温馨提示

本文的是近些年的美国数学月刊合集，我用pdfpages包把散装的pdf合并起来的，仅供个人使用。切勿传播到考研群，也不得用于商业途径。

10701. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let G be a (finite, undirected, simple) graph with vertex set V. Let $C = \{C_x : x \in V\}$ be a family of sets indexed by the vertices of G. For $X \subseteq V$, let $C_X = \cup_{x \in X} C_x$. A set $X \subseteq V$ is *C-colorable* if one can assign to each vertex $x \in X$ a "color" $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are adjacent in G.
(**a**) Prove that if $|C_X| \geq |X|$ whenever X induces a connected subgraph of G, then V is C-colorable.
(**b**) Prove that if every proper subset of V is C-colorable and if $|C_V| \geq |V|$, then V is C-colorable.
(**c**) For every connected graph G, find a family $C = \{C_x : x \in V\}$ showing that the condition $|C_V| \geq |V|$ in part (**b**) cannot be weakened to $|C_V| \geq |V| - 1$.

10702. *Proposed by Kent D. Boklan, Baltimore, MD.* What is the length of the longest nonconstant arithmetic progression of integers with the property that the kth term (for all $k \geq 1$) is a perfect kth power?

10703. *Proposed by Jean Anglesio, Garches, France.* Given triangle XYZ, let its incenter be I, its centroid C, its circumcenter O, its orthocenter H, the center of its nine-point circle W, its Gergonne point (the point of concurrency of the segments joining each vertex to the point of the incircle on the opposite side) G, and its Nagel point (the point of concurrency of the segments joining each vertex to the point of an excircle on the opposite side) N. Let S denote the intersection of the line IG with the Euler line (the line containing O, C, W, and H), and let T, U, and V denote respectively the intersections of line IG with lines NO, NW, and NH.
(**a**) Show that C lies one-third of the way from H to S (so that $SO = HO$).
(**b**) Show that $ST : SI : SU : SV = 10 : 15 : 18 : 30$.
(**c**) Show that $NO : TO = 3 : 1$, $NW : UW = 5 : 3$, and $NH = VH$. (We may now infer that $NH = 2 \cdot OI$ and that these segments are parallel.)

SOLUTIONS

A Doubly Rational Generating Function

10493 [1995, 930]. *Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA, and Christophe Reutenauer, Université du Québec, Montreal, Canada.* Fix a positive integer k. Let $f_k(m,n)$ be the number of m-tuples $a = (a_0, a_1, \ldots, a_{m-1})$ of integers satisfying: (a) $0 \leq a_i \leq n-1$ for all i, and (b) any k circularly consecutive entries of a (i.e., $a_i, a_{i+1}, \ldots, a_{i+k-1}$, where the subscripts are taken modulo m so that they lie between 0 and $m-1$) are all distinct. Show that the generating function $F_k(x,n) = \sum_{m \geq 1} f_k(m,n)x^m$ is a quotient of two polynomials in x and n.

Solution by Robin J. Chapman, University of Exeter, Exeter, U. K. Since $f_1(m,n) = n^m$, the result is immediate for $k = 1$, so we restrict attention to $k \geq 2$. In the first part of the solution, we obtain a recurrence that shows that $F_k(x,n)$ is a rational function in x for each n; we then study the dependence on n. In the second part, it is convenient to use zero as a special symbol, so we adopt an equivalent formulation using only positive integers. Thus, we note that $f_k(m,n)$ is the number of $(m+k-1)$-tuples $b = (b_1, b_2, \ldots, b_{m+k-1})$ such that (a) $1 \leq b_j \leq n$ for all j, (b) any k consecutive elements of b are all distinct, and (c) $b_j = b_{m+j}$ when $1 \leq j \leq k-1$. This number is $n(n-1)\cdots(n-k+2)$ times the number of such b also satisfying $b_j = j$ for $1 \leq j \leq k-1$. Fix n as well as k. For $c = (c_1, \ldots, c_{k-1})$, where the c_j are distinct integers, define $g(m;c) = g(m; c_1, \ldots, c_{k-1})$ to be the number of $(m+k-1)$-tuples $b = (b_1, b_2, \ldots, b_{m+k-1})$ such that $1 \leq b_j \leq n$ for all j, any k consecutive elements of b are all distinct, and $b_j = j$ and $b_{m+j} = c_j$ for

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PROBLEMS

10704. *Proposed by Wiliam G. Spohn, Jr., Ellicott City, MD.* Show that there are infinitely many pairs $((a, b, c), (a', b', c'))$ of primitive Pythagorean triples such that $|a - a'|$, $|b - b'|$, and $|c - c'|$ are all equal to 3 or 4. Examples include $((12, 5, 13), (15, 8, 17))$ and $((77, 36, 85), (80, 39, 89))$.

10705. *Proposed by D. W. Brown, Marietta, GA.* A topological space has the *fixed point property* if every continuous function from the space to itself has a fixed point. Is there a countably infinite Hausdorff space with the fixed point property?

10706. *Proposed by James G. Propp, University of Wisconsin, Madison, WI.* Given a finite sequence $(a_1, \ldots, a_n)$, define the *derived sequence* $(b_1, \ldots, b_{n+1})$ by $b_i = s - a_{i-1} - a_i$, where $s = \min_{1 \le i \le n+1}(a_{i-1} + a_i) + \max_{1 \le i \le n+1}(a_{i-1} + a_i)$ and where we interpret both a_0 and a_{n+1} as 0. Let S_0 be the sequence (1) of length 1, and for $n \ge 1$ define S_k to be the derived sequence obtained from S_{k-1}. Thus $S_1 = (1, 1)$, $S_2 = (2, 1, 2)$, $S_3 = (3, 2, 2, 3)$, and $S_4 = (5, 3, 4, 3, 5)$. Show that the middle term of S_{2n} is a perfect square.

10707. *Proposed by John Isbell, State University of New York, Buffalo, NY.* Show that
(a) no vector space over an infinite field is a finite union of proper subspaces; and
(b) no vector space over an n-element field is a union of n or fewer proper subspaces.

10708. *Proposed by the Western Maryland College Problems Group, Westminster, MD.* Let

$$f(x) = \frac{1}{4} \int_0^{\pi} \frac{1}{t} \log\left(\frac{1 - \cos(x + t)}{1 - \cos(x - t)}\right) dt$$

for $x \in (0, \pi)$.
(a) Find the Fourier sine series for f.
(b) Find the L^2 norm of f.
(c) Find $\lim_{x \to 0} f(x)$.

10709. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* Let X be a standard normal random variable, and choose $y > 0$. Show that

$$e^{-ay} < \frac{Pr(a \leq X \leq a+y)}{Pr(a-y \leq X \leq a)} < e^{-ay+(1/2)ay^3}$$

when $a > 0$. Show that the reversed inequalities hold when $a < 0$.

10710. *Proposed by Bogdan Suceava, Michigan State University, East Lansing, MI.* Let ABC be an acute triangle with incenter I, and let D, E, and F be the points where the circle inscribed in ABC touches BC, CA, and AB, respectively. Let M be the intersection of the line through A parallel to BC and DE, and let N be the intersection of the line through A parallel to BC and DF. Let P and Q be the midpoints of DM and DN, respectively. Prove that A, E, F, I, P, and Q are on the same circle.

SOLUTIONS

When O-H-I Is Isosceles

10547 [1996, 695]. *Proposed by Dan Sachelarie, ICCE Bucharest, and Vlad Sachelarie, University of Bucharest, Bucharest, Romania.* In the triangle ABC, let O be the circumcenter, H the orthocenter, and I the incenter. Prove that the triangle OHI is isosceles if and only if

$$\frac{a^3+b^3+c^3}{3abc} = \frac{R}{2r}.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria. We denote by MPV the reference D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities,* Kluwer, 1989. Neither IO nor HI is ever as large as HO [MPV, p. 288], so the only way triangle IHO can be isosceles is if $IO = HI$. Also $IO^2 = R^2 - 2Rr$ [MPV, p. 279] and $HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$ [MPV, p. 280], where s is the semiperimeter. Hence $HI = IO$ if and only if $R^2 - 2Rr = 4R^2 + 4Rr + 3r^2 - s^2$. This rearranges to $2s(s^2 - 3r^2 - 6Rr)/12Rrs = R/2r$, or, using $abc = 4Rrs$ [MPV, p. 52] and $a^3+b^3+c^3 = 2s(s^2 - 3r^2 - 4Rr)$ [MPV, p. 52], to $(a^3+b^3+c^3)/3abc = R/2r$.

Editorial comment. Another condition equivalent to $HI = IO$, given in problem E2282 [1971, 196; 1972, 397] from this MONTHLY, is that ABC has one angle equal to 60°.

Solved also by J. Anglesio (France), R. Barbara (Lebanon), F. Bellot Rosado (Spain), C. W. Dodge, J. S. Frame, Z. Franco, M. S. Klamkin (Canada), W. W. Meyer, V. Mihai (Canada), C. R. Pranesachar (India), B. Prielipp, V. Schindler (Germany), I. Sofair, M. Tabaâ (Morocco), T. V. Trif (Romania), M. Vowe (Switzerland), GCHQ Problems Group (U. K.), and the proposers.

The Divisible Differences Property

10553 [1996, 809]. *Proposed by Bjorn Poonen, Mathematical Sciences Research Institute, Berkeley, CA, Jim Propp, Massachusetts Institute of Technology, Cambridge, MA, and Richard Stong, Rice University, Houston, TX.* Say that a sequence $\langle q \rangle = q_1, q_1, q_2, \dots$ of integers has the *divisible differences property* if $(n-m)|(q_n - q_m)$ for all n and m.
(a) Show that if $\langle q \rangle$ has the divisible differences property and $\limsup |q_n|^{1/n} < e-1$, then there is a polynomial Q such that $q_n = Q(n)$.
(b) Show that there is a sequence $\langle q \rangle$ that has the divisible differences property and satisfies $\limsup |q_n|^{1/n} \leq e$, for which q_n is not given by a polynomial in n.
(c)* Is it true that $\limsup |q_n|^{1/n} \geq e$ for all non-polynomial $\langle q \rangle$ with the divisible differences property?

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PROBLEMS

10711. *Proposed by Florian Luca, Universität Bielefeld, Bielefeld, Germany.* A natural number is *perfect* if it is the sum of its proper divisors. Prove that two consecutive numbers cannot both be perfect.

10712. *Proposed by Paul Deiermann, Lindenwood University, St. Charles, MO, and Rick Mabry, Louisiana State University, Shreveport, LA.* Let $f(x)$ and $g(y)$ be twice continuously differentiable functions defined in a neighborhood of 0, and assume that $f(0) = 1$, $g(0) = f'(0) = g'(0) = 0$, $f''(0) < 0$, and $g''(0) > 0$.
(a) For sufficiently small $r > 0$, show that the curves $x = g(y)$ and $y = rf(x/r)$ have a common point (x_r, y_r) in the first quadrant with the property that, if (x, y) is any other common point, then $x_r < x$.
(b) Let $(t_r, 0)$ denote the x-intercept of the line passing through $(0, r)$ and (x_r, y_r). Show that $\lim_{r\to 0^+} t_r$ exists, and evaluate it.
(c) Is the continuity of f'' and g'' a necessary condition for $\lim_{r\to 0^+} t_r$ to exist?

10713. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.* Given a triangle with angles $A \geq B \geq C$, let a, b, and c be the lengths of the corresponding opposite sides, let r be the radius of the inscribed circle, and let R be the radius of the circumscribed circle. Show that A is acute if and only if $R + r < (b + c)/2$.

10714. *Proposed by Jet Wimp, Drexel University, Philadelphia, PA.* For $a \in (-\pi/2, \pi/2)$, define

$$c_n(t) = \frac{1}{e^{at}\cos a}\left(\frac{d}{da}\right)^n \left(e^{at}\cos a\right)$$

for every nonnegative integer n, so that $c_n(t)$ is a monic polynomial of degree n. Let G_n denote the $(n + 1)$-by-$(n + 1)$ determinant $\left|c_{j+k}(t)\right|_{j,k=0,1,\ldots,n}$. Evaluate G_n.

10715. *Proposed by Roger Cuculière, Clichy, France.* Choose $u_0 > 1$, and define $u_{n+1} = u_n + \ln u_n$ for $n \in \mathbb{N}$. Find a closed-form expression a_n such that $\lim_{n\to\infty}(u_n - a_n)/n = 0$.

10716. *Proposed by Michael L. Catalano-Johnson and Daniel Loeb, Daniel Wagner Associates, Malvern, PA.* What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

10717. *Proposed by Marcin Mazur, University of Chicago, Chicago, IL.* We say that a tetrahedron is *rigid* if it is determined by its volume, the areas of its faces, and the radius of its circumscribed sphere. We say that a tetrahedron is *very rigid* if it is determined just by the areas of its faces and the radius of its circumscribed sphere.
(**a**) Prove that every tetrahedron with faces of equal area is rigid.
(**b**) Prove that a very rigid tetrahedron with faces of equal area is regular.
(**c**)* Is every tetrahedron rigid?
(**d**)* Is every very rigid tetrahedron regular?

SOLUTIONS

Subtracting Square Roots Repeatedly

10568 [1997, 68]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let n be a nonnegative integer. The sequence defined by $x_0 = n$ and $x_{k+1} = x_k - \lceil \sqrt{x_k} \rceil$ for $k \geq 0$ converges to 0. Let $f(n)$ be the number of steps required; i.e., $x_{f(n)} = 0$ but $x_{f(n)-1} > 0$. Find a closed form for $f(n)$.

Solution by Denis Constales, University of Gent, Gent, Belgium. Every positive integer n can be written uniquely in the form $p^2 - q$, where p and q are integers satisfying $p \geq 1$ and $0 \leq q \leq 2p - 2$ (take $p = \lceil \sqrt{n} \rceil$ and $q = p^2 - n$). We call this *standard form* for n. We obtain the desired formula in terms of these parameters p and q.

Using standard form, let $n' = n - \lceil \sqrt{n} \rceil = p^2 - (q + p)$. We distinguish two cases. **Case 1:** $p - 1 \leq q \leq 2p - 2$. We rewrite n' as $(p-1)^2 - (q - (p-1))$. Since $q \geq p - 1$, this expresses n' in standard form with $p' = p - 1$ and $q' = q - (p - 1)$ (when $p > 2$). **Case 2:** $0 \leq q \leq p - 1$. Now $n' = p^2 - (q + p)$ is standard form for n' with $p' = p$ and $q' = q + p$. The next value $n'' = n' - \lceil \sqrt{n'} \rceil = p^2 - (q + 2p)$. Expressed in standard form, this is $n'' = (p-1)^2 - (q+1)$ (when $p > 2$).

We have applied the transformation once in Case 1 and twice in Case 2. Thus

$$f(p^2 - q) = \begin{cases} 2 + f((p-1)^2 - (q+1)) & \text{if } 0 \leq q \leq p - 2 \\ 1 + f((p-1)^2 - (q - p + 1)) & \text{if } p - 1 \leq q \leq 2p - 2 \end{cases}$$

whenever $p > 2$ and $0 \leq q \leq 2p - 2$. The cases $p \leq 2$ occur for $n \in \{1, 2, 3, 4\}$, where $f(n) = 1, 1, 2, 2$, respectively. With the recurrence, these initial conditions define f. Our closed form is

$$f(p^2 - q) = \begin{cases} 2p - \lfloor \log_2(p+q) \rfloor - 1 & \text{if } 0 \leq q \leq p - 1 \\ 2p - \lfloor \log_2 q \rfloor - 2 & \text{if } p \leq q \leq 2p - 2 \end{cases}$$

for integers p, q such that $1 \leq p$ and $0 \leq q \leq 2p - 2$. Also, we set $f(0) = 0$.

The proof of the formula is immediate by induction, using the recurrence in the three cases $0 \leq q \leq p - 2$, $q = p - 1$, and $p \leq q \leq 2p - 2$. The only simplification needed occurs in the second case, where $\lceil \log_2(2p - 1) \rceil = 1 + \lceil \log_2(p - 1) \rceil$, which follows immediately when $p > 1$.

Editorial comment. Robin J. Chapman and the GCHQ Problems Group expressed $f(n)$ using the single formula $f(n) = \lfloor 4n + 2^{m+3} - 3 \rfloor - (m+2)$, where $m = \lfloor \log_2(\sqrt{n} + 1) \rfloor$.

Solved also by T. Amdeberhan, K. L. Bernstein, R. J. Chapman (U. K.), D. A. Darling, M. N. Deshpande & N. N. Kasturiwale (India), K. Ferguson, R. Holzsager, W. Janous (Austria), F. Kemp, P. G. Kirmser, N. Komanda, Y. Kong, J. H. Lindsey II, W. A. Newcomb, C. R. Pranesachar (India), K. Schilling, J. H. Steelman, D. Trautman, X. Wang, D. Yuen, GCHQ Problems Group (U. K.), Westmont Problems Group, and the proposer.

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PROBLEMS

10718. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.* Let p be a prime number with $p \equiv 7 \pmod 8$, and let $L_p = \{1, 2, 3, \ldots, (p-1)/2\}$. Prove that the sum of the quadratic residues modulo p in L_p equals the sum of the quadratic nonresidues modulo p in L_p. For example, the quadratic residues in L_{23} are 1, 2, 3, 4, 6, 8, and 9, and the quadratic nonresidues in L_{23} are 5, 7, 10, and 11. Both lists sum to 33.

10719. *Proposed by Jean Anglesio, Garches, France.* Let A, I, and G be three points in the plane. Let M denote the point 2/3 of the way from A to I, and let U and V be the circles of radius $|AM|$ each of which is tangent to AI at M. Show that when G is outside both U and V, there are precisely two triangles ABC with incenter I and centroid G. Provide a Euclidean construction for them. Show that when G is in the interior of U or V, there does not exist a triangle ABC with incenter I and centroid G.

10720. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* A "binary maze" is a directed graph in which exactly two arcs lead from each vertex, one labeled 0 and one labeled 1. If $b_1, b_2, \ldots, b_m$ is any sequence of 0s and 1s and v is any vertex, let $vb_1b_2\cdots b_m$ be the vertex reached beginning at v and traversing arcs labeled $b_1, b_2, \ldots, b_m$ in order. A sequence $b_1, b_2, \ldots, b_m$ of 0s and 1s is a *universal exploration sequence* of order n if, for every strongly connected binary maze on n vertices and every vertex v, the sequence

$$v,\ vb_1,\ vb_1b_2,\ \ldots,\ vb_1b_2\cdots b_m$$

includes every vertex of the maze. For example, 01 is a universal exploration sequence of order 2, and it can be shown that 0110100 is universal of order 3.

(**a**) Prove that universal exploration sequences of all orders exist.

(**b**)* Find a good estimate for the asymptotic length of the shortest such sequence of order n.

10721. *Proposed by Daniel A. Sidney, Massachusetts Institute of Technology, Cambridge, MA.* Let $f(x) = \sin x/x$, and let m and n be nonnegative integers. Compute

$$\int_0^\infty \frac{d^m}{dx^m} f(x)\, \frac{d^n}{dx^n} f(x)\, dx.$$

10722. *Proposed by Richard F. McCoart, Loyola College, Baltimore, MD.*
(**a**) In how many ways can $2n$ indistinguishable balls be placed into n distinguishable urns, if the first r urns may contain at most $2r$ balls for each $r \in \{1, 2, \ldots, n\}$?
(**b**) Suppose that $0 \le m \le n$. In how many of the ways enumerated in part (**a**) are exactly m urns empty?

10723. *Proposed by Christopher J. Hillar, Yale University, New Haven, CT.* Let p be an odd prime. Prove that $\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{(p-1)/2} i^{p-2} \pmod{p}$.

10724. *Proposed by Serge Tabachnikov, University of Arkansas, Fayetteville, AR.*
(**a**) Let P be a convex plane polygon with vertices $A_1, \ldots, A_n$, and let l be a continuous transverse field of directions along the boundary ∂P. (This means that through every point $X \in \partial P$ there passes a line $l(X)$ that intersects the interior of P and depends continuously on X.) Let α_i and β_i be the angles between the line $l(A_i)$ and the adjacent sides $A_i A_{i-1}$ and $A_i A_{i+1}$, respectively. Assume that $\prod_1^n \sin \alpha_i = \prod_1^n \sin \beta_i$. Prove that the lines $l(X)$ cover the interior of P twice, that is, every interior point of P belongs to at least two of these lines.
(**b**) Suppose $n \ge 3$, and let P be a convex polyhedron in n-dimensional space. As in (**a**), a continuous transverse line field l is given along the boundary ∂P. This field has the property that for every $(n-2)$-dimensional face E of P there exists a hyperplane $\pi(E)$ such that all the lines $l(X)$ with $X \in E$ belong to $\pi(E)$. Prove that the lines $l(X)$ cover the interior of P twice.

SOLUTIONS

Principal Ideals in Noetherian Rings

10534 [1996, 510]. *Proposed by Paul Arne Østvær, Oslo University, Oslo, Norway.* Suppose that R is a Noetherian ring in which all maximal ideals are principal. Show that all ideals in R are principal.

Solution by Robert Gilmer, Florida State University, Tallahassee, FL. If $M = (m)$ is a maximal ideal of R, then M/M^2 is a vector space over the field R/M of dimension at most 1. Hence there are no ideals of R properly between M and M^2. From this it follows (R. Gilmer, *Multiplicative Ideal Theory,* Queen's Papers Pure Appl. Math. **90** (1992), Theorem 39.2) that $R = D_1 \oplus \cdots \oplus D_n \oplus S_1 \oplus \cdots \oplus S_m$ is a finite direct sum of Dedekind domains D_i and special primary rings S_i. To show that each ideal of R is principal, it suffices to show that the D_i and S_i have this property. For S_i this is part of the definition of a special primary ring (Gilmer, p. 200). Moreover, D_i inherits from R the property that each of its maximal ideals is principal, and a Dedekind domain is a principal ideal domain whenever all of its maximal ideals are principal.

Editorial comment. D. D. Anderson mentions a stronger result that appears in R. Gilmer and W. Heinzer, Principal ideal rings and a condition of Kummer, *J. Algebra* **83** (1983) 285–292: If R has the ascending chain condition on *principal* ideals and each maximal ideal of R is principal, then every ideal of R is principal.

Solved also by Mahalal'el ben keinan (Israel), F. Calegari (Australia), J. E. Dawson (Australia), T. H. Foregger, O. Moubinool (France), S. Sertöz (Turkey), and M. Tabaâ (Morocco).

A Telescoping Constraint

10566 [1997, 68]. *Proposed by Gerry Myerson, Macquarie University, Australia.* Let S be a finite set of cardinality $n > 1$. Let f be a real-valued function on the power set of S, and suppose that $f(A \cap B) = \min\{f(A), f(B)\}$ for all subsets A and B of S. Prove that

$$\sum (-1)^{n-|A|} f(A) = f(S) - \max f(A),$$

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PROBLEMS

10725. *Proposed by Vasile Mihai, Toronto, ON, Canada.* Fix a positive integer n. Given a permutation α of $\{1, 2, \ldots, n\}$, let $f(\alpha) = \sum_{i=1}^{n}\bigl(\alpha(i)-\alpha(i+1)\bigr)^2$, where $\alpha(n+1) = \alpha(1)$. Find the extreme values of $f(\alpha)$ as α ranges over all permutations of $\{1, 2, \ldots, n\}$.

10726. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Start in state 0. For every nonnegative integer k, stay in state k for X_k units of time, then go to state $k+1$. What is the probability of being in state s after t units of time, assuming that X_k is distributed exponentially **(a)** with mean $1/(k+1)$? **(b)** with mean $1/2^k$?

10727. *Proposed by Jean Anglesio, Garches, France.* Let m be a fixed positive integer. For a positive integer n, let $s_m(n)$ be the sum of the mth powers of the decimal digits of n. For example, $s_3(172) = 1^3 + 7^3 + 2^3 = 352$. Starting with any positive integer n_0, construct a sequence of positive integers by setting $n_k = s_m(n_{k-1})$ for every $k \geq 1$.
(a) Show that $n_0, n_1, n_2, \ldots$ is eventually periodic.
(b) Show that only finitely many periods are possible as n_0 varies.

10728. *Proposed by Titu Andreescu, American Mathematics Competitions, Lincoln, NE.* Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(x^3 + y^3 + z^3) = \bigl(f(x)\bigr)^3 + \bigl(f(y)\bigr)^3 + \bigl(f(z)\bigr)^3$$

for all integers x, y, and z.

10729. *Proposed by David P. Bellamy and Felix Lazebnik, University of Delaware, Newark, DE.* Let $I \subset \mathbb{R}$ be an open interval, and let n be a positive integer. Characterize the functions $f\colon I \to \mathbb{R}$ that have a continuous nth derivative and satisfy

$$f^{(n)} + p_1 f^{(n-1)} + \cdots + p_{n-1} f' + p_n f = 0$$

for some continuous functions $p_1, p_2, \ldots, p_n$ on I.

10730. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Fix an integer $n \geq 2$. Determine the largest constant $C(n)$ such that

$$\sum_{1 \leq i < j \leq n} (x_j - x_i)^2 \geq C(n) \cdot \min_{1 \leq i < n} (x_{i+1} - x_i)^2$$

for all real numbers $x_1 < x_2 < \cdots < x_n$.

10731. *Proposed by M. J. Pelling, London, England.* Let A be an n-by-n real symmetric matrix, and consider the quadratic form $Q(x) = x^T Ax$ for $x \in \mathbb{R}^n$. Let C be the cube $[-1, 1]^n$. Prove that $\max_{x \in C} Q(x)$ is at least as large as the sum of the positive real eigenvalues of A.

SOLUTIONS

Connected Sets of Periodic Functions

10434 [1995, 170]. *Proposed by Daniel Goffinet, Saint Étienne, France.* Let P be the set of nonconstant periodic mappings from $\mathbb{R}$ to $\mathbb{R}$, endowed with the topology derived from the supremum norm. Find the components of P.

Composite solution I by Kiran S. Kedlaya, Massachusetts Institute of Technology, Cambridge, MA, Kenneth Schilling, University of Michigan, Flint, MI, and Arlo W. Schurle, University of Guam, Mangilao, Guam. For any function $f: \mathbb{R} \to \mathbb{R}$, define $\|f\|$ to be $\sup\{|f(x)|: x \in \mathbb{R}\}$, which is taken to be ∞ when the set of values of f is unbounded.

We first show that f and g are in different components of P if $\|f - g\| = \infty$. Let $B_g = \{k \in P: \|k - g\| < \infty\}$. By the triangle inequality B_g is an open set, and if $h \notin B_g$, then the triangle inequality again shows that $\{z: \|z - h\| < 1\} \cap B_g = \emptyset$. Consequently B_g is both open and closed, and so the component of P containing any given $g \in P$ must lie in B_g.

Conversely, if $f - g$ is bounded for $f, g \in P$, then there is an arc in P joining f to g. First, suppose that f and g have a common period p. The standard path $k_t(x) = (1 - t)f(x) + tg(x)$ for $0 \leq t \leq 1$ consists of functions having p as a period, and since $\|f - g\|$ is finite, k_t depends continuously on t. There is a danger that some $k_t(x)$ is a constant function, but this can happen only if f is an affine function of g, that is, there are constants A and B with $f = Ag + B$. In this case, the function $h(x)$ that is equal to $f(x)$ except at integer multiples of p, where it is $f(x) + 1$, is at bounded distance from both f and g and is not an affine function of either. A path from f to g can be obtained by taking the standard path from f to h followed by the standard path from h to g.

Suppose now that f and g have no common period. Let r be a period of f and let s be a period of g. We wish to construct h that has both r and s as periods such that $\|f - h\|$ (and hence also $\|g - h\|$) is finite. To do this, pick an arbitrary set of coset representatives for $\mathbb{R}/(r\mathbb{Z} + s\mathbb{Z})$, define h to agree with f at these values, and extend by periodicity. Then for any x, let $x = y + rm + sn$, where y represents the coset containing x. Then

$$\begin{aligned} |h(x) - f(x)| &= |f(y) - f(y + sn)| \\ &= |f(y) - g(y) + g(y + sn) - f(y + sn)| \leq 2\,\|f - g\| \end{aligned}$$

Since f and h have common period r and $\|f - h\|$ is finite, there is a path from f to h, and since h and g have common period s and $\|h - g\|$ is finite, there is a path from h to g.

Composite solution II by Fredric D. Ancel, University of Wisconsin, Milwaukee, WI, Phil Bowers and John Bryant, The Florida State University, Tallahassee, FL, and the proposer. We assume that "mapping" means "continuous function". Then two functions in P belong to the same component if and only if they have commensurate periods. As in solution I, the components are path-components.

PROBLEMS AND SOLUTIONS

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PROBLEMS

10732. *Proposed by M. N. Deshpande, Nagpur, India.* Let n and k be positive integers with $k < n$. Select a permutation π of n objects at random, and let the random variable X_k denote the number of objects that lie in cycles of π of length less than or equal to k. Find the expected value and the variance of X_k.

10733. *Proposed by Sung Soo Kim, Hanyang University, Ansan, Korea.* Let $\{E_\alpha\}_{\alpha\in\Omega}$ be a partition of the unit interval $I = [0, 1]$ into nonempty sets that are closed in the usual topology. Is it possible that
(**a**) Ω is uncountable and E_α is uncountable for each $\alpha \in \Omega$?
(**b**) Ω is uncountable but E_α is countably infinite for each $\alpha \in \Omega$?
(**c**) Ω is countably infinite?

10734. *Proposed by Floor van Lamoen, Goes, The Netherlands.* Let ABC be a triangle with orthocenter H, incenter I, and circumcenter O. Let $[P, r]$ denote the circle with center P and radius r. Show that the radical center of $[A, CA + AB]$, $[B, AB + BC]$, and $[C, BC + CA]$ is the point obtained by reflecting H through O and then reflecting the result through I.

10735. *Proposed by Gustavus J. Simmons, Sandia Park, NM.* If L_n is the n-by-n matrix with i,j-entry equal to $\binom{i-1}{j-1}$, then $L_n^2 \equiv I_n \bmod 2$, where I_n is the n-by-n identity matrix. Show that if R_n is the n-by-n matrix with i,j-entry equal to $\binom{i-1}{n-j}$, then $R_n^3 \equiv I_n \bmod 2$.

10736. *Proposed by Mizan R. Khan, Eastern Connecticut State University, Willimantic, CT.* For a given $n \ge 2$, let $M(n) = \max\{\, |a-b| : a, b \in \{1, 2, \ldots, n\} \text{ and } ab \equiv 1 \bmod n \,\}$.
(**a**) Find a closed-form expression $U(n)$ such that $M(n) \le U(n)$ for all n, with equality in infinitely many cases.
(**b**) Show that $\lim_{n\to\infty} M(n)/n = 1$.
(**c**)* Prove or disprove that $\lim_{n\to\infty} \log\bigl(n - M(n)\bigr)/\log n = 1/2$.

10737. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.* Let m and n be positive integers with $n \geq 2m$, and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers such that

$$a_n < m + \frac{1}{2m}\left(\sum_{i=1}^{m}\binom{n}{2i}\binom{2i}{i}\right).$$

Show that there exist two different n-tuples $(\epsilon_1, \ldots, \epsilon_n)$ and $(\delta_1, \ldots, \delta_n)$, with entries 0, 1, and 2, such that $\sum_{j=1}^{n}\epsilon_j = \sum_{j=1}^{n}\delta_j \leq 2m$ and $\sum_{j=1}^{n}\epsilon_j a_j = \sum_{j=1}^{n}\delta_j a_j$.

10738. *Proposed by Radu Theodorescu, Université Laval, Sainte-Foy, PQ, Canada.* For $t > 0$, let $m_n(t) = \sum_{k=0}^{\infty} k^n e^{-t} t^k/k!$ be the nth moment of a Poisson distribution with parameter t. Let $c_n(t) = m_n(t)/n!$. A sequence $a_0, a_1, \ldots$ is *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n > 0$ and is *log-concave* if $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n > 0$.
(a) Show that $m_0(t), m_1(t), \ldots$ is log-convex.
(b) Show that $c_0(t), c_1(t), \ldots$ is not log-concave when $t < 1$.
(c) Show that $c_0(t), c_1(t), \ldots$ is log-concave when t is sufficiently large.
(d)* Is $c_0(t), c_1(t), \ldots$ log-concave when $t \geq 1$?

SOLUTIONS

Moments of Roots of Chebyshev Polynomials

10448 [1995, 360]. *Proposed by Fu-Chuen Chang, National Sun Yat-sen University, Kaohsiung, Taiwan.* Fix a positive integer n. Let $x_i = \cos\big((2i-1)\pi/(2n)\big)$ for $1 \leq i \leq n$, and let $c_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$ for $k \in \mathbb{N}$. Show that

$$c_k = \begin{cases} 0 & \text{if } k = 1, 3, \ldots, 2n-1; \\ \binom{k}{k/2}2^{-k} & \text{if } k = 0, 2, \ldots, 2n-2. \end{cases}$$

Solution I by Paul Deiermann, Louisiana State University, Shreveport, LA. When $k = 0$ and n is odd, the term for $j = (n+1)/2$ appears as 0^0, which must be taken to be 1 to arrive at the stated formula and our generalization. We show, for arbitrary integers $k \geq 0$, that

$$c_k = \begin{cases} 0 & \text{for } k \text{ odd}, \\ 2^{-k}\sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}} & \text{for } k \text{ even}, \end{cases}$$

where $m = \lfloor k/(2n) \rfloor$. The stated problem covers those k for which $m = 0$.

First note that $x_{n+1-j} = -x_j$, so the terms of the sum cancel in pairs when k is odd. We may thus restrict to the case of k even. Since $x_j = \big(e^{i\pi(2j-1)/(2n)} + e^{-i\pi(2j-1)/(2n)}\big)/2$, the binomial theorem and a summation of a finite geometric progression imply

$$\sum_{j=1}^{n} x_j^k = \sum_{j=1}^{n} 2^{-k}\left(e^{i\pi\frac{2j-1}{2n}} + e^{-i\pi\frac{2j-1}{2n}}\right)^k = 2^{-k}\sum_{j=1}^{n}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(k-2q)}e^{i\frac{2\pi}{n}(q-k/2)j}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(k-2q)}\sum_{j=1}^{n}e^{i\frac{2\pi}{n}(q-k/2)j} = 2^{-k}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(2q-k)}\sum_{u=0}^{n-1}e^{i\frac{2\pi}{n}(q-k/2)u}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(2q-k)}\begin{cases} n & \text{if } q - k/2 = pn,\ p \in \mathbb{Z}, \\ \frac{1-e^{i\pi(2q-k)}}{1-e^{i\frac{2\pi}{n}(q-k/2)}} = 0 & \text{if } n \nmid q - k/2. \end{cases}$$

Since k is even, $q - k/2 = pn$ implies $q = pn + k/2$. Then, $0 \leq q \leq k$ gives $-m \leq p \leq m$. Also, in this case, $e^{i\frac{\pi}{2n}(2q-k)} = e^{i\pi p} = (-1)^p$. Thus, we get

$$\sum_{j=1}^{n} x_j^k = 2^{-k}n\sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}}.$$

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PROBLEMS

10739. *Proposed by Oscar Ciaurri, Logroño, Spain.* Suppose that $f\colon [0,1] \to \mathbb{R}$ has a continuous second derivative with $f''(x) > 0$ on $(0,1)$, and suppose that $f(0) = 0$. Choose $a \in (0,1)$ such that $f'(a) < f(1)$. Show that there is a unique $b \in (a,1)$ such that $f'(a) = f(b)/b$.

10740. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, IL.* A connected bipartite simple graph has a unique bipartition, meaning a partition of the vertices into two independent sets. Let $\mathbf{G}$ be the set of such graphs that have no isomorphism that interchanges the two sets of the bipartition. Is there a criterion that for each $G \in \mathbf{G}$ selects a well-defined set of the bipartition?

10741. *Proposed by Tim Keller, Fair Oaks, CA.* Is there an even base b for which there exist square integers of the form $dddd_b$? By $dddd_b$, we mean the four-digit number in base b all of whose digits are d. For odd b we have the examples $1111_7 = 20^2$ and $4444_7 = 40^2$.

10742. *Proposed by Emre Alkan, University of Wisconsin, Madison, WI.* Let us say that a finite group G has the *maximal property* if, for any prime p that divides $|G|$, G has a maximal subgroup H such that $p|H|$ divides $|G|$.

(a) Show that every finite solvable group has the maximal property.

(b) Show that there are infinitely many finite groups with the maximal property that are not solvable.

(c) Show that there are infinitely many finite groups without the maximal property that are not solvable.

10743. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-La-Neuve, Belgium.* Let $p \geq 5$ be prime, and let n be an integer such that $(p+1)/2 \leq n \leq p-2$. Let $R = \sum(-1)^i\binom{n}{i}$, where the sum is taken over the quadratic residues i modulo p, and let $N = \sum(-1)^j\binom{n}{j}$, where the sum is taken over the quadratic nonresidues j modulo p. Prove that exactly one of R and N is divisible by p.

10744. *Proposed by Peter Lindqvist, Norwegian University of Science and Technology, Trondheim, Norway, and Jaak Peetre, University of Lund, Lund, Sweden.* Fix $p > 0$, and define functions $S(x)$, $C(x)$, and $T(x)$ for sufficiently small x by

$$x = \int_0^{S(x)} \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_{C(x)}^1 \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_0^{T(x)} \frac{dt}{(1+t^p)^{2/p}}.$$

Show that $S(x)^p + C(x)^p = 1$ and that $T(x) = S(x)/C(x)$. The case $p = 2$ yields the familiar trigonometric formulas.

10745. *Proposed by M. J. Pelling, London, England.* For $n \geq 1$, let $f(n)$ be the number of solutions (r, s, t) in positive integers to the Diophantine equation $rst = n(r + s + t)$.
(**a**) Prove that $f(n) = O(n^{1/2+\delta})$ for every $\delta > 0$.
(**b**)* Prove that $f(n) = O(n^{\delta})$ for every $\delta > 0$.

SOLUTIONS

Using the Walls to Find the Center

10386 [1994, 474]. *Proposed by Jordan Tabov, Bulgarian Academy of Sciences, Sofia, Bulgaria.* Let a tetrahedron with vertices A_1, A_2, A_3, A_4 have altitudes that meet in a point H. For any point P, let P_1, P_2, P_3, and P_4 be the feet of the perpendiculars from P to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Prove that there exist constants a_1, a_2, a_3, and a_4 such that one has

$$a_1\overrightarrow{PP_1} + a_2\overrightarrow{PP_2} + a_3\overrightarrow{PP_3} + a_4\overrightarrow{PP_4} = \overrightarrow{PH}$$

for every point P.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. More generally, let H and P be any two points in the space of the given tetrahedron and let P_1, P_2, P_3, P_4 be the feet of the lines through P parallel to HA_1, HA_2, HA_3, HA_4 in the faces of the tetrahedron opposite A_1, A_2, A_3, A_4, respectively. Then there exist constants a_1, a_2, a_3, a_4, independent of P, such that

$$a_1\overrightarrow{PP_1} + a_2\overrightarrow{PP_2} + a_3\overrightarrow{PP_3} + a_4\overrightarrow{PP_4} = \overrightarrow{PH}.$$

Let $\mathbf{V}$ denote the vector from an origin outside the space of the given tetrahedron to any point V in the space of the tetrahedron. Then H and P have the representations (barycentric coordinates)

$$\mathbf{H} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + x_3\mathbf{A}_3 + x_4\mathbf{A}_4 \quad (x_1 + x_2 + x_3 + x_4 = 1),$$
$$\mathbf{P} = u_1\mathbf{A}_1 + u_2\mathbf{A}_2 + u_3\mathbf{A}_3 + u_4\mathbf{A}_4 \quad (u_1 + u_2 + u_3 + u_4 = 1).$$

Since P_1 has the representation $\mathbf{P}_1 = r_2\mathbf{A}_2 + r_3\mathbf{A}_3 + r_4\mathbf{A}_4$, where $r_2 + r_3 + r_4 = 1$, we must have

$$r_2\mathbf{A}_2 + r_3\mathbf{A}_3 + r_4\mathbf{A}_4 - \mathbf{P} = \lambda_1(\mathbf{H} - \mathbf{A}_1).$$

Since $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ are independent vectors, we get $\lambda_1 = u_1/(1 - x_1)$, so that $\overrightarrow{PP_1} = (\mathbf{P}_1 - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_1)u_1/(1 - x_1)$. Similarly,

$$(\mathbf{P}_i - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_i)\frac{u_i}{1 - x_i} \quad \text{for } i = 1, 2, 3, 4.$$

Choosing $a_i = 1 - x_i$, we obtain

$$\sum a_i(\mathbf{P}_i - \mathbf{P}) = \sum u_i(\mathbf{H} - \mathbf{A}_i) = \mathbf{H} - \mathbf{P} = \overrightarrow{PH}.$$

This proof generalizes to give an analogous result for n-dimensional simplices.

Solved also by J. Anglesio (France), R. J. Chapman (U. K.), M. Golomb, K. Hanes, N. Komanda, O. P. Lossers (The Netherlands), and the proposer.

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Problem **10743** [1999; 586] in the June–July 1999 issue was misstated. Here is the corrected version.

10743. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-La-Neuve, Belgium.* Let $R = \sum(-1)^i \binom{n}{i}$, where the sum is taken over all $i \in \{0, 1, \dots, n-1\}$ such that $i+1$ is a quadratic residue modulo p, and let $N = \sum(-1)^j \binom{n}{j}$, where the sum is taken over all $j \in \{0, 1, ..., n-1\}$ such that $j+1$ is a quadratic nonresidue modulo p. Prove that exactly one of R and N is divisible by p.

PROBLEMS

10746. *Proposed by Stepan Tersian, University of Rousse, Rousse, Bulgaria.* Prove that

$$\int_0^\infty \left(e^{-y\sqrt{(s/x)^2+1}} - e^{-x\sqrt{(s/y)^2+1}}\right) \cos s\, ds = 0,$$

for all positive real numbers x and y.

10747. *Proposed by Athanasios Kalakos, Athens, Greece.* Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that are twice differentiable on an open interval containing 0, have exactly one real root, satisfy $f(1) = 1$, and satisfy $f'(f(t)) = 2f(t)$ for every $t \in \mathbb{R}$.

10748. *Proposed by Itshak Borosh, Douglas A. Hensley, and Joel Zinn, Texas A& M University, College Station, TX.* Let p and q be prime numbers, and let r be a positive integer such that $q \mid (p-1)$, $q \nmid r$, and $p > r^{q-1}$. Show that for any integers $a_1, a_2, \dots, a_r$, if $\sum_{j=1}^r a_j^{(p-1)/q} \equiv 0 \bmod p$, then $\prod_{j=1}^r a_j \equiv 0 \bmod p$.

10749. *Proposed by Alain Grigis, Université Paris 13, Villetaneuse, France.* Let ABC be a triangle with a right angle at B and an angle of $\pi/6$ at A. Consider a billiard path in the triangle that begins at A, reflects successively off side BC at P, off side AC at Q, off side AB at R, off side AC at S, and then ends at B.

(**a**) Show that AP, QR, and SB are concurrent at a point X.

(**b**) Show that the angles formed at X measure $\pi/3$.

(**c**) Show that $AX = XP + PQ + QX = XR + RS + SX = 2XB$.

10750. *Proposed by Leonard Smiley, University of Alaska, Anchorage, AK.* For a positive integer m, express $\sum_{n=1}^{\infty}(n/\gcd(m,n))x^n$ as a rational function of x.

10751. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Let n be a positive integer, and let S_n be the set of all strings $a_1a_2\cdots a_n$ of positive integers satisfying $a_1 = 1$ and $a_{i+1} - a_i \in \{1, -1, -3, -5, \ldots\}$. For example, $S_5 = \{12345, 12343, 12341, 12323, 12321, 12123, 12121\}$. Find $|S_n|$.

10752. *Proposed by Gh. Costovici, Technical University "Gh. Asachi", Iasi, Romania.* For $n \in \mathbb{N}$, let a_n and b_n be complex numbers, with each $b_n \neq 0$. Let $s_n = a_1 + a_2 + \cdots + a_n$, and let $t_n = (1 - b_1/b_{n+1})\, a_1 + (1 - b_2/b_{n+1})\, a_2 + \cdots + (1 - b_n/b_{n+1})\, a_n$.
(a) Prove that if $\lim_{n\to\infty} b_{n+1}/b_n = 1$ and $\sum_{n=1}^{\infty} |s_n - t_n|^q$ converges for some $q \in (0, 1]$, then $\sum_{n=1}^{\infty} a_n$ converges.
(b) Prove that if $\sum_{n=1}^{\infty} |b_{n+1}/b_n - 1|^r$ and $\sum_{n=1}^{\infty} |s_n - t_n|^{r/(r-1)}$ converge for some $r \in (1, \infty)$, then $\sum_{n=1}^{\infty} a_n$ converges.

SOLUTIONS

A Zeta Function over a Recurrent Sequence

10486 [1995, 841]. *Proposed by Joseph H. Silverman, Brown University, Providence, RI.* Let $a, b > 0$ and $\alpha > 1$ be real numbers, and define $Z(s) = \sum_{n\in\mathbb{Z}} \left(a\alpha^n + b\alpha^{-n}\right)^{-s}$ for complex numbers s with positive real part.
(a) Prove that $Z(s)$ has a meromorphic continuation to all of $\mathbb{C}$.
(b) Find the poles of $Z(s)$.
(c) Find the residues of $Z(s)$ at its poles.

Solution I by David Bradley, University of Maine, Orono, ME. Let σ be the real part of s. Write

$$Z(s) = (a+b)^{-s} + \sum_{n=1}^{\infty} \left(a\alpha^n + b\alpha^{-n}\right)^{-s} + \sum_{n=1}^{\infty} \left(b\alpha^n + a\alpha^{-n}\right)^{-s}. \tag{1}$$

Without loss of generality, assume that $0 < a \le b$. We first consider the case $|\alpha| > \sqrt{b/a}$. We then have the two binomial expansions

$$\left(a\alpha^n + b\alpha^{-n}\right)^{-s} = \frac{a^{-s}\alpha^{-ns}}{\left(1 + ba^{-1}\alpha^{-2n}\right)^s} = a^{-s}\alpha^{-ns}\left(\sum_{k=0}^{m-1}\binom{-s}{k}\frac{b^k}{a^k}\alpha^{-2nk} + E_{m,n}(s)\right) \tag{2}$$

and

$$\left(b\alpha^n + a\alpha^{-n}\right)^{-s} = \frac{b^{-s}\alpha^{-ns}}{\left(1 + ab^{-1}\alpha^{-2n}\right)^s} = b^{-s}\alpha^{-ns}\left(\sum_{k=0}^{m-1}\binom{-s}{k}\frac{a^k}{b^k}\alpha^{-2nk} + F_{m,n}(s)\right), \tag{3}$$

where m is a fixed positive integer and $E_{m,n}(s) = O(\alpha^{-2mn})$ and $F_{m,n}(s) = O(\alpha^{-2mn})$. Since $|\alpha| > \sqrt{b/a}$, it follows from (1)–(3) that

$$\begin{aligned} Z(s) &= (a+b)^{-s} + \sum_{k=0}^{m-1}\binom{-s}{k}\left(\frac{b^k}{a^{s+k}} + \frac{a^k}{b^{s+k}}\right)\sum_{n=1}^{\infty}\alpha^{-n(s+2k)} + O\left(\sum_{n=1}^{\infty}\alpha^{-n(\sigma+2m)}\right) \\ &= (a+b)^{-s} + \sum_{k=0}^{m-1}\binom{-s}{k}\frac{a^{-s-k}b^k + b^{-s-k}a^k}{\alpha^{s+2k} - 1} + O\left(\sum_{n=1}^{\infty}\alpha^{-n(\sigma+2m)}\right). \end{aligned} \tag{4}$$

Since $E_{m,n}(s)$ and $F_{m,n}(s)$ are analytic for $\sigma > -2m$, it follows by analytic continuation that (4) is valid for $\sigma > -2m$. Since m is an arbitrary positive integer, we conclude that $Z(s)$ has a meromorphic continuation to the entire complex plane.

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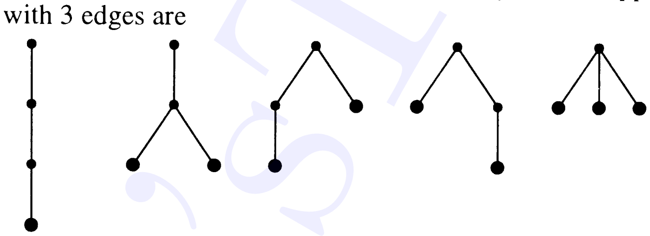
PROBLEMS

10753. *Proposed by Louis Shapiro, Howard University, Washington, DC.* An *ordered tree* is a rooted tree in which the children of each node form a sequence as opposed to a set. The 5 ordered trees with 3 edges are



The number of ordered trees with n edges is the nth Catalan number $\binom{2n}{n}/(n+1)$. Therefore, if one draws each of the ordered trees with n edges, one draws a total of $\binom{2n}{n}$ nodes. Prove that exactly half of these nodes are end-nodes (i.e., leaves with no children).

10754. *Proposed by Paul Bracken, Université de Montréal, Montréal, PQ, Canada.* Let $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, and let $\rho(s,n) = \sum_{k=n+1}^{\infty} k^{-s}$. Show that for positive integers $s \geq 2$,

$$\sum_{k=1}^{\infty} \frac{\rho(s,k)}{k} = \frac{s}{2}\zeta(s+1) - \frac{1}{2}\sum_{k=1}^{s-2} \zeta(s-k)\zeta(k+1).$$

10755. *Proposed by Jiro Fukuta, Motosu-gun, Gifu-ken, Japan.* An arbitrary circle O is drawn through vertices B and D of a convex quadrilateral $ABCD$. Let O_1 be the circle tangent to lines AB and AD and tangent to O internally at a point of O on the opposite side of line BD from A. Let O_2 be the circle tangent to lines CB and CD and tangent to O internally at a point of O on the opposite side of line BD from C. Let R_1 and R_2 be the radii of circles O_1 and O_2, respectively, and let r_1 and r_2 be the radii of the incircles of triangles ABD and CBD, respectively. Prove that the quadrilateral $ABCD$ is inscribable in a circle if and only if $r_1/R_1 + r_2/R_2 = 1$.

10756. *Proposed by Douglas Iannucci, University of the Virgin Islands, St. Thomas, VI.* Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right).$$

10757. *Proposed by Mark Kidwell, United States Naval Academy, Annapolis, MD.* Given integers $a_0, a_1, a_2, \ldots, a_n$ with $a_i \neq 0$ for $i \geq 1$, write $[a_0; a_1, a_2, \ldots, a_n]$ for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}.$$

Every positive rational number has a unique representation as $[a_0; a_1, a_2, \ldots, a_n]$ if we require that $a_0 \geq 0$, $a_i > 0$ for $1 \leq i \leq n-1$, and $a_n > 1$ (we call this the *standard representation*), but it can have other representations $[b_0; b_1, b_2, \ldots, b_m]$ if we permit negative values for some of the b_i or if we permit $b_m = 1$. For example, $11/3 = [3; 1, 2] = [3; 1, 1, 1] = [4; -3]$. Prove or disprove: If r is a positive rational number, $r = [a_0; a_1, a_2, \ldots, a_n]$ is the standard representation, and $r = [b_0; b_1, b_2, \ldots, b_m]$ is another representation, then $a_0 + a_1 + \cdots + a_n \leq |b_0| + |b_1| + \cdots + |b_m|$, with strict inequality if any of the b_i are negative.

10758. *Proposed by Mark Sapir, Vanderbilt University, Nashville, TN.* Prove that the sum of the (decimal) digits of 9^n cannot equal 9 when $n > 2$.

10759. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.* In triangle ABC, let h_a denote the altitude to the side BC and let r_a denote the exradius relative to side BC, i.e., the radius of the circle tangent to the extensions of sides AB and AC and to the side BC externally. Define h_b, h_c, r_b, and r_c correspondingly. Prove that $h_a^n r_a^n + h_b^n r_b^n + h_c^n r_c^n \leq r_a^n r_b^n + r_b^n r_c^n + r_c^n r_a^n$ for any integer n, and determine conditions for equality.

SOLUTIONS

Common Eigenvector of Commuting Matrices

10633 [1997, 975]. *Proposed by Kiran S. Kedlaya, Princeton University, Princeton, NJ.* Let S be a commuting family of n-by-n matrices over an arbitrary field. Suppose the matrices in S have a common eigenvector v, so that $Mv = \lambda_M v$ for all $M \in S$. Prove that the transposes of these matrices also have a common eigenvector with these eigenvalues, that is, a vector w satisfying $M^T w = \lambda_M w$ for all $M \in S$.

Solution by Alain Tissier, Montmermeil, France. Let K be the field. Set $\phi(M) = M - \lambda_M I$ and $\phi(S) = \{\phi(M): M \in S\}$. Thus $\phi(S)$ is a commuting family of $n \times n$ matrices over K having a common nonzero vector v such that $\phi(M)v = 0$ for all $\phi(M) \in \phi(S)$. Since $\phi(M)^T = M^T - \lambda_M I$, we have to prove only that the transposes of the matrices in $\phi(S)$ have a common nonzero vector w satisfying $\phi(M)^T w = 0$ for $\phi(M) \in \phi(S)$. Thus we may suppose that $\lambda_M = 0$ for every M.

If all matrices in S are nilpotent, then the collection of transposes is also a commuting family of nilpotent matrices. In this case there is a nonzero vector w such that $M^T w = 0$ for all $M \in S$ (section 3.3 of J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag, 1972). So we may assume that not all elements of S are nilpotent.

We proceed by induction on n. When $n = 1$ all the matrices are zero, so the conclusion is true. Take $n > 1$, and suppose the result is true for h-by-h matrices for each $h < n$. Let N

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PROBLEMS

10760. *Proposed by Bruce Reznick, University of Illinois, Urbana, IL.* A function $f\colon \mathbb{N} \to \mathbb{C}$ is *completely multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and n. Find all completely multiplicative functions f with the property that the function $F(n) = \sum_{k=1}^{n} f(k)$ is also completely multiplicative.

10761. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let G be a graph with n vertices. For each vertex v, let $f(v)$ be the maximum cardinality of an independent set of neighbors of v. Show that $\sum f(v) \le n^2/2$, where the sum is taken over all vertices of G.

10762. *Proposed by Leroy Quet, Denver, CO.* Let $x_1 = 1$, and for $m \ge 1$ let $x_{m+1} = (m + 3/2)^{-1} \sum_{k=1}^{m} x_k x_{m+1-k}$. Evaluate $\lim_{m\to\infty} x_m/x_{m+1}$.

10763. *Proposed by Jean Anglesio, Garches, France.* Let ABC be a triangle; let O be its circumcenter, H its orthocenter, I its incenter, N its Nagel point, and X, Y, Z its excenters. Let S be defined so that O is the midpoint of HS, and let T denote the midpoint of SN. It is known that the orthocenter and the nine-point center of triangle XYZ are I and O, respectively. Prove that
(a) the circumcenter of triangle XYZ is T; and
(b) the centroid of triangle XYZ is the centroid of SIN.

10764. *Proposed by Ray Redheffer, University of California, Los Angeles, CA.* Let $A = (a_{ij})$ be a real n-by-n matrix, and let x and y be real n-vectors satisfying $Ax = y$. Suppose that

$$\sum_{j\ne i} \max\{a_{ij}, 0\} < y_i \le a_{ii} + \sum_{j\ne i} \min\{a_{ij}, 0\}$$

for all $i \in \{1, 2, \ldots, n\}$. Show that $x_i > 0$ for all $i \in \{1, 2, \ldots, n\}$.

10765. *Proposed by Peter J. Ferraro, Roselle Park, NJ.* Let f_n be the nth Fibonacci number, defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \ge 1$. Fix positive integers k and n with $n \ge 2k + 1$. Prove that $\lfloor \sqrt[k]{f_n} \rfloor - \lfloor \sqrt[k]{f_{n-k}} + \sqrt[k]{f_{n-2k}} \rfloor$ is 0 unless f_n is a kth power, when it is 1.

10766. *Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let x, y, and z be nonnegative real numbers. Prove that

(a) $(x+y+z)^{x+y+z}\,x^x y^y z^z \le (x+y)^{x+y}\,(y+z)^{y+z}\,(z+x)^{z+x}$.

(b) $(x+y+z)^{(x+y+z)^2}\,x^{x^2} y^{y^2} z^{z^2} \ge (x+y)^{(x+y)^2}\,(y+z)^{(y+z)^2}\,(z+x)^{(z+x)^2}$.

SOLUTIONS

Cramer's Rule for Non-Square Matrices

10618 [1997, 768]. *Proposed by S. Lakshminarayanan, S. L. Shah, and K. Nandakumar, University of Alberta, Edmonton, Canada.* Let A be a real $m \times n$ matrix of full rank with $m < n$ and let b be a real $m \times 1$ matrix. For $1 \le i \le n$, define

$$x_i = \frac{\det(A_i^* A^T) - \det(A_i A_i^T)}{\det(AA^T)},$$

where A_i^* is obtained by replacing the ith column of A by b, and A_i is obtained by deleting the ith column of A. Show that $x = [x_1, \ldots, x_n]^T$ is a solution to the linear system $Ax = b$.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We write $A^i\langle b\rangle$ instead of A_i^* to emphasize the role of the vector b; thus $A^i\langle 0\rangle$ indicates A with its ith column zeroed out. Observe that $A_i A_i^T = A^i\langle 0\rangle A^T$, by comparing corresponding entries.

