim_{n\to+\infty} f^n(z) = \lim_{n\to+\infty}\frac{(2\cdot 4^n+1)z - 2(4^n-1)}{(1-4^n)z + (4^n+2)} = \frac{2(z-1)}{1-z} = -2.$$

Therefore,

$$\lim_{n\to+\infty} f^n(z) = \begin{cases} 1 & \text{if } z=1, \\ -2 & \text{if } z\neq 1 \end{cases}.$$

Solution 3 by David E. Manes, Oneonta, NY

We will show by induction that

$$f^{(n)}(z)=\frac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}$$

where $a_n=\dfrac{4^n-1}{3}$. If $n=1$, then $a_1=1$ and $f^{(1)}(z)=\dfrac{(2-3z)}{(z-2)}=f(z)$. Therefore, the result is true for $n=1$. Assume the positive integer $n\geq 1$ and the given formula is valid for $f^{(n)}(z)$. Then

$$
\begin{aligned}
f^{(n+1)}(z) &= f(f^{(n)}(z)) = \frac{2-3\left(\dfrac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}\right)}{\left(\dfrac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}\right)-2} = \frac{2z-2\left(\dfrac{a_n+1}{a_n}\right)-6+3\left(\dfrac{2a_n+1}{a_n}\right)z}{2-\dfrac{2a_n+1}{a_n}z-2z+2\left(\dfrac{a_n+1}{a_n}\right)} \\
&= \frac{2a_nz-2a_n-2-6a_n+6a_nz+3z}{2a_n-2a_nz-z-2a_nz+2a_n+2} = \frac{-2-8a_n+(8a_n+3)z}{-(4a_n+1)z+(4n+2)} \\
&= \frac{2+8a_n-(8a_n+3)z}{(4a_n+1)z-(4n+2)} = \frac{2+8\left(\dfrac{4^n-1}{3}\right)-\left(8\left(\dfrac{4^n-1}{3}\right)+3\right)z}{\left(4\left(\dfrac{4^n-1}{3}\right)+1\right)z-\left(4\left(\dfrac{4^n-1}{3}\right)+2\right)} \\
&= \frac{(-2+2\cdot 4^{n+1})-(1+2\cdot 4^{n+1})z}{(4^{n+1}-1)z-(4^{n+1}+2)} \\
&= \frac{2-\left(\dfrac{2\cdot 4^{n+1}+1}{4^{n+1}-1}\right)z}{z-\left(\dfrac{4^{n+1}+2}{4^{n+1}-1}\right)} = \frac{2-\left(\dfrac{\frac{2\cdot 4^{n+1}+1}{3}}{\frac{4^{n+1}-1}{3}}\right)z}{z-\left(\dfrac{\frac{4^{n+1}+2}{3}}{\frac{4^{n+1}-1}{3}}\right)} \\
&= \frac{2-\left(\dfrac{2a_{n+1}+1}{a_{n+1}}\right)z}{z-\left(\dfrac{a_{n+1}+1}{a_{n+1}}\right)}
\end{aligned}
$$

where $a_{n+1} = \dfrac{(4^{n+1}-1)}{3}$. Note that $\dfrac{4^{n+1}+2}{3} = \dfrac{4^{n+1}-1}{3}+1 = a_{n+1}+1$ and

$$
\frac{2\cdot 4^{n+1}+1}{3} = \frac{2\cdot 4^{n+1}-2}{3}+1 = 2\left(\frac{4^{n+1}-1}{3}\right)+1 = 2a_{n+1}+1.
$$

Hence, the result is true for the integer $n+1$ so that by the principle of mathematical induction the result is valid for all positive integers n.

For the limit question, note that if $f(z)=z$, then $z=1$ or $z=-2$. Therefore, one of the fixed points of f is $z=1$ so that $f^{(n)}(1)=1$ for each positive integer n and $\lim\limits_{n\to+\infty} f^{(n)}(1)=1$. Moreover, observe that

$$
\lim_{n\to+\infty}\frac{1}{a_n} = \lim_{n\to+\infty}\frac{3}{4^n-1} = 0.
$$

Therefore, if $z\neq 1$, then

$$\lim_{n\to+\infty} f^{(n)}(z) = \lim_{n\to+\infty}\left(\frac{2-\dfrac{2a_n+1}{a_n}z}{z-\dfrac{a_n+1}{a_n}}\right) = \frac{\left(2-\lim\limits_{n\to+\infty}\left(2+\dfrac{1}{a_n}\right)z\right)}{\left(z-\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{a_n}\right)\right)} = \frac{2-2z}{z-1} = -2.$$

Hence,

$$\lim_{n\to+\infty} f^{(n)}(z) = \begin{cases} 1, & \text{if z = 1,} \\ -2, & \text{if } z \neq 1. \end{cases}$$

Solution 4 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Recall the map $f(z) = \dfrac{az+b}{cz+d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives a group isomorphism between group of fractional linear transformations

$$\left\{ f : f(z) = \frac{az+b}{cz+d} \text{ where } a,b,c,d \in C \text{ and } ad-bc \neq 0 \right\}$$

under function composition and the group

$$GL(2,C) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in C \text{ and } ad-bc \neq 0 \right\}$$

under matrix multiplication.
To compute $f^n(z)$, let $M = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$. Using induction, we show

$$M^n = \frac{(-1)^n}{3}\begin{bmatrix} 2^{2n+1}+1 & -2^{2n+1}+2 \\ -4^n+1 & 4^n+2 \end{bmatrix}.$$

Observe $M^1 = \dfrac{-1}{3}\begin{bmatrix} 2^3+1 & -2^3+2 \\ -3 & 6 \end{bmatrix} = \dfrac{-1}{3}\begin{bmatrix} 9 & -6 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$.
Assume

$$M^n = \frac{(-1)^n}{3}\begin{bmatrix} 2^{2n+1}+1 & -2^{2n+1}+2 \\ -4^n+1 & 4^n+2 \end{bmatrix}$$

and observe

$$\begin{aligned} M^{n+1} &= M^n M \\ &= \frac{(-1)^n}{3}\begin{bmatrix} 2^{2n+1}+1 & -2^{2n+1}+2 \\ -4^n+1 & 4^n+2 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \\ &= \frac{(-1)^n}{3}\begin{bmatrix} -3(2^{2n+1}+1)+(-2^{2n+1}+2) & 2(2^{2n+1}+1)-2(-2^{2n+1}+2) \\ -3(-4^n+1)+(4^n+2) & 2(-4^n+1)-2(4^n+2) \end{bmatrix} \\ &= \frac{(-1)^{n+1}}{3}\begin{bmatrix} 2^{2(n+1)+1}+1 & -2^{2(n+1)+1}+2 \\ -4^{n+1}+1 & 4^{n+1}+2 \end{bmatrix}. \end{aligned}$$

Using the aforementioned group isomorphism and simplifying, we conclude

$$f^n(z) = \frac{(2^{2n+1}+1)z-2^{2n+1}+2}{(-4^n+1)z+4^n+2} = \frac{(2\cdot 4^n+1)z+(2-2\cdot 4^n)}{(1-4^n)z+(2+4^n)}.$$

Notice that the map $f^n(z)$ is undefined for $z = \dfrac{4^k+2}{4^k-1}$ where $1 \le k \le n$. Consequently $\lim\limits_{n\to+\infty} f(z)$ does not exist for these values of z. Furthermore,

$$\begin{aligned}
\lim_{n\to+\infty} f^n(z) &= \lim_{n\to+\infty} \frac{(2\cdot 4^n+1)z+(2-2\cdot 4^n)}{(1-4^n)z+(2+4^n)} \\
&= \lim_{n\to+\infty} \frac{\left(2+\frac{1}{4^n}\right)z+\left(\frac{2}{4^n}-2\right)}{\left(\dfrac{1}{4^n}-1\right)z+\left(\dfrac{2}{4^n}+1\right)} \\
&= \frac{2z-2}{-z+1} \\
&= -2\left(\frac{1-z}{1-z}\right).
\end{aligned}$$

Note $f(1)=1$ so $f^n(1)=1$ for all $n \ge 1$. It follows that

$$\lim_{n\to+\infty} f(z) = \begin{cases} \text{DNE} & \text{if } z = \dfrac{4^n+2}{4^n-1} \text{ where } n \in Z_{>0} \\ 1 & \text{if } z = 1 \\ -2 & \text{otherwise.} \end{cases}$$

$$(\text{DNE} \;=\; \text{does not exist})$$

Comment by Editor : **David Stone and John Hawkins of Georgia Southern University** stated the following in their solution: "The appearance of so many sums of powers of 4 prompts us to offer a candidate for the cutest representation of $f^{(n)}(z)$:

$$f^{(n)}(z) = \frac{(2\cdot 111\ldots\ldots 1_4+1)\,z - 2\cdot 111\ldots 1_4}{-111\ldots\ldots 1_4 z + (111\ldots 1_4+1)},$$

where each of the base 4 repunits has $n-1$ digits."

Solution 5 by Toshihiro Shimizu, Kawasaki, Japan

Let $f^n(z) = \dfrac{a_n z + b_n}{c_n z + d_n}$. Then, we have

$$\begin{aligned}
\frac{a_{n+1}z+b_{n+1}}{c_{n+1}z+d_{n+1}} &= f^{n+1}(z) \\
&= f^n\left(\frac{2-3z}{z-2}\right) \\
&= \frac{(b_n-3a_n)\,z+2\,(a_n-b_n)}{(d_n-3c_n)\,z+2\,(c_n-d_n)}
\end{aligned}$$

Therefore, we have $a_{n+1} = b_n - 3a_n$, $b_{n+1} = 2a_n - 2b_n$ and $c_{n+1} = d_n - 3c_n$, $d_{n+1} = 2c_n - 2d_n$. Since $f^0(z) = z$, $a_0 = 1, b_0 = c_0 = 0$ and $d_0 = 1$. Since $b_n = a_{n+1} + 3a_n$, we have

$$\begin{aligned}
a_{n+2} + 3a_{n+1} &= 2a_n - 2\,(a_{n+1}+3a_n) \\
a_{n+2} + 5a_{n+1} + 4a_n &= 0
\end{aligned}$$

and $a_1 = b_0 - 3a_0 = -3$. Thus, we have

$$\begin{aligned} a_n &= \frac{1}{3}(-1)^n + \frac{2}{3}(-4)^n \\ b_n &= a_{n+1} + 3a_n \\ &= \frac{1}{3}(-1)^{n+1} + \frac{2}{3}(-4)^{n+1} + (-1)^n + 2(-4)^n \\ &= \frac{2}{3}(-1)^n - \frac{2}{3}(-4)^n . \end{aligned}$$

Similarly, we have $c_{n+2} + 5c_{n+1} + 4c_n = 0$ and $c_1 = d_0 - 3c_0 = 1$. Thus, we have

$$\begin{aligned} c_n &= \frac{1}{3}(-1)^n - \frac{1}{3}(-4)^n \\ d_n &= c_{n+1} + 3c_n \\ &= \frac{2}{3}(-1)^n + \frac{1}{3}(-4)^n \end{aligned}$$

Therefore,

$$f^n(z) = \frac{((-1)^n + 2(-4)^n)z + (2(-1)^n - 2(-4)^n)}{((-1)^n - (-4)^n)z + (2(-1)^n + (-4)^n)}.$$

If $z \neq 1$, we have

$$\begin{aligned} f^n(z) &= \frac{\left(\left(\frac{1}{4}\right)^n + 2\right)z + \left(2\left(\frac{1}{4}\right)^n - 2\right)}{\left(\left(\frac{1}{4}\right)^n - 1\right)z + \left(2\left(\frac{1}{4}\right)^n + 1\right)} \\ &\to \frac{2z-2}{-z+1} \\ &= -2 \quad (n \to +\infty). \end{aligned}$$

If $z = 1$, the value of $f^n(z)$ is always 1 and its limit is also 1.

Solution 6 by Kee-Wai Lau, Hong Kong, China

It can easily be proved by induction that

$$f^n(z) = \frac{2(2^{2n}-1) - (2^{2n+1}+1)z}{(2^{2n}-1)z - 2(2^{2n-1}+1)},$$

whenever $z \notin S_n$, where $S_n = \{2\} \cup \left\{\dfrac{2(2^{2k-1}+1)}{2^{2k}-1} : k = 1, 2, 3, \cdots, n\right\}$.

Clearly, $\lim\limits_{n\to\infty} f^n(1) = 1$ and if $z \notin \mathbf{T}$, where $\mathbf{T} = \{1, 2\} \cup \left\{\dfrac{2(2^{2\mathrm{k}-1}+1}{2^{2\mathrm{k}}-1}, \mathrm{k} = 1, 2, 3 \cdots\right\}$, then $\lim\limits_{n\to\infty} f^n(z) = -2$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Brian D. Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News,VA; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy (two solutions), Rehovot, Israel; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5438: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 0$ be an integer and let $\alpha > 0$ be a real number. Prove that

$$\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} + \frac{z^{2k}}{(1-z^2)^\alpha} \geq \frac{x^k y^k}{(1-xy)^\alpha} + \frac{y^k z^k}{(1-yz)^\alpha} + \frac{x^k z^k}{(1-xz)^\alpha},$$

for $x, y, z \in (-1, 1)$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that by the Binomial theorem,

$$\frac{t^{2k}}{(1-t^2)^\alpha} = t^{2k} \sum_{j=0}^{\infty} \binom{-\alpha}{j} \left(-t^2\right)^j = \sum_{j=0}^{\infty} \binom{-\alpha}{j} t^{2k+2j}, \ -1 < t < 1,$$

where $(-1)^j \binom{-\alpha}{j} = \dfrac{\alpha(\alpha+1)\cdots(\alpha+j-1)}{j!} > 0$ for all indices $j \geq 0$.

Therefore, by the AM−GM inequality,

$$\begin{aligned}
\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} + \frac{z^{2k}}{(1-z^2)^\alpha} &= \frac{1}{2} \sum_{cycl} \left(\frac{x^{2k}}{(1-x^2)^\alpha} + \frac{y^{2k}}{(1-y^2)^\alpha} \right) \\
&= \frac{1}{2} \sum_{cycl} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} \left(x^{2k+2j} + y^{2k+2j} \right) \\
&\geq \sum_{cycl} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} |xy|^{k+y} \\
&\geq \sum_{cycl} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} (xy)^{k+y} \\
&= \sum_{cycl} \frac{(xy)^k}{(1-xy)^\alpha}, \quad \text{as claimed.}
\end{aligned}$$

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well known that for any real numbers a, b, c

$$a^2 + b^2 + c^2 \geq ab + bc + ca. \qquad (1)$$

We show that $a, b \in (-1, 1)$

$$\sqrt{(1-a^2)(1-b^2)} \leq 1 - ab. \qquad (2)$$

Suppose that to the contrary $\sqrt{(1-a^2)(1-b^2)} > 1 - ab$, by squaring both sides of the inequality, we get $1 - a^2 - b^2 + a^2 b^2 > 1 - 2ab + a^2 b^2$, which implies that

$-a^2-b^2+2ab=-(a-b)^2>0$, which is impossible, that is, (2) is proved. From (2), we can conclude that
$$\frac{1}{\sqrt{(1-a^2)(1-b^2)}}\geq\frac{1}{1-ab}. \qquad (3)$$
Now, using (1) and (3), we write

$$\frac{x^{2k}}{(1-x^2)^\alpha}+\frac{y^{2k}}{(1-y^2)^\alpha}+\frac{z^{2k}}{(1-z^2)^\alpha}$$
$$\geq \frac{x^ky^k}{((1-x^2)(1-y^2)^{\frac{\alpha}{2}}}+\frac{y^kz^k}{((1-y^2)(1-z^2))^{\frac{\alpha}{2}}}+\frac{z^kx^k}{((1-z^2)(1-x^2))^{\frac{\alpha}{2}}}$$
$$= \frac{x^ky^k}{\left(\sqrt{(1-x^2)(1-y^2)}\right)^\alpha}+\frac{y^kz^k}{\left(\sqrt{(1-y^2)(1-z^2)}\right)^\alpha}+\frac{z^kx^k}{\left(\sqrt{(1-z^2)(1-x^2)}\right)^\alpha}$$
$$\geq \frac{x^ky^k}{(1-xy)^\alpha}+\frac{y^kz^k}{(1-yz)^\alpha}+\frac{z^kx^k}{(1-zx)^\alpha}.$$

Solution 3 by Moti Levy, Rehovot, Israel

Since $\dfrac{|a|^k}{(1-|a|)^\alpha}\geq\dfrac{a^k}{(1-a)^\alpha}$, $a\in(-1,1)$ then
$$\frac{|x|^k|y|^k}{(1-|x||y|)^\alpha}+\frac{|y|^k|z|^k}{(1-|y||z|)^\alpha}+\frac{|z|^k|x|^k}{(1-|z||x|)^\alpha}\geq\frac{x^ky^k}{(1-xy)^\alpha}+\frac{y^kz^k}{(1-yz)^\alpha}+\frac{z^kx^k}{(1-zx)^\alpha}.$$

Therefore, we can assume that $x,y,z\in(0,1)$. Using the generalized binomial theorem,
$$\frac{1}{(1-u)^\alpha}=\sum_{n=0}^{\infty}\binom{n+\alpha-1}{n}u^n=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}u^n, \qquad |u|<1.$$
$$\frac{x^{2k}}{(1-x^2)^\alpha}=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}x^{2(n+k)}$$
$$\frac{x^ky^k}{(1-xy)^\alpha}=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}(xy)^{n+k}$$
By the inequality $a^2+b^2+c^2\geq ab+bc+ca, \quad a,b,c\geq 0$,
$$\left(x^{n+k}\right)^2+\left(y^{n+k}\right)^2+\left(z^{n+k}\right)^2\geq x^{n+k}y^{n+k}+y^{n+k}z^{n+k}+z^{n+k}k^{n+k}.$$

$$
\begin{aligned}
&\frac{x^{2k}}{(1-x^2)^\alpha}+\frac{y^{2k}}{(1-y^2)^\alpha}+\frac{z^{2k}}{(1-z^2)^\alpha}\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}x^{2(n+k)}+\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}y^{2(n+k)}+\ \sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}z^{2(n+k)}\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}\left(x^{2(n+k)}+y^{2(n+k)}+z^{2(n+k)}\right)\\
&\geq\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}\left(x^{n+k}y^{n+k}+y^{n+k}z^{n+k}+z^{n+k}k^{n+k}\right)\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}xy^{(n+k)}y^{(n+k)}+\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}y^{(n+k)}z^{(n+k)}+\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma(\alpha)}z^{(n+k)}k^{(n+k)}\\
&=\frac{x^ky^k}{(1-xy)^\alpha}+\frac{y^kz^k}{(1-yz)^\alpha}+\frac{z^kx^k}{(1-zx)^\alpha}.
\end{aligned}
$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We first note that

$$0<(1-x^2)(1-y^2)=(1-xy)^2-(x-y)^2\leq(1-x)^2.$$

Hence by the AM-GM inequality, we have

$$\frac{x^{2k}}{(1-x^2)^\alpha}+\frac{y^{2k}}{(1-y^2)^\alpha}\geq\frac{2|x^ky^k|}{\sqrt{(1-x^2)^\alpha(1-y^2)^\alpha}}\geq\frac{2|x^ky^k|}{(1-xy)^\alpha}.$$

Similarly,

$$
\begin{aligned}
\frac{y^{2k}}{(1-y^2)^\alpha}+\frac{z^{2k}}{(1-z^2)^\alpha}&\geq\frac{2|y^kz^k|}{(1-yz)^\alpha}\quad\text{and}\\
\frac{z^{2k}}{(1-z^2)^\alpha}+\frac{x^{2k}}{(1-x^2)^\alpha}&\geq\frac{2|z^kx^k|}{(1-zx)^\alpha}.
\end{aligned}
$$

Adding these inequalities, we easily deduce the inequality of the problem.

Also solved by Ed Gray, Highland Beach, FL; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

Mea Culpa

For a variety of reasons, mostly caused by sloppy bookkeeping, those listed below were not credited for having solved the following problems, but should have been.

5427: Paul M. Harms, North Newton, KS.

5428: Ed Gray, Highland Beach, FL;
David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

5429: Brian D. Beasley, Presbyterian College, Clinton, SC.

5431: Albert Stadler, Herrliberg, Switzerland.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
December 15, 2017

- **5457:** *Proposed by Kenneth Korbin, New York, NY*

  Given angle A with $\sin A = \dfrac{12}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

- **5458:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

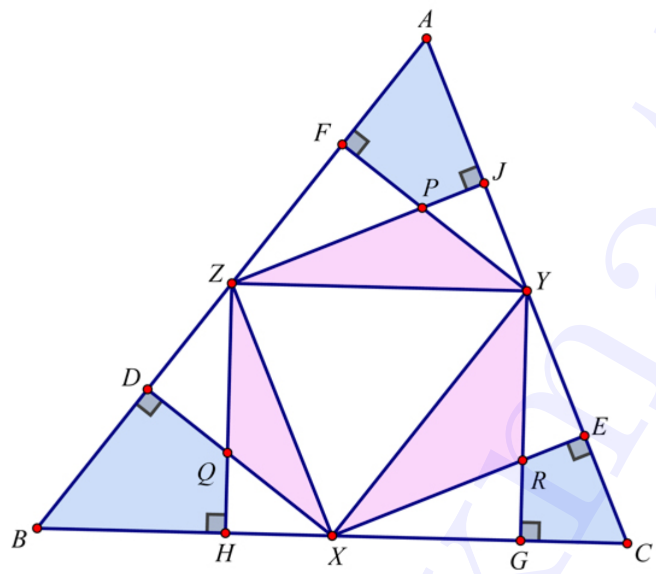
  Find two pairs of integers (a, b) from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that for all positive integers n, the number

  $$c = 537aaa\underbrace{b \ldots b}18403$$

  is composite, where there are $2n$ numbers b between a and 1 in the string above.

- **5459:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Triangle ABC is an arbitrary acute triangle. Points X, Y, and Z are midpoints of three sides of $\triangle ABC$. Line segments XD and XE are perpendiculars drawn from point X to two of the sides of $\triangle ABC$. Line segments YF and YG are perpendiculars drawn from point Y to two of the sides of $\triangle ABC$. Line segments ZJ and ZH are perpendiculars drawn from point Z to two of the sides of $\triangle ABC$. Moreover,
  $P = ZJ \cap FY$, $Q = ZH \cap DX$, and $R = YG \cap XE$. Three of the triangles, and three of the quadrilaterals in the figure are shaded. If the sum of the areas of the three shaded triangles is 5, find the sum of the areas of the three shaded quadrilaterals.



- **5460:** *Proposed by Ángel Plaza,Universidad de Las Palmas de Gran Canaria, Spain*

  If $a, b > 0$ and $x, y > 0$ then prove that

$$\frac{a^3}{ax+by} + \frac{b^3}{bx+ay} \geq \frac{a^2+b^2}{x+y}.$$

- **5461:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Compute the following sum:

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^2}.$$

- **5462:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n \geq 1$ be an integer. Calculate

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^n} \mathrm{d}x.$$

Solutions

- **5439:** *Proposed by Kenneth Korbin, New York, NY*

  Express the roots of the equation $\dfrac{(x+1)^4}{(x-1)^2} = 20x$ in closed form.
  "Closed form" means that the roots cannot be expressed in their approximate decimal equivalents.

  Solution 1 by David E. Manes, Oneonta, NY

The four roots of the equation are: $x = 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}$ and $x = 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}}$. One verifies that each of these values is a solution of the equation.
With $x \neq 1$, the equation is equivalent to $(x+1)^4 = 20x(x-1)^2$. After expanding, this equation reduces to the reciprocal equation $x^4 - 16x^3 + 46x^2 - 16x + 1 = 0$. To solve it, define the polynomial function $f(x)$ as follows and note that

$$\begin{aligned} f(x) = x^4 - 16x^3 + 46x^2 - 16x + 1 &= x^2\left(x^2 - 16x + 46 - \frac{16}{x} + \frac{1}{x^2}\right) \\ &= x^2\left[\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 46\right]. \end{aligned}$$

Let $z = x + \dfrac{1}{x}$. Then $\left(x + \dfrac{1}{x}\right)^2 = x^2 + \dfrac{1}{x^2} + 2$ implies $x^2 + \dfrac{1}{x^2} = z^2 - 2$. Therefore, $f(x) = x^2 \cdot g(z)$ where $g(z) = z^2 - 2 - 16z + 46 = z^2 - 16z + 44$. Then the roots of the reciprocal equation are the zeroes of $g(z)$ since $x = 0$ is not a solution of the equation. Using the quadratic formula, the roots of $z^2 - 16z + 44 = 0$ are $z = 8 \pm 2\sqrt{5}$. If $x + \dfrac{1}{x} = 8 + 2\sqrt{5}$, then $x^2 - (8 + 2\sqrt{5})x + 1 = 0$ with roots $x = 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}$. If $x + \dfrac{1}{x} = 8 - 2\sqrt{5}$, then $x^2 - (8 - 2\sqrt{5})x + 1 = 0$ with roots $x = 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}}$. This completes the solution.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

The equation

$$\frac{(x+1)^4}{(x-1)^2} = 20x$$

is equivalent to

$$x^4 + 4x^3 + 6x^2 + 4x + 1 = 20x^3 - 40x^2 + 20x,$$

or

$$x^4 - 16x^3 + 66x^2 - 16x + 1 = 20x^2.$$

Now,

$$x^4 - 16x^3 + 66x^2 - 16x + 1 = (x^2 - 8x + 1)^2,$$

so

$$(x^2 - 8x + 1)^2 - 20x^2 = \left[x^2 - (8 + 2\sqrt{5})x + 1\right]\left[x^2 - (8 - 2\sqrt{5})x + 1\right] = 0.$$

Thus, by the quadratic formula,

$$\begin{aligned} x &= \frac{8 + 2\sqrt{5} \pm \sqrt{(8 + 2\sqrt{5})^2 - 4}}{2} = \frac{8 + 2\sqrt{5} \pm \sqrt{80 + 32\sqrt{5}}}{2} \\ &= 4 + \sqrt{5} \pm 2\sqrt{5 + 2\sqrt{5}}, \end{aligned}$$

or

$$\begin{aligned} x &= \frac{8 - 2\sqrt{5} \pm \sqrt{(8 - 2\sqrt{5})^2 - 4}}{2} = \frac{8 - 2\sqrt{5} \pm \sqrt{80 - 32\sqrt{5}}}{2} \\ &= 4 - \sqrt{5} \pm 2\sqrt{5 - 2\sqrt{5}}. \end{aligned}$$

Solution 3 by Anna V. Tomova, Varna, Bulgaria

The possible values of the variable are those for which $x \neq 1$. The following equations are equivalent:

$$\frac{(x+1)^4}{(x-1)^2} = 20x \iff 20x(x+1)^4 - 20x(x-1)^2 = 0 \iff x^4 - 16x^3 + 46x^2 - 16x + 1 = 0.$$

Now we are looking for a representation of the left hand side of the equation as a product:

$$x^4-16x^3+46x^2-16x+1 = \left(x^2+ax+1\right)\left(x^2+bx+1\right) = x^4+(a+b)x^3+(2+ab)x^2+(a+b)x+1.$$

Therefore we have to solve the system $\begin{cases} a+b=-16 \\ ab+2=46 \end{cases} \iff \begin{cases} a+b=-16 \\ ab=44 \end{cases}$ or to solve the quadratic equation
$X^2+16X+44=0 \iff X_{1,2} = -8 \pm \sqrt{64-44} = -8 \pm 2\sqrt{5}$. Then we have:

$$x^4 - 16x^3 + 46x^2 - 16x + 1 = \left(x^2 + \left(2\sqrt{5}-8\right)x + 1\right)\left(x^2 - \left(2\sqrt{5}+8\right)x + 1\right) = 0.$$

the solutions to the problem are then:

$$x^2 + \left(2\sqrt{5}-8\right)x + 1 = 0 \iff x_{1,2} = 4 - \sqrt{5} \pm 2\sqrt{5-2\sqrt{5}};$$

$$x^2 - \left(2\sqrt{5}+8\right)x + 1 = 0 \iff x_{3,4} = 4 + \sqrt{5} \pm 2\sqrt{5+2\sqrt{5}}.$$

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University in Statesboro** made the following remark in their solution: "It's surprising that the line $y = 20x$ actually intersects the rational function four times. The line $y = 10x$, for instance, would not do so. So an interesting questions would be: *for which values of m does the equation $\frac{(x+1)^4}{(x-1)^2} = mx$ have four solutions?*" **Kenneth Korbin,** author of the problem, answered it as follows: "Possible values other than 20 would be any number 16 or greater."

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Pat Costello, Eastern Kentucky University, Richmond, KY; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Zhi Hwee Goh, Singapore, Singapore; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Henry Ricardo, Westchester Area Math Circle, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Georgia Southern University in Statesboro, GA, and the proposer.

- **5440:** *Proposed by Roger Izard, Dallas, TX*

The vertices of rectangle ABCD are labeled in clockwise order, and point F lines on line segment AB. Prove that $AD + AC > DF + FC$.

Solution 1 by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu,"George Emil Palade" School Romania

We consider the ellipse with foci D and C which passes through the points A and B. Since the point F belongs to the segment AB, we know that F is inside the ellipse. Hence, $FD + FC < AD + AC$, and we are done.

Solution 2 by Kee-Wai Lau, Hong Kong,China

We first suppose that $AF \leq BF$. We produce CB to G such that $BG = BC$. It is easy to see that $AG = AC$ and $FG = FC$.

If $AF = BF$, then DFG is a straight line. By the triangle inequality, we have $AD + AC = AD + AG > DG = DF + FG = DF + FC$ as required.

If $AF < BF$, we produce DF to meet the line AG at H. Applying the triangle inequality to triangles DAH and FHG, we obtain respectively $AD + AH > DF + HF$ and $HF + HG > FG$. Adding up the lat two inequalities, we have $AD + AG > DF + FG$ or $AD + AC > DF + FC$.

Now suppose that $AF > BF$. We produce DA to I such that $DA = IA$. Similar to the case $AF < BF$, we obtain $BC + BD > CF + FD$. Since $AD = BC$ and $BD = AC$, so again $AD + AC > DF + FC$, and this completes the solution.

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University in Statesboro** corrected the inequality to read: $AD + AC \geq DF + FC$, because equality occurs if F is either end point of the segment AB. They presented three different solution paths to the problem. In one of them they used the notion of reflection. They reflected the rectangle across the segment AB to include $AD'C'B$ as an upper rectangle, and then they reflected FC to FC'. They then argued that in triangle DAC' it is clear that $AD + AC' \geq DF + FC' \geq DC'$ because $AC' = AC$ and $FC' = FC$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Paul M. Harms, North Newton, KS; Zhi Hwee Goh, Singapore, Singapore; Ed Gray, Highland Beach, FL; David A. Huckaby, Angelo State University, San Angelo, TX; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Sachit Misra, Delhi, India; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (three solutions), Georgia Southern University, Statesboro, GA, and the proposer.

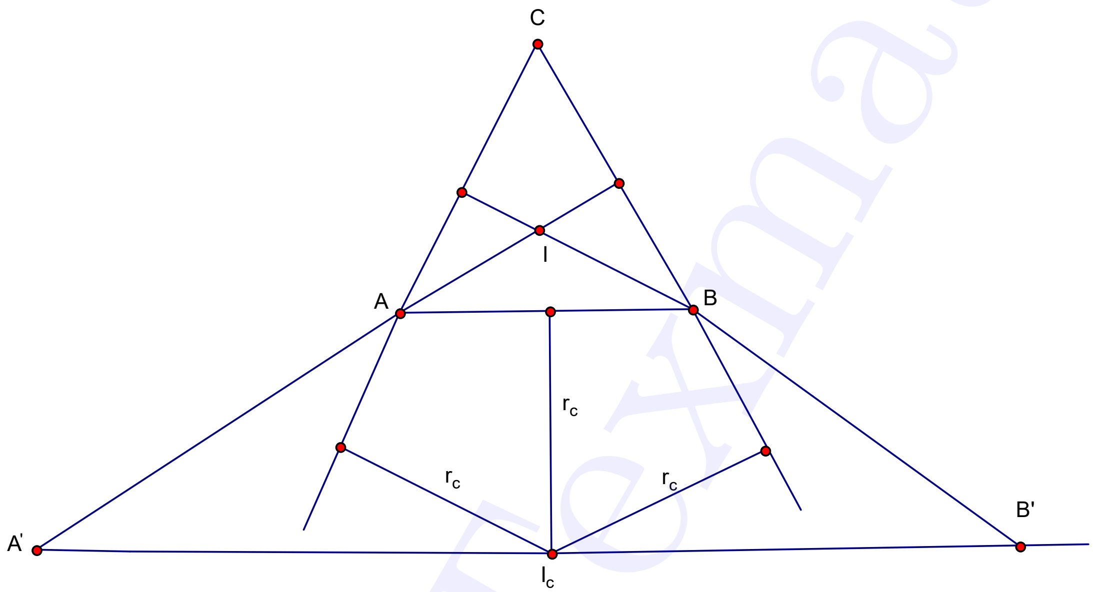
- **5441:** *Proposed by Larry G. Meyer, Fremont, OH*

In triangle ABC draw a line through the ex-center corresponding to side c so that it is parallel to side c. Extend the angle bisectors of A and B to meet the constructed lines at points A' and B' respectively. Find the length of $\overline{A'B'}$ if given either

(1) Angles A, B, C and the circumradius R
(2) Sides a, b, c

(3) The semiperimeter s, the inradius r and the exradius r_c
(4) Semiperimeter s and side c.

Solution 1 by Arkady Alt, San Jose, CA



Since $AB \parallel A'B'$ then $\triangle A'IB' \sim \triangle AIB$ and $r + r_c$ is length of height of the triangle $A'IB'$ from I to $A'B'$.

Hence, $\dfrac{A'B'}{c} = \dfrac{r + r_c}{r} \iff A'B' = \dfrac{c\,(r + r_c)}{r}$ and since

$[ABC] = rs = r_c\,(s - c) \implies \dfrac{r_c}{r} = \dfrac{s}{s - c}$ then

$A'B' = c\left(1 + \dfrac{r_c}{r}\right) = c\left(1 + \dfrac{s}{s - c}\right) = \dfrac{c\,(2s - c)}{s - c} = \dfrac{c\,(a + b)}{s - c} = \dfrac{2c\,(a + b)}{a + b - c} =$

$\dfrac{8R^2 \sin C\,(\sin A + \sin B)}{2R\,(\sin A + \sin B - \sin C)} = \dfrac{4R \sin C\,(\sin A + \sin B)}{\sin A + \sin B - \sin C}$.

Also, since $rs = r_c\,(s - c) \iff c = \dfrac{(r_c - r)\,s}{r_c}$ we obtain

$A'B' = c\left(1 + \dfrac{r_c}{r}\right) = \dfrac{(r_c - r)\,s}{r_c} \cdot \dfrac{r + r_c}{r} = \dfrac{\left(r_c^2 - r^2\right)s}{r r_c}$.

So, $A'B' = \dfrac{4R \sin C\,(\sin A + \sin B)}{\sin A + \sin B - \sin C} = \dfrac{2c\,(a + b)}{a + b - c} = \dfrac{\left(r_c^2 - r^2\right)s}{r r_c} = \dfrac{c\,(2s - c)}{s - c}$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Let the incenter of triangle ABC be I and the ex-center corresponding to side c be E_c so that CIE_c is a straight line cutting AB at D, say. Let the feet of the perpendiculars from I to AB and from E_c to AB be X and Y respectively. It is easy to see that triangle IAB and $I'A'B'$, triangles IAD and $IA'E_c$n triangles IDX and E_cDY are pairwise similar. Hence

$$A'B' = AB\frac{IA'}{IA} = AB\frac{IE_c}{ID} = AB\frac{ID + E_cD}{ID} = AB\left(1 + \frac{E_cY}{IX}\right) = AB\left(1 + \frac{r_c}{r}\right).$$

It is well known that $c = 2R\sin C, r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ and $r_c = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$. Hence the answer to part (1) is

$$A'B' = 2R\sin C\left(1 + \cot\frac{A}{2}\cot\frac{B}{2}\right).$$

It is also well-known that $r = \dfrac{[ABC]}{s}$ and $r_c = \dfrac{[ABC]}{s-c}$ were $[ABC]$ equals the area of triangle ABC. Hence the answer to part (2) is

$$A'B' = \frac{2c(a+b)}{a+b-c}.$$

and the answer to part (4) is

$$A'B' = \frac{c(2s-c)}{s-c}.$$

From $r = \dfrac{[ABC]}{s}$ and $r_c = \dfrac{ABC}{s-c}$, we obtain that $c = s\left(1 - \dfrac{r}{r_c}\right)$. The answer to part (3) is then

$$A'B' = \frac{s\left(r_c^2 - r^2\right)}{rr_c}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Toshihiro Shimizu, Kawasaki, Japan; Zhi Hwee Goh, Singapore, Singapore; Neculai Stanciu, "George Emil Palade" School, Buzău Romania and Titu Zvonaru, Comănesti, Romania; and the proposer.

- **5442:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let L_n be the n^{th} Lucas number defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Prove that for all $n \geq 0$,

$$\frac{1}{2}\begin{vmatrix} (L_n + 2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n + L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{vmatrix}$$

is the cube of a positive integer and determine its value.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The problem may be generalized as follows. For a, b positive numbers, then

$$\frac{1}{2}\begin{vmatrix} (a+2b)^2 & (a+b)^2 & b^2 \\ (a+b)^2 & (2a+b)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = \left(a^2 + 3ab + b^2\right)^3$$

which may be checked by a symbolic calculus package like Mathematica. So for the proposed expression in the problem we have

$$\frac{1}{2}\begin{vmatrix} (L_n + 2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n + L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{vmatrix} = \left(L_n^2 + 3L_nL_{n+1} + L_{n+1}^2\right)^3.$$

Solution 2 by Moti Levy, Rehovot, Israel

Let A denote our matrix,

$$A := \begin{bmatrix} (L_n + 2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n + L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{bmatrix}.$$

Using the identity related to Lucas and Fibonacci numbers,

$$L_{n+m} = L_{m+1}F_n + L_m F_{n-1},$$

the matrix A is expressed by Fibonacci numbers F_n and F_{n-1} only,

$$A = \begin{bmatrix} (7F_n + 4F_{n-1})^2 & (4F_n + 3F_{n-1})^2 & (3F_n + F_{n-1})^2 \\ (4F_n + 3F_{n-1})^2 & (5F_n + 5F_{n-1})^2 & (F_n + 2F_{n-1})^2 \\ (3F_n + F_{n-1})^2 & (F_n + 2F_{n-1})^2 & (4F_n + 3F_{n-1})^2 \end{bmatrix}.$$

From now on, our arguments do not relate to Fibonacci or Lucas numbers properties (any decent CAS, say Mathematica, can relieve us of tedious calculations). Let B be a symmetric matrix defined as follows:

$$B := \begin{bmatrix} (7x + 4y)^2 & (4x + 3y)^2 & (3x + y)^2 \\ (4x + 3y)^2 & (5x + 5y)^2 & (x + 2y)^2 \\ (3x + y)^2 & (x + 2y)^2 & (4x + 3y)^2 \end{bmatrix},$$

where x, y are real or complex numbers.
The determinant of B divided by 2 is

$$\frac{1}{2}\det B = \left(19x^2 + 31xy + 11y^2\right)^3.$$

It follows that the positive number we are seeking is $19F_n^2 + 31F_nF_{n-1} + 11F_{n-1}^2$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We replace L_{n+2} by $L_n + L_{n+1}$, expand the determinate and factor to get

$$\begin{aligned} &\frac{1}{2}\Big((L_n + 2L_{n+1})^2(2L_n + L_{n+1})^2(L_n + L_{n+1})^2 + 2(L_n + L_{n+1})^2 L_n^2 L_{n+1}^2 \\ &-(2L_n + L_{n+1})^2 L_{n+1}^4 - (L_n + 2L_{n+1})^2 L_n^4 - (L_n + L_{n+1})^6\Big) \\ = &\left(L_n^2 + 3L_nL_{n+1} + L_{n+1}^2\right)^3. \end{aligned}$$

L_n can be represented as

$L_n = f^n + (-f)^n$ with $f = \dfrac{1+\sqrt{5}}{2}$ (see: https://en.wikipedia.org/wiki/ Lucas_number).

Therefore

$$L_n^2 + 3L_nL_{n+1} + L_{n+1}^2 = f^{2n} + (-f)^{-2n} + 2(-1)^n$$

$$
\begin{aligned}
& +3f^{2n+1}+3(-f)^{-2n-1}+3(-1)^n f+3(-1)^{n+1}f^{-1} \\
& +f^{2n+2}+(-f)^{-2n-2}+2(-1)^{n+1} \\
= \;& nL_{2n}+3L_{2n+1}+L_{2n+2}+3(-1)^n\left(f-\frac{1}{f}\right) \\
= \;& \underbrace{L_{2n}+L_{2n+1}}_{=L_{2n+2}}+2L_{2n+1}+L_{2n+2}+3(-1)^n\left(f-\frac{1}{f}\right) \\
= \;& 2L_{2n+3}+3(-1)^n.
\end{aligned}
$$

So the given determinant can be represented as $(2L_{2n+3}+3(-1)^n)^3$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Zhi Hwee Goh, Singapore, Singapore; Kee-Wai Lau Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins (partial solution), Georgia Southern University, Statesboro, GA, and the proposer.

- **5443:** *Proposed by D.M. Băinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "Geroge Emil Palade" General School, Buzău, Romania*

  Compute $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x}dx.$

  Solution 1 by Ed Gray, Highland Beach, FL

  Letting $y = 6x - \dfrac{3\pi}{2}$ and changing the limits we see that:

$$
\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x}dx &= \frac{1}{36}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{y+3+\pi/2}{\cos\frac{y}{3}}dy \\
&= \frac{1}{36}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{y}{\cos\frac{y}{3}}dy + \frac{1}{36}\cdot\frac{3\pi}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos\left(\frac{y}{3}\right)}dy.
\end{aligned}
$$

  The first integral is zero because the integrand is odd, while the second integral (with the help of Wolfram-Alpha) is $\ln(27) = 3\ln(3)$. Therefore,
  $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x}dx = \frac{1}{36}\cdot\frac{3\pi}{2}\cdot 3\cdot\ln 3 = \frac{\pi\cdot\ln 3}{8}.$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $x = \arctan t$. The integral becomes

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\arctan t}{2\frac{t}{\sqrt{1+t^2}\sqrt{1+t^2}}} \frac{dt}{1+t^2} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan t}{t} dt = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\frac{\pi}{2}-\arctan\frac{1}{t}}{t} dt$$

Moreover,

$$-\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan\frac{1}{t}}{t} dt \underbrace{=}_{t=1/y} \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{2}\frac{\arctan y}{y} dy = -\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan y}{y} dy$$

It follows,

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2}\frac{\arctan t}{t} dt = \frac{\pi}{8}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{t} dt = \frac{\pi}{8}\left(\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right) = \frac{\pi}{8}\ln 3.$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Brian Bradie, Christopher Newport University, Newport News, VA; Pat Costello, Eastern Kentucky University, Richmond, KY; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Motti Levy, Rehovot, Israel; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland; Students at Taylor University {Group 1: Ellie Grace Moore, Samantha Korn, and Gwyn Terrett; Group 2: Luke Wilson, California Drage, Jonathan DeHaan}, Upland, IN; Anna V. Tomova, Varna, Bulgaria, and the proposer.

- **5444:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Solve in $\Re$ the equation $\{(x+1)^2\} = 2x^2$, where $\{a\}$ denotes the fractional part of a.

Solution 1 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Since $0 \le \{(x+1)^2\} < 1$, any solution x satisfies $0 \le 2x^2 < 1$ or equivalently $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Note that $\sqrt{2}-1 < \frac{1}{\sqrt{2}} < \sqrt{3}-1$ and observe

$$\{(x+1)^2\} = \begin{cases} (x+1)^2 & \text{if } -1 < x < 0 \\ (x+1)^2 - 1 & \text{if } 0 \le x < \sqrt{2}-1 \\ (x+1)^2 - 2 & \text{if } \sqrt{2}-1 \le x < \sqrt{3}-1. \end{cases}$$

We consider three cases.

For $-\frac{1}{\sqrt{2}} < x < 0$ the equation reduces to $(x+1)^2 = 2x^2$ or equivalent $x^2-2x-1=0$. Solving gives $x = 1 \pm \sqrt{2}$, however only $x = 1-\sqrt{2}$ lies in $\left(-\frac{1}{\sqrt{2}}, 0\right)$. Thus $x = 1-\sqrt{2}$ produces the only solution in this case.

For $0 \le x < \sqrt{2}-1$ the equation reduces to $(x+1)^2 - 1 = 2x^2$ or equivalent $x^2 - 2x = 0$. Solving gives $x = 0, 2$. We see only $x = 0$ lies in $\left[0, \sqrt{2}-1\right)$, so $x = 0$ produces the only solution in this case.

For $\sqrt{2}-1 \le x < \frac{1}{\sqrt{2}}$ the equation reduces to $(x+1)^2 - 2 = 2x^2$ or equivalent $x^2-2x+1=0$. Solving gives $x = 1$, however this does not lie in $\left[\sqrt{2}-1, \frac{1}{\sqrt{2}}\right)$. This case yields no solution.

In summary, there are two solutions, namely $x = 1 - \sqrt{2}$ and $x = 0$.

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Since $0 \le \{(x+1)^2\} < 1$ we must have $2x^2 < 1$ and so $|x| < 1/\sqrt{2}$. Hence

$$x^2 + 2x + 1 < \frac{1}{2} + \sqrt{2} + 1 < 3.$$

Thus $(x+1)^2 = x^2 + 2x + 1$ must be in $[0, 3)$.
If $x^2 + 2x + 1 \in [0, 1)$ then

$$2x^2 = \{x^2 + 2x + 1\} = x^2 + 2x + 1 \implies x^2 - 2x - 1 = 0$$

and so $x = 1 \pm \sqrt{2}$, but only $x = 1 - \sqrt{2}$ satisfies the original equation.
If $x^2 + 2x + 1 \in [1, 2)$ then

$$2x^2 = \{x^2 + 2x + 1\} = x^2 + 2x \implies x^2 - 2x = 0$$

and so $x = 0$ or $x = 2$, but only $x = 0$ satisfies the original equation.
Finally, if $x^2 + 2x + 1 \in [2, 3)$ then

$$2x^2 = \{x^2 + 2x + 1\} = x^2 + 2x - 1 \implies x^2 - 2x + 1 = 0$$

and so $x = 1$, but this does not satisfy the original equation.
Thus the only solutions are $x = 0$ and $x = 1 - \sqrt{2}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $\lfloor a \rfloor$ denote the integer part of a.

Since $\lfloor a \rfloor$ is the only integer such that $a - 1 < \lfloor a \rfloor \le a$ and $\{a\} = a - \lfloor a \rfloor$, we have that $0 \le \{a\} < 1$. Thus, $0 \le \left\{(x+1)^2\right\} = 2x^2 < 1$, so $x^2 < \dfrac{1}{2}$ and hence $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$.

Moreover, $\left\{(x+1)^2\right\} = 2x^2 \iff \lfloor (x+1)^2 \rfloor = (x+1)^2 - 2x^2 \iff \lfloor (x+1)^2 \rfloor = 1 + 2x - x^2$.
But $(x+1)^2 \ge 0 \iff 1 + 2x - x^2 = \lfloor (x+1)^2 \rfloor \ge 0$; since the graph of $f(x) = 1 + 2x - x^2$ is a concave parabola which cuts the x-axis in $x = 1 \pm \sqrt{2}$ and with vertex (absolute maximum) at $(1, 2)$, then the last obtained inequality $f(x) \ge 0$ is equivalent to $1 - \sqrt{2} \le x \le 1 + \sqrt{2}$. or what is the same, taking into account that
$-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$, 1-$\sqrt{2} \le x \le \dfrac{1}{\sqrt{2}}$.
On the other hand, $f(x) = \lfloor (x+1)^2 \rfloor \in Z$ and $0 \le f(x) \le 2$ implies $f(x) \in \{0, 1, 2\}$, that is, $x \in \{1 - \sqrt{2}, 1 + \sqrt{2}, 0, 1, 2\}$, which is equivalent, because $1 - \sqrt{2} \le x \le \dfrac{1}{\sqrt{2}}$ to x$\in \left\{1 - \sqrt{2}, 0\right\}$.

Since $\{(0+1)^2\} = 0 = 2 \cdot 0^2$ and $\left\{(1 - \sqrt{2} + 1)^2\right\} = \{6 - 4\sqrt{2}\} = 6 - 4\sqrt{2} = 2(1 - \sqrt{2})^2$, we conclude that the solutions $x \in \Re$ of the given equations $\{(x+1)^2\} = 2x^2$ are exactly $x = 1 - \sqrt{2}$ and $x = 0$.

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

From the given equation, $0 \le \left\{(x+1)^2\right\} = 2x^2 < 1$ or $0 \le x^2 < \frac{1}{2}$ or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Then, we have $0 \le (x+1)^2 < \left(1 + \frac{1}{\sqrt{2}}\right)^2 = 1 + \frac{1}{2} + \sqrt{2} < 1.5 + 1.5 = 3$. Therefore, the integer part $k = \left\lfloor (x+1)^2 \right\rfloor$ is 0 or 1 or 2.

If $k=0$, we have $2x^2=(x+1)^2$ or $x^2-2x-1=0$ or $x=1\pm\sqrt{2}$. Only $x=1-\sqrt{2}$ is valid for $k=0$.
If $k=1$, we have $2x^2+1=(x+1)^2$ or $x^2-2x=0$ or $x=0,2$. Only $x=0$ is valid for $k=1$.
If $k=2$, we have $2x^2+2=(x+1)^2$ or $x^2-2x+1=0$ or $x=1$. This is not valid for $k=2$.
Therefore, $x=1-\sqrt{2},\ 0$.

Also solved by Arkady Alt, San Jose CA; Brian Bradie, Christopher Newport University, Newport News, VA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Zhi Hwee Goh, Singapore, Singapore; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposers.

End Notes

The following should have been credited with having solved the problems below, but their names were inadvertently not listed; *mea culpa*.

Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy for problems 5433, 5437, and 5438.

Jeremiah Bartz, University of North Dakota, Grand Forks, ND for 5434.

David Stone and John Hawkins, Georgia Southern University, Statesboro, GA for 5433.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2018

- **5463:** *Proposed by Kenneth Korbin, New York, NY*

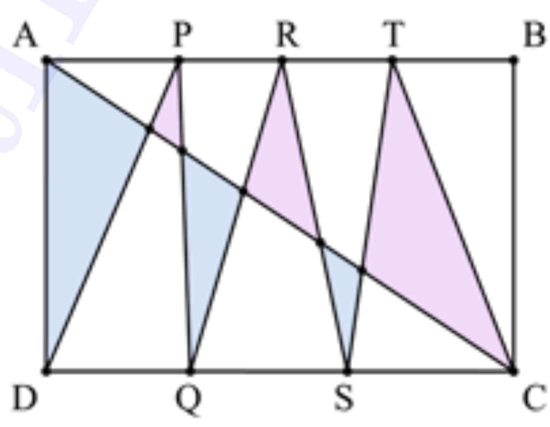
  Let N be a positive integer. Find triangular numbers x and y such that
  $x^2 + 14xy + y^2 = \left(72N^2 - 12N - 1\right)^2.$

- **5464:** *Proposed by Ed Gray, Highland Beach, FL*

  Let ABC be an equilateral triangle with side length s that is colored white on the front side and black on the back side. Its orientation is such that vertex A is at lower left, B is its apex, and C is at lower right. We take the paper at B and fold it straight down along the bisector of angle B, thus exposing part of the back side which is black. We continue to fold until the black part becomes $1/2$ of the existing figure, the other half being white. The problem is to determine the position of the fold, the distance defined by x (as a function of s) which is the distance from B to the fold.

- **5465:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Quadrilateral $ABCD$ is a rectangle with diagonal AC. Points P, R, T, Q and S are on sides AB and DC and they are connected as shown. Three of the triangles inside the rectangle are shaded pink, and three are shaded blue. Which is larger, the sum of the areas of the pink triangles or the sum of the areas of the blue triangles?



- **5466:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" School, Buzău, Romania*

  Let $f : (0, +\infty) \to (0, +\infty)$ be a continuous function. Evaluate

$$\lim_{n\to\infty} \int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right) dx.$$

- **5467:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  In an arbitrary triangle $\triangle ABC$, let a, b, c denote the lengths of the sides, R its circumradius, and let h_a, h_b, h_c respectively, denote the lengths of the corresponding altitudes. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}},$$

  and give the conditions under which equality holds.

- **5468:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all differentiable functions $f : \Re \to \Re$ with $f(0) = 1$ such that $f'(x) = f^2(-x)f(x)$, for all $x \in \Re$.

Solutions

- **5445:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of a triangle with exradii $(3, 4, 5)$.

  Solution 1 by Solution by David E. Manes, Oneonta, NY

  Denote the triangle by ABC with vertices A, B and C. Let $a = BC$, the side opposite the vertex $A, b = AC$ and $c = AB$. Let $r_a = 3$, the exradius of the circle tangent to side BC. Similarly, $r_b = 4$ is the exradius of the circle tangent to AC and $r_c = 5$ is the exradius of the circle tangent to AB. If r is the inradius of triangle ABC, then $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$ implies $\dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} = \dfrac{1}{r}$ implies $r = \dfrac{60}{47}$. If Δ is the area of triangle ABC, then $\Delta^2 = r \cdot r_a \cdot r_b \cdot r_c$. Therefore,

$$\left(\frac{60}{47}\right)(3 \cdot 4 \cdot 5) = \Delta^2 \quad \text{implies} \quad \Delta = \frac{60}{\sqrt{47}} = \frac{60\sqrt{47}}{47}.$$

  If $s = \dfrac{a+b+c}{2}$ is the semiperimeter of ABC, then $s = \dfrac{\Delta}{r}$. Therefore,

$$s = \frac{\left(\frac{60\sqrt{47}}{47}\right)}{\left(\frac{60}{47}\right)} = \sqrt{47}.$$

Using the formula $r_a = \dfrac{\Delta}{s-a}$, one obtains $a = s - \dfrac{\Delta}{r_a}$. Therefore,

$$a = \sqrt{47} - \frac{\left(\frac{60}{\sqrt{47}}\right)}{3} = \frac{47-20}{\sqrt{47}} = \frac{27\sqrt{47}}{47}.$$

Similarly,

$$b = s - \frac{\Delta}{r_b} = \sqrt{47} - \frac{\left(\frac{60}{\sqrt{47}}\right)}{4} = \frac{32\sqrt{47}}{47},$$

$$c = s - \frac{\Delta}{r_c} = \sqrt{47} - \frac{\left(\frac{60}{\sqrt{47}}\right)}{5} = \frac{35\sqrt{47}}{47}.$$

This completes the solution.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

In the published soltion by Howard Eves to problem 786 in the Journal Crux Mathematicorum (1984,10(20)), a more general result of the above problem is proved. For arbitrarily chosen positive real numbers r_1, r_2, r_3 there is one and only one triangle whose exradii are r_1, r_2, r_3, and that is the one whose sides are:

$$a = \frac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}},\ b = \frac{r_2(r_3+r_1)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}\ c = \frac{r_3(r_1+r_2)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}.$$

For the exradii values of $r_1 = 3,\ r_2 = 4$ and $r_3 = 5$ we find that

$$a = \frac{3(4+5)}{\sqrt{3\cdot 4+4\cdot 5+5\cdot 3}} = \frac{27}{\sqrt{47}},\ b = \frac{4(5+3)}{\sqrt{3\cdot 4+4\cdot 5+5\cdot 3}} = \frac{32}{\sqrt{47}}, c = \frac{35}{\sqrt{47}}.$$

Solution 3 by Ed Gray, Highland Beach, FL

Letting r be the in-radius of the given triangle, r_1, r_2, r_3 the ex-radii, s its semi-perimeter, K its area and a, b, c its side lengths, then following relationships, that were developed by Feuerbach, hold:

$$(1)\quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

$$(2)\quad K^2 = r\cdot r_1\cdot r_2\cdot r_3$$

$$(3)\quad s\cdot K = r_1\cdot r_2\cdot r_3$$

$$(4)\quad a = s - \frac{K}{r_1},\ b = s - \frac{K}{r_2}, c = s - \frac{K}{r_3}.$$

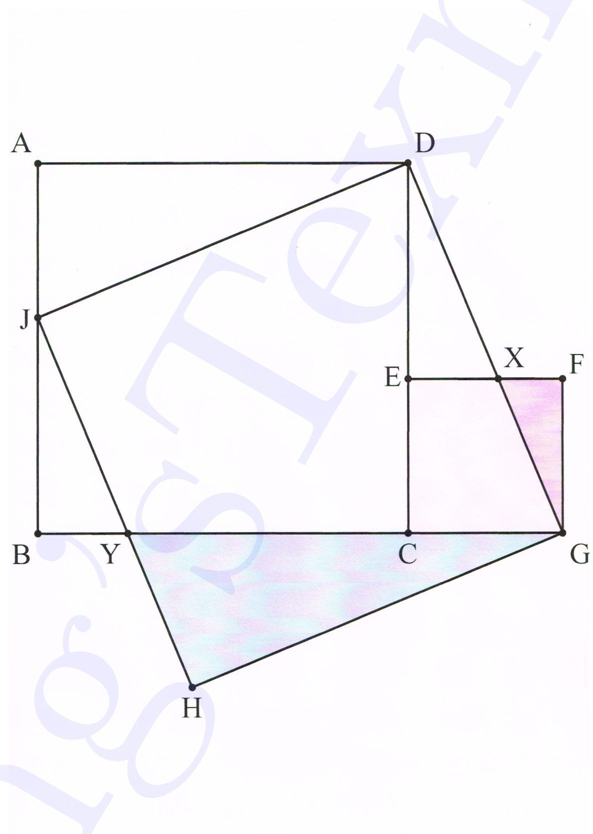
Making the substitutions we find that $a = \dfrac{27\sqrt{47}}{47},\ b = \dfrac{32\sqrt{47}}{47}, c = \dfrac{35\sqrt{47}}{47}$.

Comment by David Stone and John Hawkins of Georgia Southern University: An interesting connection to this problem (from Wolfram Mathworld) is that the curvature of the incircle equals the sum of the curvatures of the excircles: $\dfrac{1}{r} = \dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c}$ which equals $\left(\dfrac{r_br_c + r_ar_c + r_ar_b}{r_ar_br_c}\right)$. Thus the area can be written as $\Delta = \sqrt{r_ar_br_c}$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5446:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygons $ABCD, CEFG$, and $DGHJ$ are squares. Moreover, point E is on side $DC, X = DG \cap EF$, and $Y = BC \cap JH$. If GX splits square $CEFG$ in regions whose areas are in the ratio 5:19. What part of square $DGHJ$ is shaded? (Shaded region in $DGHJ$ is composed of the areas of triangle YHG and trapezoid $EXGC$.)



Solution 1 by "Get Stoked" Problem Solving Group, Mountain Lakes High School, Mountain Lakes, NJ

Since $\angle B = \angle H,\ \angle JYB = \angle GYH$,

$$\triangle JBY \sim \triangle GHY,$$

and because $\triangle JBY$ and $\triangle DAJ$ have a shared angle, $\angle B = \angle A$

$$\triangle JBY \sim \triangle DAJ,$$

and because $AD = DC, JD = GD, \angle A = \angle DCG$,

$$\triangle DAJ \cong \triangle DCG$$

and since $CG || EX$,

$$\triangle DCG \sim \triangle DEX$$

and because $\angle DEX = \angle F,\ \angle DXE = \angle GXF$,

$$\triangle DEX \sim \triangle GFX.$$

Therefore,

$$\triangle GYH \sim \triangle GXF$$

Without loss of generality, set the area of $\triangle YHG = 5$ and trapezoid $EXGC = 19$. Adding the areas of $\triangle YHG$ and trapezoid $EXGC$ and finding each side obtains:

$$\sqrt{5+19} = 2\sqrt{6}.$$

Drawing a perpendicular line from point X to CG creates line IX.
Because the area of $XFGI$ is double of that of $\triangle GXF$ and $FG = EC = XI$,

$$XF : CG = 5 : 12.$$

Since $FG = CG$, it can be concluded hat $\triangle GXF$ is a 5 -12 -13 triangle. Because $\triangle DGC \sim \triangle GXF$,

$$CG : DG = 5 : 13.$$

Now that we know the ratio between the two squares and that the ratio of the area between two similar polygons is the square of the ratio of the sides, it is apparent that

$$\frac{area(EXGC)}{area(JDGH)} = \frac{19}{24} \cdot \left(\frac{5}{13}\right)^2 = \frac{475}{4096}.$$

Adding the two pieces results in the part of square $DCHJ$ that is shaded

$$\frac{5}{24} + \frac{475}{4096} = \frac{55}{169}.$$

Solution 2 by Kenneth Korbin, New York, NY

Answer: $\dfrac{55}{169}$.

Let $\overline{XF} = 25$ and $\overline{EX} = 35$. Then each segment in the diagram will have positive integer length.

$$AD = 144,\ AJ = 60,\ JB = 89,\ BY = 35,\ YH = 65,\ HG = 156,\ GF = 60,\ FX = 25,$$

$$XE = 35,\ ED = 84,\ DJ = 156,\ JY = 91,\ YG = 169,\ CE = 60,\ CG = 60,\ XG = 65.$$

$$DX = 91.$$

Every triangle in this diagram is similar to the Pythagorean Triangle with sides $(5, 12, 13)$.

Area of square $DGHJ = (156)^2 = 24336$.

Area of triangle $YHG = \dfrac{1}{2}(65)(156) = 5070$.

Area of trapezoid $EXGC = (60)^2 - \dfrac{1}{2}(25)(60) = 2850$.

So the desired ratio is $\dfrac{5070 + 2850}{24336} = \dfrac{55}{169}$.

Also solved by Jeremiah Bartz and Nicholas Newman, University of North Dakota and Troy University respectively, Grand Forks, ND and Troy, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania; Sachit Misra, Nelhi, India; Boris Rays, Brooklyn, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5447:** *Proposed by Iuliana Trască, Scornicesti, Romanai*

Show that if x, y, and z is each a positive real number, then

$$\frac{x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3}{x^2 \cdot y^2 \cdot z^2} \geq \frac{x^3 + y^3 + z^3 + 3x \cdot y \cdot z}{2}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The stated inequality is equivalent to

$$2x^6z^3 + 2y^6x^3 + 2z^6y^3 \geq x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 + 3x^3y^3z^3. \qquad (1)$$

By the AM-GM inequality,

$$\sum_{cycl} x^6z^3 = \sum_{cycl}\left(\frac{2}{3}x^6z^3 + \frac{1}{3}y^6x^3\right) \geq \sum_{cycl}\left(x^{\frac{2}{3}\cdot 6}z^{\frac{2}{3}\cdot 6}y^{\frac{1}{3}\cdot 6}x^{\frac{1}{3}\cdot 3}\right) = \sum_{cycl} x^5y^2z^2,$$

$$\sum_{cycl} x^6z^3 \geq 3x^3y^3z^3$$

Statement (1) follows by adding these two inequalities.

Solution 2 by Arkady Alt, San Jose, CA

Note that,

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2\ y^2\ z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2} \quad \iff \quad 2\left(x^6z^3 + y^6x^3 + z^6y^3\right)$$

$$\geq \quad x^5\ y^2\ z^2 + x^2\ y^5\ z^2 + x^2\ y^2\ z^5 + 3x^3\ y^3\ z^3.$$

By AM-GM Inequality,

$$x^6z^3 + y^6x^3 + z^6y^3 \geq 3\sqrt[3]{x^6z^3 \cdot y^6x^3 \cdot z^6y^3} = 3\sqrt[3]{x^9y^9z^9} = 3x^3\ y^3\ z^3.$$

And again by AM-GM Inequality.

$$2x^6z^3 + y^6x^3 \geq 3\sqrt[3]{\left(x^6z^3\right)^2 y^6x^3} = 3\sqrt[3]{x^{15}y^6z^6} = 3x^5y^2z^2,$$

and therefore,

$$3\sum_{cyc} x^6z^3 = \sum_{cyc}\left(2x^6z^3 + y^6x^3\right) \geq \sum_{cyc} 3x^5y^2z^2 \iff \sum_{cyc} x^6z^3 \geq \sum_{cyc} x^5\ y^2\ z^2.$$

Thus, $2\sum\limits_{cyc} x^6z^3 = \sum\limits_{cyc} x^6z^3 + \sum\limits_{cyc} x^6z^3 \geq \sum\limits_{cyc} x^5\ y^2\ z^2 + 3x^3\ y^3\ z^3.$

Solution 3 by Moti Levy, Rehovot, Israel

By Muirhead inequality ($(6,3,0)$ majorizes $(5,2,2)$),

$$\sum_{sym} x^6x^3z^0 \geq \sum_{sym} x^5y^2z^2,$$

or explicitly,

$$\left(x^6z^3 + y^6x^3 + z^6y^3\right) + \left(x^6y^3 + v^6z^3 + z^6x^3\right) \geq 2\left(x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5\right). \tag{1}$$

Again, by Muirhead inequality ($(5,2,2)$ majorizes $(3,3,3)$),

$$\sum_{sym} x^5y^2z^2 \geq \sum_{sym} x^3y^3z^3$$

or explicitly,

$$x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 \geq 3x^3y^3z^3. \tag{2}$$

Given three positive numbers a, b, c. We can always assign their values to x, y and z respectively, such that $x^6z^3 + y^6x^3 + z^6y^3 \geq x^6y^3 + y^6z^3 + z^6x^3$. Hence, without loss of generality, we can assume that

$$x^6z^3 + y^6x^3 + z^6y^3 \geq x^6y^3 + y^6z^3 + z^6x^3, \tag{3}$$

then by (1), (2) and (3)

$$\begin{aligned}
2\left(x^6z^3 + y^6x^3 + z^6y^3\right) &\geq \left(x^6z^3 + y^6x^3 + z^6y^3\right) + \left(x^6y^3 + v^6z^3 + z^6x^3\right)\\
&\geq 2\left(x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5\right)\\
&\geq x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 + 3x^3y^3z^3.
\end{aligned}$$

which is equivalent to

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2y^2z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2}.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We prove the stronger inequality

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2y^2z^2} \geq x^3 + y^3 + z^3. \qquad (1)$$

Since $x^3 + y^3 + z^3 \geq 3xyz$ by the AM-GM inequality, the inequality of the problem follows immediately from (1).

By homogeneity, we assume without loss of generality that $xyz = 1$. By substituting $z = \dfrac{1}{xy}$ into (1), we deduce after some algebra that (1) is equivalent to

$$x^9 + x^9y^9 + 1 - x^9y^3 - x^6y^6 - x^3 \geq 0. \qquad (2)$$

Denote the left side of (2) by f. It can be checked readily by expanding both sides that

$$\begin{aligned}(1 + 2x^3 + x^3y^3)f \quad = \quad & x^9(1 + x^3)(1 + y^3)(1 - y^3)^2 + (1 + x^3)^2(1 - x^3)^2 + \\ & x^3(1 + y^3)(1 + x^3y^3)(1 - x^3y^3)^2,\end{aligned}$$

which is nonnegative. Thus (2) holds and this completes the solution.

Solution 5 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well-known that for all $a, b, c \geq 0$ we have $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$, and $a^3 + b^3 + c^3 \geq 3abc$. Now for all positive real numbers x, y, and z we can write

$$\frac{2\left(x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3\right)}{x^2 \cdot y^2 \cdot z^2} \geq \frac{2\left[(x^2z)^2y^2x + (y^2x)^2z^2y + (z^2y)^2x^2z\right]}{x^2y^2z^2} =$$

$$2(x^3 + y^3 + z^3) = x^3 + y^3 + z^3 + (x^3 + y^3 + z^3) \geq x^3 + y^3 + z^3 + 3xyz.$$

Now, multiplying both sides of the inequality

$$\frac{2(x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3)}{x^2 \cdot y^2 \cdot z^2} \geq x^3 + y^3 + z^3 + 3xyz,$$

by $\dfrac{1}{2}$, will give us the desired result.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; David E. Manes, Oneonta, NY; Sachit Misra, Delhi, India; Paolo Perfetti, Department of Mathematics, Tor Vergata University of Rome, Italy; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania, and the proposer.

- **5448:** *Proposed by Yubal Barrios and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Evaluate: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\sum_{\substack{0\leq i,j\leq n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j}}$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

The generating function for the central binomial coefficients is $(1-4x)^{-1/2}$; that is,

$$\sum_{i=0}^{\infty} \binom{2i}{i} x^i = \frac{1}{\sqrt{1-4x}}.$$

It follows that

$$\sum_{0\le i,j\le n, i+j=n} \binom{2i}{i}\binom{2j}{j} = \sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i}$$

is the coefficient of x^n in the function

$$\frac{1}{\sqrt{1-4x}} \cdot \frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n,$$

which is 4^n. Thus,

$$\lim_{n\to\infty} \sqrt[n]{\sum_{0\le i,j\le n, i+j=n} \binom{2i}{i}\binom{2j}{j}} = \lim_{n\to\infty} \sqrt[n]{4^n} = \lim_{n\to\infty} 4 = 4.$$

Solution 2 by Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania

$$(1+x)^0(1+x)^{2n} + (1+x)^2(1+x)^{2n-2} + (1+x)^4(1+x)^{2n-4} + \dots$$
$$\dots + (1+x)^{2n}(1+x)^0 = (2n+1)(1+x)^{2n}$$

The coefficient of x^n in LHS and RHS are equal:

$$\binom{2n}{n} + \binom{2}{1}\binom{2n-2}{n-1} + \binom{4}{2}\binom{2n-4}{n-2} + \dots + \binom{2n}{n} = (2n+1)\binom{2n}{n}$$

$$\sum_{\substack{0\le i,j\le n\\ i+j=n}} \binom{2i}{i}\binom{2j}{j} \quad = \quad (2n+1)\binom{2n}{n}$$

$$\lim_{n\to\infty} \sqrt[n]{\sum_{0\le i,j\le n} \binom{2i}{i}\binom{2j}{j}} = \lim_{n\to\infty} \sqrt[n]{(2n+1)\binom{2n}{n}}$$

$$\overbrace{=}^{CAUCHY-D'ALEMBERT} \lim_{n\to\infty} \frac{2n+3}{2n+1} \cdot \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{((n)!)^2}}$$

$$= 1 \cdot \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that that $|x| \le \dfrac{1}{4}$, $\displaystyle\sum_{i=0}^{\infty}\binom{2i}{i}x^i = \frac{1}{\sqrt{1-4x}}$, with the usual convention that $0! = 1$ and $\dbinom{0}{0} = 1$. Hence,

$$\sum_{i=0}^{\infty}(4x)^i = \frac{1}{1-4x} = \left(\sum_{i=0}^{\infty}\binom{2i}{i}x^i\right)^2 = \sum_{n=0}^{\infty}\sum_{\substack{0\le i,j\le n\\ i+j=n}}\binom{2i}{i}\binom{2j}{j}x^n.$$

Thus for nonnegative integers n,

$$\sum_{n=0}^{\infty}\sum_{\substack{0\le i,j\le n\\ i+j=n}}\binom{2i}{i}\binom{2j}{j} = 4^n,$$

so that the limit of the problem equals 4.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome Italy; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

- **5449:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the use of a computer, find the real roots of the equation

$$x^6 - 26x^3 + 55x^2 - 39x + 10 = (3x-2)\sqrt{3x-2}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We need to consider only the values of $x \ge 2/3$, since the square root is not real for $x < 2/3$.

We see that $x = 1$ and $x = 2$ are roots of the given equation Suppose that $x \notin \{1, 2\}$. We find that

$$\begin{aligned}
x^6 - 27x^3 + 55x^2 - 39x + 10 &= (x-1)(x-2)\left(x^4+3x^3+7x^2-12x+5\right)\\
&= (3x-2)\sqrt{3x-2} - x^3\\
&= \frac{(3x-2)^3 - x^6}{(3x-2)\sqrt{3x-2}+x^3}\\
&= \frac{(x-1)(x-2)\left(x^4+3x^3+7x^2-12x+4\right)}{(3x-2)\sqrt{3x-2}+x^3}, \text{ implying}\\
x^4+3x^3+7x^2-12x+5 &= \frac{x^4+3x^3+7x^2-12x+4}{(3x-2)\sqrt{3x-2}+x^3}. \qquad (1)
\end{aligned}$$

We note that $x^4 + 3x^3 + 7x^2 - 12x + 4 = x^4 + x^2(3x-2) + (3x-2)^2 \geq 0$, since $x \geq 2/3$. So (1) has no other real solutions than $x = 1$ and $x = 2$.

Solution 2 by Ed Gray, Highland Beach, FL

Define the function:

(1) $f(x) = x^6 - 26x^3 + 55x^2 - 39x + 10 - (3x-2)^{3/2} = 0$.

Consider the term $(3x-2)^{3/2}$. Since the values of $x = 1$ and $x = 2$ both provide integer values, it is worth trying these values as a first guess. In fact,

(2) $f(1) = 1 - 26 + 55 - 39 + 10 - 1 = 0$, so in fact, $x = 1$ is a root.

(3) $f(2) = 64 - 26(8) + 55(4) - 39(2) + 10 - 8 = 64 - 208 + 220 - 78 + 10 - 8 = 0$, so, in fact $x = 2$ is also a root.

It may be fruitful to utilize the derivative which is:

(4) $f'(x) = 6x^5 - 78x^2 + 110x - 39 - (3/2) * (3)\sqrt{3x-2}$.

We note that

(5) $f''(2) = (6)(32) - (78)(4) + 110)(2) - 39 - 9 = 192 - 312 + 220 - 39 - 9 = 52$.

(6) $f'(1) = 6 - 78 + 110 - 39 - 4.5 = -5.5$, so the function is 0 at $x = 1$ and $x = 2$. At $x = 1$, it is decreasing and at $x = 2$ it is increasing. Therefore, there is a point x_0 with $1 < x_0 < 2$ and $f'(x_0) = 0$. The function is increasing at $x = 2$, where the derivative is greater than 0, so if $x > 2$, the function is greater than 0. It would be good if the function stays positive for $x > 2$.

Note the second derivative is:

(7) $f''(x) = 30x^4 - 156x + 110 - (27/4) * (3x-2)^{(} - 1/2)$.

At $x = 2$, $f''(2)480 - 312 + 110 - 27/8 = 274.625$, and clearly increases as x increases. We conclude there can be no real roots greater than 2.

Now we look at the situation where $x = 1$, which is a root. $f'(1) = -5.5$. Therefore, values of $f(x)$ for x slightly less than 1 must be positive. If $x < 2/3$, we note the radical term becomes negative and complex terms will be introduced, negating the existence of real roots. We need to consider the region $2/3 < x < 1$. The value of

$$\begin{aligned} f(2/3) &= (2/3)^6 - 26((2/3)^3 + 56(3/3)^2 - 39(2/3) + 10 \\ &= 64/729 - 26(8/27) + 55(4/9) - 39(2/3) + 10 = 604/729 > 0. \end{aligned}$$

Also, $f'(2/3) = 111/243$. This is unexpectedly positive, which means the function rises from 604/729 at $x = 2/3$ as x increases, then there must be a point x_1 such that $2/3 < x_1 < 1$ and $f'(x_1) = 0$. After $x > x_1$, the derivative turns negative and the function descends to 0 at $x = 1$. Therefore, there can be no other real roots other than $x = 1$ and $x = 2$.

Solution 3 by Anna V. Tomova, Varna, Bulgaria

The decision area is: $3x - 2 \geq 0 \implies x \geq \dfrac{2}{3}$, $x^6 - 26x^3 + 55x^2 - 39x + 10 \geq 0$. We let $\sqrt{3x-2} = t \geq 0 \iff x = \dfrac{t^2+2}{3}$, and after substitution we obtain

$$t^{12} + 12t^{10} + 60t^8 - 542t^6 + 483t^4 - 729t^3 + 111t^2 + 604 = 0.$$

Looking for low-valued positive integer roots to the above equation so that we can use the factor theorem, we see that $t=1$ and $t=2$ allows us to rewrite the equation as

$$(t-1)(t-2)\left(t^{10}+3t^9+19t^8+51t^2+175t^6+423t^5+377t^4+285t^3+584t^2+453t+302\right)=0.$$

Because all of the coefficients in the third factor are positive, we see that there are no other positive roots. So, $\begin{cases} x=\dfrac{t^2+2}{3}=1 \implies x=1 \\ x=\dfrac{t^2+2}{3}=2 \implies x=2. \end{cases}$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

We note that $x=1$ and $x=2$ satisfy the given equation, and we show that those are the only real roots.

Squaring both sides of the equation and factoring yields $(x-1)(x-2)f(x)=0$, where

$$f(x)=x^{10}+3x^9+7x^8-37x^7-15x^6-49x^5+579x^4-1025x^3+820x^2-327x+54.$$

Since we must have $x\geq 2/3$ in the original equation, it suffices to show that $f(x)\neq 0$ for each $x\geq 2/3$. We write $f(x)=(g(x))^2+h(x)$, where

$$g(x)=x^5+\tfrac{3}{2}x^4+\tfrac{19}{8}x^3-\tfrac{353}{16}x^2+\tfrac{2915}{128}x-\tfrac{1603}{256}$$

and

$$h(x)=\tfrac{1463}{512}x^4+\tfrac{2463}{256}x^3+\tfrac{410783}{16384}x^2-\tfrac{684823}{16384}x+\tfrac{969335}{65536}.$$

Then $h'(x)=\frac{1463}{128}x^3+\frac{7389}{256}x^2+\frac{410783}{8192}x-\frac{684823}{16384}$. Since $h''(x)>0$ on $(0,\infty)$, h' is increasing on $(0,\infty)$. Also, $h'(2/3)>0$, so we have $h'(x)>0$ for each $x\geq 2/3$. Thus h is increasing on $[2/3,\infty)$ with $h(2/3)>0$, so $h(x)>0$ for each $x\geq 2/3$ as needed. Hence $f(x)>0$ for each $x\geq 2/3$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5450:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let k be a positive integer. Calculate

$$\int_0^1\int_0^1 \left\lfloor\frac{x}{y}\right\rfloor^k \frac{y^k}{x^k}\,dxdy,$$

where $\lfloor a\rfloor$ denotes the floor (the integer part) of a.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Reverse the order of integration, and then write

$$\begin{aligned}\int_0^1\int_0^1 \left\lfloor\frac{x}{y}\right\rfloor^k \frac{y^k}{x^k}\,dx\,dy &= \int_0^1\int_0^1 \left\lfloor\frac{x}{y}\right\rfloor^k \frac{y^k}{x^k}\,dy\,dx \\ &= \int_0^1\int_x^1 \left\lfloor\frac{x}{y}\right\rfloor^k \frac{y^k}{x^k}\,dy\,dx+\sum_{n=1}^{\infty}\int_0^1\int_{x/(n+1)}^{x/n}\left\lfloor\frac{x}{y}\right\rfloor^k \frac{y^k}{x^k}\,dy\,dx.\end{aligned}$$

For $x \le y \le 1$, $\lfloor x/y \rfloor = 0$, while for $x/(n+1) \le y \le x/n$, $\lfloor x/y \rfloor = n$, so

$$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy = \sum_{n=1}^{\infty} n^k \int_0^1 \int_{x/(n+1)}^{x/n} \frac{y^k}{x^k}\, dy\, dx.$$

Now,

$$\begin{aligned}
\int_0^1 \int_{x/(n+1)}^{x/n} \frac{y^k}{x^k}\, dy\, dx &= \frac{1}{k+1} \int_0^1 \frac{1}{x^k} \left(\frac{x^{k+1}}{n^{k+1}} - \frac{x^{k+1}}{(n+1)^{k+1}} \right) dx \\
&= \frac{1}{k+1} \left(\frac{1}{n^{k+1}} - \frac{1}{(n+1)^{k+1}} \right) \int_0^1 x\, dx \\
&= \frac{1}{2(k+1)} \left(\frac{1}{n^{k+1}} - \frac{1}{(n+1)^{k+1}} \right),
\end{aligned}$$

so

$$\begin{aligned}
\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy &= \frac{1}{2(k+1)} \sum_{n=1}^{\infty} n^k \left(\frac{1}{n^{k+1}} - \frac{1}{(n+1)^{k+1}} \right). \\
&= \frac{1}{2(k+1)} \sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{n^{k+1}}.
\end{aligned}$$

By the binomial theorem,

$$(n-1)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} n^{k-j}.$$

It follows that

$$\begin{aligned}
n^k - (n-1)^k &= \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} n^{k-j}, \\
\frac{n^k - (n-1)^k}{n^{k+1}} &= \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{n^{j+1}},
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{n^{k+1}} = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \sum_{n=1}^{\infty} \frac{1}{n^{j+1}} = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \zeta(j+1),$$

where $\zeta(x)$ denotes the Riemann zeta function. Finally,

$$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy = \frac{1}{2(k+1)} \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \zeta(j+1).$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

For any positive integer m, let us assume that $m \le \dfrac{x}{y} < m+1$, which it is equivalent to

$\dfrac{x}{m+1} \le y < \dfrac{x}{m}$. The proposed integral, say I, becomes

$$
\begin{aligned}
I &= \sum_{m=1}^{\infty} m^k \int_0^1 \frac{1}{x^k} \int_{\frac{x}{m+1}}^{\frac{x}{m}} y^k dy\, dx \\
&= \sum_{m=1}^{\infty} m^k \int_0^1 \frac{1}{x^k} \left.\frac{y^{k+1}}{k+1}\right]_{\frac{x}{m+1}}^{\frac{x}{m}} dx \\
&= \sum_{m=1}^{\infty} \frac{m^k}{2(k+1)} \left(\frac{1}{m^{k+1}} - \frac{1}{(m+1)^{k+1}}\right) \\
&= \frac{1}{2(k+1)} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{m^k}{(m+1)^{k+1}}\right) \\
&= \frac{1}{2(k+1)} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1} - \frac{\sum_{j=1}^{k}(-1)^j \binom{k}{j}}{(m+1)^{k+1-j}}\right)
\end{aligned}
$$

from where,

$$
I = \frac{\displaystyle\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\zeta(j+1)}{2(k+1)}.
$$

Solution 3 by Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome Italy

We change variables $x/y = t$, $x = u$ and the integral becomes

$$
\int_0^1 dt \int_0^t du \frac{1}{t^{k+2}} \lfloor t \rfloor^k u + \int_1^{\infty} dt \int_0^1 du \frac{1}{t^{k+2}} \lfloor t \rfloor^k u = \int_1^{\infty} dt \int_0^1 du \frac{1}{t^{k+2}} \lfloor t \rfloor^k u.
$$

The first integral is zero so we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{q=1}^{\infty} \int_q^{q+1} q^k \frac{dt}{t^{k+2}} = \frac{1}{2(k+1)} \lim_{n\to\infty} \sum_{q=1}^{n} q^k \left(\frac{1}{q^{k+1}} - \frac{1}{(q+1)^{k+1}}\right) \\
&= \frac{1}{2(k+1)} \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{q+1}\frac{(q+1-1)^k}{(q+1)^k}\right) \\
&= \frac{1}{2(k+1)} \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{(q+1)^{k+1}} \left((q+1)^k + \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{j-k} (q+1)^j\right)\right) \\
&= \frac{1}{2(k+1)} \sum_{q=1}^{\infty} \left(\underbrace{\frac{1}{q} - \frac{1}{q+1}}_{telescope} - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{j-k} (q+1)^{j-k-1}\right) \\
&= \frac{1}{2(k+1)} \left(1 - \sum_{q=1}^{\infty}\sum_{i=1}^{k} \binom{k}{i} (-1)^i (q+1)^{-i-1}\right) \\
&= \frac{1}{2(k+1)} \left(1 - \sum_{i=1}^{k} \binom{k}{i} (-1)^i (\zeta(i+1) - 1)\right) = \frac{-1}{2(k+1)} \sum_{i=1}^{k} \binom{k}{i} (-1)^i (\zeta(i+1).
\end{aligned}
$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

End Notes

Sachit Misra of Delhi, India should have been credited with having solved 5440, and **David Stone and John Hawkins of Georgia Southern University, Statesboro, GA** should have been credited for having solved 5444. Once again, *mea culpa*.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2018

- **5469:** *Proposed by Kenneth Korbin, New York, NY*

  Let x and y be positive integers that satisfy the equation $3x^2 = 7y^2 + 17$. Find a pair of larger integers that satisfy this equation expressed in terms of x and y.

- **5470:** *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

  Prove that there are an infinite number of Heronian triangles (triangles whose sides and area are natural numbers), whose side lengths are three consecutive natural numbers.

- **5471:** *Proposed by Arkady Alt, San Jose, CA*

  For natural numbers p and n where $n \geq 3$ prove that

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)(n+3)\cdots(n+p)}}.$$

- **5472:** *Proposed by Francisco Perdomo and Ángel Plaza, both at Universidad Las Palmas de Gran Canaria, Spain*

  Let α, β, and γ be the three angles in a non-right triangle. Prove that

$$\frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{1+\sin^2\beta}{\cos^2\beta} + \frac{1+\sin^2\gamma}{\cos^2\gamma} \geq \frac{1+\sin\alpha\sin\beta}{1-\sin\alpha\sin\beta} + \frac{1+\sin\beta\sin\gamma}{1-\sin\beta\sin\gamma} + \frac{1+\sin\gamma\sin\alpha}{1-\sin\gamma\sin\alpha}.$$

- **5473:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $x_1, \cdots, x_n$ be positive real numbers. Prove that for $n \geq 2$, the following inequality holds:

$$\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}.$$

  (Here the subscripts are taken modulo n)

- **5474:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $a, b \in \Re, b \neq 0$. Calculate
$$\lim_{n\to\infty} \begin{pmatrix} 1-\dfrac{a}{n^2} & \dfrac{b}{n} \\ & \\ \dfrac{b}{n} & 1+\dfrac{a}{n^2}. \end{pmatrix}^n .$$

Solutions

- **5451:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with sides $a = 8, b = 19$ and $c = 22$. The triangle has an interior point P where $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ each have positive integer length. Find $\overline{AP}$ and $\overline{BP}$, if $\overline{CP} = 4$.

Solution 1 by David E. Manes, Oneonta, NY

We will show that $\overline{BP} = 6$ and $\overline{AP} = 17$.
Using the law of cosines in ΔABC, one obtains
$$\cos\angle C = \frac{8^2+4^2-22^2}{2\cdot 8\cdot 19} = \frac{-59}{304}$$
so that $\angle C = \arccos\left(\dfrac{-59}{304}\right)$. Let $x = \overline{BP}$ and $y = \overline{AP}$. By the triangle inequality in ΔPCB, it follows that $5 \leq x \leq 11$. If $x = 5$, then
$$\cos\angle BCP = \frac{8^2+4^2-5^2}{2\cdot 8\cdot 4} = \frac{55}{64}.$$
Therefore, $\angle BCP = \arccos\left(\dfrac{55}{64}\right)$ and
$\angle PCA = \angle C - \angle BCP = \arccos\left(\dfrac{-59}{304}\right) - \arccos\left(\dfrac{55}{64}\right)$. Using the identity
$\cos(\alpha - \beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$, we get
$$\cos\angle PCA = \left(\frac{-59}{304}\right)\left(\frac{55}{64}\right) + \left(\frac{77\sqrt{15}}{304}\right)\left(\frac{3\sqrt{7\cdot 17}}{64}\right) = \frac{-3245+231\sqrt{3\cdot 5\cdot 7\cdot 17}}{304\cdot 64}.$$
Thus,
$$\begin{aligned} y^2 &= 4^2+19^2-2\cdot 4\cdot 19\cos\angle PCA = 377 - 19\left(\frac{-3245+231\sqrt{15\cdot 119}}{304\cdot 8}\right) \\ &= \frac{916864+61655-4389\sqrt{1785}}{2432}. \end{aligned}$$
Therefore,
$$y = \sqrt{\frac{978519-4389\sqrt{1785}}{2432}} \approx 18.058$$

is not an integer. Hence, $x \neq 5$.
However, if $x = 6$, then

$$\cos \angle BCP = \frac{8^2 + 4^2 - 6^2}{64} = \frac{11}{16}$$

so that $\angle BCP = \arccos\left(\frac{11}{16}\right)$ and
$\angle PCA = \angle C - \angle BCP = \arccos\left(\frac{-59}{304}\right) - \arccos\left(\frac{11}{16}\right)$. Thus,

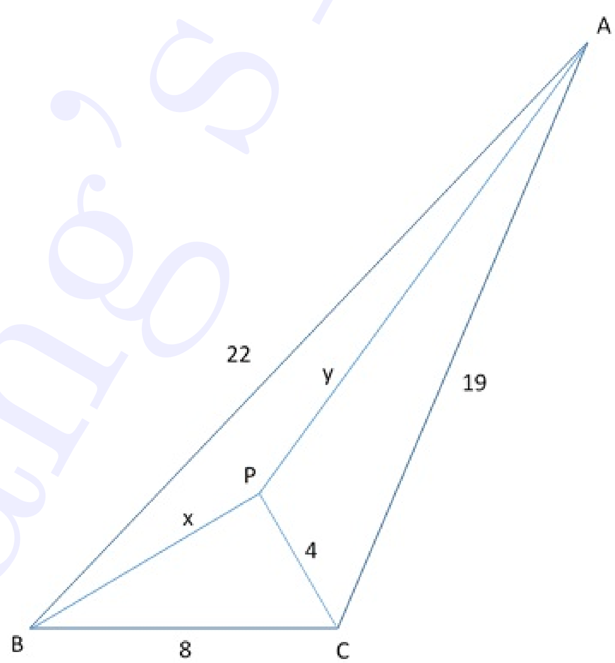
$$\begin{aligned}
\cos \angle PCA &= \cos\left[\arccos\left(\frac{-59}{304}\right) - \arccos\left(\frac{11}{16}\right)\right] \\
&= \left(\frac{-59}{304}\right)\left(\frac{11}{16}\right) + \left(\frac{77\sqrt{15}}{304}\right)\left(\frac{3\sqrt{15}}{16}\right) = \frac{-649 + 3465}{4864} \\
&= \frac{11}{19}.
\end{aligned}$$

Therefore,

$$y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19\left(\frac{11}{19}\right) = 289$$

whence $y = 17$. Hence, $x = \overline{BP} = 6$ and $y = \overline{AP} = 17$. The solution is unique since $x = 7$ does not yield an integer value for y while each of the values $x = 8, 9, 10, 11$ does not yield a triangle for ΔBPA.

Solution 2 by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel



Let $\overline{BP} = x$ and $\overline{AP} = y$. Because of the triangle inequality, $8 < x + 4, x < 8 + 4$ or $5 \leq x \leq 11$. Similarly, we have $16 \leq y \leq 22$.

These inequalities can be improved slightly using Stewart's formula for the length of cevians: if ABC is a triangle with sides $\overline{AC} = b$ and $\overline{BC} = a$ and if d is the length of a cevian from A which divides $\overline{AB}$ into segments of lengths $\overline{AB} = m$ and $\overline{AB} = n$, then:

$$d^2 = \frac{ma^2 + nb^2}{m+n} - mn$$

(this is just an easy consequence of the law of cosines). Since the maximum value of y occurs when P lies on $\overline{BC}$, by Stewart's formula, $y^2_{max} = \frac{22^2+19^2}{2} - 4^2 = 406.5 = 20.16^2$, so $y \le 20$. Similarly, the maximum value of x occurs when P lies on $\overline{AC}$, so that P divides $\overline{AC}$ into segments of lengths 4 and 15. Thus, again by Stewart's formula $x^2_{max} = \frac{15 \cdot 8^2 + 4 \cdot 22^2}{19} - 4 \cdot 15 = 92.42 \approx 9.61^2$, so that $x \le 9$. Hence:

$$5 \le x \le 9$$

$$16 \le y \le 20$$

Since P lies on a circle centered at C, and the lines $\overline{BP}$ all lie on one side of $\overline{BC}$, each length x of $\overline{BP}$ corresponds to a unique P and, therefore, to a unique value of y.
To find y for a given value of x, let $\angle BCP = \theta, \angle PCA = \phi$, and $\angle BCA = \gamma$. The cosine of γ is fixed and given by the law of cosines:

$$\cos\gamma = \frac{19^2 + 8^2 - 22^2}{2 \cdot 8 \cdot 19} = -\frac{59}{304}$$

The sine of γ is just $\sqrt{1-\cos^2\gamma}$, that is:

$$\sin\gamma = \sqrt{1 - \frac{59^2}{304^2}} = \frac{77}{304}\sqrt{15}$$

The cosine of θ for a given value of x is also given by the law of cosines:

$$\cos\theta = \frac{8^2 + 4^2 - x^2}{2 \cdot 8 \cdot 4} = \frac{80 - x^2}{64}$$

And again, $\sin\theta$ is given by $\sqrt{1-\cos^2\theta}$. Hence, the cosine of ϕ is given:

$$\cos\phi = \cos(\gamma - \theta) = -\frac{59}{304}\cos\theta + \frac{77}{304}\sqrt{15}\sin\theta$$

Thus, for any x we can calculate y, once again by the law of cosines:

$$y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19 \cdot \cos\phi = 377 - 152\cos\phi$$

Calculating y for $x = 5, 6, 7, 8, 9$ we find one integral value for y: $y = 17$ corresponding to $x = 6$.
So we have our answer:

$$\overline{AP} = 17$$

$$\overline{BP} = 6$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

We model the given triangle in the Cartesian plane by first placing A at $(19, 0)$ and C at $(0, 0)$. Then B must lie on the circles with equations

$$x^2 + y^2 = 64 \quad \text{and} \quad (x-19)^2 + y^2 = 484,$$

so we place B in the second quadrant at (d, e), where $d = -59/38$ and $e = 77\sqrt{15}/38$. Next, we seek an interior point $P = (x, y)$ such that $x^2 + y^2 = 16$, $(x-19)^2 + y^2 = m^2$, and $(x-d)^2 + (y-e)^2 = n^2$ for positive integers $m = \overline{AP}$ and $n = \overline{BP}$. Since P is interior to triangle ABC and lies on the circle with equation $x^2 + y^2 = 4$, we have $m \in \{16, 17, 18, 19, 20\}$ and $n \in \{5, 6, 7, 8, 9\}$. Solving the system

$$\begin{cases} x^2 + y^2 = 16 \\ (x-19)^2 + y^2 = m^2 \end{cases}$$

yields $x = (377 - m^2)/38$ and $y = \sqrt{-m^4 + 754m^2 - 119025}/38$. Substituting these values for x and y into $(x-d)^2 + (y-e)^2 = n^2$ for $m \in \{16, 17, 18, 19, 20\}$, we find that only $m = 17$ produces a positive integer value for n, namely $n = 6$. Hence $P = (44/19, 16\sqrt{15}/19)$ with $\overline{AP} = 17$ and $\overline{BP} = 6$.

Comment by Albert Stadler, Herrliberg, Switzerland: There is no other interior point even if we get rid of the condition that $\overline{CP} = 4$. However, letting $u = \overline{AP}, v = \overline{BP}$ and $w = \overline{CP}$ and if we permit P to lie on a side of the triangle, then $(u, v, w) = (16, 6, 7)$ is the only additional point.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Vijaya Prasad Nalluri, Rajahmundry, India; Valentin Shopov, Munich, Germany; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Students at Taylor University, Upland, IN,ne
Team 1: {Hannah Peters, Ben Robison, Stevanni McCray}
Team 2: {Hannah King, Deborah Settles, Jackson Bronkema}
Team 3: {Gwyneth Terrett, Samantha Korn, Elissa Grace Moore}, and the proposer.

- **5452:** *Proposed by Roger Izard, Dallas, TX*

Let point O be the orthocenter of a given triangle ABC. In triangle ABC let the altitude from B intersect line segment AC at E, and the altitude from C intersect line segment AB at D. If AC and AB are unequal, derive a formula which gives the square of BC in terms of AC, AB, EO, and OD.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $a = BC,\ b = CA,\ c = AB,\ d = OD,\ e = EO,\ f = EA$, and $g = AD$. Applying the Pythagorean Theorem to $\triangle ABE,\ \triangle BCE,\ \triangle OEA$ and $\triangle OAD$, and using the fact that $\triangle ABE \sim \triangle CAD$, because they are both right triangles with common angle at vertex A, we obtain:

$$\begin{aligned} c^2 &= AB^2 = BE^2 + EA^2 = BC^2 - CE^2 + EA^2 = a^2 - (b-f)^2 + f^2 = a^2 - b^2 + 2bf, \\ e^2 + f^2 &= EO^2 + EA^2 = OA^2 = OD^2 + AD^2 = d^2 + g^2, \text{ and} \end{aligned}$$

$$\frac{b}{g} = \frac{CA}{AD} = \frac{AB}{EA} = \frac{c}{f}.$$

From these two last lines, we obtain

$$\begin{aligned} e^2 + f^2 &= d^2 + \frac{b^2 f^2}{c^2} \\ c^2e^2 + c^2f^2 &= c^2d^2 + b^2f^2, \end{aligned}$$

and since $b \neq c$ by hypothesis, we see that $f^2 = \dfrac{c^2\left(e^2 - d^2\right)}{b^2 - c^2}$, and from the equality $c^2 = a^2 - b^2 + 2bf$ gives us a^2 in terms b, c, e and d. Namely,

$$a^2 = \left(b^2 + c^2 - 2bf\right) = b^2 + c^2 - 2bc\sqrt{\frac{e^2 - d^2}{b^2 - c^2}}.$$

Solution 2 by Kee-Wai Lau,Hong Kong, China

By the cosine formula, we have

$\dfrac{EO}{OA} = \sin\angle OAE = \cos\angle ACB = \dfrac{AC^2 + BC^2 - AB^2}{2(AC)(BC)}$, and similarly

$\dfrac{OD}{OA} = \sin\angle OAD = \cos\angle ABC = \dfrac{AB^2 + BC^2 - AC^2}{2(AB)(BC)}$. Hence,

$$\frac{EO}{OD} = \frac{AB(AC^2 + BC^2 - AB^2)}{AC(AB^2 + BC^2 - AC^2)}. \qquad (1)$$

Since $AC \neq AB$, so $\dfrac{EO}{OD} \neq \dfrac{AB}{AC}$ or $(AB)(OD) - (AC)(EO) \neq 0$. Solving (1) for BC^2 we obtain

$$BC^2 = \frac{(AB + AC)(AB - AC)\left((AB)(OD) + (AC)(EO)\right)}{(AB)(OD) - (AC)(EO)}.$$

Also solved by Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Vijaya Prasad Nalluri, Rajahmundry, India; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5453:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu,"George Emil Palade" School, Buzău, Romania*

If $a, b, c \in (0, 1)$ or $a, b, c \in (1, \infty)$ and m, n are positive real numbers, then prove that

$$\frac{\log_a b + \log_b c}{m + n\log_a c} + \frac{\log_b c + \log_c a}{m + n\log_b a} + \frac{\log_c a + \log_a b}{m + n\log_c b} \geq \frac{6}{m+n}$$

Solution 1 by Moti Levy, Rehovot, Israel

Let $x := \log_a b, \quad y := \log_b c, \quad z := \log_c a$. Then $xyz = \log_a b \log_b c \log_c a = 1$, and the condition $a, b, c \in (0,1)$ or $a, b, c \in (1, \infty)$ implies that $x, \ y, \ z > 0$.

The original inequality may be rephrased as:

$$\frac{x+y}{m+z^{-1}n} + \frac{y+z}{m+x^{-1}n} + \frac{z+x}{m+y^{-1}n} \geq \frac{6}{m+n}, \quad xyz = 1, \quad x, y, z > 0, \tag{1}$$

or as

$$\frac{3}{\sum_{cyc}\left(\frac{m+z^{-1}n}{x+y}\right)^{-1}} \leq \frac{m+n}{2}.$$

Since the harmonic mean is less than or equal to the geometric mean,

$$\frac{3}{\sum_{cyc}\left(\frac{m+z^{-1}n}{x+y}\right)^{-1}} \leq \sqrt[3]{\frac{m+z^{-1}n}{x+y}\frac{m+x^{-1}n}{y+z}\frac{m+y^{-1}n}{z+x}}.$$

Hence it is enough to prove (2):

$$\frac{m+z^{-1}n}{x+y}\frac{m+x^{-1}n}{y+z}\frac{m+y^{-1}n}{z+x} \leq \frac{(m+n)^3}{8},$$

$$\frac{1}{xyz}\frac{(n+mz)(n+mx)(n+my)}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8},$$

$$\frac{(n+mz)(n+mx)(n+my)}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8}. \tag{2}$$

Further simplification of (2) results in

$$\frac{n^3 + mn^2x + mn^2y + mn^2z + m^2nxy + m^2nxz + m^2nyz + m^3xyz}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8}$$

$$\frac{n^3 + mn^2(x+y+z) + m^2n(xy+yz+xz) + m^3}{(x+y)(x+z)(y+z)} \leq \frac{(m+n)^3}{8} \tag{3}$$

Equating the left and right sides of (3) shows that the inequality (3) is equivalent to (4) and (5):

$$\frac{x+y+z}{(x+y)(x+z)(y+z)} \leq \frac{3}{8}, \tag{4}$$

$$\frac{xy+yz+xz}{(x+y)(x+z)(y+z)} \leq \frac{3}{8}. \tag{5}$$

We now use the p, q, r notation:

$$p := x+y+z,$$
$$q := xy+yz+zx,$$
$$r := xyz.$$

In this notation, (4) and (5) become

$$\frac{p}{pq-r} \leq \frac{3}{8}, \tag{6}$$

$$\frac{q}{pq-r} \leq \frac{3}{8}. \tag{7}$$

In our case $r = 1$, which implies (by AM-GM inequality) that $p \geq 3$ and $q \geq 3$. Now proving (4) and (5) is straightforward:

$$\begin{aligned}
\frac{p}{pq-1} &\leq \frac{3}{8}, \\
3pq - 3 - 8p &\geq 0, \\
3pq - 3 - 8p &\geq p - 3 \geq 0.
\end{aligned}$$

$$\begin{aligned}
\frac{q}{pq-1} &\leq \frac{3}{8}, \\
3pq - 3 - 8q &\geq 0, \\
3pq - 3 - 8q &\geq q - 3 \geq 0.
\end{aligned}$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Note that since $\log_a b = \dfrac{\ln b}{\ln a}$ and $a, b, c \in (0,1)$ or $a, b, c \in (0,1)$, all the logarithms in the proposed inequality are positive, so the right-hand side is positive.

We will apply the following parametrized Nesbitt's inequality (see reference 1, theorem 7).

Let $x,\ y,\ z,\ tx + ky + lz,\ ty + kz + lx,\ tz + kx + ly$ be positive real numbers and let $-k - l < t < \dfrac{k+l}{2}$.

Then $\dfrac{x}{tx + ky + lz} + \dfrac{y}{ty + kz + lx} + \dfrac{z}{tz + kx + ly} \geq 3t + k + l.$ (1)

We will consider two inequalities, from which the stated problem will follow.

$$\frac{\log_a b}{m + n \log_a c} + \frac{\log_b c}{m + n \log_b a} + \frac{\log_c a}{m + n \log_c b} \geq \frac{3}{m+n} \qquad (2)$$

$$\frac{\log_b c}{m + n \log_a c} + \frac{\log_c a}{m + n \log_b a} + \frac{\log_a b}{m + n \log_c b} \geq \frac{3}{m+n}. \qquad (3)$$

Notice that the right-hand side of (2) is

$$RHS = \frac{\ln b}{m \ln a + n \ln c} + \frac{\ln c}{m \ln b + n \ln a} + \frac{\ln a}{m \ln c + n \ln b} \geq \frac{3}{m+n}$$

by the parametrized Nesbitt's inequality with $t = 0$, $k = m$ and $l = n$, and $x = \ln b$, $y = \ln c$, and $z = \ln a$. It also should be noticed that in the last expression we may assume that all the ln's are positive.

Now, the right-hand side of (3) is

$$RHS = \frac{\ln a \ln c}{m \ln a \ln b + n \ln b \ln c} + \frac{\ln a \ln b}{m \ln b \ln c + n \ln a \ln c} + \frac{\ln b \ln c}{m \ln a \ln c + n \ln a \ln b} \geq \frac{3}{m+n}$$

by the parametrized Nesbitt's inequality with $t = 0$, $k = m$ and $l = n$, and $x = \ln a \ln c$, $y = \ln a \ln b$, and $z = \ln b \ln c$.

References:

(1) Shanhe Wu and Ovidiu Furdui, *A note on a conjectured Nesbitt type inequality*, Taiwanese Journal of Mathematics, 15 (2) (2011), 449-456.

Solution 3 by Soumitra Mandal, Chandar Nagore, India

$$\sum_{cyc} \frac{\log_a b + \log_b c}{m + n \log_a c} = \sum_{cyc} \frac{\log b + \frac{\log a \cdot \log c}{\log b}}{m \log a + n \log c}$$

$$= \sum_{cyc} \frac{\log b}{m \log a + n \log c} + \sum_{cyc} \frac{\frac{\log a \cdot \log c}{\log b}}{m \log a + n \log c}$$

$$= \sum_{cyc} \frac{(\log b)^2}{n \log a \cdot \log b + n \log c \cdot \log b} + \sum_{cyc} \frac{\left(\frac{1}{\log b}\right)^2}{\frac{m}{\log b \cdot \log c} + \frac{n}{\log b \cdot \log a}}$$

$$\overbrace{\geq}^{BERGSTROM} \frac{(\log a + \log b + \log c)^2}{(m+n)(\log a \cdot \log b + \log b \cdot \log c + \log c \cdot \log a)} +$$

$$+ \frac{\left(\frac{1}{\log a} + \frac{1}{\log b} + \frac{1}{\log c}\right)^2}{(m+n)\left(\frac{1}{\log a \cdot \log b} + \frac{1}{\log b \cdot \log c} + \frac{1}{\log c \cdot \log a}\right)} \geq \frac{3}{m+n} + \frac{3}{m+n} = \frac{6}{m+n}$$

Editor's Comments: **Anna V. Tomova of Varna, Bulgaria** approached the solution as follows: She showed that the left hand side of the inequality can be put into the canonical form of $X + Y + \dfrac{1}{XY}$. She then showed that this canonical form has a global minimum at $(1, 1)$, forcing it to have a minimal value of 3, and working with this she produced the final result.

Bruno Salgueiro Fanego of Viveiro, Spain noted that the stated problem is a specific case of a more general result. Namely: If $x, y, z \in (0, \infty)$ and $xyz = 1$, then

$$\frac{x+y}{m + \frac{n}{z}} + \frac{y+z}{m + \frac{n}{x}} + \frac{z+x}{m + \frac{n}{y}} \geq \frac{6}{m+n}.$$

He proved the more general result, and applied it to the specific case.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray of Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Shravan Sridhar, Udupi, India; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova of Varna, Bulgaria, and the proposer.

5454: *Proposed by Arkady Alt, San Jose, CA*

Prove that for integers k and l, and for any $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, the following inequality holds:

$$k^2 \tan \alpha + l^2 \tan \beta \geq \frac{2kl}{\sin(\alpha + \beta)} - \left(k^2 + l^2\right) \cot(\alpha + \beta).$$

Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the inequality by transposing

1) $k^2\left(\dfrac{\sin a}{\cos a}+\dfrac{\cos(a+b)}{\sin(a+b)}\right)+t^2\left(\dfrac{\sin b}{\cos b}+\dfrac{\cos(a+b)}{\sin\,(a+b)}\right)\geq\dfrac{2kt}{\sin(a+b)}$

Multiplying by $\sin(a+b)$

2) $k^2\left(\dfrac{\sin a(\sin(a+b))}{\cos a}+\cos(a+b)\right)+t^2\left(\dfrac{\sin b\sin(a+b)}{\cos b}+\cos(a+b)\right)\geq 2kt$

3) $k^2\left(\dfrac{\sin a\sin(a+b)+\cos a\cos(a+b)}{\cos a}\right)+t^2\left(\dfrac{\sin b\sin(a+b)+\cos b\cos(a+b)}{\cos b}\right)\geq 2kt$

4) $k^2\left(\dfrac{\cos b}{\cos a}\right)+t^2\left(\dfrac{\cos a}{\cos b}\right)\geq 2kt$

5) $\dfrac{k^2\cos^2 b+t^2\cos^2 a}{\cos a\cos b}\geq 2kt$

6) $k^2\cos^2 b+t^2\cos^2 a\geq 2kt\cos a\cos b$, and transposing,

7) $(k\cos b-t\cos a)^2\geq 0.$.

So we retrace our steps to obtain the original inequality.

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

First we consider the case when $\alpha+\beta=\dfrac{\pi}{2}$, then $\sin(\alpha+\beta)=1$, $\cot(\alpha+\beta)=0$, and $\tan\beta=\cot\alpha$. From these we have

$$k^2\tan\alpha+t^2\tan\beta-\frac{2kl}{\sin(\alpha+\beta)}+\left(k^2+l^2\right)\cot(\alpha+\beta)=k^2\tan\alpha+l^2\cot\alpha-2lk=\left(k\sqrt{\tan\alpha}-l\sqrt{\cot\alpha}\right)^2\geq 0,$$

which completes the proof when $\alpha+\beta=\dfrac{\pi}{2}$.

Now suppose that $\alpha+\beta\neq\dfrac{\pi}{2}$. By using the identity $\cot(\alpha+\beta)=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$, we have

$$\begin{aligned}
& k^2\tan\alpha+l^2\tan\beta+(k^2+l^2)\cot(\alpha+\beta)-\frac{2kl}{\sin(\alpha+\beta)}\\
= & k^2\tan\alpha+l^2\tan\beta+(k^2+l^2)\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}\\
= & \frac{k^2\tan^2\alpha+k^2\tan\alpha\tan\beta+l^2\tan\beta\tan\alpha+l^2\tan^2\beta+(k^2+l^2)-\left(k^2+l^2\right)\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}\\
= & \frac{k^2\tan^2\alpha+l^2\tan^2\beta+(k^2+l^2)}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}\\
= & \frac{k^2(1+\tan^2\alpha)+l^2(1+\tan^2\beta)}{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}-\frac{2kl}{\sin(\alpha+\beta)}
\end{aligned}$$

$$= \frac{\dfrac{k^2\sec^2\alpha + l^2\sec^2\beta}{\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}}} {} - \frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{k^2\dfrac{\cos\beta}{\cos\alpha} + l^2\dfrac{\cos\alpha}{\cos\beta}}{\sin(\alpha+\beta)} - \frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{\left(\sqrt{k\dfrac{\cos\beta}{\cos\alpha}} - l\sqrt{k\dfrac{\cos\alpha}{\cos\beta}}\right)^2}{\sin(\alpha+\beta)} \geq 0.$$

Editor's Note: Most of the solvers mentioned that the inequality holds for all real values of k and l. **David Stone and John Hawkins of Georgia Southern University** when a bit further. They stated: "the conditions that α and β be first quadrant angles is an easy way to make $\sin(\alpha+\beta) \neq 0$ and $\tan\alpha, \tan\beta, \cot(\alpha+\beta)$ be defined and guarantee that $\cos\alpha\cos\beta\sin(\alpha+\beta) > 0$." But the proof shows that the inequality would be true for any values of α and β which satisfy these conditions.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Daniel Sitaru, "Theodor Costescu" National Economic College, Severin Mehedinti; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5455: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations:

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{abc} \\ a+b+c &= abc + \frac{8}{27}(a+b+c)^3 \end{aligned}$$

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Suppose a, b, c are real numbers satisfying our system. Consider the polynomial

$$\begin{aligned} g(x) &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \end{aligned}$$

The first equation of our original system implies $ab+ac+bc = 1$. So

$$g(x) = x^3 - \lambda x^2 + x - \mu$$

where $\lambda = a+b+c$ and $\mu = abc$. Note that the second equation of our original system can be written as $\lambda = \mu + \frac{8}{27}\lambda^3$. We make the usual substitution to get a depressed cubic:

$g(x+\lambda/3) = x^3 + px + q$ where

$$p = 1 - \frac{1}{3}\lambda^2 \text{ and } q = \frac{-2}{27}\lambda^3 + \frac{1}{3}\lambda - \mu.$$

Using $\lambda = \mu + \frac{8}{27}\lambda^3$ we have

$$q = \frac{2}{9}\lambda^3 - \frac{2}{3}\lambda$$

which we factor to get

$$q = \frac{-2}{3}\lambda\left(1 - \frac{1}{3}\lambda^2\right) = \frac{-2}{3}\lambda p.$$

The discriminant of $g(x+\lambda/3)$ is

$$\begin{aligned} D &= -4p^3 - 27q^2 \\ &= -4p^3 - 12\lambda^2 p^2 \\ &= -4p^2(p + 3\lambda^2) \\ &= -4p^2\left(1 + \frac{8}{3}\lambda^2\right) \end{aligned}$$

Note that $D \geq 0$ if and only if $p = 0$. Recall that a real cubic polynomial has three real roots if and only if its discriminant is ≥ 0. Thus $g(x+\lambda/3)$ has three real roots if and only if $p = 0$ if and only if $\lambda = \pm\sqrt{3}$. Note that when $\lambda = \pm\sqrt{3}$ we have $g(x+\lambda/3) = x^3$, and hence $g(x) = (x - \lambda/3)^3$. Therefore the only solutions to the original system are

$$a = b = c = \frac{\sqrt{3}}{3} \text{ and } a = b = c = \frac{-\sqrt{3}}{3}.$$

Solution 2 by Moti Levy, Rehovot, Israel

$\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{abc}$ implies $ab + bc + ca = 1$.

Substitution of $abc = \dfrac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ in the second equation gives

$$\begin{aligned} a + b + c - \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} - \frac{8}{27}(a+b+c)^3 &= 0, \\ \frac{(a+b)(a+c)(b+c)}{ab + ac + bc} - \frac{8}{27}(a+b+c)^3 &= 0, \\ (a+b)(a+c)(b+c) - \frac{8}{27}(a+b+c)^3 &= 0. \qquad (1) \end{aligned}$$

Let $x = a + b$, $\; y = b + c$ and $z = c + a$ then (1) becomes

$$xyz - \left(\frac{x+y+z}{3}\right)^3 = 0,$$

or $\sqrt[3]{xyz} = \dfrac{x+y+z}{3}$. The geometric mean is equal to the arithmetic mean if and only if $x = y = z$ which implies that $a = b = c$.