Extend A to a nonsingular $n \times n$ matrix $\binom{A}{C}$, where C is an $(n-m) \times n$ matrix whose rows form an orthonormal basis for the orthogonal complement of the row space of A. That is, each row of C has norm 1 and is orthogonal to all other rows of $\binom{A}{C}$. We have

$$\begin{pmatrix} A \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} AA^T & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^i\langle b\rangle \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} A^i\langle b\rangle A^T & M \\ 0 & I \end{pmatrix},$$

where I is the $(n-m) \times (n-m)$ identity matrix and M is some $n \times (n-m)$ matrix. By substituting these computations into the definition of x_i, canceling the nonzero factor $\det \binom{A}{C}^T$, and using the linearity of the determinant in its ith column, we obtain

$$x_i = \frac{\det\left(\binom{A^i\langle b\rangle}{C}\binom{A}{C}^T\right) - \det\left(\binom{A^i\langle 0\rangle}{C}\binom{A}{C}^T\right)}{\det\left(\binom{A}{C}\binom{A}{C}^T\right)} = \frac{\det\binom{A^i\langle b\rangle}{C} - \det\binom{A^i\langle 0\rangle}{C}}{\det\binom{A}{C}} = \frac{\det\binom{A}{C}^i\!\left\langle\binom{b}{0}\right\rangle}{\det\binom{A}{C}},$$

By Cramer's rule, x is the solution to the linear system $\binom{A}{C}x = \binom{b}{0}$, and hence x is a solution to $Ax = b$.

Solved also by J. Fuelberth & A. Gunawardena, J. H. Lindsey II, M. Sharma & P. G. Poonacha (India), WMC Problems Group, and the proposers.

An Identity for Strongly Connected Digraphs

10620 [1997, 870]. *Proposed by James Propp, Massachusetts Institute of Technology, Cambridge, MA.* A *digraph* on a vertex set V is a subset $A \subseteq \{(v, w)\colon v, w \in V, v \ne w\}$ and is *strongly connected* if it is possible to get from any vertex a to every other vertex e by a finite succession of arcs $(a, b), (b, c), \ldots, (d, e)$ in A. For $n \ge 1$, let E_n (respectively, O_n) denote the number of strongly connected digraphs on the vertex set $V = \{1, 2, \ldots, n\}$ with an even (respectively odd) number of arcs. Show that $E_n - O_n = (n-1)!$ for all $n \ge 1$.

Solution I by the proposer, currently at University of Wisconsin, Madison, WI. The terminology of the problem statement is somewhat nonstandard. In common usage, a digraph is

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PROBLEMS

10767. *Proposed by Bruce Dearden and Jerry Metzger, University of North Dakota, Grand Forks, ND.* For integers $n \geq 2$ and $m > 1$, how many invertible m-by-m matrices are there modulo n?

10768. *Proposed by Sung Soo Kim, Hanyang University, Ansan, Kyunggi, Korea.*
(**a**) Show that there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + g$ is not increasing for any differentiable function g.
(**b**) Show that there is a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + g$ is not increasing for any continuously differentiable function g.
(**c**) Show that, for any continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, there is a real analytic function g such that $f + g$ is increasing.

10769. *Proposed by Christian Blatter, Zürich, Switzerland.* Determine the minimum number of colors necessary to color the points of a sphere in such a way that points at spherical distance $\pi/2$ (i.e., points that subtend a right angle from the center of the sphere) get different colors.

10770. *Proposed by Călin Popescu, Louvain-la-Neuve, Belgium.* Suppose that m and n are integers with $1 < m < \phi(m) + n$, where $\phi(m)$ is the number of elements in $\{1, 2, \ldots, m\}$ that are relatively prime to m. Show that $\sum_{i=1}^{n}(-1)^i\binom{n}{i}i^m$ is divisible by m.

10771. *Proposed by Mowaffaq Hajja and Peter Walker, American University of Sharjah, Sharjah, U. A. E.* Evaluate $\int_0^1\int_0^1\int_0^1 \left(1 + u^2 + v^2 + w^2\right)^{-2} du\, dv\, dw$.

10772. *Proposed by William C. Waterhouse, Pennsylvania State University, University Park, PA.* For any ordered field K, one can define the derivative of a function $f\colon K \to K$ as usual by $f'(x) = \lim_{y\to x}(f(y) - f(x))/(y - x)$. Suppose that every $f\colon K \to K$ with derivative identically zero is constant. Prove that K is isomorphic to the field of real numbers.

10773. *Proposed by Jean Anglesio, Garches, France.* Let $a_0, a_1, \dots, a_k$ be positive integers. For $0 \le i \le k$, let p_i/q_i be the fraction in lowest terms with continued fraction expansion $[a_0, a_1, \dots, a_i]$. Find the continued fraction expansions of

$$\sqrt{\frac{p_k\, p_{k-1}}{q_k\, q_{k-1}}},\ \sqrt{\frac{p_k\, q_k}{p_{k-1}\, q_{k-1}}},\ \sqrt{\frac{p_k^2 + p_{k-1}^2}{q_k^2 + q_{k-1}^2}},\ \text{and}\ \sqrt{\frac{p_k^2 + q_k^2}{p_{k-1}^2 + q_{k-1}^2}}$$

in terms of $a_0, a_1, \dots, a_k$.

SOLUTIONS

Tracking the Incenters

10631 [1997, 975]. *Proposed by Greg Huber, University of Chicago, Chicago, IL.* Given a triangle T, let the *intriangle* of T be the triangle whose vertices are the points where the circle inscribed in T touches T. Given a triangle T_0, form a sequence of triangles $T_0, T_1, T_2, \dots$ in which each T_{n+1} is the intriangle of T_n. Let d_n be the distance between the incenters of T_n and T_{n+1}. Find $\lim_{n\to\infty} d_{n+1}/d_n$ when T_0 is not equilateral.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We show that $d_{n+1}/d_n \to 1/4$. Let A, B, C be the angles of a triangle, r its inradius, R its circumradius, and d the distance from its incenter to its circumcenter. Then

$$d^2 = R^2 - 2Rr \tag{1}$$

and

$$r = 4R\sin(A/2)\sin(B/2)\sin(C/2). \tag{2}$$

(H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited,* MAA, 1967). Now let A', B', C' be the angles of the intriangle of ABC (with A' on side BC, etc.). Then $A' = \pi/2 - A/2$, so

$$A' - \pi/3 = (-1/2)(A - \pi/3), \tag{3}$$

and similarly for B' and C'. From (3) we infer that triangle T_n approaches equilateral as $n \to \infty$. For the triangle T_n, with angles A_n, B_n, C_n, define $a_n = A_n - \pi/3$, $b_n = B_n - \pi/3$, $c_n = C_n - \pi/3$, and $S_n = a_n^2 + b_n^2 + c_n^2$. Then (3) implies that $S_{n+1}/S_n = 1/4$. Also, $a_n + b_n + c_n = 0$, so $(a_n + b_n + c_n)^2 = 0$, and therefore

$$S_n = -2(a_n b_n + b_n c_n + c_n a_n). \tag{4}$$

Now define $U_n = 1 - 8\sin(A_n/2)\sin(B_n/2)\sin(C_n/2)$. Using (1) and (2) and observing that $R_{n+1} = r_n$, we obtain

$$\left(\frac{d_{n+1}}{d_n}\right)^2 = \frac{R_{n+1}^2}{R_n^2}\,\frac{U_{n+1}}{U_n} = 16\sin^2(A_n/2)\sin^2(B_n/2)\sin^2(C_n/2)\,\frac{U_{n+1}}{U_n}. \tag{5}$$

Note that

$$\begin{aligned} 2\sin(A_n/2) &= 2\sin(a_n/2 + \pi/6) = \sqrt{3}\,\sin(a_n/2) + \cos(a_n/2) \\ &= 1 + \frac{\sqrt{3}}{2}\,a_n - \frac{1}{8}\,a_n^2 + O(a_n^3). \end{aligned}$$

Therefore

$$\begin{aligned} U_n &= 1 - \left(1 + \frac{\sqrt{3}}{2}\,a_n - \frac{1}{8}\,a_n^2 + \cdots\right)\left(1 + \frac{\sqrt{3}}{2}\,b_n - \frac{1}{8}\,b_n^2 + \cdots\right)\left(1 + \frac{\sqrt{3}}{2}\,c_n - \frac{1}{8}\,c_n^2 + \cdots\right) \\ &= \frac{1}{8}\,S_n - \frac{3}{4}\big(a_n b_n + b_n c_n + c_n a_n\big) + \text{ terms of degree 3 or higher} \\ &= \frac{1}{2}\,S_n + \text{ terms of degree 3 or higher,} \end{aligned}$$

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PROBLEMS

10774. *Proposed by Catalin Zara, Massachusetts Institute of Technology, Cambridge, MA.* Let $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$. Show that

$$\big(F(F(1998))\big)^2 + \big(F(F(1999))\big)^2 = F(F(1997))F(F(2000)).$$

10775. *Proposed by Hadi Salmasian, Sharif University of Technology, Tehran, Iran.* Suppose that G is a finite group with n elements, let m be a natural number, and define $\Gamma(G) = \sum_{g \in G} o(g)^{-m}$, where $o(g)$ denotes the order of g. Prove that $\Gamma(G) \geq \Gamma(\mathbb{Z}_n)$ with equality if and only if G is isomorphic to $\mathbb{Z}_n$.

10776. *Proposed by Yongge Tian, Concordia University, Montreal, PQ, Canada.* Suppose that A is a real m-by-n matrix. Determine the minimum rank of $A + iB$, where B ranges over all real m-by-n matrices.

10777. *Proposed by Zafar Ahmed, Bhabha Atomic Research Centre, Mumbai, India.* For nonnegative integers m and n, evaluate

$$\int_0^\infty \frac{d^m}{dx^m}\left(\frac{1}{1+x^2}\right)\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right)dx.$$

10778. *Proposed by Paul Bateman, University of Illinois, Urbana, IL, and Dennis Eichhorn, University of Arizona, Tucson, AZ.* Let k be a fixed positive integer. For each integer n, let $r_k(n)$ denote the number of solutions of $i_1^2 + i_2^2 + \cdots + i_k^2 = n$ in integers $i_1, i_2, \ldots, i_k$. Let d_k be the greatest common divisor of the infinite sequence of integers $r_k(1), r_k(2), r_k(3), \ldots$.
(**a**) Evaluate d_k.
(**b**) For each k, find the smallest positive integer m_k such that d_k is the greatest common divisor of the finite list of integers $r_k(1), r_k(2), \ldots, r_k(m_k)$.

10779. *Proposed by Andrei Jorza, "Moise Nicoara" High School, Arad, Romania.* Let $P(z) = \sum_{i=0}^n a_i z^i$ with $a_i \in \mathbb{C}$ for each $i \in \{0, 1, \ldots, n\}$. Prove that there is a $z \in \mathbb{C}$ with $|z| = 1$ and $|P(z)| \geq |a_0| + \max_{1 \leq k \leq n} |a_k| / \lfloor n/k \rfloor$.

10780.* *Proposed by Kiran Kedlaya, Massachusetts Institute of Technology, Cambridge, MA.* Let T be a triangle. Two circles in T are called *partners* if they are the incircles of two triangles with disjoint interior whose union is T. Every circle tangent to exactly two sides of T has two partners. Let $C_1, C_2, \ldots, C_6$ be disjoint circles such that C_i and C_{i+1} are partners for each $i \in \{1, 2, 3, 4, 5\}$. Show that C_6 and C_1 are partners.

SOLUTIONS

Elliptic Curves to the Rescue

10612 [1997, 665]. *Proposed by John P. Robertson, Anistics/Aon, New York, NY.* Fermat proved that there are no nontrivial 4-term arithmetic progressions all of whose terms are integer squares.

(a) Find all 5-term arithmetic progressions such that all terms but the fourth are squares.

(b) Call two arithmetic progressions *essentially different* if the ratios of corresponding terms differ. For each integer $m \geq 6$, show that there are infinitely many essentially different m-term arithmetic progressions such that the first 3 terms and the mth term are squares.

Solution by the proposer. **(a)** Let two 3-term arithmetic progressions of rational squares be *equivalent* if one is a nonzero rational multiple of the other. Each equivalence class other than $\{(0,0,0)\}$ contains exactly one progression consisting of pairwise relatively prime integers. To see this, we first multiply by the denominators to obtain an integer progression with difference d. If two terms have a common odd prime factor p, then p divides their difference, which is d or $2d$. In either case, p divides the difference of consecutive terms. Thus p and p^2 divide all three terms. If two consecutive terms are even, then the remaining term is even and a factor of 4 can be removed. If the first and third terms are even but the second is not, then modulo 4 we obtain $(0, 1, 0)$, which is not an arithmetic progression.

Call rational numbers s and t *equivalent* if $\{s, t\} \subseteq \{1, -1, \infty, 0\}$ or $s \in \{t, -1/t, (t+1)/(t-1), (1-t)/(1+t)\}$. Reflexivity is obvious; symmetry and transitivity are easily checked by cases.

We first establish a bijection between the set of equivalence classes of 3-term arithmetic progresssions and the set of equivalence classes of rational numbers.

If (a^2, b^2, c^2) is an increasing progression with a, b, c positive and pairwise relatively prime, then there are relatively prime positive integers p and q, with $p > q$ and pq even, such that $(a^2, b^2, c^2) = \big((p^2 - q^2 - 2pq)^2, (p^2 + q^2)^2, (p^2 - q^2 + 2pq)^2\big)$. This follows because $c^2 - b^2 = b^2 - a^2$ implies that c and a have the same parity, and thus $\big((c-a)/2, (c+a)/2, b\big)$ is a Pythagorean triple of pairwise relatively prime integers. Hence there exist p, q as described with $\{(c-a)/2, (c+a)/2\} = \{2pq, p^2 - q^2\}$ and $b = p^2 + q^2$. Note that setting $t = p/q$ yields $\big((t^2 - 1 - 2t)^2, (t^2 + 1)^2, (t^2 - 1 + 2t)^2\big) = (a^2, b^2, c^2)/q^4$, so these are equivalent progressions.

Similarly, if (a^2, b^2, c^2) is a decreasing progression with a, b, c positive and pairwise relatively prime, then there are relatively prime positive integers p and q, with $p > q$ and pq even, such that $(a^2, b^2, c^2) = \big((q^2 - p^2 - 2pq)^2, (p^2 + q^2)^2, (q^2 - p^2 + 2pq)^2\big)$. Setting $t = q/p$ yields $\big((t^2 - 1 - 2t)^2, (t^2 + 1)^2, (t^2 - 1 + 2t)^2\big) = (a^2, b^2, c^2)/p^4$, and again these are equivalent progressions.

With $t = 0$, we have $(1^2, 1^2, 1^2) = \big((t^2 - 1 - 2t)^2, (t^2 + 1)^2, (t^2 - 1 + 2t)^2\big)$.

Algebraic manipulation shows that under the map sending s to $\big((s^2 - 1 - 2s)^2, (s^2 + 1)^2, (s^2 - 1 + 2s)^2\big)$, the four rational numbers t, $-1/t$, $(t+1)/(t-1)$, and $(1-t)/(1+t)$ yield equivalent progressions.

Conversely, we claim that if progressions $\big((t^2 - 1 - 2t)^2, (t^2 + 1)^2, (t^2 - 1 + 2t)^2\big)$ and $\big((s^2 - 1 - 2s)^2, (s^2 + 1)^2, (s^2 - 1 + 2s)^2\big)$ are equivalent, then s and t are equivalent.

PROBLEMS AND SOLUTIONS

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PROBLEMS

10781. *Proposed by Leonard Smiley, University of Alaska, Anchorage, AK.* Prove that

$$\sum_{i=2}^{n}\binom{n}{i}i^{i-1}(n-i)^{n-i}=\sum_{i=2}^{n}\binom{n}{i-1}(i-1)^{i-1}(n-i)^{n-i},$$

where 0^0 is taken to be 1.

10782. *Proposed by Douglas Iannucci, University of the Virgin Islands, St. Thomas, VI.* Let r and s be fixed positive integers. For $n \geq 1$, let $P(r,s,n)$ be the probability that $\gcd(a_1, a_2, \ldots, a_r)$ is divisible by $\gcd(b_1, b_2, \ldots, b_s)$, where the a_i and b_j are randomly chosen integers from $\{1, 2, \ldots, n\}$. Prove that $\lim_{n\to\infty} P(r,s,n)$ exists and evaluate it.

10783. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou, China.* Let $ABCD$ be a cyclic quadrilateral such that AD is not parallel to BC. Given points E and F on the line CD, let G and H be the circumcenters of BCE and ADF. Prove that the lines AB, CD, and GH are concurrent or parallel if and only if there is a circle through A, B, E, and F.

10784. *Proposed by Alberto Facchini and Francesco Barioli, University of Padova, Padova, Italy.*
(**a**) Let F be a field, let m and n be positive integers, and let $B_1, B_2, \ldots, B_m$ be n-by-n matrices with entries in F. Suppose that $\sum_{i=1}^{m} B_i$ is nonsingular. Prove that there exists a subset $S \subset \{1, 2, \ldots, m\}$ with $|S| \leq n$ such that $\sum_{i\in S} B_i$ is nonsingular.
(**b**)* Is the same result true if F is merely a division ring? (A matrix over a division ring F is *nonsingular* if it is invertible in the ring of n-by-n matrices over F.)

10785. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Let $f_0(z) = 1$, and suppose that

$$\big(f_0(z)+f_1(z)+\cdots+f_{n-1}(z)\big)\big(f_{n-1}(z)+f_n(z)\big)z=f_n(z)$$

when $n \geq 1$. Find a formula for $f_n(z)$ that depends only on n and z.

10786. *Proposed by Leroy Quet, Denver, CO.* Let $\zeta(r)$ be the Riemann zeta function $\sum_{k=1}^{\infty} 1/k^r$. Show that

$$(r-1)^2\zeta(r) = \sum_{m=1}^{\infty}(rm+1)\binom{r+m-1}{m+1}(\zeta(m+r)-1)$$

for every integer $r \geq 2$.

10787. *Proposed by Juan Arias-de-Reyna, University of Seville, Seville, Spain.* What is the expected value of $\left(\det(v_1, v_2, \ldots, v_n)\right)^2$, if the vectors $v_1, v_2, \ldots, v_n$ are chosen independently at random from
(**a**) the unit cube $[0,1]^n$ in $\mathbb{R}^n$?
(**b**) the cube $[-1,1]^n$ in $\mathbb{R}^n$?
(**c**) the unit ball $\{(x_1,\ldots,x_n)\colon \sum_{i=1}^n x_i^2 \leq 1\}$ in $\mathbb{R}^n$?
(**d**) the generalized octahedron $\{(x_1,\ldots,x_n)\colon \sum_{i=1}^n |x_i| \leq 1\}$ in $\mathbb{R}^n$?

SOLUTIONS

On the Intersection of $\mathbb{Z}^n$ with a Hyperplane

10639 [1998, 69]. *Proposed by Warren Koepp, Texas A&M University, Commerce, TX.* Let n be a positive integer, and choose $v \in \mathbb{C}^n$. Let $H_v = \{\alpha \in \mathbb{Z}^n\colon\ v\cdot\alpha = 0\}$ denote the intersection of the hyperplane normal to v in $\mathbb{C}^n$ with the n-dimensional integer lattice.
(**a**) Find the rank of H_v as a (free abelian) subgroup of the additive group $\mathbb{Z}^n$, in terms of the coordinates of v.
(**b**) Choose $v_1,\ldots,v_k \in \mathbb{C}^n$, and let $H = \{\alpha \in \mathbb{Z}^n\colon\ v_i\cdot\alpha = 0 \text{ for all } i = 1,\ldots,k\}$ denote the intersection of the groups H_{v_i}. Show that there exists a vector $v \in \mathbb{C}v_1 + \cdots + \mathbb{C}v_k$ such that $H = H_v$.

Solution by Robin Chapman, University of Exeter, Exeter, U.K.
(**a**) The rank of H_v is $n - d_v$, where d_v is the dimension over $\mathbb{Q}$ of the subspace of $\mathbb{C}$ spanned by the entries of v. To see this, note that the rank of H_v is the same as the dimension of $K_v = \{\alpha \in \mathbb{Q}^n\colon\ v\cdot\alpha = 0\}$, since each element of K_v has a nonzero integral multiple in $K_v \cap \mathbb{Z}^n = H_v$. Also K_v is the kernel of the map $\phi\colon\ \mathbb{Q}^n \to \mathbb{C}$ given by $\phi(\alpha) = \alpha\cdot v$. Since the dimension of the image of ϕ is d_v, the dimension of K_v as a vector space over $\mathbb{Q}$ is $n - d_v$.
(**b**) We use induction to reduce to the case $k = 2$. Let L_v be the image of $\mathbb{Q}^n$ under ϕ. We show that there exists $t \in \mathbb{C}$ such that $L_{v_1} \cap tL_{v_2} = \{0\}$. If no such t exists, then for every t there exist $u_1 \in L_{v_1}$ and $u_2 \in L_{v_2}$ such that $t = u_1/u_2$. Since L_{v_1}, L_{v_2} are countable and $\mathbb{C}$ is not, this is impossible, so the claimed t exists.

Now let $v = v_1 + tv_2$. We claim that $H_v = H_{v_1} \cap H_{v_2}$. Certainly $H_v \supseteq H_{v_1} \cap H_{v_2}$. For $\alpha \in H_v$, we have $0 = \alpha\cdot v_1 + t\alpha\cdot v_2$. As $\alpha\cdot v_1 \in L_{v_1}$ and $-t\alpha\cdot v_2 \in tL_{v_2}$, both vanish. Thus $\alpha \in H_{v_1} \cap H_{v_2}$, and the proof is complete.

Solved also by J. H. Lindsey II, NSA Problems Group, and the proposer.

When a Multiple of $\pi/2$ is Close to an Integer

10640 [1998, 69]. *Proposed by Michael A. Filaseta, University of South Carolina, Columbia, SC.* Observe that $(\pi/2)b^2\sin(1/b) < \pi b/2$ for every positive integer b. Determine the six smallest positive integers satisfying $(\pi/2)b^2\sin(1/b) < \lfloor \pi b/2 \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

Solution by Allen Stenger, Tustin, CA. Since $(\pi/2)b^2\sin(1/b) - \pi b/2 \to 0$ as $b \to \infty$, we seek b so that $\pi b/2$ is slightly larger than an integer, so slightly that $(\pi/2)b^2\sin(1/b)$ is

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PROBLEMS

10788. *Proposed by Howard M. Taylor, Towson, MD.* Imagine a random walk on the nonnegative integers that begins at 1 and that takes steps according to the following rule: When located at n, the next location is chosen uniformly from $\{0, 1, \ldots, n, n+1\}$. The walk ends when it first arrives at 0.
(**a**) What is the expected number of steps in the walk?
(**b**) What is the probability that the final step of the walk is from 1 to 0?
(**c**) For $m \in \mathbb{N}$, what is the probability that the walk never exceeds m?

10789. *Proposed by Robin Chapman, University of Exeter, Exeter, U. K.* The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are defined by $x/(e^x - 1) = \sum_{k=0}^{\infty} B_k x^k/k!$. Show that

$$B_{2m} = \sum_{i=1}^{m} \frac{(-1)^{i-1}\, m!\,(m+1)!}{i(i+1)\,(m-i)!\,(m+i+1)!} \sum_{k=1}^{i} k^{2m}$$

for each positive integer m.

10790. *Proposed by Jean Anglesio, Garches, France.* Given a real number x, let T_0 be the triangle whose vertices are $(0, 0)$, $(1, x)$, and $(1, -x)$. For $n \geq 1$, let T_n be the orthic triangle of T_{n-1}, the triangle whose vertices are the feet of the altitudes of T_{n-1}. Denote by $(0, u_n)$ the vertex of T_n that is on the x-axis, and let $f(x) = \lim_{n\to\infty} u_n$. Show that $f(x)$ exists for every x and that f is a continuous but nowhere differentiable function of x.

10791. *Proposed by Antal Fekete, Memorial University of Newfoundland, St. John's, NF, Canada.* Show that

$$\left(\sum_{i=0}^{\infty} \frac{(2i+1)^n}{(2i+1)!}\right)^2 - \left(\sum_{i=0}^{\infty} \frac{(2i)^n}{(2i)!}\right)^2 \quad \text{and} \quad \left(\sum_{i=0}^{\infty} (-1)^i \frac{(2i+1)^n}{(2i+1)!}\right)^2 + \left(\sum_{i=0}^{\infty} (-1)^i \frac{(2i)^n}{(2i)!}\right)^2$$

are integers for every nonnegative integer n.

10792. *Proposed by Harold G. Diamond, University of Illinois, Urbana, IL, and Ferrell S. Wheeler, Center for Computing Sciences, Bowie, MD.* Let $\rho\colon \mathbb{R} \to \mathbb{R}$ denote the *Dickman function*: $\rho(u) = 0$ for $u < 0$, $\rho(u) = 1$ for $0 \le u \le 1$, and $\rho(u)$ is the continuous solution of the differential equation $u\rho'(u) = -\rho(u-1)$ for $u > 1$. Show that

$$\rho(v) = \rho(u) + \sum_{j<u} \frac{1}{j!} \int_1^{u/v} \cdots \int_1^{u/v} \rho\left(u - \sum_{i=1}^{j} s_i\right) \frac{ds_1}{s_1} \cdots \frac{ds_j}{s_j}$$

for $u > v > 0$.

10793. *Proposed by Florian Luca, Czech Academy of Sciences, Prague, Czech Republic.* Let $\sigma(n)$ be the sum of the divisors of the positive integer n, and let $\phi(n)$ be the number of positive integers that are less than n and relatively prime to n. Two positive integers m and n are *amicable* if $\sigma(m) = \sigma(n) = m + n$.
(a) Show that if a is a positive integer, then a and $\phi(a)$ are not amicable.
(b) Show that if a and b are positive integers with $b > 1$, then a and $(2^b - 1)a + 1$ are not amicable.
(c)* Find all positive integers a such that a and $a + 1$ are amicable.

10794. *Proposed by David S. Hough and Rodica E. Simion, The George Washington University, Washington, DC.* Let $F_{s,n} = \binom{sn}{n} / \big((s-1)n + 1\big)$. When $s = 2$, these are the Catalan numbers.
(a) When s is prime, for what values of n is $F_{s,n}$ divisible by s?
(b)* For what values of n is $F_{4,n}$ divisible by 4?
(c)* What can you say when s takes on other composite values?

SOLUTIONS

An Identity Involving Derangements

10643 [1998, 175]. *Proposed by David Callan, University of Wisconsin, Madison, WI.* Let $D_n = n! \sum_{j=0}^{n} (-1)^j / j!$ denote the nth derangement number, the number of permutations on n letters without fixed points. Show that for nonnegative integers n and k,

$$\sum_{j=0}^{k} \binom{k}{j} D_{k+n-j} = k! \sum_{j=0}^{\min\{n,k\}} \binom{k}{j} \binom{k+n-j}{k} D_{n-j}.$$

Solution I by Knut Dale and Ivar Skau, Telemark College, Bø, Norway. Let $A = \{1, \ldots, k\}$ and $B = \{k+1, \ldots, k+n\}$; we show that both sides of the identity count the set P of permutations of $A \cup B$ with no fixed point in B. Let $D_{n,k} = |P|$.

On the left side, we group P by the number of fixed points. There are $\binom{k}{j}$ ways of choosing j fixed points in A and D_{k+n-j} ways to define the permutation on the remaining points. Thus $D_{n,k} = \sum_{j=0}^{k} \binom{k}{j} D_{k+n-j}$.

The right side also counts P. Let $m = \min\{k, n\}$. Each $\pi \in P$ swaps some $A' \subseteq A$ with some $B' \subseteq B$, mapping $A - A'$ to itself and $B - B'$ to itself. Let $r = |A'| = |B'|$; we have $0 \le r \le m$. To form such a permutation, we choose A' and B' and swap them (temporarily leaving them in order), permute the resulting k elements of the "new" A arbitarily, and then permute the "new" B without leaving any of $B - B'$ fixed. For fixed r, we can do these steps in $\binom{k}{r}\binom{n}{r} k! D_{n-r,r}$ ways. Our previous formula for $D_{n-r,r}$ yields

$$D_{n,k} = k! \sum_{r=0}^{m} \binom{k}{r} \binom{n}{r} \sum_{j=0}^{r} \binom{r}{j} D_{n-j}.$$

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PROBLEMS

10795. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* A *3-dimensional lattice walk* of length n takes n successive unit steps, each in one of the six coordinate directions. How many 3-dimensional lattice walks of length n are there that begin at the origin and never go below the horizontal plane?

10796. *Proposed by Floor van Lamoen, Goes, The Netherlands.* Let ABC be a triangle, and let the feet of the altitudes dropped from A, B, C be A', B', C', respectively. Show that the Euler lines of triangles $AB'C'$, $A'BC'$, and $A'B'C$ concur at a point on the nine-point circle of ABC.

10797. *Proposed by Paul Bateman, University of Illinois, Urbana, IL, and Jeffrey Kalb, Phoenix, AZ.* Let h and k be integers with $k > 0$, $h + k > 0$, and $\gcd(h, k) = 1$. For $n \geq 1$, let $L(n)$ be the least common multiple of the n numbers $h + k, h + 2k, h + 3k, \ldots, h + nk$. Prove that

$$\lim_{n\to\infty} \frac{\log L(n)}{n} = \frac{k}{\phi(k)} \sum_{\substack{1\leq m\leq k \\ \gcd(m,k)=1}} \frac{1}{m},$$

where $\phi(k)$ is the number of integers between 1 and k that are relatively prime to k.

10798. *Proposed by Edward Neuman, Southern Illinois University, Carbondale, IL.* Given positive real numbers x and y, let A be their arithmetic mean, let G be their geometric mean, and let $L = (y - x)/(\ln y - \ln x)$ be their logarithmic mean. Prove that $A^L < G^A$ if both x and y are at least $e^{3/2}$ and that $A^L > G^A$ if both x and y are at most $e^{3/2}$.

10799. *Proposed by Curtis Herink, Mercer University, Macon, GA, and Gary Gruenhage, Auburn University, Auburn, AL.* Let κ and λ be infinite cardinals with $\kappa > \lambda$. Let X be a topological space with at least κ open sets. Show that if every open cover of X containing exactly κ open sets has a finite subcover, then every open cover of X containing exactly λ open sets has a finite subcover.

10800. *Proposed by Douglas Iannucci, University of the Virgin Islands, St. Thomas, VI.* A positive integer n is *triperfect* if the sum of its divisors is $3n$. An odd triperfect number must be a square. Prove that the square root of an odd triperfect number cannot be square-free.

10801. *Proposed by Paul R. Pudaite, Glen Ellyn, IL.* Consider the following game played by a gambler against a casino dealer: At the start of the game, the dealer places $n+1$ green balls and n red balls into a bowl. The balls are to be drawn one at a time from the bowl without replacement. The game ends when the bowl is empty. The gambler begins the game with a bankroll of 1 unit of (infinitely divisible) money. Before each ball is drawn, the gambler declares how much he bets; he may choose to bet any amount from 0 up to his entire bankroll at that point. After the gambler declares the size of his wager, the dealer chooses a ball from the bowl (not necessarily at random). If a green ball is drawn, the gambler wins an amount equal to his bet; if a red ball is drawn, he loses his bet. The gambler seeks to maximize his bankroll at the end of the game, while the dealer seeks to minimize the gambler's final bankroll. What is the gambler's final bankroll, assuming optimal play by both gambler and dealer?

SOLUTIONS

Pairs with Equal Squares

10654 [1998, 272]. *Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA.*
(**a**) Let S_n be the symmetric group on n letters, and let $f(n)$ be the number of pairs $(u, v) \in S_n \times S_n$ such that $u^2 = v^2$. Show that $f(n) = p(n)n!$, where $p(n)$ denotes the number of partitions of n.
(**b**) Generalize to other finite groups.

Solution by Richard Ehrenborg, Cornell University, Ithaca, NY. In general, the number of solutions is the order of the group G times the number of conjugacy classes C such that $C^{-1} = C$. For a solution pair (u, v), we rewrite the identity $u^2 = v^2$ as $uv^{-1} = u^{-1}v = u^{-1}vu^{-1}u = u^{-1}(uv^{-1})^{-1}u$. Hence the element $w = uv^{-1}$ is conjugate to its inverse. To obtain a solution pair, we first choose a conjugacy class C such that $C^{-1} = C$. We can choose the element w in $|C|$ ways; note that w^{-1} also belongs to C. Since G acts transitively on C by conjugation, there are $|G| / |C|$ ways to choose u such that $w = u^{-1}w^{-1}u$. Letting $v = uw^{-1}$ completes the desired pair. Thus we obtain $|G|$ solution pairs for each such conjugacy class. In the symmetric group S_n, the conjugacy classes are given by the cycle structures. A permutation and its inverse have the same cycle structure, so each conjugacy class is self-inverse. The number of conjugacy classes is the number of cycle structures, which is the number of partitions of n.

Editorial comment. The proposer noted that character theory can also be used, and Stephen Gagola took this approach.

Solved also by R. J. Chapman (U. K.), S. M. Gagola, J. H Lindsey II, and the proposer.

Another Type of Lattice Path

10658 [1998, 366]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Consider walks on the integer lattice in the plane that start at $(0, 0)$, that stay in the first quadrant (they may touch the x-axis), and such that each step is either $(2, 1)$, $(1, 2)$, or $(1, -1)$. For each nonnegative integer n, how many paths are there to $(3n, 0)$?

PROBLEMS AND SOLUTIONS

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PROBLEMS

10802. *Proposed by Doru Caragea and Viviana Ene, Constanța, Romania.* Let S be the set of monic irreducible polynomials with degree 2000 and integer coefficients. Find all $P \in S$ such that $P(a)|P(a^2)$ for every natural number a.

10803. *Proposed by Stephen Penrice, Morristown, NJ.* Let k and n be positive integers such that $k \leq n$. Consider the following method for generating a permutation π of the integers $\{1, 2, \ldots, n\}$. The values $\pi(1), \pi(2), \ldots, \pi(k)$ are determined by randomly selecting a list of k distinct integers from $\{1, 2, \ldots, n\}$, with all $n!/(n-k)!$ such lists equally likely. The remaining values are then assigned so that $\pi(k+1) < \pi(k+2) < \cdots < \pi(n)$. What is the expected value of the random variable $X_i = \pi^{-1}(i)$ for each i with $1 \leq i \leq n$? (From 1987 to 1989, the National Basketball Association used this method with $k = 3$ and $n = 7$ to determine the drafting order for teams that did not participate in playoff competition.)

10804. *Proposed by Achilleas Sinefakopoulos, University of Athens, Athens, Greece.* Let $ABCD$ be a convex quadrilateral with an incircle that contacts AB at E and CD at F. Show that $ABCD$ has a circumcircle if and only if $AE/EB = DF/FC$.

10805. *Proposed by Antal Fekete, Memorial University of Newfoundland, St. John's, NF, Canada.* Let B_n be the nth Bell number, the number of partitions of $\{1, 2, \ldots, n\}$. Let $\left[{n \atop k}\right]$ be the unsigned Stirling number of the first kind, the number of permutations of $\{1, 2, \ldots, n\}$ with k cycles. Prove that

$$\sum_{j=0}^{n-1} (-1)^j \left[{n \atop n-j}\right] B_{n-j+h} = \sum_{j=0}^{h} \binom{h}{j} B_j n^{h-j}$$

for each positive integer n and nonnegative integer h.

10806. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.* Prove that a complex number with real part $\sqrt{k+1} - \sqrt{k}$ for some positive integer k cannot be a root of unity.

10807. *Proposed by Marc Deléglise, Université Lyon, Lyon, France.* For positive parameters u and v, evaluate

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sqrt{1 + 4^n u^{2k} v^{2n-2k}}.$$

10808. *Proposed by Enrico Valdinoci, University of Texas, Austin, TX.* Prove that the series $\sum_{n=0}^{\infty} (\cos(nx))^{n^r}$ diverges for all $x \in \mathbb{R}$ if $r \le 2$ but converges for almost every $x \in \mathbb{R}$ with respect to Lebesgue measure if $r > 2$.

SOLUTIONS

Intersecting Curves

10712 [1999, 166]. *Proposed by Paul Deiermann, Lindenwood University, St. Charles, MO, and Rick Mabry, Louisiana State University, Shreveport, LA.* Let $f(x)$ and $g(y)$ be twice continuously differentiable functions defined in a neighborhood of 0, and assume that $f(0) = 1$, $g(0) = f'(0) = g'(0) = 0$, $f''(0) < 0$, and $g''(0) > 0$.
(a) For sufficiently small $r > 0$, show that the curves $x = g(y)$ and $y = rf(x/r)$ have a common point (x_r, y_r) in the first quadrant with the property that, if (x, y) is any other common point, then $x_r < x$.
(b) Let $(t_r, 0)$ denote the x-intercept of the line passing through $(0, r)$ and (x_r, y_r). Show that $\lim_{r\to 0+} t_r$ exists, and evaluate it.
(c) Is the continuity of f'' and g'' a necessary condition for $\lim_{r\to 0+} t_r$ to exist?

Solution by Alain Tissier, Montfermeil, France. The conclusions in **(a)** and **(b)** remain correct even if we do not assume continuity of f'' and g''. We retain only the continuity of the first derivative and the existence and sign of f'' and g'' at zero. We prove a generalization, weakening the hypotheses as follows: Assume that f is a continuous mapping on $[0, a]$ with $a > 0$ and that $f(x) = 1 - \lambda x^p + o(x^p)$ as $x \to 0$ for some $p > 0$ and $\lambda > 0$. Assume also that g is a continuous mapping on $[0, b]$ with $b > 0$ and that $g(y) = \mu y^q + o(y^q)$ as $y \to 0$ for some $q > 1$ and $\mu > 0$. The conditions on f and g in the problem statement imply these hypotheses with $p = q = 2$, $\lambda = -f''(0)/2$, and $\mu = g''(0)/2$.

(a) With a and b sufficiently small, we may suppose $f(x) > 0$ on $[0, a]$ and $g(y) > 0$ on $(0, b]$. Let $m > 0$ be the maximum of $f(x)$ on $[0, a]$. For each $r > 0$, let $f_r(x) = rf(x/r)$. Then f_r is a continuous mapping on $[0, ra]$, $f_r(x) = r - \lambda r^{1-p} x^p + o(x^p)$, and the maximum of f_r on $[0, ra]$ is mr. Assume that $r \le b/m$. Then $f_r(x) \le b$ on $[0, ra]$.

The function h_r defined by $h_r(x) = g(f_r(x)) - x$ is defined and continuous on $[0, ra]$, and it satisfies $h_r(0) = g(r) > 0$ and $h_r(ra) = g(rf(a)) - ra$. Since $g(rf(a)) = O(r^q)$ as $r \to 0$ and since $q > 1$, we have $g(rf(a)) = o(r)$. Hence there exists $\delta > 0$ so that $h_r(ra) \le 0$ if $r < \delta$. Assume that $r \le \delta$. The function h_r is continuous on $[0, ra]$, $h_r(0) > 0$, and $h_r(ra) \le 0$, so by the intermediate value theorem there exists $x_r > 0$ such that $h_r(x_r) = 0$ and $h_r(x) > 0$ on $[0, x_r)$. The curves $y = f_r(x)$ and $x = g(y)$ have a common point (x_r, y_r) with $y_r = f_r(x_r)$ and $x_r = g(y_r)$, and every other common point has a larger x-coordinate.

(b) We show that, in our more general setting, a finite nonzero limit exists if and only if $1/p + 1/q = 1$, and then the limit is $1/(\lambda\mu^{p-1})$. Since $0 < x_r \le ra$, we have $x_r = O(r)$ as $r \to 0$. Hence $y_r = r - \lambda r^{1-p} x_r^p + o(x_r^p) = O(r)$ as $r \to 0$. We may use this to obtain $x_r = g(y_r) = \mu y_r^q + o(y_r^q) = O(r^q)$ and $y_r = r - \lambda r^{1-p} x_r^p + o(x_r^p) = r + O(r^{1-p+pq})$ as $r \to 0$. This in turn leads to the further refinement $x_r = \mu r^q + o(r^q)$ and $y_r =$

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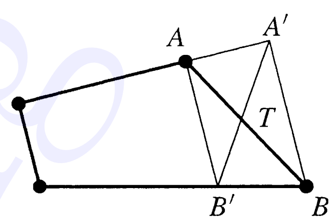
PROBLEMS

10809. *Proposed by David Beckwith, Sag Harbor, NY.* For $|x| < 1$, prove that

$$\sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{1-x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}\,.$$

10810. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.* Consider a convex quadrilateral with no parallel sides. On each side AB, select a point T as follows: Draw lines from A and B parallel to the opposite side. Let A' and B' be the new points where these lines intersect the sides neighboring AB. Let T be the point where AB intersects $A'B'$. Prove that the four points selected in this way are the corners of a parallelogram.



10811. *Proposed by Phil Tracy, Liverpool, NY.* Let G be a simple graph whose longest path has ends x and y and has length l. Let s be the sum of the degrees of x and y. Show that the distance from x to y (the length of the shortest path from x to y) is at most $\max\{l-s+2, 2\}$.

10812. *Proposed by Yehuda Pinchover and Simeon Reich, The Technion, Haifa, Israel.*
(a) Let $(V, |\cdot|)$ be a normed space. For $C = 3\sqrt{3}$, prove that the function $\rho(x, y) = |x-y|/\big(|x|\,|y|\big)$ satisfies the inequality

$$\rho(x, y) \le C\,(\rho(x, z) + \rho(z, y)) \quad \text{for all} \quad x, y, z \in V \setminus \{0\}. \qquad (*)$$

(b)* Find the smallest constant C such that $(*)$ holds for all normed spaces.

10813. *Proposed by Fred Richman, Florida Atlantic University, Boca Raton, FL.* Let F be an arbitrary field, and let V be the vector space of 2-by-2 matrices over F. Given A and B in V, let $S_{A,B} = \{C\colon AC = CB\}$. Show that the vector space $S_{A,B}$ cannot be 3-dimensional, but that every 2-dimensional subspace of V is $S_{A,B}$ for some A and B.

10814. *Proposed by Razvan Satnoianu, Oxford University, Oxford, United Kingdom.* Let P be a point in the interior of triangle ABC. Let r, s, t be the distances from P to the vertices A, B, C, respectively, and let x, y, z be the distances from P to the sides BC, CA, AB, respectively.
(**a**) Prove that $q^r + q^s + q^t + 3 \geq 2(q^x + q^y + q^z)$ for any $q \geq 1$.
(**b**) Prove that $q^{s+t} + q^{t+r} + q^{r+s} + 6 \geq q^{2x} + q^{2y} + q^{2z} + 2(q^x + q^y + q^z)$ for any $q \geq 1$.

10815. *Proposed by Barbara S. Bertram and Otto G. Ruehr, Michigan Technological University, Houghton, MI.* Let

$$F(x) = 2x \sum_{n=1}^{\infty} n e^{-xn^2}$$

for $x > 0$. Show that $F(s+t) < F(s)F(t)$ when $s, t > 0$.

SOLUTIONS

Permutation Parameters with the Same Distribution

10634 [1998, 68]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY, and Ira M. Gessel, Brandeis University, Waltham, MA.* For each permutation π of $\{1, 2, \ldots, n\}$, define

$$\begin{aligned} \operatorname{maxjump}(\pi) &= \max_{1\leq i\leq n} (\pi_i - i), \\ \operatorname{maxinv}(\pi) &= \max_{1\leq i\leq n} |\{k : \pi_k > \pi_i, k < i\}|, \text{ and} \\ \operatorname{maxrise}(\pi) &= \begin{cases} 0 & \text{if } \pi = (n\; n-1\; n-2\; \ldots\; 3\; 2\; 1), \\ \max_{1\leq i\leq n-1}(\pi_{i+1} - \pi_i) & \text{otherwise.} \end{cases} \end{aligned}$$

Show that these parameters have the same distribution.

Solution by David Callan, Madison, WI. We first show that maxjump and maxrise have the same distribution. The *standard cycle form* of a permutation lists its cycles in order so that the smallest element in each cycle occurs first and these smallest elements are in decreasing order. For example, (46)(3)(1752) is the standard cycle form for the permutation whose *word form* is 7136245. Let $\hat{\pi}$ be the permutation whose word form is obtained by writing π in standard cycle form and erasing the parentheses (4631752 in our example). The cycles of π can easily be recovered from $\hat{\pi}$; they start wherever an entry is smaller than all preceding entries. Hence $\pi \mapsto \hat{\pi}$ defines a bijection. If $\pi_i > i$, then π_i immediately follows i in $\hat{\pi}$. Conversely, if $i = \hat{\pi}_j < \hat{\pi}_{j+1}$, then $\pi_i = \hat{\pi}_{j+1}$. Thus the positive jumps of π are the same as the positive rises of $\hat{\pi}$, and $\operatorname{maxjump}(\pi) = \operatorname{maxrise}(\hat{\pi})$.

We now prove that $\operatorname{maxjump}(\pi) = \operatorname{maxinv}(\pi^{-1})$ and hence that maxjump and maxinv have the same distribution. When $j = \pi_i > i$, we seek inversions at j in π^{-1}. The values of k less than j that satisfy $\pi_k^{-1} > \pi_j^{-1}$ all belong to $\{\pi_{i+1}, \ldots, \pi_n\}$. At most $i - 1$ of the numbers less than j appear in $\{\pi_1, \ldots, \pi_{i-1}\}$, and hence at least $(j-1) - (i-1)$ of them appear later (strict inequality may hold, as when $\pi = 7136245$ and $i = 4$). For each positive jump $\pi_i = j$, there are thus at least $j - i$ inversions at j in π^{-1}, and hence $\operatorname{maxjump}(\pi) \leq \operatorname{maxinv}(\pi^{-1})$.

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PROBLEMS

10816. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* A *Motzkin path* of length n is a lattice path from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,0)$, and $(1,-1)$ that never goes below the x-axis. For $n \geq 2$, show that the number of Motzkin paths of length n with no $(1,0)$ steps on the x-axis is equal to the number of Motzkin paths of length $n-1$ with at least one $(1,0)$ step on the x-axis.

10817. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* For $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers summing to 1. Choose $j \in \{1, 2, \ldots, n-1\}$ and a real number $\alpha \geq 1$. Prove that

$$\sum_{k=1}^{n} \frac{(x_k+1)^{2\alpha}}{(x_{k+j}+1)^{\alpha}} \geq n^{1-\alpha}(n+1)^{\alpha},$$

where subscripts are taken modulo n, and determine conditions for equality.

10818. *Proposed by Cezar Joita, State University of New York, Buffalo, NY.*
(a) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} g(x) - x = \infty$ and such that the set $\{x : g(x) = x\}$ is finite and nonempty. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f \circ g = f$, then f is constant.
(b) Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is a quadratic function such that $\{x\colon g(x) = x\}$ is empty. Find a nonconstant continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g = f$.

10819. *Proposed by Olaf Krafft, Rheinisch-Westfällische Technische Hochschule, Aachen, Germany.* Let m and n be integers with $m \geq 2$ and $n \geq 1$. Show that

$$\binom{mn}{n} \geq \frac{m^{m(n-1)+1}}{(m-1)^{(m-1)(n-1)}} n^{-1/2}.$$

10820. *Proposed by M. Mirzavaziri, Ferdowsi University, Mashhad, Iran.* Let $f(m)$ be the least natural number with exactly m divisors. Find a formula for $f(m)$ in terms of $f(m/p)$, where p is the least prime divisor of m.

10821. *Proposed by Gerard J. Foschini, Bell Laboratories, Holmdel, NJ.* Find a sequence of functions $f_1, f_2, \ldots$ in $L_2[0, 1]$ that satisfies the following conditions.
(**1**) For all $\epsilon \in (0, 1)$, the space spanned by $\{f_{1,\epsilon}, f_{2,\epsilon}, \ldots\}$ is $L_2[\epsilon, 1]$, where $f_{n,\epsilon}$ is the restriction of f_n to $[\epsilon, 1]$.
(**2**) The space spanned by $\{f_1, f_2, \ldots\}$ has an infinite-dimensional orthogonal complement in $L_2[0, 1]$.

10822. *Proposed by Jeffrey Lagarias, AT&T Laboratories, Florham Park, NJ, and Jade Vinson, Princeton University, Princeton, NJ.*
(**a**) Let $f(z) = 1/(2 - z^2)$. Prove that all periodic points of f are real.
(**b**) More generally, set $f_\lambda(z) = 1/(\lambda - z^2)$. For which positive real values of λ does f_λ have only real periodic points?

SOLUTIONS

The Asymptotics of the Birthday Problem

10665 [1998, 464]. *Proposed by Jerrold R. Griggs, University of South Carolina, Columbia, SC.* For positive integers s and t, let $P(s, t)$ denote the probability that a random function $f\colon S \to T$ is injective, where S, T are sets with $|S| = s$, $|T| = t$, and, for each $x \in S$, $f(x)$ is chosen uniformly and independently from T. For example, $P(n, 365)$ approximates the probability that, in a class with n students, no two students have the same birthday.
(**a**) Show that $P(s, t) \to 0$ as $s \to \infty$ if $t \sim ks$ for some constant $k > 1$.
(**b**) What happens to $P(s, t)$ as $s \to \infty$ if $t \sim cs^2$ for some constant $c > 0$?

Solution I by Darryl K. Nester, Bluffton College, Bluffton, OH. Since $P(s, t) = \prod_{i=0}^{s-1}(t - i)/t$, we have $-\ln P(s, t) = -\sum_{i=1}^{s-1} \ln(1 - i/t)$.
(**a**) Since $-\ln(1 - x) > x$ for $x \in (0, 1)$, we have

$$-\ln P(s,t) > \sum_{i=1}^{s-1} \frac{i}{t} = \frac{(s-1)s}{2t} \sim \frac{s-1}{2k} \qquad \text{as } s \to \infty.$$

Thus $-\ln P(s, t) \to \infty$ as $s \to \infty$, which yields $P(s, t) \to 0$.
(**b**) We show that $P(s, t) \to e^{-1/(2c)}$ as $s \to \infty$. Note that $-\ln(1 - x) < x + x^2$ for $x \in (0, 1/2)$. Since $t \sim cs^2$, for all sufficiently large s we have $(s - 1)/t < 1/2$, and thus

$$\frac{(s-1)s}{2t} = \sum_{i=1}^{s-1} \frac{i}{t} < -\ln P(s,t) < \sum_{i=1}^{s-1}\left(\frac{i}{t} + \frac{i^2}{t^2}\right) = \frac{(s-1)s}{2t} + \frac{(s-1)s(2s-1)}{6t^2}.$$

For $t \sim cs^2$, both bounds are asymptotic to $1/(2c)$.

Solution II by Sung Soo Kim, Hanyang University, Ansan, Kyunggi, Korea. If $b > a$, then $(x - b)(x + b) < (x - a)(x + a)$. Thus,

$$\left(1 - \frac{s-1}{t}\right)^{s/2} \le P(s,t) = \frac{t(t-1)\cdots(t-s+1)}{t^s} \le \left(1 - \frac{s-1}{2t}\right)^s.$$

(**a**) If $t \sim ks$ for some constant $k > 1$, then $0 \le P(s, t) \le (1 - 1/(4k))^s$ for all sufficiently large s, and $P(s, t) \to 0$.
(**b**) If $t \sim cs^2$ for some constant $c > 0$, then for all sufficiently large s,

$$\left(1 - \frac{1}{\big(c - \varepsilon(s)\big)s}\right)^{s/2} \le P(s,t) \le \left(1 - \frac{1}{2\big(c + \varepsilon(s)\big)s}\right)^s,$$

where $\varepsilon(s)$ is positive and tends to 0. Both bounds tend to $e^{-1/(2c)}$.

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PROBLEMS

10823. *Proposed by George E. Andrews, Pennsylvania State University, University Park, PA.* Given $S = \{a_1, a_2, \dots, a_k\}$, where $a_i \in \mathbb{N}$ for all i and $1 \le a_1 < a_2 < \cdots < a_k \le n$, define $\phi(S)$ to be $\{a_1-1, a_2-1, \dots, a_k-1\}$ if $a_1 \ne 1$ and $\{1, 2, \dots, n\}\backslash\{a_2-1, a_3-1, \dots, a_k-1\}$ if $a_1 = 1$. Prove that $\phi^{n+1}(S) = S$ for every nonempty subset S of $\{1, 2, \dots, n\}$.

10824. *Proposed by Ho-joo Lee, Kwangwoon University, Seoul, South Korea.* Suppose that P is a point in the interior of triangle ABC such that $\angle PAB = \angle PBC = \angle PCA = 30^\circ$. Prove that ABC is equilateral.

10825. *Proposed by Carl Miller, Duke University, Durham, NC.* Given real numbers x and y, define $S_k(x, y)$ for $k \in \mathbb{Z}$ by $S_0(x, y) = x$, $S_1(x, y) = y$, and the recurrence $S_n(x, y) = S_{n-1}(x, y) + S_{n-2}(x, y)$ for all $n \in \mathbb{Z}$. Show that

$$\inf_{n\in\mathbb{Z}} |S_n(x, y)| \le \sqrt{\frac{|x^2 + xy - y^2|}{5}},$$

and determine when equality holds.

10826. *Proposed by Félix Martínez-Giménez, Universidad Politécnica de Valencia, Valencia, Spain.* Given an infinite matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$ of real numbers satisfying $0 < a_{i,j} \le a_{i,j+1}$ for all $i, j \in \mathbb{N}$, we say A satisfies condition $(*)$ if for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\sum_{i\in\mathbb{N}} a_{i,n}/a_{i,m}$ is convergent. For any real number $\theta > 0$, prove that A satisfies condition $(*)$ if and only if $A^{(\theta)}$ satisfies condition $(*)$, where $A^{(\theta)}$ is the matrix whose i, j entry is $a_{i,j}^{\theta}$.

10827. *Proposed by Ulrich Abel, Fachhochschule Giessen-Friedberg, Friedberg, Germany.* For $n \in \mathbb{N}$ and $x > 0$, let

$$f_n(x) = x^{-n} \sum_{k=1}^{n} \frac{x^k - 1}{k}.$$

Prove that $\lim_{n\to\infty} \sup_{x>1} f_n(x)$ exists.

10828. *Proposed by David Beckwith, Sag Harbor, NY.* Given a set M of natural numbers, there is a unique subset $A \subset M$, whose elements we call the *additive atoms* of M, such that every element of M can be written as $\sum_{s \in S} s$ for some $S \subset A$, while no element of A can be written as a sum of two or more distinct elements of A. For example, the additive atoms of $\{1, 2, 3, \ldots\}$ are the powers of 2. For $m > 1$, what are the additive atoms of $\{m, m+1, m+2, \ldots\}$?

10829. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* For a positive integer m, let $f(m) = \sum_{r=1}^{m} m/\gcd(m, r)$. Evaluate $f(m)$ in terms of the canonical factorization of m into a product of powers of distinct primes.

SOLUTIONS

Periodic Points and Forests

10666 [1998, 464]. *Proposed by David Callan, University of Wisconsin, Madison, WI.* Let r and n be positive integers with $r < n$, and let $[n]$ denote $\{1, 2, \ldots, n\}$. We say that $x \in [r]$ is a *periodic point* of a function $f\colon [r] \to [n]$ if $f^k(x) = x$ for some $k \ge 1$.
(a) How many of the n^r functions from $[r]$ to $[n]$ have at least one periodic point?
(b) How many of the $n(n-1)\cdots(n-r+1)$ injective functions from $[r]$ to $[n]$ have no periodic points?

Solution I by David Beckwith, Sag Harbor, NY. The answer to **(a)** is rn^{r-1}, and the answer to **(b)** is $(n-1)!/(n-1-r)!$.

Given a function $f\colon [r] \to [n]$, let S be a largest subset of $[r]$ such that the restriction of f to S is a permutation. The number of periodic points of f is $|S|$. If f has m periodic points, then S can be chosen in $\binom{r}{m}$ ways, and for each choice, f can be defined on S in $m!$ ways.

(a) Let $a'(r, n)$ denote the number of functions with no periodic point, with $a'(0, n) = 1$. Counting the functions from $[r]$ to $[n]$ by periodic points yields $\sum_{m=0}^{r} \binom{r}{m} m! a'(r-m, n) = n^r$, and thus

$$\sum_{m=0}^{r} \frac{1}{(r-m)!} a'(r-m, n) = \frac{n^r}{r!}.$$

The terms for $m > 0$ form the full summation when r is replaced with $r-1$; hence they total $n^{r-1}/(r-1)!$. Canceling these yields $a'(r, n) = n^r - rn^{r-1}$.

(b) Let $b(r, n)$ denote the number of injective functions with no periodic point. Counting the injective functions by periodic points yields $\sum_{m=0}^{r} \binom{r}{m} m! b(r-m, n-m) = \binom{n}{r} r!$, and thus

$$\sum_{m=0}^{r} \frac{1}{(r-m)!} b(r-m, n-m) = \binom{n}{r}.$$

The terms for $m > 0$ form the full summation when r is replaced with $r-1$ and n is replaced with $n-1$; hence they total $\binom{n-1}{r-1}$. Canceling these yields $b(r, n) = r!\left(\binom{n}{r} - \binom{n-1}{r-1}\right) = (n-1)!/(n-1-r)!$.