Therefore the system of equation has only two solutions:

$$a = b = c = \frac{1}{\sqrt{3}}, \qquad a = b = c = -\frac{1}{\sqrt{3}}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let $p = a + b + c,\ q = ab + bc + ca$, and $r = abc$. The first given equation becomes

$$q = 1 \tag{1}$$

and the second equations becomes

$$r = p - \frac{8p^3}{27}. \tag{2}$$

It can be checked readily that

$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 = (a-b)^2(b-c)^2(c-a)^2. \tag{3}$$

Using (1) and (2) we reduce the left side of (3) to $\dfrac{-4(p^2-3)^2(8p^2+3)}{27}$, which is non-positive. Since the right side of (3) is nonnegative, so both sides of (3) equal to zero. It follows that $p^2 = 3$ and by (2), $r = \dfrac{p}{9}$. Moreover, either $a = b$ or $b = c$ or $c = a$. By symmetry we only consider the case $a = b$. Hence either $2a+c = \sqrt{3},\ a^2+2ac = 1$, or $2a+c = -\sqrt{3},\ a^2+2ac = 1$, giving respectively $a = c = \dfrac{1}{\sqrt{3}}$ and $a = c = \dfrac{-1}{\sqrt{3}}$. Thus the solutions to the original system are precisely $(a,b,c) = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right),\ \left(\dfrac{-1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}\right)$.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Le Van, Ho Chi Minh City, Vietnam; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5456: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let k be a positive integer. Calculate

$$\lim_{x\to\infty} \mathrm{e}^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(\mathrm{e}^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Solution 1 Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

By the Taylor formula we have

$$e^x = 1 + x + x^2/2! + \cdots + x^n/n! + \int_0^x \frac{(x-t)^n}{n!}e^t dt.$$

It follows that

$$\begin{aligned}
& e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x - 1 - x - x^2/2! - \cdots - x^n/n!\right)\\
= & \sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\int_0^x \frac{(x-t)^n}{n!}e^{t-x}dt\\
= & \frac{1}{k!}\sum_{n=k}^{\infty}(-1)^n\int_0^x \frac{t^n}{(n-k)!}e^{-t}dt\\
= & \frac{1}{k!}\sum_{n=0}^{\infty}(-1)^{n+k}\int_0^x \frac{t^{n+k}}{n!}e^{-t}dt\\
= & \frac{(-1)^k}{k!}\int_0^x t^k e^{-2t}dt.
\end{aligned}$$

The limit as $x \to +\infty$ is given by

$$\frac{(-1)^k}{k!}\int_0^\infty t^k e^{-2t}dt = \frac{(-1)^k}{2^{k+1}}.$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\lim_{x\to} e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x-1-x-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\right)$$

$$= \lim_{x\to\infty} e^{-x}\sum_{n=k}^{\infty}(-1)^n\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\sum_{p=n+1}^{\infty}\frac{x^p}{p!}$$

$$= \lim_{x\to\infty} e^{-x}\frac{1}{k!}\sum_{n=k}^{\infty}(-1)^n\left(\frac{d^k}{dt^k}t^n\right)\Bigg|_{t=1}\sum_{p=n+1}^{\infty}\frac{x^p}{p!} = \frac{1}{k!}\lim_{x\to\infty} e^{-x}\frac{d^k}{dt^k}\left(\sum_{n=k}^{\infty}(-1)^n t^n\sum_{p=n+1}^{\infty}\frac{x^p}{p!}\right)\Bigg|_{t=1}$$

$$= \frac{1}{k!}\lim_{x\to\infty} e^{-x}\frac{d^k}{dt^k}\left(\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{p=0}^{n-1}(-1)^p t^p\right)\Bigg|_{t=1} = \frac{1}{k!}\lim_{x\to\infty} e^{-x}\frac{d^k}{dt^k}\left(\frac{1}{1+t}\left(\sum_{n=1}^{\infty}\frac{x^n}{n!}-\sum_{n=1}^{\infty}\frac{(-xt)^n}{n!}\right)\right)\Bigg|_{t=1}$$

$$= \frac{1}{k!}\lim_{x\to\infty}\frac{d^k}{dt^k}\left(\frac{1}{1+t}e^{-x}\left(e^x-1-(e^{-x}-1)\right)\right)\Bigg|_{t=1} = \frac{1}{k!}\lim_{x\to\infty}\frac{d^k}{dt^k}\left(\frac{1}{1+t}\left(1-e^{-x(1+t)}\right)\right)\Bigg|_{t=1}$$

$$= \frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{1}{1+t}\lim_{x\to\infty}\left(1-e^{-x(1+t)}\right)\right)\Bigg|_{t=1} = \frac{1}{k!}\left(\frac{d^k}{dt^k}\frac{1}{1+t}\right)\Bigg|_{t=1} = \frac{(-1)^k}{(1+t)^{k+1}}\Bigg|_{t=1} = \frac{(-1)^k}{2^{k+1}}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Repeated integration by parts yields

$$\int_0^x e^{-t}\frac{t^n}{n!}dt = -e^{-x}\frac{x^n}{n!}+\int_0^x d^{-t}\frac{t^{n-1}}{(n-1)!}dt = -e^{-x}\left(\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+\cdots+x\right)+\int_0^x e^{-t}dt$$

$$= -e^{-x}\left(\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+\cdots+x+1\right)+1.$$

So,

$$e^{-x}-\left(\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+\cdots+x+1\right) = e^x\int_0^x e^{-t}\frac{t^n}{n!}dt$$

and

$$e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x-\left(\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+\cdots+x+1\right)\right) = \frac{1}{k!}\sum_{n=k}^{\infty}\frac{(-1)^n}{(n-k)!}\left(\int_0^x e^{-t}t^n dt\right)$$

$$= \frac{(-1)^k}{k!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_0^x e^{-t}t^{n+k}dt\right)$$

$$= \frac{(-1)^k}{k!}\int_0^x e^{-2t}t^k dt \to \frac{(-1)^k}{k!}\int_0^{\infty} e^{-2t}t^k dt = \frac{(-1)k!}{k!2^{k+1}} = \frac{(-1)^k}{2^{k+1}}, \text{as } x \to \infty.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the given limit equals $(-1)^k 2^{-(k+1)}$.

For real x let $f(x) = e^{-x}x^k = \displaystyle\sum_{m=k}^{\infty}(-1)^{m-k}\frac{x^m}{m-k)!}$ so that

$$f^n(0) = \begin{cases} 0, & 0 \le n \le k-1 \\ (-1)^{n-k}n(n-1)\cdots(n-k+1), & n \ge k \end{cases}.$$

where $f^n(x)$ is the n th derivative of $f(x)$.

According to problem $3.89(a)$ on $\text{pp}124, 227$ of the book [Ovidiu Furdui; Limits, Series, and Fractional Part Integrals, Springer 2013] we have

$$\sum_{n=0}^{\infty} f^n(0)\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t}f(t)dt.$$

Hence,

$$\begin{aligned} e^{-x}\sum_{n=k}^{\infty}(-1)^n\binom{n}{k}\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right) &= \frac{(-1)^k}{k!}\int_0^x e^{-2t}t^k dt \\ &= \frac{(-1)^k}{2^{k+1}k!}\int_0^{2x} e^{-t}t^k dt. \end{aligned}$$

Now our result for the limit follows from the well-known fact that $\displaystyle\int_0^{\infty} e^{-t}t^k dt = k!$.

Also solved by Moti Levy, Rehovot, Israel; Anna V. Tomova, Varna, Bulgaria, and the proposer.

Editor's Comment: In Anna's solution to 5456 she acknowledged contributing conversations with **Peter Breuer and Joachim Domsta of Bulgaria.**

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2019

5523: *Proposed by Kenneth Korbin, New York, NY*

For every prime number P, there is a circle with diameter $4P^4+1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area $(2P)(2P+1)(2P-1)(2P^2-1)$. Find the sides of the triangles if $P=2$ and if $P=3$.

5524: *Proposed by Michael Brozinsky, Central Islip, NY*

A billiard table whose sides obey the law of reflection is in the shape of a right triangle ABC with legs of length a and b where $a>b$ and hypotenuse c. A ball is shot from the right angle and rebounds off the hypotenuse at point P on a path parallel to leg CB that hits let CA at point Q. Find the ratio $\dfrac{AQ}{QC}$.

5525: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu-Severin, Mehedinti, Romania*

Find real values for x and y such that:

$$4\sin^2(x+y)=1+4\cos^2 x+4\cos^2 y.$$

5526: *Proposed by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece*

The lengths of the sides of a triangle are 12, 16 and 20. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

5527: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a,b and c be positive real numbers such that $a+b+c=3$. Prove that for all real $\alpha>0$, holds:

$$\frac{1}{2}\left(\frac{1-a^{\alpha+1}b^{\alpha}}{a^{\alpha}b^{\alpha}}+\frac{1-b^{\alpha+1}c^{\alpha}}{b^{\alpha}c^{\alpha}}+\frac{1-c^{\alpha+1}a^{\alpha}}{c^{\alpha}a^{\alpha}}\right)$$

$$\leq \sqrt{\left(\frac{1-a^{\alpha+1}}{a^\alpha}+\frac{1-b^{\alpha+1}}{b^\alpha}+\frac{1-c^{\alpha+1}}{c^\alpha}\right)\left(\frac{1-a^\alpha b^\alpha c^\alpha}{a^\alpha b^\alpha c^\alpha}\right)}.$$

5528: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a>0$. Calculate $\displaystyle\int_a^\infty \int_a^\infty \frac{\mathrm{d}x\mathrm{d}y}{x^6(x^2+y^2)}$.

Solutions

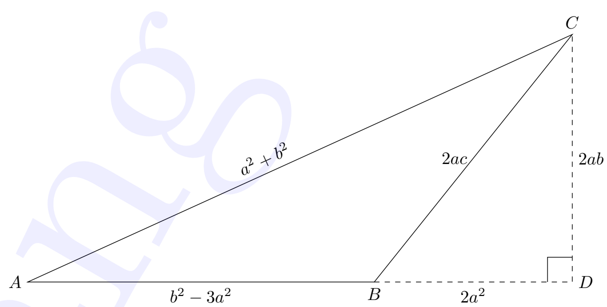
5505: *Proposed by Kenneth Korbin, New York, NY*

Given a Primitive Pythagorean Triple (a,b,c) with $b^2>3a^2$. Express in terms of a and b the sides of a Heronian Triangle with area $ab(b^2-3a^2)$.

(A Heronian Triangle is a triangle with each side length and area an integer.)

Solution 1 by Stanley Rabinowitz, Chelmsford, MA

One way of doing this would be to form an obtuse triangle ABC as shown with base of length b^2-3a^2 and altitude of length $2ab$, so that the area of $\triangle ABC$ is $ab(b^2-3a^2)$ as desired. If the line segment from B to D, the foot of the altitude from C, has length $2a^2$, then hypotenuse BC in $\triangle BDC$ would have length $2ac$, since this triangle would be similar to an a–b–c right triangle, scaled up by $2a$. Then AD would have length b^2-a^2, and by the Pythagorean Theorem, AC would have length a^2+b^2.
Thus, $\triangle ABC$ is the desired Heronian Triangle, with sides b^2-3a^2, $2a\sqrt{a^2+b^2}$, and a^2+b^2.



Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Given a primitive Pythagorean triple (a,b,c) with $b^2>3a^2$, let

$$\begin{aligned} x &= b^2-3a^2, \\ y &= 2a\sqrt{a^2+b^2}, \\ z &= a^2+b^2. \end{aligned}$$

Note that $y=2ac$ and $z=c^2$. Since $c^2-4a^2=b^2-3a^2>0$ we have $c>2a$.

We calculate

$$\begin{aligned}
-x+y+z &= -(b^2-3a^2)+2ac+(a^2+b^2) \\
&= 4a^2+2ac>0,
\end{aligned}$$

$$\begin{aligned}
x-y+z &= (b^2-3a^2)-2ac+c^2 \\
&= (b^2-3a^2)+c(c-2a)>0,
\end{aligned}$$

and

$$\begin{aligned}
x+y-z &= (b^2-3a^2)+2ac-(a^2+b^2) \\
&= 2a(c-2a)>0.
\end{aligned}$$

Thus $x+y>z$, $x+z>y$, and $y+z>x$ so (x,y,z) gives the sides of a Heronian triangle. Let s be the semiperimeter and A the area of this triangle.
By Heron's formula we have

$$A^2 = s(s-x)(s-y)(s-z).$$

We have

$$\begin{aligned}
s &= \frac{x+y+z}{2} \\
&= b^2-a^2+ac,
\end{aligned}$$

$$\begin{aligned}
s-x &= b^2-a^2+ac-(b^2-3a^2) \\
&= ac+2a^2,
\end{aligned}$$

$$\begin{aligned}
s-y &= b^2-a^2+ac-2ac \\
&= b^2-a^2-ac,
\end{aligned}$$

and

$$\begin{aligned}
s-z &= b^2-a^2+ac-(a^2+b^2) \\
&= ac-2a^2.
\end{aligned}$$

So

$$\begin{aligned}
A^2 &= (b^2-a^2+ac)(ac+2a^2)(b^2-a^2-ac)(ac-2a^2) \\
&= [(b^2-a^2)^2-(ac)^2][(ac)^2-(2a^2)^2].
\end{aligned}$$

Now

$$\begin{aligned}
(b^2-a^2)^2-(ac)^2 &= b^4-2a^2b^2+a^4-a^2(a^2+b^2) \\
&= b^4-3a^2b^2 \\
&= b^2(b^2-3a^2)
\end{aligned}$$

and

$$\begin{aligned}
(ac)^2-(2a^2)^2 &= a^2(c^2-4a^2) \\
&= a^2(b^2-3a^2)
\end{aligned}$$

so $A^2 = a^2b^2(b^2-3a^2)^2$.

Thus if (a,b,c) is a primitive Pythagorean triple with $b^2 > 3a^2$ then (x,y,z) with

$$x = b^2 - 3a^2, \quad y = 2a\sqrt{a^2+b^2}, \quad z = a^2+b^2$$

is a Heronian triangle with area $ab(b^2-3a^2)$.

N.B. For a particular (a,b,c) there can be other Heronian triangles with area $ab(b^2-3a^2)$. For example, for the primitive Pythagorean triple $(5,12,13)$ we are looking for a Heronian triangle with area 4140. The formulas above give the triangle $(69,130,169)$, but $(41,202,207)$ is another triangle with area 4140.

Solution 3 by Trey Smith, Angelo State University, San Angelo, TX

Let $x = b^2-3a^2$, $y = 2a\sqrt{a^2+b^2}$, and $z = a^2+b^2$ be the lengths of the three sides of the triangle. We first observe that all of these are positive integers; x and z obviously so, and y since $a^2+b^2=c^2$, so that

$$2a\sqrt{a^2+b^2} \quad = \quad 2a\sqrt{c^2} \quad = \quad 2ac.$$

The perimeter of the triangle is

$$\begin{aligned} & x+y+z \\ = \ & (b^2-3a^2) + (2a\sqrt{a^2+b^2}) + (a^2+b^2) \\ = \ & (c^2-4a^2) + 2ac + c^2 \\ = \ & 2c^2 + 2ac - 4a^2. \end{aligned}$$

Then the semiperimeter is $s = c^2 + ac - 2a^2$. Applying Heron's formula to find the area A, we have

$$\begin{aligned} & A^2 \\ = \ & s(s-x)(s-y)(s-z) \\ = \ & s(s-(b^2-3a^2))(s-(2a\sqrt{a^2+b^2}))(s-(a^2+b^2)) \\ = \ & s(s-(c^2-4a^2))(s-2ac)(s-c^2) \\ = \ & (c^2+ac-2a^2)((c^2+ac-2a^2)-(c^2-4a^2))((c^2+ac-2a^2)-2ac)((c^2+ac-2a^2)-c^2) \\ = \ & (c^2+ac-2a^2)(ac+2a^2)(c^2-ac-2a^2)(ac-2a^2) \\ = \ & [(c+2a)(c-a)][a(c+2a)][(c+a)(c-2a)][a(c-2a)] \\ = \ & a^2(c^2-a^2)(c^2-4a^2)^2 \\ = \ & a^2b^2(b^2-3a^2)^2. \end{aligned}$$

Thus $A = ab(b^2 - 3a^2)$.

Editor's Comment : David Stone and John Hopkins of Georgia Southern University added the following comment to their solution to this problem: "So how did we find x, y, z? We first tired the simplest possible example; $(a, b, c) = (3, 5, 12)$. After some algebra and some computer help, we found the triangle $(x, y, z) = (69, 169, 130)$ has the appropriate area. From this we conjectured the form for arbitrary x, y, z.

$$\begin{aligned} x &= 169 = 13^2 = c^2 \\ y &= 69 = 12^2 - 3 \cdot 5^2 = b^2 - 3a^2 \\ z &= 130 = 2 \cdot 5 \cdot 13 = 2ac. \end{aligned}$$

Then it only required simple algebra to verify this construction. Some Excel computations also lead us to the broader result (when $b^2 < 3a^2$). The perfect example of computing power assisting a person!"

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5506: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Find $\Omega = \det\left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100}\right]$.

Solution 1 by Michel Bataille, Ronen, France

Let $A = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$, $B = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$, $O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It is readily checked that $AB = BA = O_2$ and $A + B = 26I_2$.
Since $AB = BA$, the binomial theorem gives

$$(A + B)^{100} = \sum_{k=0}^{100} \binom{100}{k} A^k B^{100-k}. \tag{1}$$

Now, if $k \in \{1, 2, \ldots, 50\}$, then

$$A^k B^{100-k} = A^k B^k B^{100-2k} = (AB)^k B^{100-2k} = O_2 \cdot B^{100-2k} = O_2$$

(note that $A^k B^k = (AB)^k$ since $AB = BA$) and similarly, if $k \in \{51, 52, \ldots, 99\}$, then $A^k B^{100-k} = A^{2k-100}(AB)^{100-k} = O_2$.

As a result, (1) gives $(A + B)^{100} = A^{100} + B^{100}$, that is, $26^{100} I_2 = A^{100} + B^{100}$. We can conclude:

$$\Omega = \det(26^{100} I_2) = 26^{200}.$$

Solution 2 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Observe

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} = \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 5 \end{bmatrix}\right)^{100} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \left(\begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)^{99} \begin{bmatrix} 1 & 5 \end{bmatrix} = 26^{99} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$$

and

$$\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} = \left(\begin{bmatrix} 5 \\ -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \end{bmatrix}\right)^{100} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}\left(\begin{bmatrix} 5 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}\right)^{99}\begin{bmatrix} 5 & -1 \end{bmatrix} = 26^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}.$$

It follows that

$$\Omega = \det\left[26^{99}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + 26^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 26^{100} & 0 \\ 0 & 26^{100} \end{pmatrix}\right] = 26^{200}.$$

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX

Let $A = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$ and $B = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$. Matrices A and B are each symmetric, hence orthogonally diagonalizable.

Solving the equation $\det(\lambda I - A) = 0$ yields $\lambda_1 = 0$ and $\lambda_2 = 26$ as the eigenvalues of A. Solving the equation $(\lambda I - A)\overrightarrow{x} = \overrightarrow{0}$ successively for $\lambda = 0$ and $\lambda = 26$ yields $\overrightarrow{x_1} = \begin{pmatrix} \frac{-5}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{pmatrix}$ and $\overrightarrow{x_2} = \begin{pmatrix} \frac{1}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} \end{pmatrix}$ as corresponding unit eigenvectors. So
$A = \begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}$. Similarly,
$B = \begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{-5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \\ \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}$.

Since for both A and B the matrix of eigenvectors is orthogonal, we have

$$A^{100} = \begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26^{100} \end{pmatrix}\begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix} = \begin{pmatrix} 26^{99} & 5\left(26^{99}\right) \\ 5\left(26^{99}\right) & 25\left(26^{99}\right) \end{pmatrix}, \text{ and}$$

$$B^{100} = \begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{-5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26^{100} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \\ \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix} = \begin{pmatrix} 25\left(26^{99}\right) & -5\left(26^{99}\right) \\ -5\left(26^{99}\right) & 26^{99} \end{pmatrix}.$$

So $\Omega = \det\left[A^{100} + B^{100}\right] = \det\begin{pmatrix} 26^{100} & 0 \\ 0 & 26^{100} \end{pmatrix} = 26^{200}$.

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A way to calculate A^n for a 2×2 matrix is to use the Hamilton-Cayley Theorem:

$$A^2 - \mathrm{Tr}(A) \cdot A + \det A \cdot I_2 = 0.$$

For example, if we have a 2×2 matrix $A = \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$ (or $A = \begin{pmatrix} a^2 & -a \\ -a & 1 \end{pmatrix}$) with $\det A = 0$ and $\mathrm{Tr}(A) = a^2 + 1$, then the Hamilton-Cayley theorem becomes:

$$A^2 \quad = \quad \mathrm{Tr}(A) = (a^2+1)^2 A.$$

$$\begin{aligned} A^3 &= (a^2+1)A^2 = (a^2+1)^2 A, \\ &\dots \\ A^n &= (a^2+1)A^{n-1} = (a^2+1)^{n-1}A. \end{aligned}$$

So we have:

$$\begin{aligned}
\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} &= (5^2+1)^{99}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = 26^{99}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}, \\
\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} &= (5^2+1)^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = 26^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}, \\
\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100} &= 26^{99}\left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}\right) = 26^{100}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

and finally we have:

$$\Omega = \det\left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100}\right) = \det\left(26^{100}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 26^{100}.$$

Solution 5 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $c = \sqrt{26}$. We know that

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = \begin{pmatrix} -5/c & 1/c \\ 1/c & 5/c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} -5/c & 1/c \\ 1/c & 5/c \end{pmatrix} \doteq A\Lambda A^{-1}$$

$$\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1/c & -5/c \\ 5/c & 1/c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} 1/c & 5/c \\ -5/c & 1/c \end{pmatrix} \doteq B\Lambda B^{-1}$$

$$\Omega = A\Lambda^{100}A^{-1} + B\Lambda^{100}B^{-1}$$

$$A\Lambda^{100}A^{-1} = \begin{pmatrix} 26^{99} & 5\cdot 26^{99} \\ 5\cdot 26^{99} & 25\cdot 26^{99} \end{pmatrix}$$

$$B\Lambda^{100}B^{-1} = \begin{pmatrix} 25\cdot 26^{99} & -5\cdot 26^{99} \\ -5\cdot 26^{99} & 26^{99} \end{pmatrix}$$

Thus

$$\Omega = \det\begin{pmatrix} 26^{99}\cdot 26 & 0 \\ 0 & 26^{99}\cdot 26 \end{pmatrix} = 26^{200}.$$

Also solved by Arkady Alt, San Jose, CA; Ashland University Undergraduate Problem Solving Group, Ashland University, Ashland, Ohio; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J.

Bevelacqua, University of North Dakota, Grand Forks, ND; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo Sate University, San Angelo, TX; Pat Costello, Eastern Kentucky University, Richmond, KY; David Diminnie, Texas Instruments Inc., Dallas, TX; Michael Faleski, University Center, MI; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Columbia Southern University, Orange Beach, AL; Ismayil Mammadzada (student), ADA University, Baku, Azerbaijan; Pedro Pantoja, Natal/RN, Brazil; Ravi Prakash, Oxford University Press; New Delhi, India; Neculai Stanciu "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Henry Ricardo (four different proofs), Westchester Area Math Circle, NY; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Marian Ursărescu, "Roman Vodă" College, Roman, Romania; Daniel Văcaru, Pitesti, Romania, and the proposer.

5507: *Proposed by David Benko, University of South Alabama, Mobile, AL*

A car is driving forward on the real axis starting from the origin. Its position at time $0 \le t$ is $s(t)$. Its speed is a decreasing function: $v(t), 0 \le t$. Given that the drive has a finite path (that is $\lim\limits_{t\to\infty} s < \infty$), that $v(2t)/v(t)$ has a real limit c as $t \to \infty$, find all possible values of c.

Solution 1 by Moti Levy, Rehovot, Israel

We will show that the set of all possible values of c, is the interval $\left[0, \frac{1}{2}\right]$, i.e., $0 \le c \le \frac{1}{2}$.

Let us summarize the conditions on the speed function $v(t)$:
1) $v(t) \ge 0$,
2) $v(t)$ is decreasing function for all $t \ge 0$,
3) $\int_0^\infty v(t)\, dt < \infty$
4) $\lim_{t\to\infty} \frac{v(2t)}{v(t)} = c$, c is real number.

Since $v(t) \ge 0$, then clearly $c \ge 0$. Since $v(t)$ is decreasing function, then $c \le 1$. It follows that $0 \le c \le 1$.

Now we show that c can attain any value in the interval $\left[0, \frac{1}{2}\right]$.
Let r be a real number and $r > 1$. Then $v(t) = \frac{1}{1+t^r}$ satisfies all four requirements from the speed function, in particular

$$\int_0^\infty \frac{1}{1+t^r} dt < \infty, \quad \text{for} \quad r > 1,$$

and

$$\lim_{t\to\infty} \frac{v(2t)}{v(t)} = \lim_{t\to\infty} \frac{1+t^r}{1+2^r t^r} = \frac{1}{2^r} = c.$$

It follows that $c \in \left(0, \frac{1}{2}\right)$.
To see that c can attain also the value zero, choose $v(t) = e^{-t}$.

To see that c can attain also the value $\frac{1}{2}$, choose $v(t) = \begin{cases} \frac{1}{2\ln^2 2}, & \text{for } 0 \le t \le 2, \\ \frac{1}{t\ln^2 t}, & \text{for } 2 < t. \end{cases}$

Then $v(t)$ satisfies all the four requirements from the speed function, in particular

$$\int_0^\infty v(t)\,dt = \frac{1}{\ln^2 2} + \frac{1}{\ln 2},$$

and

$$\lim_{t\to\infty} \frac{v(2t)}{v(t)} = \lim_{t\to\infty} \frac{t\ln^2 t}{2t\ln^2(2t)} = \frac{1}{2}.$$

To finish the proof, we have to show that $c \notin \left(\frac{1}{2}, 1\right]$.

Suppose $\lim_{t\to\infty} \frac{v(2t)}{v(t)} = c$, then for every $\varepsilon > 0$, there is a real number t_0 such that $t > t_0$ implies $\frac{v(2t)}{v(t)} > c - \varepsilon$.

Now we define a staircase function $s(t)$, as follows:

$$s(t) := (c-\varepsilon)^k v(t_0), \quad for \;\; 2^{k-1t_0}t_0 \le t < 2^k t_0, \qquad k = 1, 2, \ldots$$

Since the function $v(t)$ is positive decreasing function for all $t \ge 0$, then $v(t) \ge s(t)$, hence

$$\int_{t_0}^\infty v(t)\,dt \ge \int_{t_0}^\infty s(t)\,dt.$$

Integrating the staircase function, we get

$$\int_{t_0}^\infty s(t)\,dt = v(t_0)\sum_{k=1}^\infty (c-\varepsilon)^k 2^{k-1} = v(t_0)(c-\varepsilon)\sum_{k=0}^\infty (2(c-\varepsilon))^k.$$

If $c - \varepsilon \ge \frac{1}{2}$ then $\int_{t_0}^\infty s(t)\,dt$ diverges and so $\int_{t_0}^\infty v(t)\,dt$ diverges.

We conclude that if $c > \frac{1}{2}$ then $\int_{t_0}^\infty v(t)\,dt$ diverges, contradicting property 3) of the speed function.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that

$$0 \le c \le \frac{1}{2} \tag{1}$$

Let $\lim\limits_{t\to\infty} s(t) = L < \infty$. Then $\lim\limits_{t\to\infty} s(2t) = L$ and $0 \le s(t) < s(2t) < L$ for $t > 0$. Hence, by L'Hôpital's rule, we have

$$1 \ge \lim_{t\to\infty} \frac{L - s(2t)}{L - s(t)} = \lim_{t\to\infty} \frac{\dfrac{ds(2t)}{dt}}{\dfrac{ds(t)}{dt}} = 2\lim_{t\to\infty} \frac{v(2t)}{v(t)} = 2c.$$

Thus (1) holds.

By taking $s(t) = 1 - e^{-t}$, $s(t) = 1 - (t+1)^{\frac{\ln(2c)}{\ln 2}}$, $s(t) = 1 - \dfrac{1}{\ln(1+e)}$ according as $c = 0$, $0 < c < \dfrac{1}{2}$, $c = \dfrac{1}{2}$ we see that each c in (1) is in fact admissible.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We claim that the set C of possible values of c is the closed interval $\left[0, \dfrac{1}{2}\right]$.

Indeed, if $v(t)=v_0 e^{-t}$, then $v(t)$ is a decreasing function , $\int_0^\infty v(t)dt<\infty$, and $\lim_{t\to\infty}\frac{v(2t)}{v(t)}=0$. So $0\in C$.

If $a\geq 1$ and $v(t)=\frac{v_0}{1+t^a\ln^2(1+t)}$ then $v(t)$ is a decreasing function, $\int_0^\infty v(t)dt<\infty$ and

$\lim_{t\to\infty}\frac{v(2t)}{v(t)}=\lim_{t\to\infty}\frac{1+t^a\ln^2(1+t)}{1+2^a t^a\ln^2(1+2t)}=\frac{1}{2^a}$. So $\left(0,\frac{1}{2}\right]\subset C$. It remains to prove that if $c>\frac{1}{2}$ then c$\notin C$.

Suppose if possible that $\lim_{t\to\infty}\frac{v(2t)}{v(t)}=c$, where $c>\frac{1}{2}$. Let $\epsilon:=\frac{c-1/2}{2}>0$. Then there is a number $T=T(\epsilon)>0$ such that $-\epsilon<\frac{v(2t)}{v(t)}-c<\epsilon$, whenever $t>T$. We conclude that

$$\int_{2T}^\infty v(t)dt=2\int_T^\infty v(2t)dt>2(c-\epsilon)\int_T^\infty v(t)dt\geq 2(c-\epsilon)\int_{2T}^\infty v(t)dt=(1+2\epsilon)\int_{2T}^\infty v(t)dt>\int_{2T}^\infty v(t)dt,$$

which is a contradiction, and the proof is complete.

Also solved by the proposer.

5508: *Proposed by Pedro Pantoja, Natal RN, Brazil*

Let a,b,c be positive real numbers such that $a+b+c=1$. Find the minimum value of

$$f(a,b,c)=\frac{a}{3ab+2b}+\frac{b}{3bc+2c}+\frac{c}{3ca+2a}.$$

Solution 1 by Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, we note that since $a,b,c>0$ and $a+b+c=1$, the Arithmetic - Geometric Mean Inequality implies that

$$\begin{aligned}
a^2+b^2+c^2&=(a+b+c)\left(a^2+b^2+c^2\right)\\
&=a^3+b^3+c^3+ab^2+bc^2+ca^2+a^2b+b^2c+c^2a\\
&=\left(a^3+ab^2\right)+\left(b^3+bc^2\right)+\left(c^3+ca^2\right)+a^2b+b^2c+c^2a\\
&\geq 2\sqrt{a^4b^2}+2\sqrt{b^4c^2}+2\sqrt{c^4a^2}+a^2b+b^2c+c^2a\\
&=3\left(a^2b+b^2c+c^2a\right). \qquad (1)
\end{aligned}$$

As a result of (1), we have

$$\begin{aligned}
1&=(a+b+c)^2\\
&=a^2+b^2+c^2+2\left(ab+bc+ca\right)\\
&\geq 3\left(a^2b+b^2c+c^2a\right)+2\left(ab+bc+ca\right)\\
&=\left(3a^2b+2ab\right)+\left(3b^2c+2bc\right)+\left(3c^2a+2ca\right). \qquad (2)
\end{aligned}$$

Then, using property (2), the convexity of $g(x) = \frac{1}{x}$ on $(0, \infty)$, and Jensen's Theorem, we obtain

$$\begin{aligned}
f(a,b,c) &= \frac{a}{3ab+2b} + \frac{b}{3bc+2c} + \frac{c}{3ca+2a} \\
&= ag(3ab+2b) + bg(3bc+2c) + cg(3ca+2a) \\
&\geq g\left[a(3ab+2b) + b(3bc+2c) + c(3ca+2a)\right] \\
&= g\left[\left(3a^2b+2ab\right) + \left(3b^2c+2bc\right) + \left(3c^2a+2ca\right)\right] \\
&= \frac{1}{(3a^2b+2ab)+(3b^2c+2bc)+(3c^2a+2ca)} \\
&\geq 1 \\
&= f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).
\end{aligned}$$

It follows that under the conditions $a, b, c > 0$ and $a + b + c = 1$, the minimum value of $f(a,b,c)$ is $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 1$.

Solution 2 by David E. Manes, Oneonta, NY

We will show that the minimum value of f is 1.
By the Arithmetic Mean-Geometric Mean inequality, we get

$$f(a,b,c) \geq 3\sqrt[3]{\frac{a}{b(3a+2)} \cdot \frac{b}{c(3b+2)} \cdot \frac{c}{a(3c+2)}} = \frac{3}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}}.$$

We again use the AM-GM inequality to obtain

$$\begin{aligned}
\sqrt[3]{(3a+2)(3b+2)(3c+2)} &\leq \frac{(3a+2)+(3b+2)+(3c+2)}{3} = \frac{3(a+b+c)+6}{3} \\
&= 3.
\end{aligned}$$

Hence,

$$\frac{1}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}} \geq \frac{1}{3}$$

so that

$$f(a,b,c) \geq \frac{3}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}} \geq 3 \cdot (1/3) = 1$$

with equality if and only if $a = b = c = \frac{1}{3}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

From Bergström's and the Arithmetic mean -Geometric mean inequalities,

$$f(a,b,c) \;=\; \frac{\left(\sqrt{\frac{a}{b}}\right)^2}{3a+2} + \frac{\left(\sqrt{\frac{b}{c}}\right)^2}{3b+2} + \frac{\left(\sqrt{\frac{c}{a}}\right)^2}{3c+2} \geq \frac{\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}\right)^2}{3a+2+3b+2+3c+2} = \left(\frac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}}{3}\right)^2$$

$$\geq \sqrt[3]{\sqrt{\frac{a}{b}}\sqrt{\frac{b}{c}}\sqrt{\frac{c}{a}}} = 1.$$

Equality is attained iff it occurs in those two inequalities, that is, iff $\dfrac{\sqrt{\frac{a}{b}}}{3a+2} = \dfrac{\sqrt{\frac{b}{c}}}{3b+2} = \dfrac{\sqrt{\frac{c}{ba}}}{3c+2}$ and $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{a}$. These last identities are true if and only if $a = b = c$, that is, if and only if $a = b = c = \dfrac{1}{3}$. In this case equality is also obtained in Bergström's inequality. So, the minimum value of $f(a,b,c)$ is 1, and this occurs if and only if $a = b = c = \dfrac{1}{3}$.

Solution 4 by Arkady Alt, San Jose,CA

Since $\left(\dfrac{a}{3a+2} - \dfrac{b}{3b+2}\right)\left(\left(-\dfrac{1}{a}\right) - \left(-\dfrac{1}{b}\right)\right) = \dfrac{2(a-b)^2}{ab(3b+2)(3a+2)} \geq 0$ then triples $\left(\dfrac{a}{3a+2}, \dfrac{b}{3b+2}, \dfrac{c}{3c+2}\right)$, $\left(-\dfrac{1}{a}, -\dfrac{1}{b}, -\dfrac{1}{c}\right)$ are agreed in order and, therefore, by the Rearrangement Inequality $\sum_{cyc} \dfrac{a}{3a+2} \cdot \left(-\dfrac{1}{a}\right) \geq \sum_{cyc} \dfrac{a}{3a+2} \cdot \left(-\dfrac{1}{b}\right) \iff$

$$\sum_{cyc} \frac{a}{(3a+2)b} \geq \sum_{cyc} \frac{a}{3a+2} \cdot \frac{1}{a} = \sum_{cyc} \frac{1}{3a+2}.$$

Also, by Cauchy Inequality $\sum_{cyc}(3a+2) \cdot \sum_{cyc} \dfrac{1}{3a+2} \geq 9 \iff 9 \cdot \sum_{cyc} \dfrac{1}{3a+2} \geq 9 \iff$ $\sum_{cyc} \dfrac{1}{3a+2} \geq 1$.Thus, $f(a,b,c) \geq 1$ and since $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$ we may conclude that $\min f(a,b,c) = 1$.

Solution 5 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Since $c = 1 - a - b$, then we have:

$$f(a,b,c) = \frac{a}{3ab+2b} + \frac{b}{3b(1-a-b)+2(1-a-b)} + \frac{1-a-b}{3(1-a-b)a+2a}.$$

That means that we may assume the function:

$$g(a,b) = \frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1)} + \frac{a+b+-1}{a(3a+3b-5)}.$$

To find the stationary points of $g(a,b)$, work out $\dfrac{\partial g}{\partial a}$ and $\dfrac{\partial g}{\partial b}$ and set both to zero . This gives two equations for two unknowns a and b. We may solve these equations for a and b (often there is more than one solution). Let (x,y) be a stationary point. If $g_{aa} > 0$ and $g_{bb} > 0$ at (x,y) then (x,y) is a minimum point . So,

$$\frac{\partial g}{\partial a} = -\frac{(a+b-1)(6a+3b-5)}{a^2(3a+3b-5)^2} - \frac{3a}{b(3a+2)^2} + \frac{b}{(3b+2)(a+b-1)^2} + \frac{1}{3ab+2b} + \frac{1}{a(3a+3b-5)}$$

$$\frac{\partial g}{\partial b} = -\frac{a}{(b^2(3a+2)} + \frac{b(3a+6b-1)}{(3b+2)^2(a+b-1)^2} - \frac{1}{(3b+2)(a+b-1)} + \frac{1}{a(3a+3b-5)} - \frac{3(a+b-1)}{a(3a+3b-5)^2},$$

and for $(a,b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have:

$$\min g(a,b) = \min\left[\frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1)} + \frac{a+b-1}{a(3a+3b-5)}\right] = 1.$$

and for $(a,b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have:

$$\min g(a,b) = \min\left[\frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1} + \frac{a+b-1}{a(3a+3b-5)}\right] = 1.$$

Solution 6 by Albert Stadler, Herrliberg, Switzerland

We will prove that the minimum value equals 1 and the minimum is assumed if and only if $a = b = c = 1/3$. To that end we must prove that

$$f(a,b,c) = \frac{a(a+b+c)}{3ab+2b(a+b+c)} + \frac{a(a+b+c)}{3ab+2b(a+b+c)} + \frac{a(a+b+c)}{3ab+2b(a+b+c)} \geq 1.$$

We clear denominators and get the equivalent inequality

$$10\sum_{cycl} a^4b^2 + 24\sum_{cycl} a^3b^3 + 18\sum_{cycl} a^4c^2 + 4\sum_{cycl} a^5c \geq 2\sum_{cycl} a^4bc + 15\sum_{cycl} a^3b^2c + 11\sum_{cycl} a^3bc^2 + 28\sum_{cycl} a^2b^2c^2. \quad (1)$$

By the (weighted)AM-GM inequality,

$$\sum_{cycl} a^4b^2 + \sum_{cycl} a^4c^2 \geq 2\sum_{cycl} a^4bc,$$

$$15\sum_{cycl} a^3b^3 = 15\sum_{cycl}\left(\frac{2}{3}a^3b^3 + \frac{1}{3}c^3a^3\right) \geq 15\sum_{cycl} a^3b^2c,$$

$$11\sum_{cycl} a^4c^2 = 11\sum_{cycl}\left(\frac{2}{3}a^4c^2 + \frac{1}{6}b^4a^2 + \frac{1}{6}c^4b^2\right) \geq 11\sum_{cycl} a^3bc^2,$$

$$9\sum_{cycl} a^4b^2 \geq 27a^2b^2c^2,$$

$$9\sum_{cycl} a^3b^3 \geq 27a^2b^2c^2,$$

$$6\sum_{cycl} a^4c^2 \geq 18a^2b^2c^2,$$

$$4\sum_{cycl} a^5c \geq 12a^2b^2c^2,$$

and (1) follows if we add the last seven inequalities. In all seven inequalities equality holds if and only if $a = b = c$.

Comment by Stanley Rabinowitz of Chelmsford, MA. Problems such as this are easily solvable by computer algebra systems these days. For example; the Mathematica command
Minimize [{a/(3a * b + 2b) + b/(3b * c + 2c) + c/(3c * a + 2a), a > 0 && b > 0 && c > 0 && $a+b+c=1$}, {a,b,c}] responds by saying that the minimum value is 1 and occurs when $a=b=c=\dfrac{1}{3}$.

Also solved by Konul Aliyeva (student), ADA University, Baku, Azerbaijan; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Tran Hong (student), Cao Lanh School, Dong Thap, Vietnam; Sanong Huayrerai, Rattanakosinsomphothow School, Nakon, Pathom, Thailand; Seyran Ibrahimov, Baku State University, Maasilli, Azerbaijan; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Stanley Rabinowitz of Chelmsford, MA; Neculai Stanciu "George Emil Palade" School, Buză, Romania and Titu Zvonaru, Comănesti, Romania; Daniel Văcaru, Pitesti, Romania; Nicusor Zlota "Traian Vuia Technical College, Focsani, Romania, and the proposer.

5509: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x,y,z be positive real numbers that add up to one and such that $0<\dfrac{x}{y},\dfrac{y}{z},\dfrac{z}{x}<\dfrac{\pi}{2}$. Prove that

$$\sqrt{x\cos\left(\frac{y}{z}\right)}+\sqrt{y\cos\left(\frac{z}{x}\right)}+\sqrt{z\cos\left(\frac{x}{y}\right)}<\frac{3}{5}\sqrt{5}.$$

Solution 1 by Michel Bataille, Rouen, France

The Cauchy-Schwarz inequality provides

$$\sqrt{x}\sqrt{\cos\left(\frac{y}{z}\right)}+\sqrt{y}\sqrt{\cos\left(\frac{z}{x}\right)}+\sqrt{z}\sqrt{\cos\left(\frac{x}{y}\right)}\le(x+y+z)^{1/2}\left(\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\right)^{1/2}.$$

Since $x+y+z=1$, it follows that the left-hand side L of the proposed inequality satisfies

$$L\le\left(\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\right)^{1/2}.$$

Thus, it suffices to show that

$$\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)<\frac{9}{5}. \tag{1}$$

Now, Jensen's inequality applied to the cosine function, which is concave on $(0,\frac{\pi}{2})$, yields

$$\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\le 3\cos\left(\frac{y/z+z/x+x/y}{3}\right). \tag{2}$$

But we have $1 = \sqrt[3]{\frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y}} \leq \frac{y/z+z/x+x/y}{3}$ (by AM-GM) and
$0 < \frac{y/z+z/x+x/y}{3} < \frac{3 \cdot \frac{\pi}{2}}{3} = \frac{\pi}{2}$, hence

$$\cos\left(\frac{y/z + z/x + x/y}{3}\right) \leq \cos(1)$$

(since the cosine function is decreasing on $(0, \frac{\pi}{2})$).
Then (2) gives $\cos\left(\frac{y}{z}\right) + \cos\left(\frac{z}{x}\right) + \cos\left(\frac{x}{y}\right) \leq 3\cos(1)$. There just remains to remark that $\cos(1) < 0.6 = \frac{3}{5}$ to obtain the desired inequality (1).

Solution 2 by Tran Hong (student), Cao Lanh School, Dong Thap, Vietnam

$$LHS \overset{BCS}{\leq} \sqrt{x+y+z}\sqrt{\cos\left(\frac{y}{z}\right) + \cos\left(\frac{z}{x}\right) + \cos\left(\frac{x}{y}\right)}$$
$$= \sqrt{1} \cdot \sqrt{\cos\left(\frac{y}{z}\right) + \cos\left(\frac{z}{x}\right) + \cos\left(\frac{x}{y}\right)} \qquad (1)$$
$$Let\ f(t) = \cos t, t \in \left(0, \frac{\pi}{2}\right) \Rightarrow f''(t) = -\cos t < 0$$
$$Using\ Jensen's\ we\ have:$$
$$f\left(\frac{y}{z}\right) + f\left(\frac{z}{x}\right) + f\left(\frac{x}{y}\right) \leq 3 \cdot f\left(\frac{\frac{y}{z} + \frac{z}{x} + \frac{x}{y}}{3}\right)$$
$$= 3\cos\left(\frac{\frac{y}{z} + \frac{z}{x} + \frac{x}{y}}{3}\right) \leq 3\cos(1).$$
$$\Rightarrow \{(1) \leq \sqrt{3\cos(1)} \approx 1,2731 < 3 \cdot \frac{\sqrt{5}}{5} \approx 1,3416.$$

Solution 3 by David E. Manes, Oneonta, NY

Let $J = \sqrt{x\cos\left(\frac{y}{z}\right)} + \sqrt{y\cos\left(\frac{z}{x}\right)} + \sqrt{z\cos\left(\frac{x}{y}\right)}$. We will show that $J \leq \sqrt{3\cos 1} < \dfrac{3}{5}\sqrt{5}$.
By the Cauchy-Schwarz inequality, one obtains

$$J = \sqrt{x\cos\left(\frac{y}{z}\right)} + \sqrt{y\cos\left(\frac{z}{x}\right)} + \sqrt{z\cos\left(\frac{x}{y}\right)} \leq \sqrt{x+y+z}\sqrt{\cos\left(\frac{y}{z}\right) + \cos\left(\frac{z}{x}\right) + \cos\left(\frac{x}{y}\right)}$$
$$= \sqrt{\cos\left(\frac{y}{z}\right) + \cos\left(\frac{z}{x}\right) + \cos\left(\frac{x}{y}\right)}.$$

At the risk of being redundant, note that

$$J \leq \sqrt{\sum_{cyc}\cos\left(\frac{y}{z}\right)} = \sqrt{(x+y+z)\sum_{cyc}\cos\left(\frac{y}{z}\right)}$$
$$= \sqrt{(x+y+z)\cos\left(\frac{y}{z}\right) + (x+y+z)\cos\left(\frac{z}{x}\right) + (x+y+z)\cos\left(\frac{x}{y}\right)}.$$

Since the cosine function is concave on the interval $(0, \pi/2)$, it follows by Jensen's inequality that for each of the following terms in the cyclic sum under the square root sign, we get

$$x\cos\left(\frac{y}{z}\right)+y\cos\left(\frac{z}{x}\right)+z\cos\left(\frac{x}{y}\right)\le\cos\left(\frac{xy}{z}+\frac{yz}{x}+\frac{xz}{y}\right)$$
$$y\cos\left(\frac{y}{z}\right)+z\cos\left(\frac{z}{x}\right)+x\cos\left(\frac{x}{y}\right)\le\cos\left(\frac{y^2}{z}+\frac{z^2}{x}+\frac{x^2}{y}\right)$$
$$z\cos\left(\frac{y}{z}\right)+x\cos\left(\frac{z}{x}\right)+y\cos\left(\frac{x}{y}\right)\le\cos(y+z+x)=\cos 1.$$

Therefore, $J\le\sqrt{\cos\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)+\cos\left(\frac{y^2}{z}+\frac{z^2}{x}+\frac{x^2}{y}\right)+\cos 1}$. For the first term, $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}$, in parentheses above, observe that using the Arithmetic Mean-Geometric Mean inequality, one obtains

$$\frac{1}{2}\left(\frac{xy}{z}+\frac{yz}{x}\right)\ge\sqrt{\frac{xy^2z}{xz}}=y,$$
$$\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}\right)\ge\sqrt{\frac{xyz^2}{xy}}=z,$$
$$\frac{1}{2}\left(\frac{zx}{y}+\frac{xy}{z}\right)\ge\sqrt{\frac{x^2yz}{yz}}=x.$$

Summing the above terms yields

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\ge x+y+z=1.\quad\mathbf{(1)}$$

Using the Cauchy-Schwarz inequality in the Engel-Titu form for the second term in parentheses in J above, one immediately obtains

$$\frac{y^2}{z}+\frac{z^2}{x}+\frac{x^2}{y}\ge\frac{(y+z+x)^2}{z+x+y}=1.\quad\mathbf{(2)}$$

Since the cosine function is decreasing on the interval $[0, \pi/2]$ and as a result of inequalities **(1)** and **(2)**, it follows that $\cos\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)\le\cos 1$ and $\cos\left(\frac{y^2}{z}+\frac{z^2}{x}+\frac{x^2}{y}\right)\le\cos 1$. Therefore,

$$J=\sqrt{x\cos\left(\frac{y}{z}\right)}+\sqrt{y\cos\left(\frac{z}{x}\right)}+\sqrt{z\cos\left(\frac{x}{y}\right)}\le\sqrt{3\cos 1}.$$

Finally, note that for each of the above steps the inequalities become equalities if and only if $x=y=z=\frac{1}{3}$.

Solution 4 by Daniel Văcaru, Pitesti, Romania

One has

$$\sqrt{x\cos\frac{y}{z}}+\sqrt{y\cos\frac{z}{x}}+\sqrt{z\cos\frac{x}{y}}\le\underbrace{\sqrt{x+y+z}}\cdot\sqrt{\cos\frac{y}{z}+\cos\frac{z}{x}+\cos\frac{x}{y}}=\sqrt{\cos\frac{y}{z}+\cos\frac{z}{x}+\cos\frac{x}{y}}=$$

$$\sqrt{\sin\left(\frac{\pi}{2}-\frac{y}{z}\right)+\sin\left(\frac{\pi}{2}-\frac{z}{x}\right)+\sin\left(\frac{\pi}{2}-\frac{x}{y}\right)} \overbrace{<} \sqrt{\left(\frac{\pi}{2}-\frac{y}{z}\right)+\left(\frac{\pi}{2}-\frac{z}{x}\right)+\left(\frac{\pi}{2}-\frac{x}{y}\right)}$$

$$= \sqrt{3\frac{\pi}{2}-\left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right)}.$$

The inequality under the brace is true because $\sin x < x$, $\forall x \in \left(0, \frac{\pi}{2}\right)$. On the other hand, one knows that $\frac{y}{z}+\frac{z}{x}+\frac{x}{y} \geq 3$ by the MA-MG inequality. Therefore one has

$$\sqrt{x\cos\frac{y}{z}}+\sqrt{\cos\frac{z}{x}}+\sqrt{\cos\frac{x}{y}} < \sqrt{3\frac{\pi}{2}-3} = \sqrt{3\cdot\left(\frac{\pi}{2}-1\right)} < \sqrt{3\cdot\left(\frac{32}{20}-1\right)} = \sqrt{\frac{3\cdot 12}{20}} = \frac{3}{\sqrt{(5}} = \frac{3}{5}\sqrt{5}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland and the proposer.

5510: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}\left[4^n\left(\zeta(2n)-1\right)-1\right],$$

where ζ denotes the Riemann zeta function.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

$$\sum_{n=1}^{\infty}(4^n(\zeta(2n)-1)-1) = \sum_{n=1}^{\infty}\left(4^n\left(\sum_{m=2}^{\infty}\frac{1}{m^{2n}}\right)-1\right) = \sum_{n=1}^{\infty}\sum_{m=3}^{\infty}\left(\frac{2}{m}\right)^{2n} =$$

$$= \sum_{m=3}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2}{m}\right)^{2n} = \sum_{m=3}^{\infty}\frac{\left(\frac{2}{m}\right)^2}{1-\left(\frac{2}{m}\right)^2} = \sum_{m=3}^{\infty}\frac{4}{m^2-4} = \sum_{m=3}^{\infty}\left(\frac{1}{m-2}-\frac{1}{m+2}\right) =$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{25}{12}.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\sum_{n=1}^{\infty}[4^n(\zeta(2n)-1)-1]=\sum_{n=1}^{\infty}[4^n\sum_{k=2}^{\infty}\frac{1}{k^{2n}}-1]=\sum_{n=1}^{\infty}\sum_{k=3}^{\infty}\frac{4^n}{k^{2n}}=$$
$$=\sum_{k=3}^{\infty}\sum_{n=1}^{\infty}\frac{4^n}{k^{2n}}=\sum_{k=3}^{\infty}\frac{4}{k^2}\frac{1}{(1-\frac{4}{k^2})}=\sum_{k=3}^{\infty}\frac{4}{k^2-4}=$$
$$=\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k-2}-\frac{1}{k-1}\right]+\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k-1}-\frac{1}{k}\right]+\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k}-\frac{1}{k+1}\right]+$$
$$+\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k+1}-\frac{1}{k+2}\right]=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$$

Solution 3 by Moti Levy, Rehovot, Israel

Let

$$S:=\sum_{n=1}^{\infty}\left(4^n\left(\zeta\left(2n\right)-1\right)-1\right),\qquad S_N:=\sum_{n=1}^{N}\left(4^n\left(\zeta\left(2n\right)-1\right)-1\right).$$

Then

$$S_N=\sum_{n=1}^{N}\left(\left(2^{2n}\sum_{k=2}^{\infty}\frac{1}{k^{2n}}\right)-1\right)=\left(\sum_{n=1}^{N}\sum_{k=2}^{\infty}\frac{2^{2n}}{k^{2n}}\right)-N$$
$$=\sum_{k=3}^{\infty}\sum_{n=1}^{N}\frac{2^{2n}}{k^{2n}}=\sum_{k=3}^{\infty}\frac{4}{k^2-4}\left(1-\left(\frac{2}{k}\right)^{2N}\right)$$

$$S=\lim_{N\to\infty}S_N=\sum_{k=3}^{\infty}\frac{4}{k^2-4}=\sum_{k=3}^{\infty}\left(\frac{1}{k-2}-\frac{1}{k+2}\right)$$
$$=\sum_{k=1}^{\infty}\frac{1}{k}-\sum_{k=5}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}.$$

Now, as a bonus, let us evaluate parametrized version of the above sum:

$$S\left(t\right):=\sum_{n=1}^{\infty}\left(t^{2n}\left(\zeta\left(2n\right)-1\right)-\frac{t^{2n}}{2^{2n}}\right),\quad S_N\left(t\right):=\sum_{n=1}^{N}\left(t^{2n}\left(\zeta\left(2n\right)-1\right)-\frac{t^{2n}}{2^{2n}}\right)$$

Then

$$S\left(t\right)_N=\sum_{n=1}^{N}\left(\left(t^{2n}\sum_{k=2}^{\infty}\frac{1}{k^{2n}}\right)-\frac{t^{2n}}{2^{2n}}\right)=\left(\sum_{n=1}^{N}\sum_{k=2}^{\infty}\frac{t^{2n}}{k^{2n}}\right)-\sum_{n=1}^{N}\frac{t^{2n}}{2^{2n}}$$
$$=\left(\sum_{k=2}^{\infty}\sum_{n=1}^{N}\frac{t^{2n}}{k^{2n}}\right)-\sum_{n=1}^{N}\frac{t^{2n}}{2^{2n}}=\sum_{k=3}^{\infty}\sum_{n=1}^{N}\frac{t^{2n}}{k^{2n}}=\sum_{k=3}^{\infty}\frac{t^2}{k^2-t^2}\left(1-\left(\frac{t}{k}\right)^{2N}\right)$$

$$S\left(t\right)=\lim_{N\to\infty}S\left(t\right)_N=\sum_{k=3}^{\infty}\frac{t^2}{k^2-t^2}$$

Let us assume that t is not a positive integer and satisfies the inequality $t > -1$, then

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - t^2} = \frac{\psi(t+1) - \psi(-t+1)}{2t},$$

where $\psi(t)$ is the Digamma function.

$$\psi(-z+1) = \psi(z) + \pi \cot(\pi z)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - t^2} = \frac{1}{2t}\left(\frac{1}{t} - \cot(\pi t)\right)$$

$$\begin{aligned} S(t) &= t^2 \frac{1}{2t}\left(\frac{1}{t} - \pi\cot(\pi t)\right) - \frac{t^2}{1^2 - t^2} - \frac{t^2}{2^2 - t^2} \\ &= \frac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)}. \end{aligned}$$

We summarize our result as follows,

$$S(t) = \begin{cases} \dfrac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)}, & |t| < 3 \\[2ex] \lim\limits_{t\to 1} \dfrac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)} = \dfrac{5}{12}, & |t| = 1, \\[2ex] \lim\limits_{t\to 2} \dfrac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)} = \dfrac{25}{12}, & |t| = 2. \end{cases}$$

Remark: The function $S(t)$, as defined above, is continuous in the interval $|t| < 3$.

Reference:
Borwein, Jonathan; Bradley, David M.; Crandall, Richard (2000). "*Computational Strategies for the Riemann Zeta Function*". J. Comp. App. Math. 121 (1–2): 247–296.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Denote the sum of the problem by S so that $S = \sum_{n=1}^{\infty}\sum_{k=3}^{\infty}\left(\frac{2}{k}\right)^{2n}$.

Since the summands are positive, so interchanging the order of summation, we have

$$S = \sum_{k=3}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2}{k}\right)^{2n} = 4\sum_{k=3}^{\infty}\frac{1}{k^2 - 4}.$$

For any integer $M \geq 3$, we have

$$4\sum_{k=3}^{\infty}\frac{1}{k^2 - 4} = \sum_{k=3}^{M}\left(\frac{1}{k-2} - \frac{1}{k+2}\right) = \frac{25}{12} - \sum_{k=M-1}^{M+2}\frac{1}{k}.$$

It follows that $S = \dfrac{25}{12}$.

Comment by Editor : **Ed Gray of Highland Beach, FL** wrote: "I didn't know any recursive formula that would help, so I did the sum by brute force, computing the sum of the first 10 terms, getting a result of 2.0828.....This aroused my curiosity, so I went to Wolfram-alpha and sought the sum for a great number of terms, like 100 and 300. It became clear that the answer is $2.0833333333\cdots$ forever. Converting this to a fraction, we get a beautiful answer of 25/24". He continued on saying that he did not actually solve the problem. This is being mentioned here as a very useful heuristic for getting a feel for the problem, and as a caveat that there are an infinite number of different ways to express a closed form representation for a specific decimal.

Also solved by Michel Bataille, Rouen, France; Bruno Salgueiro Fanego, Viveiro, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

Mea Culpa

Mary Wagner-Krankel of St. Mary's University in San Antonio, TX should have been credited with having solved problem 5500.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
April 15, 2019

5529: *Proposed by Kenneth Korbin, New York, NY*

Convex cyclic quadrilateral $ABCD$ has integer length sides and integer area. The distance from the incenter to the circumcenter is 91. Find the length of the sides.

5530: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygon $ABCD$ is an 11 by 12 rectangle ($AB > AD$). Points P, Q, R, and S are on sides AB, BC, CD, and DA, respectively, such that PR and SQ are parallel to AD and AB, respectively. Moreover, $X = PR \cap QS$. If the perimeter of rectangle $PBQX$ is 5/7 the perimeter of rectangle $SAPX$, and the perimeter of rectangle $RCQX$ is 9/10 the perimeter of rectangle $PBQX$, find the area of rectangle $SDRX$.

5531: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu-Severin, Mehedinti, Romania*

For real numbers x, y, z prove that if $x, y, z > 1$ and $xyz = 2\sqrt{2}$, then

$$x^y + y^z + z^x + y^x + z^y + x^z > 9.$$

5532: *Proposed by Arkady Alt, San Jose, CA*

Let a, b, c be positive real numbers and let $a_n = \dfrac{an+b}{an+c}, n \in N$. For any natural number m find $\displaystyle\lim_{n\to\infty} \prod_{k=n}^{nm} a_k$.