Solution II by Anchorage Math Solutions Group, University of Alaska, Anchorage, AK. We extend each function $f\colon [r] \to [n]$ to a function $f'\colon [n] \to [n]$ by letting all points of $[n] - [r]$ be fixed points. In the *functional digraph* of f', the digraph on vertex set $[n]$ containing an arc from i to j if and only if $f'(i) = j$, each element of $[n] - [r]$ is the root of a component that is a tree (except for the self-loop at the root). Any other component contains a cycle of elements from $[r]$. Thus f has no periodic point if and only if f' has only its $n - r$ tree components.

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PROBLEMS

10830. *Proposed by Floor van Lamoen, Goes, The Netherlands.* A triangle is divided by its three medians into 6 smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

10831. *Proposed by George E. Andrews, Pennsylvania State University, University Park, PA, and P. Paule and A. Riese, University of Linz, Linz, Austria.* Given positive integers m and n, let $D_{m,n}(a,b,c,d)$ be the determinant of the following matrix: On the main diagonal, there are m entries of a followed by n entries of d. The entries on the diagonal of length n above the main diagonal are all b. The entries on the diagonal of length m below the main diagonal are all c. All other entries are 0. For example

$$D_{5,3}(a,b,c,d) = \det \begin{bmatrix} a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & b \\ c & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & d \end{bmatrix}.$$

Let $g = \gcd(m,n)$, $r = m/g$, and $s = n/g$. Prove that

$$D_{m,n}(a,b,c,d) = \left(a^r d^s - (-1)^{r+s} b^s c^r\right)^g .$$

10832. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Evaluate

$$\sum_{k=1}^{\infty} \left(\frac{k^k}{k!\, e^k} - \frac{1}{\sqrt{2\pi k}} \right).$$

10833. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, IL.* Let r be a positive integer. Prove that there are infinitely many integers $k > r!$ with the property that $j!/(j-r)!$ does not divide $k!/(k-r)!$ whenever $r! < j < k$.

10834. *Proposed by Sung Soo Kim, Hanyang University, Ansan, Korea.* For $n \in \mathbb{N}$, let M_n be the set of n-by-n matrices with nonnegative entries and with zeroes on the diagonal. Define a binary operation $\circ$ on M_n by setting the i, j-entry of $A \circ B$ equal to $\min_{1 \le k \le n} a_{ik} + b_{kj}$. For $A \in M_n$, define a sequence of matrices recursively by setting $A_1 = A$ and setting $A_{k+1} = A_k \circ A$ for $k \ge 1$. Show that $A_r \circ A_s = A_s \circ A_r$ for all $r, s \in \mathbb{N}$.

10835. *Proposed by Anna Dyubina, Tel Aviv University, Tel Aviv, Israel, and Pierre de la Harpe, Université de Genève, Genève, Switzerland.* Let G be the group defined by the presentation that has an infinite sequence $b_0, b_1, b_2, \ldots$ of generators and an infinite sequence $b_1 b_0 b_1^{-1} = b_2 b_1 b_2^{-1} = b_3 b_2 b_3^{-1} = \cdots$ of relations. Show that G is not finitely generated.

10836. *Proposed by Jon A. Wellner, University of Washington, Seattle, WA.* Show that

$$4\sqrt{\pi}(1-x^2)^{3/2} \sum_{k=0}^{\infty} x^{2k} \frac{\Gamma(k+1)}{\Gamma(k+1/2)} = 4 + \sum_{k=1}^{\infty} x^{2k}(2k)! \left(\sum_{j=0}^{k} (-1)^j \frac{2^{k-j+1}}{j!\,2^j} \frac{(k-j)!}{(2k-2j)!} \right)^2$$

for all $x \in [0, 1)$.

SOLUTIONS

Bernoulli, Stirling, and Stirling

10700 [1998, 955]. *Proposed by Leroy Quet, Denver, CO.* Let $c(m,n)$ be the unsigned Stirling numbers of the first kind, the number of permutations of $\{1, 2, \ldots, m\}$ with n cycles. Let $S(m,n)$ be the Stirling numbers of the second kind, the number of partitions of $\{1, 2, \ldots, m\}$ with n blocks. Let $B(n)$ be the nth Bernoulli number, defined by $x/(e^x - 1) = \sum_{n=0}^{\infty} B(n) x^n / n!$. Show that

$$\sum_{n=1}^{r} (-1)^n \frac{n!\,S(r,n)}{n+q} = \frac{1}{(q-1)!} \sum_{n=1}^{q} B(r+n-1)c(q,n)$$

for all positive integers r and q.

Solution by Robin Chapman, University of Exeter, Exeter, U. K. We use the following three formulas:

$$\sum_{j=0}^{k-1} j^m = \sum_{i=0}^{m} \frac{B(m-i)}{i+1} \binom{m}{i} k^{i+1} \qquad \text{(for every nonnegative integer } m\text{)}, \qquad (1)$$

$$\sum_{n=1}^{q} c(q,n) x^n = q! \binom{x+q-1}{q} \qquad \text{(for every positive integer } q\text{), and} \qquad (2)$$

$$x^r = \sum_{n=1}^{r} S(r,n) n! \binom{x}{n} \qquad \text{(for every positive integer } r\text{)}, \qquad (3)$$

where $\binom{x}{t} = x(x-1)(x-2)\cdots(x-t+1)/t!$ for $x \in \mathbb{R}$ and $t \in \mathbb{N}$.

For a polynomial f, let $\hat{f}(k) = \frac{1}{k} \sum_{j=0}^{k-1} f(j)$ when k is a positive integer. For $f(x) = x^m$, (1) yields

$$\hat{f}(k) = \sum_{i=0}^{m} \frac{B(m-i)}{i+1} \binom{m}{i} k^i,$$

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PROBLEMS

10837. *Proposed by Ho-joo Lee, Kwangwoon University, Seoul, South Korea.* Let m and n be positive integers, and let $\varphi(k)$ be the number of integers in $\{1, 2, \ldots, k-1\}$ that are relatively prime to k. Prove that, for some positive integer a, each of $\varphi(a), \varphi(a+1), \varphi(a+2), \ldots, \varphi(a+n)$ is a multiple of m.

10838. *Proposed by Florian S. Pârvânescu, Slatina, Romania.* Let M be any point in the interior of triangle ABC, and let D, E, and F be points on the perimeter such that AD, BE, and CF are concurrent at M. Show that if the triangles BMD, CME, and AMF all have equal areas and equal perimeters, then ABC is equilateral.

10839. *Proposed by Beresford N. Parlett, University of California, Berkeley, CA.* Let A be a symmetric positive definite matrix with bandwidth $2b-1$. Thus, when $b=1$, A is diagonal, and when $b=2$, A is tridiagonal. Prove that the largest eigenvalue of A is no greater than the maximum of all sums of b consecutive entries on the main diagonal of A.

10840. *Proposed by Jiansheng Yang and Shulin Zhou, Peking University, Beijing, P. R. China.* Is the series $\sum_{n=1}^{\infty} x^n/(1+x^n)^n$ uniformly convergent on the interval $[0, 1]$?

10841. *Proposed by Erwin Just, Bronx Community College, Bronx, NY.* Let R be a ring with the property that, for every $x \in R$, there is an integer $n = n(x) \geq 4$ such that $x + x^2 + x^3 = x^n + x^{n+1} + x^{n+2}$.
(**a**) Prove that $x^{3n(x)-2} = x$ for every element $x \in R$.
(**b**) Prove that multiplication in R is commutative.
(**c**) Prove that every element of R has finite additive order.

10842. *Proposed by Bruce Reznick, University of Ilinois, Urbana, IL.*
(**a**) Let n be a positive integer not equal to 1, 2, 3, or 5. Show that there is at least one k with $0 \leq k \leq n$ such that $\binom{2n}{2k}$ is not divisible by $\binom{n}{k}$.
(**b**) Let m be a positive integer. Show that there is a positive integer N_m such that, whenever $n > N_m$, there is at least one k with $0 \leq k \leq n$ such that $\binom{mn}{mk}$ is not divisible by $\binom{n}{k}$.

10843. *Proposed by Andrew Vince, University of Florida, Gainesville, FL.* Define a mapping $f : [0, 1) \to [0, 1)$ by $f(x) = 2x \pmod 1$. Find $\sup_{\lambda(A)=1/2} \lambda(A \cap f^{-1}A)$, where λ denotes Lebesgue measure and the supremum is taken over all sets A that are the union of finitely many intervals and that satisfy $\lambda(A) = 1/2$.

SOLUTIONS

The Maximum Length of a Powerful Arithmetic Progression

10702 [1998, 956]. *Proposed by Kent D. Boklan, Baltimore, MD.* What is the length of the longest nonconstant arithmetic progression of integers with the property that the kth term (for all $k \geq 1$) is a perfect kth power?

Solution by John P. Robertson, St. Paul Re, New York, NY. The longest such progression has length 5.

For an example of a sequence of length 5, take the sequence $\{1, 9, 17, 25, 33\}$ and multiply each of its terms by $3^{24}5^{30}11^{24}17^{20}$.

Suppose there were such a progression of length 6. Let the second, third, and sixth terms be a^2, b^3, and c^6, respectively, so $3a^2 = 4b^3 - c^6$. If $c \neq 0$, then taking $x = 12(b/c^2)$ and $y = 36(a/c^3)$ yields $y^2 = x^3 - 432$, with x and y rational. This elliptic curve has only two rational points (L. J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, p. 247). These are $x = 12$, $y = \pm 36$, both of which produce a constant progression.

If $c = 0$, then the progression would be $\{5r, 4r, 3r, 2r, r, 0\}$ for some nonzero integer r. This would make both $4r$ and $2r$ squares, which is impossible.

Solved also by J. Manoharmayum (U. K.), M. Reid, GCHQ Problems Group (U. K.), and the proposer with N. D. Elkies.

A Union of Proper Subspaces?

10707 [1999, 67]. *Proposed by John Isbell, State University of New York, Buffalo, NY.* Show that

(a) no vector space over an infinite field is a finite union of proper subspaces; and

(b) no vector space over an n-element field is a union of n or fewer proper subspaces.

Composite solution by Julio Kuplinsky, Montclair, NJ, and Leon Mattics, Semmes, AL. Let V be a vector space over a field K. We show that, if K has at least n elements and $S_1, \ldots, S_r$ are proper subspaces of V such that $V = \bigcup_{i=1}^r S_i$, then $r \geq n + 1$. Parts **(a)** and **(b)** then follow immediately.

Let r be the smallest possible number of proper subspaces $S_1, \ldots, S_r$ of V whose union is V. Clearly $r \geq 2$. Also $S_1 \not\subseteq \bigcup_{i=2}^r S_i$ by the minimality of r. Hence we may choose $v \in S_1 - \bigcup_{i=2}^r S_i$. Similarly, we may choose $w \in S_2 - S_1$.

For $\lambda \in K$, we now have $\lambda v + w \notin S_1$, since S_1 is a subspace and $w \notin S_1$. If λ and μ are distinct elements of K such that both $\lambda v + w$ and $\mu v + w$ are in S_j, then $(\lambda - \mu)v \in S_j$. This yields $v \in S_j$, which is a contradiction.

Since K contains at least n elements, we conclude that there are at least n subspaces in the union other than S_1, and hence $r \geq n + 1$.

Editorial comment. These results have appeared previously. David Callan cites K. P. S. Bhaskara Rao and A. Ramachandra Rao, Unions and common complements of subspaces, this MONTHLY **98** (1991) 127–131. Frank Dangello, Lenny Jones, and Mike Seyfried refer to D. B. Leep and G. Myerson, Marriage, magic, and solitaire, this MONTHLY **106** (1999) 419–429. Stephen Gagola points out a similarity to his problem E 2785 [1979, 592; 1980, 672] of this MONTHLY. Robert Gilmer mentions A. Białynicki-Birula, J. Browkin, and A. Schinzel, *Colloq. Math.* **7** (1959) 31–32 and R. D. Bird, Simultaneous complements in

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PROBLEMS

10914. *Proposed by Giovanni Falcone, University of Palermo, Italy.* Given two cyclotomic polynomials Φ_m and Φ_n with $m \neq n$, find the smallest natural number k such that integer polynomials a and b with the property that $a\Phi_m + b\Phi_n$ identically equals k.

10915. *Proposed by C. P. Rupert, Durham, NC.* Given nonzero polynomials p and q in $\mathbb{Z}[x]$ satisfying $p^2 + mq \neq 0$ for $1 \leq m \leq 4$, define polynomials t_n recursively by $t_{n+2} = pt_{n+1} + qt_n$ with initial conditions $t_0 = 0$ and $t_1 = 1$. With μ denoting the Möbius function, prove for $n \geq 1$ that the polynomial $s_n \in \mathbb{Q}[x]$ defined by $s_n(x) = \prod_{d|n} t_d^{\mu(n/d)}$ actually belongs to $\mathbb{Z}[x]$.

10916. *Proposed by Gertrude Ehrlich, University of Maryland, College Park, MD.* Available are two beakers A and B, having volumes a liters and b liters, respectively, a source of water, and a drain. Water may be poured into the beakers from the source or from each other, either filling the receiving beaker or emptying the source beaker, and beakers may be emptied into the drain. Using only these operations, show that if a and b are relatively prime positive integers, then for every integer m with $1 \leq m \leq b$ it is possible to reach a state in which beaker B contains m liters.

10917. *Proposed by Jürgen Groß and Götz Trenkler, University of Dortmund, Germany.* Let P and Q be n-by-n self-adjoint, idempotent matrices, that is, $P^* = P = P^2$ and $Q^* = Q^2 = Q$. Equivalently, P and Q are orthogonal projections of the same dimension. Show that the product PQ is an orthogonal projection if and only if all nonzero eigenvalues of $P + Q$ are greater than or equal to 1.

10918. *Proposed by Matthias Beck, State University of New York, Binghamton NY.* Prove that for all positive integers a and b,
$a + (-1)^b \sum_{m=0}^{a} (-1)^{\lceil \frac{bm}{a} \rceil} \equiv b + (-1)^a \sum_{n=0}^{b} (-1)^{\lceil \frac{an}{b} \rceil} \mod 4$.

10919. *Proposed by Michael Becker, University of South Carolina Sumter, SC.* Let $H(t) = \int_0^\infty \frac{\sin(xt)}{1+x^2}\, dt$, and let $F(k) = \int_0^\infty t^{2k+1} e^{-t} H(t)\, dt$. Find a formula for $F(k)$

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PROBLEMS

10921. *Proposed by David M. Bloom, Brooklyn College CUNY, New York, NY.* Let $c_n = \binom{n}{\lfloor n/2 \rfloor}$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} = c_n c_{n+1}.$$

10922. *Proposed by Mizan R. Khan, Eastern Connecticut State University, Willimantic, CT.* For each positive integer n, let $\delta_k(n)$ denote the largest divisor of n that is relatively prime to k. Show that

$$\overline{\lim_{n\to\infty}} \quad \frac{\sigma(n)}{\sum_{k|n} \delta_k(n)} = \infty.$$

10923. *Proposed by Stephen B. Gray, Santa Monica, CA.* Given a full-dimensional simplex S in $\mathbb{R}^n$, a *step* is an affine transformation that takes S into a new simplex S' by fixing all but one vertex and moving the remaining vertex parallel to the hyperplane determined by the others.

(a) Prove that every triangle in $\mathbb{R}^2$ can be made equilateral in at most two steps.

(b) Prove that for every postive integer n there exists a positive integer N_n such that every full-dimensional simplex in $\mathbb{R}^n$ can be made regular in at most N_n steps.

10924. *Proposed by A. J. Sasane, University of Groningen, The Netherlands.* A regular polygon of 2001 sides is inscribed in a circle of unit radius. Prove that its side and all its diagonals have irrational lengths.

10925. *Proposed by David Callan, University of Wisconsin, Madison, WI.* Define a 0,1-matrix A_n with rows and column indexed by the binary n-tuples with no two consecutive 1s, such that position (u, v) is 1 if and only if v is 0 in each position where u has a 1 or has a 0 immediately preceded by a 1. A_1 and A_2 are shown. Prove that the permanent of A_n is 1.

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} 00 & 01 & 10 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

10926. *Proposed by Harold Diamond, University of Illinois, Urbana, IL.* Let x and y be real numbers with $x \neq y$ and $xy > -1$. Show, for suitable K, that $\tan^{-1} y - \tan^{-1} x$ has the continued fraction expansion

$$\cfrac{1}{K + \cfrac{1}{3K + \cfrac{4}{5K + \cfrac{9}{7K + \dots}}}}$$

(The coefficients in the numerators continue with successive squares, those in the denominators are the consecutive odd numbers.)

10927. *Proposed by Jeffrey C. Lagarias, E. M. Rains, and N. J. A. Sloane, AT&T Labs, Florham Park, NJ.* Define a sequence $\langle a \rangle$ by letting $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and for $n > 3$ letting a_n be the smallest integer among those not already used such that $\gcd(a_{n-1}, a_n) \geq 3$. The sequence begins $1, 2, 3, 6, 9, 12, 4, 8, 16, 20, 5, 10, 15, \ldots$. Prove that it is a permutation of $\mathbb{N}$.

SOLUTIONS

Continued Fractions for Some Quadratic Surds

10773 [1999, 964]. *Proposed by Jean Anglesio, Garches, France.* Let $a_0, a_1, \ldots, a_k$ be positive integers. For $0 \leq i \leq k$, let p_i/q_i be the fraction in lowest terms with continued fraction expansion $[a_0, a_1, \ldots, a_i]$. Find the continued fraction expansion of

$$\sqrt{\frac{p_k p_{k-1}}{q_k q_{k-1}}}, \quad \sqrt{\frac{p_k q_k}{p_{k-1} q_{k-1}}}, \quad \sqrt{\frac{p_k^2 + p_{k-1}^2}{q_k^2 + q_{k-1}}}, \quad \text{and} \quad \sqrt{\frac{p_k^2 + q_k^2}{p_{k-1}^2 + q_{k-1}^2}}$$

in terms of $a_0, a_1, \ldots, a_k$.

Solution by Reiner Martin, New York, N.Y. We show that the following four expansions have the desired values (overlining indicates periodic parts).

$$\begin{aligned}
\alpha &= \left[a_0, \overline{a_1, \ldots, a_{k-1}, 2a_k, a_{k-1}, \ldots, a_1, 2a_0}\,\right], \\
\beta &= \left[a_k, \overline{a_{k-1}, \ldots, a_1, 2a_0, a_1, \ldots, a_{k-1}, 2a_k}\,\right], \\
\gamma &= \left[a_0, \overline{a_1, \ldots, a_{k-1}, a_k, a_k, a_{k-1}, \ldots, a_1, 2a_0}\,\right], \\
\delta &= \left[a_k, \overline{a_{k-1}, \ldots, a_1, a_0, a_0, a_1, \ldots, a_{k-1}, 2a_k}\,\right].
\end{aligned}$$

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before May 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11474. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Valentin Vornicu, Aops-MathLinks forum, San Diego, CA.* Show that when x, y, and z are greater than 1,

$$\Gamma(x)^{x^2+2yz}\Gamma(y)^{y^2+2zx}+\Gamma(z)^{z^2+2xy}\geq(\Gamma(x)\Gamma(y)\Gamma(z))^{xy+yz+zx}.$$

11475. *Proposed by Ömer Eğecioğlu, University of California Santa Barbara, Santa Barbara, CA.* Let $h_k=\sum_{j=1}^{k}\frac{1}{j}$, and let D_n be the determinant of the $(n+1)\times(n+1)$ Hankel matrix with (i,j) entry h_{i+j+1} for $0\leq i,j\leq n$. (Thus, $D_1=-5/12$ and $D_2=1/216$.) Show that for $n\geq 1$,

$$D_n=\frac{\prod_{i=1}^{n}i!^4}{\prod_{i=1}^{2n+1}i!}\cdot\sum_{j=0}^{n}\frac{(-1)^j(n+j+1)!(n+1)h_{j+1}}{j!(j+1)!(n-j)!}.$$

11476. *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy.* Let a, b, and c be the side-lengths of a triangle, and let r be its inradius. Show

$$\frac{a^2bc}{(b+c)(b+c-a)}+\frac{b^2ca}{(c+a)(c+a-b)}+\frac{c^2ab}{(a+b)(a+b-c)}\geq 18r^2.$$

11477. *Proposed by Antonio González, Universidad de Sevilla, Seville, Spain, and José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.* Several boxes sit in a row, numbered from 0 on the left to n on the right. A frog hops from box to box, starting at time 0 in box 0. If at time t, the frog is in box k, it hops one box to the left with probability k/n and one box to the right with probability $1-k/n$. Let $p_t(k)$ be the probability that the frog launches its $(t+1)$th hop from box k. Find $\lim_{i\to\infty}p_{2i}(k)$ and $\lim_{i\to\infty}p_{2i+1}(k)$.

doi:10.4169/000298910X475032

11478. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let K be a field of characteristic zero, and let f and g be relatively prime polynomials in $K[x]$ with $\deg(g) < \deg(f)$. Suppose that for infinitely many λ in K there is a sublist of the roots of $f + \lambda g$ (counting multiplicity) that sums to 0. Show that $\deg(g) < \deg(f) - 1$ and that the sum of all the roots of f (again counting multiplicity) is 0.

11479. *Proposed by Vitaly Stakhovsky, National Center for Biotechnological Information, Bethesda, MD.* Two circles are given. The larger circle C has center O and radius R. The smaller circle c is contained in the interior of C, and has center o and radius r. Given an initial point P on C, we construct a sequence $\langle P_k \rangle$ (the *Poncelet trajectory* for C and c starting at P) of points on C: Put $P_0 = P$, and for $j \geq 1$, let P_j be the point on C to the right of o as seen from P_{j-1} on a line through P_{j-1} and tangent to c. For $j \geq 1$, let ω_j be the radian measure of the angle counterclockwise along C from P_{j-1} to P_j. Let

$$\Omega(C, c, P) = \lim_{k \to \infty} \frac{1}{2\pi k} \sum_{j=1}^{k} \omega_j.$$

(**a**) Show that $\Omega(C, c, P)$ exists for all allowed choices of C, c, and P, and that it is independent of P.
(**b**) Find a formula for $\Omega(C, c, P)$ in terms of r, R, and the distance d between O and o.

11480. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let a, b, and c be the lengths of the sides opposite vertices A, B, and C, respectively, in a nonobtuse triangle. Let h_a, h_b, and h_c be the corresponding lengths of the altitudes. Show that

$$\left(\frac{h_a}{a}\right)^2 + \left(\frac{h_b}{b}\right)^2 + \left(\frac{h_c}{c}\right)^2 \geq \frac{9}{4},$$

and determine the cases of equality.

SOLUTIONS

Powerful Polynomials

11348 [2008, 262]. *Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA.* A polynomial f over a field K is *powerful* if every irreducible factor of f has multiplicity at least 2. When q is a prime or a power of a prime, let $P_q(n)$ denote the number of monic powerful polynomials of degree n over the finite field $\mathbb{F}_q$. Show that for $n \geq 2$,

$$P_q(n) = q^{\lfloor n/2 \rfloor} + q^{\lfloor n/2 \rfloor - 1} - q^{\lfloor (n-1)/3 \rfloor}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Let $A_q(n)$ and $S_q(n)$ be the numbers of monic and monic square-free polynomials of degree n over $\mathbb{F}_q$, respectively. Introduce the ordinary generating functions:

$$\mathcal{A}_q(x) = \sum_{n=0}^{\infty} A_q(n) x^n, \quad \mathcal{P}_q(x) = \sum_{n=0}^{\infty} P_q(n) x^n, \quad \mathcal{S}_q(x) = \sum_{n=0}^{\infty} S_q(n) x^n.$$

A monic polynomial g of degree $2n$ over $\mathbb{F}_q$ is a square if and only if $g = f^2$, where f is a monic polynomial over $\mathbb{F}_q$ of degree n. Thus the ordinary generating function for monic polynomials that are squares is $\mathcal{A}_q(x^2)$. Since any polynomial can be written uniquely as a square times a square-free polynomial, $\mathcal{A}_q(x) = \mathcal{A}_q(x^2)\mathcal{S}_q(x)$. Hence $\mathcal{S}_q(x) = \frac{\mathcal{A}_q(x)}{\mathcal{A}_q(x^2)}$. A straightforward counting argument shows that $A_q(n) = q^n$, so $\mathcal{A}_q(x) = \sum_{n=0}^{\infty} q^n x^n = \frac{1}{1-qx}$, and it follows that $\mathcal{S}_q(x) = \frac{1-qx^2}{1-qx}$.

Any powerful polynomial can be written uniquely as a square times the cube of a square-free polynomial. As before, the number of cubes of square-free polynomials having degree $3n$ equals the number of square-free polynomials of degree n. Thus

$$\mathcal{P}_q(x) = \mathcal{A}_q(x^2)\mathcal{S}_q(x^3) = \frac{1}{1-qx^2}\frac{1-qx^6}{1-qx^3} = \frac{1+x+x^2+x^3}{1-qx^2} - \frac{x+x^2+x^3}{1-qx^3}.$$

Expanding,

$$\mathcal{P}_q(x) = \sum_{m=0}^{\infty} q^m(x^{2m} + x^{2m+1} + x^{2m+2} + x^{2m+3} - x^{3m+1} - x^{3m+2} - x^{3m+3}),$$

and the coefficient of x^n is as claimed.

Also solved by R. Chapman (U. K), P. Corn, O. P. Lossers (Netherlands), J. H. Smith, A. Stadler (Switzerland), B. Ward (Canada), BSI Problems Group (Germany), GCHQ Problems Group (U. K), Microsoft Research Problems Group, and the proposer.

Popoviciu's Inequality Again

11349 [2008, 262]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania.* In triangle ABC, let h_a denote the altitude to the side BC and let r_a be the *exradius* relative to side BC, which is the radius of the circle that is tangent to BC and to the extensions of AB beyond B and AC beyond C. Define h_b, h_c, r_b, and r_c similarly. Let p, r, R, and S be the semiperimeter, inradius, circumradius, and area of ABC. Let ν be a positive number. Show that

$$2(h_a^\nu r_a^\nu + h_b^\nu r_b^\nu + h_c^\nu r_c^\nu) \le r_a^\nu r_b^\nu + r_b^\nu r_c^\nu + r_c^\nu r_a^\nu + 3S^\nu\left(\frac{3p}{4R+r}\right)^\nu.$$

Solution by Elton Bojaxhiu, Albania, and Enkel Hysnelaj, Australia. Let a, b, and c be the side lengths of triangle ABC. Recall that $h_a = 2S/a$, $r_a = S/(p-a)$, and symmetrically for b and c, while $S = pr = abc/(4R) = \sqrt{p(p-a)(p-b)(p-c)}$. Putting everything in terms of a, b, and c and simplifying verifies that

$$(p-a)(p-b) + (p-b)(p-c) + (p-c)(p-a) = \frac{S(4R+r)}{p}.$$

Writing $x = 1/(p-a)$, $y = 1/(p-b)$, and $z = 1/(p-c)$, we obtain

$$\frac{3p}{4R+r} = \frac{3Sxyz}{x+y+z}, \qquad h_a r_a = \frac{2S^2 xyz}{y+z}, \qquad r_a r_b = S^2 xy.$$

Letting $f(x) = 1/x^\nu$ and plugging these in, the desired inequality is equivalent to

$$2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$
$$\le f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right).$$

Since $f(x)$ is convex for $x > 0$, this is an instance of Popoviciu's inequality (S. Savchev and T. Andreescu, *Mathematical Miniatures*, Mathematical Association of America, 2003, pp. 19–20).

Editorial comment. Pál Péter Dályay and GCHQ Problem Solving Group provided (quite different) proofs of Popoviciu's inequality. Michel Bataille noted the paper: V. Cirtoaje, "Two generalizations of Popoviciu's Inequality," *Crux Mathematicorum with Mathematical Mayhem,* vol. 31 no. 5 (2005) 313–318.

Also solved by M. Bataille (France), R. Chapman (U. K.), P. P. Dályay (Hungary), R. Stong, M. Tetiva (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

A Partially Random Permutation

11350 [2008, 262]. *Proposed by Bhavana Deshpande, Poona College of Arts, Science & Commerce Camp, Pune, India, and M. N. Deshpande, Nagpur, India.* Given a positive integer n and an integer k with $0 \le k \le n$, form a permutation $(a_1, \dots, a_n)$ of $(1, \dots, n)$ by choosing the first k positions at random and filling the remaining $n-k$ positions in ascending order. Let $E_{n,k}$ be the expected number of left-to-right maxima. (For example, $E_{3,1} = 2$, $E_{3,2} = 11/6$, and $E_{4,2} = 13/6$.) Show that $E_{n+1,k} - E_{n,k} = 1/(k+1)$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Consider the following way of generating permutations: Choose a random permutation $(b_1, \dots, b_n)$. Set $a_i = b_i$ for $1 \le i \le k$, and sort the elements $b_{k+1}, \dots, b_n$ to produce $a_{k+1}, \dots, a_n$. This is equivalent to the algorithm given in the statement, since $(b_1, \dots, b_k)$ is a random choice of the first k positions. For $j \le k$, the probability that b_j is a left-to-right maximum is the probability that b_j is the largest of $\{b_1, \dots, b_j\}$, which is $1/j$. For $j > k$, the probability that b_j becomes a left-to-right maximum of a is the probability that b_j is the largest of $\{b_1, \dots, b_k, b_j\}$, which is $1/(k+1)$. Hence

$$E_{n,k} = \left(\sum_{j=1}^{k} \frac{1}{j}\right) + \frac{n-k}{k+1},$$

from which the claim follows immediately.

Editorial comment. Christopher Carl Heckman noted that the formula for $E_{n,k}$ yields

$$E_{n,k+1} - E_{n,k} = \frac{k+1-n}{(k+1)(k+2)},$$

and Stephen Herschkorn obtained the following recurrence (free of n) for the variance $V_{n,k}$ of the number of left-to-right maxima:

$$V_{n+2,k} - 2V_{n+1,k} + V_{n,k} = \frac{2k}{(k+1)^2(k+2)}.$$

Also solved by M. Andreoli, D. Beckwith, B. Bradie, R. Chapman (U. K.), P. Corn, C. Curtis, K. David & P. Fricano, J. Ferdinands, J. Freeman, J. Guerreiro & J. Matias (Portugal), C. C. Heckman, S. J. Herschkorn, G. Keselman, J. H. Lindsey II, O. P. Lossers (Netherlands), K. McInturff, R. Mosier, D. Nacin, J. H. Nieto (Venezuela), D. Poore & B. Rice, R. Pratt, B. Schmuland (Canada), A. Stadler (Switzerland), M. Tetiva (Romania), L. Wenstrom, BSI Problems Group (Germany), CMC 328, GCHQ Problem Solving Group (U. K.), and the proposers.

Forcing Three Integers with Zero Sum

11351 [2008, 262]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Given positive integers p and q, find the least positive integer m such that among any m distinct integers in $[-p, q]$ there are three that sum to zero.

Solution by Brian Rice (student), Harvey Mudd College, Claremont, CA, and Daniel Poore (student), Pomona College, Claremont, CA. The answer is $\max\{p, q\} + c$, where $c = 3$ if p and q are even and equal, and $c = 2$ otherwise.

We may assume that $p \le q$, since the problem is symmetric with respect to negation. For the lower bound, note that $[-p, q]$ contains $q + 1$ nonnegative integers, and no three of them sum to 0. When $p = q = 2k$, we need a larger set: choose $\{-2k, \dots, -k, k, \dots, 2k\}$, which consists of the $2k + 2$ numbers with largest absolute value. The magnitudes of any two of these numbers with the same sign sum to more than the magnitude of any other, so no three sum to 0.

For the upper bound, first note that since there are q distinct nonzero absolute values, the pigeonhole principle implies that any set containing 0 and at least $q + 1$ other elements has three elements that sum to zero. Thus we need only show that if $X \subseteq [-p, q] - \{0\}$ such that no three integers in X sum to 0, then $|X| \le q + 1 + \delta$, where $\delta = 1$ if q is even and $p = q$, and $\delta = 0$ otherwise. We consider three cases.

Case 1: p and q are equal and odd. We prove by induction that $|X| \le q + 1$. For $q = 1$ this is immediate. For $q > 1$, let $Y = X \cap \{-q, -(q-1), q-1, q\}$. If $|Y| \le 2$, then $|X| \le q + 1$ by the induction hypothesis, so we may assume that $|Y| \ge 3$. Now Y has two elements with the same sign; we may assume that it has two negative numbers, so $-q \in X$. For $1 \le i \le (q-1)/2$, it follows that only one element from $\{i, q - i\}$ lies in X. Hence at most $(q + 1)/2$ positive integers are in X, with equality only if $q \in X$.

If $q \in X$, then by symmetry X contains at most $(q + 1)/2$ negative integers, so $|X| \le q + 1$, as desired. Otherwise, $q - 1 \in X$, since $|Y| \ge 3$. Now X cannot contain $-i$ and $-q + 1 + i$, for $1 \le i \le (q-3)/2$. Altogether X contains both $-q$ and $-q + 1$, at most $(q-1)/2$ positive integers, at most one from each of $(q-3)/2$ pairs of distinct negative integers summing to $-(q-1)$, and possibly the integer $-(q-1)/2$. Hence again $|X| \le q + 1$.

Case 2: p and q are equal and even. By Case 1, $|X \cap [-q + 1, q - 1]| \le q$, and X has at most two other elements. Hence $|X| \le q + 2$, as desired.

Case 3: $p \le q - 1$. If q is even, then $X - \{q\} \subseteq [-(q-1), q-1]$, so by Case 1, $|X - \{q\}| \le (q-1) + 1 = q$. If q is odd, then $X \subseteq [-q, q]$ and Case 1 yields $|X| \le q + 1$.

Also solved by D. Beckwith, C. Curtis, P. P. Dályay (Hungary), J. Ferdinands, J. H. Lindsey II, J. H. Nieto (Venezuela), T. Rucker, B. Schmuland (Canada), J. Simpson (Australia), M. Tiwari, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Taylor Remainder Limit

11352 [2008, 263]. *Proposed by Daniel Reem, The Technion-Israel Institute of Technology, Haifa, Israel.* Let I be an open interval containing the origin, and let f be a twice-differentiable function from I into $\mathbb{R}$ with continuous second derivative. Let T_2 be the Taylor polynomial of order 2 for f at 0, and let R_2 be the corresponding remainder. Show that

$$\lim_{\substack{(u,v)\to(0,0)\\ u\neq v}} \frac{R_2(u) - R_2(v)}{(u - v)\sqrt{u^2 + v^2}} = 0.$$

Solution by the BSI Problems Group, Bonn, Germany. Let $g(t) = R_2'(t)/t = (f'(t) - f'(0))/t - f''(0)$ for $t \in I \setminus \{0\}$, and $g(0) = 0$. Note that g is continuous on I and

$$R_2(u) - R_2(v) = \int_v^u R_2'(t)\,dt = \int_v^u tg(t)\,dt.$$

Without loss of generality, suppose $u > v$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\bigl(R_2(u) - R_2(v)\bigr)^2 &\le \left(\int_v^u t^2\,dt\right)\left(\int_v^u g^2(t)\,dt\right)\\ &= \left((u-v)\frac{u^2+v^2+uv}{3}\right)\left(\int_v^u g^2(t)\,dt\right).\end{aligned}$$

Since $uv \le (u^2 + v^2)/2$,

$$\frac{\bigl(R_2(u) - R_2(v)\bigr)^2}{(u-v)^2(u^2+v^2)} \le \frac{1}{2(u-v)}\int_v^u g^2(t)\,dt \le \frac{1}{2}\max\bigl\{g^2(t) : t \in [v,u]\bigr\}.$$

This tends to 0 as $(u, v) \to (0, 0)$ since g is continuous at 0 and $g(0) = 0$.

Editorial comment. The GCHQ Problem Solving Group provided a generalization. If f is k times differentiable with continuous kth derivative, and R_k is the remainder term in the Taylor approximation to f of order k at 0, then

$$\lim_{\substack{(u,v)\to(0,0)\\ u\ne v}} \frac{R_k(u) - R_k(v)}{(u-v)(u^2+v^2)^{(k-1)/2}} = 0.$$

Also solved by R. Bagby, R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, J.-P. Grivaux (France), J. Guerreiro & J. Matias (Portugal), E. A. Herman, G. Keselman, J. H. Lindsey II, O. P. Lossers (Netherlands), K. Schilling, B. Schmuland (Canada), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Integral Inequality

11353 [2008, 263]. *Proposed by Ernst Schulte-Geers, BSI, Bonn, Germany.* For $s > 0$, let $f(s) = \int_0^\infty (1 + x/s)^s e^{-x}\,dx$ and $g(s) = f(s) - \sqrt{s\pi/2}$. Show that g maps $\mathbb{R}^+$ onto $(2/3, 1)$ and is strictly decreasing on its domain.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Define $k(t) = t - \log(1 + t)$ for $t \ge 0$. Note that k is increasing, differentiable, and unbounded on $[0, \infty)$. Let h be the function on $[0, \infty)$ given by $h(u) = k^{-1}(u^2/2)$. From the limiting properties of k, it follows that $\lim_{u\to\infty} h(u) = \infty$. Note also that $u^2/2 = h(u) - \log(1 + h(u))$, so that $h'(u) = u/h(u) + u$, and thus h' is positive on $[0, \infty)$. Moreover, h is analytic in a neighborhood of 0, as it is the inverse of the function p given by $p(t) = \sqrt{2k(t)}$, which is analytic in a disk about 0. From the Lagrange inversion theorem, h has a Taylor's series expansion, and we compute $h(u) = u + (1/3)u^2 + O(u^3)$, from which it follows that $h(0) = 0$, $h'(0) = 1$, and $h''(0) = 2/3$. We claim that for $u > 0$, $h(u)^3 > u^3(1 + h(u))$. Indeed, from the definition of h this is equivalent to

$$\log(1 + h) - h + \frac{h^2}{2(1 + h)^{2/3}} > 0.$$

Equality holds at $h = 0$, so it suffices to show that the left side is a strictly increasing function of h, that is,

$$\frac{1}{1+h} - 1 + \frac{h}{(1+h)^{2/3}} - \frac{h^2}{3(1+h)^{5/3}} > 0$$

or (multiplying out and canceling a factor of h)

$$1 + \frac{2}{3}h > (1+h)^{2/3},$$

which follows from Bernoulli's inequality. This proves the claim.

Now substituting $x = sh(t/\sqrt{s})$ yields

$$f(s) = \int_0^\infty e^{-x+s\log(1+x/s)}\,dx = \sqrt{s}\int_0^\infty e^{-t^2/2}h'\left(\frac{t}{\sqrt{s}}\right)dt.$$

Since $\int_0^\infty e^{-t^2/2}\,dt = \sqrt{\pi/2}$ we have

$$g(s) = \int_0^\infty e^{-t^2/2}\sqrt{s}\left(h'\left(\frac{t}{\sqrt{s}}\right) - 1\right)dt.$$

We compute

$$\frac{d}{ds}\left(\sqrt{s}\left(h'\left(\frac{t}{\sqrt{s}}\right) - 1\right)\right) = \frac{d}{ds}\left(\frac{t}{h(t/\sqrt{s})} + t - \sqrt{s}\right) = \frac{t^3(1+h) - h^3 s^{3/2}}{2h^3 s^2},$$

and this last is negative. Here we have written h for $h(t/\sqrt{s})$ and have used $h^3 > t^3(1+h)/s^{3/2}$ from the claim proved above. It follows that g is a decreasing function of s and in fact that the integrand is decreasing. Hence the monotone convergence theorem yields

$$\lim_{s\to\infty} g(s) = \int_0^\infty e^{-t^2/2}th''(0)\,dt = h''(0) = \frac{2}{3}.$$

From the original definition and monotone convergence, we conclude that

$$\lim_{s\to 0^+} g(s) = \lim_{s\to 0^+} f(s) = \int_0^\infty e^{-x}\lim_{s\to 0^+}(1+x/s)^s\,dx = 1.$$

Thus g decreases from 1 to $2/3$ as claimed.

Also solved by R. Bagby, P. Bracken, J. Grivaux (France), F. Holland (Ireland), P. Perfetti (Italy), B. Schmuland (Canada), A. Stadler (Switzerland), B. Ward(Canada), Y. Yu, and the proposer.

An Absolute Value Sum

11354 [2008, 263]. *Proposed by Matthias Beck, San Francisco State University, San Francisco, CA, and Alexander Berkovich, University of Florida, Gainesville, FL.* Find a polynomial f in two variables such that for all pairs (s, t) of relatively prime integers,

$$\sum_{m=1}^{s-1}\sum_{n=1}^{t-1}|mt - ns| = f(s,t).$$

Solution I by Byron Schmuland, University of Alberta, Edmonton, Alberta, Canada. Let A denote the expression to be evaluated. From $-(mt - ns) = (s-m)t - (t-n)s$, we see that each nonzero contribution $|mt - ns|$ occurs once with $mt - ns > 0$ and once with $mt - ns < 0$, using symmetric indices. Therefore, it suffices to double the positive contributions: $A = 2\sum_{m=1}^{s-1}\sum_{n=1}^{\lfloor mt/s\rfloor}(mt - ns)$. The inner sum, call it A_m, evaluates to $mt\lfloor mt/s\rfloor - \frac{1}{2}s\lfloor mt/s\rfloor(\lfloor mt/s\rfloor + 1)$.

Now let $mt = sq_m + r_m$, where $0 \le r_m \le s - 1$, so $\lfloor mt/s\rfloor = (mt - r_m)/s$. Thus

$$A = 2\sum_{m=1}^{s-1} A_m = \frac{1}{6}t(s-1)(2ts - 3s - t) + \frac{1}{s}\sum_{m=1}^{s-1} r_m(s - r_m).$$

When s and t are relatively prime, the remainders r_m for $1 \le m \le s - 1$ are distinct and take on all nonzero values, so

$$\sum_{m=1}^{s-1} r_m(s - r_m) = \sum_{m=1}^{s-1} m(s - m) = \frac{1}{6}(s-1)s(s+1).$$

Summing the contributions and simplifying yields

$$\sum_{m=1}^{s-1}\sum_{n=1}^{t-1}|mt - ns| = \frac{1}{6}(s-1)(t-1)(2st - s - t - 1).$$

Solution II by Allen Stenger, Alamogordo, NM. Let s and t be relatively prime positive integers. A nonnegative integer is called *representable* if it can expressed as a linear combination of s and t with nonnegative integer coefficients; otherwise it is *nonrepresentable*. T. C. Brown and P. J.-S. Shiue (A remark related to the Frobenius problem, *Fibonacci Quarterly* **31** (1993) 32–36) showed that the sum of all nonrepresentable positive integers is

$$\frac{1}{12}(s-1)(t-1)(2st - s - t - 1). \tag{1}$$

This was recently reproved by A. Tripathi (On sums of positive integers that are not of the form $ax + by$, this MONTHLY **115** (2008) 363–364).

We show that the positive values of $mt - ns$ in our sum are the nonrepresentable positive integers. As in solution I, the negative values of $mt - ns$ are the negatives of the positive values, so the desired sum is twice (1).

Fix a positive integer a. By the Chinese remainder theorem, the integer solutions (m, n) to $mt - ns = a$ are $\{(m_0 + ks, n_0 + kt)\colon\ k \in \mathbb{Z}\}$ for a fixed solution (m_0, n_0). The nonrepresentable a are those for which no solution has $m \ge 0$ and $n \le 0$. There is one solution with $0 \le m \le s - 1$. If the corresponding n is nonpositive, then a is representable. If it is positive, then a is nonrepresentable, since increasing m requires increasing n.

Hence the positive values of $mt - ns$ with $0 \le m \le s - 1$ and $n > 0$ are the nonrepresentable numbers. If $m = 0$, then $mt - ns$ is negative, and if $n \ge t$ then $mt - ns \le (s-1)t - st < 0$. Thus the nonrepresentable numbers indeed are exactly the positive values of $mt - ns$ in our sum.

Also solved by R. Chapman (U. K.), P. Corn, P. P. Dályay (Hungary), A. Fok, J. R. Gorman, J. Guerreiro and J. Matias (Portugal), S. J. Herschkorn, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), K. Schilling, R. A. Simón (Chile), J. Simpson (Australia), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), B. Ward (Canada), H. Widmer (Switzerland), J. B. Zacharias, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before July 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11488. *Proposed by Dennis I. Merino, Southeastern Louisiana University, Hammond, LA, and Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL.*
(**a**) Show that if k is a positive odd integer, and A and B are Hermitian matrices of the same size such that $A^k + B^k = 2I$, then $2I - A - B$ is positive semidefinite.
(**b**) Find the largest positive integer p such that for all Hermitian matrices A and B of the same size, $2^{p-1}(A^p + B^p) - (A+B)^p$ is positive semidefinite.

11489. *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noei, Italy.* Let a_0, a_1, and a_2 be the side lengths, and r the inradius, of a triangle. Show that

$$\sum_{i \bmod 3} \frac{a_i^2 a_{i+1} a_{i+2}}{(a_{i+1}+a_{i+2})(a_{i+1}+a_{i+2}-a_i)} \geq 18r^2.$$

11490. *Proposed by Gábor Mészáros, Kemence, Hungary.* A semigroup S *agrees with* an ordered pair (i, j) of positive integers if $ab = b^j a^i$ whenever a and b are distinct elements of S. Find all ordered pairs (i, j) of positive integers such that if a semigroup S agrees with (i, j), then S has an idempotent element.

11491. *Proposed by Nicolae Anghel, University of North Texas, Denton, TX.* Let P be an interior point of a triangle having vertices A_0, A_1, and A_2, opposite sides of length a_0, a_1, and a_2, respectively, and circumradius R. For $j \in \{0, 1, 2\}$, let r_j be the distance from P to A_j. Show that

$$\frac{r_0}{a_0^2} + \frac{r_1}{a_1^2} + \frac{r_2}{a_2^2} \geq \frac{1}{R}.$$

11492. *Proposed by Tuan Le, student, Freemont High School, Anaheim, CA.* Show that for positive a, b, and c,

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} + \frac{\sqrt{b^3+c^3}}{b^2+c^2} + \frac{\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

doi:10.4169/000298910X480135

11493. *Proposed by Johann Cigler, Universität Wien, Vienna, Austria.* Consider the Hermite polynomials H_n, defined by

$$H_n(x,s) = \sum_{0 \le k \le n/2} \binom{n}{2k} (2k-1)!!(-s)^k x^{n-2k},$$

where $m!! = \prod_{i<m/2}(m-2i)$ for positive m, with $(-1)!! = 1$. Let L be the linear transformation from $\mathbb{Q}[x,s]$ to $\mathbb{Q}[x]$ determined by $L1 = 1$, $Lx^k s^j = x^k L s^j$ for $j, k \ge 0$, and $LH_{2n}(x,s) = 0$ for $n > 0$. (Thus, for example, $0 = LH_2(x,s) = L(x^2 - s) = x^2 - Ls$, so $Ls = x^2$.) Define the *tangent numbers*T_{2n+1} by $\tan z = \sum_{n \ge 0} T_{2n+1} z^{2n+1}/(2n+1)!$, and the *Euler numbers* E_{2n} by $\sec(z) = \sum_{n\ge 0} \frac{E_{2n}}{(2n)!} z^{2n}$.
(**a**) Show that

$$LH_{2n+1}(x,s) = (-1)^n T_{2n+1} x^{2n+1}.$$

(**b**) Show that

$$Ls^n = \frac{E_{2n}}{(2n-1)!!} x^{2n}.$$

11494. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Let A be the Glaisher-Kinkelin constant, given by

$$A = \lim_{n\to\infty} n^{-n^2/2 - n/2 - 1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.2824\ldots .$$

Show that

$$\prod_{n=1}^{\infty} \left(\frac{n!}{\sqrt{2\pi n}(n/e)^n} \right)^{(-1)^{n-1}} = \frac{A^3}{2^{7/12}\pi^{1/4}}.$$

SOLUTIONS

A Reciprocal Diophantine Equation

11355 [2008, 365]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Determine for which integers a the Diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

has infinitely many integer solutions (x, y, z) such that $\gcd(a, xyz) = 1$.

Solution by Éric Pité, Paris, France. Suppose first that a is odd. Let $x = an + 2$, $y = -(an + 1)$, and $z = a - xy = a^2n^2 + 3an + a + 2$, where n is any integer such that $xyz \ne 0$ (there are infinitely many such n). Since $x + y = 1$ and $z = a - xy = \frac{a-xy}{x+y}$, we have $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$. Also $\gcd(a, y) = 1$, and both $\gcd(a, x)$ and $\gcd(a, z)$ divide 2, but since a is odd we have $\gcd(a, xyz) = 1$.

If a is even and $\gcd(a, xyz) = 1$, then x, y, and z are odd. Now $xy + yz + zx$ is odd and cannot equal a. Hence there is no solution when a is even, and there are infinitely many when a is odd.

Also solved by D. Beckwith, B. S. Burdick, S. Casey (Ireland), R. Chapman (U. K.), K. S. Chua (Singapore), P. Corn, C. Curtis, K. Dale (Norway), D. Degiorgi (Switzerland), J. Fresán (Spain), D. Gove, E. J. Ionascu

& A. A. Stancu, I. M. Isaacs, T. Keller, K. Kneile, O. Kouba (Syria), O. P. Lossers (Netherlands), S. Meskin, A. Nakhash, J. H. Nieto (Venezuela), C. R. Pranesachar (India), K. Schilling, B. Schmuland (Canada), A. Stadler (Switzerland), R. Stong, J. V. Tejedor (Spain), M. Tetiva (Romania), V. Verdiyan (Armenia), B. Ward (Canada), BSI Problems Group (Germany), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, Northwestern Univ. Math Problem Solving Group, and the proposer.

Integral Inequalities

11360 [2008, 365]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f and g be continuous real-valued functions on $[0, 1]$ satisfying the condition $\int_0^1 f(x)g(x)\,dx = 0$. Show that $\int_0^1 f^2 \int_0^1 g^2 \geq 4\left(\int_0^1 f \int_0^1 g\right)^2$ and $\int_0^1 f^2 \left(\int_0^1 g\right)^2 + \int_0^1 g^2 \left(\int_0^1 f\right)^2 \geq 4\left(\int_0^1 f \int_0^1 g\right)^2$.

Solution by Nate Eldredge, University of California San Diego, San Diego, CA. Let $\langle u, v\rangle = \int_0^1 u(x)v(x)\,dx$. By scaling, we may assume $\langle f, f\rangle = \langle g, g\rangle = 1$. Let $a = \langle f, 1\rangle$ and $b = \langle g, 1\rangle$. The desired inequalities then read $1 \geq 4a^2b^2$ and $b^2 + a^2 \geq 4a^2b^2$. Bessel's inequality yields $1 \geq a^2 + b^2$, and $a^2 + b^2 \geq 2ab$ is trivial. Hence $1 \geq a^2 + b^2 \geq (a^2 + b^2)^2 \geq 4a^2b^2$, which proves both inequalities.

Editorial comment. Charles Kicey noted that the inequalities are best possible: let $f(x) = \sqrt{2}/2 + \cos \pi x$ and $g(x) = \sqrt{2}/2 - \cos \pi x$.

Also solved by U. Abel (Germany), K. F. Andersen (Canada), R. Bagby, A. Bahrami (Iran), M. W. Botsko, S. Casey (Ireland), R. Chapman (U. K.), H. Chen, J. Freeman, J. Grivaux (France), J. Guerreiro & J. Matias (Portugal), E. A. Herman, G. Keselman, C. Kicey, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), M. Omarjee (France), J. Rooin & M. Bayat (Iran), X. Ros (Spain), K. Schilling, B. Schmuland (Canada), A. Shafie & M. F. Roshan (Iran), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. V. Tejedor (Spain), P. Xi and Y. Yi (China), Y. Yu, L. Zhou, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposers.

Supremum of a Nonlinear Functional

11366 [2008, 462]. *Proposed by Nicolae Anghel, University of North Texas, Denton, TX.* Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $\phi(0) = 0$ and ϕ' is strictly increasing. For $a > 0$, let C_a denote the space of all continuous functions from $[0, a]$ into $\mathbb{R}$, and for $f \in C_a$, let $I(f) = \int_{x=0}^a (\phi(x)f(x) - x\phi(f(x)))\,dx$. Show that I has a finite supremum on C_a and that there exists an $f \in C_a$ at which that supremum is attained.

Solution by Eugen J. Ionascu, Columbus State University, Columbus, GA. For every $x \in [0, a]$ we let $g_x(u) = \phi(x)u - x\phi(u)$, defined for all $u \in \mathbb{R}$. The derivative is $g_x'(u) = \phi(x) - x\phi'(u)$. By the mean value theorem, $\phi(x) = \phi(x) - \phi(0) = (x - 0)\phi'(c_x)$ for some c_x between 0 and x. If $x > 0$, then $g_x'(u) = x(\phi'(c_x) - \phi'(u))$. Because ϕ' is strictly increasing, c_x is uniquely determined, and g_x attains its maximum at c_x. If $x = 0$, then $g_x \equiv 0$, and we simply define $c_0 = 0$. This gives us a function $x \mapsto c_x$ which we denote by f_0. Clearly

$$f_0(x) = \begin{cases} \phi'^{-1}\big(\phi(x)/x\big), & \text{if } 0 < x \leq a, \\ 0, & \text{if } x = 0. \end{cases}$$

This function is continuous at every positive point x, since ϕ' is continuous and strictly increasing. Also, because $0 < c_x < x$ for $x > 0$, this function is also continuous at 0. Thus, $f_0 \in C_a$. For all $f \in C_a$, we have

$$I(f) = \int_0^a g_x\big(f(x)\big)\,dx \le \int_0^a g_x\big(f_0(x)\big)\,dx = I(f_0).$$

This inequality answers both parts of the problem.

Editorial comment. Richard Bagby noted that it is not necessary to explicitly assume the continuity of ϕ'. If ϕ is differentiable everywhere, then ϕ' has the intermediate value property by Darboux's theorem, and every monotonic function on $\mathbb{R}$ with the intermediate value property is continuous.

Also solved by R. Bagby, M. W. Botsko, P. Bracken, R. Chapman (U. K.), P. J. Fitzsimmons, J.-P. Gabardo (Canada), J.-P. Grivaux (France), J. Guerreiro & J. Matias (Portugal), E. A. Herman, R. Howard, G. Keselman, J. H. Lindsey II, O. P. Lossers (Netherlands), K. Schilling, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Points Generated by the Nine Points

11370 [2008, 568]. *Proposed by Michael Goldenberg and Mark Kaplan, Baltimore Polytechnic Institute, Baltimore, MD.* Let A_0, A_1, and A_2 be the vertices of a non-equilateral triangle T. Let G and H be the centroid and orthocenter of T, respectively. Treating all indices modulo 3, let B_k be the midpoint of $A_{k-1}A_{k+1}$, let C_k be the foot of the altitude from A_k, and let D_k be the midpoint of A_kH.

The *nine-point circle* of T is the circle through all B_k, C_k, and D_k. We now introduce nine more points, each obtained by intersecting a pair of lines. (The intersection is not claimed to occur between the two points specifying a line.) Let P_k be the intersection of $B_{k-1}C_{k+1}$ and $B_{k+1}C_{k-1}$, Q_k the intersection of $C_{k-1}D_{k+1}$ and $C_{k+1}D_{k-1}$, and R_k the intersection of $C_{k-1}C_{k+1}$ and $D_{k-1}D_{k+1}$.

Let e be the line through $\{P_0, P_1, P_2\}$, and f be the line through $\{Q_0, Q_1, Q_2\}$. (By Pascal's theorem, these triples of points are collinear.) Let g be the line through $\{R_0, R_1, R_2\}$; by Desargues' theorem, these points are also collinear.

(**a**) Show that the line e is the *Euler line* of T.
(**b**) Show that g coincides with f.
(**c**) Show that f is perpendicular to e.
(**d**) Show that the intersection S of e and f is the inverse of H with respect to the nine-point circle.

Solution by the proposers. (**a**) Let k, m, n be 1, 2, 3 in some order. Applying Pappus's theorem to points B_m, C_m, A_n on line A_kA_n and to points B_n, C_n, A_m on line A_kA_m, we get that the three points P_k, G, and H, defined by $P_k = B_mC_n \cap B_nC_m$, $G = A_mB_m \cap A_nB_n$, and $H = A_mC_m \cap A_nC_n$, are collinear. So all P_k lie on the Euler line GH.

(**b**) Let N be the nine-point circle. Consider the cyclic quadrilateral $C_mC_nD_mD_n$. Because $H = C_mD_m \cap C_nD_n$, $Q_k = C_mD_n \cap C_nD_m$, and $R_k = C_mC_n \cap D_mD_n$, we conclude that points Q_k and R_k are on the polar of H with respect to N (see Theorem 6.51, p. 145, in H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited,* Mathematical Association of America, Washington, DC 1967). So f and g coincide.

(**c**) By the definition of *polar*, we have $NH \perp f$ or $e \perp f$.

(**d**) This also follows from the definition of *polar*.

Editorial comment. Most solvers proceeded analytically. Some solvers simplified the algebra by using complex numbers or determinants. Some used Maple to help.

Also solved by P. P. Dályay (Hungary), D. Gove, J.-P. Grivaux (France), R. Stong, GCHQ Problem Solving Group (U. K.).

For Grid Triangles, the Brocard Angle is Irrational in Degrees

11375 [2008, 568]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* The *first Brocard point* of a triangle ABC is that interior point Ω for which the angles ΩBC, ΩCA, and ΩAB have the same radian measure. Let ω be that measure. Regarding the triangle as a figure in the Euclidean plane $\mathbb{R}^2$, show that if the vertices belong to $\mathbb{Z} \times \mathbb{Z}$, then ω/π is irrational.

Editorial comment. The claim follows from combining several well-known results.

(a) $\cot\omega = \cot A + \cot B + \cot C = (a^2 + b^2 + c^2)/4S \geq \sqrt{3}$, where S is the area of the triangle. The first equality is shown in [**1**]; see also [**5**] and [**7**]. The second is an easy consequence of the law of sines and the law of cosines. The inequality is due to Weitzenböck [**2**], also proved in [**8**].

(b) Because the cotangent is decreasing on $(0, \pi/2)$, we conclude that $\omega \leq \pi/6$. This is also deduced in [**1**] and [**7**].

(c) The squares of the sides (by the distance formula), the area S (by Pick's Theorem), and all six trigonometric functions of the angles (by various elementary trigonometric relationships) are rational because the vertices belong to $\mathbb{Z} \times \mathbb{Z}$.

(d) Every angle in $(0, \pi/2)$ that is a rational multiple of π and has rational trigonometric functions is larger than $\pi/6$ (using Lambert's theorem; see also [**6**]); so ω cannot be a rational multiple of π.

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Jerry Minkus showed that a similar result can be obtained for triangles whose vertices lie in the set of vertices of the unit triangular tiling of the plane, except that of course equilateral triangles (for which $\omega = \pi/6$) must be excluded. He also conjectured a generalization. Given a square-free positive integer d other than 3, let the lattice L_d be defined by $\{h + k\delta\colon\ h, k \in \mathbb{Z}\}$, where $\delta = i\sqrt{d}$ when d is congruent to 1 or 2 mod 4, and $\delta = (-1 + i\sqrt{d})/2$ when $d \equiv 3 \pmod 4$. The conjecture is that if the vertices of triangle ABC lie on L_d, then the Brocard angle ω of triangle ABC is an irrational multiple of π.

Solved by R. Chapman (U. K.), P. P. Dályay (Hungary), V. V. García (Spain), J.-P. Grivaux (France), O. Kouba (Syria), J. Minkus, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group, and the proposer.

Riemann Sums Don't Converge?

11376 [2008, 664]. *Proposed by Proposed by Bogdan M. Baishanski, The Ohio State University, Columbus, OH.* Given a real number a and a positive integer n, let

$$S_n(a) = \sum_{an<k\le(a+1)n} \frac{1}{\sqrt{kn - an^2}}.$$

For which a does the sequence $\langle S_n(a)\rangle$ converge?

Solution by Vitali Stakhovsky, Rockville, MD. The sequence $\langle S_n(a)\rangle$ converges if and only if a is rational. Letting $j = k - \lfloor an\rfloor$, $an < k \le (a+1)n$ becomes $1 \le j \le n$, so $S_n(a) = n^{-1/2}\sum_{j=1}^{n}(j - \{an\})^{-1/2} = (n - n\{an\})^{-1/2} + T_n(a)$, where $T_n(a) = n^{-1/2}\sum_{j=2}^{n}(j-\{an\})^{-1/2}$. For $j \ge 2$,

$$2\left(\sqrt{j+1} - \sqrt{j}\right) < j^{-1/2} \le (j - \{an\})^{-1/2}$$
$$\le 2\left(\sqrt{j-1} - \sqrt{j-2}\right) < (j-1)^{-1/2},$$

whence

$$2n^{-1/2}\left(\sqrt{n+1} - \sqrt{2}\right) \le T_n(a) \le 2n^{-1/2}\sqrt{n-2}$$

and $\lim_{n\to\infty} T_n(a) = 2$. Thus, $S_n(a)$ converges if and only if $(n - n\{an\})^{-1/2}$ does; that is, it converges when $R_n(a) = n - n\{an\}$ has a positive limit, finite or infinite. If a is rational, then writing $a = p/q$ with p and q relatively prime yields $1 - \{an\} \ge 1/q$, so $R_n(a) \ge n/q \to \infty$ and $S_n(a)$ converges.

If a is irrational, then its continued fraction convergents p_k/q_k satisfy $0 < a - p_k/q_k < 1/q_k^2$ for even k, and $0 < p_k/q_k - a < 1/q_k^2$ for odd k. Thus for even k, $\{q_k a\} < 1/q_k$ so that $R_{q_k}(a) \ge q_k - 1$; on even k, this subsequence tends to infinity. For odd k, on the other hand, $\{q_k a\} > 1 - 1/q_k$ so that $R_{q_k}(a) \le 1$; this subsequence remains bounded. Thus $\langle R_n(a)\rangle$ has neither a positive nor infinite limit, and therefore $\langle S_n(a)\rangle$ diverges.

Editorial comment. Several solvers noted that $S_n(a)$ is a Riemann sum for the expression $\int_a^{a+1} dx/\sqrt{x-a}$, which evaluates to 2. Since the integral is improper, it need not equal the limit of its Riemann sums.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), J.-P. Grivaux (France), S. James (Canada), G. Kouba (Syria), J. H Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), P. Perfetti (Italy), É. Pité (France), M. A. Prassad (India), N. Singer, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), M. Wildon (U. K.), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), NSA Problems Group, Northwestern University Math Problem Solving Group, and the proposer.

An Infinite Product for the Exponential

11381 [2008, 665]. *Proposed by Jésus Guillera, Zaragoza, Spain, and Jonathan Sondow, New York, NY.* Show that if x is a positive real number, then

$$e^x = \prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(kx+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{1/n}.$$

Solution by BSI Problems Group, Bonn, Germany. Let f_n be the nth factor. Using

$$\log(1+kx) = \int_0^x \frac{k\,dy}{1+ky} = \int_0^x\int_0^\infty ke^{-(1+ky)t}\,dt\,dy = \int_0^\infty \frac{1-e^{-kxt}}{t}\,e^{-t}\,dt,$$

we find

$$\log f_n = \frac{1}{n}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\log(1+kx) = \frac{1}{n}\int_0^\infty \frac{(1-e^{-xt})^n}{t}\, e^{-t}\, dt.$$

For $t \geq 0$ we have

$$0 \leq \sum_{n=1}^{N}\frac{(1-e^{-xt})^n}{n} \nearrow -\log\left(1-(1-e^{-xt})\right) = xt$$

as $N \to \infty$. Hence, by the monotone convergence theorem,

$$\log\left(\prod_{n=1}^{N} f_n\right) \longrightarrow \int_0^\infty \frac{xt}{t}\, e^{-t}\, dt = x.$$

Also solved by R. Bagby, D. Beckwith, R. Chapman (U. K.), H. Chen, Y. Dumont (France), M. L. Glasser, R. Govindaraj& R. Ramanujan & R. Venkatraj (India), J. Grivaux (France), O. Kouba (Syria), O. P. Lossers (Netherlands), A. Plaza & S. Falcon (Spain), R. Pratt, N. C. Singer, A. Stadler (Switzerland), V. Stakhovsky, R. Stong, M. Tetiva (Romania), M. Vowe (Switzerland), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposers.

Can You See the Telescope?

11383 [2008, 0757]. *Proposed by Michael Nyblom, RMIT University, Melbourne, Australia.* Show that

$$\sum_{n=1}^{\infty}\cos^{-1}\left(\frac{1+\sqrt{n^2+2n}\sqrt{n^2+4n+3}}{(n+1)(n+2)}\right) = \frac{\pi}{3}.$$

Solution by Simon J. Smith, La Trobe University, Vendigo, Victoria, Australia. In fact, the answer is $\pi/6$. To see this, let

$$\theta_n = \cos^{-1}\left(\frac{1}{n+1}\right) = \sin^{-1}\left(\frac{\sqrt{n^2+2n}}{n+1}\right),$$

so that

$$\begin{aligned}
&\sum_{n=1}^{N}\cos^{-1}\left(\frac{1+\sqrt{n^2+2n}\,\sqrt{n^2+4n+3}}{(n+1)(n+2)}\right)\\
&\qquad = \sum_{n=1}^{N}\cos^{-1}\left(\cos\theta_n\cos\theta_{n+1}+\sin\theta_n\sin\theta_{n+1}\right)\\
&\qquad = \sum_{n=1}^{N}\cos^{-1}\left(\cos(\theta_{n+1}-\theta_n)\right) = \theta_{N+1}-\theta_1,
\end{aligned}$$

which converges to $\pi/2 - \pi/3 = \pi/6$ as $N \to \infty$.

Also solved by Z. Ahmed (India), B. T. Bae (Spain), R. Bagby, M. Bataille (France), D. Beckwith, M. Bello-Hernández & M. Benito (Spain), P. Bracken, B. Bradie, R. Brase, N. Caro (Brazil), R. Chapman (U. K.), H. Chen, C. Curtis, P. P. Dályay (Hungary), Y. Dumont (France), J. Freeman, A. Gewirtz (France), M. L. Glasser, M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), E. A. Herman, C. Hill, W. P. Johnson, D. Jurca, O. Kouba (Syria), V. Krasniqi (Kosova), G. Lamb, W. C. Lang, K.-W. Lau (China), O. P. Lossers (Netherlands),

G. Martin (Canada), K. McInturff, M. McMullen, R. Nandan, A. Nijenhuis, M. Omarjee (France), É. Pité (France), Á. Plaza (Spain), C. R. Pranesachar (India), M. T. Rassias (Greece), A. H. Sabuwala, V. Schindler (Germany), A. S. Shabani (Kosova), N. C. Singer, A. Stadler (Switzerland), R. Stong, J. Swenson, M. Tetiva (Romania), J. V. Tejedor (Spain), D. B. Tyler, Z. Vörös (Hungary), M. Vowe, J. B. Zacharias, BSI Problems Group (Germany), FAU Problem Solving Group, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Hofstra University Problem Solvers, Microsoft Research Problems Group, Missouri State University Problem Solving Group, NSA Problems Group, Northwestern University Math Problem Solving Group.

Angles of a Triangle

11385 [2008, 757]. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.* Let α_0, α_1, and α_2 be the radian measures of the angles of an acute triangle, and for $i \geq 3$ let $\alpha_i = \alpha_{i-3}$. Show that

$$\sum_{i=0}^{2} \frac{\alpha_i^2}{\alpha_{i+1}\alpha_{i+2}} \left(3 + 2\tan^2 \alpha_i\right)^{1/4} \geq 3\sqrt{3}.$$

Solution by Rob Brase, Lincoln, NE. We may assume $\alpha_0 \leq \alpha_1 \leq \alpha_2$. Then

$$\frac{\alpha_0^2}{\alpha_1\alpha_2} \leq \frac{\alpha_1^2}{\alpha_2\alpha_0} \leq \frac{\alpha_2^2}{\alpha_0\alpha_1} \qquad \text{and}$$
$$(2 + 2\tan^2 \alpha_0)^{1/4} \leq (2 + 2\tan^2 \alpha_1)^{1/4} \leq (2 + 2\tan^2 \alpha_2)^{1/4}.$$

By Chebyshev's inequality,

$$\sum \frac{\alpha_i^2}{\alpha_{i+1}\alpha_{i+2}} (3 + 2\tan^2 \alpha_i)^{1/4} \geq \frac{1}{3}\left[\sum \frac{\alpha_i^2}{\alpha_{i+1}\alpha_{i+2}}\right]\left[\sum (3 + 2\tan^2 \alpha_i)^{1/4}\right].$$

Calculation shows that the second derivative of $(3 + 2\tan^2\theta)^{1/4}$ is positive on $(0l\pi/2)$. Apply the AM–GM inequality to the first factor and Jensen's inequality on the second factor to obtain

$$\begin{aligned}
&\frac{1}{3}\left[\sum \frac{\alpha_i^2}{\alpha_{i+1}\alpha_{i+2}}\right]\left[\sum (3 + 2\tan^2 \alpha_i)^{1/4}\right] \\
&\quad \geq \frac{1}{3}\left[3\left(\frac{\alpha_0^2}{\alpha_1\alpha_2}\,\frac{\alpha_1^2}{\alpha_2\alpha_0}\,\frac{\alpha_2^2}{\alpha_0\alpha_1}\right)^{1/3}\right]\left[3\left(3 + 2\tan^2\left(\frac{\alpha_0+\alpha_1+\alpha_2}{3}\right)\right)^{1/4}\right] \\
&\quad = 3\left(3 + 2\tan^2\left(\frac{\pi}{3}\right)\right)^{1/4} = 3\sqrt{3}.
\end{aligned}$$

Note: equality holds only if $\alpha_0 = \alpha_1 = \alpha_2 = \pi/3$.

Also solved by B. T. Bae (Spain), D. Baralić (Serbia), M. Bataille (France), D. Beckwith, M. Can, C. Curtis, P. P. Dályay (Hungary), P. De (India), Y. Dumont (France), O. Faynshteyn (Germany), V. V. García (Spain), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), H. S. Hwang (Korea), B.-T. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), P. Perfetti (Italy), É. Pité (France), M. A. Prasad (India), S. G. Sáenz (Chile), V. Schindler (Germany), A. S. Shabani (Kosova), A. Stadler (Switzerland), R. Stong, V. Verdiyan (Armenia), Z. Vörös (Hungary), M. Vowe (Switzerland), L. Zhou, "Fejéntaláltuka Szeged" Problem Group (Hungary), GCHQ Problem Solving Group (U. K.), Hofstra University Problem Solvers, Microsoft Research Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before August 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11474. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Valentin Vornicu, Aops-MathLinks forum, San Diego, CA.* (Correction) Show that when x, y, and z are greater than 1,

$$\Gamma(x)^{x^2+2yz}\Gamma(y)^{y^2+2zx}\Gamma(z)^{z^2+2xy} \geq (\Gamma(x)\Gamma(y)\Gamma(z))^{xy+yz+zx}.$$

11483. *Proposed by Éric Pité, Paris, France.* (Correction) The word "nonnegative" should read "positive."

11495. *Proposed by Marc Chamberland, Grinnell College, Grinnell, IA.* Let a, b, and c be rational numbers such that exactly one of $a^2b + b^2c + c^2a$, $ab^2 + bc^2 + ca^2$, and $a^3 + b^3 + c^3 + 6abc$ is zero. Show that $a + b + c = 0$.

11496. *Proposed by Benjamin Bogoşel, student, West University of Timisoara, Timisoara, Romania, and Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* For a matrix X with real entries, let $s(X)$ be the sum of its entries. Prove that if A and B are $n \times n$ real matrices, then

$$n\left(s(AA^T) + s(BB^T) - s(AB^T)s(A^TB)\right) \geq$$
$$s(AA^T)(s(B))^2 + s(BB^T)(s(A))^2 - s(A)s(B)\left(s(AB^T) + s(A^TB)\right).$$

11497. *Proposed by Mihály Bencze, Brasov, Romania.* Given n real numbers $x_1, \ldots, x_n$ and a positive integer m, let $x_{n+1} = x_1$, and put

$$A = \sum_{k=1}^{n}\left(x_k^2 - x_kx_{k+1} + x_{k+1}^2\right)^m, \quad B = 3\sum_{k=1}^{n} x_k^{2m}.$$

Show that $A \leq 3^mB$ and $A \leq (3^mB/n)^n$.

doi:10.4169/000298910X480865

11498. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Let $ABCD$ be a convex quadrilateral. A line through the intersection O of the diagonals AC and BD intersects the interior of edge BC at L and the interior of AD at N. Another line through O likewise meets AB at K and CD at M. This dissects $ABCD$ into eight triangles AKO, KBO, BLO, and so on. Prove that the arithmetic mean of the reciprocals of the areas of these triangles is greater than or equal to the sum of the arithmetic and quadratic means of the reciprocals of the areas of triangles ABO, BCO, CDO, and DAO. (The quadratic mean is also known as the root mean square; it is the square root of the mean of the squares of the given numbers.)

11499. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let H_n be the nth harmonic number, given by $H_n = \sum_{k=1}^{n} 1/k$. Let

$$S_k = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\log k - (H_{kn} - H_n)\right).$$

Prove that for $k \geq 2$,

$$S_k = \frac{k-1}{2k}\log 2 + \frac{1}{2}\log k - \frac{\pi}{2k^2}\sum_{l=1}^{\lfloor k/2 \rfloor}(k+1-2l)\cot\left(\frac{(2l-1)\pi}{2k}\right).$$

11500. *Proposed by Bhavana Deshpande, Poona College, Camp Pune, Maharashtra, India, and M. N. Deshpande, Institute of Science, Nagpur, India.* We have n balls, labeled 1 through n, and n urns, also labeled 1 through n. Ball 1 is put into a randomly chosen urn. Thereafter, as j increments from 2 to n, ball j is put into urn j if that urn is empty, otherwise, it is put into a randomly chosen empty urn. Let the random variable X be the number of balls that end up in the urn bearing their own number. Show that the expected value of X is $n - H_{n-1}$.

11501. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let

$$g(z) = 1 - \frac{3}{1 - \frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of g are all nonnegative if and only if $a \geq \sqrt{3}$.

SOLUTIONS

An Unusual GCD/LCM Relationship

11346 [2008, 167]. *Proposed by Christopher Hillar, Texas A&M University, College Station, TX, and Lionel Levine, University of California, Berkeley, CA.* Let n be an integer greater than 1, and let $S = \{2, \ldots, n\}$. For each nonempty subset A of S, let $\pi(A) = \prod_{j \in A} j$. Prove that when k is a positive integer and $k < n$,

$$\prod_{i=k}^{n} \operatorname{lcm}(\{1, \ldots, \lfloor n/i \rfloor\}) = \gcd(\{\pi(A) : |A| = n-k\}).$$

(In particular, setting $k = 1$ yields $\prod_{i=1}^{n} \operatorname{lcm}(\{1, \ldots, \lfloor n/i \rfloor\}) = n!$.)

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove that both sides equal $\prod_p p^{e_p(n,k)}$, where $e_p(n,k) = \sum_{i=k}^{n} \lfloor \log_p(n/i) \rfloor$ and the product runs over all primes (only finitely many primes contribute). Let $v_p(n)$ denote the maximum r such that p^r divides n.

For the left side, letting $l(x) = \operatorname{lcm}(\{1, \dots, \lfloor x \rfloor\})$, we have $v_p(l(x)) = \lfloor \log_p x \rfloor$, since p^r divides $l(x)$ if and only if $x \geq p^r$. Hence $\prod_{i=k}^{n} l(n/i) = \prod_p p^{e_p(n,k)}$.

For the right side, let $(b_1, \dots, b_{n-1})$ be the result of putting $(v_p(2), \dots, v_p(n))$ in nonincreasing order. The number of terms with $v_p(k) \geq r$ equals the number of multiples of p^r in S, namely $\lfloor n/p^r \rfloor$. Thus $b_k \geq r$ if and only if $k \leq n/p^r$, and hence $b_k = \lfloor \log_p(n/k) \rfloor$. The smallest value of $v_p(\pi(A))$ such that $|A| = n - k$ will be achieved when A consists of exactly the elements of S corresponding to $b_k, \dots, b_{n-1}$. Hence

$$v_p(\gcd(\{\pi(A) : \ |A| = n - k\})) = \sum_{i=k}^{n-1} b_i = e_p(n,k),$$

using the fact that the term for $i = n$ in the summation for $e_p(n,k)$ always equals 0. Applying this formula over all primes shows that the right side also equals $\prod_p p^{e_p(n,k)}$.

Also solved by D. R. Bridges, J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), T. Rucker, K. Schilling, A. Stadler (Switzerland), M. Tetiva (Romania), S. Vandervelde, B. Ward (Canada), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

Some Triangle Inequalities

11363 [2008, 461]. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* Let m_a, m_b, and m_c be the lengths of the medians of a triangle T. Similarly, let I_a, I_b, I_c, h_a, h_b, and h_c be the lengths of the bisectors and altitudes of T, and let R, r, and S be the circumradius, inradius, and area of T. Show that

$$\frac{I_a I_b}{I_c} + \frac{I_b I_c}{I_a} + \frac{I_c I_a}{I_b} \geq 3(2R - r),$$

and

$$\frac{m_a I_b}{h_c} + \frac{m_b I_c}{h_a} + \frac{m_c I_a}{h_b} \geq 3^{5/4}\sqrt{S}.$$

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. We write a, b, c for the lengths of the three sides, and $s = (a + b + c)/2$ for the semiperimeter. We will write $\sum$ or $\prod$ for a three or six term sum or product, respectively, over permutations of the triangle, with three terms if the sum is formally independent of the direction of the cycle, and six if not. Thus, $\sum ab$ denotes $ab + bc + ca$ while $\sum a^2 b = a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2$. We use several results from (or easily deduced from) *Geometric Inequalities* by Bottema et. al. (Nordhoff, Groningen, 1969), including:

$$I_a = \frac{2S}{(b+c)\sin(A/2)}, \quad abc = 4Rrs, \quad \frac{r}{4R} = \prod \sin\frac{A}{2},$$
$$\sum a^2 = 2(s^2 - 4Rr - r^2), \quad \sum a^2 b = 2s(s^2 - 2Rr + r^2),$$
$$\sum a^2 b^2 c = 4Rrs(s^2 + 4Rr + r^2),$$
$$\sum a^3 b^2 = 2s(s^4 + r^4 + 6Rr^3 + 8R^2 r^2 + 2r^2 s^2 - 10Rrs^2),$$
$$\sum a^4 b = 2s(s^4 - 3r^4 - 14Rr^3 - 8R^2 r^2 - 2r^2 s^2 - 6Rrs^2).$$

The first inequality must be reversed. In fact, we will show that

$$\frac{16}{9}(2R-r) < \frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} \le 3(2R-r).$$

We begin with

$$\sum \frac{I_aI_b}{I_c} = \sum \frac{\frac{2S}{(b+c)\sin(A/2)}\,\frac{2S}{(c+a)\sin(B/2)}}{\frac{2S}{(a+b)\sin(C/2)}}$$
$$= \frac{2S}{\prod(a+b)\prod\sin(A/2)} \sum (a+b)^2\sin^2\frac{C}{2}.$$

Now

$$2\sum(a+b)^2\sin^2\frac{C}{2} = \sum(a+b)^2(1-\cos C)$$
$$= 2\sum a^2 + 2\sum ab - \sum a^2\cos C - 2\sum ab\cos C.$$

But $2\sum ab\cos C = \sum(a^2+b^2-c^2) = \sum a^2$, so

$$2\sum a^2 + 2\sum ab - 2\sum ab\cos C = (\sum a)^2 = 4s^2$$

and

$$\sum a^2\cos C = \frac{1}{abc}\sum a^3bc\cos C = \frac{1}{2abc}\sum a^2c(a^2+b^2-c^2)$$
$$= \frac{1}{2abc}\left(\sum a^4c + 2\sum a^2b^2c - \sum a^2c^3\right)$$
$$= \frac{1}{4Rr}\big[s^4 - 3r^4 - 14Rr^3 - 8R^2r^2 - 2r^2s^2 - 6Rrs^2 + 4Rr(s^2 + 4Rr + r^2) - (s^4+r^4+6Rr^3+8R^2r^2+2r^2s^2-10Rrs^2)\big]$$
$$= \frac{2Rs^2 - 4Rr^2 - r^3 - rs^2}{R}.$$

Therefore

$$2\sum(a+b)^2\sin^2\frac{C}{2} = \frac{2Rs^2 + 4Rr^2 + r^3 + rs^2}{R}.$$

Furthermore, $\prod(a+b) = \sum a^2b + 2abc = 2s(s^2+2Rr+r^2)$ and $\prod\sin(A/2) = r/(4R)$. Hence

$$\frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} = \frac{2(2Rs^2+4Rr^2+r^3+rs^2)}{s^2+2Rr+r^2}. \qquad (*)$$

Now by *Geometric Inequalities* (5.9), $4R^2+4Rr+3r^2 \ge s^2 \ge r(16R-5r)$. For our lower bound: $2Rs^2 + 36Rr^2 + 17rs^2 + 17r^3 \ge 32R^2r + 26Rr^2 + 17rs^2 + 17r^3 > 32R^2r$, so $9(2Rs^2+4Rr^2+rs^2+r^3) > 8(2Rs^2+4R^2r-rs^2-r^3) = 8(s^2+2Rr+r^2)(2R-r)$. Hence

$$\frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} > \frac{16}{9}(2R-r).$$

For our upper bound: $R \geq 2r$, so $0 \leq (R-2r)(24R+10r)r = 24R^2r - 38Rr^2 - 10r^3$, and hence $44R^2r - 10Rr^2 \geq 20R^2r + 28Rr^2 + 20r^3$. Therefore $2Rs^2 + 12R^2r \geq 44R^2r - 10Rr^2 \geq 20R^2r + 28Rr^2 + 20r^3 \geq 8Rr^2 + 5rs^2 + 5r^3$, and $3(2R-r)(s^2+2Rr+r^2) = 6Rs^2 + 12R^2r - 3rs^2 - 3r^3 \geq 4Rs^2 + 8Rr^2 + 2r^3 + 2rs^2$. This inequality, in combination with $(*)$, gives

$$\frac{I_aI_b}{I_c} + \frac{I_bI_c}{I_a} + \frac{I_cI_a}{I_b} \leq 3(2R-r).$$

Now consider the second inequality. By elementary calculus, a function of the form $f(x) = x^2 + 2\lambda/x$ achieves its minimum at $x = \lambda^{1/3}$, so $f(x) \geq 3\lambda^{2/3}$.

Letting $\lambda = \prod m_aI_b/h_c$, we have

$$\left(\sum \frac{m_aI_b}{h_c}\right)^2 = \sum \frac{m_a^2I_b^2}{h_c^2} + 2\sum \frac{m_aI_b}{h_c}\frac{m_bI_c}{h_a} = \sum\left(\frac{m_a^2I_b^2}{h_c^2} + 2\lambda\frac{h_c}{m_aI_b}\right) \geq 9\lambda^{2/3}.$$

Denote the exradii of T by r_a, r_b, and r_c. By *Geometric Inequalities* (8.21) and (6.27), we have $m_am_bm_c \geq r_ar_br_c = S^2/r = Ss$. By (8.7) we have

$$I_aI_bI_c = \frac{8a^2b^2c^2}{\prod(a+b)}\prod\cos\frac{A}{2} = \frac{8a^2b^2c^2}{\prod(a+b)}\prod\sqrt{\frac{s(s-a)}{bc}}$$
$$= \frac{8a^2b^2c^2}{\prod(a+b)}\frac{Ss}{abc} = \frac{8abcSs}{\prod(a+b)} = \frac{32RsS^2}{\prod(a+b)},$$
$$h_ah_bh_c = \prod\frac{2S}{a} = \frac{8S^3}{abc} = \frac{2S^3}{Rrs}.$$

Now

$$\lambda = Ss\frac{32RsS^2}{\prod(a+b)}\frac{Rrs}{2S^3} = \frac{16R^2rs^3}{\prod(a+b)} \quad \text{and} \quad \left(\sum\frac{m_aI_b}{h_c}\right)^2 \geq 9\left(\frac{16R^2rs^3}{\prod(a+b)}\right)^{2/3}.$$

By (5.5) and (5.1), $s^2 \geq 3r(4R+r) \geq 3r(9r) = 27r^3$, so $s \geq 3\sqrt{3}r$. By (5.8) $s^2 \leq 4R^2 + 4Rr + 3r^2$, and thus $s^2 + 2Rr + r^2 \leq 4R^2 + 6Rr + 4r^2 \leq 4R^2 + 3R^2 + R^2 = 8R^2$. Hence $\prod(a+b) = \sum a^2b + 2abc = 2s(s^2 - 2Rr + r^2) + 8Rrs = 2s(s^2 + 2Rr + r^2) \leq 2s(8R^2) = 16R^2s$. This leads to $3\sqrt{3}(\prod(a+b))^2 \leq s(16R^2s)^2 = 256R^4s^3$. Now $3^{15/2}S^3 = 3^{15/2}r^2s^3$, and

$$3^{15/2}r^3s^3 \leq 729\frac{256R^4r^2s^6}{\left(\prod(a+b)\right)^2} \Rightarrow 3^{5/2}S \leq 9\left(\frac{16R^2rs^3}{\prod(a+b)}\right)^{2/3} \leq \left(\sum\frac{m_aI_b}{h_c}\right)^2,$$

so that finally $3^{5/4}\sqrt{S} \leq \sum m_aI_b/h_c$.

Also solved by V. V. García (Spain) and R. Stong.

A Multiple of a Prime

11364 [208, 461]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let p be a prime greater than 3, and let t be the integer nearest $p/6$.

(**a**) Show that if $p = 6t+1$, then

$$(p-1)!\sum_{j=0}^{2t-1}(-1)^j\left(\frac{1}{3j+1} + \frac{1}{3j+2}\right) \equiv 0 \pmod{p}.$$

(**b**) Show that if $p = 6t - 1$, then

$$(p-1)!\left(\sum_{j=0}^{2t-1}\frac{(-1)^j}{3j+1}+\sum_{j=0}^{2t-2}\frac{(-1)^j}{3j+2}\right)\equiv 0 \pmod{p}.$$

Solution by Robin Chapman, University of Exeter, Exeter, U. K. The desired congruence in both cases is

$$(p-1)!\sum_{k=1}^{p-1}\frac{\chi(k)}{k}\equiv 0 \pmod{p}, \tag{1}$$

where

$$\chi(k)=\begin{cases} 0 & \text{if } k\equiv 0,3 \pmod{6},\\ 1 & \text{if } k\equiv 1,2 \pmod{6},\\ -1 & \text{if } k\equiv 4,5 \pmod{6}.\end{cases}$$

Note that $\chi(k)=(\zeta^k-\zeta^{-k})/\sqrt{-3}$, where $\zeta=e^{\pi i/3}=\frac{1}{2}(1+\sqrt{-3})$. Letting $F(z)=\sum_{k=1}^{p-1} z^k/k$, we have

$$\sum_{k=1}^{p-1}\frac{\chi(k)}{k}=\frac{F(\zeta)-F(\zeta^{-1})}{\sqrt{-3}}. \tag{2}$$

For the value on the right, note that $F'(z)=\sum_{k=1}^{p-1} z^{k-1}=\frac{1-z^{p-1}}{1-z}$, so $F'(1-z)=\sum_{k=0}^{p-2}(-1)^{k+1}\binom{p-1}{k+1}z^k$. Note also that $\binom{p-1}{j}\equiv(-1)^j \pmod{p}$. Hence $F'(1-z)=pG(z)+F'(z) \pmod{p}$, where G is a polynomial having integer coefficients and degree at most $p-2$. We conclude that

$$\frac{d}{dz}(F(z)-F(1-z))=-pG(z). \tag{3}$$

Let $G(z)=\sum_{k=1}^{p-1} b_k z^{k-1}$ with each $b_k\in\mathbb{Z}$. Integrating (3) from 0 to z gives

$$F(z)-F(1-z)+F(1)=-p\sum_{k=1}^{p-1}\frac{b_k}{k}z^k.$$

Setting $z=\zeta$ and using $1-\zeta=\zeta^{-1}$ yields

$$F(\zeta)-F(\zeta^{-1})=-F(1)-p\sum_{k=1}^{p-1}\frac{b_k}{k}z^k.$$

Since p is odd, $F(1)=\sum_{k=1}^{(p-1)/2}(\frac{1}{k}+\frac{1}{p-k})=\sum_{k=1}^{(p-1)/2}\frac{p}{k(p-k)}$. It follows that $(p-1)!\,F(1)$ is a multiple of p. We conclude that in the context of algebraic integers, $(p-1)!\,(F(\zeta)-F(\zeta^{-1}))\equiv 0 \pmod{p}$. Multiplying by $\sqrt{-3}$ yields a rational integer, and dividing by -3 (justified by $p>3$) and invoking (2) yields the desired congruence (1).

Editorial comment. Stong showed also that $(p-1)!\,F(\zeta)\equiv(p-1)!\,F(\zeta^{-1})\equiv 0 \pmod{p}$, which leads to $(p-1)!\sum_{k=1}^{p-1}\frac{\chi(k+s)}{k}\equiv 0 \pmod{p}$ for every integer s.

Also solved by J. H. Lindsey II, M. A. Prasad (India), A. Stadler (Switzerland), R. Tauraso (Italy), M. Tetiva (Romania), A. Wyn-Jones, GCHQ Problem Solving Group (U. K.), and the proposer.

Relating Two Integer Sequences

11365 [2008, 462]. *Proposed by Aviezri S. Fraenkel, Weizmann Institute of Science, Rehovot, Israel.* Let t be a positive integer. Let $\gamma = \sqrt{t^2+4}$, $\alpha = \frac{1}{2}(2+\gamma-t)$, and $\beta = \frac{1}{2}(2+\gamma+t)$. Show that for all positive integers n,

$$\lfloor n\beta \rfloor = \lfloor(\lfloor n\alpha \rfloor + n(t-1))\alpha\rfloor + 1 = \lfloor(\lfloor n\alpha \rfloor + n(t-1)+1)\alpha\rfloor - 1.$$

Solution I by Donald R. Bridges, Woodstock, MD. Letting $\epsilon = (\gamma - t)/2$, we have $\alpha = 1+\epsilon$ and $\beta = 1+t+\epsilon$. Note that $t^2 < \gamma^2 < (t+2)^2$, so γ and ϵ are irrational and $0 < \epsilon < 1$.

We write the expressions in terms of ϵ. For the first, $\lfloor n\beta \rfloor = n + nt + \lfloor n\epsilon \rfloor$. For the second,

$$\lfloor n\alpha \rfloor + n(t-1) = nt + \lfloor n\epsilon \rfloor,$$
$$(\lfloor n\alpha \rfloor + n(t-1))\alpha = nt + \lfloor n\epsilon \rfloor + nt\epsilon + \lfloor n\epsilon \rfloor \epsilon.$$

Squaring both sides of $\sqrt{t^2+4} = t + 2\epsilon$ yields $t\epsilon + \epsilon^2 = 1$, so $nt\epsilon + n\epsilon^2 = n$. Also, $nt\epsilon + \lfloor n\epsilon \rfloor \epsilon > nt\epsilon + (n\epsilon - 1)\epsilon$, so the floor of the last displayed expression is $nt + \lfloor n\epsilon \rfloor + n - 1$, since $0 < \epsilon < 1$. This proves the first equality.

To compute the rightmost expression in the problem statement, begin with

$$(\lfloor n\alpha \rfloor + n(t-1)+1)\alpha = nt + \lfloor n\epsilon \rfloor + 1 + nt\epsilon + \lfloor n\epsilon + 1 \rfloor \epsilon.$$

Since $nt\epsilon + \lfloor n\epsilon + 1 \rfloor \epsilon \le nt\epsilon + n\epsilon^2 + \epsilon < n+1$, we obtain the desired equality

$$\lfloor(\lfloor n\alpha \rfloor + n(t-1)+1)\alpha\rfloor = \lfloor n\beta \rfloor + 1.$$

Solution II by the proposer. First, observe that α and β are irrational numbers satisfying $1 < \alpha < \beta$ and $\alpha + \beta = \alpha\beta$, and that as a result, $\beta > 2$. It is well known that under these conditions, $A \cup B = \mathbb{N}$, where $A = \{\lfloor n\alpha \rfloor : n \ge 1\}$ and $B = \{\lfloor n\beta \rfloor : n \ge 1\}$.

Since $\beta > 2$, the set B does not contain consecutive integers. Hence each term of B lies between two consecutive terms of A. That is, for each positive integer n there exists m such that $\lfloor m\alpha \rfloor$, $\lfloor n\beta \rfloor$, and $\lfloor (m+1)\alpha \rfloor$ are consecutive integers. Given n, the problem is to determine m.

Among the integers from 1 to $\lfloor n\beta \rfloor$, exactly n lie in B, so $\lfloor n\beta \rfloor - n$ lie in A. Therefore, $m = \lfloor n\beta \rfloor - n$. Thus

$$\lfloor(\lfloor n\beta \rfloor - n)\,\alpha\rfloor, \qquad \lfloor n\beta \rfloor, \qquad \lfloor((\lfloor n\beta \rfloor - n)+1)\,\alpha\rfloor$$

are consecutive integers. It remains only to show that $\lfloor n\beta \rfloor - n = \lfloor n\alpha \rfloor + n(t-1)$. This reduces to $\left\lfloor \frac{1}{2}n(\gamma+t)\right\rfloor = \left\lfloor \frac{1}{2}n(\gamma-t)\right\rfloor + nt$, which is true.

Editorial comment. The claim that $A \cup B = \mathbb{N}$ in Solution II is well known; the proposer cited A. S. Fraenkel, How to beat your Wythoff games opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353–361. The result is so astonishing and yet easily proved that we include a short proof for the reader's pleasure.

First note that $a + b = ab$ is equivalent to $\frac{1}{a} + \frac{1}{b} = 1$. Also, $a, b > 1$. For any $k \in \mathbb{N}$, the number of terms less than k in $A \cup B$ is $\lfloor k/a \rfloor + \lfloor k/b \rfloor$, since a and b are irrational. We compute

$$\left\lfloor \frac{k}{a} \right\rfloor + \left\lfloor \frac{k}{b} \right\rfloor = \left\lfloor \frac{k}{a} \right\rfloor + \left\lfloor k\left(1 - \frac{1}{a}\right) \right\rfloor = k + \left\lfloor \frac{k}{a} \right\rfloor + \left\lfloor \frac{-k}{a} \right\rfloor = k - 1.$$

Similarly, $A \cup B$ contains k terms less than $k+1$. Hence there is exactly one term less than $k+1$ but not less than k; it equals k.

Also solved by R. Chapman (U. K.), P. Corn, C. Curtis, J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

An Exponential Inequality

11369 [2008, 567]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that for all real t, and all $\alpha \geq 2$,

$$e^{\alpha t} + e^{-\alpha t} - 2 \leq \left(e^t + e^{-t}\right)^{\alpha} - 2^{\alpha}.$$

Solution by Knut Dale, Telemark University College, Bø, Norway. For $t \in \mathbb{R}$ and $\alpha \geq 0$, let $f(t,\alpha) = ((e^t + e^{-t})^{\alpha} - 2^{\alpha}) - (e^{\alpha t} + e^{-\alpha t} - 2)$. Since $f(0,\alpha) = 0$ and $f(-t,\alpha) = f(t,\alpha)$, we need only consider $t > 0$. Write

$$\begin{aligned} f(t,\alpha) &= \alpha \int_0^t \left\{ (e^x + e^{-x})^{\alpha} \frac{\sinh x}{\cosh x} - (e^{\alpha x} - e^{-\alpha x}) \right\} dx \\ &= \alpha \int_0^t (e^x + e^{-x})^{\alpha} \big\{ g(x,1) - g(x,\alpha) \big\} dx, \end{aligned}$$

where $g(x,\alpha) = (e^{\alpha x} - e^{-\alpha x})/(e^x + e^{-x})^{\alpha}$. Let $x > 0$ and observe that $g(x,\alpha) \geq 0$, $g(x,2) = g(x,1) > 0$, and $g(x,0) = g(x,\infty) = 0$. Note that

$$\frac{\partial g(x,\alpha)}{\partial \alpha} > 0 \quad \iff \quad \frac{\ln(e^x + e^{-x}) + x}{\ln(e^x + e^{-x}) - x} > e^{2\alpha x}. \qquad (*)$$

Likewise, equivalence holds if we replace "$>$" with "$=$" or with "$<$" throughout $(*)$. Since $e^{2\alpha x}$ is an increasing function of α,

$$\frac{\ln(e^x + e^{-x}) + x}{\ln(e^x + e^{-x}) - x} = e^{2\alpha x}$$

has a unique solution α in the interval $(1,2)$. Thus, as a function of α, $g(x,\alpha)$ increases from 0 to a maximum in $(1,2)$ and then decreases towards 0. Hence $f(t,\alpha) > 0$ for $\alpha \in (0,1) \cup (2,\infty)$, $f(t,\alpha) < 0$ for $\alpha \in (1,2)$, and $f(t,\alpha) = 0$ for $\alpha \in \{0,1,2\}$.

Editorial comment. Grahame Bennett (Indiana University) provided an instructive solution including a general context for this inequality. That solution is now incorporated into a paper, appearing in the current issue of this MONTHLY (see p. 334).

Also solved by F. Alayont, K. Andersen (Canada), R. Bagby, G. Bennett, D. & J. Borwein (Canada), P. Bourdon, P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), K. Endo, G. C. Greubel, J. Grivaux (France), J. A. Grzesik, S. J. Herschkorn, M. Hildebrand, F. Holland (Ireland), A. Incognito & T. Mengesha, V. K. Jenner (Switzerland), O. Kouba (Syria), K.-W Lau (China), W. R. Livingston, O. P. Lossers (Netherlands), K. McInturff, K. Nagasaki (Japan), T. Nakata (Japan), O. Padé (Israel), P. Perfetti (Italy), Á. Plaza & J. M. Pacheco (Spain), D. S. Ross, V. Rutherfoord, B. Schmuland (Canada), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), M. Thaler (Australia), J. Vinuesa (Spain), Z. Vörös (Hungary), T. Wilkerson, Y. Yu, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before September 30, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11502. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* For a triangle with area F, semiperimeter s, inradius r, circumradius R, and heights h_a, h_b, and h_c, show that

$$5(h_a + h_b + h_c) \geq \frac{2Fs}{Rr} + 18r \geq \frac{10r(5R - r)}{R}.$$

11503. *Proposed by K. S. Bhanu, Institute of Science, Nagpur, India, and M. N. Deshpande, Nagpur, India.* We toss an unbiased coin to obtain a sequence of heads and tails, continuing until r heads have occurred. In this sequence, there will be some number R of runs (runs of heads or runs of tails) and some number X of isolated heads. (Thus, with $r = 4$, the sequence $HHTHTTH$ yields $R = 5$ and $X = 2$.) Find the covariance of R and X in terms of r.

11504. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let N be a positive integer and x a positive real number. Prove that

$$\sum_{m=0}^{N} \frac{1}{m!} \left(\sum_{k=1}^{N-m+1} \frac{x^k}{k} \right)^m \geq 1 + x + \cdots + x^N.$$

11505. *Proposed by Bruce Burdick, Roger Williams University, Bristol, RI.* Define $\{a_n\}$ to be the periodic sequence given by $a_1 = a_3 = 1$, $a_2 = 2$, $a_4 = a_6 = -1$, $a_5 = -2$, and $a_n = a_{n-6}$ for $n \geq 7$. Let $\{F_n\}$ be the Fibonacci sequence with $F_1 = F_2 = 1$. Show that

$$\sum_{k=1}^{\infty} \frac{a_k F_k F_{2k-1}}{2k-1} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1} F_{kn+3k-1}} = \frac{\pi}{4}.$$

doi:10.4169/000298910X486003

11506. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Show that for positive integers m and n with $m + n < mn$, and for positive a and b,

$$\sin\left(\frac{\pi}{n}\right)\int_{x=0}^{\infty}\frac{x^{1/n}}{x+a}\,\frac{b^{1/m}-x^{1/m}}{b-x}\,dx = \sin\left(\frac{\pi}{m}\right)\int_{x=0}^{\infty}\frac{x^{1/m}}{x+b}\,\frac{a^{1/n}-x^{1/n}}{a-x}\,dx.$$

11507. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be a positive integer and let R be a plane region of perimeter 1. Inside R there are a finite number of line segments the sum of whose lengths is greater than n. Prove that there exists a line that intersects at least $2n + 1$ of the segments.

11508. *Proposed by Mihály Bencze, Brasov, Romania.* Prove that for all positive integers k there are infinitely many positive integers n such that $kn + 1$ and $(k + 1)n + 1$ are both perfect squares.

SOLUTIONS

Special Divisors of Factorials

11358 [2008, 365]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bârlad, Romania.* Let d be a square-free positive integer greater than 1. Show that there are infinitely many positive integers n such that $dn^2 + 1$ divides $n!$.

Solution I by O. P. Lossers, Technical University of Eindhoven, Eindhoven, The Netherlands. The condition that d is square-free is unnecessary. Consider the following factorization: $x^{105} + 1 = p(x)q(x)r(x)$ (obtained from the factorization of $x^{210} - 1$ in irreducible cyclotomic polynomials), where

$$\begin{aligned}
p(x) &= 1 + x - x^5 - x^6 - x^7 - x^8 + x^{10} + x^{11} + x^{12} + x^{13}\\
&\quad + x^{14} - x^{16} - x^{17} - x^{18} - x^{19} + x^{23} + x^{24},\\
q(x) &= 1 - x^3 + x^6 - x^9 + x^{12} + x^{21} - x^{24} + x^{27} - x^{30} + x^{33},\\
r(x) &= 1 - x + x^2 + x^5 - x^6 + 2x^7 - x^8 + x^9 + x^{12} - x^{13} + x^{14} - x^{15}\\
&\quad + x^{16} - x^{17} - x^{20} - x^{22} - x^{24} - x^{26} - x^{28} - x^{31} + x^{32} - x^{33}\\
&\quad + x^{34} - x^{35} + x^{36} + x^{39} - x^{40} + 2x^{41} - x^{42} + x^{43} + x^{46} - x^{47} + x^{48}.
\end{aligned}$$

For $x \geq 2$, we have $p(x) < x^{25} < q(x) < x^{34} < r(x) < x^{52}$. Taking $n = a^{105}d^{52}$, where $a \geq 1$ is any integer, we have

$$dn^2 + 1 = (da^2)^{105} + 1 = p(da^2)q(da^2)q(da^2).$$

This product divides $n!$, since the three factors are different and all three are less than $a^{104}d^{52}$, which is at most n.

Solution II by João Guerreiro, student, Insituto Superior Técnico, Lisbon, Portugal. We prove the claim for every positive integer d. Let

$$n = dk^2(d+1)^2 + k(d+1) + 1,$$

where k is a positive integer greater than 1. We claim that all such n have the property that $dn^2 + 1$ divides $n!$. With n so defined, we put $m = dn^2 + 1$ and factor the expression for m, obtaining

$$\begin{aligned} m &= d[dk^2(d+1)^2 + k(d+1) + 1]^2 + 1 \\ &= [dk^2(d+1)^2 + 1][d^2k^2(d+1)^2 + d + 2dk(d+1)] + dk^2(d+1)^2 + 1 \\ &= [dk^2(d+1)^2 + 1](d+1)[d^2k^2(d+1) + 2dk + 1] \end{aligned}$$

For $k > 1$, we also have

$$d + 1 < d^2k^2(d+1) + 2dk + 1 < dk^2(d+1)^2 + 1 < n.$$

Since these quantities are distinct integers less than n, their product m divides $n!$.

Solution III by GCHQ Problem Solving Group, Cheltenham, U. K. The Pell equation $y^2 - dn^2 = 1$ has infinitely many positive integer solutions (y, n). If (Y, N) is any solution, then an infinite family of solutions is generated using $(y - n\sqrt{d}) = (Y - N\sqrt{d})^r$ for $r \in \mathbb{N}$. This follows immediately from the standard result that, if (y_0, n_0) is the smallest positive solution, then $(y - n\sqrt{d}) = (y_0 - n_0\sqrt{d})^k$ for $k \in \mathbb{N}$ generates all positive solutions.

For some solution (Y, N) with $Y > 3\sqrt{d}$, generate solutions (y_r, n_r) as above. Use only odd r, so that Y divides y_r. Also make r large enough so that $y_r > 2Y^2$.

Let (y, n) be the solution given by any such r. Let $s = y/Y$, so $s > 2Y$. Since $y^2 = dn^2 + 1 < \frac{9}{4}dn^2$, we have $y < \frac{3}{2}n\sqrt{d}$. Dividing by Y and using $Y > 3\sqrt{d}$ yields $s < n/2$.

Since $y^2 = Y^2s^2$, we have $(dn^2 + 1) \mid Y \cdot 2Y \cdot s \cdot 2s$. Since $2Y < s < n/2$, these four factors are distinct and less than n. Thus their product divides $n!$.

Editorial comment. Most solvers used solutions to the Pell equation. John P. Robertson proved a more general result: whenever d and c are integers not both 0, there are infinitely many positive integers n such that $dn^2 + c$ divides $n!$. The proposer generalized this further: if a, b, and c are not all 0, then there are infinitely many positive integers n such that $an^2 + bn + c$ divides $n!$. The proposer asks whether the result extends to polynomials of higher degree.

Also solved by S. Casey (Ireland), R. Chapman (U. K.), K. Dale (Norway), P. W. Lindstrom, U. Milutinović (Slovenia), J. P. Robertson, B. Schmuland (Canada), N. C. Singer, A. Stadler (Switzerland), R. Stong, Microsoft Research Problems Group, and the proposer.

A Weighted Sum in a Triangle

11368 [2008, 462]. *Proposed by Wei-Dong Jiang, Weihai Vocational College, Weihai, ShanDong, China.* For a triangle of area 1, let a, b, c be the lengths of its sides. Let $s = (a + b + c)/2$. Show that the weighted average of $(s - a)^2$, $(s - b)^2$, and $(s - c)^2$, weighted by the angles opposite a, b, and c respectively, is at least $1/\sqrt{3}$.

Solution by Richard Stong. We begin with a computational lemma.

Lemma. *If $x, y, z > 0$ and $xy + yz + zx = 1$, then*

$$\frac{yz \arctan x}{x} + \frac{zx \arctan y}{y} + \frac{xy \arctan z}{z} \geq \frac{2\pi}{\sqrt{3}}. \tag{1}$$

Proof. Let $\alpha = (2\pi\sqrt{3} - 3)/8 \approx 0.98$, $\beta = \alpha - 3/4 > 0$. Calculus shows that for $t \geq 0$, we have $(\arctan t)/t \geq \alpha - \beta t^2$. Using this, we conclude that the left side of (1) is at least $(xy + yz + zx)\alpha - (x^2yz + y^2zx + z^2xy)\beta$. Applying

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca)$$

(a corollary of the Cauchy–Schwarz inequality), we get

$$x^2yz + y^2zx + z^2xy = (xy)(xz) + (yz)(yx) + (zx)(zy) \le \frac{1}{3}(xy + yz + zx)^2.$$

Since $xy + yz + zx = 1$, the left side of (1) is at least $\alpha - \beta/3 = 2\pi/\sqrt{3}$. $\square$

Now consider a triangle $\triangle ABC$. Let a, b, c be the lengths of the sides opposite A, B, C, respectively. Let $x = r/(s-a)$, $y = r/(a-b)$, and $z = r/(s-c)$, where r is the inradius. The tangents to the incircle from vertex A have length $s - a$, so $A = 2\arctan x$, and symmetrically for the other two vertices. By Heron's formula, the area K is given by $K^2 = s(s-a)(s-b)(s-c)$. But also $K = rs$, so

$$\frac{yz}{x} = \frac{r(s-a)}{(s-b)(s-c)} = \frac{rs(s-a)^2}{K^2} = (s-a)^2,$$

and two other similar equations. Thus the desired inequality follows from the lemma.

Editorial comment. Some solvers pointed out that the problem concerns a *weighted sum* not a *weighted average,* and that the weighted average version is false.

Also solved by J. Grivaux (France), K. McInturff, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

Glaisher–Kinkelin Infinite Product

11371 [2008, 567]. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.* Let A denote the *Glaisher-Kinkelin* constant, given by

$$A = \lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.2824\cdots.$$

Evaluate in closed form

$$A^6 \prod_{n=1}^{\infty} \left(e^{-1}(1+1/n)^n\right)^{(-1)^n}.$$

Solution by Richard Stong, Center for Communications Research, San Diego CA. The terms in the infinite product tend to 1, so it suffices to show that the even-numbered partial products converge. Using Stirling's formula and the definition of A, we obtain

$$\sum_{k=1}^{N} \log k = \log N! = N\log N - N + \frac{1}{2}\log N + \frac{1}{2}\log(2\pi) + \mathrm{O}(1/N),$$

$$\sum_{k=1}^{N} k\log k = \frac{N(N+1)}{2}\log N - \frac{N^2}{4} + \frac{1}{12}\log N + \log A + \mathrm{O}(1/N).$$

Therefore

$$\sum_{k=1}^{2N} (-1)^k \log k = 2\sum_{k=1}^{N} \log(2k) - \sum_{k=1}^{2N} \log k = \frac{1}{2}\log N + \frac{1}{2}\log\pi + \mathrm{O}(1/N),$$

$$\sum_{k=1}^{2N} (-1)^k k\log k = 2\sum_{k=1}^{N} 2k\log(2k) - \sum_{k=1}^{2N} k\log k$$

$$= N\log N + N\log 2 + \frac{1}{4}\log N - \frac{1}{12}\log 2 + 3\log A + \mathrm{O}(1/N).$$

Subtracting twice the second equation from the first yields $\sum_{k=1}^{2N}(-1)^{k-1}(2k-1)\log k = -2N\log(2N) - 6\log A + \frac{1}{6}\log(2\pi^3) + O(1/N)$. Therefore

$$\log\left[\prod_{k=1}^{2N}\left(e^{-1}\left(1+\frac{1}{k}\right)^k\right)^{(-1)^k}\right] = \sum_{k=1}^{2N}(-1)^k\big(k\log(k+1) - k\log k - 1\big)$$

$$= \sum_{k=1}^{2N}(-1)^k\big(k\log(k+1) - k\log k\big)$$

$$= 2N\log(2N+1) + \sum_{k=1}^{2N}(-1)^{k-1}(2k-1)\log k$$

$$= 2N\log\left(\frac{2N+1}{2N}\right) - 6\log A + \frac{1}{6}\log(2\pi^3) + O(1/N).$$

The first term on the right tends to 1. Exponentiate both sides and multiply this result by A^6 to see that the desired limit is $2^{1/6}e\sqrt{\pi}$.

Also solved by J. Borwein (Canada), B. Bradie, B. S. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), G. C. Greubel, J. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, W. R. Livingston, P. Perfetti (Italy), A. Stadler (Switzerland), M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Fibonacci Fixed Points

11373 [2008, 568]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Let S_n be the symmetric group on $\{1,\dots,n\}$. By the canonical cycle decomposition of an element π of S_n, we mean the cycle decomposition of π in which the largest entry of each cycle is at the beginning of that cycle, and the cycles are arranged in increasing order of their first elements.

Let $\psi_n\colon S_n \to S_n$ be the mapping that associates to each $\pi \in S_n$ the permutation whose one-line representation is obtained by removing the parentheses from the canonical cycle decomposition of π. (Thus the permutation $\binom{12345}{34521}$ has one-line representation 34521 and canonical cycle representation (42)(513) and is mapped by ψ_5 to 42513.) Describe the fixed points of ψ_n and find their number.

Solution by John H. Lindsey II, Cambridge, MA. Let $f(n)$ be the number of fixed points of ψ_n.

If π is a fixed point of ψ_n such that $\pi(n) = n$, where $n \ge 1$, then $\pi|_{\{1,\dots,n-1\}}$ is a fixed point of ψ_{n-1}. Conversely, every fixed point π of ψ_{n-1} may be extended to a fixed point of ψ_n by setting $\pi(n) = n$. Hence there are $f(n-1)$ fixed points π of ψ_n with $\pi(n) = n$.

Let π be a fixed point of ψ_n such that $\pi(n) < n$, where $n \ge 2$. Since n is the largest element of its cycle, this cycle in the canonical representation appears as $(n, \pi(n), \dots)$. Thus the one-line representation of $\psi_n(\pi)$ ends with $n, \pi(n), \dots$. Since π is a fixed point of ψ_n and the one-line representation of π ends with $\pi(n)$, it must end with $n, \pi(n)$. Thus $\pi(n-1) = n$, and the cycle of π containing n has only the two elements n and $n-1$. Furthermore, $\pi|_{\{1,\dots,n-2\}}$ is a fixed point of ψ_{n-2}, and conversely every fixed point of ψ_{n-2} yields a fixed point of ψ_n by adding the cycle $(n, n-1)$.

Thus $f(n) = f(n-1) + f(n-2)$ for $n \ge 2$, with $f(0) = f(1) = 1$, so $f(n)$ is the $(n+1)$st Fibonacci number, and the fixed points of ψ_n are products of disjoint transpositions of consecutive integers.

Editorial comment. Marian Tetiva pointed out that a related problem, also proposed by Emeric Deutsch, appeared as problem 1525 in the June 1997 issue of Mathematics Magazine, solved by José Nieto on pages 227–228 of the June 1998 issue (volume 71). That problem asks about the fixed points for a similar mapping in which the canonical representation for permutations puts the smallest entry of each cycle last, with the cycles in increasing order. There are 2^{n-1} fixed points for that mapping.