5533: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find the value of the sum

$$\sum_{n=1}^{+\infty} \frac{n^2\alpha^n}{(n-1)!}$$

for any real number $\alpha > 0$. (Here, $0! = 1! = 1$).

5534: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy$.

Solutions

5511: *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with perimeter $58+14\sqrt{11}$ is inscribed in a circle with diameter $17+7\sqrt{11}$. Find its dimensions if each of its sides is of the form $a+b\sqrt{11}$ where a and b are positive integers.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

We define four points

$$\begin{aligned} P &= \left(\frac{b}{2}, \frac{\sqrt{d^2-b^2}}{2}\right) \\ Q &= \left(-\frac{b}{2}, \frac{\sqrt{d^2-b^2}}{2}\right) \\ R &= \left(-\frac{a}{2}, -\frac{\sqrt{d^2-a^2}}{2}\right) \\ S &= \left(\frac{a}{2}, -\frac{\sqrt{d^2-a^2}}{2}\right) \end{aligned}$$

where $a=20+6\sqrt{11}$, $b=12+2\sqrt{11}$, $c=13+3\sqrt{11}$, and $d=17+7\sqrt{11}$. Note that $\overline{PQ}$ and $\overline{SR}$ are parallel and that $OP=OQ=OR=OS=d/2$. Thus $PQRS$ is a trapezoid inscribed in the circle with center $O=(0,0)$ and diameter d.
We have $PQ=b$, $SR=a$, and $PS=QR$. Now

$$(PS)^2 = \left(\frac{b}{2}-\frac{a}{2}\right)^2 + \left(\frac{\sqrt{d^2-b^2}}{2}+\frac{\sqrt{d^2-a^2}}{2}\right)^2.$$

Thus

$$\begin{aligned} 4(PS)^2 &= (b-a)^2 + (\sqrt{d^2-b^2}+\sqrt{d^2-a^2})^2 \\ &= 2d^2-2ab+2\sqrt{d^2-b^2}\sqrt{d^2-a^2} \end{aligned}$$

so

$$2(PS)^2 = d^2-ab+\sqrt{d^2-b^2}\sqrt{d^2-a^2}. \qquad (1)$$

We have

$$\begin{aligned} (d^2-b^2)(d^2-a^2) &= 16300+4800\sqrt{11} \\ &= (80+30\sqrt{11})^2 \end{aligned}$$

and

$$d^2 - ab = 456 + 126\sqrt{11}.$$

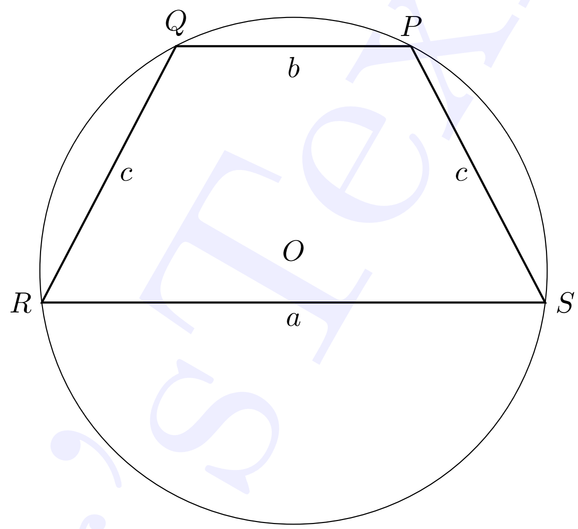
Therefore, by (1), we have

$$\begin{aligned}(PS)^2 &= 268 + 78\sqrt{11} \\ &= c^2.\end{aligned}$$

Finally, the perimeter of $PQRS$ is

$$\begin{aligned}PQ + QR + RS + SP &= b + c + a + c \\ &= 58 + 14\sqrt{11}.\end{aligned}$$

Thus the desired trapezoid has parallel sides of lengths $a = 20 + 6\sqrt{11}$ and $b = 12 + 2\sqrt{11}$ and the other two sides of length $c = 13 + 3\sqrt{11}$.
Here it is.



Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the dimensions of the trapezoid are $12 + 2\sqrt{11},\ 20 + 6\sqrt{11},\ 13 + 3\sqrt{11}$, and $13 + 3\sqrt{11}$.

Let $ABCD$ be the trapezoid with $AB||CD$. Since it is inscribed in a circle so it is in fact isosceles. Let $AB = x,\ CD = y,\ AD = BC = z,\ BD = w$, with $x \geq y$.

Since the perimeter of the trapezoid is $58 + 14\sqrt{11}$ so

$$x + y + 2z = 58 + 14\sqrt{11} \tag{1}$$

Applying the cosine formula respectively to triangles ABD and CDB, we obtain $\cos\angle DAB = \dfrac{x^2 + z^2 - w^2}{2xz}$ and $\cos\angle DCB = \dfrac{y^2 + z^2 - w^2}{2yz}$. From $\angle DAB + \angle DCB = \pi$, we have $\cos\angle DAB = -\cos\angle DCB$, and deduce that

$$w^2 = xy + z^2. \tag{2}$$

Let h be the length of the perpendicular from D to AB, and d be the diameter of the circumcircle of $\triangle ABD$. By the Pythagorean theorem, we have $h^2 = z^2 - \left(\dfrac{x-y}{2}\right)^2 = \dfrac{(2z+x-y)(2z-x+y)}{4}$. Applying the sine formula to triangle ABD, we have $d = \dfrac{w}{\sin \angle DAB} = \dfrac{w}{\left(\dfrac{h}{z}\right)}$ or $dh = zw$. Since $d = 17 + 7\sqrt{11}$, by (2) we obtain from $d^2h^2 - z^2w^2 = 0$ that

$$(414 + 119\sqrt{11})(2z + x - y)(2z - x + y)(2z - x + y) - 2z^2(xy + z^2) = 0. \qquad (3)$$

Let $x = p + q\sqrt{11}$ and $y = r + s\sqrt{11}$, where p, q, r, s are positive integers. We substitute z of (1) into (3) so that the left side of (3) equals $f + h\sqrt{11}$ where f and g are integers depending only on p, q, r, s. Thus, (3) holds if and only if $f = g = 0$.
From (1), we see that both p and r do not exceed 56 and that both q and s do not exceed 12. By a compute search, we find that (3) holds if and only if $p = 20, q = 6, r = 12, s = 2$. Hence our solution for the dimensions of the trapezoid.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Since the trapezoid must be isosceles, we denote the lengths of the bases by x and y (with $x \le y$) and the length of each leg by z. Then (see [1]) the radius r of the circle is given by

$$r = z\sqrt{\frac{xy + z^2}{4z^2 - (x-y)^2}}.$$

We write $x = a + b\sqrt{11}$ and $y = c + d\sqrt{11}$, where a, b, c, and d are positive integers. Then $z = (58 + 14\sqrt{11} - x - y)/2 = e + f\sqrt{11}$, with $e = 29 - a/2 - c/2$ and $f = 7 - b/2 - d/2$ also positive integers. Using $r = (17 + 7\sqrt{11})/2$, we eventually obtain $t + u\sqrt{11} = v + w\sqrt{11}$, where:

$$t = 414(4e^2 + 44f^2 - a^2 + 2ac - c^2 - 11b^2 + 22bd - 11d^2) + 1309(8ef - 2ab + 2ad + 2bc - 2cd);$$

$$u = 414(8ef - 2ab + 2ad + 2bc - 2cd) + 119(4e^2 + 44f^2 - a^2 + 2ac - c^2 - 11b^2 + 22bd - 11d^2);$$

$$v = (2e^2 + 22f^2)(ac + 11bd + e^2 + 11f^2) + 44ef(ad + bc + 2ef);$$

$$w = (2e^2 + 22f^2)(ad + bc + 2ef) + 4ef(ac + 11bd + e^2 + 11f^2).$$

Setting $t = v$ and $u = w$, we obtain the following results via computer search:

$$x = 12 + 2\sqrt{11}, \;\; y = 20 + 6\sqrt{11}, \;\text{ and }\; z = 13 + 3\sqrt{11}.$$

[1] https://en.wikipedia.org/wiki/Isosceles_trapezoid

Editor's comments : Computers were called into service on this problem and that seemed to bother some of the solvers. **David Stone and John Hawkins of Georgia Southern University** also obtained the correct result and described their solution as follows: "We do not have an algebraic derivation for this result. Instead, we wrote algebraic conditions on the sides of the trapezoids, used the fact that there are only finitely many chords of the form $a + b\sqrt{11}$, (i.e. possibilities for the sides), then used a BASIC program to test them and find the solution amongst all possibilities." They went on to say that “another approach would be using analytic geometry. This lead us to the same end game−−write a program to check all possible values.”

Ken Korbin, proposer of the problem, attached a note to the problem giving us some insights into how he constructed it. He stated: Begin with a circle with diameter K^3 with $K \geq 3$. It is possible to inscribe in this circle a trapezoid with parallel sides of lengths $2K^2$ and $6K^2-32$, and with each slant side of length K^3-8K. He then checked this statement by computing:

$$\text{Arcsin}\left(\frac{2K^2}{K^3}\right)+2\text{Arcsin}\left(\frac{K^3-8K}{K^3}\right)=\text{Arcsin}\left(\frac{6K^2-32}{K^3}\right).$$

For this trapezoid, Perimeter $= 2K^3+8K^2-16K-32$.

In this problem, let

$$K = 1+\sqrt{11} \approx 4.3166$$

$$\text{Diameter } = K^3 = 34+14\sqrt{11} \text{ and the sides of the trapezoid are:}$$

$$24+4\sqrt{11},\ 40+12\sqrt{11},\ 26+6\sqrt{11} \text{ and } 26+6\sqrt{11}.$$

Divide each length by 2 to get diameter $17+7\sqrt{11}$ and sides $\begin{cases} 12+2\sqrt{11} \\ 20+6\sqrt{11} \\ 13+3\sqrt{11} \\ 13+3\sqrt{11} \end{cases}$

Perimeter $= 58+14\sqrt{11}$

Check to see if this trapezoid can be inscribed in a circle with diameter $= 17+7\sqrt{11}$.

Check:

$$\text{Arcsin}\left(\frac{12+2\sqrt{11}}{17+7\sqrt{11}}\right)+2\text{Arcsin}\left(\frac{13+3\sqrt{11}}{17+7\sqrt{11}}\right)=\text{Arcsin}\left(\frac{20+6\sqrt{11}}{17+7\sqrt{11}}\right).$$

Also solved by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5512: *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

If $a_k > 0$, $(k = 1, 2, \ldots, n)$ then $\dfrac{n}{\displaystyle\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k}} - \dfrac{n}{\displaystyle\sum_{k=1}^{n} \frac{1}{a_k}} \geq \dfrac{2}{n+1}.$

Solution 1 by Moti Levy, Rehovot, Israel

$$F(x) := \begin{cases} \frac{2}{n+1}, & for \quad x = 0, \\ \frac{n}{\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}} - \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k x}}, & for \quad x > 0. \end{cases} \tag{1}$$

Note that $F(x)$ is continuous at $x=0$, since $\lim_{x\to 0} F(x) = F(0)$.

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{n}{\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}} - \lim_{x\to 0} \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k x}}$$
$$= \frac{n}{\sum_{k=1}^{n} \frac{1}{\frac{1}{k}}} - 0 = \frac{2}{n+1}.$$

Thus, in terms of $F(x)$, our original inequality is $F(1) \geq F(0)$.
Now we show that $\frac{dF}{dx} \geq 0$ for $x>0$ (i.e., $F(x)$ is monotone increasing for $x>0$),

$$\frac{dF}{dx} = -\frac{n}{\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2} \frac{d\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)}{dx} + \frac{n}{\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2} \frac{d\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)}{dx} \tag{2}$$

$$= n\left(\frac{\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}}{\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2} - \frac{\sum_{k=1}^{n} \frac{a_k}{(a_k x)^2}}{\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2}\right). \tag{3}$$

$\frac{dF}{dx} \geq 0$ is equivalent to

$$\frac{\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}}{\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2} \geq \frac{\sum_{k=1}^{n} \frac{a_k}{(a_k x)^2}}{\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2},$$

or to

$$\left(\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2 \geq \left(\sum_{k=1}^{n} \frac{a_k}{(a_k x)^2}\right)\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2.$$

We simplify by multiplying both sides by x^2 and obtain,

$$\left(\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq \left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2 \tag{4}$$

But (5) is a direct consequence of the Cauchy-Schwarz inequality.
We conclude that $F(x) \geq F(0)$ in the interval $[0,1]$, hence $F(1) \geq F(0)$.

Solution 2 by Michel Bataille, Rouen, France

Since $\frac{2}{n+1} = \frac{n}{\sum_{k=1}^{n} k}$, it suffices to show that more generally

$$\left(\sum (a_k+b_k)^{-1}\right)^{-1} \geq \left(\sum a_k^{-1}\right)^{-1} + \left(\sum b_k^{-1}\right)^{-1} \qquad (1)$$

holds whenever $a_k, b_k > 0$, $(k=1,2,\ldots,n)$. [The problem is the particular case $b_k = \frac{1}{k}$.]
(Here and in what follows $\sum$ means $\sum_{k=1}^{n}$.)
We propose two proofs of (1).

Proof 1:
If p is a negative real number, then

$$\left(\sum (a_k+b_k)^p\right)^{1/p} \geq \left(\sum a_k^p\right)^{1/p} + \left(\sum b_k^p\right)^{1/p}.$$

(see G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, C.U.P., 1934, p. 30). Taking $p=-1$ gives (1).

Proof 2:
Let $x_k = \frac{1}{a_k},\ y_k = \frac{1}{b_k}$ and $X_n = \sum x_k,\ Y_n = \sum y_k$. It is readily seen that (1) rewrites as $L_n \leq R_n$ where $L_n = \sum \frac{x_k y_k}{x_k + y_k}$ and $R_n = \frac{X_n Y_n}{X_n + Y_n}$.
Since $4L_n = \sum \frac{(x_k+y_k)^2-(x_k-y_k)^2}{x_k+y_k} = X_n + Y_n - Z_n$ where $Z_n = \sum \frac{(x_k-y_k)^2}{x_k+y_k}$, the inequality $L_n \leq R_n$ is successively equivalent to

$$X_n + Y_n - Z_n \leq \frac{4X_nY_n}{X_n+Y_n}$$

$$(X_n+Y_n)^2 - 4X_nY_n \leq (X_n+Y_n)Z_n$$
$$(X_n - Y_n)^2 \leq (X_n+Y_n)Z_n$$
$$\left(\sum(x_k-y_k)\right)^2 \leq \left(\sum(x_k+y_k)\right)\left(\sum \frac{(x_k-y_k)^2}{x_k+y_k}\right).$$

With $u_k = \sqrt{x_k+y_k},\ v_k = \frac{x_k-y_k}{\sqrt{x_k+y_k}}$, the latter is just $(\sum u_k v_k)^2 \leq (\sum u_k^2)(\sum v_k^2)$, which holds by the Cauchy-Schwarz inequality. Thus $L_n \leq R_n$ holds as well and we are done.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University of Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5513: *Proposed by Michael Brozinsky, Central Islip, NY*

In an $n \times n \times n$ cube partitioned into n^3 congruent cubes by $n-1$ equally spaced planes parallel to each pair of parallel faces, there are 20 times as many non-cubic rectangular parallelepipeds that could be formed as were cubic parallelepipeds. What is n?

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The number of cubic parallelepipeds equals

$$\sum_{r=1}^{n}(n+1-r)^3 = \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

while the number of non-cubic rectangular parallelepipeds equals

$$\sum_{\substack{1\leq r,s,t\leq n \\ r,s,t \text{ not all equal}}} (n+1-r)(n+1-s)(n+1-t) = \left(\sum_{r=1}^{n}(n+1-r)\right)^3 - \sum_{r=1}^{n}(n+1-r)^3 =$$

$$= \left(\sum_{r=1}^{n} r\right)^3 - \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^3 - \left(\frac{n(n+1)}{2}\right)^2.$$

Therefore

$$\left(\frac{n(n+1)}{2}\right)^3 - \left(\frac{n(n+1)}{2}\right)^2 = 20\left(\frac{n(n+1)}{2}\right)^2,$$

which implies that $\frac{n(n+1)}{2} = 21$ and finally, $n = 6$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Clearly the total number of parallelepipeds is $\binom{n+1}{2}^3 = \dfrac{n^3(n+1)^3}{8}$.

It can be counted readily that the number of $(n-k)\times(n-k)\times(n-k)$ cubic parallelepipeds equals $(k+1)^3$ for $k = 0, 1, 2, \cdots, n-1$. Hence the total number of cubic parallelepipeds equals $\sum_{k=0}^{n-1}(k+1)^3 = \dfrac{n^2(n+1)^2}{4}$. So according to the given conditions of the problem we have

$$\frac{n^3(n+1)^3}{8} - \frac{n^2(n+1)^2}{4} = 20\left(\frac{n^2(n+1)^2}{4}\right),$$

which reduces to the equation $n^2 + n - 42 = 0$. It follows that $n = 6$.

Also solved by the proposer.

5514: *Proposed by D. M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a \in \left(0, \dfrac{\pi}{2}\right)$ and $b = \arcsin a$, then calculate $\lim\limits_{n\to\infty} \sqrt[n]{n!}\left(\sin\left(\dfrac{b\cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right)$.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $\dfrac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \doteq q_n$.

Result I. $\lim\limits_{n\to\infty} q_n = 1$.

It is equivalent to

$$\lim_{n\to\infty} \frac{\ln(2n+1)!!}{n+1} - \frac{\ln(2n-1)!!}{n} = 0 \tag{1}$$

that is

$$\lim_{n\to\infty} \frac{1}{n+1}\ln\frac{(2n+2)!}{2^{n+1}(n+1)!} - \frac{1}{n}\ln\frac{(2n)!}{2^n n!} = 0$$

Let's break the above limit as

$$\lim_{n\to\infty}\left(\frac{\ln(2n+2)!}{n+1} - \frac{\ln(2n)!}{n}\right) + \lim_{n\to\infty}\left(\frac{\ln(2^{-n-1})}{n+1} - \frac{\ln(2^{-n})}{n}\right) + \\ + \lim_{n\to\infty}\left(\frac{-\ln((n+1)!)}{n+1} + \frac{\ln n!}{n}\right)$$

$$\lim_{n\to\infty}\left[\frac{\ln(2n+2)!}{n+1} - \frac{\ln(2n)!}{n}\right] = \lim_{n\to\infty}\left[\frac{\ln(2n)!}{n+1} - \frac{\ln(2n)!}{n} + \frac{\ln(2n+2)}{n+1} + \frac{\ln(2n+1)}{n+1}\right] =$$
$$= \lim_{n\to\infty} -\frac{\ln(2n)!}{n(n+1)} \underbrace{=}_{C.S.} \lim_{n\to\infty} -\frac{\ln(2n+2)! - \ln(2n)!}{(n+1)(n+2) - n(n+1)} =$$
$$= \lim_{n\to\infty} -\frac{\ln((2n+2)(2n+1))}{2n+2} = 0$$

C.S. stands for Cesàro–Stolz.

$$\frac{\ln(2^{-n-1})}{n+1}-\frac{\ln(2^{-n})}{n}=0$$

$$\lim_{n\to\infty}\frac{-\ln((n+1)!)}{n+1}+\frac{\ln n!}{n}=\lim_{n\to\infty}\frac{\ln n!}{n}-\frac{-\ln((n)!)}{n+1}-\lim_{n\to\infty}\frac{-\ln(n+1)}{n+1}=$$
$$=\lim_{n\to\infty}\frac{\ln n!}{n(n+1)}\underbrace{=}_{C.S.}\lim_{n\to\infty}\frac{\ln((n+1)!)-\ln n!}{(n+1)(n+2)-n(n+1)}=\lim_{n\to\infty}\frac{\ln(n+1)}{2n+2}=0$$

Result II.

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\underbrace{=}_{C.S.}\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^n=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}=\frac{1}{e}$$

Result III.

$$\lim_{n\to\infty}n(q_n-1)=\lim_{n\to\infty}n\ln q_n\cdot\frac{q_n-1}{\ln q_n}=\lim_{n\to\infty}n\ln q_n=$$
$$=\lim_{n\to\infty}n\left[\frac{\ln(2n+1)!!}{n+1}-\frac{\ln(2n-1)!!}{n}\right]=\lim_{n\to\infty}\ln\frac{(2n+1)!!}{(2n-1)!!}-\frac{\ln(2n+1)!!}{n+1}=$$
$$=\lim_{n\to\infty}\frac{(n+1)\ln(2n+1)-\ln(2n+1)!!}{n+1}\underbrace{=}_{C.S.}$$
$$=\lim_{n\to\infty}\frac{(n+2)\ln(2n+3)-(n+1)\ln(2n+1)-\ln((2n+1)!!)+\ln((2n-1)!!)}{n+2-n-1}=$$
$$=\lim_{n\to\infty}(n+2)\ln\left(1+\frac{2}{2n+1}\right)=1$$

The limit we are searching is

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}n\left[\sin(bq_n)-\sin b\right]=\lim_{n\to\infty}\frac{n}{e}2\left[\sin\frac{b(q_n-1)}{2}\underbrace{\cos\frac{b(q_n+1)}{2}}_{\to\sqrt{1-a^2}}\right]=$$
$$=\lim_{n\to\infty}\frac{2}{e}\sqrt{1-a^2}n(q_n-1)\frac{\sin\frac{b(q_n-1)}{2}}{q_n-1}=\frac{b}{e}\sqrt{1-a^2}\lim_{n\to\infty}n(q_n-1)=$$
$$=\frac{b}{e}\sqrt{1-a^2}$$

$|a|\le 1$ should have been in the statement.

Solution by 2 Moti Levy, Rehovot, Israel

$$(2n-1)!!=\frac{(2n)!}{2^n n!}.\qquad(1)$$

Using the Stirling 's asymptotic formula, we have

$$n!\sim\frac{n^n}{e^n}.\qquad(2)$$

Applying (2) to (1) yields

$$\sqrt[n]{n!} \sim \frac{n}{e},$$

$$\sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}, \qquad \sqrt[n+1]{(2n+1)!!} \sim \frac{2n+2}{e},$$

$$\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \sim \frac{2n+2}{e}\frac{e}{2n} = 1 + \frac{1}{n}.$$

$$\sqrt[n]{n!}\left(\sin\left(b\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right) \sim \frac{n}{e}\left(\sin\left(b\left(1+\frac{1}{n}\right)\right) - \sin b\right)$$

$$= 2\frac{n}{e}\sin\frac{b\left(1+\frac{1}{n}\right)-b}{2}\cos\frac{b\left(1+\frac{1}{n}\right)+b}{2}$$

$$= \frac{b}{e}\left(\frac{\sin\frac{b}{2n}}{\frac{b}{2n}}\right)\cos\left(b\left(1+\frac{1}{2n}\right)\right) \to \frac{b\cos b}{e}.$$

We conclude that $\lim_{n\to\infty}\sqrt[n]{n!}\left(\sin\left(b\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right) = \dfrac{b\cos b}{e}$.

Also solved by Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5515: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Prove that

$$\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 \geq 1.$$

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

Let $n \geq 1$ and $1 \leq k \leq n$. Then, since $f(x) = \sqrt{x}$ is concave on $(0,\infty)$, Jensen's

Theorem implies that

$$
\begin{aligned}
\sqrt{\frac{1}{n^2}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2} &= \sqrt{2}\sqrt{\frac{\frac{1}{n^2}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2}{2}} \\
&= \sqrt{2} f\left(\frac{\frac{1}{n^2}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2}{2}\right) \\
&\geq \sqrt{2}\frac{f\left(\frac{1}{n^2}\right)+f\left[\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2\right]}{2} \\
&= \frac{\frac{1}{n}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}}{\sqrt{2}}.
\end{aligned}
$$

If we let $i=k-1$ for $k=1,\ldots,n$ and use the known result that

$$
\sum_{i=0}^{m}\begin{pmatrix} m \\ i \end{pmatrix}=2^m
$$

for $m \geq 0$, it follows that

$$
\begin{aligned}
\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2} &\geq \frac{1}{\sqrt{2}}\left[\sum_{k=1}^{n}\frac{1}{n}+\sum_{k=1}^{n}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}\right] \\
&= \frac{1}{\sqrt{2}}\left[1+\sum_{i=0}^{n-1}\begin{pmatrix} n-1 \\ i \end{pmatrix}\right] \\
&= \frac{1}{\sqrt{2}}\left(1+2^{n-1}\right).
\end{aligned}
$$

Further, the Arithmetic - Geometric Mean Inequality yields

$$
\begin{aligned}
1+2^{n-1} &\geq 2\sqrt{2^{n-1}} \\
&= 2\cdot 2^{\frac{n-1}{2}} \\
&= 2^{\frac{n+1}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2} &\geq \frac{1}{\sqrt{2}}\left(1+2^{n-1}\right) \\
&\geq \frac{1}{\sqrt{2}}2^{\frac{n+1}{2}}. \\
&= 2^{\frac{n}{2}},
\end{aligned}
$$

and we have

$$
\begin{aligned}
\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2}\right)^2 &\geq \frac{1}{2^n}\left(2^{\frac{n}{2}}\right)^2 \\
&= 1.
\end{aligned}
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The statement hold true for $n=1$. Let $n>1$. Then

$$\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 \geq \frac{1}{2^n}\left(\sum_{k=1}^{n}\binom{n-1}{k-1}\right)^2 = \frac{1}{2^n}\left(2^{n-1}\right)^2 = 2^{n-2} \geq 1,$$

as claimed.

Solution 3 by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

For $n=1$ the equality holds. For $n>1$, we have

$$\begin{aligned}
\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 &> \frac{1}{2^n}\left(\sum_{k=1}^{n}\binom{n-1}{k-1}\right)^2 \\
&= \frac{1}{2^n}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}\right)^2 \\
&= \frac{1}{2^n}\left(2^{n-1}\right)^2 \\
&= \frac{2^{2n-2}}{2^n} \\
&= 2^{n-2} \\
&\geq 1.
\end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Henry W. Gould, West Virginia University, Morgantown, WV with Scott H. Brown, Auburn University, Montgomery, AL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Adrian Naco, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy Ioannis D.Sfikas, National and Kapodistrian University of Athens, Greece; Ramiz Valizada (student of Yagub N. Aliyev), ADA University, Baku, Azerbaijan, and the proposer.

5516: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\displaystyle\sum_{n=1}^{\infty} n\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)$.

Solution 1 by Michel Bataille, Rouen, France

Let $S=\sum\limits_{n=1}^{\infty} n\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)$. We claim that $S=\frac{1}{4}-\frac{\pi^2}{12}$.

We have $\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}=\sum\limits_{k=n+1}^{\infty}\frac{1}{k^3}$ and

$$\frac{1}{n^2}=\sum_{k=n+1}^{\infty}\left(\frac{1}{(k-1)^2}-\frac{1}{k^2}\right)=\sum_{k=n+1}^{\infty}\frac{2k-1}{k^2(k-1)^2},$$

hence

$$\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} - \frac{1}{2n^2} = \sum_{k=n+1}^{\infty} \left(\frac{1}{k^3} - \frac{2k-1}{2k^2(k-1)^2} \right) = \sum_{k=n+1}^{\infty} \frac{2-3k}{2k^3(k-1)^2}$$

and so $S = \sum\limits_{n=1}^{\infty} n a_n$ with $a_n = \sum\limits_{k=n+1}^{\infty} \frac{2-3k}{2k^3(k-1)^2}$.

Now, let $S_N = \sum\limits_{n=1}^{N} n a_n$ where N is an integer with $N > 2$. Summing by parts gives

$$S_N = \sum_{n=1}^{N-1} \frac{n(n+1)}{2}(a_n - a_{n+1}) + \frac{N(N+1)}{2}\, a_N. \tag{1}$$

But, as $k \to \infty$, we have

$$\begin{aligned} \frac{2-3k}{2k^3(k-1)^2} &= \frac{1}{k^3} - \frac{2k(1-\frac{1}{2k})}{2k^4(1-\frac{1}{k})^2} \\ &= \frac{1}{k^3} - \frac{1}{k^3}\left(1 - \frac{1}{2k}\right)\left(1 - \frac{1}{k}\right)^{-2} \\ &= \frac{1}{k^3}\left(1 - \left(1 - \frac{1}{2k}\right)\left(1 + \frac{2}{k} + o(1/k)\right)\right) \\ &= \frac{1}{k^3}\left(-\frac{3}{2k} + o(1/k)\right) \sim \frac{-3}{2k^4} \end{aligned}$$

and so

$$a_N \sim -\frac{3}{2} \sum_{k=N+1}^{\infty} \frac{1}{k^4} \sim -\frac{3}{2} \cdot \frac{1}{3N^3} = -\frac{1}{2N^3} \qquad (N \to \infty).$$

It readily follows that $\lim\limits_{N\to\infty} \frac{N(N+1)}{2}\, a_N = 0$.

On the other hand, $a_n - a_{n+1} = \frac{2-3(n+1)}{2(n+1)^3 n^2} = \frac{1}{(n+1)^3} - \frac{2n+1}{2n^2(n+1)^2}$ and a simple calculation yields

$$\frac{n(n+1)}{2}(a_n - a_{n+1}) = \frac{-(3n+1)}{4n(n+1)^2} = -\frac{1}{4} \cdot \left(\frac{1}{n} - \frac{1}{n+1} + \frac{2}{(n+1)^2} \right).$$

As a result, we obtain

$$\sum_{n=1}^{N-1} \frac{n(n+1)}{2}(a_n - a_{n+1}) = -\frac{1}{4}\left(\sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \sum_{n=1}^{N-1} \frac{2}{(n+1)^2} \right).$$

Since $\lim\limits_{N\to\infty} \sum\limits_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ and $\lim\limits_{N\to\infty} \sum\limits_{n=1}^{N-1} \frac{2}{(n+1)^2} = 2\left(\frac{\pi^2}{6} - 1\right)$, we deduce (using (1)) that

$$S = \lim_{N\to\infty} S_N = -\frac{1}{4}\left(1 + \frac{\pi^2}{3} - 2\right) = \frac{1}{4} - \frac{\pi^2}{12},$$

as claimed.

Solution 2 by Paola Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The sum is

$$\lim_{N\to\infty}\sum_{n=1}^{N} n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^3}-\frac{1}{2n^2}\right)$$

Summation by parts

$$\sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=2}^{n} A_{k-1}(b_{k-1}-b_k), \qquad A_n = \sum_{k=1}^{n} a_k$$

gives

$$\frac{N(N+1)}{2}\left(\sum_{k=N+1}^{\infty}\frac{1}{k^3}-\frac{1}{2N^2}\right)+\sum_{k=2}^{N}\frac{k(k-1)}{2}\left(\frac{1}{2k^2}-\frac{1}{2(k-1)^2}+\frac{1}{k^3}\right)=$$

$$=\frac{N(N+1)}{2}\underbrace{\Big(\sum_{k=N+1}^{\infty}\frac{1}{k^3}-\frac{1}{2N^2}\Big)}_{=A}+\sum_{k=2}^{N}\left(-\frac{1}{2k^2}+\frac{1}{4k}-\frac{1}{4(k-1)}\right)$$

We know that

$$\frac{1}{2(N+1)^2}-\frac{1}{2N^2}=\int_{N+1}^{\infty}\frac{dx}{x^3}-\frac{1}{2N^2}<A<\int_{N}^{\infty}\frac{dx}{x^3}-\frac{1}{2N^2}=0$$

and

$$\lim_{N\to\infty}\frac{N(N+1)}{2}\Big(\frac{1}{2(N+1)^2}-\frac{1}{2N^2}\Big)=\lim_{N\to\infty}\frac{-4N-2}{2N(N+1)}=0$$

thus

$$\lim_{N\to\infty}\frac{N(N+1)}{2}\left(\int_{N+1}^{\infty}\frac{dx}{x^3}-\frac{1}{2N^2}\right)=0$$

while

$$\lim_{N\to\infty}\sum_{k=2}^{N}\left(-\frac{1}{2k^2}+\frac{1}{4k}-\frac{1}{4(k-1)}\right)=-\frac{\pi^2}{12}+\frac{1}{2}-\frac{1}{4}=\frac{3-\pi^2}{12}$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that

$$\sum_{n=1}^{\infty} n(\left(\zeta(3)-1-\frac{1}{2^3}-\cdots\frac{1}{n^3}-\frac{1}{2n^2}\right)=\frac{3-\pi^2}{12} \tag{1}$$

For $x>0$, denote by $f(x)$ the function $\dfrac{1}{x^3}$, so that

$$\zeta(3)-1-\frac{1}{2^3}-\ldots-\frac{1}{n^3}=\sum_{k=n+1}^{\infty}f(k).$$

It can be proved readily by induction that for positive integers M, we have

$$2\sum_{n=1}^{M} n\left(\sum_{k=n+1}^{\infty} f(k)-\frac{1}{2n^2}\right)=M(M+1)\sum_{n=M+2}^{\infty} f(n)+\frac{1}{M+1}-\sum_{n=1}^{M+1}\frac{1}{n^2}. \qquad (2)$$

Since $f(x)$ is strictly decreasing, so

$$\frac{1}{2(M+2)^2}=\int_{M+2}^{\infty} f(x)dx<\sum_{n=M+2}^{\infty} f(n)<\int_{M+1}^{\infty} f(x)dx=\frac{1}{2(M+1)^2}.$$

It follows that

$$\lim_{M\to\infty} M(M+1)\sum_{n=M+2}^{\infty} f(n)=\frac{1}{2} \qquad (3)$$

Now (1) follows from $(2),(3)$ and the facts that $\lim\limits_{M\to\infty}\frac{1}{M+1}=0$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

Also solved by Ed Gray, Highland Beach, FL (partial solution); Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

Stanley Rabinowitz of Chelmsford, MA should have been credited with having solved 5506. Like several of the other readers, he generalized problem 5506 and I had marked his solution to this generalization for publication. It was inadvertently omitted from the January issue of the column, and so it is being listed here.

Solution to 5506 by Stanley Rabinowitz, Chelmsford, MA

We will find the more general solution: $\Omega_n=\det\left[\begin{pmatrix}1 & c\\ c & c^2\end{pmatrix}^n+\begin{pmatrix}c^2 & -c\\ -c & 1\end{pmatrix}^n\right]$.

Let $\mathbf{A}=\begin{pmatrix}1 & c\\ c & c^2\end{pmatrix}$, $\mathbf{B}$=$\begin{pmatrix}c^2 & -c\\ -c & 1\end{pmatrix}$, and $\mathbf{S}=\mathbf{A}+\mathbf{B}$. Although matrix multiplication is not commutative, it is associative. In the expansion of $(\mathbf{A}+\mathbf{B})^n$, every term except $\mathbf{A}^n$ and $\mathbf{B}^n$has an $\mathbf{A}$ next to a $\mathbf{B}$. Since $\mathbf{AB}=\mathbf{BA}=\mathbf{0}$, therefore $\mathbf{S}^n=\mathbf{A}^n+\mathbf{B}^n$.

Thus, $\Omega_n=\det[\mathbf{A}^n+\mathbf{B}^n]=\det[\mathbf{S}^n]=(\det[\mathbf{S}])^n=\begin{vmatrix}1+c^2 & 0\\ 0 & c^2+1\end{vmatrix}^n=(c^2+1)^{2n}$.

The solution by Paul M. Harms of North Newton, KS to 5506 was received by yours truly three weeks after he had mailed it. His method of solution is also unique.

Let A be the matrix in the problem with two elements of 5 and let B be the matrix in the problem with two elements of -5. We have

$$A^2=26A,\ A^3=26A^2=26^2A,\ldots,\ A^{100}=26^{99}A.$$

In a similar manner, $B^2=26B,\ B^3=26^2B,\ \ldots,B^{100}=26^{99}B$. Then the matrix $A^{100}+B^{100}$ has the number $26^{99}(1+25)=26^{100}$ along the main diagonal and

$26^{99}(5+(-5)) = 0$ for the other two elements . The value of the determinate in the problem is then $(26^{100})^2 = 26^{200}$.

G. C. Greubel of Newport News, VA should have been credited for solving 5505 and 5510. His solution to 5505 also developed the generalization stated above, and his solution for 5510 also generalized the problem showing:

$$S(a) = \sum_{n=1}^{\infty} \left[a^{2n} \left(\zeta(2n) - 1\right) - \sum_{k=2}^{a} \left(\frac{a}{k}\right)^{2n} \right],$$

can be reduced to

$$S(a) = \frac{a}{2} \sum_{k=1}^{2a} \frac{1}{k} = \frac{a}{2}\, H_{2a},$$

where H_n is the n^{th} Harmonic number. By setting $a = 2, 3, 4$ he quickly determined that

$$\sum_{k=1}^{\infty} \left[4^n \left(\zeta(2n) - 1\right) - 1\right] = H_4 = \frac{25}{12},$$

$$\sum_{k=1}^{\infty} \left[9^n \left(\zeta(2n) - 1\right) - 1 - \left(\frac{3}{2}\right)^{2n} \right] = H_6 = \frac{49}{20}, \text{ and}$$

$$\sum_{k=1}^{\infty} \left[16^n \left(\zeta(2n) - 1\right) - 1 - 4^n - \left(\frac{4}{3}\right)^{2n} \right] = H_8 = \frac{761}{280}.$$

Editor′s Comments

Late solutions were received to the following problems;

5506: Aydin Javadov, (student of Yagub Aliyev), ADA University, Baku, Azerbaijan.

5508: Rasul Balayev, Ilkin Guluzada, Nuru Nurdil, and Leyla Shamoyeva (students of Yagub Aliyev), ADA University, Baku, Azerbaijan.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2019

5535: *Proposed by Kenneth Korbin, New York, NY*

Given positive angles A and B with $A + B = 180^o$. A circle with radius 3 and a circle of radius 4 are each tangent to both sides of $\angle A$. The circles are also tangent to each other Find $\sin A$.

5536: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu,"George Emil Palade" School, Buzău, Romania*

If $a \in (0,1)$ then calculate $\lim\limits_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\sin\left(\dfrac{a \cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right) - \sin a\right)$.

5537: *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X, Y be two real-valued continuous random variables on the real line with associated mean, median and mode $\overline{x}, \tilde{x}, \hat{x}$, and $\overline{y}, \tilde{y}, \hat{y}$, respectively. For each of the following conditions, show that there are variables X,Y satisfying them or prove such random variables do not exist.

$$\begin{array}{llllcllll}
(i) & \overline{x} \le \overline{y}, & \tilde{x} \le \tilde{y}, & \hat{x} \le \hat{y}, & \quad & (v) & \overline{x} > \overline{y}, & \tilde{x} \le \tilde{y}, & \hat{x} \le \hat{y} \\
(ii) & \overline{x} \le \overline{y}, & \tilde{x} \le \tilde{y}, & \hat{x} > \hat{y}, & & (vi) & \overline{x} > \overline{y}, & \tilde{x} \le \tilde{y}, & \hat{x} > \hat{y} \\
(iii) & \overline{x} \le \overline{y}, & \tilde{x} > \tilde{y}, & \hat{x} \le \hat{y}, & & (vii) & \overline{x} > \overline{y}, & \tilde{x} > \tilde{y}, & \hat{x} \le \hat{y} \\
iv) & \overline{x} \le \overline{y}, & \tilde{x} > \tilde{y}, & \hat{x} > \hat{y}, & & (viii) & \overline{x} > \overline{y}, & \tilde{x} > \tilde{y}, & \hat{x} > \hat{y}
\end{array}$$

5538: *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve for all real numbers $x \neq \dfrac{\pi}{2}(2k+1)$, $k \in Z$.

$$2 - 2019x = e^{\tan x} + 3^{\sin x} + \tan^{-1} x.$$

5539: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let α, β, γ be nonzero real numbers. Find the minimum value of

$$\left(\sum_{cyclic} \left(\frac{1 + \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha} \right)^3 \right)^{1/3}$$

5540: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A \in M_2(\Re)$ be a matrix which has real eigenvalues. Prove that if $\sin A$ is similar to A then $\sin A = A$.

Solutions

5517: *Proposed by Kenneth Korbin, New York, NY*

Find positive integers (a, b, c) such that
$\arccos\left(\dfrac{a}{1331}\right) = \arccos\left(\dfrac{b}{1331}\right) + \arccos\left(\dfrac{c}{1331}\right)$ with $a < b < c$.

Solution 1 by David E. Manes, Oneonta, NY

If $(a, b, c) = (370, 869, 1210)$, then

$$\arccos\left(\frac{370}{1331}\right) = \arccos\left(\frac{869}{1331}\right) + \arccos\left(\frac{1210}{1331}\right) \approx 1.28\,909\,899\,845.$$

Writing the arccosine equation in terms of the cosine function and using the identity $\sin(\arccos x) = \sqrt{1 - x^2}$, one obtains

$$\cos\left(\arccos\left(\frac{a}{1331}\right)\right) = \cos\left[\arccos\left(\frac{b}{1331}\right) + \arccos\left(\frac{c}{1331}\right)\right].$$

Therefore,

$$\begin{aligned}
\frac{a}{1331} &= \cos\left(\arccos\left(\frac{b}{1331}\right)\right)\cos\left(\arccos\left(\frac{c}{1331}\right)\right) - \sin\left(\arccos\left(\frac{b}{1331}\right)\right)\sin\left(\arccos\left(\frac{c}{1331}\right)\right) \\
&= \left(\frac{b}{1331}\right)\left(\frac{c}{1331}\right) - \sqrt{1 - \left(\frac{b}{1331}\right)^2}\sqrt{1 - \left(\frac{c}{1331}\right)^2} \\
&= \frac{bc}{(1331)^2} - \sqrt{1 - \frac{b^2}{(1331)^2} - \frac{c^2}{(1331)^2} + \frac{b^2c^2}{(1331)^4}} \\
&= \frac{bc}{(1331)^2} - \frac{\sqrt{(1331)^4 - (1331)^2b^2 - (1331)^2c^2 + b^2c^2}}{(1331)^2}.
\end{aligned}$$

Thus,

$$\frac{\sqrt{(1331)^4 - (1331)^2b^2 - (1331)^2c^2 + b^2c^2}}{1331} = \frac{bc}{1331} - a.$$

Squaring both sides of this equation yields

$$\frac{(1331)^4-(1331)^2b^2-(1331)^2c^2+b^2c^2}{(1331)^2}=a^2+\frac{b^2c^2}{(1331)^2}-\frac{2abc}{1331},$$
$$(1331)^4-(1331)^2b^2-(1331)^2c^2=(1331)^2a^2-2abc(1331),$$
$$(1331)^3-1331b-1331c=1331a^2-2abc.$$

Writing this equation as a quadratic in a, we get

$$1331a^2-(2bc)a+1331\left(b^2+c^2-(1331)^2\right)=0.$$

Therefore, by the quadratic formula

$$a=\frac{2bc\pm\sqrt{4b^2c^2-4(1331)^2\left(b^2+c^2-(1331)^2\right)}}{2(1331)}.$$

This equation reduces to

$$a=\frac{bc}{1331}\pm\sqrt{\left(\frac{bc}{1331}\right)^2-(b^2+c^2-(1331)^2)}.$$

Noting that $1331=11^3$, we choose values for b and c that are divisible by powers of 11. We summarize the results.

1. If $b=79\cdot 11=869$ and $c=10\cdot 11^2=1210$, then

$$a=\left(\frac{(79\cdot 11)\left(10\cdot 11^2\right)}{1331}\right)\pm\sqrt{((79\cdot 10)^2-((79\cdot 11)^2+(10\cdot 11^2)^2-(1331)^2))}$$
$$=790\pm 420.$$

If $a=790+420=1210=c$, then this root is extraneous. If $a=790-420=370$, then $a<b<c$ and

$$\arccos\left(\frac{370}{1331}\right)=\arccos\left(\frac{869}{1331}\right)+\arccos\left(\frac{1210}{1331}\right)\approx 1.28\,909\,899\,845.$$

All of the following solutions are obtained from integer values for b' and c' such that a' is also an integer that satisfies the above equation for a with b' and c' substituted for b and c, respectively. The integer values for a', b' and c' do not satisfy the parameters of the problem. The values for a, b and c are then obtained as a permutation of a', b' and c' such that $a<b<c$ and the inverse cosine equation is satisfied.

2. If $b'=49\cdot 11=539$ and $c'=6\cdot 11^2=726$, then $a'=294\pm 1020$. Therefore, define a, b and c so that $a=539$, $b=726$ and $c=294+1020=1314$. Then $a<b<c$ and

$$\arccos\left(\frac{539}{1331}\right)=\arccos\left(\frac{726}{1331}\right)+\arccos\left(\frac{1314}{1331}\right)\approx 1.15\,386\,269\,047.$$

3. If $b'=89\cdot 11=979$ and $c'=4\cdot 11^2=484$, then $a'=356\pm 840$. Define the values of a, b and c so that $a=484$, $b=979$ and $c=356+840=1196$. Then $a<b<c$ and

$$\arccos\left(\frac{484}{1331}\right)=\arccos\left(\frac{979}{1331}\right)+\arccos\left(\frac{1196}{1331}\right)\approx 1.19\,862\,779\,283.$$

4. If $b' = 103 \cdot 11 = 1133$ and $c' = 3 \cdot 11^2 = 363$, then $a' = 309 \pm 672$. The values of a, b and c are then $a = 363$, $b = 309 + 672 = 981$ and $c = 1133$. Then $a < b < c$ and

$$\arccos\left(\frac{363}{1331}\right) = \arccos\left(\frac{981}{1331}\right) + \arccos\left(\frac{1133}{1331}\right) \approx 1.29\,456\,969\,603.$$

5. If $b' = 113 \cdot 11 = 1243$ and $c' = 2 \cdot 11^2 = 242$, then $a' = 226 \pm 468$. The values of a, b and c are now $a = 242$, $b = 226 + 468 = 694$ and $c = 1243$. Then $a < b < c$ and

$$\arccos\left(\frac{242}{1331}\right) = \arccos\left(\frac{694}{1331}\right) + \arccos\left(\frac{1243}{1331}\right) \approx 1.38\,796\,118\,98.$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

A computer program found the following 8 integer solutions (a, b, c) of

$$\arccos\frac{a}{1331} = \arccos\frac{b}{1331} + \arccos\frac{c}{1331}$$

with $0 < a < b < c$:

$$\begin{gathered}(121, 359, 1309)\\(242, 694, 1243)\\(253, 847, 1169)\\(363, 981, 1133)\\(370, 869, 1210)\\(484, 979, 1196)\\(539, 726, 1314)\\(605, 781, 1315)\end{gathered}$$

Remark: Unfortunately, I don't know a systematic way to find these solutions without a computer.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\frac{a}{11^3} = \cos\left(\arccos\left(\frac{a}{11^3}\right)\right) = \cos\left(\arccos\left(\frac{b}{11^3}\right) + \arccos\left(\frac{c}{11^3}\right)\right) =$$

$$= \frac{b}{11^3} \cdot \frac{c}{11^3} - \sqrt{1 - \frac{b^2}{11^6}} \cdot \sqrt{1 - \frac{c^2}{11^6}},$$

which implies

$$\left(\frac{a}{11^3} - \frac{bc}{11^6}\right)^2 = \left(1 - \frac{b^2}{11^6}\right)\left(1 - \frac{c^2}{11^6}\right),$$

or equivalently,

$$11^3\left(a^2 + b^2 + c^2\right) = 11^9 + 2abc. \tag{*}$$

An exhaustive computer search in the range $0 < a < b < c \leq 1331$ reveals that $(*)$ implies $(a, b, c) \in \{(121, 359, 1309), (242, 694, 1243), (253, 847, 1169), (363, 981, 1133), (370, 869, 1210), (484, 979, 1196), (539, 726, 1314), (605, 781, 1315)\}$.

Note: $(*)$ is equivalent to $\left(\dfrac{a}{11^3}\right)^2 + \left(\dfrac{b}{11^3}\right)^2 + \left(\dfrac{a}{11^3}\right)^2 = 1 + 2\dfrac{a}{11^3}\cdot\dfrac{b}{11^3}\cdot\dfrac{c}{11^3}$.

The Diophantine equation

$$x^2 + y^3 = 2 + z^2 = 1 + 2xyz$$

has been extensively studied in the literature, see the references

[1] L. J. Mordell, On the Integer Solutions of the Equation $x^2 + y^2 + z^2 + 2xyz = n$ Journal of the London Mathematical Society, Volumes 1-28, Issue 4, 1 October 1953, Pages 500-510, https://doi.org/10.1112/jlms/s1-28.4.500

[2] A. Oppenheim, "On the Diophantine Equation $x^2 + y^2 + z^2 + 2xyz = 1$." The American Mathematical Monthly, vol. 64, no. 2, 1957, pp. 101-103. DOI: 10.2307/2310390.

A. Oppenheim provides the general solution of $x^2 + y^2 + z^2 + 2xyz = 1$ in rational integers. The given problem asks for the solutions in rational numbers whereby the denominators equal 11^3.

Editor's comment: Ken Korbin, proposer of the problem, included in his solution some algebraic expressions and a geometric interpretation of the problem that gives us some insight into how he constructed the problem.

The expressions he listed are:

$$\begin{array}{ll} a) & 121N \\ b) & \left|\left(22N^2 - 1331\right)\right| \\ c) & \left|\left(363N - 4N^3\right)\right| \quad \text{with} \end{array}$$

$$0 < N < \frac{11\sqrt{2}}{2} \quad \text{or} \quad \frac{11\sqrt{3}}{2} < \mathrm{N} < 11.$$

So if $N = 19$, then $(a, b, c) = (370, 869, 1210)$, and if $N = 7$, then $(a, b, c) = (253, 847, 1169)$, and if $N = 6$, then $(a, b, c) = (539, 726, 1314)$, etc.

Suppose $(a, b, c) = (370, 869, 1210)$. Arrange four points A, B, C, D in a circular arrangement with the vertices being in a clockwise direction. Connecting the segments $\overline{AB}, \overline{BD}, \overline{AC}$, and $\overline{CD}$ gives us a diagram that resembles a butterfly.

Ken then stated that for this triplet, (a, b, c), there is a convex cyclic quadrilateral A, B, C, D with

$$\begin{array}{rcl} \overline{AC} & = & \text{Diameter} = 1331, \\ \overline{AB} & = & 1210, \\ \text{Diagonal } \overline{\mathrm{BD}} & = & 869, \text{ and } \overline{\mathrm{CD}} = 370. \end{array}$$

Also solved by, Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Ed Gray, Highland Beach, FL; David Stone and John Hawkins of Georgia Southern University, Statesboro GA, and the proposer.

5518: *Proposed by Roger Izard, Dallas, TX*

Let triangle PQR be equilateral and let it intersect another triangle ABC at points U, U', W, W', V, V' such that WU', UV', VW' are equal in length, and triangles $AU'W, BV'U, CW'V$ are equal in area (see Figure 1). Show that triangle ABC must then also be equilateral

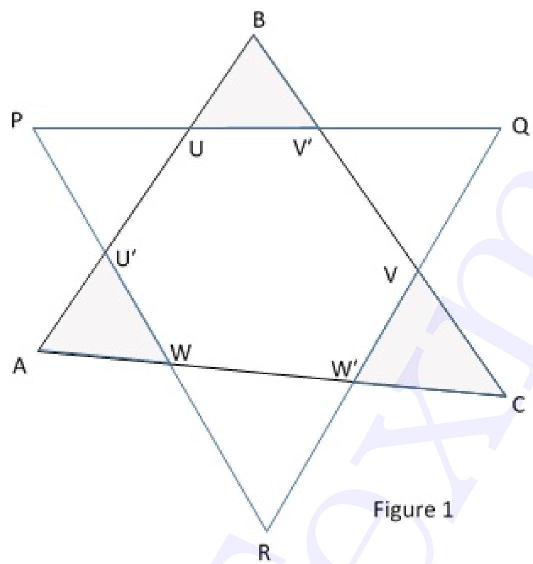

Figure 1

Solution 1 by Kee-Wai Lau, Hong Kong, China

Without loss of generality let $WU' = UV' = VW' = 1$. Let $\angle PU'U = \alpha, \angle RW'W = \beta, \angle QV'V = \gamma$. It is easy to check that $\angle BUV' = \dfrac{2\pi}{3} - \alpha,\ \angle BV'U = \gamma,\ \angle UBV' = \dfrac{\pi}{3} - \gamma + \alpha$. Applying the sine formula to triangle $BV'U$, we have $BU = \dfrac{\sin\gamma}{\sin\left(\frac{\pi}{3} - \gamma + \alpha\right)},\ BV' = \dfrac{\sin\left(\frac{\pi}{3} + \alpha\right)}{\sin\left(\frac{\pi}{3} - \gamma + \alpha\right)}$. Hence the area of triangle $BV'U$ equals $\dfrac{\sin\gamma\sin\left(\frac{\pi}{3} + \alpha\right)}{2\sin\left(\frac{\pi}{3} - \gamma + \alpha\right)} = \dfrac{1}{2\left(\cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right)\right)}$, using the formula $\sin x - y) = \sin x\cos y - \cos x\sin y$. Similarly the areas of triangles $AU'W$ and $CW'V$ are respectively $\dfrac{1}{2\left(\cot\alpha - \cot\left(\frac{\pi}{3} + \beta\right)\right)}$ and $\dfrac{1}{2\left(\cot\beta - \cot\left(\frac{\pi}{3} + \gamma\right)\right)}$. Given that these areas are equal, so,

$$\cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right) = \cot\alpha - \cot\left(\frac{\pi}{3} + \beta\right) = \cot\beta - \cot\left(\frac{\pi}{3} + \gamma\right).$$

We only consider the case $\alpha \geq \beta$, since the case $\alpha \leq \beta$ can be treated similarly. We have

$$\cot\alpha - \cot\left(\frac{\pi}{3} + \beta\right) = \cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right) \geq \cot\gamma - \cot\left(\frac{\pi}{3} + \beta\right),$$

so that $\gamma \geq \alpha$. Hence

$$\cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right) = \cot\beta - \cot\left(\frac{\pi}{3} + \gamma\right) \geq \cot\beta - \cot\left(\frac{\pi}{3} + \alpha\right),$$

implying $\beta \geq \gamma$. It follows that $\beta = \gamma\alpha$. Thus $\angle UBV' = \dfrac{\pi}{3}$, and similarly $\angle WAU' = \angle VCW' = \dfrac{\pi}{3}$. This shows that triangle ABC is also equilateral.

Solution 2 by Michael Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel

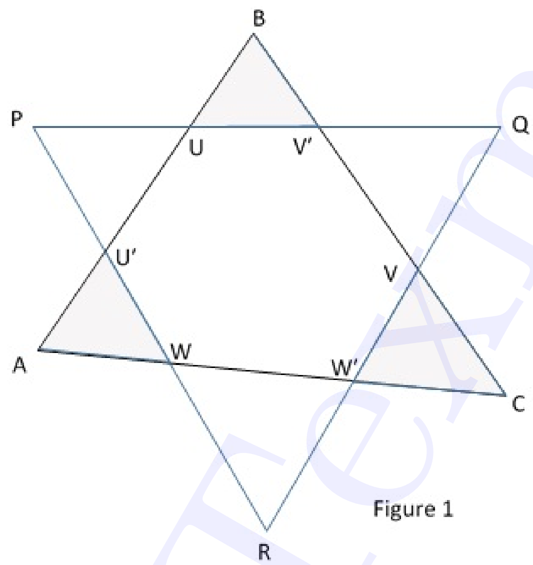

Figure 1

We can turn this into an equivalent problem in the following way. First, slide triangle $AU'W'$ along AB so that U' and U coincide and $CW'V$ along CB so that V' and V coincide (see Figure 2). Since PQR is an equilateral triangle the angles WUV and $W'VU$ are both $60°$. But since also $WU' = UV' = VW'$, the points W' and W must also coincide so that we have an equilateral triangle UVW inscribed in another triangle ABC (the latter is a triangle since AW and $W'C$ are always parallel to AB so that AWC is a straight line, while AUB and CVB are just segments of the original lines $AU'UB$ and $CVV'B$, respectively).