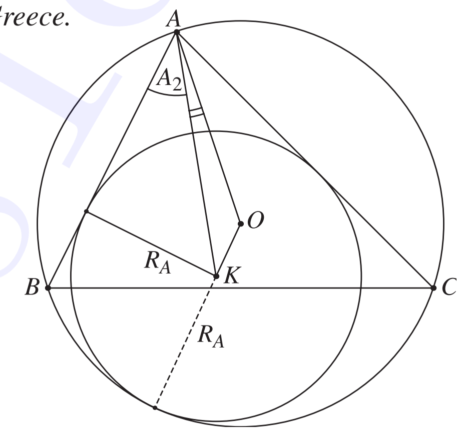
Also solved by R. Bagby, D. Beckwith, J. C. Binz (Switzerland), R. Chapman (U. K.), M. T. Clay, P. Corn, C. Curtis, P. P. Dályay (Hungary), K. David & P. Fricano, M. N. Deshpande & K. Laghale (India), A. Incognito, C. Lanski, O. P. Lossers (Netherlands), R. Martin (Germany), J. H. Nieto (Venezuela), R. Pratt, M. Reid, K. Schilling, E. Schmeichel, B. Schmuland (Canada), P. Spanoudakis (U. K.), R. Stong, J. Swenson, R. Tauraso (Italy), M. Tetiva (Romania), BSI Problems Group (Germany), Szeged Problem Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Houghton College Problem Solving Group, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Circle Radii Related to a Triangle

11386 [2008, 757]. *Proposed by Greg Markowsky, Somerville, MA.* Consider a triangle ABC. Let $\mathcal{O}$ be the circumcircle of ABC, r the radius of the incircle, and s the semiperimeter. Let arc (BC) be the arc of $\mathcal{O}$ opposite A, and define arc (CA) and arc (AB) similarly. Let $\mathcal{O}_A$ be the circle tangent to AB and AC and internally tangent to $\mathcal{O}$ along arc (BC), and let R_A be its radius. Define $\mathcal{O}_B$, $\mathcal{O}_C$, R_B, and R_C similarly. Show that

$$\frac{1}{aR_A}+\frac{1}{bR_B}+\frac{1}{cR_C}=\frac{s^2}{rabc}.$$

Solution by George Apostolopoulos, Greece. Let K be the center of $\mathcal{O}_A$, so $AK = R_A/\sin(A/2)$. Also $AO = R$, $OK = R - R_A$, and $\angle OAK = (\angle B - \angle C)/2$.



The law of cosines gives $OK^2 = AO^2 + AK^2 - 2 \cdot OA \cdot AK \cdot \cos\angle OAK$, or put another way,

$$(R-R_A)^2 = R^2 + \frac{R_A^2}{\sin^2(A/2)} - 2R\frac{R_A}{\sin(A/2)}\cos\frac{B-C}{2}.$$

Therefore

$$-2R + R_A = \frac{R_A}{\sin^2(A/2)} - \frac{2R\cos((B-C)/2)}{\cos((B+C)/2)}.$$

Equivalently,

$$R_A = \frac{4R\sin(A/2)\sin(B/2)\sin(C/2)}{\cos^2(A/2)} = \frac{r}{\cos^2(A/2)} = \frac{rbc}{s(s-a)}.$$

Similarly, $R_B = rca/(s(s-b))$ and $R_C = rab/(s(s-c))$. Thus

$$\frac{1}{aR_A}+\frac{1}{bR_B}+\frac{1}{cR_C} = \frac{s(s-a)}{rabc}+\frac{s(s-b)}{rabc}+\frac{s(s-c)}{rabc} = \frac{3s^2-s(a+b+c)}{rabc} = \frac{s^2}{rabc}.$$

Also solved by B. T. Bae (Spain), D. Baralíc (Serbia), M. Bataille (France), M. Can, C. Curtis, P. P. Dályay (Hungary), P. De (India), Y. Dumont (France), O. Faynshteyn (Germany), V. V. García (Spain), M. Goldenberg & M. Kaplan, J. Grivaux (France), J. G. Heuver (Canada), E. J. Ionascu, B. T. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, A. Nijenhuis, P. Nüesch (Switzerland), V. Schindler (Germany), E. A. Smith, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), G. Tsapakidis (Greece), D. Vacaru (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), J. B. Zacharias & K. Greeson, L. Zhou, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), and the proposer.

Complex Hermitian Matrix

11396 [2008, 856]. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.* For complex z, let $H_n(z)$ denote the $n \times n$ Hermitian matrix whose diagonal elements all equal 1 and whose above-diagonal elements all equal z. For $n \geq 2$, find all z such that $H_n(z)$ is positive semi-definite.

Solution by Mark Wildon, University of Bristol, Bristol, U. K. If z is real, then $\sum_1^n \mathbf{e}_k$ is an eigenvector of $H_n(z)$ with corresponding eigenvalue $1+(n-1)z$, while for $k \in \{2, \dots, n\}$, $\mathbf{e}_1 - \mathbf{e}_k$ is an eigenvector of $H_n(z)$ with corresponding eigenvalue $1-z$. This shows that $1+(n-1)z$ and $1-z$ are the only eigenvalues of $H_n(z)$. Hence $H_n(z)$ is positive semi-definite if and only if

$$-\frac{1}{n-1} \leq z \leq 1.$$

Now suppose that z is not real. By replacing $H_n(z)$ with its transpose, we may assume that $\operatorname{Im} z > 0$. Under this assumption, we shall prove that $H_n(z)$ is positive semi-definite if and only if

$$\arg(z-1) \geq \frac{\arg z}{n} + \frac{(n-1)\pi}{n}$$

(where for $\operatorname{Im} z > 0$ we take $0 < \arg z < \pi$).

We first claim that $H_n(z)$ is singular if and only if

$$\left(\frac{1-z}{1-\overline{z}}\right)^n = \frac{z}{\overline{z}}.$$

Let $w = (1-z)/(1-\overline{z})$ and $v \in \mathbb{C}^n$. The difference between the ith and $(i+1)$st components of $vH_n(z)$ is $(1-z)v_i - (1-\overline{z})v_{i+1}$. Hence $vH_n(z)$ has constant components if and only if v is a scalar multiple of the vector $(1, w, w^2, \dots, w^{n-1})$. The first component of $(1, w, w^2, \dots, w^{n-1})H_n(z)$ is

$$1+\overline{z}(w+w^2+\cdots+w^{n-1}) = \frac{\overline{z}(w^n-w)+w-1}{w-1} = \frac{\overline{z}w^n - z}{w-1}.$$

This vanishes if and only if $w^n = z/\overline{z}$, which proves the claim.

If $\lambda \in \mathbb{R}$ is an eigenvalue of $H_n(z)$ then (provided $\lambda \neq 1$), the matrix $H_n(z/(1-\lambda))$ is singular. Hence $H_n(z)$ has an eigenvalue less than zero if and only if there exists a $\lambda < 0$ such that

$$\left(\frac{1-\lambda-z}{1-\lambda-\overline{z}}\right)^n = \frac{z}{\overline{z}},$$

or equivalently, if and only if

$$\arg(z+\lambda-1) = \frac{\arg z}{n} + \frac{k\pi}{n}$$

for some $\lambda < 0$ and $k \in \mathbb{Z}$. As λ decreases from 0 to $-\infty$, the argument of $z+\lambda-1$ increases from $\arg(z-1)$ to π. Hence if this equation has a solution, it has one with $k = n-1$ and a solution exists if and only if

$$\arg(z-1) < \frac{\arg z}{n} + \frac{(n-1)\pi}{n}.$$

This proves our claimed criterion for $H_n(z)$ to be positive semi-definite.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), A. Stadler (Switzerland), R. Stong, T. Tam, S. E. Thiel, F. Vial (Chile), E. I. Verriest, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

A Max and Min Inequality

11397 [2008, 948]. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Suppose a, b, c, x, y, z are positive numbers such that $a+b+c = x+y+z$ and $abc = xyz$. Show that if $\max\{x, y, z\} \geq \max\{a, b, c\}$, then $\min\{x, y, z\} \geq \min\{a, b, c\}$.

Solution by Marian Tetiva, Bîrlad, Romania. For $q = ab+ac+bc$ and $r = xy+xz+yz$, the identity

$$(t-a)(t-b)(t-c) - (t-x)(t-y)(t-z) = (q-r)t \tag{1}$$

follows from the hypotheses. If $x = \max\{x, y, z\}$ and $z = \min\{x, y, z\}$, for example, and we replace t in (1) by x and z, then we obtain

$$(x-a)(x-b)(x-c) = (q-r)x, \qquad (z-a)(z-b)(z-c) = (q-r)z,$$

respectively. Since $x \geq \max\{a, b, c\}$, it follows that $q-r = (x-a)(x-b)(x-c)/x \geq 0$, which implies $(z-a)(z-b)(z-c) = (q-r)z \geq 0$. This implies that $z \geq \min\{a, b, c\}$.

Editorial comment. Richard Stong remarked that if $a \leq b \leq c$ and $x \leq y \leq z$, then $a \leq x \leq y \leq b \leq c \leq z$.

Also solved by B. M. Abrego, M. Afshar (Iran), K. Andersen (Canada), D. Beckwith, D. Borwein (Canada), R. Brase, P. Budney, R. Chapman (U. K.), J. Christopher, P. Corn, C. Curtis, L. W. Cusick, P. P. Dályay (Hungary), Y. Dumont (France), D. Fleischman, T. Forgács, J. Freeman, D. Grinberg, J. Grivaux (France), E. Hysnelaj & E. Bojaxhiu (Australia & Albania), B.-T. Iordache (Romania), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Nyenhuis (Canada), E. Pité (France), C. Pohoata (Romania), M. A. Prasad (India), R. E. Rogers, J. Schaer (Canada), B. Schmuland (Canada), R. A. Simón (Chile), A. Stadler (Switzerland), J. Steinig (Switzerland), R. Stong, S. E. Thiel, V. Verdiyan (Armenia), E. I. Verriest, J. Vinuesa (Spain), Z. Vörös (Hungary), S. Wagon, H. Widmer (Switzerland), Y. Yu, J. B. Zacharias, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, Northwestern University Math Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before October 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11496 (April, 2010, p. 370) Correction: On the left, square $s(AA^T)$ and $s(BB^T)$.

11509. *Proposed by William Stanford, University of Illinois-Chicago, Chicago, IL.* Let m be a positive integer. Prove that

$$\sum_{k=m}^{m^2-m+1} \frac{\binom{m^2-2m+1}{k-m}}{k\binom{m^2}{k}} = \frac{1}{m\binom{2m-1}{m}}.$$

11510. *Proposed by Vlad Matei, student, University of Bucharest, Bucharest, Romania.* Prove that if I is the n-by-n identity matrix, A is an n-by-n matrix with rational entries, $A \neq I$, p is prime with $p \equiv 3 \pmod 4$, and $p > n+1$, then $A^p + A \neq 2I$.

11511. *Proposed by Retkes Zoltan, Szeged, Hungary.* For a triangle ABC, let f_A denote the distance from A to the intersection of the line bisecting angle BAC with edge BC, and define f_B and f_C similarly. Prove that ABC is equilateral if and only if $f_A = f_B = f_C$.

11512. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let N be a nonnegative integer. For $x \geq 0$, prove that

$$\sum_{m=0}^{N} \frac{1}{m!}\left(\sum_{k=1}^{N-m+1} \frac{x^k}{k}\right)^m \geq 1 + x + \cdots + x^N.$$

11513. *Proposed by Pál Péter Dályay, Szeged, Hungary.* For a triangle with area F, semiperimeter s, inradius r, circumradius R, and heights h_a, h_b, and h_c, show that

$$5(h_a + h_b + h_c) \geq \frac{2sF}{Rr} + 18r \geq \frac{10r(5R-r)}{R}.$$

doi:10.4169/000298910X492862

11514. *Proposed by Mihaly Bencze, Brasov, Romania.* Let k be a positive integer, and let $a_1, \dots, a_n$ be positive numbers such that $\sum_{i=1}^n a_i^k = 1$. Show that

$$\sum_{i=1}^n a_i + \frac{1}{\prod_{i=1}^n a_i} \geq n^{1-1/k} + n^{n/k}.$$

11515. *Proposed by Estelle L. Basor, American Institute of Mathematics, Palo Alto, CA, Steven N. Evans, University of California, Berkeley, CA, and Kent E. Morrison, California Polytechnic State University, San Luis Obispo, CA.* Find a closed-form expression for

$$\sum_{n=1}^{\infty} 4^n \sin^4\left(2^{-n}\theta\right).$$

SOLUTIONS

An Old Four-Squares Chestnut

11374 [2008, 568]. *Proposed by Harley Flanders and Hugh L. Montgomery, University of Michigan, Ann Arbor, MI.* Let a, b, c, and m be positive integers such that $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$.

Solution by Afonso Bandeira and Joel Moreira, Universidade de Coimbra, Portugal, and João Guerreiro, Instituto Superior Técnico, Portugal. Viewing the equation modulo 4 shows that 4 divides m. Let $n = m/4$. Now suppose there is a solution with $n > 1$. Let (a, b, c) be such a solution where $a + b + c$ is minimal. Name the values so that $a \geq b \geq c$.

Now a is a solution to the quadratic equation $x^2 - x(4bcn) + (b^2 + c^2 + 1) = 0$. By Vieta's formula, another solution is a', where $a' = 4bcn - a$. If $a' \geq a$, then $a^2 + b^2 + c^2 + 1 = 4abcn \geq 2a^2$, and so $a^2 \leq b^2 + c^2 + 1 \leq 2b^2 + 1$. Now $a^2 < a^2 + 1 \leq 2b^2 + 2 \leq 4b^2$, so $a < 2b$. This yields $4abcn > 2a^2cn \geq 4a^2 \geq a^2 + b^2 + c^2 + 1$, which contradicts (a, b, c) being a solution.

Thus (a', b, c) is a solution that contradicts the minimality of $a + b + c$. We conclude that $n > 1$ is impossible, so $n = 1$ and $m = 4$.

Editorial comment. We print this proof because of its brevity. A. Hurwitz showed in Über eine Aufgabe der unbestimmten Analysis, *Arch. Math. Phys.* **3** (1907) 185–196, that $x_1^2 + x_2^2 + \cdots + x_n^2 = kx_1x_2 \dots x_n$ has no solution in positive integers if $k > n$, from which the present claim follows directly. This reference was supplied by each of S. Gao, W. C. Jagy, J. H. Jaroma, and J. P. Robertson. A new proof of Hurwitz's theorem may be found in S. Gao, C. Caliskan, and S. Rong, Some properties of n-dimensional generalized Markoff equation, *Congr. Numer.* **177** (2005) 217–221.

Also solved by R. Chapman (U.K.), J. Christopher, P. Corn, S. Gao, H. S. Hwang & K. J. Kim (Korea), I. M. Isaacs, W. C. Jagy, J. H. Jaroma, O. Kouba (Syria), O. P. Lossers (Netherlands), É. Pité (France), C. R. Pranesachar (India), J. P. Robertson, B. Schmuland (Canada), N. C. Singer, R. Stong, H. T. Tang, M. Tetiva (Romania), Fisher Problem Group, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposers.

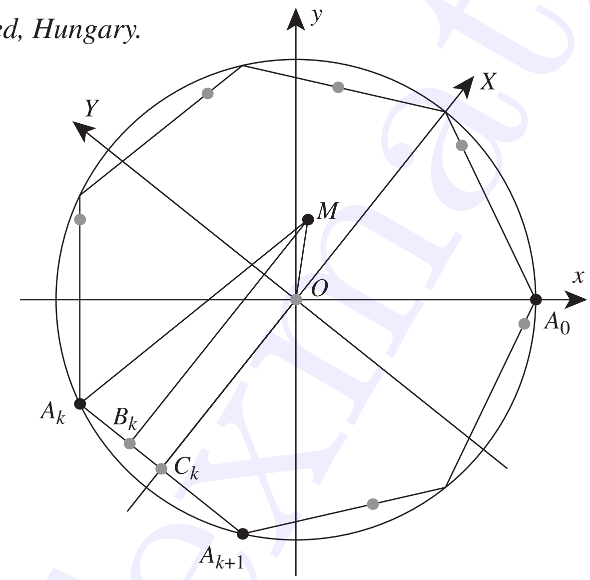
Perpendicular Half-Area

11392 [2008, 855]. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let the consecutive vertices of a regular n-gon P

be denoted $A_0, \dots, A_{n-1}$, in order, and let $A_n = A_0$. Let M be a point such that for $0 \le k < n$ the perpendicular projections of M onto each line $A_k A_{k+1}$ lie interior to the segment (A_k, A_{k+1}). Let B_k be the projection of M onto $A_k A_{k+1}$. Show that

$$\sum_{k=0}^{n-1} \operatorname{Area}(\triangle(M A_k B_k)) = \frac{1}{2}\operatorname{Area}(P).$$

Solution by Pál Péter Dályay, Szeged, Hungary. Select as unit of length the radius of the circumcircle of the regular n-gon. Use the coordinate system xOy in the plane so that vertex A_k has coordinates (x_k, y_k) with $x_k = \cos(2k\pi/n)$ and $y_k = \sin(2k\pi/n)$ for $0 \le k \le n$. Let M have coordinates $(\rho\cos\phi, \rho\sin\phi)$. Fix one index k. If the axes of coordinate system xOy are rotated by the angle $\big((2k+1)\pi\big)/n - \pi$, then we obtain the axes of a new coordinate system XOY. Note $\angle XOM = \phi - \big((2k+1)\pi\big)/n + \pi$. Let H_k be the point where $B_k M$ crosses OY, and let C_k be the midpoint of the segment $A_k A_{k+1}$. Since the axes of the coordinate system XOY are parallel to $B_k M$ and $A_k A_{k+1}$, respectively, we have



$$\begin{aligned} A_k B_k = A_k C_k - B_k C_k &= \frac{1}{2} A_k A_{k+1} - \rho \sin\left(\phi - \frac{(2k+1)\pi}{n} + \pi\right) \\ &= \sin\left(\frac{\pi}{n}\right) + \rho\sin\left(\phi - \frac{(2k+1)\pi}{n}\right), \\ B_k M = C_k O + H_k M &= C_k O + \rho\cos\left(\phi - \frac{(2k+1)\pi}{n} + \pi\right) \\ &= \cos\left(\frac{\pi}{n}\right) - \rho\cos\left(\phi - \frac{(2k+1)\pi}{n}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} &2\operatorname{Area}\big(\triangle(M A_k B_k)\big) = A_k B_k \cdot B_k M \\ &= \left(\sin\left(\frac{\pi}{n}\right) + \rho\sin\left(\phi - \frac{(2k+1)\pi}{n}\right)\right)\left(\cos\left(\frac{\pi}{n}\right) - \rho\cos\left(\phi - \frac{(2k+1)\pi}{n}\right)\right) \\ &= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) + \rho\sin\left(\phi - \frac{2(k+1)\pi}{n}\right) - \frac{\rho^2}{2}\sin\left(2\phi - \frac{2(2k+1)\pi}{n}\right). \end{aligned} \tag{1}$$

Recall that for $\alpha, \beta \in \mathbb{R}$ and $\beta \ne 2s\pi$ with $s \in \mathbb{Z}$,

$$\sum_{k=0}^{n-1} \sin(\alpha + k\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)} \sin\left(\alpha + \frac{1}{2}(n-1)\beta\right).$$

Thus, since $n \geq 3$ implies that $2\pi/n \neq 2s\pi$ with $s \in \mathbb{Z}$, and since $\phi - 2(k+1)\pi/n = (\phi - 2\pi/n) - 2k\pi/n$, we have

$$\sum_{k=0}^{n-1} \sin\left(\left(\phi - \frac{2(k+1)\pi}{n}\right)\right) = 0 = \sum_{k=0}^{n-1} \sin\left(2\phi - \frac{2(2k+1)\pi}{n}\right).$$

Summing both sides of (1) over k, we obtain the required result:

$$2\sum_{k=0}^{n-1} \operatorname{Area}\big(\triangle(MA_kB_k)\big) = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right),$$

and this last expression gives the area of P.

Also solved by M. Bataille (France), D. Beckwith, R. Chapman (U.K.), C. Curtis, J. Freeman, D. Grinberg, J.-P. Grivaux (France), K. Hanes, E. A. Herman, S. Hitotumatu (Japan), E. J. Ionascu, L. R. King, P. T. Krasopoulos (Greece), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mihai (Canada), C. R. Pranesachar (India), M. A. Prasad (India), R. A. Russell, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), J. Vinuesa (Spain), A. Vorobyov, Z. Vörös (Hungary), M. Vowe (Switzerland), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

Concurrent Lines

11393 [2008, 856]. *Proposed by Cosmin Pohoata, student, National College "Tudor Vianu," Bucharest, Romania.* In triangle ABC, let M and Q be points on segment AB, and similarly let N and R be points on AC, and P and S, points on BC. Let d_1 be the line through M, N, d_2 the line through P, Q, and d_3 the line through R, S. Let $\rho(X, Y, Z)$ denote the ratio of the length of XZ to that of XY. Let $m = \rho(M, A, B)$, $n = \rho(N, A, C)$, $p = \rho(P, B, C)$, $q = \rho(Q, B, A)$, $r = \rho(R, C, A)$, and $s = \rho(S, C, B)$. Prove that the lines (d_1, d_2, d_3) are concurrent if and only if $mpr + nqs + mq + nr + ps = 1$.

Solution by Michel Bataille, Rouen, France. We use barycentric coordinates relative to (A, B, C), and accordingly we write $U(u_1, u_2, u_3)$ as an abbreviation for "$U = (u_1A + u_2B + u_3C)/(u_1 + u_2 + u_3)$." (When $u_1 + u_2 + u_3 = 0$ we obtain a "point at infinity"). With this convention we have $M(m, 1, 0)$, $N(n, 0, 1)$, $P(0, p, 1)$, $Q(1, q, 0)$, $R(1, 0, r)$, and $S(0, 1, s)$. The equation of line d_1 is

$$\begin{vmatrix} x & m & n \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} = 0, \qquad \text{that is,} \qquad x = my + nz.$$

Similarly, the equation of line d_2 is $y = pz + qx$, and the equation of line d_3 is $z = rx + sy$. These three lines are parallel (concurrent at a point at infinity) or concurrent (literally) if and only if

$$\begin{vmatrix} -1 & q & r \\ m & -1 & s \\ n & p & -1 \end{vmatrix} = 0.$$

This can be rewritten as

$$mpr + nqs + mq + nr + ps = 1, \qquad (*)$$

so this is a necessary condition for concurrence of d_1, d_2, d_3.

Conversely, suppose that $(*)$ holds. If d_1, d_2, d_3 were parallel, then the point at infinity on d_1, namely $(n - m, -1 - n, 1 + m)$, would also lie on d_2 and d_3. This means

$mq = 1 + n + p + mp + qn$ and $nr = 1 + s + m + sn + rm$. Since m, n, q, r, s are nonnegative, it follows that $mq + nr \geq 2$. But $mq + nr \leq 1$ follows from $(*)$. This contradiction shows that d_1, d_2, d_3 cannot be parallel, and must instead be concurrent.

Also solved by R. Chapman (U.K.), P. P. Dályay (Hungary), M. Goldenberg & M. Kaplan, D. Grinberg, J. Grivaux (France), S. Hitotumatu (Japan), B.-T. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, R. Nandan, C. R. Pranesachar (India), R. Stong, M. Tetiva (Romania), R. S. Tiberio, A. Vorobyov, Z. Vörös (Hungary), J. B. Zacharias, GCHQ Problem Solving Group (U.K.), and the proposer.

Jensenoid Inequalities

11399 [2008, 948]. *Proposed by Biaggi Ricceri, University of Catania, Catania, Italy.* Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with finite nonzero measure M, and let $p > 0$. Let f be a lower semicontinuous function on $\mathbb{R}$ with the property that f has no global minimum, but for each $\lambda > 0$, the function $t \mapsto f(t) + \lambda|t|^p$ does have a unique global minimum. Show that exactly one of the two following assertions holds:
(a) For every $u \in L^p(\Omega)$ that is not essentially constant,

$$Mf\left(\left(\frac{1}{M}\int_\Omega |u(x)|^p \, d\mu\right)^{1/p}\right) < \int_\Omega f(u(x))\, d\mu,$$

and $f(t) < f(s)$ whenever $t > 0$ and $-t \leq s < t$.
(b) For every $u \in L^p(\Omega)$ that is not essentially constant,

$$Mf\left(-\left(\frac{1}{M}\int_\Omega |u(x)|^p \, d\mu\right)^{1/p}\right) < \int_\Omega f(u(x))\, d\mu,$$

and $f(-t) < f(s)$ whenever $t > 0$ and $-t < s \leq t$.

Solution by Julien Grivaux, student, Université Pierre et Marie Curie, Paris, France. First note that we may assume that $p = 1$. Indeed, let $\theta\colon \mathbb{R} \to \mathbb{R}$ be defined by $\theta(t) = \operatorname{signum}(t)|t|^p$, and let $\widetilde{f}(t) = f(\theta^{-1}(t))$ and $\widetilde{u}(t) = \theta(u(t))$. Then

$$\int_\Omega \widetilde{f}\big(\widetilde{u}(t)\big) = \int_\Omega f\big(u(t)\big) \qquad \text{and} \qquad \widetilde{f}\left(\pm\int_\Omega |\widetilde{u}|\right) = f\left(\pm\left(\int_\Omega |u|^p\right)^{1/p}\right).$$

We may also assume without loss of generality that $M = 1$.

For $\lambda > 0$, let $\phi(\lambda)$ be the unique value where the function $t \mapsto f(t) + \lambda|t|$ reaches its minimum.

Lemma 1. *The function ϕ is continuous on $(0, \infty)$.*

Proof. Let λ be positive and let $\langle\lambda_n\rangle$ be a sequence of positive numbers converging to λ. Letting $t_n = \phi(\lambda_n)$, we have $f(t) + \lambda_n|t| \geq f(t_n) + \lambda_n|t_n|$. Let λ_0 be such that $0 < \lambda_0 < \lambda$ and $m = \inf_{\mathbb{R}}(f(t) + \lambda_0|t|)$. Now

$$f(t_n) + \lambda_n|t_n| = f(t_n) + \lambda_0|t_n| + (\lambda_n - \lambda_0)|t_n| \geq m + (\lambda_n - \lambda_0)|t_n|.$$

This proves that for all t, $(\lambda_n - \lambda_0)|t_n| \leq f(t) + \lambda_n|t| - m$, so that for n large enough that $\lambda_n - \lambda_0 > \frac{1}{2}(\lambda - \lambda_0)$, taking $t = 0$ gives $|t_n| < 2(f(0) - m)/(\lambda - \lambda_0)$. Thus $\langle t_n\rangle$ is bounded. Let t' be a limit point of $\langle t_n\rangle$. There exists a subsequence $\langle t_{\psi(n)}\rangle$ which converges to t'. For all t in $\mathbb{R}$, $f(t) + \lambda_n|t| \geq f(t_n) + \lambda_n|t_n|$. By lower semicontinuity, for all t,

$$\begin{aligned} f(t) + \lambda|t| &= \liminf\left[f(t) + \lambda_{\psi(n)}|t|\right] \geq \liminf\left[f(t_{\psi(n)}) + \lambda_{\psi(n)}\left|t_{\psi(n)}\right|\right] \\ &= \liminf f(t_{\psi(n)}) + \lambda|t'| \geq f(t') + \lambda|t'|. \end{aligned}$$

By the uniqueness of the minimum, $t' = \phi(\lambda)$. Since $\langle t_n \rangle$ is bounded we conclude that $\langle t_n \rangle$ converges to $\phi(\lambda)$. This shows that ϕ is continuous. ■

Lemma 2. $\lim_{\lambda\to+\infty} \phi(\lambda) = 0$ *and* $\lim_{\lambda\to 0^+} |\phi(\lambda)| = +\infty$.

Proof. Let $\langle \lambda_n \rangle$ be a sequence such that $\lim_{n\to\infty} \lambda_n = +\infty$, and let $t_n = \phi(\lambda_n)$. For $t \in \mathbb{R}$, we have $f(t_n)/\lambda_n + |t_n| \le f(t)/\lambda_n + |t|$, and in particular $f(t_n)/\lambda_n + |t_n| \le f(0)/\lambda_n$. Let λ_0 be a fixed positive value, and let $m = \inf_{\mathbb{R}}[f(t) + \lambda_0|t|]$. Now $f(t_n) \ge m - \lambda_0|t_n|$, so $(1 - \lambda_0/\lambda_n)|t_n| \le \big(f(0) - m\big)/\lambda_n$. Therefore $\lim_{n\to\infty} t_n = 0$.

For the other claim of the lemma, let $\langle \lambda_n \rangle$ be a positive sequence that tends to zero, let $t_n = \phi(\lambda_n)$, and let t' be a limit point of $\langle t_n \rangle$ (if one exists). The argument of Lemma 1 proves that for any real t, $f(t) \ge f(t')$. That makes $f(t')$ a global minimum for f, contrary to the hypothesis. Since $\langle t_n \rangle$ has no limit point, $\lim_{n\to\infty} |t_n| = +\infty$. ■

From these two lemmas, we see that the range of ϕ contains $(0, \infty)$ or $(-\infty, 0)$ (but not both). We will show that in the first case conclusion (a) holds. Similarly, the second case leads to (b).

Assume the range contains $(0, \infty)$, and let $m(\lambda) = \inf_{\mathbb{R}} \big(f(t) + \lambda|t|\big)$. Now $f(t) \ge \sup_\lambda \big(m(\lambda) - \lambda|t|\big)$. If $t = \phi(\lambda)$, then $f(\phi(\lambda)) = m(\lambda) - \lambda|\phi(\lambda)|$. Thus f is the pointwise supremum of a family of affine functions on $(0, \infty)$, so f is convex there. We claim that f is actually *strictly* convex. Indeed, if f is affine on some interval $[a, b]$ with $0 < a < b$, then we can choose λ such that the function f_λ given by $f_\lambda(t) = f(t) + \lambda|t|$ reaches its infimum at a point of (a, b). Since f_λ is is affine on this interval, it is minimized at an interior point only if it is constant on that interval, which contradicts the uniqueness of the minimum point.

Let s, t be given with $t > 0$ and $-t \le s < t$. There exists λ such that $t = \phi(\lambda)$. Thus

$$f(s) + \lambda|s| > f(t) + \lambda|t| \ge f(t) + \lambda|s|.$$

We obtain $f(s) > f(t)$. (If $-t \le s \le t$, we obtain $f(s) \ge f(t)$.) For the integral inequality, we have $-|u(x)| \le u(x) \le |u(x)|$. So $f(u(x)) \ge f\big(|u(x)|\big)$. Since f is convex, Jensen's inequality yields

$$\int_\Omega f(u) \ge \int_\Omega f\big(|u|\big) \ge f\left(\int_\Omega |u|\right).$$

It is a strict inequality since u is not essentially constant and f is strictly convex.

Also solved by R. Stong.

Squares On Graphs

11402 [2008, 949]. *Proposed by Doru Catalin Barboianu, Infarom Publishing, Craiova, Romania* Let $f : [0, 1] \to [0, \infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval (0,1) on the x-axis and the other two vertices on the graph of f.

Solution by Byron Schmuland and Peter Hooper, University of Alberta, Edmonton, AB, Canada. Extend f by letting $f(x) = 0$ for $x \ge 1$. Define $g(x) = f(x + f(x)) - f(x)$ for $x \ge 0$. If there exists $x \in (0, 1)$ with $g(x) = 0$, then a square as required exists with vertices

$$(x, 0), \quad (x + f(x), 0), \quad (x, f(x)), \quad (x + f(x), f(x)).$$

Now g is continuous, so to show that such x exists we will show that $y, z \in (0,1)$ exist with $g(y) \geq 0$ and $g(z) \leq 0$. Let z be a value where f takes its maximum. Then $f(z) \geq f(z + f(z))$, so that $g(z) \leq 0$. Since $0 + f(0) = 0 < z < z + f(z)$, by continuity there is a value $y \in (0, z)$ so that $y + f(y) = z$. Hence $g(y) = f(y + f(y)) - f(y) = f(z) - f(y) \geq 0$.

Editorial comment. Pál Péter Dálylay (Hungary) noted a generalization: Given any $p > 0$, there exists a rectangle with base-to-height ratio p having two vertices on the x-axis and the other two vertices on the graph of f.

Also solved by B. M. Ábrego & S. Fernández-Merchant, F. D. Ancel, K. F. Andersen (Canada), R. Bagby, N. Caro (Brazil), D. Chakerian, R. Chapman (U.K.), B. Cipra, P. Corn, C. Curtis, P. P. Dálylay (Hungary), C. Diminnie & R. Zarnowski, P. J. Fitzsimmons, D. Fleischman, T. Forgács, O. Geupel (Germany), D. Grinberg, J. Grivaux (France), J. M. Groah, E. A. Herman, S. J. Herschkorn, E. J. Ionascu, A. Kumar & C. Gibbard (U.S.A. & Canada), S. C. Locke, O. P. Lossers (Netherlands), R. Martin (Germany), K. McInturff, M. McMullen, M. D. Meyerson R. Mortini M. J. Nielsen, M. Nyenhuis (Canada), Á. Plaza & S. Falcón (Spain), K. A. Ross, T. Rucker, J. Schaer (Canada), K. Schilling, E. Shrader, A. Stadler (Switzerland), R. Stong, B. Taber, M. Tetiva (Romania), T. Thomas (U.K.), J. B. Zacharias & K. Greeson, BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Lafayette College Problem Group, Microsoft Research Problems Group, Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposer.

A Trig Series Rate

11410 [2009, 83]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* For $0 < \phi < \pi/2$, find

$$\lim_{x\to 0} x^{-2}\left(\frac{1}{2}\log\cos\phi + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{\sin^2(nx)}{(nx)^2}\sin^2(n\phi)\right).$$

Solution by Otto B. Ruehr, Michigan Technological University, Houghton, MI. We begin with three elementary identities. The first is

$$\sum_{n=1}^{\infty} r^n \sin^2 n\phi = \frac{r(r+1)\sin^2\phi}{(1-r)\big[(1-r)^2 + 4r\sin^2\phi\big]}. \tag{i}$$

This is derived by writing $\sin^2 n\phi$ in terms of exponentials and summing the resulting geometric series. Now divide (i) by r and integrate with respect to r to get

$$\sum_{n=1}^{\infty}\frac{r^n}{n}\sin^2 n\phi = \frac{1}{4}\log\left[\frac{(1-r)^2 + 4r\sin^2\phi}{(1-r)^2}\right]. \tag{ii}$$

Differentiate (i) with respect to r to obtain

$$\sum_{n=1}^{\infty} nr^{n-1}\sin^2 n\phi = \frac{1}{2(1-r)^2} - \frac{1}{2}\left[\frac{(r-1)^2 - 2(r^2+1)\sin^2\phi}{[(1-r)^2 + 4r\sin^2\phi]^2}\right]. \tag{iii}$$

The limit at $r = -1$ in (ii) gives us

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin^2 n\phi = -\frac{1}{2}\log\cos\phi.$$

Now we can write the requested limit as

$$\lim_{x\to 0} x^{-2} \lim_{r\to -1^+}\sum_{n=1}^{\infty}\frac{r^n}{n}\left[1 - \frac{\sin^2 nx}{n^2x^2}\right]\sin^2 n\phi.$$

Here we have anticipated the divergent series that would result if the $\lim_{x\to 0}$ were taken directly. Since the series as written is convergent, by the regularity of the Abel summation process it is equal to its Abel sum. Now, for $|r| < 1$, we can bring the outer limit under the sum, which yields

$$\lim_{r\to -1^+} \frac{1}{3}\sum_{n=1}^{\infty} nr^n \sin^2 n\phi.$$

From (iii) we obtain $\frac{1}{24}\tan^2\phi$ as the desired limit.

Also solved by R. Bagby, D. H. Bailey & J. M. Borwein (Canada), D. Beckwith, P. Bracken, R. Chapman (U.K.), H. Chen, P. P. Dályay (Hungary), J. Grivaux (France), F. Holland (Ireland), K. L. Joiner, G. Keselman, A. Stadler (Switzerland), R. Stong, E. I. Verriest, and the proposer.

A Minimum Determinant

11415 [2009, 180]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let $(A_1, \ldots, A_n)$ be a list of n positive-definite 2×2 matrices of complex numbers. Let G be the group of all unitary 2×2 complex matrices, and define the function F on the Cartesian product G^n by

$$F(U) = F(U_1, \ldots, U_n) = \det\left(\sum_{k=1}^{n} U_k^* A_k U_k\right).$$

Show that

$$\min_{U\in G^n} F(U) = \sum_{k=1}^{n}\sigma_1(A_k)\cdot\sum_{k=1}^{n}\sigma_2(A_k),$$

where $\sigma_1(A_j)$ and $\sigma_2(A_j)$ denote the greatest and least eigenvalue of A_j, respectively.

Solution by Roger A. Horn, University of Utah, Salt Lake City, UT. It suffices to assume that the matrices A_i are positive semidefinite and therefore Hermitian. Let $A = \sum_{i=1}^n U_i^* A_i U_i$, $\alpha = \sum_{i=1}^n \sigma_1(A_i)$, and $\beta = \sum_{i=1}^n \sigma_2(A_i)$. Note that $\alpha \ge \beta \ge 0$ and $(\alpha+\beta)/2 \ge \beta$. Let $\lambda = \sigma_1(A)$ and $\mu = \sigma_2(A)$, so that $\lambda \ge \mu$ and $\lambda + \mu = \operatorname{tr}(A) = \sum_{i=1}^n \operatorname{tr}(U_i^* A_i U_i) = \sum_{i=1}^n \operatorname{tr} A_i = \sum_{i=1}^n(\sigma_1(A_i)+\sigma_2(A_i)) = \alpha+\beta$.

For Hermitian matrices C and D, Weyl's inequality ensures that $\sigma_2(C)+\sigma_2(D) \le \sigma_2(C+D)$. From this along with the definition of A it follows that $\mu = \sigma_2(A) \ge \sum_{i=1}^n \sigma_2(U_i^* A_i U_i) = \sum_{i=1}^n \sigma_2(A_i) = \beta$. Since $\det A = \lambda\mu$, we want to determine $\min\{\lambda\mu : \lambda+\mu = \alpha+\beta \text{ and } \lambda \ge \mu \ge \beta\}$. That is, for $f(\mu) = (\alpha+\beta-\mu)\mu$, we require $\min\{f(\mu) : \beta \le \mu \le \frac{1}{2}(\alpha+\beta)\}$. Clearly, $f'(\mu) = \alpha+\beta-2\mu \ge 0$ for $\mu \in [\beta, \frac{1}{2}(\alpha+\beta)]$, so the minimum value of $f(\mu)$ is $f(\beta) = \alpha\beta$.

If the unitary matrices are chosen such that $U_i^* A_i U_i = \operatorname{diag}(\sigma_1(A_i), \sigma_2(A_i))$ for $i = 1, \ldots, n$, then $A = \operatorname{diag}(\alpha, \beta)$, and it follows that $\det(A) = \alpha\beta$.

Also solved by R. Chapman (U.K.), M. J. Englefield (Australia), J.-P. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Stong, M. Tetiva (Romania), E. I. Verriest, GCHQ Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before December 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11516. *Proposed by Elton Bojaxhiu, Albania, and Enkel Hysnelaj, Australia.* Let $\mathcal{T}$ be the set of all nonequilateral triangles. For T in $\mathcal{T}$, let O be the circumcenter, Q the incenter, and G the centroid. Show that $\inf_{T\in\mathcal{T}} \angle OGQ = \pi/2$.

11517. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f be a three-times differentiable real-valued function on $[a, b]$ with $f(a) = f(b)$. Prove that

$$\left|\int_a^{(a+b)/2} f(x)\,dx - \int_{(a+b)/2}^{b} f(x)\,dx\right| \le \frac{(b-a)^4}{192} \sup_{x\in[a,b]} |f'''(x)|.$$

11518. *Proposed by Mihaly Bencze, Brasov, Romania.* Suppose $n \ge 2$ and let $\lambda_1, \dots, \lambda_n$ be positive numbers such that $\sum_{k=1}^{n} 1/\lambda_k = 1$. Prove that

$$\frac{\zeta(\lambda_1)}{\lambda_1} + \sum_{k=2}^{n} \frac{1}{\lambda_k}\left(\zeta(\lambda_k) - \sum_{j=1}^{k-1} j^{-\lambda_k}\right) \ge \frac{1}{(n-1)(n-1)!}.$$

11519. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m},$$

where H_n denotes the nth harmonic number.

11520. *Proposed by Peter Ash, Cambridge Math Learning, Bedford, MA.* Let n and k be integers with $1 \le k \le n$, and let A be a set of n real numbers. For i with $1 \le i \le n$, let S_i be the set of all subsets of A with i elements, and let $\sigma_i = \sum_{s\in S_i} \max(s)$. Express the kth smallest element of A as a linear combination of $\sigma_0, \dots, \sigma_n$.

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11521. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be a positive integer and let $A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n$ be points on the unit two-dimensional sphere $\mathbb{S}_2$. Let $d(X, Y)$ denote the geodesic distance on the sphere from X to Y, and let $e(X, Y)$ be the Euclidean distance across the chord from X to Y. Show that

(**a**) There exists $P \in \mathbb{S}_2$ such that $\sum_{i=1}^n d(P, A_i) = \sum_{i=1}^n d(P, B_i) = \sum_{i=1}^n d(P, C_i)$.
(**b**) There exists $Q \in \mathbb{S}_2$ such that $\sum_{i=1}^n e(Q, A_i) = \sum_{i=1}^n e(Q, B_i)$.
(**c**) There exist a positive integer n, and points $A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n$ on $\mathbb{S}_2$, such that for all $R \in \mathbb{S}_2$, $\sum_{i=1}^n e(R, A_i)$, $\sum_{i=1}^n e(R, B_i)$, and $\sum_{i=1}^n e(R, C_i)$ are not all equal. (That is, part (b) cannot be strengthened to read like part (a).)

11522. *Proposed by Moubinool Omarjee, Lycée Jean Lurçat, Paris, France.* Let E be the set of all real 4-tuples (a, b, c, d) such that if $x, y \in \mathbb{R}$, then $(ax + by)^2 + (cx + dy)^2 \le x^2 + y^2$. Find the volume of E in $\mathbb{R}^4$.

SOLUTIONS

Cevian Subtriangles

11404 [2009, 83]. *Proposed by Raimond Struble, North Carolina State at Raleigh, Raleigh, NC.* Any three non-concurrent cevians of a triangle create a subtriangle. Identify the sets of non-concurrent cevians which create a subtriangle whose incenter coincides with the incenter of the primary triangle. (A *cevian* of a triangle is a line segment joining a vertex to an interior point of the opposite edge.)

Solution by M. J. Englefield, Monash University, Victoria, Australia. Label the vertices of the primary triangle ABC in counterclockwise order, and let I be the incenter. The following construction identifies the required triples of cevians. Take an arbitrary cevian AA' not passing through I and consider the circle κ centered at I tangent to AA', say at P_A. There are two points on κ for which the line joining them to B is tangent to κ. Choose for P_B the one that is counterclockwise from P_A on κ, and take B' to be the intersection of the line through B and P_B with AC. Similarly choose P_C to lie counterclockwise from P_B on κ, and let C' be the intersection of AB with the tangent from C to κ at P_C. By construction, κ is the incircle of the subtriangle.

Editorial comment. Little attention has been given to the subtriangle that is the topic of this problem. If the non-concurrent cevians divide the sides of $\triangle ABC$ in ratios λ, μ, ν, Routh's theorem gives the area of the subtriangle as $(\lambda\mu\nu - 1)^2/((\lambda\mu + \lambda + 1)(\mu\nu + \mu + 1)(\nu\lambda + \nu + 1))$ times the area of ABC. It is also known (H. Bailey, Areas and centroids for triangles within triangles, *Math. Mag.* **75** (2002) 371) that the centroids of the two triangles coincide if and only if $\lambda = \mu = \nu$.

Also solved by R. Chapman (U. K.), C. Curtis, J. H. Lindsey II, M. D. Meyerson, J. Schaer (Canada), R. A. Simon (Chile), R. Stong, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

A Limit of an Alternating Series

11412 [2009, 179]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let f be a monotone decreasing function on $[0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$. Define F on $(0, \infty)$ by $F(x) = \sum_{n=0}^{\infty} (-1)^n f(nx)$.
(**a**) Show that if f is continuous at 0 and convex on $[0, \infty)$, then $\lim_{x\to 0^+} F(x) = f(0)/2$.

(**b**) Show that the same conclusion holds if we drop the second condition on f from (**a**) and instead require that f have a continuous second derivative on $[0, \infty)$ such that $\int_0^\infty |f''(x)|\,dx < \infty$.
(**c**) Dropping the conditions of (**a**) and (**b**), find a monotone decreasing function f on $[0, \infty)$ with $f(0) > 0$ such that

$$\limsup_{x\to 0^+} \sup_{0<y<x} F(y) = f(0), \qquad \limsup_{x\to 0^+} \inf_{0<y<x} F(y) = 0.$$

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. For f a monotone decreasing function on $[0, \infty)$ with $\lim_{x\to\infty} f(x) = 0$, define

$$F(x) = \sum_{n=0}^{\infty} (-1)^n f(nx) = \sum_{n=0}^{\infty} [f(2nx) - f((2n+1)x)], \qquad x > 0.$$

By the alternating series test, the series defines $F(x)$ with $0 \le F(x) \le f(0)$.

(**a**) If f is convex, then for each $x > 0$, the difference $f(kx) - f((k+1)x)$ is a nonincreasing function of the positive integer k. Therefore, we have

$$F(x) \ge \sum_{n=0}^{\infty} [f((2n+1)x) - f((2n+2)x)] = f(0) - F(x),$$

as well as

$$F(x) \le f(0) - f(x) + \sum_{n=1}^{\infty} [f((2n-1)x) - f(2nx)] = 2f(0) - f(x) - F(x).$$

Thus we see that $f(0) \le 2F(x) \le 2f(0) - f(x)$ for all $x > 0$ when f is convex. In particular, $\lim_{x\to 0^+} F(x) = \frac{1}{2} f(0)$ if $f(x)$ is also continuous at the origin.

(**b**) Suppose that instead of assuming that $f(x)$ is convex, we assume that $f \in C^2[0, \infty)$ with $\int_0^\infty |f''(x)|\,dx < \infty$. Observe that since $f(x) \to 0$ as $x \to \infty$, we may write

$$\begin{aligned}
F(x) &= \frac{1}{2} f(0) + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n [f(nx) - f((n+1)x)] \\
&= \frac{1}{2} f(0) + \frac{1}{2} \sum_{n=0}^{\infty} \left[\int_{(2n+1)x}^{(2n+2)x} f'(t)\,dt - \int_{2nx}^{(2n+1)x} f'(t)\,dt \right] \\
&= \frac{1}{2} f(0) + \frac{1}{2} \sum_{n=0}^{\infty} \int_{2nx}^{(2n+1)x} \left(\int_0^x f''(s+t)\,ds \right) dt \\
&= \frac{1}{2} f(0) + \frac{1}{2} \int_0^x \left(\sum_{n=0}^{\infty} \int_{2nx}^{(2n+1)x} f''(s+t)\,dt \right) ds.
\end{aligned}$$

This implies that

$$\left| F(x) - \frac{1}{2} f(0) \right| \le \frac{x}{2} \int_0^\infty |f''(t)|\,dt,$$

so once again $F(x) \to \frac{1}{2} f(0)$ as $x \to 0$ from the right.

(**c**) A simple choice of a monotone decreasing function f with $f(0) > 0$ for which

$$\limsup_{x\to 0^+} F(x) = f(0), \qquad \liminf_{x\to 0^+} F(x) = 0$$

is given by $f(x) = 1$ for $0 \le x < 1$ and $f(x) = 0$ for $1 \le x < \infty$. For each positive integer k, we then have $F(1/(2k)) = 1$ and $F(1/(2k+1)) = 0$.

Also solved by M. Bello-Hernández & M. Benito (Spain), N. Caro (Colombia), R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, J. Grivaux (France), J. H. Lindsey II, O. P. Lossers (Netherlands), K. Schilling, R. Stong, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

A Definite Hyperbolic

11418 [2009, 276]. *Proposed by George Lamb, Tucson, AZ.* Find

$$\int_{-\infty}^{\infty} \frac{t^2 \operatorname{sech}^2 t}{a - \tanh t}\, dt$$

for complex a with $|a| > 1$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. The answer is $\frac{1}{12}\big(\operatorname{Log}^3(\frac{a+1}{a-1}) + \pi^2 \operatorname{Log}(\frac{a+1}{a-1})\big)$, where Log is the principal branch of the logarithm defined on the complex plane cut along the negative real numbers. The formula is valid for every complex number a with $a \notin [-1, 1]$.

For $a \notin [-1, 1]$ the integral is convergent. Denote its value by $I(a)$. Compute

$$\begin{aligned} I(a) &= \int_{-\infty}^{\infty} \frac{t^2\, dt}{(a\cosh t - \sinh t)\cosh t} = \int_{-\infty}^{\infty} \frac{4t^2 e^{2t}\, dt}{((a-1)e^{2t} + a + 1)(e^{2t} + 1)} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 e^x\, dx}{((a-1)e^x + a + 1)(e^x + 1)} = \frac{1}{2(a-1)} J\left(\frac{a+1}{a-1}\right), \end{aligned}$$

with

$$J(b) = \int_{-\infty}^{\infty} \frac{x^2 e^x\, dx}{(e^x + b)(e^x + 1)}.$$

In order to evaluate $J(b)$ for $b \in \mathbb{C} \setminus (-\infty, 0]$, let

$$F(z) = \frac{(z^3 + \pi^2 z)e^z}{(1 - e^z)(b - e^z)}.$$

For large positive R, consider the contour γ_R consisting of a positively oriented rectangle $ABCD$ with vertices A, B, C, D at $-R - i\pi$, $R - i\pi$, $R + i\pi$, and $-R + i\pi$, respectively. The only points inside the rectangle γ_R where the denominator of F vanishes are 0 and $\operatorname{Log} b$, but 0 is a removable singularity for F and $\operatorname{Log} b$ is a simple pole with residue

$$\operatorname{Res}(F, \operatorname{Log} b) = \frac{\operatorname{Log}^3 b + \pi^2 \operatorname{Log} b}{b - 1}.$$

The residue formula says that

$$\int_{\gamma_R} F(z)\, dz = \frac{2\pi i}{b - 1}(\operatorname{Log}^3 b + \pi^2 \operatorname{Log} b).$$

However,

$$\begin{aligned}\int_{AB} F(z)\,dz + \int_{CD} F(z)\,dz &= \int_{-R}^{R} F(x-i\pi)\,dx - \int_{-R}^{R} F(x+i\pi)\,dx \\ &= \int_{-R}^{R} \frac{\big((x+i\pi)^3 + \pi^2(x+i\pi) - (x-i\pi)^3 - \pi^2(x-i\pi)\big)e^x}{(1+e^x)(b+e^x)}\,dx \\ &= 6\pi i \int_{-R}^{R} \frac{x^2 e^x\,dx}{(1+e^x)(b+e^x)},\end{aligned}$$

so

$$\lim_{R\to\infty}\left(\int_{AB} F(z)\,dz + \int_{CD} F(z)\,dz\right) = 6\pi i \int_{-\infty}^{\infty} \frac{x^2 e^x\,dx}{(1+e^x)(b+e^x)} = 6\pi i\,J(b).$$

Next, $\int_{BC} F(z)\,dz = i\int_{-\pi}^{\pi} F(R+it)\,dt$, so if $R > 1+|b|$, then

$$\left|\int_{BC} F(z)\,dz\right| \le 2\pi \sup_{t\in[-\pi,\pi]} |F(R+it)| \le 2\pi \frac{\sqrt{R^2+\pi^2}(R^2+2\pi^2)e^R}{(e^R-1)(e^R-|b|)}.$$

Therefore, $\lim_{R\to\infty}\int_{BC} F(z)\,dz = 0$. Similarly, $\lim_{R\to\infty}\int_{DA} F(z)\,dz = 0$. Combining our results, we conclude that

$$6\pi i\,J(b) = \frac{2\pi i}{b-1}(\mathrm{Log}^3 b + \pi^2 \mathrm{Log}\, b),$$

or, equivalently,

$$J(b) = \frac{1}{3(b-1)}(\mathrm{Log}^3 b + \pi^2 \mathrm{Log}\, b).$$

Therefore, as claimed, we get

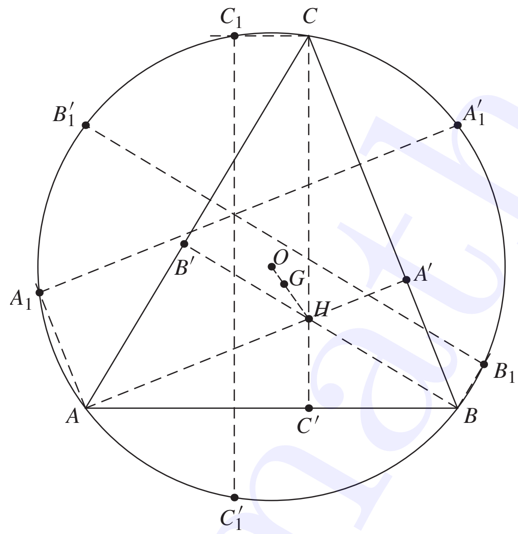
$$I(a) = \frac{a-1}{2} J\left(\frac{a+1}{a-1}\right) = \frac{1}{12}\left(\mathrm{Log}^3\left(\frac{a+1}{a-1}\right) + \pi^2 \mathrm{Log}\left(\frac{a+1}{a-1}\right)\right).$$

Also solved by R. Bagby, D. H. Bailey & J. M. Borwein (U.S.A. & Canada), D. Beckwith, R. Chapman (U. K.), H. Chen, P. Corn, Y. Dumont (France), M. L. Glasser, J. Grivaux (France), J. A. Grzesik, K. McInturff, L. A. Medina, P. Perfetti (Italy), Á. Plaza (Spain), O. G. Ruehr, A. Stadler (Switzerland), V. Stakhovsky, R. Stong, N. Thornber, GCHQ Problem Solving Group (U. K.), and the proposer.

A Triangle Construction

11419 [2009, 276]. *Proposed by Vasile Mihai, Belleville, Ontario, Canada.* Let G be the centroid, H the orthocenter, O the circumcenter, and P the circumcircle of a triangle ABC that is neither isosceles nor right.

Let A', B', and C' be the *orthic points* of ABC, that is, the respective feet of the altitudes from A, B, and C. Let A_1 be the point on P such that AA_1 is parallel to BC, and define B_1, C_1 similarly. Let A_1' be the point on P such that A_1A_1' is parallel to AA', and define B_1', C_1' similarly (see sketch).



Show that

(**a**) A_1A_1', B_1B_1', and C_1C_1' are concurrent at the point I opposite H from O on the Euler line HO.

(**b**) A_1A', B_1B', and C_1C' are concurrent at the centroid G.

(**c**) the circumcircles of OA_1A_1', OB_1B_1', and OC_1C_1' (which are clearly concurrent at O) are concurrent at a second point K lying on HO, and $|OH| \cdot |OK| = abc/p$, where a, b, and c are the edge lengths of ABC, and p is the perimeter of $A_1B_1C_1$.

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL.

(**a**) Each of the lines A_1A_1', B_1B_1', and C_1C_1' is the reflection of an altitude in the perpendicular bisector of the corresponding side, and these bisectors each contain the circumcenter O. Since the altitudes intersect at the orthocenter H, these reflected lines intersect at the reflection of H in O.

(**b**) Let D be the midpoint of BC. Since AA_1 and BC are parallel and $AA_1 = 2 \cdot DA'$, the lines A_1A' and AD intersect at a point that divides each of A_1A' and AD in the ratio $2:1$. This point is the centroid G of triangle ABC. The same holds for B_1B' and C_1C'.

(**c**) The inverse of the line A_1A_1' in the circumcircle P is the circle OA_1A_1'. This circle contains the inverse K of I in P. The same holds for the lines B_1B_1' and C_1C_1'. Note that $|OH| \cdot |OK| = |OI| \cdot |OK| = R^2$, where R is the circumradius.

If ABC is acute, then the angles of $A_1B_1C_1$ are $\pi - 2A$, $\pi - 2B$, and $\pi - 2C$. The perimeter p of triangle $A_1B_1C_1$ is given by

$$
\begin{aligned}
p &= 2R(\sin 2A + \sin 2B + \sin 2C) = 2a\cos A + 2b\cos B + 2c\cos C \\
&= \frac{a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)}{abc} \\
&= \frac{16\Delta^2}{abc} = \left(\frac{abc}{R}\right)^2 \cdot \frac{1}{abc} = \frac{abc}{R^2}.
\end{aligned}
$$

Therefore, $R^2 = abc/p$.

This formula is correct only for acute triangles. If angle A is obtuse, the angles of triangle $A_1B_1C_1$ are $2A - \pi$, $2B$, and $2C$.

Also solved by M. Bataille (France), J. Cade, R. Chapman (U. K.), P. P. Dálya y (Hungary), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), J. G. Heuver (Canada), L. R. King, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Minkus, C. R. Pranesachar (India), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Matrix Normality

11422 [2009, 277]. *Proposed by Christopher Hillar, The Mathematical Sciences Research Institute, Berkeley, CA.* Let H be a real $n \times n$ symmetric matrix with distinct eigenvalues, and let A be a real matrix of the same size. Let $H_0 = H$, $H_1 = AH_0 - H_0A$, and $H_2 = AH_1 - H_1A$. Show that if H_1 and H_2 are symmetric, then $AA^t = A^tA$; that is, A is normal.

Solution by Patrick Corn, St. Mary's College of Maryland, St. Mary's City, MD. If we conjugate H_0, H_1, H_2, and A by the same orthogonal matrix, then the hypotheses, definitions, and conclusion remain unchanged. There exists an orthogonal matrix that diagonalizes H_0, since H_0 is a real, symmetric matrix. Without loss of generality, then, we may assume that H_0 is diagonal with distinct entries.

Since H_1 is symmetric, it follows that $AH_0 - H_0A = (AH_0 - H_0A)^t = H_0A^t - A^tH_0$, and thus $(A + A^t)H_0 = H_0(A + A^t)$. Since the matrix $A + A^t$ commutes with H_0, it must be diagonal. Now write $A = D + S$, where $D = (1/2)(A + A^t)$ is diagonal and $S = (1/2)(A - A^t)$ is skew-symmetric.

Since H_2 is symmetric, we have $H_1(A + A^t) = (A + A^t)H_1$, and $H_1D = DH_1$. That is, $(AH_0 - H_0A)D = D(AH_0 - H_0A)$. Since D and H_0 commute, $AH_0 - H_0A = SH_0 - H_0S$, and then $(SH_0 - H_0S)D = D(SH_0 - H_0S)$, so $H_0(DS - SD) = (DS - SD)H_0$. Thus $DS - SD$ commutes with H_0, so it must be diagonal. However, DS and SD both have zero diagonals, since S does, and therefore $DS = SD$.

Expanding and using $DS = SD$, we conclude that

$$AA^t - A^tA = (D + S)(D^t + S^t) - (D^t + S^t)(D + S) = 2(SD - DS) = 0.$$

This gives the desired result.

Also solved by R. Chapman (U. K.), C. Curtis, P. P. Dályay (Hungary), A. Fok, S. M. Gagola Jr., M. Goldenberg & M. Kaplan, D. Grinberg, J.-P. Grivaux (France), E. A. Herman, R. Howard, O. Kouba (Syria), C. Lanski, J. H. Lindsey II, O. P. Lossers (Netherlands), A. Muchlis (Indonesia), J. Simons (U. K.), J. H. Smith, R. Stong, E. I. Verriest, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

A Lobachevsky Integral

11423 [2009, 277]. *Proposed by Gregory Minton, D. E. Shaw Research, LLC, New York, NY.* Show that if n and m are positive integers with $n \geq m$ and $n - m$ even, then $\int_{x=0}^{\infty} x^{-m} \sin^n x \, dx$ is a rational multiple of π.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. We use induction on m. Let $I(n, m) = \int_0^{\infty} x^{-m} \sin^n x \, dx$. First, for any odd positive integer $n = 2k + 1$, we recall that

$$\sin^{2k+1} x = \frac{1}{2^{2k}} \sum_{i=0}^{k} (-1)^{k-i} \binom{2k+1}{i} \sin\big((2k - 2i + 1)x\big)$$

and

$$\int_0^{\infty} \frac{\sin(ax)}{x} \, dx = \frac{\pi}{2}$$

for $a > 0$. Hence

$$I(2k + 1, 1) = \frac{1}{2^{2k+1}} \sum_{i=0}^{k} (-1)^{k-i} \binom{2k+1}{i} \pi$$

is a rational multiple of π. For $m = 2$, note that integration by parts gives

$$I(n, 2) = n \int_0^\infty \frac{\sin^{n-1} x \, \cos x}{x} \, dx.$$

Using the product to sum formula for sine and cosine, for $n = 2k$ we can expand $\sin^{2k-1} x \cos x$ as

$$\frac{1}{2^{2k-1}} \sum_{i=0}^{k-1} (-1)^{k-i+1} \binom{2k-1}{i} \Big(\sin\big((2k-2i)x\big) + \sin\big((2k-2i-2)x\big) \Big),$$

so

$$I(2k, 2) = \frac{k}{2^{2k-2}} \left(\frac{1}{2} \binom{2k-1}{k-1} + \sum_{i=0}^{k-2} (-1)^{k-i+1} \binom{2k-1}{i} \right) \pi$$

is also a rational multiple of π. For $m \geq 2$, integrating by parts twice leads to

$$I(n, m+1) = -\frac{n^2}{m(m-1)} I(n, m-1) + \frac{n(n-1)}{m(m-1)} I(n-2, m-1).$$

When $n - (m+1)$ is even and nonnegative, the right side is a rational multiple of π by the induction hypothesis. Therefore, the left side is also such a multiple, which completes the proof.

Editorial comment. The integrals $I(n, m)$ were apparently first considered by N. I. Lobachevskiĭ, Probabilité des résultats moyens tirés d'observations répétées, *J. Reine Angew. Math.* **24** (1842) 164–170.

T. Hayashi, in "On the integral $\int_0^\infty \frac{\sin^n x}{x^m} \, dx$," *Nieuw Arch. Wiskd.* (2) **14** (1923) 13–18, gave the following explicit evaluation:

$$I(n, m) = \frac{\pi(-1)^{(n-m)/2}}{2^{n-m+1}(m-1)!} \sum_{0 \leq j \leq (n-1)/2} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j \right)^{m-1}$$

which for $m = 1$ or 2 simplifies to

$$I(2k+1, 1) = \frac{\pi}{2^{k+1}} \frac{(2k-1)!!}{k!} = \frac{\pi}{2^{2k+1}} \binom{2k}{k}$$

$$I(2k, 2) = \frac{\pi}{2^k} \frac{(2k-3)!!}{(k-1)!} = \frac{\pi}{2^{2k-1}} \binom{2k-2}{k-1},$$

and these more than suffice for the current problem.

Also solved by K. F. Andersen (Canada), R. Bagby, M. Bataille (France), D. Beckwith, D. Borwein (Canada), K. N. Boyadzhiev, R. Buchanan, R. Chapman (U. K.), P. Corn, J. Dai & C. Goff, P. P. Dályay (Hungary), Y. Dumont (France), G. C. Greubel, J. Grivaux (France), J. A. Grzesik, E. A. Herman, G. Keselman, J. Kolk (Netherlands), T. Konstantopoulis (U. K.), O. Kouba (Syria), I. E. Leonard (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), Y. Mikata, M. Omarjee (France), É. Pité (France), Á. Plaza (Spain), R. E. Rogers, O. G. Ruehr, J. Simons (U. K.), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), N. Thornber, E. I. Verriest, Z. Vörös (Hungary), M. Vowe (Switzerland), H. Widmer (Switzerland), L. Zhou, Columbus State University Problem Solvers, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before February 28, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11523. *Proposed by Timothy Chow, Princeton, NJ.* Given boxes 1 through n, put balls in k randomly chosen boxes. The *score* of a permutation π of $\{1, \dots, n\}$ is the least i such that box $\pi(i)$ has a ball. Thus, if $\pi = (3, 4, 1, 5, 2)$ with $(n, k) = (5, 2)$, and boxes 1 and 4 have balls, then π has score 2.
(**a**) A permutation π is *fair* if, regardless of the value of k, the probability that π scores lower than the identity permutation equals the probability that it scores higher. Show that π is fair if and only if for each i in $[1, n]$, either $\pi(i) > i$ and $\pi^{-1}(i) > i$, or $\pi(i) \le i$ and $\pi^{-1}(i) \le i$.
(**b**) Let $f(n)$ be the number of fair permutations of $\{1, \dots, n\}$, with the convention that $f(0) = 1$. Show that $\sum_{n=0}^{\infty} f(n)x^n/n! = e^x \sec(x)$.
(**c**) Assume now that $n = m^3$ with $m \ge 2$, and the boxes are arranged in m rows of length m^2. Alice scans the top row left to right, then the row below it, and so on, until she finds a box with a ball in it. Bob scans the leftmost column top to bottom, then the next column, and so on. They start simultaneously and both check one box per second. For which k are Alice and Bob equally likely to be the first to discover a ball?

11524. *Proposed by H. A. ShahAli, Tehran, Iran.* A vector v in $\mathbb{R}^n$ is *short* if $\|v\| \le 1$.
(**a**) Given six short vectors in $\mathbb{R}^2$ that sum to zero, show that some three of them have a short sum.
(**b**)* Let $f(n)$ be the least M such that, for any finite set T of short vectors in $\mathbb{R}^n$ that sum to 0, and any integer k with $1 \le k \le |T|$, there is a k-element subset S of T such that $\|\sum_{v \in S} v\| \le M$. The result of part (**a**) suggests $f(2) = 1$. Find $f(n)$ for $n \ge 2$.

11525. *Proposed by Grigory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.*
(**a**) Prove that for each $n \ge 3$ there is a set of regular n-gons in the plane such that every line contains a side of exactly one polygon from this set.

doi:10.4169/000298910X515820

(**b**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly one circle from the set?
(**c**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly two circles from the set?
(**d**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly three circles from the set?

11526. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Prove that there is no function f from $\mathbb{R}^3$ to $\mathbb{R}^2$ with the property that $\|f(x) - f(y)\| \geq \|x - y\|$ for all $x, y \in \mathbb{R}^3$.

11527. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* Prove that in an acute triangle with sides of length a, b, c, inradius r, and circumradius R,

$$\frac{a^2}{b^2+c^2-a^2} + \frac{b^2}{c^2+a^2-b^2} + \frac{c^2}{a^2+b^2-c^2} \geq \frac{3}{2} \cdot \frac{R}{r}.$$

11528. *Proposed by Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let p, a, and b be positive integers with $a < b$. Consider a sequence $\langle x_n \rangle$ defined by the recurrence $nx_{n+1} = (n + 1/p)x_n$ and an initial condition $x_1 \neq 0$. Evaluate

$$\lim_{n\to\infty} \frac{x_{an} + x_{an+1} + \cdots + x_{bn}}{nx_{an}}.$$

11529. *Proposed by Walter Blumberg, Coral Springs, FL.* For $n \geq 1$, let $A_n = \left(3\sum_{k=1}^{n} \left\lfloor \frac{k^2}{n} \right\rfloor\right) - n^2$. Let p and q be distinct primes with $p \equiv q \pmod 4$. Show that $A_{pq} = A_p + A_q - 2$.

SOLUTIONS

Splitting Elements of Set Systems

11372 [2008, 568]. *Proposed by Jennifer Vandenbussche and Douglas B. West, University of Illinois at Urbana-Champaign, Urbana, IL.* In a family of finite sets, let a *splitting element* be an element that belongs to at least two of the sets and is omitted by at least two of the sets. Determine the maximum size of a family of subsets of $\{1, \ldots, n\}$ for which there is no splitting element.

Solution by David Gove, California State University, Bakersfield, CA. The maximum size is $n + 1$. Consider a largest such family. Removing x from the sets it lies in and adding it to the others yields another such family. Hence we may assume that each element appears in at most one of the sets. If any of the sets has more than one element, then we can obtain a bigger family by replacing that set by its singleton subsets. Thus the family consisting of the empty set and the singleton sets is a largest such family.

Editorial comment. By the argument above, there are 2^n extremal families. Marian Tetiva sent a thorough discussion of a more general problem. Let $g_s(n)$ be the maximum size of a family of subsets of $\{1, \ldots, n\}$ such that every element appears in at most s sets or avoids at most s sets; the stated problem is $g_1(n) = n + 1$, and clearly $g_0(n) = 1$. By the complementation argument above, we may equivalently seek the largest family such that every element appears in at most s sets. Tetiva proved a bound and conjectured equality. Intuitively, the idea is that one should take all the small sets until the bound on the number of appearances of each element is reached. For example, if $s = \sum_{i=1}^{k} \binom{n-1}{i-1}$, then one should take all the sets of size at most k. When s is not of this form, the exact solution is more difficult.

Also solved by D. Beckwith, P. Corn, D. L. Craft, C. Curtis, P. P. Dálylay (Hungary), K. David & P. Fricano, D. Degiorgi (Switzerland), J. Gately, J. Guerreiro (Portugal), H. S. Hwang (Korea), K. Kneile, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), M. D. Meyerson, J. H. Nieto (Venezuela), R. E. Prather, T. Rucker, V. Rutherfoord, K. Schilling, E. Schmeichel, B. Schmuland (Canada), R. Stong, J. Swenson, M. Tetiva (Romania), B. Tomper, Fisher Problem Group, Szeged Problem Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Houghton College Problem Solving Group, Microsoft Research Problems Group, NSA Problems Group, and the proposers.

A Determinant Generated by a Polynomial

11377 [2008, 664]. *Proposed by Christopher Hillar, Texas A&M University, College Station, TX and Lionel Levine, Massachusetts Institute of Technology, Cambridge, MA.* Given a monic polynomial p of degree n with complex coefficients, let A_p be the $(n+1) \times (n+1)$ matrix with $p(-i+j)$ in position (i, j), and let D_p be the determinant of A_p. Show that D_p depends only on n, and find its value in terms of n.

Solution by John H. Lindsey II, Cambridge, MA. The value of D_p is $(n!)^{n+1}$, which we prove by induction on n. The result is trivial when $n = 0$. For $n > 0$, use indices $0, \dots, n$ for the rows and columns of A_p. In A_p, let C_j be column j and R_i be row i. Given a function f, define Δf by $\Delta f(k) = f(k+1) - f(k)$. By induction on n, if f is a monic polynomial of degree n, then $\Delta^n f(x) = n!$ for all x.

Replacing C_n with $\sum_{j=0}^{n} \binom{n}{j}(-1)^{n-j} C_j$ does not change the determinant, but it turns the ith entry of column n into $\Delta^n p(-i)$, which equals $n!$. Now for $0 \le i \le n-1$ in order, subtract the next row from R_i, replacing R_i with $R_i - R_{i+1}$. This puts 0 in the last column, except for the last row. For $j < n$, the new entry $a'_{i,j}$ is $p(-i+j) - p(-i-1+j)$, which equals $\Delta p(-i-1+j)$. Since Δp has leading coefficient n, the upper left $(n-1)$-by-$(n-1)$ block has the form nA_f, where $f(x) = (1/n)\Delta p(x-1)$.

Since f is a monic polynomial with degree $n-1$, by the induction hypothesis $D_f = (n-1)!^n$. Expanding the altered D_p down the last column yields $D_p = n!\, n^n (n-1)!^n = n!^{n+1}$.

Editorial comment. Solvers used a variety of methods, including Vandermonde determinants. Roger Horn proved a substantial generalization. Given a matrix A, let $p(A)$ denote the entrywise application of the polynomial p to A; that is, the (i, j)-entry of $p(A)$ is $p(a_{i,j})$. For $x \in \mathbb{C}^{n+1}$, let $A(x)$ be the matrix given by $a_{i,j} = x_i + j - 1$. If p is a monic polynomial of degree n, then

$$\det p(A(x)) = \left(\prod_{i>j}(x_i - x_j)\right) \frac{(-1)^{\lfloor (n+1)/2 \rfloor (n!)^n}}{\prod_{i=1}^{n-1} i!}, \tag{1}$$

which depends only on x and n, not p. The originally stated problem is the case $x = (0, -1, \dots, -n)^T$.

Also solved by D. Beckwith, R. Chapman (U. K.), P. Corn, P. P. Dálylay (Hungary), J. Grivaux (France), J. Hartman, C. C. Heckman, R. A. Horn, R. Howard, G. Keselman, O. Kouba (Syria), S. C. Locke, O. P. Lossers (Netherlands), K. McInturff, J. H. Nieto (Venezuela), É. Pité (France), C. R. Pranesachar (India), M. A. Prasad (India), N. C. Singer, J. H. Smith, A. Stadler (Switzerland), V. Stakhovsky, R. Stong, T. Tam, M. Tetiva (Romania), B. Tomper, M. Vowe (Switzerland), L. Zhou, BSI Problems Group (Germany), FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

The Column Space of a Very Nilpotent Matrix

11379 [2008, 664]. *Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Technische Universität Dortmund, Dortmund, Germany.* Let A be a complex matrix of order n whose square is the zero matrix. Show that $\mathcal{R}(A + A^*) = \mathcal{R}(A) + \mathcal{R}(A^*)$, where $\mathcal{R}(\cdot)$ denotes the column space of a matrix argument.

Solution by M. Andreoli, Miami Dade College, Miami, FL. Note first that $A^2 = 0$ implies $\mathcal{R}(A) \subseteq \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the nullspace of A. This holds because $y = Ax$ implies $Ay = A^2x = 0$.

For $y \in \mathcal{R}(A + A^*)$, there exists x such that $y = (A + A^*)x = Ax + A^*x$. Hence $y \in \mathcal{R}(A) + \mathcal{R}(A^*)$, and we conclude that $\mathcal{R}(A + A^*) \subseteq \mathcal{R}(A) + \mathcal{R}(A^*)$.

Conversely, for $y \in \mathcal{R}(A) + \mathcal{R}(A^*)$, there exist vectors x_1 and x_2 such that $y = Ax_1 + A^*x_2$. Since $\mathcal{N}(A^*)$ and $\mathcal{R}(A)$ are orthogonal complements in $\mathbb{C}^n$, there exist vectors $u \in \mathcal{R}(A)$ and $v \in \mathcal{N}(A^*)$ such that $x_1 - x_2 = u + v$. Since $\mathcal{R}(A) \subseteq \mathcal{N}(A)$, we have $u \in \mathcal{N}(A)$. Letting $x = x_1 - u = x_2 + v$, we have

$$\begin{aligned}(A + A^*)x = Ax + A^*x \;&=\; A(x_1 - u) + A^*(x_2 + v)\\ &= Ax_1 - Au + A^*x_2 + A^*v \;=\; Ax_1 + A^*x_2 \;=\; y.\end{aligned}$$

Thus $y \in \mathcal{R}(A + A^*)$, and hence $\mathcal{R}(A) + \mathcal{R}(A^*) \subseteq \mathcal{R}(A + A^*)$.

Also solved by M. Bataille (France), P. Budney, R. Chapman (U. K.), P. Corn, C.-K. Fok, J. Freeman, J.-P. Grivaux (France), J. Hartman, E. A. Herman, R. A. Horn, O. Kouba (Syria), C. Lanski, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), I. Pinelis, É. Pité (France), N. C. Singer, J. H. Smith, R. Stong, J. Stuart, F. Vrabec (Austria), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposers.

A Generalized Binomial Coefficient

11380 [2008, 665]. *Proposed by Hugh Montgomery, University of Michigan, Ann Arbor, MI, and Harold S. Shapiro, Royal Institute of Technology, Stockholm, Sweden.* For $x \in \mathbb{R}$, let $\binom{x}{k} = \frac{1}{k!}\prod_{j=0}^{k-1}(x - j)$. For $k \geq 1$, let a_k be the numerator and q_k the denominator of the rational number $\binom{-1/3}{k}$ expressed as a reduced fraction with $q_k > 0$.
(**a**) Show that q_k is a power of 3.
(**b**) Show that a_k is odd if and only if k is a sum of distinct powers of 4.

Solution by Stephen M. Gagola Jr., Kent State University, Kent, OH. We prove more generally that if $m > 1$ and $m + 1$ is a power of a prime p, and the rational number $\binom{-1/m}{k}$ has numerator a_k and denominator q_k in lowest terms with $q_k > 0$, then
(**a**$'$) all prime factors of q_k divide m, and
(**b**$'$) $p \nmid a_k$ if and only if k is a sum of distinct powers of $m + 1$.

The stated problem is the case $m = 3$, where $p = 2$.
(**a**$'$) For clarity, let $c_k = \binom{-1/m}{k} = a_k/q_k$. In the formal power series ring $\mathbb{Q}[[x]]$,

$$(1 + x)^{-1/m} = \sum_{k=0}^{\infty} c_k x^k. \tag{1}$$

Therefore,

$$\sum_{k=0}^{\infty}(-1)^k x^k = (1 + x)^{-1} = \left(\sum_{k=0}^{\infty} c_k x^k\right)^m = \sum_{k=0}^{\infty}\sum c_{i_1}\cdots c_{i_m} x^k, \tag{2}$$

where the inner sum extends over all m-tuples $(i_1, \dots, i_m)$ of nonnegative integers summing to k. Exactly m such m-tuples have k as an entry. Equating coefficients of x^k in (2) then yields

$$mc_k + \sum c_{i_1} \cdots c_{i_m} = (-1)^k, \tag{3}$$

where the sum extends over m-tuples $(i_1, \dots, i_m)$ with sum k and entries less than k.

Let $R_m = \bigcup_{i \geq 0}(1/m^i)\mathbb{Z}$. Note that R_m is the subring of $\mathbb{Q}$ consisting of all rational numbers whose denominators factor into primes dividing m. Also, $c_0 = 1$, so $c_0 \in R_m$. Since m is a unit of R_m, (3) yields $c_k \in R_m$ for all k, inductively. Thus (**a**$'$) follows.

(**b**$'$) View (1) and (2) above in the formal power series ring $R_m[[x]]$. We write $f(x) \equiv g(x)$ when $f(x) - g(x) = ph(x)$ for some power series $h(x) \in R_m[[x]]$. Since $f(x)^p \equiv f(x^p)$ for all $f(x) \in R_m[[x]]$, also $f(x)^{m+1} \equiv f(x^{m+1})$. Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty} c_k x^k = (1+x)^{-1/m} &= (1+x)((1+x)^{-1/m})^{m+1} = (1+x)\left(\sum_{k=0}^{\infty} c_k x^k\right)^{m+1} \\ &\equiv (1+x)\sum_{k=0}^{\infty} c_k x^{(m+1)k} = \sum_{k=0}^{\infty}(c_k x^{(m+1)k} + c_k x^{(m+1)k+1}). \end{aligned} \tag{4}$$

We conclude that $c_k \equiv 0 \bmod p$ if k is not congruent to 0 or 1 modulo $m+1$, and the same holds for a_k.

Note that $c_0 = 1$ and $c_1 = -1/m \equiv 1 \bmod p$. Hence p divides neither a_0 nor a_1. For $k > 1$, if $m+1$ divides k or $k-1$, then write $k = (m+1)k' + \epsilon$, where $\epsilon \in \{0, 1\}$. Note that k is a sum of distinct powers of $m+1$ if and only if k' is. The congruence in (4) implies that $c_k \equiv c_{k'} \bmod p$, and (**b**$'$) follows by induction.

Also solved by R. Chapman (U. K.), H. Chen, P. Corn, P. P. Dályay (Hungary), Y. Dumont (France), E. Errthum, S. M. Gagola Jr., J. H. Lindsey II, O. P. Lossers (Netherlands), J. Minkus, M. A. Prasad (India), B. Schmuland (Canada), N. C. Singer, J. H. Smith, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

Convergence of a Prime-denominated Series

11384 [2008, 757]. *Proposed by Moubinool Omarjee, Lycée Jean-Lurçat, Paris, France.* Let p_n denote the nth prime. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{p_n}$$

converges.

Solution by Greg Martin, University of British Columbia, Vancouver, CA. Let $S_N = \sum_{n=1}^{N} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{p_n}$. It suffices to show that the subsequence $\{S_{M^2-1}\colon M \geq 1\}$ converges, since S_N is between S_{M^2-1} and $S_{(M+1)^2-1}$ for N between M^2-1 and $(M+1)^2-1$. However, $S_{M^2-1} = \sum_{m=2}^{M} T_m$, where

$$T_m = \sum_{n=(m-1)^2}^{m^2-1} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{p_n} = (-1)^{m-1} \sum_{n=(m-1)^2}^{m^2-1} \frac{1}{p_n}.$$

Since $\{T_m : m \geq 1\}$ alternates in sign, it suffices to show that $\lim T_m = 0$, by the alternating series test. Using the crude inequality $p_n > n$, we obtain

$$|T_m| < \sum_{n=(m-1)^2}^{m^2-1} \frac{1}{n} < \frac{1}{(m-1)^2} \sum_{n=(m-1)^2}^{m^2-1} 1 = \frac{2m-1}{(m-1)^2},$$

and thus $\lim T_m = 0$.

A similar proof works if the sequence of primes is replaced by an arbitrary sequence q satisfying $q_n/\sqrt{n} \to \infty$.

Editorial comment. Many solvers used detailed information about the distribution of the prime numbers, but the proof above shows that this is unnecessary.

Also solved by R. Bagby, H. Chen, P. P. Dályay (Hungary), Y. Dumont (France), V. V. Garcia (Spain), S. James (Canada), O. Kouba (Syria), K. Y. Li (China), J. Oelschlager, P. Perfetti (Italy), É. Pité (France), Á. Plaza (Spain), C. R. Pranesachar (India), M. T. Rassias (Greece), V. Schindler (Germany), B. Schmuland (Canada), N. C. Singer, A. Stadler (Switzerland), R. Stong, T. Tam, R. Tauraso (Italy) & M. Lerma, D. B. Tyler, J. Vinuesa (Spain), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), and the proposer.

Capturing Eigenvalues in an Interval

11387 [2008, 758]. *Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Technische Universität Dortmund, Dortmund, Germany.* Let $\mathcal{C}_{n,n}$ denote the set of $n \times n$ complex matrices. Determine the shortest interval $[a, b]$ such that if P and Q in $\mathcal{C}_{n,n}$ are nonzero orthogonal projectors, that is, Hermitian idempotent matrices, then all eigenvalues of $PQ + QP$ belong to $[a, b]$.

Solution I by O. P. Lossers, Eindhoven University of Technology, Einhoven, The Netherlands. The eigenvalues of $PQ + QP$ lie in $[-\frac{1}{4}, 2]$. The matrix $P + Q$ is Hermitian, and hence there is an orthonormal basis of its eigenvectors. The eigenvalues of $P + Q$ are real and in $[0, 2]$, since $|(P + Q)x| \leq |Px| + |Qx| \leq 2|x|$. The matrix $PQ + QP$ equals $(P + Q)^2 - (P + Q)$ and thus has the same eigenvectors as $P + Q$, with eigenvalues of the form $\lambda^2 - \lambda$ with $0 \leq \lambda \leq 2$. It follows that the eigenvalues of $PQ + QP$ lie in $[-1/4, 2]$.

The maximum is attained when P and Q both equal the identity matrix, while the minimum is attained for the projections on two lines intersecting at an angle of $\pi/3$.

Solution II by Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL. Since $|(PQ + QP)x| \leq |PQx| + |QPx| \leq 2|x|$ for all x, the eigenvalues are at most 2. For the lower bound, write $X \geq Y$ if X and Y are Hermitian and $X - Y$ is positive semidefinite. Note that

$$0 \leq \left(P + Q - \frac{1}{2}I\right)^2 = P^2 + Q^2 + \frac{1}{4}I + PQ + QP - P - Q = \frac{1}{4}I + PQ + QP.$$

It follows that $PQ + QP \geq -\frac{1}{4}I$, and therefore each eigenvalue of $PQ + QP$ is at least $-\frac{1}{4}$.

For the extreme cases, taking $P = Q = I$ gives the largest eigenvalue 2. Setting $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $Q = \frac{1}{4}\left[\begin{smallmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{smallmatrix}\right]$ yields $-\frac{1}{4}$ as an eigenvalue.

Editorial comment. The part $PQ + QP \geq -\frac{1}{4}I$ of this problem appeared in F. Zhang, *Linear Algebra: Challenging Problems for Students* (2nd ed.), Johns Hopkins University Press, Baltimore, 2009, p. 81.

Also solved by R. Chapman (U. K.), J. Freeman, J.-P. Grivaux (France), J. Hartman, E. A. Herman, O. Kouba (Syria), T. Laffey & H. Šmigoc (Ireland), J. H. Lindsey II, M. Omarjee (France), R. Stong, S. E. Thiel, N. Thornber, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), and the proposers.

Distinct Multisets with the Same Pairwise Sums

11389 [2008, 758]. *Proposed by Elizabeth R. Chen and Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Given a multiset $A = \{a_1, \dots, a_n\}$ of n real numbers (not necessarily distinct), define the sumset $S(A)$ of A to be $\{a_i + a_j : 1 \le i < j \le n\}$, a multiset with $n(n-1)/2$ not necessarily distinct elements. For instance, if $A = \{1, 1, 2, 3\}$, then $S(A) = \{2, 3, 3, 4, 4, 5\}$.
(a) When n is a power of 2 with $n \ge 2$, show that there are two distinct multisets A_1 and A_2 such that $S(A_1) = S(A_2)$.
(b) When n is a power of 2 with $n \ge 4$, show that if r distinct multisets $A_1, \dots, A_r$ all have the same sumset, then $r \le n - 2$.
(c*) When n is a power of 2 with $n \ge 4$, can there be as many as 3 distinct multisets with the same sumset?

(Distinct multisets are known to have distinct sumsets when n is not a power of 2.)

Solution by BSI Problems Group, Bonn, Germany.
(a) We recursively construct multisets A_m and B_m of size 2^m for $m \ge 0$. For $m \ge 0$, choose arbitrary positive c_m. Let $A_0 = \{0\}$ and $B_0 = \{c_0\}$. For $m > 0$, let $A_m = A_{m-1} \cup \{b + c_m : b \in B_{m-1}\}$ and $B_m = B_{m-1} \cup \{a + c_m : a \in A_{m-1}\}$. Inductively, $|A_m| = |B_m| = 2^m$ and $S(A_m) = S(B_m)$. Also $\min A_m = 0 < \min B_m$, which yields $A_m \ne B_m$.

(b) First we prove three claims. Let $A = \{a_1, \dots, a_n\}$ with $a_1 \le \cdots \le a_n$, and let $S(A) = \{s_1, \dots, s_{n(n-1)/2}\}$ with $s_1 \le \cdots \le s_{n(n-1)/2}$.

Claim 1: $a_2 + a_3 \in \{s_3, \dots, s_n\}$. Since $a_1 + a_2 \le a_1 + a_3 \le a_2 + a_3$, we have $a_2 + a_3 \ge s_3$. Also, the only sums that can be strictly smaller than $a_2 + a_3$ are $\{a_1 + a_i : 2 \le i \le n\}$. Thus $a_2 + a_3 \le s_n$.

Claim 2: Let $B = \{b_1, \dots, b_n\}$ with $b_1 \le \cdots \le b_n$. If $a_1 = b_1$ and $S(A) = S(B)$, then $A = B$. We prove $a_i = b_i$ by induction on i. Let $A(i) = \{a_1, \dots, a_i\}$ and $B(i) = \{b_1, \dots, b_i\}$. If $A(i-1) = B(i-1)$, then $a_1 + a_i$ and $b_1 + b_i$ are both minimal among $S(A) - S(A(i-1))$. Thus $a_{i+1} = b_{i+1}$.

Claim 3: Let $B = \{b_1, \dots, b_n\}$ with $b_1 \le \cdots \le b_n$. If $a_2 + a_3 = b_2 + b_3$ and $S(A) = S(B)$, then $A = B$. Since the two smallest sums from the two sets are equal, $a_1 + a_2 = s_1 = b_1 + b_2$ and $a_1 + a_3 = s_2 = b_1 + b_3$. With the hypothesis $a_2 + a_3 = b_2 + b_3$, we have $a_1 = b_1$. Claim 2 now applies.

Given these claims, let $A^1, \dots, A^{n-1}$ be multisets of size n having the same sumset. Write $A^k = \{a_1^{(k)}, \dots, a_n^{(k)}\}$ with $a_1^{(k)} \le \cdots \le a_n^{(k)}$. By Claim 1, there are at most $n - 2$ values for the sum of the second and third smallest elements. By the pigeonhole principle, there exist distinct k and l such that $a_2^{(k)} + a_3^{(k)} = a_2^{(l)} + a_3^{(l)}$. By Claim 3, $A_k = A_l$. Thus at most $n - 2$ multisets can have the same sumset.

(c) The answer is yes. Let $A = \{0, 4, 4, 4, 6, 6, 6, 10\}$, $B = \{2, 2, 2, 4, 6, 8, 8, 8\}$, and $C = \{1, 3, 3, 3, 7, 7, 7, 9\}$. With exponents denoting multiplicity, $S(A)$, $S(B)$, and $S(C)$ all equal $\{4^{(3)}, 6^{(3)}, 8^{(3)}, 10^{(10)}, 12^{(3)}, 14^{(3)}, 16^{(3)}\}$.

Editorial comment. The GCHQ Problem Solving Group solved part (a) by letting A_1 be the set of nonnegative integers less than $2n$ whose binary expansion has an even number of ones and setting $A_2 = \{0, 1, \dots, 2n - 1\} - A_1$. This results from the construction given above by setting $c_m = 2^m$.

For part (c), Daniele Degiorgi gave the example $A = \{0, 6, 7, 9, 11, 13, 14, 20\}$, $B = \{1, 5, 6, 8, 12, 14, 15, 19\}$, and $C = \{2, 4, 5, 9, 11, 15, 16, 18\}$, showing that it can be solved with sets (i.e., multisets with no repeated elements).

It remains open whether there are quadruples of multisets of size greater than 2 with the same sumset, or whether there are triples of multisets of any size greater than 2 other than 8 with the same sumset. Richard Stong showed that the search for such triples can be restricted to multisets whose size is an odd power of 2.

Also solved by D. Degiorgi (Switzerland), R. Stong, and the GCHQ Problem Solving Group (U. K.). Parts (a) and (b) solved also by O. P. Lossers (Netherlands), M. A. Prasad (India), Microsoft Research Problems Group, and the proposers.

Tetrahedral Cevians

11405 [2009, 82]. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Let P be an interior point of a tetrahedron $ABCD$. When X is a vertex, let X' be the intersection of the opposite face with the line through X and P. Let XP denote the length of the line segment from X to P.
(**a**) Show that $PA \cdot PB \cdot PC \cdot PD \geq 81 PA' \cdot PB' \cdot PC' \cdot PD'$, with equality if and only if P is the centroid of $ABCD$.
(**b**) When X is a vertex, let X'' be the foot of the perpendicular from P to the plane of the face opposite X. Show that $PA \cdot PB \cdot PC \cdot PD = 81 PA'' \cdot PB'' \cdot PC'' \cdot PD''$ if and only if the tetrahedron is regular and P is its centroid.

Solution by Kit Hanes, Bellingham, WA. We will consider the more general case of an n-simplex with vertices $A_0, \ldots, A_n$. Let P be a point in the interior, and let A_i' be the point where the line $A_i P$ meets the face opposite A_i. We will show that $\prod_{k=0}^n PA_k \geq n^{n+1} \prod_{k=0}^n PA_k'$, with equality if and only if P is the centroid of the simplex. Let $P = a_0 A_0 + \cdots + a_n A_n$ where $a_0 + \cdots + a_n = 1$ and each a_i is positive. For each j, A_j' is a convex combination of the A_i with A_j omitted and P is a convex linear combination of A_j and A_j'. Hence $P = a_j A_j + (1 - a_j) A_j'$. Hence $PA_j / PA_j' = (1 - a_j)/a_j$. The inequality of (**a**) is equivalent to $\prod_{j=0}^n (1 - a_j) \geq n^{n+1} \prod_{j=0}^n a_j$. This inequality follows by applying the arithmetic-geometric mean inequality

$$\frac{1 - a_j}{n} = \frac{a_0 + \cdots + \widehat{a_j} + \cdots + a_n}{n} \geq \sqrt[n]{a_0 \ldots \widehat{a_j} \ldots a_n}$$

to each term separately and taking the product. (Here, the hats indicate that the hatted term is to be skipped.) Equality holds if and only if all the a_i are equal, and hence $a_i = 1/(n+1)$ for all i and P is the centroid of the simplex. For part (**b**), note that $PA_i' \geq PA_i''$ with equality if and only if $A_i' = A_i''$, i.e., if and only if the line PA_i is an altitude of the simplex. Hence the stated equality holds exactly when P is both the centroid and the orthocenter of the simplex. That this is equivalent to the simplex being regular is half of Problem 11087 from this MONTHLY, December, 2005.

Editorial comment. Part (a) of this problem is the generalization from triangles to tetrahedra of Problem 11325, this MONTHLY, November, 2007.

Also solved by S. Amghibech (Canada), M. Bataille (France), M. Can, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. Grivaux (France), K. Hanes, J. G. Heuver (Canada), B.-T. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Schaer (Canada), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before March 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11501. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* (Correction) Let

$$g(z) = 1 - \frac{3}{\frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of g about 0 are all nonnegative if and only if $a \geq \sqrt{3}$.

11530. *Proposed by Pál Peter Dályay, Szeged, Hungary.* Let A be an $m \times m$ matrix with nonnegative entries $a_{i,j}$ and with the property that there exists a permutation σ of $\{1, \ldots, m\}$ for which $\prod_{i=1}^{m} a_{i,\sigma(i)} \geq 1$. Show that the union over $n \geq 1$ of the set of entries of A^n is bounded if and only if some positive power of A is the identity matrix.

11531. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC and let λ_1, λ_2, λ_3 be positive real numbers. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Show that

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3 \left(\frac{|MA|}{\lambda_1} + \frac{|MB|}{\lambda_2} + \frac{|MC|}{\lambda_3} \right).$$

(Here, for $V = A, B, C$, $|MV|$ denotes the length of the line segment MV.)

11532. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Find all prime numbers p such that there exists a 2×2 matrix A with integer entries, other than the identity matrix I, for which $A^p + A^{p-1} + \cdots + A = pI$.

doi:10.4169/000298910X521724

11533. *Proposed by Erwin Just (emeritus), Bronx Community College of the City College of New York, Bronx, NY.* Let t be a positive integer and let R be a ring, not necessarily having an identity element, such that $x + x^{2t+1} = x^{2t} + x^{10t+1}$ for each x in R. Prove that R is a Boolean ring, that is, $x = x^2$ for all x in R.

11534. *Proposed by Christopher Hillar, Mathematical Sciences Research Institute, Berkeley, CA.* Let k and n be positive integers with $k < n$. Characterize the $n \times n$ real matrices M with the property that for all $v \in \mathbb{R}^n$ with at most k nonzero entries, Mv also has at most k nonzero entries.

11535. *Proposed by Marian Tetiva, Bîrlad, Romania.* Let f be a continuously differentiable function on [0, 1]. Let $A = f(1)$ and let $B = \int_0^1 x^{-1/2} f(x)\,dx$. Evaluate

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)\,dx - \sum_{k=1}^{n}\left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right)$$

in terms of A and B.

11536. *Proposed by Mihaly Bencze, Brasov, Romania.* Let K, L, and M denote the respective midpoints of sides AB, BC, and CA in triangle ABC, and let P be a point in the plane of ABC other than K, L, or M. Show that

$$\frac{|AB|}{|PK|} + \frac{|BC|}{|PL|} + \frac{|CA|}{|PM|} \ge \frac{|AB| \cdot |BC| \cdot |CA|}{4|PK| \cdot |PL| \cdot |PM|},$$

where $|UV|$ denotes the length of segment UV.

SOLUTIONS

The Number of k-cycles in a Random Permutation

11378 [2008, 664]. *Proposed by Daniel Troy (Emeritus), Purdue University–Calumet, Hammond, IN.* Let n be a positive integer, and let $U_1, \ldots, U_n$ be random variables defined by one of the following two processes:

- **A:** Select a permutation of $\{1, \ldots, n\}$ at random, with each permutation of equal probability. Then take U_k to be the number of k-cycles in the chosen permutation.
- **B:** Repeatedly select an integer at random from $\{1, \ldots, M\}$ with uniform distribution, where M starts at n and at each stage in the process decreases by the value of the last number selected, until the sum of the selected numbers is n. Then take U_k to be the number of times the randomly chosen integer took the value k.

Show that the probability distribution of $(U_1, \ldots, U_n)$ is the same for both processes.

Solution by O.P. Lossers, Eindhoven University of Technology, Netherlands. First we introduce a standard notation for the permutations: in each cycle put the lowest number in front, and list the cycles with the first elements in decreasing order. Next we count the permutations of n objects where the last cycle has length k. The last cycle starts with 1, and the other $k - 1$ elements are arbitrary, in any order. Hence there are $(n-1)!/(n-k)!$ ways to fill the last cycle, and then the permutation can be completed in $(n-k)!$ ways. Hence the number of permutations in which the last cycle has length k is $(n-1)!$, independent of k. It follows that the length of the last cycle is uniformly distributed, and the remaining cycles are produced by the same process on the remaining $n - k$ elements. Hence the production of cycle lengths from back to front under process A emulates process B.

Editorial comment. Erich Bach noted that the use of process B to generate the cycle lengths of random permutations has appeared before, such as in E. Bach, Exact Analysis of a Priority Queue Algorithm for Random Variate Generation, *Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 1994, 48–56.

Also solved by E. Bach, D. Beckwith, R. Chapman (U. K.), S. J. Herschkorn, J. H. Lindsey II, R. Martin (Germany), J. H. Nieto (Venezuela), M. A. Prasad (India), K. Schilling, J. H. Smith, P. Spanoudakis (U. K.), R. Stong, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

When $HI = IO$

11398 [2008, 948]. *Proposed by Stanley Huang, Jiangzhen Middle School, Huaining, China.* Assume acute triangle ABC has its middle-sized angle at A. Suppose further that the incenter I is equidistant from the circumcenter O and the orthocenter H. Show that angle A has measure 60 degrees and that the circumradius of IBC is the same as that of ABC.

Composite solution by the Editors. The restriction to acute triangles appears to be unnecessary.

V. V. Garcia (Huelva, Spain) pointed to Problem E2282, this MONTHLY, April 1972, pp. 397–8, where it is shown that (excepting only equilateral triangles, for which $IO = 0$, and not excluding right or obtuse triangles) HI/IO is (1) less than 1, (2) equal to 1, or (3) greater than 1, according as the middle-sized angle of the triangle is (1) greater than, (2) equal to, or (3) less than $60°$. Geometrically, this means that with respect to the perpendicular bisector λ of the Euler segment, I is (1) on the H side of λ, (2) on λ, or (3) on the O side of λ. Thus when I is equidistant from O and H, i.e., on λ, the middle-sized angle must be $60°$.

The second claim of this problem is too humble. Actually, when angle A has measure $60°$, the reflection $\mathcal{C}'$ of the circumcircle $\mathcal{C}$ of ABC across BC, which of course has the same radius, contains not only I (making it the circumcircle of BIC) but also O and H. A proof of this expanded claim was submitted to this MONTHLY in 1998 by W. W. Meyer as part of a solution to Problem 10547. Here, we will give a proof based on the solution by Jerry Minkus (San Francisco, CA): Let the angles at A, B, and C be α, β, and γ, respectively. We have shown that $\alpha = 60°$.

Claim. I lies on $\mathcal{C}'$. *Proof.* Designate the midpoint of BC as M. Let P be the point on the opposite side of BC from A at which the perpendicular bisector of BC meets $\mathcal{C}$. Triangles BPM and CPM are congruent, so arcs BP and CP are congruent. Therefore angles BAP and CAP are congruent. Thus AP is the angle bisector of BAC, and therefore AP contains I.

It is known that $R^2 - IO^2 = 2Rr$, which may also be observed by constructing the diameter of $\mathcal{C}$ through I. Thus $IA \cdot IP = (R + OI) \cdot (R - OI) = R^2 - OI^2 = 2Rr$. Since $IA = r/\sin(\alpha/2)$, we have $IP = 2R\sin(\alpha/2)$. Similarly, BP and CP are also equal to $2R\sin(\alpha/2)$. Hence B, C, and I all lie on a circle about P. When $E\alpha = 60°$, the radius of that circle is R, because $\sin(60°/2) = 1/2$. Hence P is the reflection in BC of O, and the circle just referenced containing B, C, and I is the circle $\mathcal{C}'$.

Claim. O lies on $\mathcal{C}'$. *Proof.* O and P are reflections of each other in BC.

Claim. H lies on $\mathcal{C}'$. *Proof.* Note that $AH = 2R\cos\alpha$. This may be seen by extending ray CO to meet $\mathcal{C}$, say at Q. Then since CQ is a diameter, its length is $2R$, angle CBQ is right, and $\angle BQC = \angle BAC = \alpha$, so $BQ = 2R\cos\alpha$. Now BQ is parallel to AH, and similarly, AQ is parallel to BH. Thus $AHBQ$ is a parallelogram and $AH = BQ = 2R\cos\alpha$. Here we have $\alpha = 60°$ and $\cos 60° = 1/2$, so $AH = R$. We may conclude that $AOPH$ is a parallelogram, since AH is parallel to OP and of the same length. (It is in fact a rhombus.) It follows that $PH = AO = R$. Thus as claimed H lies on $\mathcal{C}'$.

Editorial comment. The Blundon result from E2282 may be strengthened in an interesting way due to Francisco Bellot Rosado (Spain), who submitted it to this MONTHLY in 1998 as part of a solution to Problem 10547: Let G denote the centroid of the triangle. The incenter I always lies inside the circle whose diameter is GH, because the angle GIH is always obtuse. Since the perpendicular bisector λ of the Euler segment OH divides the circle of Bellot Rosado into a larger and a smaller piece, I is (1) in the larger piece, (2) on line λ, or (3) in the smaller piece, according as the middle-sized angle of ABC is (1) greater than, (2) equal to, or (3) less than 60°.

Also solved by M. Bataille (France), R. Chapman (U. K.), C. Curtis, Y. Dumont (France), D. Fleischman, V. V. Garcia (Spain), D. Grinberg, J.-P. Grivaux (France), E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), O. Kouba (Syria), J. H. Lindsey II, J. Minkus, R. Stong, M. Tetiva (Romania), D. Vacaru (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), J. B. Zacharias & K. Greeson, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Alternating Series

11409 [2009, 83]. *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.* For positive real α and β, let

$$S(\alpha, \beta, N) = \sum_{n=2}^{N} n\log(n)(-1)^n \prod_{k=2}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)}.$$

Show that if $\beta > \alpha$, then $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. Let $\omega_k = k\log k$. Write

$$a_n = \omega_n \prod_{k=2}^{n} \frac{\alpha + \omega_k}{\beta + \omega_{k+1}} = b_n \prod_{k=3}^{n} \left(1 - \frac{\beta - \alpha}{\beta + \omega_k}\right), \quad \text{where} \quad b_n = \frac{(\alpha + \omega_2)\omega_n}{\beta + \omega_{n+1}}, \quad (1)$$

and suppose $\beta > \alpha$. We will prove that

$$\sum_{n=2}^{\infty} (-1)^n a_n \quad \text{converges,}$$

so $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists. By the alternating series test of Leibniz, and noting $a_n > 0$, it suffices to prove

(i) $a_{n+1}/a_n < 1$ for all sufficiently large n, and

(ii) $a_n \to 0$ as $n \to \infty$.

(i) From the definition of a_n in (1),

$$\frac{a_{n+1}}{a_n} = \frac{\omega_{n+1}(\alpha + \omega_{n+1})}{\omega_n(\beta + \omega_{n+2})},$$

so $a_{n+1}/a_n < 1$ is equivalent to $\omega_{n+1}\alpha + (\omega_{n+1}^2 - \omega_n\omega_{n+2}) < \omega_n\beta$. Calculation shows $\omega_{n+1}^2 - \omega_n\omega_{n+2} = (\log n)^2 + \log n + 1 + o(1)$. Because $\beta > \alpha$ and $\omega_{n+1} \sim \omega_n = n\log n$, the required result follows.

(ii) Because $\lim_{n\to\infty} b_n$ exists, to show $\lim_{n\to\infty} a_n = 0$ it suffices to show that the infinite product

$$\prod_{k=3}^{\infty} \left(1 - \frac{\beta - \alpha}{\beta + \omega_k}\right) \quad (2)$$

diverges to zero. Recall that if $0 < c_k < 1$ for all k and $\sum_{k=1}^{\infty} c_k$ diverges, then $\prod_{k=1}^{\infty}(1 - c_k)$ diverges to 0. In the present case, the divergence of

$$\sum_{k=3}^{\infty} \frac{1}{\omega_k} = \sum_{k=3}^{\infty} \frac{1}{k \log k}$$

shows that the infinite product in (2) diverges to 0. (That the sum diverges is well known, as it follows from the integral test or Cauchy condensation test.)

Also solved by S. Amghibech (Canada), P. Bracken, R. Chapman (U. K.), P. P. Dálymay (Hungary), D. Grinberg, J. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

A Fix for a Triangle Inequality

11413 [2009, 179]. *Proposed by Mihály Bencze, Brasov, Romania.* Let θ_i for $1 \le i \le 5$ be nonnegative, with $\sum_1^3 \theta_i = \pi$, $\theta_4 = \theta_1$, and $\theta_5 = \theta_2$. Let $S = \sum_{i=1}^3 \sin\theta_i$. Show that

$$S \le \frac{3\sqrt{3}}{2} - 4 \max_{1 \le i \le 3}\left(\sin^2\left(\frac{1}{(4)}(\theta_i - \theta_{i+1})\right)\cos\left(\frac{1}{2}\theta_{i+2}\right) + \sqrt{3}\sin^2\left(\frac{1}{12}(\pi - 3\theta_{i+2})\right)\right).$$

Solution by Richard Stong, San Diego, CA. (The originally published statement had a misprint, with "2" where "(4)" now stands.) If $A, B, C \ge 0$ with $A + B + C = \pi$, then

$$S = \sin A + \sin B + \sin C = 4\cos(A/2)\cos(B/2)\cos(C/2).$$

Hence

$$\begin{aligned} S + 4\sin^2((A - B)/4)\cos(C/2) &= 4\cos^2((A + B)/4)\cos(C/2) \\ &= 4\cos^2(\pi - C)/4)\cos(C/2). \end{aligned}$$

Applying the identity

$$4\cos(x + 2y)\cos^2(x - y) + 8\sin^2 y \cos x = 4\cos^3 x - 4\sin^2 y\cos(x - 2y)$$

with $x = \pi/6$ and $y = (\pi - 3C)/12$, we have

$$4\cos\frac{C}{2}\cos^2\frac{\pi - C}{4} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} = \frac{3\sqrt{3}}{2} - 4\sin^2\frac{\pi - 3C}{12}\cos\frac{2\pi - 3C}{6}$$

or, combined with the above,

$$S + 4\sin^2\frac{A - B}{4}\cos\frac{C}{2} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} = \frac{3\sqrt{3}}{2} - 4\sin^2\frac{\pi - 3C}{12}\cos\frac{2\pi - 3C}{6}.$$

Since $0 \le C \le \pi$, the last cosine is nonnegative, and hence

$$S + 4\sin^2\frac{A - B}{4}\cos\frac{C}{2} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} \le \frac{3\sqrt{3}}{2}.$$

Apply this result three times, taking (A, B, C) to be $(\theta_1, \theta_2, \theta_3)$, then $(\theta_2, \theta_3, \theta_1)$, and finally $(\theta_3, \theta_1, \theta_2)$, to obtain the desired result.

Editorial comment. Some solvers corrected the problem by showing that it holds as originally printed but with the inequality reversed.

Also solved by R. Bagby, P. P. Dályay (Hungary), J. H. Lindsey II, GCHQ Problem Solving Group (U. K.), and Microsoft Research Problems Group.

Blundon's Inequality Improved

11414 [2009, 179]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let ABC be a triangle with largest angle at A, let A also denote the measure of that angle, let $c = \cot(A/2)$, and let s, r, and R be the semi-perimeter, inradius, and circumradius of the triangle, respectively. Show that Blundon's Inequality $s \le 2R + r(3\sqrt{3} - 4)$ can be strengthened to

$$s \le 2R + r\left(3\sqrt{3} - 4 - \frac{(\sqrt{3} - c)^3}{4c}\right).$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Lemma. *If a, b, c are positive real numbers such that $a + b + c = abc$ and $c = \min\{a, b, c\}$, then $(a - 1)(b - 1)(c - 1) \le 6\sqrt{3} - 10 - (\sqrt{3} - c)^3/(2c)$.*

Proof. Note that $ba = (abc)/c > (abc)/(a + b + c) = 1$, and similarly $bc > 1$ and $ca > 1$. Thus at most one of the numbers a, b, c can be less than 1. Hence $a \ge 1$ and $b \ge 1$. The equality $a + b + c = abc$ yields $c = (a + b)/(ab - 1)$. We must show that if $a, b \ge 1$ and $ab > 1$, then $f(a, b) \le 0$, where

$$f(a, b) = (a - 1)(b - 1)\left(\frac{a + b}{ab - 1} - 1\right) - (6\sqrt{3} - 10) - \frac{\left(\sqrt{3}(ab - 1) - (a + b)\right)^3}{2(ab - 1)^2(a + b)}.$$

Put $a = 1 + x$ and $b = 1 + y$ with $x, y \ge 0$, and rewrite the function as

$$f(1 + x, 1 + y) = -2(x + y + 2)(x + y + xy)\left(x^2y^2 + (6\sqrt{3} - 12)xy + (6\sqrt{3} - 10)(x + y)\right).$$

Observe $x + y \ge 2\sqrt{xy}$ and substitute $t = \sqrt{xy}$ to reduce the inequality to $p(t) \ge 0$ for all $t \ge 0$, where $p(t) = t^4 + (6\sqrt{3} - 12)t^2 + (12\sqrt{3} - 20)t$. This follows from the factorization $p(t) = t(t - (\sqrt{3} - 1))^2(t + 2\sqrt{3} - 2)$. ∎

In triangle ABC, the numbers $a = \cot(A/2)$, $b = \cot(B/2)$, and $c = \cot(C/2)$ satisfy $a + b + c = s/r = abc$, $ab + bc + ca = (4R + r)/r$, and $c = \min\{a, b, c\}$. By the lemma,

$$\begin{aligned} s &= \frac{r}{2}\left[(a - 1)(b - 1)(c - 1) + ab + bc + ca + 1\right] \\ &\ge \frac{r}{2}\left(6\sqrt{3} - 10 - \frac{(\sqrt{3} - c)^3}{2c} + \frac{4R + r}{r} + 1\right) \\ &= 2R + r\left(3\sqrt{3} - 4 - \frac{(\sqrt{3} - c)^3}{4c}\right). \end{aligned}$$

Equality holds if and only if the triangle is equilateral.

Editorial comment. Richard Stong proved the stronger inequality

$$s \le 2R + r\left(3\sqrt{3} - 4 - \frac{9(2-\sqrt{3})}{8}\,\frac{(\sqrt{3}-c)^2}{c^2}\right).$$

Also solved by J. H. Lindsey II, C. R. Pranesachar (India), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Closed-Form Definite Integral

11416 [2009, 180]. *Proposed by Yaming Yu, University of California Irvine, Irvine, CA.* Let f be the decreasing function on $(0, \infty)$ that satisfies

$$f(x)e^{-f(x)} = xe^{-x}.$$

(To visualize, draw a graph of the function xe^{-x} and a horizontal line that is tangent to it or crosses it at two points; if one of these points is x, then the other is $f(x)$.) Show that

$$\int_0^\infty x^{-1/6}(f(x))^{1/6}\,dx = \frac{2\pi^2}{3}.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. From the definition of f, we have $f(x)/x = e^{f(x)-x}$. Writing $u = f(x)/x$ and eliminating $f(x)$ gives $x = \log u/(u-1)$, so that as x increases from 0 to ∞, u decreases from ∞ to 0. The integral to be computed, call it A, can then be written as $A = \int_0^\infty u^\alpha(x)\,dx$ (with $\alpha = 1/6$). Integrating first by parts and then changing variables from x to u in the resulting integral gives

$$A = \int_{x=0}^\infty u^\alpha(x)\,dx = xu^\alpha(x)\Big|_{x=0}^\infty + \alpha\int_{u=0}^\infty \frac{u^{\alpha-1}\log u}{u-1}\,du.$$

Here we could refer to Gradshteyn & Ryzhik (formula 4.254.1) and Abramowitz & Stegun (formula 6.4.7). In this special case, though, there is a simpler solution. For $0 < \alpha < 1$ the integral converges. The first term on the right-hand side is zero because it is equal to $u^\alpha \log(u)/(u-1)|_{u=\infty}^0$. Split the second term into two parts:

$$-\alpha\int_0^1 \frac{u^{\alpha-1}\log u}{1-u}\,du + \alpha\int_1^\infty \frac{u^{\alpha-2}\log u}{1-1/u}\,du.$$

Expand $(1-u)^{-1}$ and $(1-1/u)^{-1}$ as geometric series, then integrate:

$$\alpha\sum_{n=0}^\infty \frac{1}{(n+\alpha)^2} + \alpha\sum_{n=1}^\infty \frac{1}{(n+1-\alpha)^2}.$$

Using the Hurwitz zeta function notation $\zeta(s,a) = \sum_{n=0}^\infty (n+a)^{-s}$, for arbitrary α in $(0,1)$ this can be written as $\alpha(\zeta(2,\alpha) + \zeta(2,1-\alpha))$. Starting with the known fact that $\zeta(2) = \zeta(2,1) = \pi^2/6$, elementary calculations give $\zeta(2,1/2) = 3\zeta(2)$ and $\zeta(2,1/3) + \zeta(2,2/3) = 8\zeta(2)$, so that $\zeta(2,1/6) + \zeta(2,5/6)$ is given by

$$\sum_{k=1}^6 \zeta(2,k/6) - \sum_{k=1}^2 \zeta(2,k/3) - \zeta(2,1/2) - \zeta(2,1) = (36-8-3-1)\zeta(2) = 4\pi^2.$$

The required sum is thus $4\alpha\pi^2 = 2\pi^2/3$.

Also solved by R. Bagby, D. Beckwith, P. Bracken, B. S. Burdick, P. Corn, L. Gerber, M. L. Glasser, J. Grivaux (France), E. A. Herman, F. Holland & T. Carroll (Ireland), K. McInturff, O. G. Ruehr, V. Rutherfoord, R. Stong, J. B. Zacharias, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Integral-Derivative Inequality

11417 [2009, 180]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanţa, Romania.* Let f be a continuously differentiable real-valued function on [0, 1] such that $\int_{1/3}^{2/3} f(x)\,dx = 0$. Show that $\int_0^1 (f'(x))^2\,dx \ge 27\left(\int_0^1 f(x)\,dx\right)^2$.