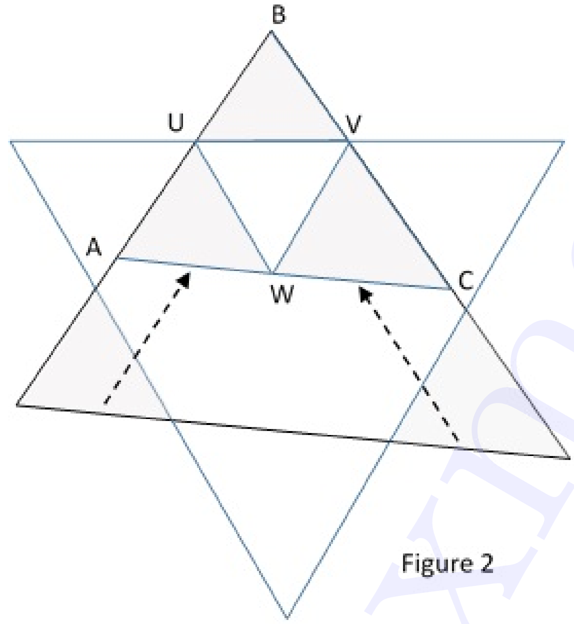

Figure 2

So the new problem can be stated as follows:
If a triangle ABC is circumscribed about an equilateral triangle UVW so that the areas AUW, BVU, CWV are equal, then ABC must also be equilateral.
But we can still do better. Suppose the common area of AUW, BVU, CWV is K, then the locus of all points A such that $AUW = K$ is a line parallel to UW. Similarly, the locus of all points B such that $BVU = K$ is a line parallel to UV. This line is also at the same distance from UV as the previous line is from UW. Finally, the locus of points C such that $CWV = K$ is again a line parallel to VW and at the same distance from VW as the previous line is from UV. These three parallel lines thus form another equilateral triangle XYZ whose sides are parallel to those of UVW and equidistant from them. So, the triangle ABC will circumscribe UVW and be inscribed in XYZ (see Figure 3). As a terminological convention, we will say that ABC is situated *between* UVW and XYZ

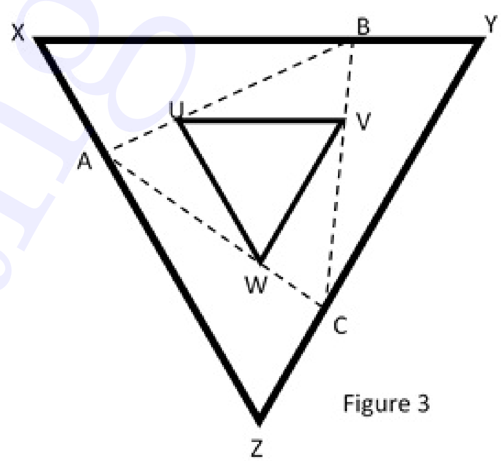

Figure 3

With that, we can formulate yet another problem equivalent to the first:
If UVW and XYZ are concentric equilateral triangles whose respective sides are parallel (UVW inside XYZ), then any triangle ABC situated between UVW and XYZ must be

equilateral.[1]
Now it is easy to show that if there is any such triangle ABC at all, there is at least one which is equilateral. Indeed, there is exactly one such equilateral triangle or there are exactly two: there cannot be more than two, but there can be none.
On each side of UVW draw circular arcs so that each side subtends an angle of $60°$. If the circles do not touch the sides of XYZ then can be no triangle such as ABC, for its angles, such as UAW, would all have to be less than $60°$ since their vertices would have to fall outside the circles (see Figure 4, left).
If the circles are tangent to the sides of XYZ (see Figure 4, right), then joining UA and UB, we that angle $XAU = XBU = 60°$ so that AUB is a straight line parallel to YZ. Similarly BVC and AWC are straight lines, so we obtain in this way one equilateral triangle ABC situated between UVW and XYZ. Moreover, there can be no other such triangle, for the angles of any other triangle, such as the angle $UA'W$ would fall outside the circle and therefore would, again, all be less than $60°$.

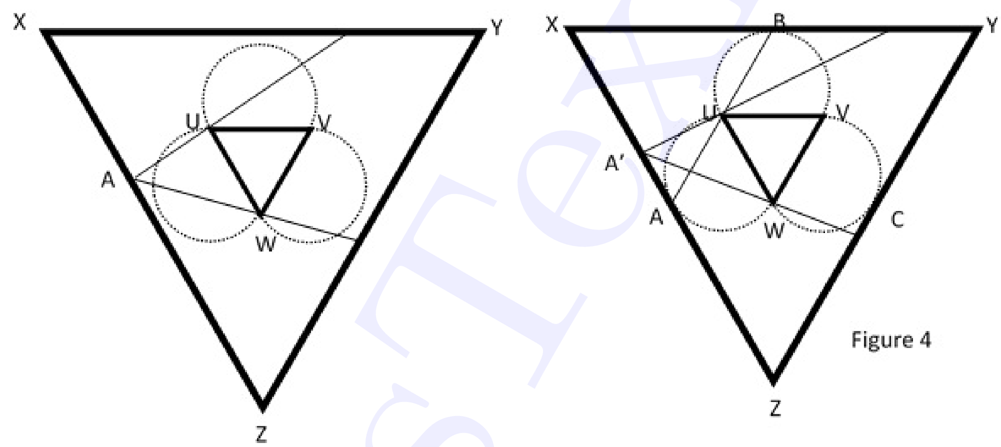

Figure 4

Consider then the last case in which the circles intersect each of the sides of XYZ in two points such as A and A' and B and B' in Figure 5.

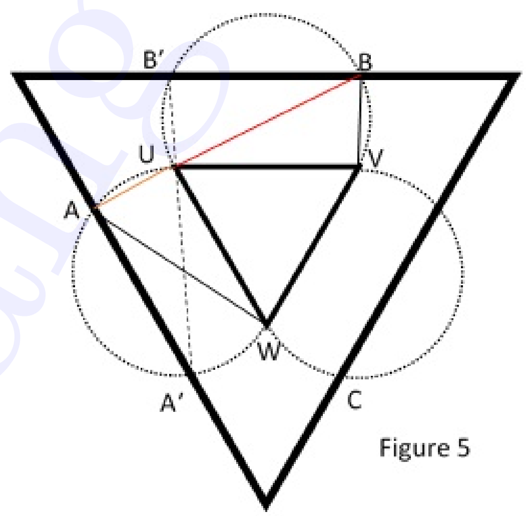

Figure 5

[1]All triangle centers of equilateral triangles coincide, so one can speak about concentric triangles without further ado. The two conditions assure that the corresponding sides of the two triangles are the same distance from one another

Join AU and UB. The angles at B and A are of course, by construction, both $60°$. Then since the distance between the sides of the two equilateral triangles is the same for all three sides (and because all the circles are obviously congruent) the arcs $BV, B'U, AU, A'W$ are all equal, so that also angle $AWU = BUV$. Hence, $AUW = 120° - AWU = 120° - BUV$, and therefore,

$$BUV + VUW + AUW = BUV + 60° + 120° - BUV = 180°$$

so that AUB is a straight line. Joining and extending AW and BV to the point C, we obtain an equilateral triangle situated between UVW and XYZ. A second equilateral triangle can be obtained by repeating the process beginning with points A' and B'. Now, to finish the proof, note that any other line from, say, XY to XZ via U must either begin from $B'B$ and end on $A'A$ or must begin outside $B'B$ and end outside $A'A$ (see Figure 6).

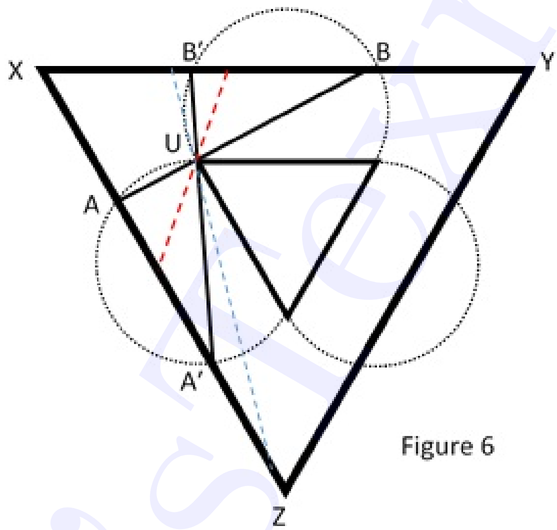

Figure 6

But, in the first case, a triangle ABC situated between UVW andXYZ would have all of its angles greater than $60°$ while, in the second case, all of the angles would be less than $60°$, which is impossible in either case (see Figure 7)

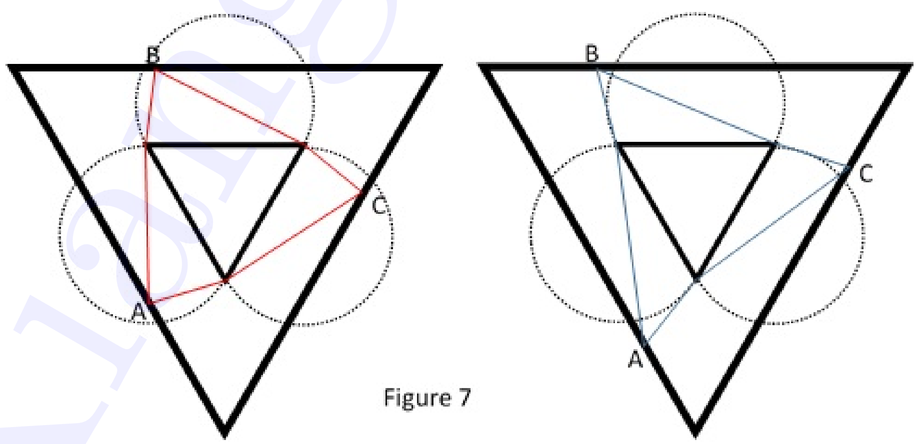

Figure 7

Thus, we can state the final version of the theorem as follows:
Given two concentric equilateral triangles whose sides are parallel there can only be either 0, 1, or 2 triangles situated between the two equilateral triangles, and in every case the triangles are themselves equilateral

Solution 3 by Ed Gray of Highland Beach, FL

$Editor's\ comment$**:** Following is a heuristic argument using the contrapositive of the statement.

Given equilateral triangle PQR, we have a triangle erected on each side of PQR such that each of the three triangles has equal area and equal base. As a consequence, each of the three must have equal altitudes. (We will attempt the proof going the other way). Let A, B, and C be the apex of each triangle and D, E, and F be points on PQ, QR, and RP respectively which are the feet of the three altitudes. (That is, the altitudes are BD, CF, and AD). If we connect A and B, this line intersects PQ at point U and PR at point U'. If we connect A with C, this line intersects PR at W and QR at W'. Finally, if we connect B with C, this line intersects PQ at V' and QR at V. There is a severe restriction that the lengths WU' on PR, UV' on PQ, and VW' on RQ must all be of the same length. The plan is to show that this can only happen if the points D, E, and F are the mid-points of PQ, QR, and RP respectively. First, we show that indeed, if D, E, and F are the mid-points, then the triangle ABC will be equilateral.

If we picture a coordinate system with $R = (0, 0), P = \left(\dfrac{-s}{2}, \dfrac{\sqrt{3}s}{2}\right)$, and $Q = \left(\dfrac{s}{2}, \dfrac{\sqrt{3}s}{2}\right)$, $s =$ side length of triangle PQR, the slope of PR is $-\sqrt{3}$, so the slope of AF is $\dfrac{1}{\sqrt{3}}$; similarly, the slope of QR is $+\sqrt{3}$, so the slope of EC is $\dfrac{-1}{\sqrt{3}}$. Since AF and EC are the same length, the difference in coordinates between C and E and A and F would be the same. It follows that AC is parallel to PQ. Similarly, BC is parallel to PR and AB is parallel to RQ. It follows that the angles A, B, and C are the same as P, Q and $R = 60^\circ$ and ABC is equilateral. Clearly, $UV' = VW' = WU'$.

Suppose now, at least one altitude foot is not the mid-point of an equilateral triangle side. For instance, if D is closer to P than Q, it is clear that BA is less than BC. If we try to compensate, say, by moving F closer to R, then AC will be the smaller side.

Also solved by the proposer.

5519: *Proposed by Titu Zvonaru, Comănesti, Romania*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{2abc}{a^3 + b^3 + c^3} \geq \frac{11}{3}. \qquad (1)$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{2abc}{a^3 + b^3 + c^3} - \frac{11}{3} = \frac{f(a, b, c)}{3a^2b^2c^2(a^3 + b^3 + c^3)}.$$

Then

$f(a, b, c) = 3a^5b^4 + 3a^2b^7 + 3a^7c^2 - 11a^5b^2c^2 + 3a^4b^3c^2 - 11a^2b^5c^2 + 6a^3b^3c^3 +$
$3a^2b^4c^3 + 3a^3b^2c^4 + 3b^5c^4 + 3a^4c^5 - 11a^2b^2c^5 + 3b^2c^7,$

and $f(a,b,c)$ is a cyclic polynomial in the variables a,b,c. We may assume without loss of generality that $a=\min(a,b,c)$. Therefore there are numbers $u\geq 0, v\geq 0, w\geq 0$ such that either $a=u, b=u+v, c=u+v+w$ or $a=u, c=u+v, b=u+v+w$.

A brute force calculation reveals that

$f(u,u+v,u+v+w)$
$=30u^7v^2+152u^6v^3+353u^5v^4+480u^4v^5+401u^3v^6+200u^2v^7+54uv^8+6v^9+30u^7vw$
$+264u^6v^2w+850u^5v^3w+1461u^4v^4w+1470u^3v^5w+856u^2v^6w+264uv^7w+33v^8w$
$+30u^7w^2+228u^6vw^2+885u^5v^2w^2+1892u^4v^3w^2+2311u^3v^4w^2+14589u^2v^5w^2+567uv^6w^2$
$+81v^7w^2+58u^6w^3+388u^5vw^3+1197u^4v^2w^3+1930u^3v^3w^3+1648u^2v^4w^3+702uv^5w^3$
$+117v^6w^3+71u^5w^4+396u^4vw^4+918u^3v^2w^4+1025u^2v^3w^4+540uv^4w^4+108v^5w^4$
$+55u^4w^5+230u^3vw^5+367u^2v^2w^5+252uv^2w^5+63v^4w^5+21u^3w^6+63u^2vw^6+$
$63u^2w^6+21v^3w^6+3u^2w^7+6uvw^7+3v^2w^7$

and

$f(u,u+v,u+v+w)$
$=30u^7v^2+152u^6v^3+353u^5v^4+480u^4v^5+401u^3v^6+200u^2v^7+54uv^8+6v^9+30u^7vw$
$+192u^6v^2w+562u^5v^3w+939u^4v^4w+936u^3v^5w+544u^2v^6w+168uv^7w+21v^8w$
$+30u^7w^2+156u^6vw^2+453u^5v^2w^2+848u^4v^3w^2+976u^3v^4w^2+653u^2v^5w^2+231uv^6w^2$
$+33v^7w^2+58u^6w^3+244u^5vw^3+513u^4v^2w^3+634u^3v^3w^3+457u^2v^4w^3+180uv^5w^3$
$+30v^6w^3+71u^5w^4+234u^4vw^4+309u^3v^2w^4+203u^2v^3w^4+75uv^4w^4+15v^5w^4$
$+55u^4w^5+116u^3vw^5+70u^2v^2w^5+12uv^2w^5+3v^4w^5+21u^3w^6+21u^2vw^6+3u^2w^7.$

All coefficients are positive. Therefore $f(a,b,c)\geq 0$, if $a\geq 0, b\geq 0, c\geq 0$.

Note: Let $f(a,b,c)$ be a cyclic real polynomial in the variables a,b,c (that is $f(a,b,c)=f(b,c,a)=f(c,a,b)$), which is claimed to be nonnegative for $a\geq 0, b\geq 0, c\geq 0$. It has happened to me multiple times that I was unable to apply the AM-GM inequality directly to prove that $f(a,b,c)\geq 0$ (assuming that $a\geq 0, b\geq, c\geq 0$). However the following brute force approach was mostly successful: due to the fact that $f(a,b,c)$ is cyclic one may assume that $a=\min(a,b,c)$. Then there are nonnegative variables u,v,w such that either $(a,b,c)=(u,u+v,u+v+w)$ or $(a,b,c)=(u,u+v+w,u+v)$. Then $f(u,u+v,u+v+w)$ and $f(u,u+v+w,u+v)$ are polynomials in u,v,w, and when multiplied out one sees (very often) that all coefficients are positive (as in the case above), showing that $f(a,b,c)\geq 0$ if $a\geq 0, b\geq 0, c\geq 0$. Multiplying out requires a computational effort, no doubt about that, but it is a purely mechanical task and does not require any creativity. Computer algebra systems are a very useful assistant for this specific computation.

Solution 2 by Moti Levy, Rehovot, Israel

Since the inequality is homogenous, then we may assume without loss of generality that $a+b+c=1$.
By Titu's lemma,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

Hence it is enough to prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2}+\frac{2abc}{a^3+b^3+c^3}\geq\frac{11}{3}. \tag{1}$$

Now we use the p,q,r notation: $p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$

The following equations and inequalities are well known:

$$a^2+b^2+c^2=p^2-2q, \tag{2}$$
$$a^3+b^3+c^3=p^3-3pq+3r, \tag{3}$$
$$r\le\frac{pq}{9}, \tag{4}$$
$$r\le\frac{p^3}{27}. \tag{5}$$

Using (2), (3) and setting $p=1$, inequality (1) can be rewritten as follows,

$$\frac{1}{1-2q}+\frac{2r}{1-3q+3r}-\frac{11}{3}\ge 0.$$

By inequality (4) $r\le\frac{q}{9}$, so that

$$\begin{aligned}\frac{1}{1-2q}+\frac{2r}{1-3q+3r}-\frac{11}{3}&\ge\frac{1}{1-18r}+\frac{2r}{1-24r}-\frac{11}{3}\\&=\frac{8\left(r-\frac{1}{27}\right)\left(r-\frac{2}{45}\right)}{3\left(r-\frac{1}{18}\right)\left(r-\frac{1}{24}\right)}.\end{aligned}$$

By inequality (5), $r\le\frac{1}{27}$ and it follows immediately that $\frac{8\left(r-\frac{1}{27}\right)\left(r-\frac{2}{45}\right)}{3\left(r-\frac{1}{18}\right)\left(r-\frac{1}{24}\right)}\ge 0.$

Solution 3 by Michel Bataille, Rouen France

Let L denote the left-hand side of the inequality and let $x=\frac{a}{b},\ y=\frac{b}{c},\ z=\frac{c}{a}$. Then, $x,y,z>0,\ xyz=1$ and

$$L=x^2+y^2+z^2+\frac{2}{\frac{x}{z}+\frac{y}{x}+\frac{z}{y}}.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)^2\le(x^2+y^2+z^2)\left(\frac{1}{z^2}+\frac{1}{x^2}+\frac{1}{y^2}\right)=(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2).$$

It follows that

$$L\ge x^2+y^2+z^2+\frac{2}{\sqrt{(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2)}}$$

and it is sufficient to prove that

$$u+v+w+\frac{2}{\sqrt{(u+v+w)(uv+vw+wu)}}\ge\frac{11}{3} \tag{1}$$

whenever $u,v,w>0$ and $uvw=1$.
Now, from $(u+v+w)^2=u^2+v^2+w^2+2(uv+vw+wu)\ge 3(uv+vw+wu)$ we deduce that $(u+v+w)^3\ge 3(u+v+w)(uv+vw+wu)$ so that (1) will certainly hold if

$$u+v+w+\frac{2\sqrt{3}}{(u+v+w)^{3/2}}\ge\frac{11}{3}. \tag{2}$$

To prove (2), we consider the function f defined on $(0,\infty)$ by $f(x)=x+2\sqrt{3}x^{-3/2}$. From the derivative $f'(x)=x^{-5/2}(x^{5/2}-3^{3/2})$ we deduce that f is increasing on

$[3^{3/5},\infty)$. Since $u+v+w \geq 3\sqrt[3]{uvw} = 3 > 3^{3/5}$, the inequality $f(u+v+w) \geq f(3) = \frac{11}{3}$ holds and (2) follows.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Two inequalities will be used.

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a^2+b^2+c^2}{\sqrt[2]{abc}} \geq 3\frac{a^2+b^2+c^2}{ab+bc+ca}$$

The L.H.S. is

$$a^2c+b^2a+c^2b \geq (a+b+c)(abc)^{2/3}$$

and

$$\frac{a^2c+a^2c+b^2a}{3} \geq a^{\frac{5}{3}}(bc)^{\frac{2}{3}}, \quad \frac{b^2a+b^2a+c^2b}{3} \geq (ac)^{\frac{2}{3}}b^{\frac{5}{3}}, \quad \frac{a^2c+b^2a+c^2b}{3} \geq (ab)^{\frac{2}{3}}c^{\frac{5}{3}}$$

Moreover

$$a^{\frac{5}{3}}(bc)^{\frac{2}{3}}+(ac)^{\frac{2}{3}}b^{\frac{5}{3}}+(ab)^{\frac{2}{3}}c^{\frac{5}{3}} = (a+b+c)(abc)^{2/3}$$

The R.H.S. follows trivially by $ab+bc+ca \geq 3(abc)^{\frac{2}{3}}$

It suffices to prove

$$3\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{2abc}{a^3+b^3+c^3} \geq \frac{11}{3} \tag{1}$$

Let's change variables $a+b+c=3u$, $ab+bc+ca=3v^2$, $abc=w^3$. The inequality (1) is

$$3\frac{9u^2-6v^2}{3v^2}+\frac{2w^3}{27u^3-27uv^2+3w^3} \geq \frac{11}{3}$$

$$\frac{3(-3v^2w^3+3u^2w^3-56u^3v^2+29v^4u+27u^5)}{v^2(w^3+9u^3-9uv^2)} \geq 0$$

This is a linear increasing function of w^3 because $u^2 \geq v^2$ by the AGM thus the inequality holds true if and only if it holds true for the minimum value of the variable w^3. Once fixed the values of (u,v), the minimum value occurs when $c=0$ (or cyclic) or $c=b$ (or cyclic).

Let $c=0$. The inequality becomes

$$\frac{9a^5+9a^2b^3+9b^2a^3+9b^5-11ab^4-11a^4b}{3ab(a^3+b^3)} \geq 0$$

$$9a^5+9b^2a^3 \geq 18a^4b, \qquad 9b^5+9b^3a^2 \geq 18b^4a$$

If $c=b$ the inequality becomes

$$\frac{(9a^3-4a^2b-10ab^2+14b^3)(-b+a)^2}{3b(2a+b)(a^3+2b^3)}$$

$$\frac{10}{3}b^3+\frac{10}{3}b^3+\frac{10}{3}a^3 \geq 10ab^2$$

$$\frac{4}{3}a^3+\frac{4}{3}a^3+\frac{4}{3}b^3 \geq 4a^2b$$

and the proof is complete.

Solution 5 by Adrian Naco, Polytechnic University, Tirana, Albania

Firstly let us prove the following inequality, using the well known ABC method:

$$\frac{2abc}{a^3+b^3+c^3}+\frac{2}{3}\geq\frac{ab+bc+ca}{a^2+b^2+c^2}, \qquad (2)$$

The last inequality is equivalent to the following one,

$$f(a,b,c)=abc\left(\sum_{cyc}^{a,b,c}a^2\right)+\frac{2}{3}\left(\sum_{cyc}^{a,b,c}a^3\right)\left(\sum_{cyc}^{a,b,c}a^2\right)-\left(\sum_{cyc}^{a,b,c}a^3\right)\left(\sum_{cyc}^{a,b,c}ab\right)\geq 0$$

Since the expression on the left of the last inequality is of the third degree and is a symmetrical one in terms of a, b, c, based on the ABC method for solving inequalities in three variables, the minimum value is attainable when,

$$(a-b)(b-c)(c-a)=0 \quad \text{or/and} \quad abc=0.$$

WLOG let check first the case $a=b$ and then the case $a=0$.
Considering $a=b$ and doing easy manipulations, the inequality (2) is transformed equivalently to the following inequalities,

$$\frac{2a^2c}{2a^3+c^3}+\frac{2}{3}\geq\frac{a^2+2ac}{2a^2+c^2} \quad \Leftrightarrow \quad \frac{3a^2c+4a^3+2c^3}{6a^3+3c^3}\geq\frac{a^2+2ac}{2a^2+c^2}$$

$$\begin{aligned}&\Leftrightarrow \quad (3a^2c+4a^3+2c^3)(2a^2+c^2)\geq(6a^3+3c^3)(a^2+2ac)\\ &\Leftrightarrow \quad 2(a-c)^4(a+c)\geq 0\end{aligned}$$

The last inequality is true since a and b are positive real numbers.
If $a=0$ and doing easy manipulations the inequality (2) is transformed equivalently to the following inequalities,

$$\frac{2}{3}\geq\frac{bc}{b^2+c^2} \quad \Leftrightarrow \quad 2(b^2+c^2)\geq\frac{3}{b}c \quad \Leftrightarrow \quad \frac{3}{2}(b+c)^2+\frac{1}{2}(b^2+c^2)\geq 0.$$

The last inequality is true, since b and c are positive real numbers.
Referring to the book, *Secrets in Inequalities*,
Vol.1, Pham Kim Hung, 2007, BIL Publishing House, Page 193-195,
it has been proved the following inequality,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\geq 12, \qquad (3).$$

Using the inequalities (2) and (3), the given inequality (1), the statement of the problem, is transformed to the following inequalities,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{2abc}{a^3+b^3+c^3}\ge\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}-\frac{4}{3}=$$

$$=\frac{7}{9}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+\frac{2}{9}\left[\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\right]-\frac{4}{3}\ge$$

$$\ge\frac{7}{9}\cdot 3\cdot\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}}+\frac{8}{3}-\frac{4}{3}=\frac{11}{3}.$$

Equality is obtained for $a=b=c$.

Solution 6 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The equality given in the statement of the problem is equivalent to

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-3\ge\frac{2\left(a+b^3+c^3\right)-2abc}{3(a^3+b^3+c^3)},\ \text{or}$$

$$\frac{(a^2-b^2)^2}{a^2b^2}+\frac{(a^2-b^2)(c^2-b^2)}{a^2c^2}\ge\frac{2(a+b+c)((a-b)^2+(c-a)(c-b))}{3(a^3+b^3+c^3)},\ \text{or}$$

$$(a-b)^2\left(\frac{(a+b)^2}{a^2b^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)}\right)+(c-a)(c-b)\left(\frac{(a+c)(b+c)}{a^cc^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)}\right)\ge 0.$$

Finally, we only need to prove that

$$\frac{(a+b)^2}{a^2b^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)}\ge\frac{(a+b)^2}{a^2b^2}-\frac{2(a+b+\frac{a+b}{2})}{3(a^3+b^3)}=\frac{(a+b)^2}{a^2b^2}-\frac{1}{a^2-ab+b^2}>0$$

and

$$\frac{(a+c)(b+c)}{a^2c^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)}\ge\frac{ab}{a^2b^2}-\frac{2(a+b+\frac{a+b}{2})}{3(a^3+b^3)}=\frac{1}{ab}-\frac{1}{a^2-ab+b^2}\ge 0.$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Daniel Văcaru, Pitesti, Romania, and the proposer.

5520: *Proposed by Raquel León (student) and Angel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{2n}\binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k}\frac{1}{k+1}=0.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We first need the following well-known indentities:

$$\sum_{k=0}^{\infty} \binom{2k}{k} t^k = \frac{1}{\sqrt{1-4t}}, \tag{1}$$

$$(1-t)^{-n-1} = \sum_{m=0}^{\infty} \binom{n+m}{n} t^m, \tag{2}$$

for $|t| < \dfrac{1}{4}$.

Let x, y be real numbees satisfying $|x| \leq \dfrac{1}{2}$ and $|y| < \dfrac{1}{4}$. By substituting $t = xy(1-y)^{-2}$ into (1) and then using (2), we have

$$\begin{aligned}
(1-2(1+2x)y+y^2)^{1/2} &= (1-y)^{-1}\left(1-4xy(1-y)^{-2}\right)^{-1/2} \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} x^k y^k (1-y)^{-2k-1} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{2k}{k} \binom{2k+m}{2k} x^k y^{m+k}
\end{aligned}$$

Replacing y by $-y$ we have

$$(1+2(1+2x)y+y^2)^{-1/2} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{m+k} \binom{2k}{k} \binom{2k+m}{2k} x^k y^{m+k}.$$

Hence,

$$\begin{aligned}
(1-2(1+2x)y+y^2)^{-1/2} &+ (1+2(1+2x)y+y^2)^{-1/2} \\
&= 2\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{2n+k}{2k} \binom{2k}{k} x^k y^{2n} \\
&= 2\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{2n+k}{2k} \binom{2n}{k} x^k y^{2n}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n} \binom{2n+k}{k} \binom{2n}{k} \frac{(-1)^k}{2^k} \frac{1}{k+1} \right) y^{2n} \\
&= 2\int_{-1/2}^{0} \sum_{k=0}^{2n} \sum_{n=0}^{\infty} \binom{2n+k}{k} \binom{2n}{k} x^k y^{2n} dx \\
&= \int_{-1/2}^{0} \left(1-2(1+2x)y+y^2\right)^{-1/2} dx + \int_{-1/2}^{0} \left(1+2(1+2x)y+y^2\right)^{-1/2} dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{1+y^2}-1+y}{2y} + \frac{1+y-\sqrt{1+y^2}}{2y} \\
&= 1,
\end{aligned}
$$

whenever $|y| < \dfrac{1}{4}$. It follows that $\displaystyle\sum_{k=0}^{2n} \binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k}\frac{1}{k+1} = 0$ as desired.

Solution 2 by G.C. Greubel, Newport News, VA

Consider the series

$$S_n(x) = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}\frac{x^k}{k+1}.$$

Now consider the generating function of this series. This will be determined in the following. One component that will be required is the use of Catalan numbers, C_n, and the generating function

$$\sum_{n=0}^{\infty} C_n\, t^n = \frac{1-\sqrt{1-4t}}{2t}. \tag{6}$$

With in mind, then:

$$
\begin{aligned}
\sum_{n=0}^{\infty} S_n(x)\, t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}\frac{x^k\, t^n}{k+1} \\
&= \sum_{n,k=0}^{\infty} \binom{n+2k}{k}\binom{n+k}{k}\frac{(xt)^k\, t^n}{k+1} \\
&= \sum_{n,k=0}^{\infty} \frac{(n+2k)!\,(xt)^k\, t^n}{n!\,k!\,(k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!\,(xt)^k}{k!\,(k+1)!} \cdot \sum_{n=0}^{\infty} \frac{(2k+1)_n\, t^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!\,(xt)^k}{k!\,(k+1)!}\,(1-t)^{-2k-1} \\
&= \frac{1}{1-t}\sum_{k=0}^{\infty} C_k \left(\frac{xt}{(1-t)^2}\right)^k \\
&= \frac{1}{1-t}\,\frac{(1-t)^2}{2xt}\left(1-\sqrt{1-\frac{4xt}{(1-t)^2}}\right) \\
&= \frac{1}{2xt}\left(1-t-\sqrt{1-2(1+2x)t+t^2}\right).
\end{aligned}
$$

Since a generating function of the series has been established consider the reduction when $x = -1/2$. This reduces to

$$\sum_{n=0}^{\infty} S_n \left(-\frac{1}{2}\right) t^n = 1 - \frac{1-\sqrt{1+t^2}}{t}. \tag{7}$$

which, by use of (1), becomes

$$\sum_{n=0}^{\infty} S_n \left(-\frac{1}{2}\right) t^n = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n C_n}{2^{2n+1}} t^{2n+1}.$$

By considering even and odd terms it can be stated that

$$S_m \left(-\frac{1}{2}\right) = \begin{cases} 1 & m=0 \\ \frac{(-1)^n C_n}{2^{2n+1}} & m = 2n+1 \\ 0 & m = 2n, n \geq 1 \end{cases}.$$

This leads to

$$\sum_{k=0}^{2n} \binom{2n+k}{k} \binom{2n}{k} \frac{(-1)^k}{2^k (k+1)} = 0$$

$$\sum_{k=0}^{2n+1} \binom{2n+k+1}{k} \binom{2n+1}{k} \frac{(-1)^k}{2^k (k+1)} = \frac{(-1)^n C_n}{2^{2n+1}}.$$

Editor's comment: The proposers of this problem also used the notion of a generating function They stated: "We will show that the generating function of the sequence $(a_m)_{m\geq 0}$ with $a_m = \displaystyle\sum_{k=0}^{m} \binom{m+k}{k}\binom{m}{k}\left(\frac{-1}{2}\right)^k \frac{1}{k+1}$ only has odd terms, from where the result follows."

Also solved by Michele Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Moti Levi, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5521: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a > 0$ be a real number. If f is an odd non-constant real function having second derivative in the interval $[-a, a]$ and $f'(-a) = f'(a) = 0$, then prove that there exists a point $c \in (-a, a)$ such that

$$\frac{1}{2} f''(c) \geq \frac{|f(a)|}{a^2}$$

Solution 1 by Michel Bataille, Rouen, France

We shall use the following lemma: Let u, v be real numbers with $u < v$ and let $g : [u, v] \to R$. If g has a second derivative in $[u, v]$ and $g'(u) = g'(v) = 0$, then for some $c_0 \in (u, v)$,

$$|g''(c_0)| \geq \frac{4|g(v) - g(u)|}{(v-u)^2}.$$

To deduce the required result, we take $u=-a, v=a, g=f$; because f is odd, this yields $|f''(c_0)| \geq \dfrac{4|f(a)-f(-a)|}{(a+a)^2} = \dfrac{2|f(a)|}{a^2}$ for some $c_0 \in (u,v)$. If $f''(c_0) \geq 0$, we take $c = c_0$. Otherwise, we have $f''(-c_0) = -f''(c_0) = |f''(c_0)|$ (note that f'' is odd) and we take $c = -c_0$.

Proof of the lemma. Let $m = \dfrac{u+v}{2}$; for some c_1, c_2 in the interval (u,v), we have

$$g(m) - g(u) = (m-u)g'(u) + \frac{(m-u)^2}{2}g''(c_1) = \frac{(v-u)^2}{8}g''(c_1)$$

and

$$g(m) - g(v) = (m-v)g'(v) + \frac{(m-v)^2}{2}g''(c_2) = \frac{(v-u)^2}{8}g''(c_2)$$

(from the Taylor-Lagrange formula), hence

$$\begin{aligned} |g(v)-g(u)| \leq |g(m)-g(v)| + |g(m)-g(u)| &\leq \frac{(v-u)^2}{8}(|g''(c_1)| + |g''(c_2)|) \\ &\leq \frac{(v-u)^2}{4}\max(|g''(c_1)|, |g''(c_2)|). \end{aligned}$$

Now, taking $c_0 = c_1$ if $|g''(c_1)| \geq |g''(c_2)|$ and $c_0 = c_2$ otherwise, we obtain $|g''(c_0)| \geq \dfrac{4|g(v)-g(u)|}{(v-u)^2}$.

Note. The hypothesis f non constant is not necessary: if f is odd and constant, then f is the zero function and the result remains true.

Solution 2 by Moti Levy, Rehovot, Israel

Let us assume that the statement $\exists c \in (-a,a)$ such that $f''(c) \geq \frac{2}{a^2}|f(a)|$ is false, then we have

$$f''(x) < \frac{2}{a^2}|f(a)| \quad for\ all\ x \in (-a,a). \tag{1}$$

The second derivative of an odd function is odd, i.e., $f''(x)$ is an odd function,

$$-f''(x) = f''(-x). \tag{2}$$

Inequality (1) is valid if we replace x by $-x$, hence

$$f''(-x) < \frac{2}{a^2}|f(a)| \quad for\ all\ x \in (-a,a). \tag{3}$$

By (2) and (3) we have

$$f''(x) > -\frac{2}{a^2}|f(a)|. \tag{4}$$

Equations (1) and (4) imply that that

$$\left|f''(x)\right| < \frac{2}{a^2}|f(a)| \quad for\ all\ x \in (-a,a). \tag{5}$$

By integration by parts,

$$f\left(t\right)-f\left(-a\right)=\int_{-a}^{t}f^{'}\left(x\right)dx=\left.xf^{'}\left(x\right)\right]_{-a}^{t}-\int_{-a}^{t}xf^{''}\left(x\right)dx. \tag{6}$$
$$=tf^{'}\left(t\right)-\int_{-a}^{t}xf^{''}\left(x\right)dx.$$

Setting $t=a$ in (6),

$$f\left(a\right)-f\left(-a\right)=af^{'}\left(a\right)-\int_{-a}^{a}xf^{''}\left(x\right)dx.$$

Noting that $f\left(a\right)=-f\left(-a\right)$ and $f^{'}\left(a\right)=0$, we get

$$2f\left(a\right)=-\int_{-a}^{a}xf^{''}\left(x\right)dx$$

and by taking the absolute value of both sides,

$$2\left|f\left(a\right)\right|=\left|\int_{-a}^{a}xf^{''}\left(x\right)dx\right|\leq\int_{-a}^{a}\left|x\right|\left|f^{''}\left(x\right)\right|dx.$$

Now we use (5) and $\int_{-a}^{a}|x|\,dx=a^2$ to obtain

$$\int_{-a}^{a}|x|\left|f''(x)\right|dx<\frac{2}{a^2}\left|f(a)\right|\int_{-a}^{a}|x|\,dx=2|f(a)|.$$

We arrived at the absurd $|f(a)|<|f(a)|$, therefore our assumption is false and we deduce that indeed there exists a point $c\in(-a,a)$ such that $f''(c)\geq\dfrac{2}{a^2}\left|f(a)\right|$.

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We find $c\in(-a,a)$ such that $\dfrac{1}{2}f''(c)=\dfrac{|f(a)|}{a^2}$. By Taylor's theorem, there exists $c_1\in(0,a)$ such that:

$$0=f(0)=f(a)-f'(a)a+\frac{f''(c_1)}{2}a^2=f(a)+\frac{f(c_1)}{2}a^2,$$

which gives: $\dfrac{f^{''}(c_1)}{2}=-\dfrac{f(a)}{a^2}$. Similarly, there exists $c_2\in(-a,0)$, such that:

$$0=f(0)=f(-a)+f^{'}(-a)a+\frac{f^{''}(c_2)}{2}a^2=-f(-a)+\frac{f^{''}(c_2)}{2}a^2,$$

which gives: $\dfrac{f''(c_2)}{2}=\dfrac{f(a)}{a^2}$. So if $f(a)>0$, then $\dfrac{f^{''}(c_2)}{2}=\dfrac{|f(a)|}{a^2}$, and if $f(a)<0$, then $\dfrac{f^{''}(c_1)}{2}=\dfrac{|f(a)|}{a^2}$.

Solution 4 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Since f is odd on $[-a,+a]$ we have

$$2f\left(a\right)=f\left(a\right)-f\left(-a\right)=\int_{-a}^{a}f'\left(t\right)dt.$$

By the condition $f'(a) = f'(-a) = 0$, integration by parts yields

$$2f(a) = -\int_{-a}^{a} t f''(t)\, dt.$$

If we assume that $f''(t) < 2a^{-2}|f(a)|$, for all $t \in (-a, +a)$, it would follow the contradiction

$$2|f(a)| = \left| -\int_{-a}^{a} t f''(t)\, dt \right| < 2a^{-2}|f(a)| \int_{-a}^{a} |t|\, dt < 2|f(a)|.$$

Remark: In my opinion, it is not necessary to propose that f is non-constant. The only constant odd function is the zero function. In this case the inequality $f''(c) \geq 2a^{-2}|f(a)|$ is valid, for all $c \in (-a, +a)$.

Also solved by Kee Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vegata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5522: *Proposed by Ovidiu Furdui and Cornel Vălean from Technical University of Cluj-Napoca, Cluj-Napoca, Romania and Timiş, Romania, respectively*

Calculate

$$\int_0^1 \int_0^1 \frac{\log(1-x) - \log(1-y)}{x-y} dxdy.$$

Solution 1 by G.M. Greubel, Newport News, VA

First consider the integrand by expanding the logarithms into power series as follows.

$$\begin{aligned} f(x,y) &= \frac{\ln(1-x) - \ln(1-y)}{x-y} \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n - y^n}{x-y} \\ &= -\sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n} x^{n-r-1} y^r. \end{aligned}$$

Now, integration with respect to x and y yields

$$\begin{aligned} I &= \int_0^1 \int_0^1 \frac{\ln(1-x) - \ln(1-y)}{x-y}\, dx\, dy \\ &= -\sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n} \int_0^1 \int_0^1 x^{n-r-1} y^r\, dx\, dy \\ &= -\sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n(r+1)(n-r)} \\ &= -\sum_{n=1}^{\infty} \frac{S_n}{n}, \end{aligned}$$

where S_n is the sum, and evaluation, given by

$$
\begin{aligned}
S_n &= \sum_{r=0}^{n-1} \frac{1}{(r+1)(n-r)} \\
&= \frac{1}{n+1} \sum_{r=0}^{n-1} \left(\frac{1}{r+1} + \frac{1}{n-r} \right) \\
&= \frac{1}{n+1} \left(\sum_{r=1}^{n} \frac{1}{r} + \sum_{r=0}^{n-1} \frac{1}{n-r} \right) \\
&= \frac{2\, H_n}{n+1}.
\end{aligned}
$$

Here, H_n denotes the harmonic number. Returning to the integral it is determined that

$$
\begin{aligned}
I &= -\sum_{n=1}^{\infty} \frac{2\, H_n}{n\,(n+1)} \\
&= 2 \sum_{n=1}^{\infty} \left(\frac{H_n}{n+1} - \frac{H_n}{n} \right) \\
&= 2 \left(\sum_{n=2}^{\infty} \frac{H_n - \frac{1}{n}}{n} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) \\
&= -2\,\zeta(2) = -\frac{\pi^2}{3}.
\end{aligned}
$$

It can now be stated that

$$
\int_0^1 \int_0^1 \frac{\ln(1-x) - \ln(1-y)}{x-y}\, dx\, dy = -2\,\zeta(2) = -\frac{\pi^2}{3}.
$$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We have

$$
\begin{aligned}
I &= \int_0^1 \int_0^1 \frac{\log(1-x) - \log(1-y)}{x-y} dxdy = -\int_0^1 \int_0^1 \frac{\sum\limits_{n=1}^{+\infty} \frac{x^n}{n} - \sum\limits_{n=1}^{+\infty} \frac{y^n}{n}}{x-y} dxdy \\
&= -\int_0^1 \int_0^1 \frac{1}{x-y} \sum_{n=1}^{+\infty} \frac{x^n - y^n}{n} dxdy \\
&= -\int_0^1 \int_0^1 \sum_{n=1}^{+\infty} \frac{(x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)}{n(x-y)} dxdy \\
&= -\int_0^1 \int_0^1 \sum_{n=1}^{+\infty} \frac{1}{n} \left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right) dxdy
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^1 \left[\sum_{n=1}^{+\infty} \frac{1}{n} \int_0^1 \left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right) dx\right] dy \\
&= -\int_0^1 \sum_{n=1}^{+\infty} \frac{1}{n} \left[\frac{1}{n} + \frac{y}{n-1} + \cdots + \frac{y^{n-2}}{2} + y^{n-1}\right] dy \\
&= -\sum_{n=1}^{+\infty} \frac{1}{n} \int_0^1 \frac{1}{n} \left[\frac{1}{n} + \frac{y}{n-1} + \cdots + \frac{y^{n-2}}{2} + y^{n-1}\right] dy \\
&= -\sum_{n=1}^{+\infty} \frac{1}{n} \left[\frac{1}{n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots + \frac{1}{3(n-2)} + \frac{1}{2(n-1)} + \frac{1}{n}\right] \\
&= -\sum_{n=1}^{+\infty} \frac{2}{n} \left[\frac{1}{n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots\right] \\
&= -\sum_{n=1}^{+\infty} \frac{2}{n^2} - \sum_{n=1}^{+\infty} \frac{1}{n(n-1)} - \sum_{n=1}^{+\infty} \frac{2}{3n(n-2)} - \cdots \\
&\cong -\frac{\pi^2}{3}.
\end{aligned}
$$

So,

$$
I \cong -\frac{\pi^2}{3} \cong -3.28986813.
$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$
\begin{aligned}
\int_0^1 \int_0^1 \frac{\log(1-x) - \log(1-y)}{x-y} dxdy &= \int_0^1 \int_0^1 \frac{-\sum_{n=1}^{\infty} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{y^n}{n}}{x-y} dxdy \\
&= \int_0^1 \int_0^1 \frac{\sum_{n=1}^{\infty} \frac{y^n - x^n}{n}}{x-y} dxdy \\
&= -\int_0^1 \int_0^1 \sum_{n=1}^{\infty} \frac{1}{n} \frac{y^n - x^n}{y-x} dxdy \\
&= -\int_0^1 \int_0^1 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^k y^{n-k-1} dxdy
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_0^1 x^k dx \int_0^1 y^{n-k-1} dy \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{1}{n-k} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{n+1} \left(\frac{1}{k+1} + \frac{1}{n-k} \right) \\
&= -\sum_{k=0}^{\infty} \frac{1}{n(n+1)} \sum_{k=0}^{n-1} \left(\frac{1}{k+1} + \frac{1}{n-k} \right) \\
&= -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(-\sum_{k=0}^{n-1} \frac{1}{k+1} + \sum_{k=0}^{n-1} \frac{1}{n-k} \right) \\
&= -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\sum_{k=0}^{n-1} \frac{1}{k+1} + \sum_{k=0}^{n-1} \frac{1}{n-k} \right) \\
&= -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(H_n + H_n \right) \\
&= -2 \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} = -\frac{\pi^2}{3},
\end{aligned}
$$

where this last identity follows from entry 55.2.7 on page 361 of the book by E.R. Hansen *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, NJ, 1975; $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(x+n)} \left[\psi(x+n) - \psi(x+1)\right] = \frac{1}{x^2} \left[\frac{\pi^2 x}{6} - \gamma - \psi(x+1) \right]$ is valid for all positive integers, and in this particular case, when $x = 1$.

Solution 4 by Moti Levy, Rehovot, Israel

Define

$$
I(\alpha) := \int_0^1 \int_0^1 \frac{\ln(1-\alpha x) - \ln(1-\alpha y)}{x-y} dx dy.
$$

We will differentiate under the integral sign.

$$
\begin{aligned}
\frac{dI}{d\alpha} &= \int_0^1 \int_0^1 \frac{\partial \left(\frac{\ln(1-\alpha x) - \ln(1-\alpha y)}{x-y} \right)}{\partial \alpha} dx dy = -\int_0^1 \int_0^1 \frac{1}{(1-\alpha x)(1-\alpha y)} dx dy \\
&= -\left(\int_0^1 \frac{1}{1-\alpha x} dx \right) \left(\int_0^1 \frac{1}{1-\alpha y} dy \right) = -\left(\frac{\ln(1-\alpha)}{\alpha} \right)^2, \qquad \alpha \le 1.
\end{aligned}
$$

$$I(\alpha)=\int_0^{\alpha}\left(-\frac{1}{t^2}\right)\ln^2(1-t)\,dt+\text{constant}$$

But since $I(0)=\int_0^1\int_0^1\frac{\ln(1)-\ln(1)}{x-y}dxdy=0$, then the constant is zero.
Setting $\alpha=1$, we get

$$I(1)=\int_0^1\ln^2(1-t)\left(\frac{-1}{t^2}\right)dt$$

By integration by parts,

$$I(1)=-2\int_0^1\frac{1}{1-t}\frac{1}{t}\ln(1-t)\,dt.$$

By change of variable $x=-\ln(1-t)$,

$$I(1)=-2\int_0^{\infty}\frac{x}{e^x-1}dx=-2\zeta(2)=-\frac{\pi^2}{3}.$$

Excerpt from Richard P. Feynman, the American theoretical physicist, book "Surely You're Joking, Mr. Feynman:"

"One thing I never did learn was contour integration. I had learned to do integrals by various methods show in a book that my high school physics teacher Mr. Bader had given me. The book also showed how to differentiate parameters under the integral sign - It's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals. The result was that, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me."

Also solved by Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Kee -Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Addendum

A late solution by G.C. Greubel of Newport News, VA was received for problem 5515.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2019

5541: *Proposed by Kenneth Korbin, New York, NY*

A convex cyclic quadrilateral has inradius r and circumradius R. The distance from the incenter to the circumcenter is 169. Find positive integers r and R.

5542: *Proposed by Michel Bataille, Rouen, France*

Evaluate in closed form: $\cos\dfrac{\pi}{13}+\cos\dfrac{3\pi}{13}-\cos\dfrac{4\pi}{13}$.
(Closed form means that the answer should not be expressed as a decimal equivalent.)

5543: *Proposed by Titu Zvonaru, Comănesti, Romania*

Let $ABDC$ be a convex quadrilateral such that
$\angle ABC=\angle BCA=25^\circ, \angle CBD=\angle ADC=45^\circ$. Compute the value of $\angle DAC$. (Note the order of the vertices.)

5544: *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve in $\Re$:

$$\begin{cases}\tan^{-1}x = & \tan y+\tan z\\ \tan^{-1}y = & \tan x+\tan z\\ \tan^{-1}z = & \tan x+\tan y\end{cases}$$

5545: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let p,q be two twin primes. Show that

$$1+4\left(\sum_{j=1}^{\frac{p-1}{2}}\left\lfloor\frac{jq}{p}\right\rfloor+\sum_{k=1}^{\frac{q-1}{2}}\left\lfloor\frac{kp}{q}\right\rfloor\right)$$

is a perfect square and determine it. (Here $\lfloor x\rfloor$ represents the integer part of x).

5546: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate
$$\sum_{n=1}^{\infty}(-1)^{\lfloor\frac{n}{2}\rfloor}\left(\mathrm{e}^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\right).$$

Solutions

5523: *Proposed by Kenneth Korbin, New York, NY*

For every prime number P, there is a circle with diameter $4P^4+1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area $(2P)(2P+1)(2P-1)(2P^2-1)$. Find the sides of the triangles if $P=2$ and if $P=3$.

Solution 1 by Ed Gray, Highland Beach, FL

Case 1. $P=2$. Then Area $=4\cdot 5\cdot 3\cdot 7=2^2\cdot 3^1\cdot 5^1\cdot 7^1=420$.
By Brahmaguptas formula, $A^2=s(s-a)(s-b)(s-c)$, where a, b, and c are the sides, and s is the semi-perimeter. We note that $(s-a)+(s-b)+(s-c)=3s-2s=s$. So we seek a factor, s, and three other factors whose sum is s.
$A^2=(2^4)(3^2)(5^2)(7^2)=2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7$. A discerning eye sees that
$49=20+20+9$, so
$s=7\cdot 7$
$s-a=2\cdot 2\cdot 5$, so $a=49-20=29$.
$s-b=2\cdot 2\cdot 5$, so $b=49-20=29$.
$s-c=3\cdot 3$, so $c=49-9=40$.
Each side is less than $4P^4+1=65$, and the triangle inequality holds.

Case 2. $P=3$. The area $=6\cdot 5\cdot 7\cdot 17=(2^1)(3^1)(5^1)(7^1)(17^1)=3570$.
$A^2=s(s-a)(s-b)(s-c)=(2^2)(3^2)(5^2)(7^2)(17^2)=2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 17\cdot 17$.
Let $s=2\cdot 3\cdot 5\cdot 7=210$. Then
$s-a=7\cdot 17=119,\ a=210-119=91$.
$s-b=5\cdot 17=85,\ b=210-85=125$.
$s-c=2\cdot 3=6,\quad c=210-6=204$.
Each side is less than $4P^4+1=325$, and the triangle inequality holds.

Solution 2 by David E. Manes, Oneonta, NY

Given triangle $\triangle ABC$ with side lengths a, b and c opposite the respective vertices A, B and C. Moreover, assume that the triangle has area $[ABC]=(2P)(2P+1)(2P-1)(2P^2-1)$ and is inscribed in a circle with diameter $4P^4+1$, where P is a prime. If $P=2$, then the area $[ABC]=4\cdot 5\cdot 3\cdot 7=420$ and the circle has diameter $4\cdot 2^4+1=65$. Therefore, the radius R of the circumscribed circle has value $R=32.5$. The formula relating the radius R, the area $[ABC]$ and the side lengths a, b and c is $R=abc/(4[ABC])$. With $R=32.5$, $[ABC]=420$, one obtains

$abc = 4R[ABC] = 4(32.5)(420) = 54600 = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13$. Using the prime factorization of 54600, we then assign values to a, b and c so that $[ABC] = 420$. If $a = 3 \cdot 13 = 39$, $b = 5^2 = 25$ and $c = 2^3 \cdot 7 = 56$, then the semi-perimeter s of $\triangle ABC$ is given by $s = (a+b+c)/2 = (39+25+56)/2 = 60$ and Heron's formula for the area yields

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{60 \cdot 21 \cdot 35 \cdot 4} = 420.$$

Accordingly, if $P = 2$, then the triangle with integer length sides 25, 39 and 56 is inscribed in a circle with diameter $4P^4 + 1 = 65$ and has area $(2P)(2P+1)(2P-1)(2P^2-1) = 420$.

If $P = 3$, then $\triangle ABC$ has area $[ABC] = (2P)(2P+1)(2P-1)(2P^2-1) = 6 \cdot 7 \cdot 5 \cdot 17 = 3570$ and is inscribed in a circle with diameter $4P^4 + 1 = 4 \cdot 3^4 + 1 = 325$, whence the radius R of the circumscribed circle is $R = 162.5$. Therefore, the product of the side lengths a, b and c satisfies the equation $abc = 4R[ABC] = 4(162.5)(3570) = 2320500 = 2^2 \cdot 3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$. For this case, let $a = 2^2 \cdot 3 \cdot 17 = 204$, $b = 5^3 = 125$ and $c = 7 \cdot 13 = 91$. Then the semi-perimeter $s = (a+b+c)/2 = (204+125+91)/2 = 210$ so that the area of $\triangle ABC$ is given by

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{210 \cdot 6 \cdot 85 \cdot 119} = 3570.$$

Therefore, if $P = 3$, then the triangle with integer side lengths 91, 125 and 204 is inscribed in a circle with diameter $4P^4 + 1 = 325$ and the triangle $\triangle ABC$ has area $[ABC]$ given by $[ABC] = (2P)(2P+1)(2P-1)(2P^2-1) = 3570$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The condition that P be prime is not necessary.

Note: This problem is similar to SSM Problem 5356 published May 2015. Our solution is based on Brian Beasley's solution (December 2015) to that problem.

We show that for a positive integer P, the triangle with sides given by

$$\begin{aligned} a &= (2P+1)\left(2P^2 - 2P + 1\right) = 4P^3 - 2P^2 + 1 \\ b &= (2P-1)\left(2P^2 + 2P + 1\right) = 4P^3 + 2P^2 - 1 \\ c &= 4P\left(2P^2 + 1\right) = 8P^3 - 4P \end{aligned}$$

has area $2P(2P+1)(2P-1)\left(2P^2-1\right)$ and can be inscribed in a circle with diameter $4P^4 + 1$.

In particular:

For $P = 2$, the sides of the triangle are $25, 39$ and 56; the diameter of the circle is 65 and the area of the triangle is 420.

For $P = 3$, the sides of the triangle are $91, 125$ and 204; the diameter of the circle is 325 and the area of the triangle is 3570.

We do not know whether our formula produces all such triangles. We used a computer program to determine that it does produce the unique triangle for each positive integer P from 1 through 12.

SOLUTION:

We let a, b, and c be the sides of the triangle, A its area, and R its circumradius. It is known that R is given by $R = \dfrac{abc}{4A}$.

Thus since we are given $A = 2P(2P+1)(2P-1)\left(2P^2-1\right)$ and diameter $4P^4+1$, we have

$$\begin{aligned}
abc &= 4AR = 4 \cdot 2P(2P+1)(2P-1)\left(2P^2-1\right)\frac{4P^4+1}{2} \\
&= 4P(2P+1)(2P-1)\left(2P^2-1\right)\left(4P^4+1\right) \\
&= 4P(2P+1)(2P-1)\left(2P^2-1\right)\left(2P^2+2P+1\right)\left(2P^2-2P+1\right).
\end{aligned}$$

We found a, b, c (as given above) by judiciously selecting the above factors of abc so that $1 \le a, b, c \le 2R = 4P^4+1$ and the sum of any two of them exceeds the third.

It is easy to verify that our a, b, and c are ≥ 1.

We must show that a, b, c satisfy the requirements of the problem. Note that

$$\begin{aligned}
a+b+c &= 16P^3 - 4P = 4P(2P-1)(2P+1); \\
a+b-c &= 4P > 0, so\ a+b > c; \\
a+c-b &= 8P^3 - 4P^2 - 4P + 2 = 2(2P-1)\left(2P^2-1\right) > 0, \text{so } a+c > b; \\
b+c-a &= 8P^3 + 4P^2 - 4P - 2 = 2(2P+1)\left(2P^2-1\right) > 0, \text{so } b+c > a.
\end{aligned}$$

This shows that a, b, c do form a triangle. It also puts us in position to calculate the area by Herons Formula;

$$\begin{aligned}
A^2 &= \frac{1}{16}(a+b+c)(a+b-c)(a+c-b)(b+c-a) \\
&= \frac{1}{16}4P(2P-1)(2P+1)(4P)\left[2(2P-1)\right]\left(2P^2-1\right)\Big]\left[2(2P+1)\left(2P^2-1\right)\right] \\
&= 4P^2(2P-1)^2(2P+1)^2\left(2P^2-1\right)^2.
\end{aligned}$$

Therefore, $A - 2P(2P-1)(2P_1)(2P^2-1)$, as desired.

Finally, we calculate the diameter of the circumscribed circle:

$$\begin{aligned}
D &= 2R = \frac{abc}{2A} = \frac{4P(2P+1)(2P-1)\left(2P^2-1\right)\left(2P^2+2P+1\right)\left(2P^2-2P+1\right)}{4P(2P-1)(2P+1)\left(2P^2-1\right)} \\
&= \left(2P^2+2P+1\right)\left(2P^2-2P+1\right) = 4P^4+1, \text{as desired.}
\end{aligned}$$

Because the sides a, b, c produce the appropriate circumradius, we know that the sides actually fit into the circle: each is $\le D$.

Here are the results for $P = 1, 2, \ldots, 12$. Each of these is the unique triangle satisfying the given conditions.

P	a	b	c	$Area$	$Diameter$
1	3	5	4	6	5
2	25	39	56	420	65
3	91	125	204	3570	325
4	225	287	496	15624	1025
5	451	549	980	48510	2501
6	793	935	1704	121836	5185
7	1275	1469	2716	264810	9605
8	1921	2175	4064	518160	16385
9	2755	3077	5796	936054	26245
10	3801	4199	7960	1588020	40001
11	5083	5565	10604	2560866	58565
12	6625	7199	13776	3960600	82945

Comment: There are other ways to factor
$abc = 4P(2P+1)(2P-1)\left(2P^2-1\right)\left(2P^2+2P+1\right)\left(2P^2-2P+1\right)$ so the sides form a triangle, but which do not give the desired area. For example, with $P=2$, the sides $50, 39, 28$ form a triangle whose area is not the desired area (420).

Ditto for $35, 39, 40$.

Also these triangles do not have the desired circumradius of $4P^4+1$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Kee-Wai Lau, Hong Kong, China, and the proposer.

5524: *Proposed by Michael Brozinsky, Central Islip, NY*

A billiard table whose sides obey the law of reflection is in the shape of a right triangle ABC with legs of length a and b where $a > b$ and hypotenuse c. A ball is shot from the right angle and rebounds off the hypotenuse at point P on a path parallel to leg CB that hits CAat point Q. Find the ratio $\dfrac{\overline{AQ}}{\overline{QC}}$.

Solution 1 by Ed Gray, Highland Beach,FL

Usually in a triangle, especially right triangles, sides are labeled with small letters, and the vertices are labeled with capital letters, the same letter being used to designate a side being opposite a vertex. To make this problem work, the drawing must be as follows (not withstanding rotations). The right angle C is at lower right, the hypotenuse is c. Vertex A is North of C, and B is at the left of C. However, $\overline{AC} > \overline{BC}$ to accommodate the law of reflection. So, if $a > b, \overline{AC} = a$, and $\overline{BC} = b$.

Let D be a point on the hypotenuse such that $\overline{DC}$ is perpendicular to the hypotenuse. Point P is on the hypotenuse where the ball strikes and $\overline{BD} < \overline{BP}$, (i.e., P is between D and A). Let $\overline{PF}$ be the normal to the hypotenuse where F is a point on $\overline{AC}$.

Let $r =$ the angle of incidence $= CPF$. The angle of reflection $= r = \angle FPQ$. Since $\angle PQC$ is a right angle, then $\angle QCP = 90^\circ - 2r$. Note that $\angle PCD = r$ since $\overline{PF} \parallel \overline{CD}$ and alternate interior angles are equal.

Therefore,

$$\angle DCB + r + (90^\circ - 2r) = \angle ACB = 90^\circ, \text{so } \angle DCB = r,$$

and $\angle DBC = 90^\circ - r,\ \angle APQ = 90^\circ - r$ by corresponding angles, so $\angle BAC = r$.

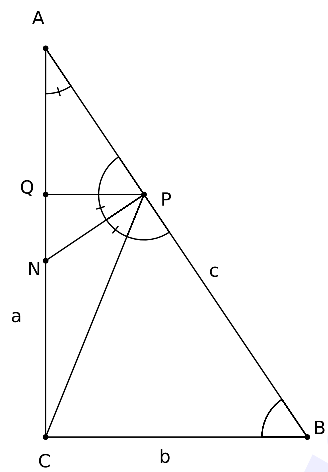
Then $\tan(\angle APQ) = \tan(90^\circ - r) = \dfrac{1}{\tan(r)} = \dfrac{\overline{AQ}}{\overline{PQ}}$, and $\tan(\angle QPC) = \tan(2r) = \dfrac{\overline{CQ}}{\overline{PQ}}$.

So, $\dfrac{\overline{AQ}}{\overline{CQ}} = \dfrac{\dfrac{1}{\tan(r)}}{\tan(2r)} = \dfrac{1 - \tan^2(r)}{2\tan^2(r)}$. From $\triangle ABC, \tan(r) = \dfrac{b}{a}$. So, $\dfrac{\overline{AQ}}{\overline{CQ}} = \dfrac{a^2 - b^2}{2b^2}$.

$Editor's\ comment$: Ed's comment that nonstandard labeling was being used in this problem is absolutely correct. I wrote to the proposer and he acknowledged the mix up, but stated that everything will still work out with standard notation but then we must state that $a < b$.

Solution 2 by Michel Bataille, Rouen, France

Since PQ is parallel to BC, we have $\angle PQA = 90^\circ$, hence $\angle QPA = B$. Let the perpendicular to AB at P intersect the line AC at N. Then $\angle NPC = \angle QPN = 90^\circ - B = A$ and $\angle ACP = \angle QCP = 90^\circ - 2A$. Thus, we must have $A \leq 45^\circ$ and so $B \geq 45^\circ \geq A$. Therefore the longest leg is $a = CA$ while $b = CB$ (see figure).
Now, $\angle PCB = 2A$ and $\angle CPB = B = \angle PBC$, from which we deduce $PB = 2CB\cos B = 2b\cos B = 2b \cdot \frac{b}{c} = \frac{2b^2}{c}$. It follows that $AP = c - \frac{2b^2}{c} = \frac{c^2-2b^2}{c} = \frac{a^2-b^2}{c}$ and so $\frac{AP}{PB} = \frac{a^2-b^2}{2b^2}$. Since PQ is parallel to BC, we have $\frac{AQ}{QC} = \frac{AP}{PB}$ and we can conclude that $\frac{AQ}{QC} = \frac{a^2-b^2}{2b^2}$.



Also solved by Kenneth Korbin, NY, NY; David Stone and John Hawkins, Georgia Southern University Statesboro GA, and the proposer.

5525: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu-Severin, Mehedinti, Romania*

Find real values for x and y such that:

$$4\sin^2(x+y) = 1 + 4\cos^2 x + 4\cos^2 y.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Put $u = e^{2ix}, v = e^{2iy}$. Then the given equation reads as

$$\begin{aligned} 0 &= \left(e^{2ix+2iy} + e^{-2ix-2iy} - 2\right) + 1 + \left(e^{2ix} + e^{-2ix} + 2\right) + \left(e^{2iy} + e^{-2iy} + 2\right) = \\ &= u\frac{1}{uv} + u + \frac{1}{u} + v + \frac{1}{v} + 3 = \frac{(uv+u+1)(uv+v+1)}{uv}. \end{aligned}$$

So either $v = -\frac{1}{u} - 1$ or $\frac{1}{v} = -u - 1$. If x and y run through the real numbers v and $\frac{1}{v}$ represent circles in the complex plane with radius 1 and center 0, while $-u-1$ and $\frac{-1}{u} - 1$ represent circles with radius 1 and center -1. Therefore $(u,v) \in \left\{\left(e^{2\pi i/3}, e^{2\pi i/3}\right), \left(e^{-2\pi i/3}, e^{-2\pi i/3}\right)\right\}$ which translates to $x \equiv y \equiv \pm\frac{\pi}{3} (\mathrm{mod}\ \pi)$.

Solution 2 by Michael C. Faleski, University Center, MI

Let's rewrite the statement of the problem using several trigonometric indentities. This leads to

$$4(\sin x \cos y + \sin x \cos y)^2 = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4(\sin^2 x \cos^2 y + \sin^2 y \cos^2 x + 2\sin x \sin y \cos x \cos y) = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4\left((1-\cos^2 x)\cos^2 y + \cos^2 x(1-\cos^2 y) + 2\sin x \sin y \cos x \cos y\right) = 1 + 4\cos^2 x + 4\cos^2 y$$

$$-8\cos^2 x \cos^2 y + 8\sin x \sin y \cos x \cos y = 1$$

$$-8\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2y)\right) + 2\sin 2x \sin 2y = 1$$

$$-2(1 + \cos 2x + \cos 2y + \cos 2x \cos 2y) + 2\sin 2x \sin 2y = 1$$

$$-2 - 2\cos 2x - 2\cos 2y - 2\cos 2x \cos 2y + 2\sin 2x \sin 2y = 1$$

$$-2\cos 2x - 2\cos 2y - 2(\cos 2x \cos 2y - \sin 2x \sin 2y) = 3$$

$$\cos 2x + \cos 2y + \cos(2x+2y) = -\frac{3}{2}.$$

And now we use $\cos a = \cos b = 2\cos\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right)$
to produce $2\cos(x+y)\cos(x-y) + (2\cos^2(x+y) - 1) = -\frac{3}{2}$,
and so we have $2\cos^2(x+y) + 2\cos(x-y)\cos(x+y) + \frac{1}{2} = 0$, or
$\cos^2(x+y) + \cos(x-y)\cos(x+y) + \frac{1}{4} = 0$.
We will now use the quadratic formula to solve for $\cos(x+y)$.

$$\cos(x+y) = \frac{-\cos(x-y) \pm \sqrt{\cos^2(x-y) - 1}}{2}.$$

As we are required to have real solutions, this means that
$\cos^2(x-y) - 1 \geq 0 \longrightarrow \cos^2(x-y) \geq 1$. This condition is only true for
$\cos^2(x-y) = 1 \longrightarrow \cos(x-y) = 1$.
Letting $y = x - a$, we find $\cos a = 1 \longrightarrow a = 2n\pi, \forall n \in Z$.

$$\cos(x+y) = -\frac{\cos(x-y)}{2} = -\frac{1}{2}.$$

Since $y = \pm 2n\pi$, then for $0 \leq x \leq 2\pi, x = y$. Hence, $\cos 2x = -\frac{1}{2}$, which leads to
$2x = \frac{2}{3}\pi, \frac{4}{3}\pi \longrightarrow x = \left(\frac{1}{3}\pi, \frac{2}{3}\pi\right)$. So, for $0 \leq x, y \leq 2\pi$, $(x, y) = \left(\frac{1}{3}\pi, \frac{1}{3}\pi\right), \left(\frac{2}{3}\pi, \frac{2}{3}\pi\right)$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$4\sin^2((x+y) = 1+4\cos^2 x+4\cos^2 y \iff 4\left(1-\cos^2(x+y)\right) = 1+2\cos(2x)+2+2\cos(2y)$$

$$\iff \quad 4 - 4\cos^2(x+y) = 5 + 4\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)$$

$$\iff \quad 0 = 4 - 4\cos^2(x+y) + 4\cos(x+y)\cos(x-y) + 1$$

$$\iff \quad 0=(2\cos(x+y)+\cos(x-y))^2-\cos^2(x-y)+1$$

$$\iff \quad 0=(2\cos(x+y)+\cos(x-y))^2+\sin^2(x-y)$$

$$\iff \quad 2\cos(x+y)+\cos(x-y)=0=\sin(x-y) \iff x-y=k\pi,\ k\ \in Z$$

$$\cos(x+y)+\cos(k\pi)=0 \iff x-y=k\pi;\quad \cos(x+y)=\frac{(-1)^{k+1}}{2},\ k\in Z$$

$$\iff \quad x-y=k\pi;\ x+y=\arccos\frac{(-1)^{k+1}}{2},\ \in Z$$

$$\iff \quad x=\frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2}+k\pi\right),\ y=\frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2}-k\pi\right),\ k\in Z.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

Since $\sin(x+y)=\sin x\cos y+\cos x\sin y$, so the given equation is equivalent to $1-8\sin x\cos x\sin y\cos y+8\cos^2 x\cos^2 y=0$. Clearly $\cos x\neq 0$ and $\cos y\neq 0$. So dividing both sides of the last equation by $\cos^2 x\cos^2 y$, we obtain $\sec^2 x\sec^2 y-8\tan x\tan y+8=0$ or $(1+\tan^2 x)(1+\tan^2 y)-8\tan x\tan y+8=0$, or

$$(\tan x-\tan y)^2+(\tan x\tan y-3)^2=0.$$

Thus $\tan x=\tan y$ and $\tan x\tan y=3$, so that $\tan x=\tan y=\sqrt{3}$ or $\tan x=\tan y=-\sqrt{3}$. It follows that

$$(x,y)=\left(\frac{\pi}{3}+m\pi,\frac{\pi}{3}+n\pi\right),\ \left(\frac{2\pi}{3}+m\pi,\frac{2\pi}{3}+n\pi\right),$$

where m and n are arbitrary integers.

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Using $\cos(2x)=2\cos^2(x)-1=1-2\sin^2(x)$ we see that the equation

$$4\sin^2(x+y)=1+4\cos^2(x)+4\cos^2(y)$$

is equivalent to

$$0=3+2\cos(2x+2y)+2\cos(2x)+2\cos(2y)=:f(x,y).$$

Using $\sin(2a)+\sin(2b)=2\sin(a+b)\cos(a-b)$ we obtain

$$\begin{aligned}\operatorname{grad} f(x,y) &= -4\cdot(\sin(2x+2y)+\sin(2x),\sin(2x+2y)+\sin(2y))\\ &= -8\cdot(\sin(2x+y)\cos y,\sin(x+2y)\cos x).\end{aligned}$$

Therefore, $\operatorname{grad} f(x,y)=(0,0)$ happens if

• $2x=\pi\ (\mathrm{mod}\ 2\pi)$ and $2y=\pi\ (\mathrm{mod}\ 2\pi)$. The critical points $\left(\frac{2n+1}{2}\pi,\frac{2m+1}{2}\pi\right)$ with integers n,m satisfy

$$f\left(\frac{2n+1}{2}\pi,\frac{2m+1}{2}\pi\right)=3+2\cdot 1+2(-1)^{n+1}+2(-1)^{m+1}>0.$$

- $2x = \pi \,(\mathrm{mod}\ 2\pi)$ and $2x + y = 0 \,(\mathrm{mod}\ \pi)$. The critical points $\left(\dfrac{2n+1}{2}\pi, m\pi - (2n+1)\pi\right)$ with integers n, m satisfy

$$f\left(\frac{2n+1}{2}\pi, m\pi - (2n+1)\pi\right) = 3 + 2 \cdot (-1) + 2(-1)^{n+1} + 2 \cdot 1 > 0.$$

- 2y=π (mod 2π) and $x + 2y = 0$ (mod π) is symmetrical to the preceding case.
- $2x + y = 0\,(\mathrm{mod}\pi)$ and $x + 2y = 0$ ($\mathrm{mod}\pi$). This implies $3x + 3y = (n+m)\pi$ and $x - y = (n-m)\pi$ with integers n, m. We infer that $(x, y) = \dfrac{\pi}{3}(2n-m, 2m-n)$ are the remaining critical points of f.

$$\begin{aligned} & f\left(\frac{2n-m}{3}\pi, \frac{2m-n}{3}\pi\right) \\ = \;& 3 + 2\cos\frac{2(n+m)\pi}{3} + 2\cos\frac{(4n-2m)\pi}{3} + 2\cos\frac{(4m-2n)\pi}{3} \\ = \;& 3 + 2\left(2\cos^2\frac{(n+m)\pi}{3} - 1\right) + 4\cos\frac{(n+m)\pi}{3}\cos(n-m)\pi \\ = \;& 1 + 4\cos^2\frac{N\pi}{3} + 4(-1)^N\cos\frac{N\pi}{3} = \left(1 + 2(-1)^N\cos\frac{N\pi}{3}\right)^2 \geq 0 \end{aligned}$$

with $N := n + m$. Consequently, the function value is equal to zero iff N is not a multiple of 3.

In total, we have $f(x, y) \geq 0$ on R^2 and $f(x, y) = 0$ if and only if $(x, y) = (2n-m, 2m-n)\dfrac{\pi}{3}$, for all integers n, m satisfying $n + m \neq 0$ (mod 3). The solutions of the above trigonometric identity are exactly the zeros of f.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Marian Ursărescu, "Roman Vodă College," Roman, Romania, and the proposer.

5526: *Proposed by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece*

The lengths of the sides of a triangle are 12, 16 and 20. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We claim that there is exactly one straight line which simultaneously halves the area and the perimeter of the triangle.

If the line passes through the sides of length 12 and 16, and its intersection with side 12 is x units from the acute angle on that side, then the line cuts off a right triangle of base $12 - x$

and height $12+x$. The area of this triangle is $\dfrac{144-x^2}{2}$. Setting this equal to 48, we would have $x=\pm 4\sqrt{3}$, but the construction of this line requires $0 \le x \le 4$, so there is no such line that cuts the triangle's area in half.

If the line passes through the sides of length 16 and 20, and its intersection with side 16 is x units from the acute angle on that side, then it cuts off a triangle with base x and height $\dfrac{3}{5}(24-x)$. The area of this triangle is $\dfrac{1}{2}x\dfrac{3}{5}(24-x)=\dfrac{3}{10}x(24-x)$, which takes a maximum value $\dfrac{432}{10}<48$ at $x=12$, so no such line can cut the triangle's area in half.

The remaining case is a line through sides 12 and 20. Let the line intersect side 12 at a point x units from the right angle. Then it cuts off a triangle of base $12-x$ and height $\dfrac{4}{5}(12+x)$, which has area $\dfrac{2}{5}(144-x^2)$. Setting this equal to 48, we find that $x=\pm 2\sqrt{6}$, but $0 \le x \le 8$ by the construction of the line, so we have one solution, $x=2\sqrt{6}$.

Comment: this problem is not new. It was discussed (for instance) in an internet site called "Problem of the Month", run by the University of Regina in Regina, Saskatchewan, Canada.

The problem of the month of April 2012 stated
(see http://mathcentral.uregina.ca/mp/previous2011/apr12sol.php):

Recall that the incenter I of a triangle is the point where the three internal angle bisectors meet. Prove that any line through I that divides the area of the triangle in half also divides its perimeter in half; conversely, any line through I that divides the perimeter of the triangle in half also divides its area in half.

In the solution the problem editor referred to a theorem of Verena Haider which states that for any triangle ABC and any line l, l divides the area and the perimeter of $\triangle ABC$ in the same ratio if and only if it passes through the triangle's incenter. Furthermore the problem editor made the statement that it is not hard to prove that every triangle has exactly one, two, or three bisecting lines, and no other values are possible, and provided a few references.

Solution 2 by Adrian Naco, Polytechnic University, Tirana, Albania

Let be a right angle triangle ABC where $AB=20, AC=12, BC=16$.

Case 1. The straight line intersect the sides AC and AB in the points M and N respectively. Let us sign $AM=x, AN=y$.
The area of the traiangle AMN (we sign the area of the triangle by $[AMN]$) is half the area of the triangle ABC ($[ABC]$), that is'

$$\begin{aligned}
[AMN]=\frac{1}{2}[ABC] \qquad &\Rightarrow \qquad \frac{AM\cdot AN\cdot \sin\angle MAN}{2}=\frac{1}{2}\cdot\frac{AC\cdot AB\cdot\sin\angle CAB}{2}\\
&\Rightarrow \qquad \frac{xy\sin\angle MAN}{2}=\frac{1}{2}\cdot\frac{12\cdot 20\cdot\sin\angle CAB}{2}\\
&\Rightarrow \qquad xy=120
\end{aligned}$$

Furthermore, the straight line MN halve the perimeter of the triangle ABC, that is, $x+y=24$. So, the lengths x, y of the respective sides AM and AN of the triangle AMN are roots of the following quadratic equation,

$$\begin{aligned} t^2-24t+120=0 \quad &\Rightarrow \quad \{x,y\}=\{10,14\} \\ &\Rightarrow \quad x=AM=10,\ \ y=AN=14 \end{aligned}$$

Case 2. The straight line intersect the sides BC and AB in the points M and N respectively. Let us sign $BM=x, BN=y$.
The area of the traiangle BMN is half the area of the triangle ABC, that is,

$$\begin{aligned} [BMN]=\frac{1}{2}[ABC] \quad &\Rightarrow \quad \frac{BM\cdot BN\cdot \sin\angle MBN}{2}=\frac{1}{2}\cdot\frac{BC\cdot AB\cdot\sin\angle CBA}{2} \\ &\Rightarrow \quad \frac{xy\sin\angle MBN}{2}=\frac{1}{2}\cdot\frac{16\cdot 20\cdot\sin\angle CAB}{2} \\ &\Rightarrow \quad xy=160 \end{aligned}$$

Furthermore, the straight line MN halve the perimeter of the triangle ABC that is, $x+y=24$. So, the lengths x, y of the respective sides BM and BN of the triangle BMN are roots of the following quadratic equation,

$$t^2-24t+160=0 \quad \Leftrightarrow \quad (t-12)^2+16=0$$

which have no solution. So this case is not possible. Case 3. The straight line intersect the sides AC and BC in the points M and N respectively. Let us sign $CM=x, CN=y$.
The area of the traiangle CMN is half the area of the triangle ABC, that is,

$$\begin{aligned} [CMN]=\frac{1}{2}[ABC] \quad &\Rightarrow \quad \frac{CM\cdot CN\cdot \sin\angle MCN}{2}=\frac{1}{2}\cdot\frac{AC\cdot BC\cdot\sin\angle ACB}{2} \\ &\Rightarrow \quad \frac{xy\sin\angle MCN}{2}=\frac{1}{2}\cdot\frac{12\cdot 16\cdot\sin\angle ACB}{2} \\ &\Rightarrow \quad xy=96 \end{aligned}$$

Furthermore, the straight line MN halve the perimeter of the triangle ABC, that is, $x+y=24$. So, the lengths x, y of the respective sides CM and CN of the triangle CMN are roots of the following quadratic equation,

$$\begin{aligned} t^2-24t+96=0 \quad &\Rightarrow \quad \{x,y\}=\{12-2\sqrt{7}, 12+2\sqrt{7}\} \\ &\Rightarrow \quad x=CM=12-2\sqrt{7},\ \ y=CN=12+2\sqrt{7} \end{aligned}$$

This case is not possible since $12+2\sqrt{7}>16=BC$.

Finally, the only possible case is when the straight line intersect the sides $AC=12$ and $AB=20$ in the respective points M and N such that $AM=10$ and $AN=14$.

Editor's comment: The proposer, **Ioannis D. Sfikas of National and Kapodistrian University in Athens, Greece** accompanied his solution with an interesting discussion of the problem's history. He wrote the following:

Comments. An interesting issue arising from classical Euclidean geometry concerns the existence of lines called "equalizers" that bisect both the area and the perimeter of a triangle. The search for such lines can be seen as a trivial process, but this abstains from the real picture. The complete study concerning the special case of a triangle was conducted by Kontokostas (2010). The possibility of the existence of an equalizer that can be applied to an arbitrary planar shape is an important parameter. However, a general method may not exist in order to solve this problem.

In general, an equalizer can be applied to any body and that is a fact that came up from a useful topology theorem: the Ham-Sandwich Theorem, also called the Stone-Tukey Theorem (after Arthur H. Stone and John W. Tukey). The theorem states that, given $d \geq 2$ measurable solids in $\Re^d$, it is possible to bisect all of them in half with a single $(d-1)$-dimensional hyperplane. In other words, the Ham-Sandwich Theorem provides the following paraphrased statement: *Take a sandwich made of a slice of ham and two slices of bread. No matter where one places the pieces of the sandwich in the kitchen, or house, or universe, so long as one's knife is long enough one can cut all three pieces in half in only one pass.* Proving the theorem for $d = 2$ (known as the *Pancake Theorem)* is simple and can be found in Courant and Robbins (1996, p. 267).

In 1994, Alexander Shen, professor at the Independent University of Moscow, published in *The Mathematical Intelligencer* a selection of problems, known as "coffin problems," which were offered to "undesirable" applicants at the entrance examinations at the Department of Mechanics and Mathematics (Mekh-mat) of Moscow University at 1970s and 1980s. Four examinations were held at the Mekh-Mat: written math, oral math, literature essay composition, and oral physics (Frenkel, 2013, p. 28). These problems appear to resemble greatly with the Olympiad problems. It should be noted that these problems also differ from the Olympiad problems by being, in many cases, either false or poorly stated. Their solution does not require knowledge of a higher level of mathematics, but require, however, ingenuity, creativity and unorthodox attitudes. Solutions to these problems were thoroughly analyzed by Ilan Vardi (2005a, 2005b, 2005c).

The Mathematics Department of Moscow State University, the most prestigious mathematics school in Russia, had at that time been actively trying to keep Jewish students (and other "undesirables") from enrolling in the department (Vershik, 1994, p. 5). One of the methods they used for doing this was to give the unwanted students a different set of problems on their oral exam. These problems were carefully designed to have elementary solutions (so that the Department could avoid scandals) that were nearly impossible to find. Any student who failed to answer could be easily rejected, so this system was an effective method of controlling admissions. These kinds of math problems were informally referred to as "Jewish" problems or "coffins." Coffins is the literal translation from Russian (Khovanova and Radul, 2012, p. 815). These problems along with their solutions were, of course, kept as a secret, but Valera Senderov and his friends had managed to collect a list. In 1975, they approached us to solve these problems, so that they could train the Jewish students following these mathematical ideas. *Problem 5* of Shen's catalogue, which had been proposed by Podkolzin in 1978, states: *Draw a straight line that halves the area and perimeter of a triangle.* A solution was included in the first chapter of Mikhail Shifman's book (2005, pp. 50-51).

The Canadian Mathematical Olympiad is an annual premier national advanced mathematics competition sponsored by the Canadian Mathematical Society. In 1985, 17th Canadian Mathematical Olympiad was held, and the first problem was:

17th Canadian Mathematical Olympiad 1985, Problem 1
A triangle has sides 6, 8, 10. Show that there is a unique line, which bisects the area and the perimeter.

The solution to the above problem is given in detail by Doob (1993, p. 169). The same subject seems to appear as *Problem 9* at the Canadian mathematical magazine *Crux Mathematicorum* destined for students. Readers are invited to search for the number of equalizers included on a right triangle whose sides differ from those presented in Problem 1 (Woodrow, 1991, p. 72):

Problem 9, Crux Mathematicorum 1991

The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

A solution to the magazine's *Problem 9* was given by Michael Selby from the University of Windsor. A solution was also already given to *Problem 1* of the Canadian Mathematical Olympiad stating that the questioned right triangle contains only one equalizer. The solution of the particular problem doesn't abstain from *Problem 1.* A relative problem was also proposed by the Flemish Mathematical Olympiad in 2004 in Belgium. It states:

Flanders Mathematics Olympiad 2004, Problem 1

Consider a triangle with side lengths 501 m, 668 m, 835 m. How many lines can be drawn with the property that such a line halves both area and perimeter?

[1] Courant, Richard and Robbins, Herbert (1996). *What is Mathematics? An elementary approach to ideas and methods,* second edition. Oxford, England: Oxford University Press.

[2] Doob, Michael (1993). *The Canadian Mathematical Olympiad (1969-1993): celebrating the first twenty-five years.* Canadian Mathematical Society.

[3] Frenkel, Edward (2013). *Love and Math: the heart of hidden reality* . BasicBooks.

[4] Khovanova, Tanya and Radul, Alexey (2012). Killer problems. *The American Mathematical Monthly,* 119 (10): 815-823.

[5] Kodokostas, Dimitrios (2010). Triangle equalizers. *Mathematics Magazine,* 83 (2): 141-146.

[6] Shen, Alexander (1994). Entrance examinations to the Mekh-mat. *The Mathematical Intelligencer,* 16 (4): 6-10.

[7] Shifman, Mikhail (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics. World Scientific.

[8] Vardi, Ilan (2005a). Mekh-mat entrance examination problems. In Shifman, Mikhail A. (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics, pp. 22-95. World Scientific.

[9] Vardi, Ilan (2005b). Solutions to the year 2000 International Mathematical Olympiad. In Shifman, Mikhail A. (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics, pp. 96-121. World Scientific.

[10] Vardi, Ilan (2005c). My role as an outsider. In Shifman, Mikhail A. (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics, pp. 122-125. World Scientific.

[11] Vershik, Anatoly (1994). Admission to the mathematics faculty in Russia in the 1970s and 1980s. *Mathematical Intelligencer,* 16 (4): 4-5.

[12] Woodrow, Robert E. (March 1991). The Olympiad Corner: No 123. *Crux Mathematicorum,* 17(3), 65-74.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5527: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b and c be positive real numbers such that $a+b+c=3$. Prove that for all real $\alpha>0$, holds:

$$\frac{1}{2}\left(\frac{1-a^{\alpha+1}b^{\alpha}}{a^{\alpha}b^{\alpha}}+\frac{1-b^{\alpha+1}c^{\alpha}}{b^{\alpha}c^{\alpha}}+\frac{1-c^{\alpha+1}a^{\alpha}}{c^{\alpha}a^{\alpha}}\right)$$
$$\leq\sqrt{\left(\frac{1-a^{\alpha+1}}{a^{\alpha}}+\frac{1-b^{\alpha+1}}{b^{\alpha}}+\frac{1-c^{\alpha+1}}{c^{\alpha}}\right)\left(\frac{1-a^{\alpha}b^{\alpha}c^{\alpha}}{a^{\alpha}b^{\alpha}c^{\alpha}}\right)}.$$

Editor′s comment : A mistake was detected in the statement of the problem by **Michel Bataille of Rouen, France.** He noticed the following:

The inequality easily rewrites as

$$A:=a^{\alpha}+b^{\alpha}+c^{\alpha}-3a^{\alpha}b^{\alpha}c^{\alpha}\leq B:=2\sqrt{(1-a^{\alpha}b^{\alpha}c^{\alpha})(a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+a^{\alpha}c^{\alpha}-3a^{\alpha}b^{\alpha}c^{\alpha})}.\quad (1)$$

We take $a=\frac{1}{2},\ b=1,\ c=\frac{3}{2}$ and first consider the case $\alpha=2$. We obtain $A=1.8125$ and $B=\frac{\sqrt{154}}{8}=1.55....$, hence (1) does not hold.
In the case $\alpha=1$, we find $A=0.75$ and $B=\frac{\sqrt{2}}{2}=0.707...$, hence (1) does not hold.
In the case $\alpha=1/2$, $A=0.333..$ and $B=0.327..$, hence (1) does not hold.
However, we prove the reverse inequality in the case $\alpha=1$, that is,

$$3-3abc\geq 2\sqrt{(1-abc)(ab+bc+ca-3abc)}.\quad (2)$$

Since $3=a+b+c\geq 3\sqrt[3]{abc}$, we have $1-abc\geq 0$ and (2) will certainly holds if $3\sqrt{1-abc}\geq 2\sqrt{ab+bc+ca-3abc}$ or, squaring and arranging,

$$9+3abc-4(ab+bc+ca)\geq 0.\quad (3)$$

Now, From Schur's inequality $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)\geq 0$, we obtain

$$(a+b+c)(ab+bc+ca)-3abc\leq (a+b+c)\left((a+b+c)^2-3(ab+bc+ca)\right)+6abc$$

or since $a+b+c=3$, $3(ab+bc+ca)-3abc\leq 3(9-3(ab+bc+ca))+6abc$, that is, $4(ab+bc+ca)\leq 9+3abc$ and (3) holds.
Perhaps the reverse inequality does hold when $\alpha>0, \alpha\neq 1$ but I have not been able to find a proof.

Editor again : With respect to the above, the solution to this problem remains open.

5528: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a>0$. Calculate $\displaystyle\int_a^\infty\int_a^\infty \frac{\mathrm{d}x\mathrm{d}y}{x^6(x^2+y^2)}$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We easily verify that

$$\int\int\frac{dxdy}{x^6(x^2+y^2)}=\frac{1}{6xy^5}-\frac{1}{18x^3y^3}+\frac{1}{30x^5y}+\frac{\arctan\frac{x}{y}}{6y^6}-\frac{\arctan\frac{y}{x}}{6x^6}+C.$$

Therefore,

$$\int_a^\infty\int_a^\infty\frac{dxdy}{x^6(x^2+y^2)}=\frac{13}{90a^6}.$$

Solution 2 by Michael C. Faleski, University Center, MI

We start by evaluating the y−integral using trigonometric substitution with $y=x\tan\theta, dy=x\sec^2\theta d\theta$ to give

$$\int_a^\infty\frac{dy}{x^6(x^2+y^2)}\to\int\frac{1}{x^6}\left(\frac{x}{x^2}\right)d\theta\to\frac{1}{x^7}\tan^{-1}\left(\frac{y}{x}\right)\Big|_a^\infty=\frac{1}{x^7}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{a}{x}\right)\right).$$

This quantity is now integrated with respect to x by braking it into two terms written as

$$\int_a^\infty\frac{\pi}{2}\frac{dx}{x^7}-\int_a^\infty\frac{\tan^{-1}\left(\frac{a}{x}\right)}{x^7}dx.$$

The first term evaluates as

$$\int_a^\infty\frac{\pi dx}{2x^7}=-\frac{\pi}{12}\frac{1}{x^6}\Big|_a^\infty=\frac{\pi}{12a^6}.$$

For the second term, we start with integration by parts using $u=-\tan^{-1}\left(\frac{a}{x}\right)\to du=\dfrac{a}{x^2+a^2}$ and $dv=\dfrac{1}{x^7}dx\to v=-\dfrac{1}{6x^6}$ which yields

$$\frac{\tan^{-1}\left(\frac{a}{x}\right)}{6x^6}\Bigg|_a^\infty-\left(-\frac{a}{6}\right)\int_a^\infty\frac{dx}{x^6(x^2+a^2)}=\left(0-\frac{\pi}{24a^6}\right)+\frac{a}{6}\int_a^\infty\frac{dx}{x^6(x^2+a^2)}.$$

For the last term, one approach would be to make a $u-$substitution 0f $x = a\tan\theta \to dx = a\sec^2\theta d\theta$ leading to

$$\frac{a}{6}a\int_a^\infty \frac{dx}{x^6(x^2+a^2)} \to \frac{a}{6}\int_{\pi/4}^{\pi/2}\frac{1}{a^8}\frac{a\sec^2\theta\ d\theta}{\tan^6\theta\sec^2\theta} =$$

We can use (which is easily shown using $\cot^2 x = (\csc 2x - 1)$ repeatedly) that

$$\int \cot^6 x dx = \frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \frac{\cot x}{1} - x + C.$$

For our scenario, we have

$$\frac{1}{6a^6}\int_{\pi/4}^{\pi/2}\cot^6\theta d\theta = \frac{1}{6a^6}\left(-\frac{\cot^5\theta}{5} + \frac{\cot^3\theta}{3} - \frac{\cot\theta}{1} - \theta\right)\Bigg|_{\pi/4}^{\pi/2} = \frac{1}{6a^6}\left(\frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}\right).$$

So finally, putting putting all of the numerical terms together yields:

$$\int_a^\infty\int_a^\infty \frac{dxdy}{x^6(x^2+y^2)} = \frac{\pi}{12a^6} - \frac{\pi}{24a^6} + \frac{1}{6a^6}\left(\frac{13}{15} - \frac{\pi}{4}\right) = \frac{13}{90a^6}.$$

Solution 3 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We show that, for $a > 0$,

$$I := \int_a^\infty\int_a^\infty \frac{dxdy}{x^6\left(x^2+y^2\right)} = \frac{13}{90a^6}.$$

Integrating both sides of the identity

$$\frac{1}{x^6\left(x^2+y^2\right)} + \frac{1}{y^6\left(x^2+y^2\right)} = \frac{x^6+y^6}{x^6y^6\left(x^2+y^2\right)} = \frac{x^4-x^2y^2+y^4}{x^6y^6} = \frac{1}{x^2y^6} - \frac{1}{x^4y^4} + \frac{1}{x^6y^2}$$

we conclude that

$$\begin{aligned} 2I &= \int_a^\infty\int_a^\infty\left(\frac{1}{x^2y^6} - \frac{1}{x^4y^4} + \frac{1}{x^6y^2}\right)dxdy \\ &= \left(\frac{1}{x\cdot 5y^5} - \frac{1}{3x^3\cdot 3y^3} + \frac{1}{5x^5\cdot y}\right)\Bigg|_{x=a}^\infty\Bigg|_{y=a}^\infty \\ &= -\left(\frac{1}{5} - \frac{1}{9} + \frac{1}{5}\right)\frac{1}{a^6} = \frac{13}{45a^6}. \end{aligned}$$

Solution 4 by Brian Bradie, Christopher Newport University, Newport, News, VA

Let $a > 0$, n be a positive integer, and consider

$$\int_a^\infty\int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)}.$$

With the substitutions $u = x/a$ and $v = y/a$,

$$\begin{aligned}
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)} &= \frac{1}{a^n}\int_1^\infty \int_1^\infty \frac{du\,dv}{u^n(u^2+v^2)} \\
&= \frac{1}{a^n}\int_1^\infty \frac{1}{u^n}\left.\frac{\tan^{-1}(v/u)}{u}\right|_1^\infty du \\
&= \frac{1}{a^n}\int_1^\infty \frac{1}{u^{n+1}}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{u}\right) du \\
&= \frac{1}{a^n}\int_1^\infty \frac{1}{u^{n+1}}\tan^{-1} u\,du.
\end{aligned}$$

By integration by parts, we next find

$$\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)} = \frac{1}{na^n}\left(\frac{\pi}{4} + \int_1^\infty \frac{u^{-n}}{1+u^2}\,du\right);$$

the substitution $u = 1/w$ then yields

$$\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)} = \frac{1}{na^n}\left(\frac{\pi}{4} + \int_0^1 \frac{u^n}{1+u^2}\,du\right).$$

Let

$$I_n = \int_0^1 \frac{u^n}{1+u^2}\,du$$

Then

$$\begin{aligned}
I_1 &= \int_0^1 \frac{u}{1+u^2}\,du = \frac{1}{2}\ln 2; \\
I_2 &= \int_0^1 \frac{u^2}{1+u^2}\,du = \int_0^1 \left(1 - \frac{1}{1+u^2}\right) du = 1 - \frac{\pi}{4};
\end{aligned}$$

and, for $n > 2$,

$$I_n = \frac{1}{n-1} - \int_0^1 \frac{u^{n-2}}{1+u^2}\,du = \frac{1}{n-1} - I_{n-2}.$$

Thus,

$$\begin{aligned}
I_3 &= \frac{1}{2} - I_1 = \frac{1}{2} - \frac{1}{2}\ln 2; \\
I_4 &= \frac{1}{3} - I_2 = \frac{\pi}{4} - \frac{2}{3}; \\
I_5 &= \frac{1}{4} - I_3 = \frac{1}{2}\ln 2 - \frac{1}{4}; \text{ and} \\
I_6 &= \frac{1}{5} - I_4 = \frac{13}{15} - \frac{\pi}{4}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x(x^2+y^2)} &= \frac{1}{a}\left(\frac{\pi}{4}+\frac{1}{2}\ln 2\right)\\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^2(x^2+y^2)} &= \frac{1}{2a^2}\left(\frac{\pi}{4}+1-\frac{\pi}{4}\right)=\frac{1}{2a^2}\\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^3(x^2+y^2)} &= \frac{1}{3a^3}\left(\frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}\ln 2\right)\\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^4(x^2+y^2)} &= \frac{1}{4a^4}\left(\frac{\pi}{4}+\frac{\pi}{4}-\frac{2}{3}\right)=\frac{1}{4a^4}\left(\frac{\pi}{2}-\frac{2}{3}\right)\\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^5(x^2+y^2)} &= \frac{1}{5a^5}\left(\frac{\pi}{4}+\frac{1}{2}\ln 2-\frac{1}{4}\right)\\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^6(x^2+y^2)} &= \frac{1}{6a^6}\left(\frac{\pi}{4}+\frac{13}{15}-\frac{\pi}{4}\right)=\frac{13}{90a^6}
\end{aligned}$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

We show that the integral of the problem, denoted by I, equals $\dfrac{13}{90a^6}$.

Since

$$\int_a^\infty \frac{dy}{x^2+y^2}=\frac{1}{x}\left[\arctan\left(\frac{y}{x}\right)\right]_a^\infty=\frac{\arctan\left(\frac{x}{a}\right)}{x}\ \text{for } x>0,\ \text{so } I=\int_a^\infty \frac{\arctan\left(\frac{x}{a}\right)}{x^7}.$$

By the substitution $t=\dfrac{x}{a}$, we obtain $I=\dfrac{1}{a^6}\displaystyle\int_1^\infty \frac{\arctan t}{t^7}dt$. Integrating by parts, we obtain $I=\dfrac{\pi}{24a^6}+\dfrac{J}{6a^6}$, where $J=\displaystyle\int_1^\infty \frac{dt}{(1+t^2)t^6}$. We now substitute $t=\cot\theta$ to reduce J to the standard integral $\displaystyle\int_0^{\pi/4}\tan^6\theta d\theta$. which equals $\dfrac{13}{15}-\dfrac{\pi}{4}$. Hence our result for I.

Also solved by Michel Bataille, Rouen, France; Pat Costello, Eastern Kentucky University, Richmond, KY; Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University, Athens, Greece, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
Sept. 15, 2019

5547: *Proposed by Kenneth Korbin, New York, NY*

Given Heronian Triangle ABC with $\overline{AC} = 10201$ and $\overline{BC} = 10301$. Observe that the sum of the digits of $\overline{AC}$ is 4 and the sum of the digits of BC is 5. Find $\overline{AB}$ if the sum of its digits is 3.

(An Heronian Triangle is one whose side lengths and area are integers.)

5548: *Proposed by Michel Bataille, Reoun, France*

Given nonzero real numbers p and q, solve the system

$$\begin{cases} 2p^2x^3 - 2pqxy^2 - (2p-1)x = y \\ \\ 2q^2y^3 - 2pqx^2y + (2q+1)y = x \end{cases}$$

5549: *Proposed by Arkady Alt, San Jose, CA*

Let P be an arbitrary point in $\triangle\, ABC$ that has side lengths a, b, and c .

a) Find minimal value of

$$F(P) := \frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)};$$

b) Prove the inequality $\dfrac{a^2}{d_a(P)} + \dfrac{b^2}{d_b(P)} + \dfrac{c^2}{d_c(P)} \geq 36r$, where r is the inradius.

5550: *Proposed by Ángel Plaza, University of the Las Palmas de Gran Canaria, Spain*

Prove that

$$\sum_{n=4}^{\infty} \sum_{k=2}^{n-2} \frac{1}{k\binom{n}{k}} = \frac{1}{2}.$$

5551: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $n \geq 2$ be positive real numbers. Prove that the following inequality holds:

$$1 + \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \frac{\left(\sqrt{\alpha_i \alpha_{j+1}} - \sqrt{\alpha_j \alpha_{i+1}}\right)^2}{\alpha_i \alpha_j} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{\alpha_{k+1}}{\alpha_k} \right)^2 \right)^{1/2}$$

(Here the subscripts are taken modulo n.)

5552: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : \Re \to \Re$ such that $f'(x) - f(-x) = e^x, \forall x \in \Re$, with $f(0) = 0$.

Solutions

5529: *Proposed by Kenneth Korbin, New York, NY*

Convex cyclic quadrilateral $ABCD$ has integer length sides and integer area. The distance from the incenter to the circumcenter is 91. Find the length of the sides.

Solution 1 by David E. Manes, Oneonta, NY

Let $ABCD$ be a bicentric quadrilateral with inradius r and circumradius R and side lengths $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Then $a + c = b + d$ since the quadrilateral has an inscribed circle. Denote the diagonals $AC = p$ and $BD = q$. Finally, let $D = 2R$ represent the diameter of the circumscribed circle. If $x = 91$ denotes the distance between the incenter and the circumcenter, then Fuss' theorem gives a relation between r, R and $x = 91$; namely;

$$\frac{1}{(R-x)^2} + \frac{1}{(R+x)^2} = \frac{1}{r^2}.$$

Solving this equation for r, one obtains

$$r = \frac{R^2 - x^2}{\sqrt{2(R^2 + x^2)}} = \frac{R^2 - 91^2}{\sqrt{2(R^2 + 91^2)}}.$$

Substituting values for $R > 91$ in this equation, one quickly finds that if $R = 221$, then $r = 120$. Therefore,
$pq = 2r\left(r + \sqrt{4R^2 + r^2}\right) = 2(120)\left(120 + \sqrt{4 \cdot 221^2 + 120^2}\right) = 138720.$
Consider the quadrilateral with side lengths $a = AB = 170$, $b = BC = 408$, $c = CD = 408$ and $d = DA = 170$. Then $a + c = b + d = 578 = s$, the semi-perimeter of $ABCD$. Moreover,

$$a^2 + c^2 = 170^2 + 408^2 = b^2 + d^2 = D^2 = 442^2 = p^2;$$

hence, the quadrilateral is a kite. It consists of two congruent right triangles with a common hypotenuse, the diameter D of the circumscribed circle which is also the diagonal $p = AC$. For the given side lengths, note that the circumradius R is given by

$$R = \frac{1}{2}\sqrt{a^2 + c^2} = \frac{1}{2}\sqrt{170^2 + 408^2} = \frac{1}{2}\sqrt{b^2 + d^2} = 221.$$

and the inradius r is given by

$$r = \frac{pq}{2\sqrt{pq + 4R^2}} = \frac{138720}{2\sqrt{138720 + 4(221)^2}} = 120.$$

Since the quadrilateral $ABCD$ is a kite, the two diagonals $p = AC$ and $q = BD$ are perpendicular so that $\sin\theta = 1$, where θ is the angle between p and q. Therefore, the following formulas for the area K of $ABCD$ all agree:

$$\begin{aligned} K &= \sqrt{(s-a)^2(s-c)^2} = \sqrt{(578-170)^2(578-408)^2} \\ &= \sqrt{abcd} = \sqrt{(ab)^2} = ab = 170 \cdot 408 \\ &= r\left(r + \sqrt{4R^2 + r^2}\right)\sin\theta = 120\left(120 + \sqrt{4 \cdot 221^2 + 120^2}\right) \\ &= 69360. \end{aligned}$$

Finally, the four sides a, b, c, d of a bicentric quadrilateral with inradius $r = 120$, circumradius $R = 221$ and semi-perimeter $s = 578$ are the four roots of the quartic equation

$$y^4 - 2sy^3 + \left(s^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}\right)y^2 - 2rs\left(\sqrt{4R^2 + r^2} + r\right)y + r^2s^2 = 0.$$

Therefore,

$$y^4 - 1156y^3 + 472804y^2 - 80180160y + 4810809600 = 0,$$

$$(y - 408)^2(y - 170)^2 = 0.$$

Hence, the roots are 170 and 408, each of multiplicity two. This completes the solution.

Solution 2 by Ed Gray, Highland Beach, FL

We start with Fuss' Theorem which says: Given $R =$ circumradius, $r =$ inradius,, $x =$ distance between the incenter and the circumcenter, then:

1. $$\frac{1}{(R+x)^2} + \frac{1}{(R-x)^2} = \frac{1}{r^2}$$

2. $$\frac{(R-x)^2 + (R+x)^2}{(R+x)^2 \cdot (R-x)^2} = \frac{1}{r^2}$$

3. $$\frac{R^2 - 2Rx + x^2 + R^2 + 2Rx + x^2}{(R^2 - x^2)^2} = \frac{1}{r^2}$$

4. $2r^2 \cdot (R^2 + x^2) = (R^2 - x^2)^2$

5. $2 \cdot r^2 \cdot R^2 + 2 \cdot r^2 \cdot x^2 = R^4 - 2 \cdot R^2 \cdot x^2 + x^4$

   Writing (5) as a quadratic in R^2,

6. $R^4 - (2 \cdot r^2 + 2 \cdot x^2)R^2 + x^4 - 2 \cdot r^2 \cdot x^2 = 0,$ with solution

7. $\quad 2R^2 = 2r^2 + 2x^2 + \sqrt{4r^4 + 8 \cdot r^2 \cdot x^2 + 4x^4 - 4(x^4 - 2 \cdot r^2 \cdot x^2)}$

The + sign is used to ensure$R \geq r\sqrt{2}$.

8. $\quad 2R^2 = 2(r^2 + x^2) + \sqrt{4r^4 + 16 \cdot r^2 \cdot x^2}$, and

9. $\quad R^2 = r^2 + x^2 + r\sqrt{r^2 + 4x^2}$

Letting $x = 91$, consider the discriminant:

10. $\quad D^2 = r^2 + 33124$

11. $\quad (D - r)(D + r) = 2^2 \cdot 7^2 \cdot 13^2$

$(D - r)$ and $(D + r)$ must have the same parity since their sum is even. Since their product is even, each factor is even. $D - r$ must be less than $2 \cdot 7 \cdot 13, D + r$ must be greater than $2 \cdot 7 \cdot 13$. We try for a solution assuming that r is an integer. The possible values for $D - r$ are, $2, 14, 26, 98$. Since $x = 91$, the disparity between r and R cannot be exceedingly large. Accordingly, we start with the largest value for $D - r$.

12. $\quad D - r = 98$

13. $\quad D + r = 338$

14. $\quad 2D = 436, D = 218, r = 120$. Substituting these values into (9),

15. $\quad R^2 = 14400 + 8281 + 120 \cdot 218 = 48841$. Then:

16. $\quad R = 221$.

To get an idea of the character of the sides, we co-ordinate the quantities in a Cartesian coordinate system. For convenience, we put the circumcenter, O, at the origin, $(0, 0)$. The incenter, I, will be on the positive $y-$axis and have coordinates $(0, 91)$. With $r = 120$, notice that the incircle has its extreme point on the $y-$ axis with coordinates $(0, 211)$. The upper extreme for the circumcenter is $(0, 221)$, so that they only differ by 10. The lower extreme for the incircle has coordinates $(0, -29)$. Clearly, picturing the sides shows the quadrilateral must have two long sides for the lower two, and two much shorter sides for the upper two, suggesting a kite-like shape for the quadrilateral. In fact, we will pursue this concept, placing vertex A at $(0, 221)$, vertex C at $(0, -221)$. Vertex B will have $x > 0, y > 0$, Vertex D will have $x < 0, y > 0$. We have $AB = AD, BC + DC$, and, of course, as in all bi-centric quadrilaterals, $AB + CD = BD + CB$.

Now consider the side AB. It is tangent to the incircle at point T, so that IT is perpendicular to AB. Triangle AIT is a right triangle, with hypotenuse $AI = 130$, leg $IT = r = 120$.
So that $AT = 50$. Let $\angle TAI = t$. We note that $\cos(t) = 5/13, \sin(t) = 12/13$. The equation of side AB is $y = mx + b$, where
$m = \tan(t - 90) = \dfrac{\sin(t - 90)}{\cos(t - 90)} = \dfrac{(\sin(t) \cdot \cos(90) - \cos(t) \cdot \sin(90)}{\cos(t) \cdot \cos(90) + \sin(t) \cdot \sin(90)} = \dfrac{(-5/13)}{(12/13)} = -\dfrac{5}{12}$.
When $x = 0, y = 221$, so the equation of the chord AB is: 17. $y = 2215x/12$. The equation of the circumcircle is: 18. $x^2 + y^2 = 48841$.

The coordinates of vertex B can be found by solving (17), (18) simultaneously.

19. $x^2 + (215x/12)^2 = 48841$
20. $x^2 + 488412210 \cdot x/12 + (25x^2)/144 = 48841$
21. $x^2(1 + 25/144) = 2210x/12$
22. $(169/144)x = 2210/12$
23. $x = (2210/12) \cdot (144/169) = (12) \cdot (13.07692308) = 156.9230769$
24. $y = 2215(156.9230769)/12 = 22165.38461538 = 155.6153846$
25. The coordinates of vertex $B = (156.9230769, 155.6153846)$

Using the distance formula, we can compute the length of side AB.

26. $AB = \sqrt{(156.92307690)^2 + (221155.6153846)^2}$
27. $AB = \sqrt{24624.85206 + 4275.147931}$
28. $AB = \sqrt{28900)} = 170.$
29. We can now compute the length of chord BC by the Law of Cosines, using $\triangle ABC$.

We have: $(BC)^2 = 170^2 + 442^2 2 \cdot 170 \cdot 442 \cdot (5/13)$ 30.
$(BC)^2 = 28900 + 19536457800 = 166464$

31. $BC = 408.$

The sides appear to be $170, 170, 408, 408$. As noted, integer area did not come into play, Explicitly. We show that, indeed, the area is an integer by using Brahmaguptas formula:

32. $A = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{408 \cdot 408 \cdot 170 \cdot 170} = 408 \cdot 170 = 69, 360.$
33. As a check, $r = A/s = 69, 369/578 = 120.$

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For clarity we label the lengths of the sides of quadrilateral $ABCD$ so AB has length a, BC has length b, CD has length c, and DA has length d.

We will show that a quadrilateral with sides $a = d = 408$ and $b = c = 170$ is a cyclic quadrilateral with integer area and distance from its incenter to its circumcenter is 91.

We were not able to show that this is the only such quadrilateral.

Because our quadrilateral is given to have an incenter, there must be an inscribed circle, tangent to all four sides (hence, known as a tangential quadrilateral).

Such a cyclic, tangential, quadrilateral is termed a bicentric quadrilateral. (Wikipedia: *https://en.wikipedia.org/wiki/Bicentric_quadrilateral*).
FromWikipedia(URL $https://en.wikipedia.org/wiki/Pitot\_theorem$), we find the following:

The **Pitot theorem,** named after the French engineer Henri Pitot, states that in a tangential quadrilateral the two sums of lengths of opposite sides are the same. Both sums of lengths equal the semiperimeter of the quadrilateral.

A convex quadrilateral $ABCD$ with sides a, b, c, d is bicentric if and only if opposite sides satisfy Pitot's theorem for tangential quadrilaterals and the cyclic quadrilateral property that opposite angles are supplementary; that is,
opposite sides equal: $a + c = b + d$
opposite angles are supplementary: $A + B = C + D = \pi$.

For a bicentric quadrilateral, Fuss' Theorem gives a relation between the inradius r, the circumradius R and the distance x between the incenter and the circumcenter:

$$\frac{1}{(R-x)^2}+\frac{1}{(R+x)^2}=\frac{1}{r^2}$$

Some relevant facts about a bicentric quadrilateral:
(1) the area is given by $A=\sqrt{abcd}$
(2) the inradius is given by $r=\frac{A}{s}=\frac{\sqrt{abcd}}{s}=\frac{\sqrt{abcd}}{a+c}=\frac{\sqrt{abcd}}{b+d}$.

By (2), the inradius of our quadrilateral must be rational. There are no *a priori* restrictions on the circumradius R.

However, well first look for integer values for r and R.

Substituting our known value, $x=91$, into Fuss' Theorem and solving for r yields
$r=\frac{R^2-91^2}{\sqrt{(R+91)^2+(R-91)^2}}=\frac{R^2-91^2}{\sqrt{2(R^2+91^2)}}$.
At the worst, the quantity inside the radical must be the square of a rational; well impose the condition that it be the square of an integer:
$2(R^2+91^2)=z^2$ so $r=\frac{R^2-91^2}{z}$.
Thus, z must be even; say $z=2w$:
$2(R^2+91^2)=(2w)^2=4w^2$.
$\mathrm{R}^2+91^2=2w^2$
(3) $R^2-2w^2=-91^2$.

This is a Pell-like equation. With some initial assistance from Excel, we can find infinitely many solutions in integers. Because R must be larger than 91, the smallest valid solution is $R=221, w=169, soz=338$.
This yields an integer value for $r: r=120$.

Now the fun begins we must find values for a,b,c,d.

We want
$120=\frac{\sqrt{abcd}}{a+c}=\frac{abcd}{b+d}$,
$120^2(a+c)^2=abcd$ and $120^2(b+d)^2=abcd.$

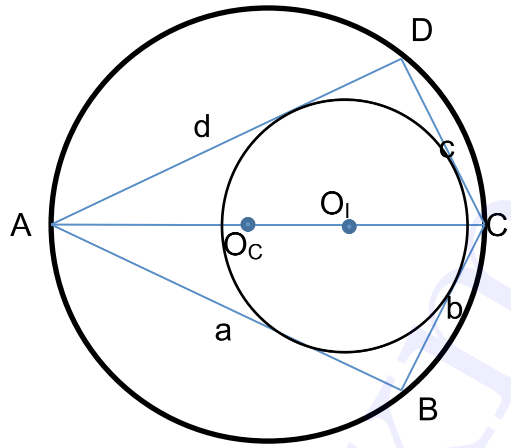
Using the prime factorization of 120 and applying some ingenuity, we find that the values $a=d=408$ and $b=c=170$. satisfy the conditions.
This would make our quadrilateral a (convex) kite, which is automatically a tangential quadrilateral.

However, the lengths of the sides by themselves do not completely specify a quadrilateral. We must proscribe its shape.

Noting that
$a=d=408=34\cdot 12$
and
$b=c=170=34\cdot 5$, we build the quadrilateral so the principal diagonal
$AC=442=34\cdot 13$.

This forces ABC and ADC to be right triangles (scaled-up copies of the 5-12-13 triangle), with AC being a diameter of the circumcircle.

Hence, our quadrilateral is inscribed in a circle, hence is cyclic and bicentric, as required. The difference x between the incenter and the circumcenter must equal 91 by our derivation of R, r.



In fact, the only kite which is cyclic is one formed by two congruent right triangles joined along the hypotenuse (= the diameter). Its sometimes known as a *right kite.*

Comment: We make no claim that our solution is unique. For instance, even after R and r were determined, the conditions $r^2(a+c)^2 = abcd$ and $r^2(b+d)^2 = abcd$ could admit other solutions (although a computer search found none).

Moreover, the Pell equation $R^2 - 2w^2 = -91^2$ has infinitely many solutions.
Using $\begin{pmatrix} R_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 221 \\ 169 \end{pmatrix}$ or $\begin{pmatrix} 299 \\ 221 \end{pmatrix}$ or $\begin{pmatrix} 637 \\ 455 \end{pmatrix}$ as a base, we can generate infinitely many more solutions by the recursive scheme
$\begin{pmatrix} R_{k+1} \\ w_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} R_k \\ w_k \end{pmatrix}$.

Of the solutions that we have checked, each produces a rational, non-integer value for the inradius r (which is acceptable but makes it much more difficult to find a, b, c, d). So there could be many other solutions to the problem.

Also solved by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece and the proposer.

5530: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygon $ABCD$ is an 11 by 12 rectangle ($AB > AD$). Points P, Q, R, and S are on sides AB, BC, CD, and DA, respectively, such that PR and SQ are parallel to AD and AB, respectively. Moreover, $X = PR \cap QS$. If the perimeter of rectangle $PBQX$ is 5/7 the perimeter of rectangle $SAPX$, and the perimeter of rectangle $RCQX$ is 9/10 the perimeter of rectangle $PBQX$, find the area of rectangle $SDRX$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $u = SD$ and $v = DR$. Then $AS = AD - SD = 11 - u$ and $RC = DC - DR = 12 - v$. Since $BQ = AS$, $PB = RC$, and $QC = SD$, the perimeter of rectangles $PBQX$, $SAPX$, and $RCQX$ are, respectively,

$2(PB+BQ) = 2(12-v+11-u)$, $2(AS+AP) = 2(11-u+v)$, and
$2(RC+QC) = 2(12-v+u)$.

Hence,

$$2(12-v+11-u) = \left(\frac{5}{7}\right) 2(11-u+v), \text{ and } 2(12-v+u) = \left(\frac{9}{10}\right) 2(12-v+11-u),$$

which implies $(u,v) = (5,8)$, so the area of rectangle $SDRX$ is $SD \cdot DR = uv = 40$.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX

From the given dimensions of rectangle $ABCD$, we have $PX + RX = 11$ and $QX + SX = 12$. Since the perimeter of rectangle $PBQX$ is $\frac{5}{7}$ the perimeter of rectangle $SAPX$, we have $BQ + PX + BP + QX = \frac{5}{7}(AP + SX + AS + PX)$, that is, $2PX + 2QX = \frac{5}{7}(2SX + 2PX)$ or $PX + QX = \frac{5}{7}(SX + PX)$. Similarly, since the perimeter of rectangle $RCQX$ is $\frac{9}{10}$ the perimeter of rectangle $PBQX$, we have $QX + RX = \frac{9}{10}(QX + PX)$.