Solution by Moubinool Omarjee, Paris, France. Let $h(x)$ be the continuous, piecewise linear function given by

$$h(x) = \begin{cases} -x, & 0 \le x \le 1/3, \\ 2x-1, & 1/3 \le x \le 2/3, \\ 1-x, & 2/3 \le x \le 1. \end{cases}$$

Integrating by parts gives

$$\int_0^1 h(x) f'(x)\,dx = \int_0^1 f(x)\,dx - 3\int_{1/3}^{2/3} f(x)\,dx = \int_0^1 f(x)\,dx,$$

and we compute that

$$\int_0^1 h(x)^2\,dx = \frac{1}{27}.$$

Hence the Cauchy-Schwarz inequality applied to h and f' reads

$$\int_0^1 (f'(x))^2\,dx \ge 27\left(\int_0^1 f(x)\,dx\right)^2,$$

as desired.

Editorial comment. Several solvers remarked that this problem generalizes with essentially the same proof. In the simplest form, suppose that $\phi(x)$ is an integrable function with $\int_0^1 \phi(x)\,dx = 1$, and define $h(x) = -x + \int_0^x \phi(t)\,dt$ and $C = \int_0^1 h(x)^2\,dx$. For any continuously differentiable real-valued function f on [0, 1] such that $\int_0^1 f(x)\phi(x)\,dx = 0$, one has

$$C\int_0^1 (f'(x))^2\,dx \ge \left(\int_0^1 f(x)\,dx\right)^2.$$

More generally, this holds with $\phi(x)\,dx$ replaced by a signed Borel measure.

Also solved by K. F. Andersen (Canada), R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, O. Geupel (Germany), J. Grivaux (France), G. Keselman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), D. S. Ross, R. Tauraso (Italy), P. Venkataramana, E. I. Verriest, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), St. John's University Problem Solving Group, and the proposers.

Gamma Products

11426 [2009, 365]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Find

$$\frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)},$$

where Γ denotes the usual gamma function, given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$.

Solution by Matthew A. Carlton, Cal Poly State University, San Luis Obispo, CA. The multiplication formula for the gamma function may be written as

$$\Gamma(z) = 2\sqrt{\pi}\cdot 2^{-2z}\cdot\frac{\Gamma(2z)}{\Gamma(z+1/2)}.$$

Apply this with z equal to each of the six values in the original expression, e.g.

$$\Gamma(1/14) = 2\sqrt{\pi}\cdot 2^{-1/7}\cdot\frac{\Gamma(1/7)}{\Gamma(4/7)}.$$

The numerator of the original expression can then be written

$$\begin{aligned}(2\sqrt{\pi}\,)^3\cdot 2^{-1/7-9/7-11/7}&\cdot\frac{\Gamma(1/7)\Gamma(9/7)\Gamma((11/7)}{\Gamma(4/7)\Gamma(8/7)\Gamma(9/7)}\\ &= 8\pi^{3/2}\cdot\frac{1}{8}\cdot\frac{\Gamma(1/7)\cdot 4/7\Gamma(4/7)}{\Gamma(4/7)\cdot 1/7\Gamma(1/7)} = 4\pi^{3/2}.\end{aligned}$$

Similarly, the denominator simplifies to $2\pi^{3/2}$. Thus the quotient is 2.

Editorial comment. Some solvers provided generalizations. The most interesting and complete was from Albert Stadler (Switzerland). Let p be an odd prime, and denote the Legendre symbol by $\left(\frac{k}{p}\right)$. Then

$$\prod_{k=1}^{p}\Gamma\left(\frac{2k-1}{2p}\right)^{\left(\frac{2k-1}{p}\right)} = \begin{cases}1, & \text{if } p\equiv 1 \pmod 8,\\ \varepsilon(p)^{h(p)}, & \text{if } p\equiv 5 \pmod 8,\\ 2^{-\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\frac{k}{p}}, & \text{if } p\equiv 7 \pmod 8,\end{cases} \qquad (*)$$

where $\varepsilon(p)$ denotes the fundamental unit and $h(p)$ the class number of the real quadratic field $\mathbb{Q}(\sqrt{p}\,)$. The case $p \equiv 3 \pmod 8$ was not resolved. The fundamental unit $\varepsilon(p) = (x + y\sqrt{p}\,)/2$ is a solution of Pell's equation $x^2 - py^2 = 4$ with the property that both x and y are positive and y is minimal. The result asked for here is the case $p = 7$. Other examples ($p = 5$, 13, 17):

$$\frac{\Gamma(1/10)\Gamma(9/10)}{\Gamma(3/10)\Gamma(7/10)} = \frac{3+\sqrt{5}}{2},$$

$$\frac{\Gamma(1/26)\Gamma(3/26)\Gamma(9/26)\Gamma(17/26)\Gamma(23/26)\Gamma(25/26)}{\Gamma(5/26)\Gamma(7/26)\Gamma(11/26)\Gamma(15/26)\Gamma(19/26)\Gamma(21/26)} = \frac{11+3\sqrt{13}}{2},$$

$$\frac{\Gamma(1/34)\Gamma(9/34)\Gamma(13/34)\Gamma(15/34)\Gamma(19/34)\Gamma(21/34)\Gamma(25/34)\Gamma(33/34)}{\Gamma(3/34)\Gamma(5/34)\Gamma(7/34)\Gamma(11/34)\Gamma(23/34)\Gamma(27/34)\Gamma(29/34)\Gamma(31/34)} = 1.$$

Since the values in $(*)$ are algebraic numbers, we have a corollary: If p is an odd prime $\not\equiv 3 \pmod 8$, then the $p-1$ numbers $\Gamma((2k-1)/(2p))$, $1 \le k \le p$, $k \ne (p-1)/2$, are algebraically dependent.

Also solved by Z. Ahmed & M. A. Prasad (India), K. F. Andersen (Canada), R. Bagby, B. Bauldry, D. Beckwith, P. Bracken, M. A. Carlton, R. Chapman (U. K.), H. Chen, C. K. Cook, P. Costello, P. P. Dálylay (Hungary), F. Flores & F. Mawyer, M. R. Gopal, D. Gove, G. C. Greubel, D. Grinberg, J. Grivaux (France), J. A. Grzesik, C. C. Heckman, E. A. Herman, D. Hou, R. Howard, E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), G. Keselman, T. Konstantopoulis (U. K.), O. Kouba (Syria), V. Krasniqi (Kosova), H. Kwong, G. Lamb, O. P. Lossers (Netherlands), R. Martin (Germany), K. McInturff, A. Nijenhuis, O. Padé (Israel), R. Padma (India), C. R. Pranesachar (India), H. Riesel (Sweden), I. Rusodimos, O. A. Saleh & S. Byrd, A. S. Shabani (Kosova), M. A. Shayib, N. C. Singer, A. Stadler (Switzerland), R. Stong, T. Tam, R. Tauraso (Italy), Z. Vörös (Hungary), M. Vowe (Switzerland), Z. Wenlong (China), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

An Equilateral Condition

11427 [2009, 365]. *Proposed by Viorel Bǎndilǎ, C.A. Rosetti High School, Bucharest, Romania.* In a triangle ABC, let m be the length of the median from A, l the length of the angle bisector from B, and h the length of the altitude from C. Let a, b, and c be the lengths of the edges opposite A, B, and C, respectively. Show that ABC is equilateral if and only if $a^2+m^2=b^2+l^2=c^2+h^2$.

Solution by Bianca-Teodora Iordache, student, "Carol I" High School, Craiova, Romania. If ABC is equilateral, then $a=b=c$ and $m=l=h$, so the equations hold. We must prove the converse. Let m_a, l_a, and h_a denote the lengths of the median, angle bisector, and altitude, respectively, corresponding to the edge a, and define similar notation for edges b and c. We must prove that

$$a^2+m_a^2=b^2+l_b^2=c^2+h_c^2 \quad\Longrightarrow\quad a=b=c.$$

Claim 1. $a^2+m_a^2 \le b^2+m_b^2 \Longleftrightarrow a \le b$. Indeed,

$$a^2+m_a^2=a^2+\frac{2(b^2+c^2)-a^2}{4}=\frac{3a^2+2b^2+2c^2}{4}.$$

Hence $a^2+m_a^2 \le b^2+m_b^2 \Longleftrightarrow 3a^2+2b^2+2c^2 \le 3b^2+2a^2+2c^2 \Longleftrightarrow a^2 \le b^2 \Longleftrightarrow a \le b$.

Claim 2. $a^2+h_a^2 \le b^2+h_b^2 \Longleftrightarrow a \le b$. Using $h_a=2S/a$, where S is the area of ABC, we have $a^2+h_a^2=a^2+4S^2/a^2$, so

$$a^2+h_a^2 \le b^2+h_b^2 \Longleftrightarrow (b^2-a^2)\frac{a^2b^2-4S^2}{a^2b^2} \ge 0 \Longleftrightarrow b \ge a.$$

Also recall that $h_a \le l_a \le m_a$ and similarly for b, c. Next suppose that $a^2+m_a^2=b^2+l_b^2=c^2+h_c^2$. We have $a^2+m_a^2=c^2+h_c^2 \le c^2+m_c^2$, so $a \le c$ from Claim 1. We have $c^2+h_c^2=b^2+l_b^2 \ge b^2+h_b^2$, so $b \le c$ by Claim 2. From the Heron formula, $16S^2=(a+b+c)(-a+b+c)(a-b+c)(a+b-c)=2\sum a^2b^2-\sum a^4$, using $\sum$ for sums over cyclic permutations of the triangle. Now $a^2+m_a^2=c^2+h_c^2$ so

$$\frac{3a^2+2b^2+2c^2}{4}=c^2+\frac{2\sum a^2b^2-\sum a^4}{4c^2},$$

so $c^2(3a^2+2b^2-2c^2)=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$ and thus

$$c^2(c^2-a^2)=(b^2-a^2)^2. \tag{1}$$

Since $c^2 \ge b^2 > b^2-a^2$ and $c^2-a^2 \ge b^2-a^2$, for equality in (1) we must have $c^2-a^2=b^2-a^2=0$. This shows $a=c$ and $a=b$ as required.

Also solved by R. Bagby, M. Bataille (France), H. Caerols (Chile), R. Chapman (U. K.), G. Crandall, P. P. Dálylay (Hungary), D. Fleischman, D. Gove, J. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, J. McHugh, J. Minkus, M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before April 30, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11537. *Proposed by Lang Withers, Jr., MITRE, McClean, VA.* Let p be a prime and a be a positive integer. Let X be a random variable having a Poisson distribution with mean a, and let M be the pth moment of X. Prove that $M \equiv 2a \pmod{p}$.

11538. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Prove that a finite commutative ring in which every element can be written as a product of two (not necessarily distinct) elements has a multiplicative identity.

11539. *Proposed by William C. Jagy, MSRI, Berkeley, CA.* Let E be the set of all positive integers not divisible by 2 or 3 or by any prime q represented by the quadratic form $4u^2 + 2uv + 7v^2$. (Thus, the first few members of E are 1, 5, 11, 17, 23, and 25.) Show that $4x^2 + 2xy + 7y^2 + z^3$ is not in $\{2n^3, -2n^3, 32n^3, -32n^3\}$ for $n \in E$ and $x, y, z \in \mathbb{Z}$.

11540. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be an integer greater than 1, other than 4. Let p and q be positive integers less than n and relatively prime to n. Let $a = \frac{\cos(2\pi p/n)}{\cos(2\pi q/n)}$. Show that if a^k is rational for some positive integer k, then a^k is either 1 or -1.

11541. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Let $|MA|$, $|MB|$, and $|MC|$ be the distances from M to A, B, and C. Show that

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \le \frac{3}{2}.$$

11542. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania.* Show that for $x, y, z > 1$, and for positive

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α, β, γ,

$$(2x^2 + yz)\Gamma(x) + (2y^2 + zx)\Gamma(y) + (2z^2 + xy)\Gamma(z)$$
$$\geq (x + y + z)(x\Gamma(x) + y\Gamma(y) + z\Gamma(z)),$$

and

$$B(x,\alpha)^{x^2+2yz} B(y,\beta)^{y^2+2zx} B(z,\gamma)^{z^2+2xy}$$
$$\geq (B(x,\alpha)B(y,\beta)B(z,\gamma))^{xy+yz+zx}.$$

Here, $B(x,\alpha)$ is Euler's beta function, defined by $B(x,\alpha) = \int_0^1 t^{x-1}(1-t)^{\alpha-1}dt$.

11543. *Proposed by Richard Stong, Center for Communications Research, San Diego, CA.* Let x, y, z be positive numbers with $xyz = 1$. Show that $(x^5 + y^5 + z^5)^2 \geq 3(x^7 + y^7 + z^7)$.

SOLUTIONS

A Euclidean Path

11390 [2008, 855]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Let G be the undirected graph on the vertex set V of all pairs (a, b) of relatively prime integers, with edges linking (a, b) to $(a + kab, b)$ and $(a, b + kab)$ for all integers k.

(**a**) Show that for all (a, b) in V, there is a path joining (a, b) and $(1, 1)$.

(**b**)* Call an edge linking (a, b) to $(a + kab, b)$ or $(a, b + kab)$ *positive* if $k > 0$, and *negative* if $k < 0$. Let the *reversal number* of a path from $(1, 1)$ to (a, b) be one more that the number of sign changes along the path, and let the *reversal value* of (a, b) be the minimal reversal number over all paths from $(1, 1)$ to (a, b). Are there pairs of arbitrarily high reversal value?

Solution by M. D. Meyerson and M. E. Kidwell, U.S. Naval Academy.

(**a**) Suppose first that a and b are positive; we may assume that $a < b$. Let $c = b - a$. Note that b and c are relatively prime (if d divides both, then it also divides a); hence there are integers m and n such that $mb + nc = 1$. We may choose m positive and n negative, since increasing m by c and decreasing n by b does not change $mb + nc$. We can link (a, b) to (a, c) via two negative edges, since $(a, b - mab) = (a, b - a(1 - nc)) = (a, b - a + nac) = (a, c + nac)$. We can similarly link (b, a) to (c, a) via two negative edges. By the Euclidean algorithm, we can thus reach $(1, 1)$ via only negative edges.

If $ab = 0$ then there is a negative edge from one of $(-1, 1)$, $(1, -1)$, or $(1, 1)$ to (a, b).

If exactly one of $\{a, b\}$ is negative, then we can add $(-2)ab$ to the negative component of (a, b) to reach a pair with positive components via a negative edge, followed by linking as above to $(1, 1)$. If both a and b are negative, then to make at least one coordinate positive we must use a sufficiently large positive multiple of their product, after which we can reach $(1, 1)$ via only negative edges. This process misses four points, $(0, \pm 1)$ and $(\pm 1, 0)$, which can easily be linked to $(1, 1)$ via at most two edges.

(**b**)* By the process in part (a), we can reach $(1, 1)$ via only negative edges unless a and b are both negative, in which case we only need to use one positive edge to start after which we can reach $(1, 1)$ using only negative edges. Thus there is always a path from (a,b) to $(1,1)$ with reversal number at most 2, so there are no pairs (a,b) of arbitrarily high reversal value.

Editorial comment. The sign of an edge is well defined; if the link can be viewed from both ends, then the corresponding choices for k are equal and thus have the same sign.

Both parts also solved by P. Corn, K. Schilling, B. Schmuland (Canada), R. Stong, A. Vorobyov, and the Texas State University Problem Solvers Group. Part (a) also solved by D. Klyve & C. Storm, M. A. Prasad (India), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

A Congruence for Vanishing Modular Sums

11391 [2008, 855]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let p be a positive prime and s a positive integer. Let n and k be integers such that $n \ge k \ge p^s - p^{s-1}$, and let $x_1, \dots, x_n$ be integers. For $1 \le j \le n$, let m_j be the number of expressions of the form $x_{i_1} + \cdots + x_{i_j}$ with $1 \le i_1 < \cdots < i_j \le n$ that evaluate to 0 modulo p, and let n_j denote the number of such expressions that do not. (Set $m_0 = 1$ and $n_0 = 0$.) Apart from the cases $(s,k) = (1, p-1)$ and $s = p = k = 2$, show that

$$\sum_{j=0}^{k} (-1)^j \binom{n-k+j}{j} m_{k-j} \equiv 0 \pmod{p^s},$$

and show that the same congruence holds with n_{k-j} in place of m_{k-j}.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove a much stronger statement. Let $X = \{x_1, \dots, x_n\}$, and let q_j be the number of j-element subsets of X whose sum is congruent to a modulo p. For $n \ge k \ge 1 + s(p-1)$, we prove that

$$\sum_{j=0}^{k} (-1)^j \binom{n-k+j}{j} q_{k-j} \equiv 0 \pmod{p^s}, \tag{1}$$

except in the excluded cases. The desired result for m_{k-j} is the case $a = 0$, and the result for n_{k-j} follows by summing the remaining residue classes.

We first show that it suffices to prove the case $n = k$, which reduces to

$$\sum_{j=0}^{k} (-1)^j q_{k-j} \equiv 0 \pmod{p^s} \tag{2}$$

for $k \ge 1 + s(p-1)$. Assume (2), then, and let $[n]$ denote $\{1, \dots, n\}$. For $S \subseteq [n]$, let $S^* = \{T \subseteq S \colon \sum_{i \in T} x_i \equiv a \pmod{p}\}$. For general n and k, (2) implies, modulo p^s,

$$\begin{aligned} 0 &\equiv \sum_{\substack{S \subseteq [n] \\ |S| = k}} \left(\sum_{j=0}^{k} (-1)^j \sum_{\substack{T \in S^* \\ |T| = k-j}} 1 \right) \equiv \sum_{j=0}^{k} (-1)^j \sum_{\substack{T \in [n]^* \\ |T| = k-j}} \sum_{\substack{T \subseteq S \subseteq [n] \\ |S| = k}} 1 \\ &\equiv \sum_{j=0}^{k} (-1)^j \sum_{\substack{T \in [n]^* \\ |T| = k-j}} \binom{n-(k-j)}{j} \equiv \sum_{j=0}^{k} (-1)^j \binom{n-k+j}{j} q_{k-j}. \end{aligned}$$

This proves that (1) follows from (2). To prove (2), we work in the ring $\mathbb{Z}[t]/(t^p - 1)$, where $t^p = 1$. In this ring, let

$$f(t) = \prod_{x \in X} (1 - t^x) = (1-t)^k \prod_{x \in X} (1 + t + \cdots + t^{x-1}).$$

The terms in the expansion of f have the form $(-1)^{|Y|}\prod_{y\in Y} t^y$, where $Y \subseteq X$. For fixed a, $(-1)^j q_j$ is the contribution to the coefficient of t^a in the expansion of f due to Y of size j and sum congruent to a mod p, and $\sum_{j=0}^{k}(-1)^j q_j = [t^a]f(t)$. We now show that each coefficient of f is a multiple of p^s, from which (2) follows. To see that each coefficient is a multiple of p^s, we show that when $k > (p-1)s$, every coefficient of $(1-t)^k$ is a multiple of p^s.

First we construct a polynomial $h(t)$ such that $(1-t)^p = p\cdot(1-t)h(t)$. For $p = 2$ we have $(1-t)^2 = 1-2t+t^2 = 2-2t = 2(1-t)$. For odd p, we have

$$(1-t)^p = 1 + \sum_{k=1}^{p-1}\binom{p}{k}(-1)^k t^k - t^p = \sum_{k=1}^{(p-1)/2}\binom{p}{k}(-1)^k t^k(1-t^{p-2k})$$

$$= p\cdot(1-t)\sum_{k=1}^{(p-1)/2}\binom{p}{k}/p(-1)^k t^k(1+t+\cdots+t^{p-2k-1}).$$

Now induction on s and the previous result imply when $k > s(p-1)$ that $(1-t)^k = p^s\cdot(1-t)^{k-s(p-1)}h_s(t)$ for some polynomial $h_s(t)$.

Also solved by R. Chapman (U.K.), D. Grinberg, J. H. Lindsey II, and the proposer.

Runs Versus Isolated Heads in Coin Tossing

11394 [2008, 856]. *Proposed by K. S. Bhanu, Institute of Science, Nagpur, India, and M. N. Deshpande, Nagpur, India.* A fair coin is tossed n times, with $n \geq 2$. Let R be the resulting number of runs of the same face, and X the number of isolated heads. Show that the covariance of the random variables R and X is $n/8$.

Solution by Michael Andreoli, Miami Dade College, Miami, FL. Define binary n-tuples U and V by letting $U_k = 1$ if and only if an isolated head occurs at toss k, and $V_k = 1$ if and only if a run begins at toss k. Now $X = \sum_k U_k$ and $R = \sum_k V_k$. Because $E(U_k) = P(U_k = 1)$, we have $E(U_1) = E(U_n) = 1/4$ and $E(U_k) = 1/8$ for $2 \leq k \leq n-1$. Similarly, $E(V_1) = 1$ and $E(V_k) = 1/2$ for $2 \leq k \leq n$. It follows that $E(X) = (n+2)/8$ and $E(R) = (n+1)/2$.

Because $E(U_i V_j) = P(U_i = 1 \text{ and } V_j = 1)$, we obtain

- $E(U_1V_1) = E(U_1V_2) = 1/4$ and $E(U_1V_j) = 1/8$ for $3 \leq j \leq n$;
- $E(U_nV_1) = E(U_nV_n) = 1/4$ and $E(U_nV_j) = 1/8$ for $2 \leq j \leq n-1$; and
- for $2 \leq i \leq n-1$ and $1 \leq j \leq n$,

$$E(U_iV_j) = \begin{cases} 1/8 & \text{if } j \in \{1, i, i+1\}; \\ 1/16 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(XR) = \sum_{i=1}^{n}\sum_{j=1}^{n} E(U_iV_j) = \sum_{j=1}^{n} E(U_1V_j) + \sum_{i=2}^{n-1}\sum_{j=1}^{n} E(U_iV_j) + \sum_{j=1}^{n} E(U_nV_j)$$

$$= \frac{n+2}{8} + \frac{(n-2)(n+3)}{16} + \frac{n+2}{8} = \frac{n^2+5n+2}{16}.$$

It follows that

$$\mathrm{Cov}(XR) = E(XR) - E(X)E(R) = \frac{n^2+5n+2}{16} - \frac{n+2}{8}\cdot\frac{n+1}{2} = \frac{n}{8}.$$

Also solved by D. Beckwith, M. A. Carlton, N. Caro (Brazil), R. Chapman (U.K.), M. P. Cohen, C. Curtis, P. J. Fitzsimmons, N. Grivaux (France), C. C. Heckman, S. J. Herschkorn, G. Keselman, J. H. Lindsey II, K. McInturff, E. Orney & S. Van Gulck (Belgium), A. Plaza & J. J. Gonzalez (Spain), M. A. Prasad (India), R. Pratt & E. Lada, K. Schilling, B. Schmuland (Canada), A. Stadler (Switzerland), J. H. Steelman, R. Stong, R. Tauraso (Italy), Armstrong Problem Solvers, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

Finite Subgroups of Continuous Bijections of [0,1]

11395 [2008, 856]. *Proposed by M. Farrokhi D.G., University of Tsukuba, Tsukuba Ibakari, Japan.* Prove that if H is a finite subgroup of the group G of all continuous bijections of $[0, 1]$ to itself, then the order of H is 1 or 2.

Solution by Jeffrey Bergen, DePaul University, Chicago, IL. If $g \in G$, then g is continuous and injective. Hence g is monotonic, by the intermediate value theorem. Therefore, either (i) $g(0) = 0$ and $g(1) = 1$ or (ii) $g(0) = 1$ and $g(1) = 0$.

Set $g^2 = g \circ g$ and $g^{n+1} = g \circ g^n$ for $n > 1$. If $g(0) = 0$ and $g(a) > a$ for some $a \in [0, 1]$, then the sequence $a, g(a), g^2(a), \ldots$ is increasing. Similarly, if $g(0) = 0$ and $g(a) < a$, then $a, g(a), g^2(a), \ldots$ is decreasing. Therefore, if $g(0) = 0$ and $g(x) \neq x$ for some $x \in [0, 1]$, then g does not have finite order. We conclude that if $g \in H$ and $g(0) = 0$, then g is the identity map.

Next, if $f_1, f_2 \in H$ with $f_1(0) = f_2(0) = 1$, then $f_1 \circ f_2 \in H$ with $f_1 \circ f_2(0) = 0$. Our previous argument shows that $f_1 \circ f_2(x) = x$, and so both f_2 and f_1 are inverses of f_1. Since inverses are unique in a group, it follows that $f_1 = f_2$. As a result, H contains at most one element other than the identity map, and so H has order either 1 or 2, as claimed.

Also solved by M. Barr (Canada), M. Bataille (France), D. R. Bridges, P. Budney, B. S. Burdick, N. Caro & F. Valenzuela (Brazil), R. Chapman (U.K.), L. Comerford, P. Corn, P. P. Dályay (Hungary), D. Grinberg, J. P. Grivaux (France), K. Hanes, E. A. Herman, S. P. Herschkorn, E. J. Ionascu, J. Konienczny, O. Kouba (Syria), J. Kujawa & K. Shankar, J. H. Lindsey II, O. P. Lossers (Netherlands), A. Magidin, R. Martin (Germany), S. Metcalfe, V. Pambuccian, J. W. Pfeffer, E. Pité (France), J. Schaer (Canada), B. Schmuland (Canada), N. C. Singer, V. Stakhovsky, J. H. Steelman, R. Stong, T. Tam, M. Tetiva (Romania) J. Vinuesa (Venezuela), G. Wene, M. Wildon (UK), N. Wodarz, Armstrong Problem Solvers, BSI Problems Group (Germany), Szeged Problem Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U.K.), McDaniel College Problems Group, Microsoft Research Problems Group, Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposer.

A Riemann (Zeta) Sum

11400 [2008, 948]. *Proposed by Paul Bracken, University of Texas–Pan American, Edinburg, TX.* Let ζ be the Riemann zeta function. Evaluate $\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)}$ in closed form.

Solution by Oliver Guepel, Brühl, NRW, Germany. The sum is $\log(2\pi) - \frac{1}{2}$. Since summation of absolutely convergent series can be interchanged, we have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^{2n} n(n+1)}$$
$$= 1 + \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{k^2}\right)^n - \sum_{k=2}^{\infty} \left(k^2 \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{k^2}\right)^{n+1} \right)$$

$$= 1 - \sum_{k=2}^{\infty} \log\left(1 - \frac{1}{k^2}\right) + \sum_{k=2}^{\infty}\left(1 + k^2 \log\left(1 - \frac{1}{k^2}\right)\right)$$

$$= 1 + \lim_{n\to\infty} \sum_{k=2}^{n} \left[1 + (k^2 - 1)\left(\log(k+1) - 2\log k + \log(k-1)\right)\right].$$

With $f(n) = n^2 - 1$ and $g(n) = \log n$, this last line can be written as

$$1 + \lim_{n\to\infty} \sum_{k=2}^{n} (1 + f(k)(g(k+1) - 2g(k) + g(k-1))).$$

Now put $h(n) = f(n-1)g(n) - f(n)g(n-1)$. In general, $h(n+1) - h(n) = f(n)(g(n+1) - 2g(n) + g(n-1)) - g(n)(f(n+1) - 2f(n) + f(n-1))$. Here, the second difference of f is identically 2, so

$$f(n)(g(n+1) - 2g(n) + g(n-1)) = h(n+1) - h(n) + 2\log n.$$

Thus

$$1 + \sum_{k=2}^{n} (1 + f(k)(g(k+1) - 2g(k) + g(k-1)))$$

$$= n + \sum_{k=2}^{n} (h(k+1) - h(k) + 2\log k) = n + h(n+1) - h(2) + 2\log(n!)$$

$$= n + (n^2 - 1)\log(n+1) - (n^2 + 2n)\log n + 2\log(n!).$$

A straightforward application of Stirling's formula yields $\log 2\pi - \frac{1}{2}$ as the limit. It also follows now from

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)} = \log 2\pi - 1$$

(this MONTHLY **94** (1987), p. 467) that we have the rational sum

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(n+1)(2n+1)} = \frac{1}{2}.$$

Also solved by K. F. Andersen (Canada), R. Bagby, M. Bataille (France), D. Beckwith, B. S. Burdick, R. Chapman (U.K.), H. Chen, P. Corn, G. Crandall, P. P. Dályay (Hungary), B. E. Davis, Y. Dumont (France), O. Furdui (Romania), M. L. Glasser, G. C. Greubel, J. Grivaux (France), N. Grossman, J. A. Grzesik, E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), G. Keselman, O. Kouba (Syria), G. Lamb, O. P. Lossers (Netherlands), K. McInturff, M. Omarjee (France), P. Perfetti (Italy), E. Pité (France), Á. Plaza & S. Falcón (Spain), C. Pohoata (Romania), M. A. Prasad (India), P. R. Refolio (Spain), O. G. Ruehr, V. Rutherfoord, B. Schmuland (Canada), N. C. Singer, S. Singh, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. R. Teske, M. Tetiva (Romania), J. Vinuesa (Spain), M. Vowe (Switzerland), BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

A Characterization of the Identity Matrix

11401 [2008, 949]. *Proposed by Marius Cavachi, "Ovidius" University of Constanţa, Constanţa, Romania.* Let A be a nonsingular square matrix with integer entries. Suppose that for every positive integer k, there is a matrix X with integer entries such that $X^k = A$. Show that A must be the identity matrix.

Solution by Microsoft Research Problems Group, c/o Peter Montgomery, Redmond, WA. For $k \in \mathbb{N}$, let X_k be an integer matrix such that $X_k^k = A$. Let p be a prime that does not divide $\det A$. Viewing $X_k \bmod p$ as an element of the general linear group G over the field $\mathbb{F}_p$, Legendre's theorem implies that $X_k^{|G|} \equiv I \bmod p$ for all k. Setting $k = |G|$ yields $A = X_{|G|}^{|G|} \equiv I \bmod p$. That is, all entries of $A - I$ are divisible by p. Since there are infinitely many choices for p, we obtain $A = I$.

Also solved by P. Budney, N. Caro (Brazil), R. Chapman (U.K.), P. P. Dályay (Hungary), D. Grinberg, J. Grivaux (France), E. A. Herman, J. Konieczny, K. Koo, T. Laffey & H. Šmigoc (Ireland), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Nakhash, S. Pierce, E. Pité (France), C. Pohoata (Romania), V. Rutherfoord, R. A. Simon (Chile), N. C. Singer, R. Stong, T. Tam, M. Tetiva (Romania), T. Thomas (U.K.), Z. Vörös (Hungary), J. Young, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

A Double Factorial Sum

11406 [2009, 82]. *Proposed by A. A. Dzhumadil'daeva, Almaty, Republics Physics and Mathematics School, Almaty, Kazakhstan.* Let $n!!$ denote the product of all positive integers not greater than n and congruent to n mod 2, and let $0!! = (-1)!! = 1$. Thus, $7!! = 105$ and $8!! = 384$. For positive integer n, find

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2(n-i)-1)!!$$

in closed form.

Solution I by Kenneth F. Andersen, University of Alberta, Edmonton, Alberta, Canada. The sum is $2^n n!$. To see this, let $f(x) = (1-2x)^{-1/2}$ and $g(x) = (1-2x)^{-1}$ for $|x| < 1/2$. Induction shows that the ith derivatives of f and g are given by

$$\begin{aligned} f^{(i)}(x) &= (2i-1)!!\,(1-2x)^{-1/2-i} \\ g^{(i)}(x) &= 2^i i!\,(1-2x)^{-1-i} \end{aligned} \tag{3}$$

for each nonnegative integer i. In particular, $f^{(i)}(0) = (2i-1)!!$, so

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2n-2i-1)!! = \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(0) f^{(n-i)}(0).$$

Since $g = f^2$, the Leibniz rule for the nth derivative of a product shows that the latter sum is $g^{(n)}(0)$. In view of (3), this equals $2^n n!$.

Solution II by Ulrich Abel, University of Applied Sciences Giessen-Friedberg, Friedberg, Germany. First note that

$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2n-2i}{n-i} = [z^n]\left(\sum_{i=0}^{\infty} \binom{2i}{i} z^i\right)^2 = [z^n]((1-4z)^{-1/2})^2 = 4^n.$$

Using $(2k-1)!! = (2k)!/(2^k k!)$, the original sum becomes

$$\sum_{i=0}^{n} \binom{n}{i} \frac{(2i)!}{2^i i!} \frac{(2n-2i)!}{2^{n-i}(n-i)!} = n!\,2^{-n} \sum_{i=0}^{n} \binom{2i}{i}\binom{2n-2i}{n-i} = n!\,2^{-n} 4^n = n!\,2^n.$$

Also solved by 65 other readers.

Some Intermediate Value Variants

11429 [2009, 365]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* For a continuous real-valued function ϕ on $[0, 1]$, let $T\phi$ be the function mapping $[0, 1] \to \mathbb{R}$ given by $T\phi(t) = \phi(t) - \int_0^t \phi(u)\,du$, and similarly define S by $S\phi(t) = t\phi(t) - \int_0^t u\phi(u)\,du$. Show that if f and g are continuous real-valued functions on $[0, 1]$, then there exist numbers a, b, and c in $(0, 1)$ such that each of the following is true:

$$Tf(a) = Sf(a),$$

$$Tg(b)\int_{u=0}^{1} f(u)\,du = Tf(b)\int_{u=0}^{1} g(u)\,du,$$

$$Sg(c)\int_{u=0}^{1} f(u)\,du = Sf(c)\int_{u=0}^{1} g(u)\,du.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA.

Lemma. *If h is continuous on $[0, \alpha]$, and $h(\alpha) = 0$, then there exists $a \in (0, \alpha)$ such that $h(a) = \int_0^a h(u)\,du$.*

Proof. Let $H(t) = e^{-t}\int_0^t h(u)\,du$. Note that $H(0) = 0$, and H is continuously differentiable with $H'(t) = e^{-t}(h(t) - \int_0^t h(u)\,du)$. Thus it suffices to find an $a \in (0, \alpha)$ with $H'(a) = 0$. If no such a exists, then $H(t)$ is monotone, and hence $J(t) = H(t)^2$ is monotone increasing and in particular $J(\alpha) > 0$. This gives the contradiction $J'(\alpha) = 2H(\alpha)H'(\alpha) = -2e^{-2\alpha}\left(\int_0^\alpha h(u)\,du\right)^2 = -2J(\alpha)^2 < 0$. ∎

Let $F = \int_0^1 f(t)\,dt$, $G = \int_0^1 g(t)\,dt$. Applying the lemma to $h(t) = (1-t)f(t)$ with $\alpha = 1$ gives $a \in (0, 1)$ such that $(1-a)f(a) = \int_0^a (1-u)f(u)\,du$ or $Tf(a) = Sf(a)$. Applying the lemma to $h(t) = f(t)G - g(t)F$, and noting that $\int_0^1 h(t)\,dt = 0$ implies the existence of some $\alpha \in (0, 1)$ with $h(\alpha) = 0$, gives $b \in (0, 1)$ such that

$$f(b)G - g(b)F = \int_0^b f(u)\,du\,G - \int_0^b g(u)\,du\,F,$$

or $Tf(b)G = Tg(b)F$. Applying the lemma to $h(t) = tf(t)G - tg(t)F$, and noting that the α found in the previous case still works, gives $c \in (0, 1)$ such that

$$cf(c)G - cg(c)F = \int_0^c uf(u)\,du\,G - \int_0^c ug(u)\,du\,F$$

or $Sf(c)G = Sg(c)F$.

Also solved by K. F. Andersen (Canada), R. Bagby, R. Chapman (U.K.), W. J. Cowieson, P. P. Dályay (Hungary), E. A. Herman, B.-I. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, P. Perfetti (Italy), GCHQ Problem Solving Group, and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before May 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11544. *Proposed by Max A. Alekseyev, University of South Carolina, Columbia, SC, and Frank Ruskey, University of Victoria, Victoria, BC, Canada.* Prove that if m is a positive integer, then

$$\sum_{k=0}^{m-1} \varphi(2k+1) \left\lfloor \frac{m+k}{2k+1} \right\rfloor = m^2.$$

Here φ denotes the Euler totient function.

11545. *Proposed by Manuel Kauers, Research Institute for Symbolic Computation, Linz, Austria, and Sheng-Lan Ko, National Taiwan University, Taipei, Taiwan.* Find a closed-form expression for

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} s(n+k,k),$$

where s refers to the (signed) Stirling numbers of the first kind.

11546. *Proposed by Kieren MacMillan, Toronto, Canada, and Jonathan Sondow, New York, NY.* Let d, k, and q be positive integers, with k odd. Find the highest power of 2 that divides $\sum_{n=1}^{2^d k} n^q$.

11547. *Proposed by Francisco Javier García Capitán, I.E.S Álvarez Cubero, Priego de Córdoba, Spain, and Juan Bosco Romero Márquez, University of Valladolid, Spain.* Let the altitude AD of triangle ABC be produced to meet the circumcircle again at E. Let K, L, M, and N be the projections of D onto the lines BA, AC, CE, and EB, and let P, Q, R, and S be the intersections of the diagonals of $DKAL$, $DLCM$, $DMEN$, and $DNBK$, respectively. Let $|XY|$ denote the distance from X to Y, and let α, β, γ be the

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radian measure of angles BAC, CBA, ACB, respectively. Show that $PQRS$ is a rhombus and that $|QS|^2/|PR|^2 = 1 + \cos(2\beta)\cos(2\gamma)/\sin^2\alpha$.

11548. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f be a twice-differentiable real-valued function with continuous second derivative, and suppose that $f(0) = 0$. Show that

$$\int_{-1}^{1} (f''(x))^2\,dx \geq 10\left(\int_{-1}^{1} f(x)\,dx\right)^2.$$

11549. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Determine all continuous functions f on $\mathbb{R}$ such that for all x,

$$f(f(f(x))) - 3f(x) + 2x = 0.$$

11550. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Let G be a point inside triangle ABC. Let α, β, γ be the radian measures of angles BGC, CGA, AGB, respectively. Let O, R, S be the triangle's circumcenter, circumradius, and area, respectively. Let $|XY|$ be the distance from X to Y. Prove that

$$|GA|\cdot|GB|\cdot|GC|(|GA|\sin\alpha + |GB|\sin\beta + |GC|\sin\gamma) = 2S(R^2 - |GO|^2).$$

SOLUTIONS

A Consequence of Wolstenholme's Theorem

11382 [2008, 665]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For $k \geq 1$, let H_k be the kth harmonic number, defined by $H_k = \sum_{j=1}^{k} 1/j$. Show that if p is prime and $p > 5$, then

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2} \pmod{p^2}.$$

(Two rationals are congruent modulo d if their difference can be expressed as a reduced fraction of the form da/b with b relatively prime to a and d.)

Solution by Douglas B. Tyler, Raytheon, Torrance, CA. Let $S = \{1, 2, \dots, p-1\}$. All summations are over $k \in S$. Note that

$$3\left(\sum \frac{H_k}{k^2} - \sum \frac{H_k^2}{k}\right) = \sum \left(H_k - \frac{1}{k}\right)^3 - \sum H_k^3 + \sum \frac{1}{k^3}.$$

Since $H_k - \frac{1}{k} = H_{k-1}$, the right side telescopes to $-H_{p-1}^3 + \sum \frac{1}{k^3}$. Since $p > 3$, it suffices to show that H_{p-1}^3 and $\sum \frac{1}{k^3}$ are both congruent to 0 modulo p^2.

Modulo p, the reciprocals of the elements of S form a permutation of S, so $H_{p-1} = \sum k^{-1} \equiv \sum k = \frac{1}{2}p(p-1) \equiv 0 \pmod{p}$. Thus $H_{p-1}^3 \equiv 0 \bmod p^3$.

By reversing the index in one copy of the sum, modulo p^2 we have

$$2\sum \frac{1}{k^3} = \sum \frac{p^3 - 3p^2k + 3pk^2}{k^3(p-k)^3} \equiv \sum \frac{3pk^2}{k^3(p-k)^3} = 3p\sum \frac{1}{k(p-k)^3}.$$

It remains to show $\sum \frac{1}{k(p-k)^3} \equiv 0 \bmod p$. This sum is congruent to $\sum \frac{1}{-k^4}$. Modulo p, the reciprocals of the fourth powers of S form a permutation of the fourth powers of S, so $\sum \frac{1}{k^4} = \sum k^4 \bmod p$. It is well known that the sum over S of the rth powers is a polynomial of degree $r+1$ in p. In fact, $\sum k^4 = \frac{p^5}{5} - \frac{p^4}{2} + \frac{p^3}{3} - \frac{p}{30}$, easily proved by induction. With no constant term, the polynomial has value $0 \bmod p$ when $p > 5$.

Editorial comment. That $H_{p-1} \equiv 0 \bmod p$, and that $\sum_{k=1}^{p-1} k^{-3} \equiv 0 \bmod p^2$, could have been established by an appeal to Wolstenholme's theorem.

Also solved by R. Chapman (U. K.), P. Corn, P. P. Dályay (Hungary), Y. Dumont (France), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), N. C. Singer, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

Groups with Arbitrarily Sparse Squares

11388 [2008, 758]. *Proposed by M. Farrokhi D.G., University of Tsukuba, Tsukuba Ibakari, Japan.* Given a group G, let G^2 denote the set of all squares in G. Show that for each natural number n there exists a finite group G such that the cardinality of G is n times the cardinality of G^2.

Solution by Richard Stong, San Diego, CA. When G has odd order, every element is a square, so $|G|/|G^2| = 1$. For order 2, only the identity is a square, so $|G|/|G^2| = 2$.

Let p be an odd prime, and let s be the largest integer such that $p \equiv 1 \bmod 2^s$. The multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ of nonzero congruence classes modulo p is cyclic of order $p-1$ and has an element a of order 2^s. Hence $a^{2^{s-1}} \equiv -1 \bmod p$, and no smaller power of a satisfies this congruence. Now consider the group H_p with presentation

$$H_p = \langle x, y : x^p = y^{2^{s+1}} = 1, yxy^{-1} = x^a \rangle.$$

Every element of this group can be written uniquely as $x^b y^c$ for $b \in \mathbb{Z}/p\mathbb{Z}$ and $c \in \mathbb{Z}/2^{s+1}\mathbb{Z}$, and the multiplication law is

$$x^{b_1} y^{c_1} x^{b_2} y^{c_2} = x^{b_1 + a^{c_1} b_2} y^{c_1 + c_2}$$

with operations in the exponents of x and y taken mod p and mod 2^{s+1}, respectively. Setting $b = b_1 = b_2$ and $c = c_1 = c_2$, we see that the squares in H_p are precisely the elements of the form $x^{b(1+a^c)} y^{2c}$. Hence, if $x^\beta y^\gamma = (x^b y^c)^2$, then γ is even and either $c = \gamma/2$ or $c = \gamma/2 + 2^s$. Since $a^{2^s} = 1$, both possibilities give the same value of $1 + a^c$. If $\gamma \neq 2^s$ (that is, if $c \neq 2^{s-1}$), then $1 + a^c$ is nonzero and all choices of β give squares. If $\gamma = 2^s$, then $c = \pm 2^{s-1}$ and $1 + a^{c_2} = 0$, so only $\beta = 0$ gives a square. Thus $|H_p^2| = (2^s - 1)p + 1$. Note that $p \equiv 1 + 2^s \bmod 2^{s+1}$, so $(2^s - 1)p + 1$ is indeed a multiple of 2^{s+1}. Hence

$$\frac{|H_p|}{|H_p^2|} = \frac{2^{s+1} p}{(2^s - 1)p + 1} = \frac{p}{r_p},$$

where r_p is the integer $((2^s - 1)p + 1)/(2^{s+1})$ and $r_p < p$.

If G and H are finite, then the set of squares in $G \times H$ is $G^2 \times H^2$, so

$$\frac{|G \times H|}{|(G \times H)^2|} = \frac{|G|}{|G^2|} \cdot \frac{|H|}{|H^2|}.$$

The result now follows by induction on n. We have given examples for $n = 1$ and $n = 2$, so consider $n \geq 3$. When n is even, let $G_{n/2}$ be an example with $|G_{n/2}|/|G_{n/2}^2| =$

$n/2$; now $G_{n/2} \times \mathbb{Z}/2\mathbb{Z}$ is the desired example for G_n. When n is odd, let p be an odd prime divisor of n, let $m = nr_n/p < n$ (with r_n as above), and let G_m be an example with $|G_m|/|G_m^2| = m$. Now $G_m \times H_p$ is the desired example for G_n.

Also solved by A. J. Bevelacqua, R. Martin (Germany), L. Reid, D. B. Tyler, NSA Problems Group, and the proposer.

A Nonexistent Ring

11407 [2009, 82]. *Proposed by Erwin Just (emeritus), Bronx Community College of the City University of New York, New York, NY.* Let p be a prime greater than 3. Does there exists a ring with more than one element (not necessarily having a multiplicative identity) such that for all x in the ring, $\sum_{i=1}^{p} x^{2i-1} = 0$?

Solution by O.P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We prove that no such ring R exists by showing that the assumption $\sum_{i=1}^{p} x^{2i-1} = 0$ for all x yields $R = \{0\}$, contradicting the hypothesis that $|R| \geq 2$. Multiplying by x^2 yields $\sum_{i=1}^{p} x^{2i+1} = 0$, and then $x^{2p+1} = x$ by subtraction. Now $x^{4p} = x^{2p-1}x^{2p+1} = x^{2p-1}x = x^{2p}$. We conclude that all positive even powers of x^p are equal. Next compute

$$0 = \sum_{i=1}^{p} \left(x^{2p}\right)^{2i-1} = \sum_{i=1}^{p} x^{2(2i-1)p} = px^{2p}.$$

Since $x^{2p+1} = x$, we have $px = px^{2p+1} = \left(px^{2p}\right)x = 0x = 0$. Thus $(x+x)^p = x^p + x^p$. Now

$$2x = (2x)^{2p+1} = 2x[(x+x)^p]^2 = 2x(x^p + x^p)^2 = 2x4x^{2p} = 8x^{2p+1} = 8x.$$

Therefore, $6x = 8x - 2x = 0$, and we already know that $px = 0$. Therefore, $0 = \gcd(6, p)x = x$. Since x is an arbitrary element of R, it follows that $R = \{0\}$.

Also solved by E. P. Amendariz, N. Caro (Colombia), R. Chapman (U. K.), Y. Ge (Austria), D. Grinberg, J. H. Lindsey II, A. Sh. Shabani (Kosova), R. Stong, C. T. Stretch (Ireland), N. Vonessen, FAU Problem Solving Group, NSA Problem Group, and the proposer.

Summing to kth Powers

11408 [2009, 83]. *Proposed by Marius Cavachi, "Ovidius" University of Constanţa, Constanţa, Romania.* Let k be a fixed integer greater than 1. Prove that there exists an integer n greater than 1, and distinct integers $a_1, \ldots, a_n$ all greater than 1, such that both $\sum_{j=1}^{n} a_j$ and $\sum_{j=1}^{n} \varphi(a_j)$ are kth powers of a positive integer. Here φ denotes Euler's totient function.

Solution by C. R. Pranesachar, Indian Institute of Science, Bangalore, India. We first choose a and b such that $2a + 6b = (2k+2)^k$ and $a + 2b = (2k)^k$, both kth powers of integers. Solving the linear system yields $a = 3(2k)^k - (2k+2)^k = 2^k(3k^k - (k+1)^k)$ and $b = \frac{1}{2}((2k+2)^k - 2(2k)^k) = 2^{k-1}((k+1)^k - 2k^k)$. Since $2 < (1 + \frac{1}{k})^k < 3$ for $k > 1$, it follows that a and b are positive integers. Express the even integers $2a$ and $2b$ as sums of distinct positive powers of 2:

$$2a = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_l}, \qquad 1 \leq r_1 < r_2 < \cdots < r_l;$$
$$2b = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_m}, \qquad 1 \leq s_1 < s_2 < \cdots < s_m.$$

Let $a_i = 2^{r_i}$ for $1 \le i \le l$ and $a_{l+j} = 3 \cdot 2^{s_j}$ for $1 \le j \le m$. Let $n = l + m$, and consider $a_1, \ldots, a_n$, which are clearly distinct. Note that $\sum_{j=1}^{n} a_j = 2a + 6b = (2k+2)^k$. Since $\varphi(2^r) = 2^{r-1}$ and $\varphi(3 \cdot 2^r) = 2^r$,

$$\sum_{h=1}^{n} \varphi(a_h) = \sum_{i=1}^{l} 2^{r_i - 1} + \sum_{j=1}^{m} 2^{s_i} = a + 2b = (2k)^k.$$

Editorial comment. The GCHQ Problem Solving Group used distinct powers of 3, distinct numbers of the form $3 \cdot 2^r$, and distinct powers of 2 to show that there are distinct numbers $a_1, \ldots, a_n$, all greater than 1, such that $\sum_{j=1}^{n} a_j = s$ and $\sum_{j=1}^{n} \phi(a_j) = t$, provided that $s/2 < t < 8s/15$.

Also solved by P. P. Dályay (Hungary), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

An Inequality

11430 [2009, 366]. *Proposed by He Yi, Macao University of Science and Technology, Macao, China.* For real $x_1, \ldots, x_n$, show that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Solution by Kenneth F. Andersen, University of Alberta, Edmonton, AB, Canada. Letting $x_0 = 1$, we have

$$\sum_{j=1}^{n} \frac{x_j^2}{(1+x_1^2+\cdots+x_j^2)^2} \le \sum_{j=1}^{n} \left[\frac{1}{x_0^2+x_1^2+\cdots+x_{j-1}^2} - \frac{1}{x_0^2+x_1^2+\cdots+x_j^2} \right]$$
$$= 1 - \frac{1}{1+x_1^2+\cdots+x_n^2} < 1.$$

The Cauchy–Schwarz inequality shows that, as required,

$$\sum_{j=1}^{n} \frac{x_j}{1+x_1^2+\cdots+x_j^2} \le \left[\sum_{j=1}^{n} 1 \right]^{1/2} \left[\sum_{j=1}^{n} \frac{x_j^2}{(1+x_1^2+\cdots+x_j^2)^2} \right]^{1/2} < \sqrt{n}.$$

Editorial comment. This problem is known. (1) It was a Romanian proposal for the IMO 2001; two solutions are on page 676 of *The IMO Compendium* (Springer, 2006). (2) It was part of the Indian Team Selection Test for the 2002 IMO; a solution was published in *Crux Mathematicorum with Mathematical Mayhem* **35** (2009) 98. (3) It was Problem 1242 in *Elementa der Mathematik* **63** (2008) 103.

Also solved by A. Alt, M. S. Ashbaugh & S. G. Saenz (U.S.A. & Chile), R. Bagby, M. Bataille (France), D. Borwein (Canada), P. Bracken, M. Can, R. Chapman (U. K.), H. Chen, L. Csete (Hungary), P. P. Dályay (Hungary), J. Fabrykowski & T. Smotzer, O. Geupel (Germany), J. Grivaux (France), E. Hysnelaj & E. Bojaxhiu (Australia & Albania), Y. H. Kim (Korea), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Moreira (Portugal), P. Perfetti (Italy), C. Pohoata (Romania), M. A. Prasad (India), A. Pytel (Poland), H. Ricardo, C. R. & S. Selvaraj, J. Simons (U. K.), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), D. Vacaru (Romania), E. I. Verriest, M. Vowe (Switzerland), A. P. Yogananda (India), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Shur and Definite

11431 [2009, 336]. *Proposed by Finbarr Holland and Stephen Wills, University College Cork, Cork, Ireland.* A matrix is *Schur invertible* if all its entries are nonzero, and the *Schur inverse* is the matrix obtained by taking the reciprocal of each entry. Show that an $n \times n$ complex matrix A with all entries nonzero has the property that it and its Schur inverse are both nonnegative definite if and only if there are nonzero complex numbers $a_1, \ldots, a_n$ such that for $1 \le j, k \le n$, the (j,k)-entry of A is $a_j \overline{a_k}$.

Solution by Éric Pité, Paris, France. Let A be an $n \times n$ complex matrix with all entries nonzero such that it and its Schur inverse are both nonnegative definite. Such an A is a Gramian matrix, i.e., there exist $v_1, \ldots, v_n \in \mathbb{C}^n$ such that $a_{j,k} = \langle v_j, v_k \rangle$ for all (j,k).

Using the Cauchy-Schwarz inequality, for $1 \le j, k \le n$ we have

$$|a_{j,k}|^2 \le \|v_j\|^2 \|v_k\|^2 = a_{j,j} a_{k,k}.$$

The Schur inverse is also Gramian, so $1/|a_{j,k}|^2 \le 1/(a_{j,j}a_{k,k})$ as well. Hence in all these applications of the Cauchy-Schwarz inequality we have equality. It follows that the vectors $v_1, \ldots, v_n$ are all proportional. Hence we can write $v_j = \overline{a_j} u$ for some common unit vector u and complex numbers $a_1, \ldots, a_n$ and the (j,k)-entry of A is $a_j \overline{a_k}$.

The converse is clear: if y is the vector $(a_1, \ldots, a_n)$, then $A = y\overline{y}^T$ and $v^T A v = |\langle y, v \rangle|^2 \ge 0$, so A is nonnegative definite, and similarly for its Schur inverse.

Also solved by P. Budney, R. Chapman (U. K.), P. P. Dályay (Hungary), N. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Muchlis (Indonesia), R. Stong, M. Tetiva (Romania), Con Amore Problem Group (Denmark), and the proposer.

Interior Evaluation and Boundary Evaluation

11432 [2009, 463]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let P be a polynomial of degree n with complex coefficients and with $P(0) = 0$. Show that for any complex α with $|\alpha| < 1$ there exist complex numbers $z_1, \ldots, z_{n+2}$, all of norm 1, such that $P(\alpha) = P(z_1) + \cdots + P(z_{n+2})$.

Solution I by O. P. Lossers, Technical University of Eindhoven, Eindhoven, The Netherlands. We prove something stronger. Given α we prove the existence of $z_1, \ldots, z_{n+2}$ such that $|z_j| = 1$ and $z_1^k + \cdots + z_{n+2}^k = \alpha^k$ for $1 \le k \le n$. Thus, for every polynomial P of degree n with $P(0) = 0$, we have $P(\alpha) = \sum_{j=1}^{n+2} P(z_j)$.

To any list of numbers $(z_1, \ldots, z_{n+2})$ we associate the polynomial Q given by $Q(z) = \prod_{k=1}^{n+2}(z - z_j)$, and numbers π_k given by $\pi_k = \sum_{j=1}^{n+2} z_j^k$. The numbers π_k and the coefficients c_j in the expansion $Q(z) = \sum_{j=0}^{n+2}(-1)^j c_j z^{n+2-j}$ are related by the Newton identities: $c_0 = 1$, and

$$k(-1)^k c_k + \pi_k c_0 - \pi_{k-1} c_1 + \cdots + (-1)^{k-1} \pi_1 c_{k-1} = 0 \quad \text{for } 1 \le k \le n+2.$$

We want $\pi_k = \alpha^k$ for $1 \le k \le n$. This can only happen if $c_1 = \alpha$ and $c_j = 0$ for $2 \le j \le n$. We must therefore choose $Q(z)$ of the form $z^{n+2} - \alpha z^{n+1} + Az + B$. We take $Q(z) = z^{n+2} - \alpha z^{n+1} - \overline{\alpha} z + 1$. With this choice of Q, each z_j satisfies $z^{n+1} = (\overline{\alpha} z - 1)/(z - \alpha)$. The expression on the right side of this equation is the value at z of a Möbius transformation that maps the inside of the unit disk to the outside and vice versa, so $|z_j| = 1$ for $1 \le j \le n+2$.

Solution II by Richard Stong. We prove something stronger. If k is any integer ≥ 2, then there exist $z_1, \ldots, z_k$ of norm 1 with $P(\alpha) = P(z_1) + \cdots + P(z_k)$. Let $B =$

$\{P(z)\colon |z| = 1\}$ and $F = \{P(z)\colon |z| \le 1\}$. Both sets are closed and bounded, and since P is an open map (L. Ahlfors, *Complex Analysis*, Corollary 1, p. 132), the boundary ∂F of F is a subset of B. Also, F and B are both path connected, since both are the continuous image of a path connected set.