So we have the following system of four equations in four unknowns:

$$\begin{cases} PX & & +RX & & = 11 \\ & QX & & +SX & = 12 \\ \frac{2}{7}PX & +QX & & -\frac{5}{7}SX & = 0 \\ -\frac{9}{10}PX & +\frac{1}{10}QX & +RX & & = 0 \end{cases}$$

Solving this systems yields $PX = 6$, $QX = 4$, $RX = 5$, and $SX = 8$, whence the area of rectangle $SDRX$ is $(RX)(SX) = (5)(8) = 40$.

Also solved by Ashland University Undergraduate Problem Solving Group, Ashland, Ohio; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.)

5531: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu-Severin, Mehedinti, Romania*

For real numbers x, y, z prove that if $x, y, z > 1$ and $xyz = 2\sqrt{2}$, then

$$x^y + y^z + z^x + y^x + z^y + x^z > 9.$$

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

If $f(t) = t \ln\left(\frac{2\sqrt{2}}{t}\right)$, with $t > 1$, the $f'(t) = \ln\left(\frac{2\sqrt{2}}{t}\right) - 1$, and for $f'(t_k) = 0$, we have $t_k = \frac{2\sqrt{2}}{e} > 1$. So, we have: $f(t) \geq f\left(\frac{2\sqrt{2}}{e}\right) = \frac{2\sqrt{2}}{e} \ln\left(2\sqrt{2} \cdot \frac{e}{2\sqrt{2}}\right) = \frac{2\sqrt{2}}{e}$.

Furthermore, we have:

$$x^y \geq 1 + y\ln x, \quad y^x \geq 1 + x\ln y, \quad y^x \geq 1 + z\ln y,$$

$$z^y \geq 1 + y\ln z, \quad z^x \geq 1 + x\ln z, \quad x^z \geq 1 + z\ln z,$$

So, we have:

$$\begin{aligned}
x^y + y^z + z^x + y^x + z^y + x^z &\geq 6 + x\ln(yz) + y\ln(xz) + z\ln(xy) \\
&= 6 + x\ln\left(\frac{2\sqrt{2}}{x}\right) + y\ln\left(\frac{2\sqrt{2}}{y}\right) + z\ln\left(\frac{2\sqrt{2}}{z}\right) \\
&\geq 6 + 3\cdot\frac{2\sqrt{2}}{e} > 9.
\end{aligned}$$

Solution 2 by Adrian Naco, Polytechnic University of Tirana, Albania

Since, $x > 1, y > 1$, and using the Bernoulli inequality, we have that

$$x^y = [1 + (x-1)]^y > 1 + y(x-1). \qquad (2)$$

Acting analogously it implies that,

$$x^y + y^z + z^x + y^x + z^y + x^z > 6 + 2(xy + yz + zx) - 2(x + y + z). \qquad (3)$$

To prove the given inequality (1), it is enough to prove the following equivalent inequalities,

$6 + 2(xy + yz + zx) - 2(x + y + z) > 9$ or equivalently $(xy + yz + zx) - (x + y + z) > \dfrac{3}{2}$

Let

$$f(x, y, z) = (xy + yz + zx) - (x + y + z) - \frac{3}{2}. \qquad g(x, y, z) = xyz - 2\sqrt{2}$$

and using Langrange Multipliers method, we have that,

$$F(x, y, z) = f(x, y, z) - \lambda g(x, y, z) = (xy + yz + zx) - (x + y + z) - \frac{3}{2} - \lambda(xyz - 2\sqrt{2}).$$

$$F_x = y + z - 1 - \lambda yz = 0$$

$$F_y = x + z - 1 - \lambda xz = 0$$

$$F_z = x + y - 1 - \lambda xy = 0$$

$$F_\lambda = -xyz + 2\sqrt{2} = 0$$

Subtracting side by side, each couple of the last three first equations, we get the following:

$$(z-1)(x-y) = 0$$

$$(y-1)(x-z) = 0$$

$$(x-1)(z-y) = 0$$

$$xyz = 2\sqrt{2}$$

So, $x = y = z = \sqrt{2}$, is the only solution (since $x > 1, y > 1, z > 1$). Finally,

$$\min f(x, y, z) = f(\sqrt{2}; \sqrt{2}; \sqrt{2}) = 2 + 2 + 2 - 3\sqrt{2} - \frac{3}{2} = \frac{3}{2}(3 - 2\sqrt{2}) > 0.$$

Note. Even if we consider the case when $x = 1$, we have that,

$$f(1, y, z) = y + yz + z - 1 - z - y - \frac{3}{2} = 2\sqrt{2}) - \frac{5}{2} > f(\sqrt{2}; \sqrt{2}; \sqrt{2}) = \frac{3}{2}(3 - 2\sqrt{2}) > 0.$$

Solution 3 by Moti Levy, Rehovot, Israel

Let $f(u, v) := u^v + v^u, \quad u, v > 1$. By verifying that the Hessian of $f(u, v)$ is positive semi-definite, it becomes evident that $f(u, v)$ is convex function in the domain $u, v > 1$.

$$Hess\,(u^v + v^u) = \begin{bmatrix} u^{v-2}(v-1)v + v^u \ln^2 v & u^{v-1} + v^{u-1} + vu^{v-1}\ln u + uv^{u-1}\ln v \\ u^{v-1} + v^{u-1} + vu^{v-1}\ln u + uv^{u-1}\ln v & (u-1)uv^{u-2} + u^v \ln^2 v \end{bmatrix}$$

Then by Jensen's inequality

$$\begin{aligned} & x^y + y^z + z^x + y^x + z^y + x^z \\ & = f(x, y) + f(y, z) + f(z, x) \geq 3f\left(\frac{x+y+z}{3}, \frac{y+z+x}{3}\right) \end{aligned} \quad (1)$$

By AM-GM inequality,

$$xyz = 2\sqrt{2} \quad \Longrightarrow \quad \frac{x+y+z}{3} \geq \sqrt[3]{2\sqrt{2}} = \sqrt{2}. \quad (2)$$

Inequalities (1) and (2) imply the required result,

$$x^y + y^z + z^x + y^x + z^y + x^z \geq 3f\left(\sqrt{2}, \sqrt{2}\right) = 6\left(\sqrt{2}\right)^{\sqrt{2}} > 9.$$

Also solved by Khaled Abd Imouti, Zaki Al Arzousi School, Damascus, Syria, (communicated to SSM by Daniel Sitaru of Romania); Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Tran Hong (student), Cao Lang School, Dong Thap, Vietnam (communicated to SSM by Daniel Sitaru of Romania) and the proposer.

5532: *Proposed by Arkady Alt, San Jose, CA*

Let a, b, c be positive real numbers and let $a_n = \dfrac{an+b}{an+c}, n \in N$. For any natural number m find $\displaystyle\lim_{n\to\infty} \prod_{k=n}^{nm} a_k$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

For large n,

$$a_n = \frac{an+b}{an+c} = \frac{1 + \frac{b}{an}}{1 + \frac{c}{an}} \sim 1 + \frac{b-c}{an}.$$

Thus,

$$\ln \prod_{k=n}^{mn} a_k = \sum_{k=n}^{mn} \ln a_k \sim \sum_{k=n}^{mn} \frac{b-c}{an} = \frac{b-c}{a}\left(H_{mn} - H_{n-1}\right),$$

where H_n denotes the nth Harmonic number. Now,

$$H_n \sim \ln n + \gamma,$$

where γ is the Euler-Mascheroni constant, so

$$H_{mn} - H_{n-1} \sim \ln \frac{mn}{n-1}$$

and

$$\ln \prod_{k=n}^{mn} a_k \sim \frac{b-c}{a} \ln \frac{mn}{n-1}.$$

Thus,

$$\lim_{n\to\infty} \ln \prod_{k=n}^{mn} a_k = \frac{b-c}{a} \ln m,$$

and

$$\lim_{n\to\infty} \prod_{k=n}^{mn} a_k = \exp\left(\frac{b-c}{a} \ln m\right) = m^{(b-c)/a}.$$

Solution 2 by Moti Levy, Rehovot, Israel

We rewrite the product as

$$\prod_{k=n}^{mn} a_k = \prod_{k=n}^{mn} \left(1 + \frac{\alpha}{k+\beta}\right), \qquad \alpha = \frac{b-c}{a}, \quad \beta = \frac{c}{a}.$$

$$\ln \prod_{k=n}^{mn} a_k = \sum_{k=n}^{mn} \ln\left(1 + \frac{\alpha}{k+\beta}\right) = \sum_{k=n}^{mn} \left(\frac{\alpha}{k+\beta} + O\left(\frac{1}{k^2}\right)\right) = \sum_{k=n}^{mn} \left(\frac{\alpha}{k} + O\left(\frac{1}{k^2}\right)\right)$$

$$\lim_{n\to\infty} \ln \prod_{k=n}^{mn} a_k = \lim_{n\to\infty} \sum_{k=n}^{mn} \frac{\alpha}{k} = \alpha \lim_{n\to\infty} \sum_{k=n}^{mn} \frac{1}{k}$$

$$\sum_{k=n}^{mn} \frac{1}{k} = \frac{1}{n} \sum_{k=0}^{(m-1)n} \frac{1}{1+\frac{k}{n}}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{(m-1)n} \frac{1}{1+\frac{k}{n}} = \int_0^{m-1} \frac{1}{1+x} dx = \ln m.$$

$$\lim_{n\to\infty} \ln \prod_{k=n}^{mn} a_k = \alpha \ln m,$$

hence

$$\prod_{k=n}^{mn} a_k = m^{\alpha} = m^{\frac{b-c}{a}}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

The Euler gamma function $\Gamma(x)$ satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$. Therefore

$$\prod_{k=n}^{mn} a_k = \prod_{k=n}^{mn} \frac{k+\frac{b}{a}}{k+\frac{c}{a}} = \frac{\Gamma\left(mn+1+\frac{b}{a}\right)}{\Gamma\left(n+\frac{b}{a}\right)} \cdot \frac{\Gamma\left(n+\frac{b}{a}\right)}{\Gamma\left(mn+1+\frac{c}{a}\right)}.$$

Stirling's asymptotic formula for the Euler gamma function states that
$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left(1+O\left(\frac{1}{x}\right)\right)$, as $x \to \infty$. So,

$$\prod_{k=n}^{mn} a_k \sim \frac{\sqrt{\dfrac{2\pi}{mn+1+\dfrac{b}{a}}}\left(\dfrac{mn+1+\dfrac{b}{a}}{e}\right)^{mn+1+\frac{b}{a}}}{\sqrt{\dfrac{2\pi}{n+\dfrac{b}{a}}}\left(\dfrac{n+\dfrac{b}{a}}{e}\right)^{n+\frac{b}{a}}} \cdot \frac{\sqrt{\dfrac{2\pi}{n+\dfrac{c}{a}}}\left(\dfrac{n+\dfrac{c}{a}}{e}\right)^{n+\frac{c}{a}}}{\sqrt{\dfrac{2\pi}{mn+1+\dfrac{c}{a}}}\left(\dfrac{mn+1+\dfrac{c}{a}}{e}\right)^{mn+1+\frac{c}{a}}} \sim$$

$$\sim \frac{\left(mn+1+\dfrac{b}{a}\right)^{\frac{b}{a}}}{\left(n+\dfrac{b}{a}\right)^{\frac{b}{a}}} \cdot \frac{\left(n+1+\dfrac{c}{a}\right)^{\frac{c}{a}}}{\left(mn+1+\dfrac{c}{a}\right)^{\frac{c}{a}}} \sim m^{\frac{b-c}{a}} \text{ as } n \to \infty.$$

Solution 4 by Michel Bataille, Rouen, France

We show that the required limit is $m^{(b-c)/a}$.

We shall use the following well-known result about the Gamma function: if s is a positive real number, then

$$\lim_{n\to\infty} \frac{n! \cdot n^s}{s(s+1)(s+2)\cdots(s+n)} = \Gamma(s).$$

For $n \geq 2$, we have

$$\prod_{k=n}^{nm}(ak+b) = a^{nm-n+1}\prod_{k=n}^{nm}\left(\frac{b}{a}+k\right) = a^{nm-n+1} \cdot \frac{\prod_{k=0}^{nm}\left(\frac{b}{a}+k\right)}{\prod_{k=0}^{n-1}\left(\frac{b}{a}+k\right)}$$

so that, as $n \to \infty$,

$$\prod_{k=n}^{nm}(ak+b) \;\sim\; a^{nm-n+1} \cdot \frac{(nm)!(nm)^{b/a}}{\Gamma(b/a)} \cdot \frac{\Gamma(b/a)}{(n-1)!(n-1)^{b/a}} = K_{m,n} \cdot \left(\frac{nm}{n-1}\right)^{b/a}$$

where $K_{m,n} = a^{nm-n+1} \cdot \frac{(nm)!}{(n-1)!}$.

Similarly, $\prod\limits_{k=n}^{nm}(ak+c) \;\sim\; K_{m,n} \cdot \left(\frac{nm}{n-1}\right)^{c/a}$ and it follows that

$$\prod_{k=n}^{nm} a_k \;\sim\; \left(\frac{nm}{n-1}\right)^{(b-c)/a}.$$

Since $\lim\limits_{n\to\infty} \frac{nm}{n-1} = m$, we obtain that $\lim\limits_{n\to\infty} \prod\limits_{k=n}^{nm} a_k = m^{(b-c)/a}$.

Solution 5 by Kee-Wai Lau, Hong Kong, China

We have $\ln a_k = \ln\left(1 + \dfrac{b-c}{ak+c}\right) = \dfrac{b-c}{ak+c} + O\left(\dfrac{1}{k^2}\right)$ as $k \to \infty$, where the constant implied by O depends at most on a, b, c. Hence

$$\sum_{k=n}^{mn} \ln a_k = (b-c)\sum_{k=n}^{nm} \frac{1}{ak+c} + O\left(\frac{1}{n}\right).$$

For $x > 0$, let $f(x)$ be the decreasing function $\dfrac{1}{ax+c}$ so that

$$\frac{1}{a}\ln\left(\frac{anm+c}{an+c}\right) = \int_n^{nm} \frac{dx}{ax+c} < \sum_{k=n}^{nm} \frac{1}{ak+c} < \int_{n-1}^{nm} \frac{dx}{ax+c} = \frac{1}{a}\ln\left(\frac{anm+c}{an+c-a}\right).$$

It follows that $\lim\limits_{n\to\infty} \sum\limits_{k=n}^{nm} \dfrac{1}{ak+c} = \dfrac{\ln m}{a}$. Thus

$$\lim_{n\to\infty} \prod_{ki=n}^{nm} a_k = e^{\lim\limits_{n\to\infty} \sum\limits_{k=n}^{nm} \frac{1}{ak+c}} = m^{(b-c)/a}.$$

Also solved by Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News,VA; and the proposer.

5533: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find the value of the sum

$$\sum_{n=1}^{+\infty} \frac{n^2\alpha^n}{(n-1)!}$$

for any real number $\alpha > 0$. (Here, $0! = 1! = 1$).

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

The solution is

$$\left(\alpha^3 + 3\alpha^2 + \alpha\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!}$$

for all real α. To avoid encountering the disputed expression 0^0 in our work, we note first that for $\alpha = 0$,

$$\left(\alpha^3 + 3\alpha^2 + \alpha\right) e^{\alpha} = 0 = \sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!}.$$

For $\alpha \neq 0$, we proceed as follows. Since

$$e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!},$$

we have

$$\alpha e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{\alpha^n}{(n-1)!}.$$

Then, if we differentiate with respect to α, we obtain

$$(\alpha+1)\, e^{\alpha} = \sum_{n=1}^{\infty} \frac{n\alpha^{n-1}}{(n-1)!}$$

and hence,

$$\left(\alpha^2+\alpha\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n\alpha^{n}}{(n-1)!}.$$

Differentiate again with respect to α to get

$$\left(\alpha^2+3\alpha+1\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n^2\alpha^{n-1}}{(n-1)!}$$

and therefore,

$$\left(\alpha^3+3\alpha^2+\alpha\right) e^{\alpha} = \sum_{n=1}^{\infty} \frac{n^2\alpha^{n}}{(n-1)!}.$$

Comment: Once we know the answer, we can verify this result directly as follows. As noted above, when $\alpha = 0$,

$$\left(\alpha^3+3\alpha^2+\alpha\right) e^{\alpha} = 0 = \sum_{n=1}^{\infty} \frac{n^2\alpha^{n}}{(n-1)!}.$$

For $\alpha \neq 0$,

$$\begin{aligned}
\left(\alpha^3+3\alpha^2+\alpha\right) e^{\alpha} &= \left(\alpha^3+3\alpha^2+\alpha\right) \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{\alpha^{n+2}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{3\alpha^{n+1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{\alpha^{n}}{(n-1)!} \\
&= \sum_{n=3}^{\infty} \frac{\alpha^{n}}{(n-3)!} + \sum_{n=2}^{\infty} \frac{3\alpha^{n}}{(n-2)!} + \sum_{n=1}^{\infty} \frac{\alpha^{n}}{(n-1)!} \\
&= 3\alpha^2 + \left(\alpha+\alpha^2\right) + \sum_{n=3}^{\infty} \left[\frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}\right] \alpha^n \\
&= \alpha + 4\alpha^2 + \sum_{n=3}^{\infty} \frac{(n-2)\,(n-1) + 3\,(n-1) + 1}{(n-1)!} \alpha^n \\
&= \alpha + 4\alpha^2 + \sum_{n=3}^{\infty} \frac{n^2\alpha^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!}.
\end{aligned}$$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We have:

$$n^2 = (n-1)(n-2) + 3(n-1) + 1,$$

for $n \in N$ with $n \geq 1$. So we have:

$$\frac{n^2\alpha^n}{(n-1)!} = \frac{\alpha^n}{(n-3)!} + \frac{3\alpha^n}{(n-2)!} + \frac{\alpha^n}{(n-1)!},$$

and

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{n^2\alpha^n}{(n-1)!} &= \alpha^3 \sum_{n=1}^{+\infty} \frac{\alpha^{n-3}}{(n-3)!} + 3\alpha^2 \sum_{n=1}^{+\infty} \frac{\alpha^{n-2}}{(n-2)!} + \alpha \sum_{n=1}^{+\infty} \frac{\alpha^{n-1}}{(n-1)!} \\ &= \alpha^3 e^\alpha + 3\alpha^2 e^\alpha + \alpha e^\alpha = (\alpha^3 + 3\alpha^2 + \alpha)e^\alpha. \end{aligned}$$

Solution 3 by Moti Levy, Rehovot, Israel

Let $F(z)$ be the generating function of the sequence $\left(\frac{n^2}{(n-1)!}\right)_{n=1}^{\infty}$,

$$F(z) := \sum_{n=1}^{\infty} \frac{n^2}{(n-1)!} z^n.$$

Then by two repeated integrations, one may write,

$$\int_0^z \frac{1}{v} \int_0^v \frac{1}{u} F(u)\, du = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^n = ze^z.$$

Now we can express $F(z)$ by

$$F(z) = z \frac{d\left(z \frac{d(ze^z)}{dz}\right)}{dz} = z\left(z^2 + 3z + 1\right)e^z.$$

We conclude that

$$\sum_{n=1}^{\infty} \frac{n^2\alpha^n}{(n-1)!} = \alpha\left(\alpha^2 + 3\alpha + 1\right)e^\alpha, \qquad for \quad \alpha \in C.$$

Remark: the value of the sum holds true for any complex number α. There is no reason to restrict to positive real numbers.

Solution 4 by Henry Ricardo, Westchester Area Math Circle, NY

We start with the power series expansion $e^z = \sum_{n=0}^{\infty} z^n/n!$, convergent for all complex numbers z and note that the series may be differentiated term-by-term.

Then

$$\frac{d}{dz}(e^z) = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!}, \qquad z\frac{d}{dz}(e^z) = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!},$$

$$\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\} = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{(n-1)!}, \qquad z\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\} = \sum_{n=1}^{\infty} \frac{nz^n}{(n-1)!},$$

$$\frac{d}{dz}\left[z\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\}\right] = \sum_{n=1}^{\infty} \frac{n^2 z^{n-1}}{(n-1)!},$$

and, finally,

$$z\,\frac{d}{dz}\left[z\,\frac{d}{dz}\left\{z\frac{d}{dz}(e^z)\right\}\right]=\sum_{n=1}^{\infty}\frac{n^2z^n}{(n-1)!}. \qquad (*)$$

After some tedious but simple differentiations and multiplications, the left-hand side of $(*)$ becomes $ze^z(z^2+3z+1)$. Letting $z=\alpha\in C$ in $(*)$ gives us

$$\sum_{n=1}^{\infty}\frac{n^2\alpha^n}{(n-1)!}\;=\;\alpha e^{\alpha}(\alpha^2+3\alpha+1).$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

Since $\dfrac{n^2}{(n-1)!}=\dfrac{1}{(n-3)!}+\dfrac{3}{(n-2)!}+\dfrac{1}{(n-1)!}$ for $n\geq 3$, so

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{n^2\alpha^n}{(n-1)!}&=\alpha+4\alpha^2+\sum_{n=3}^{+\infty}\frac{\alpha^n}{(n-3)!}+3\sum_{n=3}^{+\infty}\frac{\alpha^n}{(n-2)!}+\sum_{n=3}^{+\infty}\frac{\alpha^n}{(n-1)!}\\&=\alpha+4\alpha^2+\alpha^3e^{\alpha}+3\alpha^2(e^{\alpha}-1)+\alpha(e^{\alpha}-1-\alpha)\\&=\alpha e^{\alpha}\left(\alpha^2+3\alpha+1\right).\end{aligned}$$

Solution 6 by Arkady Alt, San Jose, CA

Since $e^x=\sum\limits_{n=1}^{+\infty}\dfrac{x^{n-1}}{(n-1)!}$ then $(xe^x)'=\left(\sum\limits_{n=1}^{+\infty}\dfrac{x^n}{(n-1)!}\right)'\iff e^x+xe^x=\sum\limits_{n=1}^{+\infty}\dfrac{nx^{n-1}}{(n-1)!}$

and, therefore, $\left(xe^x+x^2e^x\right)'=\left(\sum\limits_{n=1}^{+\infty}\dfrac{nx^n}{(n-1)!}\right)'\iff e^x\left(x^2+3x+1\right)=\sum\limits_{n=1}^{+\infty}\dfrac{n^2x^{n-1}}{(n-1)!}$.

Hence, $\sum\limits_{n=1}^{+\infty}\dfrac{n^2\alpha^n}{(n-1)!}=\alpha e^{\alpha}\left(\alpha^2+3\alpha+1\right)$

Editor's Comment: **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA,** generalized the procedure used in (4) and (6) above, and showed that $\sum\limits_{n=1}^{\infty}\dfrac{n^3\alpha^n}{(n-1)!}=(\alpha^4+6\alpha^3+7\alpha^2+\alpha)e^n$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Naren Bhandari, Bajura School, Nepal, India; Brian Bradie, Christopher Newport University, Newport, News, VA; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News,VA; David E. Manes, Oneonta, NY; Adrian Naco, Polytechnic University of Tirana, Albania; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ravi Prakash, Oxford University Press,

New Delhi, India; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University, Statesboro, GA, and the proposer.

5534: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy = \int_0^1 \int_0^1 (x+y)\ln(1-(1-x)(1-y))dxdy =$$

$$= -\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 \int_0^1 (x+y)(1-x)^k(1-y)^k dxdy = -2\sum_{k=1}^{\infty} \frac{1}{k} \frac{k!}{(k+2)!} \frac{1}{(k+1)} =$$

$$= -2\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2(k+2)} = -2\sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{(k+2)^2} - \frac{1}{2(k+2)}\right) = 2\left(\frac{1}{2}+\frac{1}{4}-\frac{\pi^1}{6}+1\right) =$$

$$= \frac{\pi^2}{3} - \frac{7}{2}$$

where we have used that for natural numbers m and n,
$\int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We prove that $\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy = \frac{\pi^2}{3} - \frac{7}{2}$. By symmetry we have:

$$I = \int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy. = \int_0^1 2\int_0^1 \ln(x-xy+y)dxdy.$$

and integration by parts we have:

$$\int_0^1 \ln(x-xy+y)dxdy = -\int_0^1 \frac{y(1-x)}{(1-x)y+x}dy = -1 + \int_0^1 \frac{dy}{y+\frac{x}{1-x}} = -1 - \frac{x\ln x}{1-x}.$$

So we have;

$$I = -2\int_0^1 x\left(1+\frac{x\ln x}{1-x}\right)dx = -1 - 2\int_0^1 \frac{x^2\ln x}{1-x}dx,$$

and if $x = e^t$, then:

$$I = -1 + 2\int_0^{+\infty} \frac{te^{-3t}}{1-e^{-t}}dt \quad = \quad -1 + 2\int_0^{+\infty} te^{-3t}\sum_{n\geq 0} e^{-nt}dt$$

$$\begin{aligned} &= -1+2\sum_{n\geq 0}\int_0^{+\infty} te^{-(n+3)t}dt = -1+2\sum_{n\geq 0}\frac{1}{(n+3)^2} \\ &= -1+2\left(\sum_{n\geq 0}\frac{1}{n^2}-1-\frac{1}{4}\right) \\ &= -1+2\left(\frac{\pi^2}{6}-\frac{5}{4}\right)=\frac{\pi^2}{3}-\frac{7}{2}. \end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News,VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the proposer.

Mea Culpa

Received from Ed Gray, Highland Beach, FL.

"I have been reviewing my solution to 5523 which you published in the last column. I regret to say that the case for P = 2 is not correct. The problem is that the formula for the circumscribed circle, R, is not satisfied. R = abc/4A. If you recall, we got excited about discovering more than 1 solution, later found to be incorrect. You sent a note asking if there could be three solutions? P=2, area = 420, sides (25,39,56), diameter 65. And if so, are there still others? The answer is that there is only 1 solution, the one you sent. I would be most happy if you printed my error in the next column."

Arkady Alt of San Jose, CA should have been credited with having solved problem 5525. Mea Culpa.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
December 15, 2019

- **5553:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with sides $(x, x, 57)$ has the same area as a triangle with sides $(x+1, x+1, 55)$. Find x.

- **5554:** *Proposed by Michel Bataille, Rouen, France*

  Find all pairs of complex numbers (a, b) such that the polynomial $x^5 + x^2 + ax + b$ has two roots of multiplicity 2.

- **5555:** *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy*

  Show that $x^x - 1 \le x^{1-x^2} e^{x-1} (x-1)$ for $0 < x \le 1$.

- **5556:** *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $\alpha_k = \dfrac{k + \sqrt{k^2+4}}{2}$. Evaluate $\displaystyle\lim_{k\to\infty} \frac{\prod_{n=1}^{\infty}\left(1 + \dfrac{(k-1)\alpha_k + 1}{\alpha_k^n + \alpha_k}\right)}{\alpha_k}$.

- **5557:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $n \ge 2$ be an integer. If for all $k \in \{1, 2, \ldots, n\}$ we have

  $$A_k = \begin{pmatrix} k+1 & k \\ k+3 & k+2 \end{pmatrix},$$

  compute the value of $\displaystyle\sum_{1 \le i < j \le n} \det(A_i + A_j)$.

- **5558:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all continuous functions $f : \Re \to \Re$ such that

$$\int_{-x}^{0} f(t)dt + \int_{0}^{x} tf(x-t)dt = x, \forall x \in \Re.$$

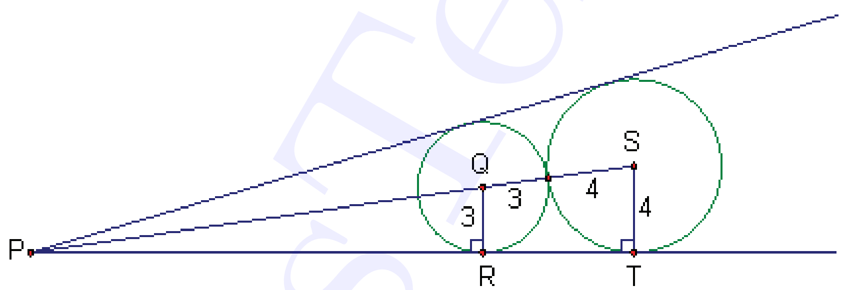
Solutions

- **5535:** *Proposed by Kenneth Korbin, New York, NY*

Given positive angles A and B with $A + B = 180^o$. A circle with radius 3 and a circle of radius 4 are each tangent to both sides of $\angle A$. The circles are also tangent to each other Find $\sin A$.

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

See the figure below, in which angle QPR is $\dfrac{A}{2}$.



We have $\dfrac{QR}{PQ} = \dfrac{ST}{PS}$, that is, $\dfrac{3}{PQ} = \dfrac{4}{PQ+7}$, whence $PQ = 21$. So $\sin\left(\dfrac{A}{2}\right) = \dfrac{1}{7}$ and $\cos\left(\dfrac{A}{2}\right) = \sqrt{1 - \left(\dfrac{1}{7}\right)^2} = \dfrac{4\sqrt{3}}{7}$.

So $\sin A = 2\sin\left(\dfrac{A}{2}\right)\cos\left(\dfrac{A}{2}\right) = 2\left(\dfrac{1}{7}\right)\left(\dfrac{4\sqrt{3}}{7}\right) = \dfrac{8\sqrt{3}}{49}$.

Solution 2 by David E. Manes, Oneonta, NY

The value of $\sin A$ is $8\sqrt{3}/49$.

Let X, Y denote the centers of the circles with radii 3 and 4, respectively. From vertex A, draw the line through the centers X and Y. This line splits the circles and the angle into two equal parts so that it is the angle bisector of $\angle A$. Construct the radius vector XR from the center of the circle with radius 3 to the point of tangency R with angle A. Similarly, YS is the radius vector from the circle of radius 4 to the point of tangency S of angle A. Then triangles AXR and AYS are similar right triangles with right angles at points R and S, respectively. If x denotes the hypotenuse AX of $\triangle AXR$, then $x+7$ is the hypotenuse AY of $\triangle AYS$. By the similarity of the two right triangles, it follows that the ratio of corresponding sides are equal. Therefore, $AX/XR = AY/YS$ or $x/3 = (x+7)/4$

implies $x = 21$. Let s denote the side length AR. Then $s^2 + 3^2 = 21^2$ or $s = 12\sqrt{3}$. Therefore,

$$\sin(A/2) = XR/AX = 3/21 = 1/7 \quad \text{and} \quad \cos(A/2) = AR/AX = 12\sqrt{3}/21 = 4\sqrt{3}/7.$$

Hence,

$$\sin A = 2\sin(A/2)\cos(A/2) = 2(1/7)(4\sqrt{3}/7) = 8\sqrt{3}/49 = \sin B.$$

Solution 3 by Ed Gray, Highland Beach, FL

Let:

$\angle A = \angle DAC$, where AC lies on the x-axis, and the coordinates of vertex $A = (0, 0)$.

Let O = center of circle with radius 3, O' = center of circle with radius 4. The angle bisector passes through both circle centers. Let OP be perpendicular to AC, and $AP = x$. The coordinates of $O = (x, 3)$. Let $O'Q$ be perpendicular to AC, and $PQ = y$.

$AQ = x+y$, and the coordinates of $O' = (x+y, 4)$. The distance from O to $O' = 3+4 = 7$.

(1) $\tan(A/2) = 3/x = 4/(x + y)$.

(2) $4x = 3x + 3y$, and $x = 3y$.

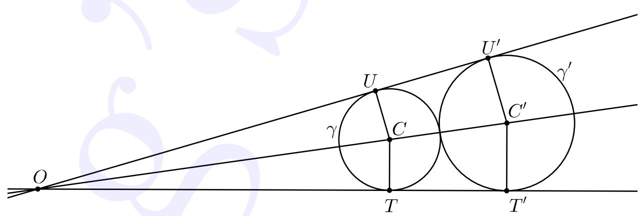
(3) Let T have coordinates $(x + y, 3)$, so that OT is parallel to AC.

(4) OTO' is a right triangle with legs of 1 and y, and hypotenuse of 7.

(5) Then $y^2 + 1 = 49, y^2 = 48$, and $y = 4\sqrt{3}, x = 3y = 12\sqrt{3}$.

(6) $\sin(A) = \sin[2(A/2)] = 2\sin(A/2)\cos(A/2) = 2\left(\frac{1}{7}\right)\left(\frac{y}{7}\right) = \frac{2y}{49} = \frac{8\sqrt{3}}{49}$.

Solution 4 by Michel Bataille, Rouen, France



Let γ and γ' be the circles with radii 3 and 4, respectively. The circle γ (resp. γ') is tangent to the sides of $\angle A$ at T and U (resp. at T' and U') [see figure]. Note that the centres C and C' of γ and γ' lie on the internal bisector of $\angle A$. Let O be the vertex of $\angle A$. The homothety with centre O and scale factor $\frac{4}{3}$ transforms γ into γ' and C into C'. Thus, we have $\frac{OC'}{OC} = \frac{4}{3}$ and, since γ and γ' are tangent to each other, $CC' = 4 + 3 = 7$. It follows that

$$\frac{OC'}{4} = \frac{OC}{3} = \frac{OC' - OC}{4 - 3} = \frac{CC'}{1} = 7.$$

As a result, we obtain $OC = 21$ and so $\sin\frac{A}{2} = \frac{CT}{OC} = \frac{3}{21} = \frac{1}{7}$. In addition, since $0 < \frac{A}{2} < 90°$, we have $\cos\frac{A}{2} > 0$ hence $\cos\frac{A}{2} = \sqrt{1 - \sin^2\frac{A}{2}} = \frac{4\sqrt{3}}{7}$. We can now conclude that

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{8\sqrt{3}}{49}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5536:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a \in (0,1)$ then calculate $\lim\limits_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\sin\left(\dfrac{a\cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-\sin a\right)$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

By Stirling's approximation,

$$n! \sim \frac{n^n}{e^n},$$

so

$$\sqrt[n]{n!} \sim \frac{n}{e} \quad \text{and} \quad \sqrt[n+1]{(n+1)!} \sim \frac{n+1}{e}.$$

Moreover,

$$(2n-1)!! = \frac{(2n)!}{2^n n!} \sim \frac{(2n)^{2n}/e^{2n}}{2^n n^n/e^n} = \frac{2^n n^n}{e^n},$$

so

$$\sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}.$$

It follows that

$$\sqrt[n]{(2n-1)!!}\left(\sin \frac{a\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - \sin a\right) \sim \frac{2n}{e}\left(\sin a\left(1+\frac{1}{n}\right) - \sin a\right).$$

Using the identity

$$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$$

with

$$A = a\left(1+\frac{1}{n}\right) \quad \text{and} \quad B = a,$$

we find

$$\sin a\left(1+\frac{1}{n}\right) - \sin a = 2\sin\frac{a}{2n}\cos\left(a+\frac{1}{2n}\right).$$

Thus,

$$\begin{aligned}\sqrt[n]{(2n-1)!!}\left(\sin \frac{a\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - \sin a\right) &\sim \frac{4n}{e}\sin\frac{a}{2n}\cos\left(a+\frac{1}{2n}\right) \\ &= \frac{2a}{e}\frac{\sin\frac{a}{2n}}{\frac{a}{2n}}\cos\left(a+\frac{1}{2n}\right).\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\sin\frac{a\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-\sin a\right) &= \lim_{n\to\infty}\frac{2a}{e}\frac{\sin\frac{a}{2n}}{\frac{a}{2n}}\cos\left(a+\frac{1}{2n}\right)\\
&= \frac{2a}{e}\cos a.
\end{aligned}$$

Note the restriction $a \in (0,1)$ is not necessary.

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The solution is $\displaystyle\lim_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\sin\left(\frac{a\cdot\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-\sin a\right)=\frac{2a\cos a}{e}$.

Note first that by Stirling formula $\sqrt[n]{(2n-1)!!}\sim\dfrac{2n}{e}$, and also that $\dfrac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\to 1$, for $n\to\infty$, and therefore, by Taylor expassion of $\sin ax$ at $x=1$, it follows that the proposed limit, say L, is

$$\begin{aligned}
L &= \frac{2}{e}\lim_{n\to\infty}\frac{-\frac{1}{2}(x-1)^2\left(a^2\sin(a)\right)+a(x-1)\cos(a)+\sin(a)-\sin(a)}{\frac{1}{n}}\\
&= \frac{2}{e}\lim_{n\to\infty}\frac{-\frac{1}{2}(x-1)^2\left(a^2\sin(a)\right)+a(x-1)\cos(a)}{\frac{1}{n}}\\
&= \frac{2a\cos a}{e},
\end{aligned}$$

where we have used $x=\dfrac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$, so, by the Stolz-Cezaro Lemma,

$$\lim_{n\to\infty}\frac{x-1}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\frac{\sqrt[n]{n!}}{n}}=e\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}=1$$

and consequently $\displaystyle\lim_{n\to\infty}\frac{(x-1)^2}{\frac{1}{n}}=0$, and the conclusion follows.

Solution 3 by Michel Bataille, Rouen, France

The required limit is $\dfrac{2a\cos a}{e}$.

Recall the well-known asymptotic expansion of $\ln(n!)$ as $n\to\infty$:

$$\ln(n!)=n\ln(n)-n+o(n) \qquad (1).$$

From (1), we deduce $\sqrt[n]{n!}\sim\frac{n}{e}$ as $n\to\infty$ [because $\sqrt[n]{n!}=e^{\frac{\ln(n!)}{n}}=e^{\ln(n)-1+o(1)}=\frac{n}{e}\cdot e^{o(1)}$ so that $\lim_{n\to\infty}\frac{e}{n}\cdot\sqrt[n]{n!}=1$]. It follows that

$$\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$$

as $n \to \infty$. Indeed, since $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{2^n n!}$, we have

$$\sqrt[n]{(2n-1)!!} = \frac{\left(\sqrt[2n]{(2n)!}\right)^2}{2\sqrt[n]{n!}} \sim \frac{1}{2}\cdot\frac{(2n/e)^2}{n/e} = \frac{2n}{e}.$$

To address the second factor, we first remark that $u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$ satisfies $u_n \sim \frac{n+1}{e}\cdot\frac{e}{n} = \frac{n+1}{n}$ so that $\lim\limits_{n\to\infty} u_n = 1$. Since $\ln(x) \sim x-1$ as $x \to 1$, it follows that

$$\begin{aligned}
u_n - 1 \sim \ln(u_n) &= \frac{1}{n+1}\left(\ln(n+1)+\ln(n!)\right) - \frac{1}{n}\ln(n!)\\
&= \frac{1}{n+1}\left(\ln(n+1) - \frac{1}{n}\ln(n!)\right)\\
&= \frac{1}{n}\left(1+\frac{1}{n}\right)^{-1}\left(\ln\left(1+\frac{1}{n}\right) + \ln(n) - \frac{1}{n}\ln(n!)\right)
\end{aligned}$$

and so $u_n - 1 \sim \frac{1}{n}$ as $n \to \infty$ (note that (1) gives $\lim\limits_{n\to\infty}\left(\ln(n) - \frac{1}{n}\ln(n!)\right) = 1$).
Now, since $\sin x \sim x$ as $x \to 0$, we obtain

$$\sin(au_n) - \sin a = 2\sin\frac{a(u_n-1)}{2}\cos\frac{a(u_n+1)}{2} \sim (2\cos a)\cdot\frac{a(u_n-1)}{2} \sim (a\cos a)\cdot\frac{1}{n}$$

as $n \to \infty$ and deduce that the desired limit is

$$\lim_{n\to\infty}\frac{2n}{e}\cdot(a\cos a)\cdot\frac{1}{n} = \frac{2a\cos a}{e}.$$

Editor's comment: The statement that there is no need to restrict a to $(0,1)$ was also noted in the solution submitted by **Moti Levy of Rehovot Israel.** Indeed, the result is valid for $a \in \mathbb{C}$.

Also solved by Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland and the proposer.

- **5537:** *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X, Y be two real-valued continuous random variables on the real line with associated mean, median and mode $\overline{x}, \tilde{x}, \hat{x}$, and $\overline{y}, \tilde{y}, \hat{y}$, respectively. For each of the following conditions, show that there are variables X,Y satisfying them or prove such random variables do not exist.

(i)	$\overline{x} \le \overline{y}$,	$\tilde{x} \le \tilde{y}$,	$\hat{x} \le \hat{y}$,	*(v)*	$\overline{x} > \overline{y}$,	$\tilde{x} \le \tilde{y}$,	$\hat{x} \le \hat{y}$
(ii)	$\overline{x} \le \overline{y}$,	$\tilde{x} \le \tilde{y}$,	$\hat{x} > \hat{y}$,	*(vi)*	$\overline{x} > \overline{y}$,	$\tilde{x} \le \tilde{y}$,	$\hat{x} > \hat{y}$
(iii)	$\overline{x} \le \overline{y}$,	$\tilde{x} > \tilde{y}$,	$\hat{x} \le \hat{y}$,	*(vii)*	$\overline{x} > \overline{y}$,	$\tilde{x} > \tilde{y}$,	$\hat{x} \le \hat{y}$
iv)	$\overline{x} \le \overline{y}$,	$\tilde{x} > \tilde{y}$,	$\hat{x} > \hat{y}$,	*(viii)*	$\overline{x} > \overline{y}$,	$\tilde{x} > \tilde{y}$,	$\hat{x} > \hat{y}$

Solution 1 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We provide examples that satisfy each of the 8 conditions. They can all happen.

Note that examples satisfying conditions (i) through (iv) with strict inequality satisfy conditions $(viii), (vii), (vi)$, and (v) respectively, if X and Y are reversed. For example, if X and Y satisfy condition (i), then Y and X satisfy condition $(viii)$; (ii) and (vii), (iii) and (vi), and (iv) and (v). So we only need four examples.

We'll define the random variables, X, Y_1, Y_2, Y_3, Y_4.

The probability density function for X:

$$f_X(t) = \begin{cases} 0, & t < 0; \\ 2.5t, & 0 \le t \le .8; \\ 10(1-t), & .8 \le t \le 1; \\ 0, & 1 < t. \end{cases}$$

It is straightforward to verify that $\int_{-\infty}^{\infty} f_X(t)\, dt = 1$.

Then the cumulative distribution function is $F_X(x) = \int_{-\infty}^{x} f_x(t)\, dt$.

The <u>mean</u> of X is
$\overline{X} = \int_{-\infty}^{\infty} t f_X(t) dt = \int_0^{.8} t(2.5t)dt + \int_{.8}^{1} 10(1-t)t dt = \frac{32}{75} + \frac{13}{75} = .6.$
To find the <u>median</u> of X, we must find the value for x which makes
$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \frac{1}{2}.$
By the definition of the pdf, this spot must occur before $x = .8$. So either by geometry or solving $\int_0^x 2.5t dt = \frac{1}{2}$, we find that the median is $\tilde{x} = \sqrt{\frac{2}{5}} = \frac{\sqrt{10}}{6} \approx .63246.$
The maximum value of the pdf f_X is .4, which occurs at $x = .8$.
That is the <u>mode</u> of X is $\hat{x} = .8$.

Conditions $(viii)$ and (i).

We define Y_1 by the density function

$$f_{Y_1}(t) = \begin{cases} 0, & t < -.5; \\ 4(t+.5), & -.5 \le t \le 0; \\ 4(.5-t), & 0 \le t \le .5; \\ 0, & .5 < t. \end{cases}$$

As above, we calculate our three measures:
Mean of $Y_1 : \overline{y_1} = 0.$
Median of $Y_1 : \tilde{y_1} = 0.$
Mode of $Y_1 : \hat{y_1} = 0.$
We see that
$\overline{x} = .6 > \overline{y_1} = 0.$
$\tilde{x} = .6325 > \tilde{y_1} = 0.$
$\hat{x} = .8 > \hat{y_1} = 0.$
Thus, X and Y_1 satisfy condition $(viii)$. Reversing X and Y_1 gives an example which satisfies condition (i)

Conditions (iv) and (v).

We define Y_2 by the density function

$$f_{Y2}(t) = \begin{cases} 0, & t < .12; \\ 4(t - .12), & .12 \le t \le .62; \\ 4(1.12 - t), & .62 \le t \le 1.12; \\ 0, & 1.12 < t. \end{cases}$$

As above, we calculate our three measures:
Mean of $Y_2 : \overline{y_2} = .62$.
Median of $Y_2 : \tilde{y_2} = .62$.
Mode of $Y_2 : \hat{y_2} = .62$
We see that
$\overline{x} = .6 < \overline{y_2} = .62$
$\tilde{x} = .6325 > \tilde{y_2} = .62$
$\hat{x} = .8 > \hat{y_2} = .62$.
Thus, X and Y_2 satisfy condition (iv),

Reversing X and Y_2 gives an example which satisfies condition (v)

Conditions (ii) and (vii).

We define Y_3 by the density function

$$f_{Y_3}(t) = \begin{cases} 0, & t < .2; \\ 4(t - .2), & .2 \le t \le .7 \\ 4(1.2 - t), & .7 \le t \le 1.2; \\ 0, & 1.2 < t. \end{cases}$$

As above, we calculate our three measures:
Mean of $Y_3 : \overline{y_3} = .7$.
Median of $Y_3 : \tilde{y_3} = .7$.
Mode of $Y_3 : \hat{y_3} = .7$.

We see that
$\overline{x} = .6 < \overline{y_3} = .7$
$\tilde{x} = .6325 < \tilde{y_3} = .7$
$\hat{x} = .8 > \hat{y_3} = .7$.
Thus, X and Y_3 satisfy condition (ii), Reversing X and Y_3 gives an example which satisfies condition (vii)

Conditions (iii) and (vi).

We define Y_4 by the density function, which is piecewise continuous and defined for all real numbers. Thus the cumulative distribution function for Y_4 is also continuous and defined everywhere. Thus Y_4 is a continuous random variable.

$$
f_{Y_4}(t) = \begin{cases} 0, & t < .47; \\ 10/3, & .47 \le t \le .62 \\ \\ 8.33/3, & .62 \le t \le .8 \\ \\ 2000000(t-8), & .8 \le t \le .80001 \\ \\ 2000000(.80002 - t), & .80001 \le t \le .80002 \\ \\ 0, & .80002 < t. \\ \end{cases}
$$

It is more tedious, but we calculate our three measures:
Mean of $Y_4 : \overline{y_4} = .62736$.
Median of $Y_4 : \tilde{y_4} = .62$.
Mode of $Y_4 : \hat{y_4} = .80001$

We see that
$\overline{x} = .6 < \overline{y_4} = .62736$
$\tilde{x} = .6325 > \tilde{y_4} = .62$
$\hat{x} = .8 < \hat{y_4} = .80001$.
Thus, X and Y_4 satisfy condition (iii)

Reversing X and Y_4 gives an example which satisfies condition (vi).

So for each condition $(i) \ldots (viii)$, we have an example satisfying it.

Note: If a random variable that has a continuous probability density function is desired, the following can be used for the definition of Y_4, (but the mathematics to compute its mean and median is much more tedious):

$$
f_{Y_4}(t) = \begin{cases} 0, & t < .45; \\ \frac{29167}{9303}(t - .45), & .45 \le t < .46 \\ \\ \frac{29167}{9303}, & .46 \le t \le .61 \\ \\ \left(\frac{833}{3} - \frac{2916700}{9303}\right)(t - .61) + \frac{29167}{9303}, & .61 < t < .62 \\ \\ \frac{8.33}{3}, & .62 < t \le .8 \\ \\ \frac{5167000}{3}(t - .8) + \frac{8.33}{3}, & .8 \le t \le .80001 \\ \\ \frac{60}{.000025835}(.80001 - t) + 20, & .8000 \le t \le .80001 + \frac{.000025835}{3} \\ \\ 0, & t > .80001 + \frac{.000025835}{3} \\ \end{cases}
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Let $0 < r < s < 1, u > 0, v > 0, w > 0$ such that

$$\begin{aligned}&(i)\ \ ur + v(s-r)/2 = 1,\\ &(ii)\ \ v(s-r)/2 + w(1-s) = 1.\end{aligned}$$

Let $p(x)$ be the continuous probability density function that is zero in $]-\infty, 0] \cup [1, \infty[$ and piecewise linear in $[0,1]$ such that the graph of $p(x)$ consists of line segments joining the points $(0,0)$ and $(r/2, u), (r/2, u)$ and $(r, 0), (r, 0)$ and $((r+s)/2, v), ((r+s)/2, v)$ and $(s, 0), (s, 0)$ and $((s+1)/2, w), ((s+1)/2, w)$ and $(1, 0)$. $p(x)$ is a probability density function with three "peaks."

Let X be the random variable whose probability density function is $p(x)$. Let r, s, v be given. We solve (i) and (ii) for u and w and find:

$$u = \frac{2 - v(s-r)}{2r},$$

$$w = \frac{2 - v(s-r)}{2(1-s)}.$$

Clearly, $v < \dfrac{2}{s-r}$, since both $u > 0$ and $v > 0$.

We next calculate the mean, median and mode of X, and express these quantities in terms of r, s, v:

$$\begin{aligned}\overline{x} \;&=\; \int_0^1 xp(x)dx = \int_0^{r/2} x\left(\frac{2u}{r}x\right)dx + \int_{r/2}^{r} x\left(-\frac{2u}{r}(x-r)\right)dx + \int_r^{(r+s)/2} x\left(\frac{2v}{s-r}(x-r)\right)dx +\\ &+\; \int_{(r+s)/2}^{s} x\left(-\frac{2v}{s-r}(x-s)\right)dx + \int_s^{(s+1)/2} x\left(\frac{2w}{1-s}(x-s)\right)dx + \int_{(s+1)/2}^{1} x\left(-\frac{2w}{1-s}(x-1)\right)dx =\end{aligned}$$

$$= \frac{1}{4}\left(r^2u - r^2v + s^2v + w - s^2w\right) = \frac{1+r+s}{4} + v\frac{(s-r)(r+s-1)}{8}.$$

Clearly, $\tilde{x} = \dfrac{r+s}{2}$, since $\displaystyle\int_0^{\tilde{x}} p(x)dx = \frac{1}{2}$ by (i).

The mode $\hat{x}$ is defined as the value $\hat{x}$ for which we have $p(\hat{x}) = \max(u, v, w)$.

We have

$\hat{x} = \dfrac{r}{2}$, if $u > v$ and $u > w$ which is equivalent to $v < \dfrac{2}{r+s}$ and $r + s < 1$.

$\hat{x} = \dfrac{r+s}{2}$, if $v > w$ and $v > u$ which is equivalent to $v > \dfrac{2}{r+s}$ and $v > \dfrac{2}{2-r-s}$.

$\hat{x} = \dfrac{s+1}{2}$, if $w > u$ and $w > v$ which is equivalent to $v < \dfrac{2}{2-r-s}$ and $r + s > 1$.

We have three free parameters at our disposal, namely r, s, v, we can play with. It turns out that by a suitable choice of these parameters all 8 variants can be realized as is evidenced by the subsequent table:

case	r	s	u	v	w	Mean	Median	Mode
(i)	0.296	0.615	0.528	5.289	0.406	0.459	0.456	0.456
	0.301	0.728	0.513	3.961	0.567	0.513	0.515	0.515
(ii)	0.301	0.509	0.547	8.031	0.336	0.413	0.405	0.405
	0.143	0.824	2.459	1.904	1.998	0.486	0.484	0.072
(iii)	0.407	0.881	2.214	0.418	7.571	0.579	0.644	0.941
	0.299	0.944	1.449	1.757	7.739	0.595	0.622	0.972
(iv)	0.502	0.953	1.652	0.758	17.640	0.633	0.728	0.977
	0.536	0.849	0.234	5.590	0.829	0.680	0.693	0.693
(v)	0.155	0.720	2.970	1.910	1.644	0.452	0.438	0.078
	0.260	0.630	0.462	4.756	0.325	0.448	0.445	0.445
(vi)	0.364	0.845	1.224	2.306	2.874	0.581	0.605	0.923
	0.485	0.744	1.603	1.719	3.037	0.570	0.615	0.872
(vii)	0.250	0.619	2.460	2.087	1.614	0.455	0.435	0.125
	0.269	0.452	3.507	0.620	1.721	0.426	0.361	0.135
(viii)	0.595	0.763	0.879	5.675	2.208	0.632	0.679	0.679
	0.371	0.752	1.401	2.521	2.096	0.546	0.562	0.562

The table was generated by a computer program that selected values for r, s, v randomly, thereby creating instances of the random variables X and Y, until a pair of random variables was found for each of the eight cases.

Editor's comment: This problem asked us to determine if certain relationships can exist between the mean, median, and mode in two sets of data that are subject to certain constrains. If the constraints on the data are relaxed, and by focusing on the mean, median, and mode on small finite sets of data, one can easily determine the validity of the relationships in this question.

Also solved by the proposer.

- **5538:** *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve for all real numbers $x \neq \dfrac{\pi}{2}(2k+1)$, $k \in \mathbb{Z}$.

$$2 - 2019x = e^{\tan x} + 3^{\sin x} + \tan^{-1} x.$$

Solution 1 by Michel Bataille, Rouen, France

For $k \in \mathbb{Z}$, let I_k denote the open interval $\left(\frac{\pi}{2}(2k-1), \frac{\pi}{2}(2k+1)\right)$. We first show that the equation has no solution in I_k for $k \geq 1$.
If $t \in I_k$ is a solution to the equation, then we have $2 - 3^{\sin t} - \tan^{-1} t = e^{\tan t} + 2019t$ and so

$$2019t < 2019t + e^{\tan t} = 2 - 3^{\sin t} - \tan^{-1} t < 2 - \frac{1}{3} + \frac{\pi}{2}$$

(since $3^{\sin t} \geq 3^{-1}$ and $\tan^{-1} t > -\frac{\pi}{2}$). It follows that $t < \frac{1}{2019}\left(\frac{5}{3} + \frac{\pi}{2}\right) < \frac{\pi}{2}$ and so we must have $k \leq 0$.

Now, we consider the function f defined by $f(x) = e^{\tan x} + 3^{\sin x} + \tan^{-1} x + 2019x$ whose derivative is $f'(x) = (1 + \tan^2 x)e^{\tan x} + (\ln 3)(\cos x)3^{\sin x} + \frac{1}{1+x^2} + 2019$.

Since $|(\ln 3)(\cos x)3^{\sin x}| \leq (\ln 3)3^{\sin x} \leq 3 \ln 3$, we have $(\ln 3)(\cos x)3^{\sin x} + 2019 > 0$, hence $f'(x) > 0$. It follows that the restriction f_k of f to the interval I_k, which is continuous and strictly increasing, is a bijection from I_k onto the interval (α_k, β_k) where $\alpha_k = \lim\limits_{x \to \frac{\pi}{2}(2k-1)} f_k(x)$ and $\beta_k = \lim\limits_{x \to \frac{\pi}{2}(2k+1)} f_k(x)$. Since $e^{\tan x}$ tends to 0 when $\tan x$ tends to $-\infty$ and to ∞ when $\tan x$ tends to ∞, it is readily seen that for $k \leq 0$, $\alpha_k < 0$ and $\beta_k = \infty$. Thus, the equation $f_k(x) = 2$ has a unique solution x_k in I_k for $k \leq 0$; in particular $x_0 = 0$. Therefore the given equation has infinitely many solutions, the numbers $x_k = f_k^{-1}(2)$ for $k \leq 0$.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that there are infinitely many solutions, one of which is zero and the rest of which are negative real numbers.

For each integer k, let D_k be the interval $(\pi(2k-1)/2, \pi(2k+1)/2)$. Let $D = \cup_{k=-\infty}^{\infty} D_k$. Define $f(x) = 3^{\sin x} + \tan^{-1} x + 2019x - 2$ for each x in $\mathbb{R}$, and define $g(x) = f(x) + e^{\tan x}$ for each x in D. Then

$$f'(x) = (\cos x)3^{\sin x}(\ln 3) + 1/(1 + x^2) + 2019$$

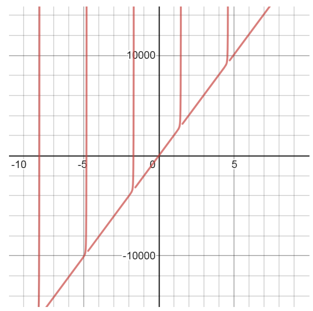
and $g'(x) = f'(x) + (\sec^2 x)e^{\tan x}$. Since $|(\cos x)3^{\sin x}(\ln 3)| \leq 3 \ln 3$ for all real numbers x, we have $f'(x) > 0$ on $\mathbb{R}$ and $g'(x) > 0$ on D. Thus f is increasing on $\mathbb{R}$, while g is increasing on each D_k.

Next, we note that on each D_k,

$$\lim_{x \to \frac{\pi}{2}(2k-1)^+} g(x) = f\left(\frac{\pi}{2}(2k-1)\right) \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}(2k+1)^-} g(x) = \infty.$$

Then there is exactly one zero of $g(x)$ in D_k if and only if $f\left(\frac{\pi}{2}(2k-1)\right) < 0$. Since $f(-\pi/2) < 0$ and $f(\pi/2) > 0$, we have exactly one zero x_k of $g(x)$ in D_k if and only if k is a non-positive integer. In particular, $x_0 = 0$, $x_{-1} \approx -1.693068317$, $x_{-2} \approx -4.820854357$, $x_{-3} \approx -7.956873841$, etc.

Graph of $g(x) = e^{\tan x} + 3^{\sin x} + \tan^{-1} x + 2019x - 2$:



Also solved by Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5539:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let α, β, γ be nonzero real numbers. Find the minimum value of

$$\left(\sum_{cyclic}\left(\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\right)^3\right)^{1/3}$$

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

If we let: $a=\tan\alpha,\ b=\tan\beta$, and $c=\tan\gamma$ then,

$$\sin^2\alpha=\frac{\tan^2\alpha}{1+\tan^2\alpha}=\frac{a^2}{1+a^2}$$

$$\sin^2\beta=\frac{\tan^2\beta}{1+\tan^2\beta}=\frac{b^2}{1+b^2}\qquad\qquad \sin^2\gamma=\frac{\tan^2\gamma}{1+\tan^2\gamma}=\frac{c^2}{1+c^2}.$$

Since $0\le\sin^2 x\le 1$, then $0\le\dfrac{\tan^2 x}{1+\tan^2 x}\le 1$ for $x\in\{\alpha,\beta,\gamma\}$. So, we have:

$$\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}=\frac{(1+a^2)(1+b^2)+a^2b^2}{a^2(1+b^2)}=\frac{1+a^2}{a^2}+\frac{b^2}{1+b^2}\ge 1+\frac{1+a^2}{a^2}.$$

Since $\lim\limits_{a\to\pm\infty}\left(1+\dfrac{1+a^2}{a^2}\right)=2$, then $\dfrac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\ge 3$, and:

$$\left[\sum_{cyclic}\left(\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\right)^3\right]^{1/3}\ge 3\sqrt[3]{3}\approx 4.32674871.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the expression of the problem by E. We show that the minimum of E is $2\sqrt[3]{3}$.

Since

$$\sum_{cyclic}\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\ =\ \sum_{cyclic}\frac{1}{\sin^2\alpha}+\sum_{cyclic}\sin^2\beta$$

$$
\begin{aligned}
&= \sum_{cyclic}\left(\frac{1}{\sin^2\alpha}+\sin^2\alpha\right)\\
&= \sum_{cyclic}\left(\left(\frac{1}{\sin\alpha}-\sin\alpha\right)^2+2\right)\\
&\geq 6,
\end{aligned}
$$

so by Hölder's inequality, we have $E \geq 3^{-2/3}\displaystyle\sum_{cyclic}\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha} \geq 2\sqrt[3]{3}$.

When $\alpha=\beta=\gamma=\dfrac{\pi}{2}$, we obtain $E = 2\sqrt[3]{3}$ and hence our claimed minimum.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We claim that the minimum value equals $2\sqrt[3]{3}$ and is assumed for $\alpha=\beta=\gamma=\dfrac{\pi}{2}$.

Let $u=\sin\alpha, v=\sin\beta, w=\sin\gamma$. The by the AM-GM inequality,

$$
\sqrt[3)]{\sum_{cyclic}\left(\frac{1+\sin^2\alpha^2\beta}{\sin^2\alpha}\right)} = \sqrt[3]{\left(\frac{1}{u}+v\right)^3+\left(\frac{1}{v}+w\right)^3+\left(\frac{1}{w}+u\right)^3} \geq
$$

$$
\geq \sqrt[3]{\left(2\sqrt{\frac{v}{u}}\right)^3+\left(2\sqrt{\frac{w}{v}}\right)^3+\left(2\sqrt{\frac{u}{w}}\right)^3} \geq 2\sqrt[3]{3}\sqrt[3]{\frac{v}{u}\cdot\frac{w}{v}\cdot\frac{u}{w}} = 2\sqrt[3]{3}.
$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Moti Levy Rehovot, Israel; David E. Manes, Oneonta, NY, and the proposer.

- **5540:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A \in M_2(\Re)$ be a matrix which has real eigenvalues. Prove that if $\sin A$ is similar to A then $\sin A = A$.

Solution 1 by Moti Levy, Rehovot, Israel

The matrix $A \in M_2(\mathbb{R})$, with real eigenvalues must be similar (according to the Jordan canonical form) to $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, or to $\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$.

If $A = P^{-1}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P$ then $A^n = P^{-1}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P$ and it follows that

$$\sin A = \sum_{n=1}^{\infty} a_n A^n = P^{-1}\begin{bmatrix} \sum_{n=1}^{\infty} a_n\lambda_1^n & 0 \\ 0 & \sum_{n=1}^{\infty} a_n\lambda_2^n \end{bmatrix} P = P^{-1}\begin{bmatrix} \sin\lambda_1 & 0 \\ 0 & \sin\lambda_2 \end{bmatrix} P. \quad (1)$$

If $A = P^{-1}\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} P$ then $A^n = P^{-1}\begin{bmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_2^n \end{bmatrix} P$ and it follows that

$$\sin A = \sum_{n=1}^{\infty} a_n A^n = P^{-1}\begin{bmatrix} \sum_{n=1}^{\infty} a_n\lambda_1^n & \sum_{n=1}^{\infty} n a_n\lambda_1^{n-1} \\ 0 & \sum_{n=1}^{\infty} a_n\lambda_2^n \end{bmatrix} P = P^{-1}\begin{bmatrix} \sin\lambda_1 & \cos\lambda_1 \\ 0 & \sin\lambda_1 \end{bmatrix} P. \quad (2)$$

Similar matrices have the same eigenvalues, hence from (1)

$$\sin\lambda_1 = \lambda_1,$$
$$\sin\lambda_2 = \lambda_2,$$

which implies $\lambda_1 = \lambda_2 = 0$. In this case $A = \sin A = P^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Similarly, it follows from (2) that

$$\sin\lambda_1 = \lambda_1,$$
$$\cos\lambda_1 = 1,$$

which implies $\lambda_1 = 0$. In this case $A = \sin A = P^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Solution 2 by Michel Bataille, Rouen, France

Let λ_1, λ_2 be the eigenvalues of A. First, we suppose that $\lambda_1 \neq \lambda_2$ and we show that $\sin A$ cannot be similar to A in that case. Since its eigenvalues are distinct, the matrix A is diagonalizable, that is, $A = PDP^{-1}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $P \in GL_2(\mathbb{R})$. Then

$$\sin A = P(\sin D)P^{-1} = P\begin{pmatrix} \sin(\lambda_1) & 0 \\ 0 & \sin(\lambda_2) \end{pmatrix} P^{-1}$$

so that the eigenvalues of $\sin A$ are $\sin(\lambda_1)$ and $\sin(\lambda_2)$. If $\sin A$ were similar to A, then we would have $\{\lambda_1, \lambda_2\} = \{\sin(\lambda_1), \sin(\lambda_2)\}$. However, $\sin(\lambda_1) = \lambda_1$, $\sin(\lambda_2) = \lambda_2$ implies $\lambda_1 = \lambda_2 (= 0)$ contradicting $\lambda_1 \neq \lambda_2$. Nor can the remaining possibility $\sin(\lambda_1) = \lambda_2$, $\sin(\lambda_2) = \lambda_1$ occur; indeed, in that case $\lambda_1, \lambda_2 \in [-1, 1]$ and $\sin(\sin(\lambda_1)) = \lambda_1$, $\sin(\sin(\lambda_2)) = \lambda_2$. But the function $\phi : x \mapsto x - \sin(\sin x)$ is strictly increasing, hence injective, on $[-1, 1]$ (its derivative $x \mapsto 1 - (\cos x)\cos(\sin x)$ is nonnegative and vanishes only at 0 since $0 < \cos x < 1$ for $x \in [-1, 1]$, $x \neq 0$). Thus, from $\phi(\lambda_1) = \phi(\lambda_2)$ we deduce $\lambda_1 = \lambda_2$, again a contradiction.

Suppose now that A has a unique eigenvalue λ.

If A is diagonalizable, then $A = \lambda I_2$ and so $\sin A = (\sin\lambda)I_2$. If $\sin A$ is similar to A, then

$\sin\lambda=\lambda$, hence $\lambda=0$ and we conclude that $\sin A = A(=O_2)$.
If A is not diagonalizable, the A is similar to its Jordan form $J=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$: $A=QJQ^{-1}$ for some matrix $Q\in GL_2(\mathbb{R})$. Since $J^n=\begin{pmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{pmatrix}$ for any positive integer n (easy induction), we obtain that

$$\sin A = Q\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,J^{2n+1}\right)Q^{-1}=Q\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\begin{pmatrix}\lambda^{2n+1} & (2n+1)\lambda^{2n}\\ 0 & \lambda^{2n+1}\end{pmatrix}\right]Q^{-1},$$

that is,

$$\sin A = Q\begin{pmatrix}\sin(\lambda) & \cos(\lambda)\\ 0 & \sin(\lambda)\end{pmatrix}Q^{-1}.$$

Now, if $\sin A$ is similar to A, then $\sin\lambda=\lambda$, hence $\lambda=0$ and therefore $A=Q\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}Q^{-1}=\sin A$.
We conclude that $\sin A=A$ whenever $\sin A$ is similar to A.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let λ_1 and λ_2 be the real eigenvalues of A, so that the eigenvalues of $\sin A$ are $\sin\lambda_1$ and $\sin\lambda_2$. Since $\sin A$ and A are similar, they have the same eigenvalues. Thus either a) $\sin\lambda_1=\lambda_1, \sin\lambda_2=\lambda_2$ or b) $\sin\lambda_1=\lambda_2, \sin\lambda_2=\lambda_1$.

For case a) let $f(x)=x-\sin x$, where x is any real number. We have $f'(x)=1-/cosx$, so that $f(x)$ is strictly increasing for $x\in(-1,0)\cup(0,1)$. Since $f(0)=0$ and $f(x)$ is nondecreasing in general, so $f(x)=0$ if and only if $x=0$. It follows that $\lambda_1=\lambda_2=0$.

For case b), we have $\sin(\sin\lambda_1)=\lambda_1$ and $\sin(\sin\lambda_2)=\lambda_2$. For real numbers x let $g(x)=x=\sin(\sin x)$ so that $g'(x)=1-\cos x\cos(\sin x)$. Similar to a), we see that $g(x)=0$ if and only if $x=0$. Again $\lambda_1=\lambda_2=0$.

It is known ([1] p.200, Theorem 4.11) that if A has equal eigenvalues λ, then $=(\cos\lambda)A+(\sin\lambda-\lambda\cos\lambda)I_2$, where I_2 is the identity matrix of order 2.

Since $\lambda=0$, so $\sin A=A$, as desired.

Reference 1. V. Pop, O. Furdui: Square Matrices of Order 2, Springer, 2017

Solution 4 by Albert Stadler, Herrliberg, Switzerland

Let a,b be the eigenvalues of A which are assumed to be real. Any matrix A (with real or complex entries) is similar to an upper triangular matrix whose diagonal entries are the eigenvalues of A, i.e. there is an invertible 2 by 2 matrix T such that

$$T^{-1}AT=\begin{pmatrix}a & *\\ 0 & b\end{pmatrix}.$$

We conclude that

$$T^{-1}\sin AT=T^{-1}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}A^k\right)T=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(T^{-1}AT\right)^k=\begin{pmatrix}\sin a & *\\ 0 & \sin b\end{pmatrix}.$$

By assumption A and $\sin A$ are similar. Similar matrices have the same eigenvalues. Therefore $\{a, b\} = \{\sin a, \sin b\}$. So either $a = \sin a$ and $b = \sin b$ or $a = \sin b$ and $b = \sin a$.

In the first case we have $a = b = 0$, since a and b are real. In the second case we have $a = \sin \sin a$ and $b = \sin \sin b$ which implies again $a = b = 0$. (Note that for $x \neq 0 |\sin x| < |x|$.)

Thus $A = T \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} T^{-1}$. which implies that A^k is the null-matrix for all $k > 1$ and therefore

$$\sin A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^k = A.$$

Also solved by the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2020

- **5559:** *Proposed by Kenneth Korbin, New York, NY*

  For every positive integer N there are two Pythagorean triangles with area $(N)(N+1)(2N+1)(2N-1)(4N+1)(4N^2+2N+1)$. Find the sides of the triangles if $N = 4$.

- **5560:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Square ABCD (in clockwise order) with all sides equal to x has point E as the midpoint of side AB. The right triangle EBC is folded along segment EC so that what was previously corner B is now at point B' which is at a distance d from side AD. Find d and also the distance of B' from AB.

- **5561:** *Proposed by Pedro Pantoja, Natal/RN, Brazil*

  Calculate the *exact value* of:

  $$\cos\frac{5\pi}{28} + \cos\frac{13\pi}{28} - \cos\frac{17\pi}{28}.$$

- **5562:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  Prove: If $a, b, c \geq 1$, then

  $$e^{ab} + e^{bc} + e^{ca} > 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}.$$

- **5563:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the aid of a computer, find the value of

  $$\sum_{n=1}^{+\infty} \frac{15}{25n^2 + 45n - 36}.$$

- **5564:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $a > 0$ and let $f : [0, a] \to \Re$ be a Riemann integrable function. Calculate

$$\lim_{n\to\infty} \int_0^a \frac{f(x)}{1 + nx^n} \mathrm{d}x.$$

Solutions

5541: *Proposed by Kenneth Korbin, New York, NY*

A convex cyclic quadrilateral has inradius r and circumradius R. The distance from the incenter to the circumcenter is 169. Find positive integers r and R.

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Fuss' theorem gives a relation between the inradius r, the circumradius R, and the distance d between the incenter I and the circumcenter O, for any bicentric quadrilateral. The relation is:

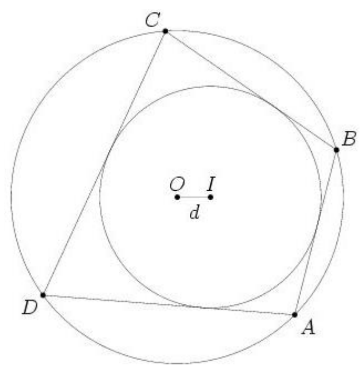
$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}, \qquad (1)$$

or equivalently:

$$2r^2(R^2 + d^2) = (R^2 - d^2)^2.$$

It was derived by Nicolaus Fuss (1755-1826) in 1792. Solving for d yields:

$$d = \sqrt{R^2 + h^2 - r\sqrt{4R^2 + h^2}}.$$



Since $d = 169 = 13^2$, then we may assume the relation (1) as a Diophantine equation:

$$\frac{1}{(R+13^2)^2} + \frac{1}{(R-13^2)^2} = \frac{1}{r^2},$$

with $r, R > 0$ or:

$$r^2 = \frac{(R^2 - 13^4)^2}{2(R^2 + 13^4)}$$

We may assume the Diophantine equation:

$$2(R^2 + 13^4) = y^2,$$

and:

$$R = -\frac{169}{2}\left[(\sqrt{2}+1)(3-2\sqrt{2})^n - (\sqrt{2}-1)(3+2\sqrt{2})^n\right],$$

$$R = \frac{169}{2}\left[(\sqrt{2}+2)(3-2\sqrt{2})^n - (\sqrt{2}-2)(3+2\sqrt{2})^n\right],$$

for $n \geq 1$ and $n \in N$. So, we have: $r = r(n)$ which must be an integer. By calculations, we have:

$$r = 28560 \text{ and } R = 40391.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

By Fuss' theorem (https://en.wikipedia.org/wiki/Bicentric-quadrilateral),

$$\frac{1}{(R-x)^2}+\frac{1}{(R+x)^2}=\frac{1}{r^2},$$

or equivalently

$$r=\frac{R^2-x^2}{\sqrt{2(R^2+x^2)}},$$

where r is the inradius, R the circumradius and x the distance between the incenter and the circumcenter of the bicentric quadrilateral.

By assumption, $x=169$ and r is an integer. Therefore $\sqrt{2(R^2+x^2)}$ is a (rational) integer. We note that

$$2r=\frac{2R^2+2x^2-4^2}{\sqrt{2(R^2+x^2)}}=\sqrt{2(R^2+x^2)}-\frac{4x^2}{\sqrt{2(R^2+x^2)}},$$

which implies that $\sqrt{2(R^2+x^2)}$ divides $2^2 13^4$. We conclude that $2(R^2+x^2)\in$ $\{4, 16, 676, 2704, 114244, 456976, 19307236, 77228944, 3262922884, 13051691536\}$.

The only feasible value for R is $R=40391$ which leads to $r=28560$.

Solution 3 by Ed Gray of Highland Beach, FL

Editor′s comment: I am taking the liberty of jumping into the middle of Ed's solution. Like those above, his solution started off using Fuss' Formula, and immediately substituted $d=169$ into it. After some algebra he obtained that

$$R^4-(2r^2+57122)R^2=815730721-57122r^2,$$

that he solved as a quadratic in R^2. Solving this he obtained that $R^2=r^2+28561\pm r\sqrt{r^2+114244}$. Letting $e^2=r^2+114244$, he continued on as follows:

Then $114244=2\cdot 2\cdot 13^4=e^2-r^2=(e-r)(e+r)$. The sum of the factors $e-r$ and $e+r$ is even and equals $2e$. Therefore the factors are both even or both odd. Since their product is even, they both must be even. There are only 2 possibilities:

(i) $e-r=2$ and $e+r=2\cdot 28561=57122$. Then $2e=57124, e=28562$, and $r=28560$. From the equation $R^2=r^2+28561\pm -r\sqrt{r^2+114244})$,

$$R^2=815673600+28561\pm(28560)(28562)=815702161\pm 815730720.$$

Clearly the negative sign is not viable.

So $R^2=815702161+815730720=1631432881$ and $R=40391$. The solution pair (r,R) is $(28560, 40391)$, and they satisfy Fuss' Theorem.

(ii) $e-r=2\cdot 13=26$ and $e+r=2\cdot(13^3)=4394$. Then $2e=4420, e=2210$, and $r=2184$. From the equation $R^2=r^2+28561\pm -r\sqrt{r^2+114244})$ we see that $R^2=4769856+28561\pm(2184)(2210)$. However, in this is the case, then R^2 will end in 7, and so R cannot be an integer.

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5542: *Proposed by Michel Bataille, Rouen, France*

Evaluate in closed form: $\cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13}$.
(Closed form means that the answer should not be expressed as a decimal equivalent.)

Solution 1 by David E. Manes, Oneonta, NY

We will show that $\cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13} = \sqrt{\dfrac{7+\sqrt{13}}{8}} = \dfrac{1+\sqrt{13}}{4}$. To do so, we assume the following identities:

$$\cos^2\left(\frac{\pi}{13}\right) + \cos^2\left(\frac{3\pi}{13}\right) + \cos^2\left(\frac{4\pi}{13}\right) = \frac{11+\sqrt{13}}{8},$$

$$\sum_{k=1}^{n}\cos\left(\frac{2k\pi}{2n+1}\right) = \cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cdots + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2},$$

where n is a positive integer. Let $C = \cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13}$. Then

$$\begin{aligned}
C^2 &= \cos^2\left(\frac{\pi}{13}\right) + \cos^2\left(\frac{3\pi}{13}\right) + \cos^2\left(\frac{4\pi}{13}\right) + 2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} - 2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13}\\
&= \frac{11+\sqrt{13}}{8} + 2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} - 2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13}.
\end{aligned}$$

By the product-to-sum formulas, one finds

$$\begin{aligned}
2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} &= \cos\frac{4\pi}{13} + \cos\frac{2\pi}{13}\\
-2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} &= -\cos\frac{5\pi}{13} - \cos\frac{3\pi}{13}\\
-2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13} &= -\cos\frac{7\pi}{13} - \cos\frac{\pi}{13}.
\end{aligned}$$

Using the addition formula for $\cos(\pi - x) = -\cos x$, we get

$$\begin{aligned}
-\cos\frac{5\pi}{13} &= \cos\left(\pi - \frac{5\pi}{13}\right) = \cos\frac{8\pi}{13}, -\cos\frac{3\pi}{13} = \cos\frac{10\pi}{13},\\
-\cos\frac{7\pi}{13} &= \cos\left(\pi - \frac{7\pi}{13}\right) = \cos\frac{6\pi}{13}, -\cos\frac{\pi}{13} = \cos\frac{12\pi}{13}.
\end{aligned}$$

Therefore, rearranging the terms, one obtains,

$$2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} - 2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13} = \sum_{k=1}^{6}\cos\left(\frac{2k\pi}{13}\right) = -\frac{1}{2}.$$

Therefore,

$$C^2 = \frac{11+\sqrt{13}}{8} - \frac{1}{2} = \frac{7+\sqrt{13}}{8}, \text{ whence } C = \sqrt{\frac{7+\sqrt{13}}{8}}.$$

Note that $\sqrt{\dfrac{7+\sqrt{13}}{8}} = \dfrac{1+\sqrt{13}}{4}$ since $\left(\dfrac{1+\sqrt{13}}{4}\right)^2 = \dfrac{7+\sqrt{13}}{8}$.

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Let $x = \cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13}$, $\theta = \dfrac{\pi}{13}$ and $c = \cos\theta$. If $c_k = \cos k\theta$, then $c_{2k} = 2c_k^2 - 1$, $2c_pc_q = c_{p+q} + c_{p-q}$, and $c_{13+k} = c_{13-k}$. Notice that $x > 0$. Therefore

$$\begin{aligned} x^2 &= c_1^2 + c_3^2 + c_4^2 + 2c_1c_3 - 2c_1c_4 - 2c_3c_4 \\ &= \frac{c_2+1}{2} + \frac{c_6+1}{2} + \frac{c_8+1}{2} + c_4 + c_2 - c_5 - c_3 - c_7 - c_1, \\ x^2 + x &= \frac{1}{2}(3 + 3c_2 - 2c_5 + c_6 - 2c_7 + c_8) \\ &= \frac{1}{2}(3 + 3c_2 + 3c_6 + 3c_8). \end{aligned}$$

Now, if $y = c_2 + c_6 + c_8$, then

$$\begin{aligned} y^2 &= \frac{c_4+1}{2} + \frac{c_{12}+1}{2} + \frac{c_{16}+1}{2} + c_8 + c_4 + c_{10} + c_6 + c_{14} + c_2 \\ 2y^2 &= 3c_4 + 3c_{10} + 3c_{12} + 2y + 3 \\ 2y^2 &= 3(c_4 + c_{10} + c_{12}) + 2y + 3. \end{aligned}$$

Now, since $c_2 + c_4 + c_6 + c_8 + c_{10} + c_{10} + c_{12} = -\frac{1}{2}$ because $c_2 + c_4 + c_6 + c_8 + c_{10} + c_{10} + c_{12} = \Re\left(\sum_{k=1}^{6} e^{(2k\pi i)/13}\right)$ and applying the sum of a geometric series, we get $-\dfrac{1}{2}$. Then, $c_4 + c_{10} + c_{12} = -\dfrac{1}{2} - (c_2 + c_6 + c_8) = -\dfrac{1}{2} - y$ and so, $2y^2 = 3(-\frac{1}{2} - y) + 2y + 3$ from where, since $y > 0$, $y = \dfrac{-1+\sqrt{13}}{4}$.
Finally, by solving $x^2 + x = \dfrac{1}{2}(3 + 3y)$ and, since $x > 0$ it is obtained $x = \dfrac{1+\sqrt{13}}{4}$.

Solution 3 by Andrea Fanchini, Cantú, Italy

Let p be an odd prime number. Then we know that

$$g_p = \sum_{k=0}^{p-1} \exp(2\pi k^2/p)$$

is a *quadratic Gaussian sum*, where $g_p = \sqrt{p}$ or $i\sqrt{p}$ according to whether $p \equiv 1$ or $p \equiv 3 \pmod 4$. So $g_{13} = \sqrt{13}$. Therefore,

$$\begin{aligned} \sqrt{13} = 1 + e^{2\pi i/13} &+ e^{8\pi i/13} + e^{-8\pi i/13} + e^{6\pi i/13} + e^{-2\pi i/13} + e^{-6\pi i/13} \\ &+ e^{-6\pi i/13} + e^{-2\pi i/13} + e^{6\pi i/13} + e^{-8\pi i/13} + e^{8\pi i/13} + e^{2\pi i/13}. \end{aligned}$$

Recalling that $e^{ix} + e^{-ix} = 2\cos x$ we then have:

$$\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} = \frac{\sqrt{13}-1}{4}, \qquad (1)$$

Now we consider the sum of cosines with arguments in arithmetic progression.

$$\sum_{k=0}^{n-1} \cos(a+kd) = \frac{\sin(nd/2)}{\sin(d/2)} \cos\left(a + \frac{(n-1)d}{2}\right)$$

where $a, d \in R, d \neq 0$, and that n is a positive integer.

In our case, we set $a = d = \frac{2\pi}{13}$ and $n = 6$, then

$$\begin{aligned}
\cos\frac{2\pi}{13} + \cos\frac{4\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13} &= \frac{\sin\frac{6\pi}{13}\cos\frac{7\pi}{13}}{\sin\frac{\pi}{13}} \\
&= \frac{-2\sin\frac{6\pi}{13}\cos\frac{6\pi}{13}}{2\sin\frac{\pi}{13}} \\
&= -\frac{\sin\frac{12\pi}{13}}{2\sin\frac{\pi}{13}} \\
&= -\frac{\sin\frac{\pi}{13}}{2\sin\frac{\pi}{13}} \\
&= -\frac{1}{2}.
\end{aligned}$$

Substituting the sum in (1) into this last expression we obtain;

$$\cos\frac{4\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13} = -\frac{1}{2} - \frac{\sqrt{13}-1}{4} = -\frac{\sqrt{13}+1}{4}.$$

So finally we have:

$$-\cos\frac{12\pi}{13} - \cos\frac{10\pi}{13} - \cos\frac{4\pi}{13} = \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} = \frac{\sqrt{13}+1}{4}.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that

$$\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{3\pi}{13} = \frac{1+\sqrt{13}}{4}. \qquad (1)$$

Denote the left side of (1) by x, which is clearly positive. So (1) will follow from

$$4x^2 - 2x - 3 = 0. \qquad (2)$$

Let $\theta = \dfrac{\pi}{13}$ and $i = \sqrt{-1}$. Since $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\cos^2 3\theta = \dfrac{1+\cos 6\theta}{2}$,

$\cos^2 4\theta = \dfrac{1-\cos 5\theta}{2}$, $2\cos\theta\cos 3\theta = \cos 2\theta + \cos 4\theta$, $2\cos\theta\cos 4\theta = \cos 3\theta + \cos 5\theta$, and

$2\cos 3\theta\cos 4\theta = \cos\theta - \cos 6\theta$, so

$$4x^2 - 2x - 3 = 3 + 6\sum_{k=1}^{6}(-1)^k\cos(k\theta) = 3\sum_{k=0}^{12}(-1)^k\cos(k\theta)$$

$$= 3Re\sum_{k=0}^{12}(-1)^k e^{ik\theta} = 3Re\left(\frac{1+e^{13i\theta}}{1+e^{i\theta}}\right) = 0.$$

This proves (2) and completes the solution.

Solution 5 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the given expression equals $\dfrac{1+\sqrt{13}}{4}$.

Let $a = \cos(\pi/13)$, $b = \cos(3\pi/13)$, and $c = \cos(4\pi/13)$. Using the multiple-angle formulas for cosine, we have $b = 4a^3 - 3a$ and $c = 8a^4 - 8a^2 + 1$. Then $a + b - c = (1 + \sqrt{13})/4$ if and only if

$$[4(-8a^4 + 4a^3 + 8a^2 - 2a - 1) - 1]^2 = 13.$$

This in turn holds if and only if $f(a)(16a^2 - 8a - 12) = 0$, where

$$f(x) = 64x^6 - 32x^5 - 80x^4 + 32x^3 + 24x^2 - 6x - 1.$$

Using another multiple-angle formula for cosine, namely $\cos(13\theta) = 4096r^{13} - 13312r^{11} + 16640r^9 - 9984r^7 + 2912r^5 - 364r^3 + 13r = (r+1)[f(r)]^2 - 1$ with $r = \cos\theta$, we have

$$(a+1)[f(a)]^2 = 0.$$

Since $a \neq -1$, we conclude $f(a) = 0$, completing the proof.

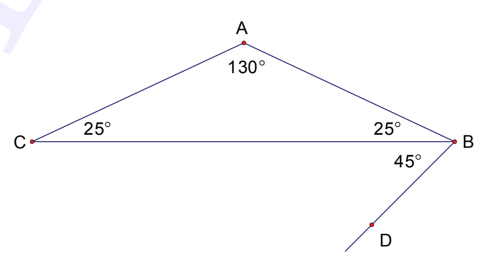
Also solved by Brian Bradie, Christopher Newport University, Newport News,VA; Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5543: *Proposed by Titu Zvonaru, Comănesti, Romania*

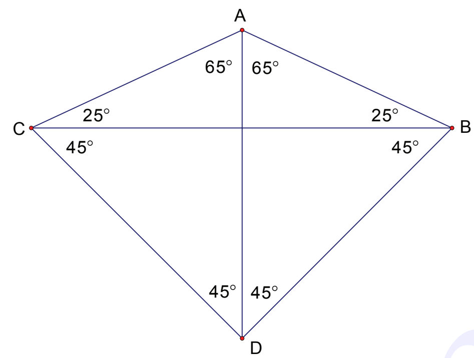
Let $ABDC$ be a convex quadrilateral such that $\angle ABC = \angle BCA = 25°, \angle CBD = \angle ADC = 45°$. Compute the value of $\angle DAC$. (Note the order of the vertices.)

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

From the given facts $\angle ABC = \angle BCA = 25°$ and $\angle CBD = 45°$, we know that $\angle CAB = 130°$ and that D lies on the ray emanating from point B at a $45°$ angle from BC, as shown in the figure below.

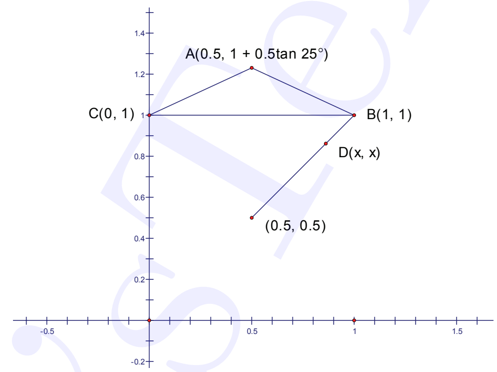


From the additional given fact $\angle ADC = 45°$, by inspection one solution is the kite shown in the figure below, in which $\angle DAC = 65°$.



It is clear that if D is farther from B (on the ray emanating from point B at a $45°$ angle from BC), then $\angle ADC < 45°$, and that as D moves closer to B that $\angle ADC > 45°$ and increases, reaches a maximum, and then decreases to approach $\angle ABC = 25°$ as D approaches B. This implies that there is one more location for point D such that $\angle ADC = 45°$.

To find it, let us place the points on a coordinate grid. See the figure below.



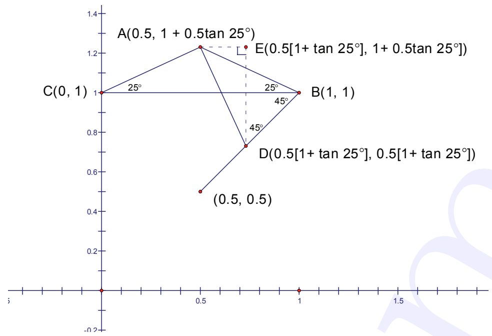
Consider $\angle ADC = 45°$ to be an inscribed angle of the circle passing through the points A, C, and the first location for D, namely $(\frac{1}{2}, \frac{1}{2})$. The other point where this circle intersects the line $y = x$ is the second location for D.

If the circle has center (h, k) and radius r, we have $(0-h)^2 + (1-k)^2 = (\frac{1}{2} - h)^2 + (1 + \frac{1}{2}\tan 25° - k)^2 = (\frac{1}{2} - h)^2 + (\frac{1}{2} - k)^2$. The second equation gives $1 + \frac{1}{2}\tan 25° - k = k - \frac{1}{2}$, whence $k = \frac{1}{4}(3 + \tan 25°)$. Using this value for k in the first equation gives $(0-h)^2 + (1 - \frac{1}{4}[3 + \tan 25°])^2 = (\frac{1}{2} - h)^2 + (1 + \frac{1}{2}\tan 25° - \frac{1}{4}[3 + \tan 25°])^2$. Expanding terms and solving for h yields $h = \frac{1}{4}(1 + \tan 25°)$.

To find r^2, we substitute the point $C(0, 1)$ into the equation of the circle $(x - \frac{1}{4}[1 + \tan 25°])^2 + (y - \frac{1}{4}[3 + \tan 25°])^2 = r^2$. Doing this and expanding terms yields $r^2 = \frac{1}{8}(1 + \tan^2 25°)$. So the equation of the circle is $(x - \frac{1}{4}[1 + \tan 25°])^2 + (y - \frac{1}{4}[3 + \tan 25°])^2 = \frac{1}{8}(1 + \tan^2 25°)$.

The circle intersects the line $y = x$ when $(x - \frac{1}{4}[1 + \tan 25°])^2 + (x - \frac{1}{4}[3 + \tan 25°])^2 = \frac{1}{8}(1 + \tan^2 25°)$. Expanding this yields the quadratic equation $2x^2 - (2 + \tan 25°)x + \frac{1}{2}(1 + \tan°) = 0$. The quadratic formula yields the two solutions $x = \frac{1}{2}$, which we already know, and $x = \frac{1}{2}(1 + \tan 25°)$.

Referring to the figure below, we see that
$\tan \angle ADE = \frac{AE}{DE} = \frac{\frac{1}{2}(1+\tan 25^\circ)-\frac{1}{2}}{1+\frac{1}{2}\tan 25^\circ-\frac{1}{2}(1+\tan 25^\circ)} = \tan 25^\circ$ so that $\angle ADE = 25^\circ$. So $\angle ADB = 25^\circ + 45^\circ = 70^\circ$, whence $\angle DAB = 180^\circ - 70^\circ - 70^\circ = 40^\circ$, so that $\angle DAC = 130^\circ - 40^\circ = 90^\circ$.



So the two possible values of $\angle DAC$ are 65° and 90°.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

The triangle ABC is isosceles and BCD is right isosceles, making AD the angle bisector of $\angle BAC$. So $\angle DCA = 65^\circ$. Furthermore, there are two possible answers since there are two positions for D.

Let F lie on BD, such that CF and BD are perpendicular. Let the circumcircle of ACF intersect BD at E. Now, $\angle EAC = 90^\circ$, or $\angle BAE = 40^\circ$, or $\angle AEB = \angle ABE$. So, $AB = AE$. But $AB = AC$. So, $AE = AC$ and ∠AEC=∠AFC=45°. So, E and F both satisfy the conditions imposed on D and in each case, we have $\angle DAC = 90^\circ, 65^\circ$, respectively (as D=E,F).

Solution 3 by Ed Gray, Highland Beach, FL

From the information given, triangle ABC is isosceles, with $AB = AC$. To enhance the lucidity of the calculations, we assign the value of 2.0 to each of these sides. We define $\angle DAC = x, \angle BAD = a, \angle BCD = c$, and $\angle BDA = b$.

(**1**) In triangle ABC, by the Law of Sines, $\dfrac{BC}{\sin(130^\circ)} = \dfrac{2}{\sin(c)}$; $BC = 3.625231148$

(**2**) In triangle CBD, $c + 45^\circ + b + 45^\circ = 180^\circ$, so $b + c = 90^\circ$.

(**3**) In triangle CBD, by the Law of Sines, $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{BD}{\sin(c)}$, or

(**4**) $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{BD}{\cos(b)}$.

(**5**) In triangle ABD, $\dfrac{BD}{\sin(a)} = \dfrac{2}{\sin(b)}$, $BD = \dfrac{2\sin(a)}{\sin(b)}$

(**6**) In triangle $ABD, a + 70^\circ + b = 180^\circ, a + b = 110^\circ, a = 110^\circ - b$.

From Equation **(4),**

(**7**) $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{2\cdot\sin(a)}{\sin(b)\cdot\cos(b)}$,

(**8**) $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{2\cdot\sin(110^\circ - b)}{\sin(b)\cdot\cos(b)}$

Substituting BC from Equation (1), we have a trigonometric equation for b.

(**9**) $1.812615574 \cdot \sin(b) \cdot \cos(b) =$
$[\sin(110°) \cdot \cos(b) - \cos(110°) \cdot \sin(b)] \cdot [\sin(b) \cdot \cos(45°) + \cos(b) \cdot \sin(45°)]$.

Since $\cos(45°) = \sin(45°) = \dfrac{\sqrt{2}}{2}$, we divide both sides by 0.707106781

(**10**) $2.563425529 \cdot \sin(b) \cdot \cos(b) =$
$\sin(110) \cdot \sin(b) \cdot \cos(b) + \sin(110°) \cdot \cos^2(b) - \cos(110°) \cdot \sin^2(b) - \cos(110°) \cdot \sin(b) \cdot \cos(b)$

(**11**) $2.563425529 \cdot \sin(b) \cdot \cos(b) = 0.939692621 \cdot \sin(b) \cdot \cos(b) + 0.939692621 \cdot \cos^2(b) +$
$0.342020143 \cdot \sin^2(b) + 0.342020143 \cdot \sin(b) \cdot \cos(b)$

(**12**) $1.281712765 \cdot \sin(b) \cdot \cos(b) = 0.342020143 \cdot \sin^2(b) + 0.939692621 \cdot \cos^2(b)$.

Squaring,

(**13**) $1.6427876 \cdot \sin^2(b) \cdot \cos^2(b) =$
$0.116977778 \cdot \sin^4(b) + 0.642787609 \cdot \sin^2(b) \cdot \cos^2(b) + 0.8883022222 \cdot \cos^4(b)$,

(**14**) $\cos^2(b) = 1 - \sin^2(b)$

(**15**) $\cos^4(b) = 1 - 2 \cdot \sin^2(b) + \sin^4(b)$

(**16**) $0.116977778 \cdot \sin^4(\mathrm{b}) - \sin^2(\mathrm{b}) \cdot \cos^2(\mathrm{b}) + 0.883022222(1 - 2 \cdot \sin^2(\mathrm{b}) + \sin^4(\mathrm{b})) = 0$

(**17**) $0.116977778 \cdot \sin^4(b) - \sin^2(b)(1 - \sin^2(b)) + 0.883022222 - 1.766044444 \cdot \sin^2(b) +$
$0.883022222 \cdot \sin^4(b) = 0$

(**18**) $2 \cdot \sin^4(b) - 2.766044444 \cdot \sin^2(b) + 0.883022222 = 0$

This is a quadratic equation in $\sin^2(b)$, with solutions:

(**19**) $4 \cdot \sin^2(b) = 2.766044444 \pm \sqrt{7.651001866 - 7.064177776)}$, or

(**20**) $4 \cdot \sin^2(b) = 2.766044444 \pm 7.66044444$

(**21**) So $\sin^2(b_1) = \dfrac{2}{4}$, $\sin(b_1) = 0.707106781, b_1 = 45°$.

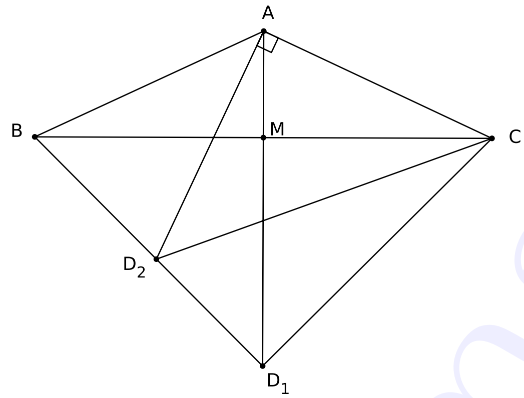
(**22**) $\sin^2(b_2) = \dfrac{3.532088888}{4} = 0.883022222$, $\sin(b_2) = 0.939692621, b_2 = 70°$.

When $b = 45°, a = 65°, x = 65°$.

When $b = 70°, a = 40°, x = 90°$.

Editor's comment: The following remark followed this solution: "I must admit having 2 answers is a surprise,..., however both solutions satisfy Equation **(12),** which is a good sign, because that is the fundamental equation and no extraneous root was introduced by squaring."

Solution 4 by Michel Bataille, Rouen, France



We consider ΔABC, which we suppose positively oriented, and let M be the midpoint of BC (see figure). Since $\angle ABC = \angle BCA$, AM is the perpendicular bisector of BC.

First, let D_1 be the image of B under the rotation with centre M and angle $+90°$. Then, ABD_1C is a convex quadrilateral and $\angle CBD_1 = \angle AD_1C = 45°$.

Second, let D_2 on BD_1 be such that $\angle CAD_2 = 90°$. Since $\angle BAC = 130°$, we have $\angle BAD_2 = 40°$. Also, $\angle ABD_2 = 25° + 45° = 70°$ and so $\angle AD_2B = 180° - 40° - 70° = 70° = \angle ABD_2$. It follows that $AD_2 = AB = AC$ and the triangle CAD_2 is right-angled at A and isosceles. As a result the quadrilateral ABD_2C is convex with $\angle AD_2C = 45° = \angle CBD_2$.

Thus, we have found two candidates D_1, D_2 for the vertex D. There cannot be more: indeed, because of the convexity of $ABDC$, D must be on the ray BD_1 (to ensure that $\angle CBD = 45°$) and on the arc of circle, locus of the points P such that $\angle(\overrightarrow{PC}, \overrightarrow{PA}) = +45°$ (to ensure that $\angle ADC = 45°$). We conclude that the answer to the problem is twofold: if $D = D_1$, then $\angle DAC = \frac{1}{2}\angle BAC = 65°$; if $D = D_2$, then $\angle DAC = 90°$.

Also solved by Andrea Fanchini, Cantú, Italy; Kee-Wai Lau, Hong Kong, China; Raquel Rosado, Hallie Kaiser, Mitch DeJong, and Caleb Edington, students at Taylor University, Upland, IN; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5544: *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve in $\Re$:

$$\begin{cases} \tan^{-1} x = & \tan y + \tan z \\ \tan^{-1} y = & \tan x + \tan z \\ \tan^{-1} z = & \tan x + \tan y \end{cases}$$

Solution by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Adding the equations we have:

$$\sum_{cyc} (2\tan x - \tan^{-1} x) = 0.$$

Let $f(x) = \left|2\tan x - \tan^{-1} x\right|$, for ever $\left\{x \in \Re : k\pi - \dfrac{\pi}{2} < x < k\pi + \dfrac{\pi}{2} \text{ and } k \in Z\right\}$.

Then $f'(x) = \dfrac{2}{\cos^2 x} - \dfrac{1}{x^2+1} > 0$ for every $x \in \Re$. So, $f(x)$ is an increasing monotonic function and $f(x) \geq f(0) = 0$, since equality holds if $x = 0$.

Similarly, $f(y) \geq f(0) = 0$ and $f(z) \geq f(0) = 0$, since equality holds if $y = z = 0$.
Then, the only real solution is $x = y = z = 0$.

Also solved by Ed Gray, Highland Beach, FL and the proposer.

5545: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let p, q be two twin primes. Show that

$$1 + 4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right)$$

is a perfect square and determine it. (Here $\lfloor x \rfloor$ represents the integer part of x).

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The integers p and q are odd (since they are twin primes) and so their difference is two. Let $x = (p+q)/2$. Then $\min(p,q) = x-1,\ \max(p,q) = x+1$.

We consider the rectangle R with vertices $A(0,0), B(p/2,0), C(p/2,q/2), D(0,q/2)$ in the Euclidean plane. The number of lattice points that are strictly inside R equals

$$L = \frac{p-1}{2} \cdot \frac{q-1}{2}.$$

There are no lattice points on the diagonal AC, since p and q are relatively prime.
Clearly L equals the number of lattice points strictly inside the triangle ABC plus the number of lattice points strictly inside the triangle CDA. Therefore

$$L = \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor.$$

We conclude that

$$\begin{aligned} 1 + 4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor\right) &= 1 + 4L = 1 + (p-1)(q-1) = 1 + (x-2)x = (x-1)^2 = \\ &= (\min(p,q))^2. \end{aligned}$$

Solution 2 by Charles Diminnie and Simon Pfeil, Angelo State University, San Angelo, TX

We will assume only that p is odd, $p \geq 3$, and $q = p+2$. It is unnecessary to restrict p and/or q to be prime. To begin, if $j = 1, 2, \ldots, \dfrac{p-1}{2}$, then

$$
\begin{aligned}
j &< \frac{jq}{p} \\
&= \frac{j\left(p+2\right)}{p} \\
&= j + \frac{2j}{p} \\
&\leq j + \left(\frac{2}{p}\right)\left(\frac{p-1}{2}\right) \\
&= j + \frac{p-1}{p} \\
&< j+1.
\end{aligned}
$$

Hence, $\left\lfloor \frac{jq}{p} \right\rfloor = j$ for $j = 1, 2, \ldots, \frac{p-1}{2}$.

Further, for $k = 1, 2, \ldots, \frac{q-1}{2}$,

$$
\begin{aligned}
k &> \frac{kp}{q} \\
&= \frac{k\left(q-2\right)}{q} \\
&= k - \frac{2k}{q} \\
&\geq k - \left(\frac{2}{q}\right)\left(\frac{q-1}{2}\right) \\
&= k - \frac{q-1}{q} \\
&> k-1.
\end{aligned}
$$

Therefore, $\left\lfloor \frac{kp}{q} \right\rfloor = k-1$ for $k = 1, 2, \ldots, \frac{q-1}{2} = \frac{p+1}{2}$.

Using the known result that

$$
\sum_{i=1}^{n} i = \frac{n\left(n+1\right)}{2}
$$

for $n \geq 1$, we obtain

$$
\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor &= \sum_{j=1}^{\frac{p-1}{2}} j \\
&= \left(\frac{1}{2}\right)\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right) \\
&= \frac{p^2-1}{8}
\end{aligned}
$$

and

$$\begin{aligned}
\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor &= \sum_{k=1}^{\frac{p+1}{2}} (k-1) \\
&= \sum_{k=2}^{\frac{p+1}{2}} (k-1) \\
&= \sum_{i=1}^{\frac{p-1}{2}} i \\
&= \frac{p^2-1}{8},
\end{aligned}$$

(substituting $i = k-1$ in the last sum.)

As a result,

$$\begin{aligned}
&1 + 4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) \\
&= 1 + 4\left(\frac{p^2-1}{8} + \frac{p^2-1}{8}\right) \\
&= 1 + \left(p^2-1\right) \\
&= p^2.
\end{aligned}$$

Solution 3 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

For any relatively prime odd integers $p, q \geq 3$ we have

$$\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) = \frac{p-1}{2} \cdot \frac{q-1}{2}$$

by, for example, Theorem 86 of Nagell's *Number Theory*. (The proof is standard and elementary: Consider the set of integer points (j, k) with $1 \leq j \leq (p-1)/2$ and $1 \leq k \leq (q-1)/2$. There are $\frac{p-1}{2}\frac{q-1}{2}$ such points. None of these are on the line $py = qx$. The number of points below the line is $\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor$ while the number of points above is $\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor$.)

Now suppose p and q are twin primes with $p < q$. Then p and q are relatively prime odd integers ≥ 3 with $q = p + 2$. So

$$\begin{aligned}
1 + 4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) &= 1 + 4 \cdot \frac{p-1}{2} \cdot \frac{q-1}{2} \\
&= 1 + 4 \cdot \frac{p-1}{2} \cdot \frac{p+1}{2} \\
&= p^2.
\end{aligned}$$

(Note that we only need p and q to be consecutive odd integers ≥ 3 in this argument.)

Solution 4 by Brian Bradie, Christopher Newport University, Newport News,VA

Without loss of generality, suppose p is the smaller of the two primes. Then $p \geq 3$, and $p+2=q$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor &= \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor j\left(1+\frac{2}{p}\right) \right\rfloor = \sum_{j=1}^{\frac{p-1}{2}} j \\
&= \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8}, \\
\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor &= \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor k\left(1-\frac{2}{q}\right) \right\rfloor = \sum_{k=1}^{\frac{q-1}{2}} (k-1) \\
&= \frac{\frac{q-3}{2}\cdot\frac{q-1}{2}}{2} = \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8},
\end{aligned}$$

and

$$\begin{aligned}
1+4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) &= 1+4\left(\frac{p^2-1}{8}+\frac{p^2-1}{8}\right) \\
&= p^2.
\end{aligned}$$

Solution 5 by Moti Levy, Rehovot, Israel

Without loss of generality, suppose p is the smaller of the two primes. Then $p \geq 3$, and $p+2=q$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor &= \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor j\left(1+\frac{2}{p}\right) \right\rfloor = \sum_{j=1}^{\frac{p-1}{2}} j \\
&= \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8}, \\
\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor &= \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor k\left(1-\frac{2}{q}\right) \right\rfloor = \sum_{k=1}^{\frac{q-1}{2}} (k-1) \\
&= \frac{\frac{q-3}{2}\cdot\frac{q-1}{2}}{2} = \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8},
\end{aligned}$$

and

$$\begin{aligned}
1+4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) &= 1+4\left(\frac{p^2-1}{8}+\frac{p^2-1}{8}\right) \\
&= p^2.
\end{aligned}$$

Solution 6 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We show the slightly more general formula

$$1+4\left(\sum_{j=1}^{(n-1)/2}\left\lfloor j\frac{n+2}{n}\right\rfloor+\sum_{k=1}^{(n+1)/2}\left\lfloor k\frac{n}{n+2}\right\rfloor\right)=n^2 \qquad (n=3,5,7,9,\ldots).$$

Proof: Let $n\geq 3$ be an odd integer. Since $j<j\frac{n+2}{n}=j+\frac{2j}{n}<j+1$, for $1\leq j\leq (n-1)/2$, and $k-1<k-\frac{2k}{n+2}=k\frac{n}{n+2}<k$, for $1\leq k\leq (n+1)/2$, we conclude that

$$\begin{aligned} & 1+4\left(\sum_{j=1}^{(n-1)/2}\left\lfloor j\frac{n+2}{n}\right\rfloor+\sum_{k=1}^{(n+1)/2}\left\lfloor k\frac{n}{n+2}\right\rfloor\right)\\ =\; & 1+4\left(\sum_{j=1}^{(n-1)/2} j+\sum_{k=1}^{(n+1)/2}(k-1)\right)\\ =\; & 1+4\frac{n-1}{2}\frac{n+1}{2}=n^2. \end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5546: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}(-1)^{\left\lfloor \frac{n}{2}\right\rfloor}\left(\mathrm{e}^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\right).$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $e^x-1-\dfrac{x}{1!}-\dfrac{x^2}{2!}-\ldots-\dfrac{x^n}{n!}=\displaystyle\sum_{k=n+1}^{\infty}\frac{x^k}{k!}$, the proposed series, say S, is absolutely convergent, and

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}(-1)^{\lfloor \frac{n}{2}\rfloor}\sum_{k=n+1}^{\infty}\frac{x^k}{k!}\\
&= \sum_{k=2}^{\infty}\frac{x^k}{k!}\sum_{n=1}^{k-1}(-1)^{\lfloor \frac{n}{2}\rfloor}\\
&= \sum_{k=1}^{\infty}\frac{x^k}{k!}\cos\left(\frac{(k-2)\pi}{2}\right)\\
&= \sum_{k=1}^{\infty}(-1)^{n+1}\frac{x^{2k}}{(2k)!}\\
&= 1-\cos x.
\end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\sum_{n=1}^{\infty}(-1)^{\lfloor \frac{n}{2}\rfloor}\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\ldots-\frac{x^n}{n!}\right)=\sum_{n=1}^{\infty}((-1)^{\lfloor \frac{n}{2}\rfloor}\left(\sum_{k=1}^{\infty}\frac{x^k}{k!}-1-\frac{x}{1!}-\frac{x^2}{2!}-\ldots-\frac{x^n}{n!}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^{\lfloor \frac{n}{2}\rfloor}\left(\sum_{k=n+1}^{\infty}\frac{x^k}{k!}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=n+1}^{\infty}(-1)^{\lfloor \frac{n}{2}\rfloor}\frac{x^k}{k!}\right)=\sum_{k=2}^{\infty}\left(\sum_{n=1}^{k-1}(-1)^{\lfloor \frac{n}{2}\rfloor}\frac{x^k}{k!}\right)$$

$$=1\frac{x^2}{2!}+0\frac{x^3}{3!}-1\frac{x^4}{4!}+0\frac{x^5}{5!}+1\frac{x^6}{6!}+\ldots=-\sum_{i=1}^{\infty}(-1)^i\frac{x^{2i}}{i!}=1-\sum_{i=0}^{\infty}(-1)^i\frac{x^{2i}}{i!}=1-\cos x.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the given sum equals $1-\cos x$.

Let $f(x)=\sin x+\cos x$, so that

$$f^{(n)}(x)=\begin{cases}\sin x+\cos x & n\equiv 0 \pmod 4\\ \cos x-\sin x & n\equiv 1 \pmod 4\\ -\sin x-\cos x & n\equiv 2 \pmod 4\\ -\cos x+\sin x & n\equiv 3 \pmod 4\end{cases}$$

It follows that the given sum $\displaystyle\sum_{n=1}^{\infty}f^{(n)}(0)\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\right)$.

According to entry 3.89 (a) on pp. 154, 227 of [1], we have

$$\sum_{n=1}^{\infty}f^{(n)}(0)\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\right)=\int_0^x e^{x-t}f(t)dt$$

which equals $1-\cos x$, by standard integration. Our claimed result now follows easily.

Reference:

1. O. Furdui: *Limits, Series, and Fractional Part Integrals, Springer, 2013.*

Solution 4 by Michel Bataille, Rouen, France

For every nonnegative integer n and any real number x, let
$R_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} = \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$ and let $f(x) = \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_n(x)$ be the required sum. We show that $f(x) = 1 - \cos x$.

Let $A > 0$ and $x \in [-A, A]$. Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}e^c}{(n+1)!}$ for some c beteween 0 and x (Taylor-Lagrange relation), we see that

$$|R_n(x)| \leq \frac{A^{n+1}}{(n+1)!} \cdot e^A.$$

It follows that the series $\sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_n(x)$ is uniformly convergent on any interval $[-A, A]$ $(A > 0)$. Since the derivative $R_n'(x)$ is equal to R_{n-1} $(n \in N)$, the same is true of the series $\sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_n'(x) = \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-1}(x)$. As a result, we have

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-1}(x) = e^x - 1 + \sum_{n=2}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-1}(x)$$

for any $x \in R$.

Likewise, f' is differentiable on R and for any real number x,

$$\begin{aligned}
f''(x) &= e^x + \sum_{n=2}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-2}(x) \\
&= e^x - R_0(x) + \sum_{n=3}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-2}(x) \\
&= 1 + \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n+2}{2} \rfloor} R_n(x) \\
&= 1 + \sum_{n=1}^{\infty} (-1)^{1+\lfloor \frac{n}{2} \rfloor} R_n(x) = 1 - f(x).
\end{aligned}$$

Thus, f is the solution to the differential equation $y'' + y = 1$ satisfying $f(0) = 0 = f'(0)$. Solving is classical and we readily obtain $f(x) = 1 - \cos x$.

Solution 5 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The sum of the series is $1 - \cos(x)$.

Recall the Maclaurin series for $\cos(x)$: $\cos(x) = \sum_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!}$.

As expected, we'll also use the Maclaurin series representation for the exponential function:
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = E_k + R_k$, for any $k \geq 1$,

where $E_k = \sum_{n=0}^{k} \frac{x^n}{n!}$ is the k^{th} partial sum and $R_k = \sum_{n=k+1}^{\infty} \frac{x^n}{n!}$ is the remainder.

Because the series converges, we know that the sequence $\{R_i\}_{k \geq 1}$ has limit 0.

Note also that $e^x - E_k = R_k$, and $E_{k+1} - E_k = \dfrac{x^{k+1}}{(k+1)!}$.

Consider the partial sums of our given series:
let $S_m = \sum_{n=1}^{m} (-1)^{\lfloor \frac{n}{2} \rfloor} \left(e^x - 1 = \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^n}{n!} \right) = \sum_{n=1}^{m} (-1)^{\lfloor \frac{n}{2} \rfloor} (e^x - E_n)$.

We compute the first few partial sums.

To simplify the calculations, we first handle the sign term:
its pattern is $1, -1, -1, 1, 1, -1, -1, 1, 1, -1, \ldots$
This "block of four" pattern suggests that it will be productive to consider pairing consecutive terms (although we cannot be content with just carrying out a regrouping of a series without a guarantee of convergence).

$S_1 = e^x - E_1 = e^x - 1 - \dfrac{x}{1!}$

$S_2 = (e^x - E_1) - (e^x - E_2) = E_2 - E_1 = \dfrac{x^2}{2!}$

$S_3 = (e^x - E_1) - (e^x - E_2) - (e^x - E_3) = E_2 - E_1 = S_2 - R_2$

$S_4 = (e^x - E_1) - (e^x - E_2) - (e^x - E_3) + (e^x - E_4) = S_2 - (E_4 - E_3) = \dfrac{x^2}{2!} - \dfrac{x^4}{4!}$

$S_5 = S_4 + (e^x - E_5) = S_4 + R_5$

$S_6 = S_4 + (e^x - E_5) - (e^x - E_6) = S_4 + (E_6 - E_5) = \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!}$

$S_7 = S_6 + (e^x - E_7) = S_6 + R_7$

$S_8 = S_6 + (e^x - E_7) - (e^x - E_8) = S_6 + (E_8 - E_7) = \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \dfrac{x^8}{8!}$.

Inductively, we can show that, for even subscripts

$S_{4k} = \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \ldots \dfrac{x^{4k}}{(4k)!}$

$S_{4k+2} = \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \ldots + \dfrac{x^{4k-2}}{(4k-2)!}$

and for odd subscripts

$S_{4k+1} = S_{4k} + R_{4k+1}$

$S_{4k+3} = S_{4k+2} - R_{4k+3}$.

We see that the subsequence $\{S_{2k}\}_{k \geq 1}$ has as its limit the Maclaurin series for $1 - \cos(x)$.
If we had a priori knowledge that our given series is convergent, this would guarantee that our series has sum $1 - \cos(x)$.

However, looking at the odd-subscript partial sums will give us enough information to draw that conclusion. The subsequence $\{S_{2k+1}\}_{k \geq 1}$ has as the same limit as $\{S_{2k}\}_{k \geq 1}$ because the sequence $R_n \longrightarrow 0$.

Therefore, the limit of the sequence of partial sums, i.e. the sum of the given series, is $1 - \cos(x)$.

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Brian Bradie, Christopher Newport University, Newport News, VA; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.