Lemma. *For any $p, q \in F$ there exist $w, z \in B$ such that $p + q = w + z$.*

Proof. Let $m = \frac{1}{2}(p+q)$. It will suffice to show that $B \cap (2m - B) \ne \emptyset$, because given $w \in B \cap (2m - B)$, we make take $z = 2m - w$ and have $w, z \in B$ with $w + z = 2m = p + q$. Observe next that $\partial(2m - F) \subseteq (2m - B)$. Now $\partial(F \cup (2m - F)) \ne \emptyset$. If $\partial F \cap \partial(2m - F) \ne \emptyset$, we are done. Otherwise, after replacing u by $2m - u$ if necessary, we may assume the existence of u such that $u \in \partial F$, $u \notin 2m - F$. Thus $u \in B$, $u \notin 2m - F$, $2m - u \in \partial(2m - F)$, and $2m - u \notin F$. On the other hand, $p \in F \cap (2m - F)$ because $2m - p = q$. Since $2m - F$ is path connected, there is a path in $2m - F$ from $2m - u$ to p. Since $2m - u \notin F$ and $p \in F$, there is a v along the path such that $v \in \partial F$, whence $v \in (2m - F) \cap B$. Finally, since B too is path connected, there is a path in B from $u \notin 2m - F$ to v, and it contains a w in $\partial(2m - F)$. This puts $w \in (2m - B) \cap B$. ∎

Now taking $p = P(\alpha)$ and $q = P(0) = 0$ in the lemma, we get $P(\alpha) = P(z_1) + P(w)$, where z_1 and w have norm 1. Next, taking $p = P(w)$ and $q = 0$, we get $P(w) = P(z_2) + P(w')$, where again z_2 and w' have norm 1. Continuing in this way, we see that for any $k \ge 2$ we can write $P(\alpha) = P(z_1) + \cdots + P(z_k)$ with all z_j of norm 1.

Also solved by R. Chapman (U. K.), O. Kouba (Syria), J. Schaer (Canada), J. Simons (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

A Triangle Inequality

11435 [2009, 463]. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.* In a triangle T, let a, b, and c be the lengths of the sides, r the inradius, and R the circumradius. Show that

$$\frac{a^2bc}{(a+b)(a+c)} + \frac{b^2ca}{(b+c)(b+a)} + \frac{c^2ab}{(c+a)(c+b)} \le \frac{9}{2}rR.$$

Solution by Chip Curtis, Missouri State Southern University, Joplin, MO. Write K for the area of T and s for the semiperimeter. Then $r = K/s$ and $R = abc/(4K)$, so $rR = abc/(4s) = abc/(2(a+b+c))$. The claimed inequality is equivalent to

$$abc\left[\frac{a}{(a+b)(a+c)} + \frac{b}{(b+c)(b+a)} + \frac{c}{(c+a)(c+b)}\right] \le \frac{9abc}{4(a+b+c)}$$

which simplifies to $(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \ge 6abc$. In this last form, it follows from the AM–GM inequality.

Editorial comment. The problem was published with a misprint: $9/4$ in place of $9/2$. We regret the oversight.

Also solved by A. Alt, R. Bagby, M. Bataille (France), E. Braune (Austria), M. Can, R. Chapman (U. K.), L. Csete (Hungary), P. P. Dályay (Hungary), S. Dangc, V. V. García (Spain), M. Goldenberg & M. Kaplan, M. R. Gopal, D. Grinberg, J.-P. Grivaux (France), S. Hitotumatu (Japan), E. Hysnelaj & E. Bojaxhiu (Australia & Albania), B.-T. Iordache (Romania), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Mabuchi (Japan), J. Minkus, D. J. Moore, R. Nandan, M. D. Nguyen (Vietnam), P. E. Nuesch

(Switzerland), J. Oelschlager, G. T. Prăjitură, C. R. Pranesachar (India), J. Rooin & A. Asadbeygi (Iran), S. G. Saenz (Chile), I. A. Sakmar, C. R. & S. Selvaraj, J. Simons (U. K.), E. A. Smith, S. Song (Korea), A. Stadler (Switzerland), R. Stong, W. Szpunar-Łojasiewicz, R. Tauraso (Italy), M. Tetiva (Romania), B. Tomper, E. I. Verriest, Z. Vörös (Hungary), M. Vowe (Switzerland), J. B. Zacharias, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Partition by a Function

11439 [2009, 547]. *Proposed by Stephen Herschkorn, Rutgers University, New Brunswick, NJ.* Let f be a continuous function from $[0, 1]$ into $[0, 1]$ such that $f(0) = f(1) = 0$. Let G be the set of all (x, y) in the square $[0, 1] \times [0, 1]$ so that $f(x) = f(y)$.
(**a**) Show that G need not be connected.
(**b**)* Must $(0, 1)$ and $(0, 0)$ be in the same connected component of G?

Composite solution by Armenak Petrosyan (student), Yerevan State University, Yerevan, Armenia, and Richard Stong, San Diego, CA.
(**a**) Let f be the piecewise linear function whose graph joins the points $(0, 0)$, $(1/6, 1)$, $(1/3, 1/2)$, $(1/2, 1)$, $(2/3, 0)$, $(5/6, 1/2)$, and $(1, 0)$. This f has a strict local minimum at $x = 1/3$ and a strict local maximum at $x = 5/6$ with $f(1/3) = f(5/6) = 1/2$. Thus $(1/3, 5/6)$ is an isolated point of G, so G is not connected.
(**b**) We claim that $(0, 1)$ and $(0, 0)$ are in the same component of G. Let $D = \{(x, x)\colon\ 0 \le x \le 1\}$. If $(0, 1)$ and D are in different components of G, then there are disjoint open sets U, V in the square $S = [0, 1] \times [0, 1]$ such that $(0, 1) \in U$, $D \subset V$, and $G \subset U \cup V$. Let $C_1 = G \cap U$ and $C_2 = G \cap V$. Since C_1 and C_2 are both open in G, they are also both closed, hence compact. We may further assume that C_1 lies entirely above the line $y = x$. For each point $p \in C_1$, choose an open square centered at p with sides parallel to the axes, not lying along any edge of S, and with closure disjoint from C_2. These squares form an open cover of C_1, so there is a finite subcover. Let F be the union of the closed squares corresponding to this subcover. Let F' be the intersection of S with the boundary of F. Now F is closed, lies above $y = x$, and contains C_1 in its interior and C_2 in its complement. Also, F' consists of line segments. From F' we define a graph H whose vertices are the intersections of these line segments with each other or with the boundary of S; vertices of H are adjacent when connected by a segment contained in F'. Vertices have degree 1, 2, or 4, with degree 1 only on the boundary of S.

Since $(0, 1) \in F$ and $D \cap F = \emptyset$, toggling membership in F at vertices of H along the left edge of S implies that the number of vertices of degree 1 on the left edge of S is odd, and similarly along the top edge. Since each component of a graph has an even number of vertices of odd degree, some component contains vertices of degree 1 on both of these edges, and hence H must contain at least one path joining these edges. However, the function ϕ on S given by $\phi(x, y) = f(x) - f(y)$ is continuous, nonnegative on the top edge and nonpositive on the left edge. Thus some point (x, y) on this path must have $\phi(x, y) = 0$. Such a point lies in G, contrary to our construction. Thus $(0, 1)$ and D lie in the same component of G.

Editorial comment. A second approach to solving part (**b**) builds from the case where f is piecewise linear (essentially the "Two Men of Tibet" problem; see P. Zeitz, *The Art and Craft of Problem Solving*, John Wiley & Sons, 1999).

Also solved by D. Ray, V. Rutherfoord, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary). Part (**a**) solved by R. Chapman (U. K.), W. J. Cowieson, M. D. Meyerson, J. H. Nieto (Venezuela), A. Pytel (Poland), Fisher Problem Solving Group, GCHQ Problem Solving Group (U. K.), and Microsoft Research Problems Group.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before June 30, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11551. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Given a finite set S of closed bounded convex sets in $\mathbb{R}^n$ having positive volume, prove that there exists a finite set X of points in $\mathbb{R}^n$ such that each $A \in S$ contains at least one element of X and any $A, B \in S$ with the same volume contain the same number of elements of X.

11552. *Proposed by Weidong Jiang, Weihai Vocational College, Weihai, China.* In triangle ABC, let A_1, B_1, C_1 be the points opposite A, B, C at which the angle bisectors of the triangle meet the opposite sides. Let R and r be the circumradius and inradius of ABC. Let a, b, c be the lengths of the sides opposite A, B, C, and let a_1, b_1, c_1 be the lengths of the line segments B_1C_1, C_1A_1, A_1B_1. Prove that

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq 1 + \frac{r}{R}.$$

11553. *Proposed by Mihály Bencze, Brasov, Romania.* For a positive integer k, let $\alpha(k)$ be the largest odd divisor of k. Prove that for each positive integer n,

$$\frac{n(n+1)}{3} \leq \sum_{k=1}^{n} \frac{n-k+1}{k}\alpha(k) \leq \frac{n(n+3)}{3}.$$

11554. *Proposed by Zhang Yun, Xi'an Jiao Tong University Sunshine High School, Xi'an, China.* In triangle ABC, let I be the incenter, and let A', B', C' be the reflections of I through sides BC, CA, AB, respectively. Prove that the lines AA', BB', and CC' are concurrent.

11555. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous real-valued function on $[0, 1]$ such that $\int_0^1 f(x)\,dx = 0$. Prove that there exists c in the interval $(0, 1)$ such that $c^2 f(c) = \int_0^c (x + x^2) f(x)\,dx$.

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11556. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* For positive real numbers a, b, c, d, show that

$$\frac{9}{a(b+c+d)}+\frac{9}{b(c+d+a)}+\frac{9}{c(d+a+b)}+\frac{9}{d(a+b+c)}$$
$$\geq \frac{16}{(a+b)(c+d)}+\frac{16}{(a+c)(b+d)}+\frac{16}{(a+d)(b+c)}.$$

11557. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let S be a finite set of circles in the Cartesian plane having the property that any two circles in S intersect in exactly two points, each circle encloses the origin, but no three circles share a common point. Construct a graph G by taking as the vertices the set of all intersection points of circles in S, with edges corresponding to arcs of a circle in S connecting vertices without passing through any intermediate vertex. (Thus, with four circles, there are 12 vertices and 24 edges.) Show that the resulting graph contains a Hamiltonian path.

SOLUTIONS

An Arctan Series

11438 [2009, 464]. *Proposed by David H. Bailey, Lawrence Berkeley National Laboratory, Berkeley, CA, Jonathan M. Borwein, University of Newcastle, Newcastle, Australia and Dalhousie University, Halifax, Canada, and Jörg Waldvogel, Swiss Federal Institute of Technology ETH, Zurich, Switzerland.* Let

$$P(x)=\sum_{k=1}^{\infty}\arctan\left(\frac{x-1}{(k+x+1)\sqrt{k+1}+(k+2)\sqrt{k+x}}\right).$$

(**a**) Find a closed-form expression for $P(n)$ when n is a nonnegative integer.
(**b**) Show that $\lim_{x\to-1^+}P(x)$ exists, and find a closed-form expression for it.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.
(**a**) Notice that

$$\frac{\frac{1}{\sqrt{k+1}}-\frac{1}{\sqrt{k+n}}}{1+\frac{1}{\sqrt{k+1}}\cdot\frac{1}{\sqrt{k+n}}}=\frac{\sqrt{k+n}-\sqrt{k+1}}{1+\sqrt{k+1}\cdot\sqrt{k+n}}.$$

Rationalizing the numerator gives

$$\frac{\frac{1}{\sqrt{k+1}}-\frac{1}{\sqrt{k+n}}}{1+\frac{1}{\sqrt{k+1}}\cdot\frac{1}{\sqrt{k+n}}}=\frac{n-1}{(k+n+1)\sqrt{k+1}+(k+2)\sqrt{k+n}}.$$

From the identity

$$\arctan\alpha-\arctan\beta=\arctan\left(\frac{\alpha-\beta}{1+\alpha\beta}\right),$$

we obtain

$$P(n) = \sum_{k=1}^{\infty} \arctan\left(\frac{n-1}{(k+n+1)\sqrt{k+1}+(k+2)\sqrt{k+n}}\right)$$
$$= \sum_{k=1}^{\infty}\left(\arctan\frac{1}{\sqrt{k+1}} - \arctan\frac{1}{\sqrt{k+n}}\right).$$

Clearly, $P(1) = 0$. The series for $P(0)$ telescopes to give

$$P(0) = -\arctan 1 + \lim_{k\to\infty} \arctan\frac{1}{\sqrt{k+1}} = -\frac{\pi}{4}.$$

In general, for $n \geq 2$, the series telescopes into the form

$$P(n) = \sum_{k=2}^{n} \arctan\frac{1}{\sqrt{k}}.$$

(**b**) Now use the inequality $\arctan t < t$ for $t > 0$. If $k \geq 2$ and $x \geq -1$, then

$$\arctan\left(\frac{|x-1|}{(k+x+1)\sqrt{k+1}+(k+2)\sqrt{k+x}}\right) \leq \frac{|x|+1}{k\sqrt{k+1}+(k+2)\sqrt{k-1}}.$$

By the Weierstrass M-test, the series $P(x)$ converges uniformly, and therefore it is continuous for $x > -1$. As in (**a**), we have

$$P(x) = \sum_{k=1}^{\infty}\left(\arctan\frac{1}{\sqrt{k+1}} - \arctan\frac{1}{\sqrt{k+x}}\right),$$

so

$$P(x+1) = P(x) + \arctan\frac{1}{\sqrt{1+x}}.$$

Thus,

$$\lim_{x\to -1^+} P(x) = \lim_{x\to -1^+}\left(P(x+1) - \arctan\frac{1}{\sqrt{1+x}}\right) = P(0) - \frac{\pi}{2} = -\frac{3\pi}{4}.$$

Editorial comment. The proposers report that they discovered the value $-3\pi/4$ experimentally. They ask whether there are more general closed forms for P, say at half-integers.

Also solved by R. Bagby, N. Bagis (Greece) & M. L. Glasser, D. Beckwith, M. Benito, Ó. Ciaurri, E. Fernández & L. Roncal (Spain), M. Chamberland, R. Chapman (U.K.), Y. Dumont (France), M. Goldenberg & M. Kaplan, O. Kouba (Syria), G. Lamb, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), A. H. Sabuwala, R. Stong, M. Tetiva (Romania), M. Vowe (Switzerland), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposers.

A Vector Differential Equation

11440 [2009, 547]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Consider the vector differential equation

$$\mathbf{x}''(t) = p(t, \mathbf{x}(t), \mathbf{x}'(t))\mathbf{x}'(t) \times \left(\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|}\right) \tag{1}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$, $\|\mathbf{u}\|$ denotes the usual Euclidean norm of a vector $\mathbf{u}$, $\times$ is the standard cross-product, and p and its first partial derivatives are real-valued and continuous.

(**a**) Show that all solutions to (1) are defined on all of $\mathbb{R}$.

(**b**) Show that any nonconstant solution tends to infinity as $t \to +\infty$.

(**c**) Show that for any nonzero solution $\mathbf{x}(t)$, $\lim_{t\to+\infty} \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|}$ exists.

Solution by Robin Chapman, University of Exeter, Exeter, U.K.

(**a**) Consider a nonconstant solution of (1) on an open interval I. From (1),

$$\mathbf{x}(t) \cdot \mathbf{x}''(t) = \mathbf{x}'(t) \cdot \mathbf{x}''(t) = 0$$

on I. Therefore,

$$\frac{d}{dt}(\mathbf{x}'(t) \cdot \mathbf{x}'(t)) = 2\mathbf{x}'(t) \cdot \mathbf{x}''(t) = 0,$$

which implies $\mathbf{x}'(t) \cdot \mathbf{x}'(t) = A$, where A is a constant. Certainly $A \geq 0$. If $A = 0$, then $\mathbf{x}'(t) = 0$ on I, so $\mathbf{x}(t)$ would have to be constant. Hence $A > 0$. Next,

$$\frac{d}{dt}(\mathbf{x}(t) \cdot \mathbf{x}'(t)) = \mathbf{x}'(t) \cdot \mathbf{x}'(t) + \mathbf{x}(t) \cdot \mathbf{x}''(t) = A.$$

This implies that $\mathbf{x}(t) \cdot \mathbf{x}'(t) = At + B$ for some constant B. Also, $\frac{d}{dt}(\mathbf{x}(t) \cdot \mathbf{x}(t)) = 2\mathbf{x}(t) \cdot \mathbf{x}'(t) = 2At + 2B$. This in turn implies that

$$\mathbf{x}(t) \cdot \mathbf{x}(t) = At^2 + 2Bt + C,$$

where C is a constant.

By the Cauchy–Schwarz inequality,

$$(\mathbf{x}(t) \cdot \mathbf{x}'(t))^2 \leq (\mathbf{x}(t) \cdot \mathbf{x}(t))(\mathbf{x}'(t) \cdot \mathbf{x}'(t)).$$

Upon substituting the results above, this becomes

$$(At + B)^2 \leq A(At^2 + 2Bt + C).$$

Thus $B^2 \leq AC$. If $B^2 = AC$, then $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are linearly dependent, so $\mathbf{x}'(t) \times (\mathbf{x}(t)/||\mathbf{x}(t)||) = 0$. Thus by (1), $\mathbf{x}''(t) = \mathbf{0}$, so $\mathbf{x}'(t)$ is constant. In this case the solution has the form $\mathbf{x}(t) = (t + k)\mathbf{u}$ for a fixed $k \in \mathbb{R}$ and vector $\mathbf{u}$; this extends to all of $\mathbb{R}$. Moreover, $\mathbf{x}(t)/||\mathbf{x}(t)|| = \mathbf{u}/||\mathbf{u}||$ for $t \neq k$.

Suppose now that $B^2 < AC$. We then have

$$\mathbf{x}(t) \cdot \mathbf{x}(t) = \frac{(At + B)^2 + (AC - B^2)}{A} \geq \frac{AC - B^2}{A} > 0.$$

Therefore, a problem with the differential equation in (1) being ill-defined when $\mathbf{x}(t) = 0$ does not arise. From the theory of ordinary differential equations, a solution on an open interval I with an endpoint a extends to a larger open interval J containing a provided neither $\mathbf{x}(t)$ nor $\mathbf{x}'(t)$ tends to infinity as $t \to a$. Our formulas for $\mathbf{x}(t) \cdot \mathbf{x}(t)$ and $\mathbf{x}'(t) \cdot \mathbf{x}'(t)$ prevent this. Hence $\mathbf{x}(t)$ extends to a solution on all of $\mathbb{R}$.

(**b**), (**c**) Since $\mathbf{x}(t) \cdot \mathbf{x}(t) \to \infty$, it follows that $\mathbf{x}(t) \to \infty$ as $t \to \infty$. Let $\mathbf{y}(t) = \mathbf{x}(t)/||\mathbf{x}(t)||$. Now $\frac{d}{dt}||\mathbf{x}(t)|| = \frac{\mathbf{x}(t)\cdot\mathbf{x}'(t)}{||\mathbf{x}(t)||}$. Hence,

$$\mathbf{y}'(t) = \frac{\mathbf{x}'(t)}{||\mathbf{x}(t)||} - \frac{(\mathbf{x}(t) \cdot \mathbf{x}'(t))\,\mathbf{x}(t)}{||\mathbf{x}(t)||^3} = \frac{(\mathbf{x}(t) \cdot \mathbf{x}(t))\,\mathbf{x}'(t) - (\mathbf{x}(t) \cdot \mathbf{x}'(t))\mathbf{x}(t)}{||\mathbf{x}(t)||^3}.$$

Consequently,

$$\begin{aligned}||\mathbf{y}'(t)||^2 &= \frac{||\mathbf{x}(t)||^2||\mathbf{x}'(t)||^2 - (\mathbf{x}(t)\cdot\mathbf{x}'(t))^2}{||\mathbf{x}(t)||^4} \\ &= \frac{A(At^2+2Bt+C)-(At+B)^2}{(At^2+2Bt+C)^2} = \frac{AC-B^2}{(At^2+2Bt+C)^2}.\end{aligned}$$

Hence $||\mathbf{y}'(t)|| = O(1/t^2)$ as $t \to \infty$. The integral $\int_0^\infty \mathbf{y}'(s)\,ds$ then converges absolutely, and so

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{y}'(s)\,ds \to \mathbf{y}(0) + \int_0^\infty \mathbf{y}'(s)\,ds,$$

a finite limit as $t \to \infty$.

Also solved by R. Bagby, W. J. Cowieson, H. Guggenheimer, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Simons (U.K.), R. Stong, N. Thornber, and the proposer.

Triangles in a Subdivided Polygon

11441 [2009, 547]. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Let $n \ge 4$, let $A_0, \dots, A_{n-1}$ be the vertices of a convex polygon, and for each i let B_i be a point in the interior of the segment A_iA_{i+1}. (Here, and throughout, indices of points are taken modulo n.) Let C_i denote the intersection of diagonals $B_{i-2}B_i$ and $B_{i-1}B_{i+1}$. Let $a(p,q,r)$ denote the area of the triangle with vertices p, q, r. Show that

$$\sum_{i=0}^{n-1} \frac{1}{a(A_i, B_i, B_{i-1})} \ge \sum_{i=0}^{n-1} \frac{1}{a(C_i, B_i, B_{i-1})}.$$

Solution by Jim Simons, Cheltenham, U.K. Fix all the B_j (and therefore all the C_j), and all the A_j except A_i and A_{i+1} for some particular i, and consider varying the line A_iA_{i+1} through B_i. Let $|B_{i-1}B_{i+1}| = l$, $\alpha = \angle B_iB_{i-1}B_{i+1}$,$\beta = \angle B_iB_{i-1}A_i$, $\gamma = \angle B_iB_{i+1}B_{i-1}$, $\delta = \angle B_iB_{i+1}A_{i+1}$ and $\theta = \angle A_iB_iB_{i-1}$, so that $\alpha+\gamma-\theta = \angle A_{i+1}B_iB_{i+1}$. Now

$$|B_{i-1}B_i| = \frac{l\sin\gamma}{\sin(\alpha+\gamma)} \qquad \text{and} \qquad |B_iB_{i+1}| = \frac{l\sin\alpha}{\sin(\alpha+\gamma)}.$$

If h_i is the distance of A_i from the line $B_{i-1}B_i$, then

$$h_i = \frac{|B_{i-1}B_i|}{\cot\beta+\cot\theta} \qquad \text{and} \qquad h_{i+1} = \frac{|B_iB_{i+1}|}{\cot\delta+\cot(\alpha+\gamma-\theta)}.$$

Writing Δ for $1/a(A_iB_iB_{i-1}) + 1/a(A_{i+1}B_{i+1}B_i)$, we conclude that

$$\Delta = \frac{2\sin^2(\alpha+\gamma)}{l^2\sin^2\gamma}(\cot\beta+\cot\theta) + \frac{2\sin^2(\alpha+\gamma)}{l^2\sin^2\alpha}\big(\cot\delta+\cot(\alpha+\gamma-\theta)\big).$$

Differentiating with respect to θ here gives

$$\frac{d\Delta}{d\theta} = \frac{2\sin^2(\alpha+\gamma)}{l^2}\left(\frac{-1}{\sin^2\gamma\sin^2\theta} + \frac{1}{\sin^2\alpha\sin^2(\alpha+\gamma-\theta)}\right).$$

Thus $d\Delta/d\theta = 0$ when $\sin\gamma\sin\theta = \pm\sin\alpha\sin(\alpha+\gamma-\theta)$. Since the sines are all positive, the only valid case is when $\sin\gamma\sin\theta = \sin\alpha\sin(\alpha+\gamma-\theta)$, and this gives a minimum of Δ since $\Delta \to \infty$ as $\theta \to 0$ and as $\theta \to \alpha + \beta$. Therefore Δ is minimized when $\sin\gamma \sin\theta = \sin\alpha\big(\sin(\alpha+\gamma)\cos\theta - \sin\theta\cos(\alpha+\gamma)\big)$, or equivalently, when

$$\begin{aligned}\tan\theta &= \frac{\sin\alpha\sin(\alpha+\gamma)}{\sin\gamma + \sin\alpha\sin(\alpha+\gamma)} = \frac{\sin\alpha\sin(\alpha+\gamma)}{\sin\gamma + \sin\alpha\cos\alpha\cos\gamma - \sin^2\alpha\sin\gamma} \\ &= \frac{\sin\alpha\sin(\alpha+\gamma)}{\sin\gamma\cos^2\alpha + \sin\alpha\cos\alpha\cos\gamma} = \tan\alpha.\end{aligned}$$

This occurs when A_iA_{i+1} is parallel to $B_{i-1}B_{i+1}$. Thus in a configuration that minimizes $\sum_{i=0}^{n-1} 1/a(A_i, B_i, B_{i-1})$ for a given value of $\sum_{i=0}^{n-1} 1/a(C_i, B_i, B_{i-1})$, every A_iA_{i+1} is parallel to the corresponding $B_{i-1}B_{i+1}$. In that case every $A_iB_{i-1}C_iB_i$ is a parallelogram, so that every $a(A_i, B_i, B_{i-1}) = a(C_i, B_i, B_{i-1})$, and therefore

$$\sum_{i=0}^{n-1}\frac{1}{a(A_i, B_i, B_{i-1})} = \sum_{i=0}^{n-1}\frac{1}{a(C_i, B_i, B_{i-1})}.$$

Also solved by R. Chapman (U.K.), P. P. Dályay (Hungary), J. H. Lindsey II, Á. Plaza & S. Falcón (Spain), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposer.

That's Sum Inequality

11442 [2009, 547]. *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Universidad Politécnica de Cataluña, Barcelona, Spain.* Let $\langle a_k\rangle$ be a sequence of positive numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

$$\left[\left(\sum_{k=1}^{n}\frac{a_k}{1+a_k}\right)\left(\sum_{k=1}^{n}\frac{1}{a_k(1+a_k)}\right)\right]^{1/2} \le \frac{1}{4}\left(\frac{a_1+a_n}{\sqrt{a_1a_n}}\right).$$

Solution by Jim Simons. Cheltenham, U.K. This is an extraordinarily weak inequality! The left side exceeds 1 for the first time when $n = 9$, at which point the right side exceeds 10^{25}. To see that it is true, we first note that $a_k > 2^k$. To prove this by induction, note $a_1 = 3 > 2$ and $a_2 = 5 > 4$; beyond that, $a_k > 4$ and $a_{k+1} > 2a_k$. (A stronger bound of $a_k \ge 2(3/2)^{2^{k-1}}$ is also easy.) Since $a_k/(1+a_k) < 1$, we have $\sum_{k=1}^{n} a_k/(1+a_k) < n$. Since $1 + a_k \ge 4$ and $a_k > 2^k$, we have $1/(a_k(1+a_k)) < 2^{-k-2}$ and $\sum_{k=1}^{n} 1/a_k(1+a_k) < 1/4$. Combining these, we see that the left side is less than $\sqrt{n}/2$. For $n \ge 8$, the right side satisfies

$$\frac{1}{4}\left(\frac{a_1+a_n}{\sqrt{a_1a_n}}\right) > \frac{\sqrt{a_n}}{4\sqrt{3}} > \frac{2^{n/2}}{4\sqrt{3}} > \sqrt{\frac{n}{2}}.$$

Direct calculation shows that the inequality holds for smaller n, the closest call being at $n = 3$.

Also solved by R. Chapman (U.K.), L. Csete (Hungary), P. P. Dályay (Hungary), J.-P. Grivaux (France), E. Hysnelaj (Australia) & E. Bojaxhiu (Germany), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), K. McInturff, P. Perfetti (Italy), C. R. & S. Selvaraj, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

An Integral with Fractional Parts

11447 [2009, 647]. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Let a be a positive number, and let g be a continuous, positive, increasing function on [0, 1].

Prove that

$$\lim_{n\to\infty}\int_0^1 \left\{\frac{n}{x}\right\}^a g(x)\,dx = \frac{1}{a+1}\int_0^1 g(x)\,dx,$$

where $a > 0$ and $\{x\}$ denotes the fractional part of x.

Solution by Ralph Howard, University of South Carolina, Columbia, SC. The result holds in greater generality; we claim that:

If $\beta : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function that is periodic with period 1, so that β satisfies $\beta(z+1) = \beta(z)$, and if $g \in L^1([0,1])$, then

$$\lim_{n\to\infty}\int_0^1 \beta\left(\frac{n}{x}\right) g(x)\,dx = \int_0^1 \beta(z)\,dz \int_0^1 g(x)\,dx. \tag{1}$$

Assuming (1) and taking $\beta(z) = \{z\}^a$, which has period one and is bounded for $a > 0$, we have

$$\lim_{n\to\infty}\int_{x=0}^1 \left\{\frac{n}{x}\right\}^a g(x)\,dx = \int_0^1 z^a\,dz \int_{x=0}^1 g(x)\,dx = \frac{1}{a+1}\int_{x=0}^1 g(x)\,dx.$$

For the proof of (1), we extend the Riemann–Lebesgue lemma:

Lemma. *If $f \in L^1(\mathbb{R})$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function such that $\beta(z+1) = \beta(z)$, then*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \beta(ny) f(y)\,dy = \int_0^1 \beta(z)\,dz \int_{-\infty}^{\infty} f(y)\,dy. \tag{2}$$

The proof of this lemma proceeds just as in one of the standard proofs of the Riemann-Lebesgue lemma: It is easy to check that it holds for $f = \chi_{[a,b]}$, the characteristic function of an interval. By linearity, it then holds for finite linear combinations of characteristic functions of intervals, that is, for step functions. However, step functions are dense in $L^1(\mathbb{R})$, so the result holds for all $f \in L^1(\mathbb{R})$ by approximation.

To obtain (1) from the lemma, let $g \in L^1([0,1])$, and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(y) = y^{-2}g(1/y)$ for $y \geq 1$ and $f(y) = 0$ otherwise. Letting $y = 1/x$ in the change of variable formula yields

$$\int_{-\infty}^{\infty} f(y)\,dy = \int_1^{\infty} y^{-2} g(1/y)\,dy = \int_0^1 g(x)\,dx.$$

This equation holds as well with absolute value bars on the integrands, and therefore $f \in L^1(\mathbb{R})$. The same change of variable yields

$$\int_{-\infty}^{\infty} \beta(ny) f(y)\,dy = \int_1^{\infty} \beta(ny)\frac{g(1/y)}{y^2}\,dy = \int_0^1 \beta\left(\frac{n}{x}\right) g(x)\,dx.$$

The required result is now an application of the lemma.

Also solved by K. F. Andersen (Canada), M. R. Avidon, R. Bagby, D. Borwein (Canada), R. Chapman (U.K.), W. J. Cowieson, M. Eyvasi (Iran), P. J. Fitzsimmons, J.-P. Grivaux (France), E. A. Herman, M. Kochanski (U.K.), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. S. Miller, M. Omarjee (France), E. Omey (Belgium), P. Perfetti (Italy), Á. Plaza & S. Falcón (Spain), K. Schilling, J. Simons (U.K.), A. Stadler (Switzerland), A. Stenger, R. Stong, J. V. Tejedor (Spain), M. Tetiva (Romania), E. I. Verriest, L. Zhou, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

Asymptotics of a Product

11456 [2009, 747]. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France.* Find

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right).$$

Solution by Oliver Geupel, Brühl, Germany. We use Stirling's formula, which says that $n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}$, together with the infinite product

$$\cosh z = \prod_{m=1}^{\infty} \left[1 + \frac{4z^2}{(2m-1)^2\pi^2}\right].$$

(Abramowitz & Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover 1972, Formula 4.5.69, p. 85).

We have

$$\begin{aligned}
\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) &= \lim_{n\to\infty} n \prod_{m=1}^{n} \frac{(2m-1)^2+4}{(2m)^2} \\
&= \lim_{n\to\infty} n \prod_{m=1}^{n} \frac{(2m-1)^2}{(2m)^2} \cdot \prod_{m=1}^{n} \frac{(2m-1)^2+4}{(2m-1)^2} \\
&= \lim_{n\to\infty} \frac{n\cdot(2n)!^2}{16^n \cdot n!^4} \cdot \prod_{m=1}^{n} \left[1 + \frac{4\pi^2}{(2m-1)^2\pi^2}\right] \\
&= \lim_{n\to\infty} \frac{n \cdot 2\pi \cdot (2n)^{4n+1} e^{-4n}}{16^n \cdot (2\pi)^2 \cdot n^{4n+2} \cdot e^{-4n}} \cdot \cosh(\pi) = \frac{\cosh(\pi)}{\pi}.
\end{aligned}$$

Editorial comment. A generalization was provided by Jerry Minkus (San Francisco, CA): Let b be a positive integer, and c a nonzero constant such that $\prod_{m=1}^{b}(m^2 - bm + c) \neq 0$. Letting $\Delta = b^2/4 - c$, we have

$$\lim_{n\to\infty} n(n-1)(n-2)\cdots(n-b+1) \prod_{m=1}^{n} \left(1 - \frac{b}{m} + \frac{c}{m^2}\right)$$

$$= \frac{1}{c} \prod_{m=1}^{b} (m^2 - bm + c) \cdot \frac{1}{\Gamma(b/2 - \sqrt{\Delta}\,)\,\Gamma(b/2 + \sqrt{\Delta}\,)}.$$

Several other solvers provided the generalization with $b = 1$ but general c.

Also solved by R. A. Agnew, K. F. Andersen (Canada), M. S. Ashbaugh (U.S.A.) & F. V. Prado (Chile), R. Bagby, D. H. Bailey (U.S.A.) & J. M. Borwein (Canada), M. Bataille (France), K. A. Beres, P. Bracken, B. S. Burdick, M. A. Carlton, R. Chapman (U.K.), H. Chen, P. P. Dályay (Hungary), O. Furdui (Romania), M. Goldenberg & M. Kaplan, G. C. Greubel, J. Grivaux (France), W.-P. Heidorn (Germany), E. A. Herman, F. Holland (Ireland), A. Ilić (Serbia), M. E. H. Ismail, T. Konstantopoulos (U.K.), O. Kouba (Syria), O. P. Lossers (Netherlands), S. de Luxán & Á. Plaza (Spain), R. Martin (Germany), V. S. Miller, J. Minkus, B. Mulansky (Germany), M. Muldoon (Canada), D. K. Nester, M. Omarjee (France), J. Posch, H. Riesel (Sweden), O. G. Ruehr, B. Schmuland (Canada), J. Simons (U.K.), N. C. Singer, A. Stenger, R. Stong, T. Tam, R. Tauraso (Italy), J. V. Tejedor (Spain), M. Tetiva (Romania), D. B. Tyler, M. Vowe (Switzerland), S. Wagon, B. Wallace, T. Wiandt, P. Xi (China), S. Xiao, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before July 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11558. *Proposed by Andrew McFarland, Płock, Poland.* Given four concentric circles, find a necessary and sufficient condition that there be a rectangle with one corner on each circle.

11559. *Proposed by Michel Bataille, Rouen, France.* For positive p and $x \in (0, 1)$, define the sequence $\langle x_n \rangle$ by $x_0 = 1$, $x_1 = x$, and, for $n \geq 1$,

$$x_{n+1} = \frac{p x_{n-1} x_n + (1-p) x_n^2}{(1+p) x_{n-1} - p x_n}.$$

Find positive real numbers α, β such that $\lim_{n\to\infty} n^\alpha x_n = \beta$.

11560. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.*
(**a**) The diagonals of a convex pentagon $P_0P_1P_2P_3P_4$ divide it into 11 regions, of which 10 are triangular. Of these 10, five have two vertices on the diagonal P_0P_2. Prove that if each of these has rational area, then the other five triangles, and the original pentagon, all have rational areas.
(**b**) Let $P_0, P_1, \ldots, P_{n-1}$, $n \geq 5$ be points in the plane. Suppose no three are collinear, and, interpreting indices on P_k as periodic modulo n, suppose that for all k, $P_{k-1}P_{k+1}$ is not parallel to P_kP_{k+2}. Let Q_k be the intersection of $P_{k-1}P_{k+1}$ with P_kP_{k+2}. Let α_k be the area of triangle $P_kQ_kP_{k+1}$, and let β_k be the area of triangle $P_{k+1}Q_kQ_{k+1}$. For $0 \leq j \leq 2n-1$, let

$$\gamma_j = \begin{cases} \alpha_{j/2}, & \text{if } j \text{ is even;} \\ \beta_{(j-1)/2}, & \text{if } j \text{ is odd.} \end{cases}$$

Interpreting indices on γ_j as periodic modulo $2n$, find the least m such that if m consecutive γ_j are rational, then all are rational.

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11561. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania.* Let $f_1, \dots, f_n$ be continuous real valued functions on $[0, 1]$, none identically zero, such that $\int_0^1 f_i(x) f_j(x)\,dx = 0$ if $i \neq j$. Prove that

$$\prod_{k=1}^{n} \int_0^1 f_k^2(x)\,dx \geq n^n \left(\prod_{k=1}^{n} \int_0^1 f_k(x)\,dx \right)^2,$$

$$\sum_{k=1}^{n} \int_0^1 f_k^2(x)\,dx \geq \left(\sum_{k=1}^{n} \int_0^1 f_k(x)\,dx \right)^2, \quad \text{and}$$

$$\sum_{k=1}^{n} \frac{\int_0^1 f_k^2(x)\,dx}{\left(\int_0^1 f_k(x)\,dx \right)^2} \geq n^2.$$

11562. *Proposed by Pál Péter Dályay, Szeged, Hungary.* For positive a, b, c, and z, let $\Psi_{a,b,c}(z) = \Gamma((za + b + c)/(z + 2))$, where Γ denotes the gamma function. Show that $\Psi_{a,b,c}(z)\Psi_{b,c,a}(z)\Psi_{c,a,b}(z)$ is increasing in z for $z \geq 1$.

11563. *Proposed by Vlad Matei (student), University of Bucharest, Bucharest, Romania.* For each integer $k \geq 2$, find all nonconstant f in $\mathbb{Z}[x]$ such that for every prime p, $f(p)$ has no nontrivial kth-power divisor.

SOLUTIONS

Explaining a Polynomial

11403 [2008, 949]. *Proposed by Yaming Yu, University of California–Irvine, Irvine, CA.* Let n be an integer greater than 1, and let f_n be the polynomial given by

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x + j).$$

Find the degree of f_n.

Solution by Nicolás Caro, IMPA, Rio de Janeiro, Brazil, and independently by Cosmin Pohoata, Tudor Vianu National College, Bucharest, Romania. The degree of f_n is $\lfloor n/2 \rfloor$. This follows immediately from the stronger statement that the coefficient of x^r in $f_n(x)$ is the number of derangements of $[n]$ with r cycles, since each cycle must have at least two elements. Here $[n] = \{1, \dots, n\}$, and a derangement is a permutation with no fixed points.

Let $c(n, k)$ be the number of permutations of $[n]$ with k cycles (the unsigned Stirling number of the first kind). The well-known generating function for these numbers is given by $\sum_{k=1}^{n} c(n, k) x^k = \prod_{j=0}^{n-1} (x + j)$ (provable in many ways, including induction on n). Thus

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \sum_{k=1}^{i} c(i, k) x^k = \sum_{\ell=0}^{n} \binom{n}{\ell} (-x)^{\ell} \sum_{k=0}^{n-\ell} c(n - \ell, k) x^k$$
$$= \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{\ell} \sum_{r=\ell}^{n} c(n-\ell, r-\ell) x^r = \sum_{r=0}^{n} \sum_{\ell=0}^{r} \binom{n}{\ell} (-1)^{\ell} c(n-\ell, r-\ell) x^r.$$

The coefficient of x^r in this expression is precisely the inclusion-exclusion formula to count permutations with r orbits that have no fixed points. The universe is the set of permutations with r orbits, and the ith of the n sets to be avoided is the set of permutations in which element i is a fixed point.

Editorial comment. Let $d(n, r)$ be the number of derangements of $[n]$ with r cycles. The fact that the coefficient of x^r in $f_n(x)$ is $d(n, r)$ can also be proved by induction on n using Pascal's formula and the recurrence $d(n, r) = (n-1)[d(n-2, r-1) + d(n-1, r)]$.

O. P. Lossers (and others) gave a short proof of the degree statement by observing that $f_n(x)$ is the nth derivative (with respect to t) of the product $e^{-tx}(1-t)^{-x}$, evaluated at $t = 0$.

This polynomial appears explicitly in *An Introduction to Combinatorial Analysis*, chapter 4 section 4, pp. 72–74 by Riordan (Wiley, 1958). It is also mentioned in *Advanced Combinatorics*, chapter VI, section 6.7, p. 256, by Comtet (Reidel, 1974) with some references to previous non-combinatorial appearances in articles by Tricomi and Carlitz.

Also solved by T. Amdeberhan & S. B. Ekhad, R. Bagby, D. Beckwith, R. Chapman (U.K.), P. Corn, P. P. Dályay (Hungary), O. Geupel (Germany), D. Grinberg, J. Grivaux (France), S. J. Herschkorn, E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), D. E. Knuth, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), Á. Plaza & S. Falcón (Spain), M. A. Prasad (India), R. Pratt, O. G. Ruehr, B. Schmuland (Canada), A. Stadler (Switzerland), J. H. Steelman, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

Mean Inequalities

11434 [2009, 463]. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.* Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $x_1, \ldots, x_n$ be distinct real numbers, and let $p_1, \ldots, p_n$ be positive numbers summing to 1. Let

$$S = \frac{\sum_{k=1}^{n} p_k x_k^3 - \left(\sum_{k=1}^{n} p_k x_k\right)^3}{3\left(\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{k=1}^{n} p_k x_k\right)^2\right)}.$$

Show that $\min\{x_1, \ldots, x_n\} \leq S \leq \max\{x_1, \ldots, x_n\}$.

Solution by Jim Simons, Cheltenham, U.K. Consider a probability distribution on the real line that takes value x_j with probability p_j for $1 \leq j \leq n$. Write μ_i' for the ith moment about 0 and μ_i for the ith moment about the mean μ_1'. Now

$$S = \frac{\mu_3' - {\mu_1'}^3}{3(\mu_2' - {\mu_1'}^2)} = \frac{\mu_3 + 3\mu_1'\mu_2}{3\mu_2} = \mu_1' + \frac{\mu_3}{3\mu_2}.$$

From this we obtain inequalities stronger than those proposed:

$$\frac{1}{3}\min\{x_1, \ldots, x_n\} + \frac{2}{3}\mu_1' \leq S \leq \frac{1}{3}\max\{x_1, \ldots, x_n\} + \frac{2}{3}\mu_1'.$$

Also solved by R. Bagby, R. Chapman (U.K.), M. P. Cohen, W. J. Cowieson, P. P. Dályay (Hungary), H. Dehghan (Iran), P. J. Fitzsimmons, D. Grinberg, E. A. Herman, T. Konstantopoulos (U.K.), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Posch, K. Schilling, B. Schmuland (Canada), R. Stong, M. Tetiva (Romania), B. Tomper, BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

A Circumradius Equation

11443 [2009, 548]. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.* Consider a triangle ABC with circumcenter O and circumradius R. Denote the distances from O to the sides AB, BC, CA, respectively, by x, y, z. Show that if ABC is acute then $R^3 - (x^2 + y^2 + z^2)R = 2xyz$, and $(x^2 + y^2 + z^2)R - R^3 = 2xyz$ otherwise.

Solution by Philip Benjamin, Berkeley College, Woodland Park, NJ. We first prove the identity

$$1 - \left(\cos^2 A + \cos^2 B + \cos^2 C\right) = 2\cos A \cos B \cos C. \qquad (*)$$

Indeed, $C = \pi - (A + B)$, so $\cos C = -\cos(A + B) = \sin A \sin B - \cos A \cos B$. Isolating $\sin A \sin B$ and squaring yields $\cos^2 A \cos^2 B + 2\cos A \cos B \cos C + \cos^2 C = \sin^2 A \sin^2 B = 1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B$. This simplifies to $(*)$.

Let the side lengths a, b, and c be opposite angles A, B, and C, respectively, so $a/\sin A = b/\sin B = c/\sin C = 2R$. The perpendicular from O to side AB bisects AB, so we have a right triangle with side lengths x, $c/2$, and R. Since $c = 2R\sin C$, we conclude that $x = R|\cos C|$. Similarly $y = R|\cos A|$ and $z = R|\cos B|$. If $\triangle ABC$ is acute, then the three cosines are positive, so multiplying $(*)$ by R^3 produces the desired result. Otherwise, say angle C is right or obtuse. Now $x = -R\cos C$ and the other two cosines are positive. Again, multiplying $(*)$ by R^3 produces the desired result.

Editorial comment. A similar problem was proposed in *Crux Mathematicorum with Mathematical Mayhem,* December, 2008, Problem 3395.

Also solved by A. Alt, H. Bailey, M. Bataille (France), D. Beckwith, R. Chapman (U.K.), L. Csete (Hungary), C. Curtis, P. P. Dályay (Hungary), P. De (India), D. Fleischman, V. V. García (Spain), M. Garner & J. Zacharias, O. Geupel (Germany), M. Goldenberg & M. Kaplan, M. R. Gopal, D. Gove, J.-P. Grivaux (France), L. Herot, J. G. Heuver (Canada), E. Hysnelaj (Australia) & E. Bojaxhiu (Germany), Y. K. Jeon (Korea), G. A. Kandall, Y. H. Kim (Korea), L. R. King, B. Klotzsche, T. Konstantopoulos (U.K.), O. Kouba (Syria), K.-W. Lau (China), J. C. Linders (Netherlands), J. H. Lindsey II, O. P. Lossers (Netherlands), J. McHugh, J. Minkus, J. H. Nieto (Venezuela), P. Nüesch (Switzerland), J. Oelschlager, M. Omarjee (France), J. Posch, C. R. & S. Selvaraj, R. A. Simon (Chile), J. Simons (U.K.), R. Stong, M. Tetiva (Romania), B. Tomper, Z. Vörös (Hungary), M. Vowe (Switzerland), H. Widmer (Switzerland), S. Xiao (Canada), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

Extrema

11449 [2009, 647]. *Proposed by Michel Bataille, Rouen, France.* (corrected) Find the maximum and minimum values of

$$\frac{(a^3 + b^3 + c^3)^2}{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}$$

given that $a + b \ge c > 0$, $b + c \ge a > 0$, and $c + a \ge b > 0$.

Solution by Jim Simons, Cheltenham, U.K. Call this big expression X. Since X is homogeneous, we may assume $a^2 + b^2 + c^2 = 1$. The feasible region then consists of a triangular patch on the positive octant of the unit sphere, excluding the vertices (where one of a, b, c is zero), but including the interiors of the sides (where two of a, b, c are equal). Using spherical polar coordinates, we may set $(a, b, c) =$

$(\cos\alpha, \sin\alpha\cos\theta, \sin\alpha\sin\theta)$, where, since a, b, c are positive, θ is uniquely determined and $0 < \theta < \pi/2$. Now

$$X = \frac{\left(\cos^3\alpha + \sin^3\alpha(\cos^3\theta + \sin^3\theta)\right)^2}{\sin^2\alpha\left(\cos^2\alpha + \sin^2\alpha\sin^2\theta\right)\left(\cos^2\alpha + \sin^2\alpha\cos^2\theta\right)}$$
$$= \frac{\left(\cos^3\alpha + \sin^3\alpha(\cos^3\theta + \sin^3\theta)\right)^2}{\sin^2\alpha\left(\cos^4\alpha + \cos^2\alpha\sin^2\alpha + \sin^4\alpha\sin^2\theta\cos^2\theta\right)}.$$

If $f(\theta) = \cos^3\theta + \sin^3\theta$, then $f'(\theta) = 3\cos\theta\sin\theta(\sin\theta - \cos\theta)$. In the feasible region for θ, this is positive for $\theta > \pi/4$ and negative for $\theta < \pi/4$. Thus f, and with it, the numerator of X for fixed α, is less at $\theta = \pi/4$ than at any other feasible θ. Similarly, if $g(\theta) = \sin^2\theta\cos^2\theta$, g, and with it, the denominator of X for fixed α, is increasing in θ for $\theta < \pi/4$ and decreasing in θ for $\theta > \pi/4$. Thus X is, for fixed α, smallest at $\theta = \pi/4$, and greatest at an edge of the feasible region. By symmetry, the minimum value of X is $9/8$, attained when $a = b = c$.

From the foregoing, the maximum value of X on the closure of the feasible region occurs at a point where, with respect to any translation into spherical coordinates, θ is extremal. The only such points are the corners of the region. At $(a, b, c) = (2^{-1/2}, 2^{-1/2}, 0)$, $X = 2$. However, this maximum is not attained because these corners are not in the feasible region.

Also solved by R. Agnew, A. Alt, R. Bagby, D. Beckwith, H. Caerols & R. Pellicer (Chile), R. Chapman (U.K.), H. Chen, C. Curtis, Y. Dumont (France), D. Fleischman, J.-P. Grivaux (France), E. A. Herman, F. Holland (Ireland), T. Konstantopoulos (U.K.), O. Kouba (Syria), A. Lenskold, J. H. Lindsey II, P. Perfetti (Italy), N. C. Singer, R. Stong, T. Tam, R. Tauraso (Italy), M. Tetiva (Romania), E. I. Verriest, Z. Vörös (Hungary), S. Wagon, G. D. White, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

Max Min Coordinate Difference

11450 [2009, 647]. *Proposed by Cosmin Pohoata (student), National College "Tudor Vianu," Bucharest, Romania.* Let A be the unit ball in $\mathbb{R}^n$. Find

$$\max_{\mathbf{a}\in A}\left\{\min_{1\le i<j\le n}\left|a_i - a_j\right|\right\}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let M_n denote the desired maximum. It is implicit in the statement of the problem that $n \ge 2$. We show that $M_n = \sqrt{12/(n(n^2-1))}$.

Let $(a_1, \dots, a_n)$ be an element of A at which the maximum is achieved, and let $M_n = \min\{|a_i - a_j|\colon\ 1 \le i < j \le n\}$. There is a permutation σ of the set $\{1, 2, \dots, n\}$ such that $a_{\sigma(1)} \le a_{\sigma(2)} \le \cdots \le a_{\sigma(n)}$. Write for simplicity $b_j = a_{\sigma(j)}$. For $j > i$, we then have

$$b_j - b_i = \sum_{k=i+1}^{j}(b_k - b_{k-1}) \ge (j-i)M_n.$$

From this we conclude that $|b_j - b_i| \geq |j-i|\, M_n$ for $1 \leq i,\, j \leq n$. Therefore

$$
\begin{aligned}
M_n^2 \sum_{1\leq i,j\leq n} (j-i)^2 &\leq \sum_{1\leq i,j\leq n} (b_j - b_i)^2 = \sum_{1\leq i,j\leq n} (a_j - a_i)^2 \\
&= \sum_{1\leq i,j\leq n} (a_j^2 + a_i^2 - 2a_i a_j) \\
&\leq 2n \sum_{k=1}^{n} a_k^2 - 2\left(\sum_{k=1}^{n} a_k\right)^2 \leq 2n,
\end{aligned}
$$

since $\sum_{k=1}^{n} a_k^2 \leq 1$ when $(a_1, \ldots, a_n) \in A$. On the other hand,

$$
\sum_{1\leq i,j\leq n} (j-i)^2 = 2n \sum_{k=1}^{n} k^2 - 2\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2(n^2-1)}{6}.
$$

It follows that $M_n^2 \leq 12/(n(n^2-1))$, so $M_n \leq \sqrt{12/(n(n^2-1))}$.

Conversely, if we consider $(a_1^{(0)}, a_2^{(0)}, \ldots, a_n^{(0)})$ defined by

$$
a_k^{(0)} = \sqrt{\frac{12}{n(n^2-1)}}\left(k - \frac{n+1}{2}\right), \qquad k = 1, 2, \ldots, n,
$$

then $(a_1^{(0)}, \ldots, a_n^{(0)}) \in A$ and

$$
\min_{1\leq i<j\leq n} \left|a_i^{(0)} - a_j^{(0)}\right| = \sqrt{\frac{12}{n(n^2-1)}}.
$$

Thus $M_n \geq \sqrt{12/(n(n^2-1))}$.

Editorial comment. Marian Tetiva (Romania) notes that a stronger form of this problem appeared as Problem E2032, this MONTHLY **76** (1969) 691–692, proposed by D. S. Mitrinović. See also Problem 3.9.9 in Mitrinović, *Analytic Inequalities* (Springer-Verlag, 1970).

Also solved by A. Alt, R. F. de Andrade, M. R. Avidon, R. Bagby, D. Beckwith, J. Cade, R. Chapman (U.K.), L. Comerford, W. J. Cowieson, P. P. Dályay (Hungary), A. Fielbaum (Chile), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, A. Ilić (Serbia), T. Konstantopoulos (U.K.), J. Kuplinsky, J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, D. Ray, K. Schilling, B. Schmuland (Canada), J. Simons (U.K.), R. Stong, M. Tetiva (Romania), E. I. Verriest, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

A Cauchy–Schwarz Puzzle

11458 [2009, 747]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Let $a_1, \ldots, a_n$ be nonnegative and let r be a positive integer. Show that

$$
\left(\sum_{1\leq i,j\leq n} \frac{i^r j^r a_i a_j}{i+j-1}\right)^2 \leq \sum_{m=1}^{n} m^{r-1} a_m \sum_{1\leq i,j,k\leq n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.
$$

Solution by Francisco Vial, student, Pontificia Universidad Católica de Chile, Santiago, Chile. Let $f(x) := \sum_{i=1}^{n} i^r a_i x^{i-1}$, so

$$\int_0^1 f(x)\,dx = \sum_{m=1}^{n} m^{r-1}\,a_m,$$

$$\int_0^1 f^2(x)\,dx = \int_0^1 \left(\sum_{1\le i,j\le n} i^r j^r\, a_i a_j\, x^{i+j-2} \right) dx = \sum_{1\le i,j\le n} \frac{i^r j^r\, a_i a_j}{i+j-1}, \qquad \text{and}$$

$$\int_0^1 f^3(x)\,dx = \int_0^1 \left(\sum_{1\le i,j,k\le n} i^r j^r k^r a_i a_j a_k x^{i+j+k-3} \right) dx = \sum_{1\le i,j,k\le n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

The stated inequality is equivalent to

$$\left(\int_0^1 f^2(x)\,dx \right)^2 \le \left(\int_0^1 f(x)\,dx \right) \left(\int_0^1 f^3(x)\,dx \right),$$

which follows by applying the Cauchy–Schwarz inequality to $f(x)^{1/2}$ and $f(x)^{3/2}$.

Remarks. Because $a_1, \dots, a_n$ are nonnegative, $f(x)$ is nonnegative and continuous on $[0, 1]$, so $f(x)^{1/2}$ and $f(x)^{3/2}$ are real and well defined. The parameter r need not be an integer.

Also solved by M. R. Avidon, R. Chapman (U.K.), P. P. Dályay (Hungary), D. Grinberg, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Simons (U.K.), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposers.

An Orthocenter Inequality

11461 [2009, 844]. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.* Let a, b, and c be the lengths of the sides opposite vertices A, B, and C of an acute triangle. Let H be the orthocenter. Let d_a be the distance from H to side BC, and similarly for d_b and d_c. Show that

$$\frac{1}{d_a + d_b + d_c} \ge \frac{2}{3} \left(\frac{3}{abc} \left(\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} \right) \right)^{1/4}.$$

Solution by Michael Vowe, Fachhochschule Nordwestschweiz, Muttenz, Switzerland. Let R be the circumradius, r the inradius, F the area, and s the semiperimeter. From $d_a = 2R\cos B\cos C$, $d_b = 2R\cos C\cos A$, $d_c = 2R\cos A\cos B$, we obtain

$$d_a + d_b + d_c = 2R\big(\cos A\cos B + \cos B\cos C + \cos C\cos A\big) \le 2r\left(1 + \frac{r}{R}\right)$$

(see 6.10, p. 181, in D. Mitrinovic et al., *Recent Advances in Geometric Inequalities,* Dordrecht, 1989). From Jensen's inequality for concave functions (here, the square root), we have

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le 3 \cdot \sqrt{\frac{1}{3}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)} = \sqrt{\frac{6s}{abc}}.$$

From $abc = 4RF = 4Rrs$ and $s^2 \geq 27r^2$ (6.1, p. 180, ibid.) we get

$$\frac{2}{3}\left(\frac{3}{abc}\left(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}\right)\right)^{1/4} \leq \frac{2}{3}\left(\frac{3\sqrt{6}s}{(abc)^{3/2}}\right)^{1/4}$$

$$\leq \frac{2}{3}\left(\frac{3\sqrt{6}}{(4Rr)^{3/2}}\cdot\frac{1}{3\sqrt{3}\,r}\right)^{1/4} = \frac{2^{3/8}}{3}\cdot\frac{1}{R^{3/8}}\cdot\frac{1}{r^{5/8}}.$$

Thus it suffices to prove

$$\frac{1}{2r\left(1+\dfrac{r}{R}\right)} \geq \frac{2^{3/8}}{3}\cdot\frac{1}{R^{3/8}}\cdot\frac{1}{r^{5/8}}.$$

Writing $x = r/R$, this means we must prove $x^{3/8}(1+x) \leq 3/(2\cdot 2^{3/8})$ for $0 < x \leq 1/2$. The function $f(x) = x^{3/8}(1+x)$ is increasing on $[0, 1/2]$, though, and we are done.

Equality holds only if $x = 1/2$, or equivalently, $R = 2r$, which makes the triangle equilateral.

Also solved by P. P. Dályay (Hungary), O. Faynshteyn (Germany), K.-W. Lau (China), C. R. Pranesachar (India), R. Stong, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

An Erroneous Claim

11465 [2009, 845]. *Proposed by Pantelimon George Popescu, Polytechnic University of Bucharest, Bucharest, Romania, and José Luis Díaz-Barrero, Polytechnic University of Catalonia, Barcelona, Spain.* Consider three simple closed curves in the plane, of lengths p_1, p_2, and p_3, enclosing areas A_1, A_2, and A_3, respectively. Show that if $p_3 = p_1 + p_2$ and $A_3 = A_1 + A_2$, then $8\pi A_3 \leq p_3^2$.

Solution by the Texas State University Problem Solvers Group, San Marcos, TX. The problem as stated is false. Consider the following counterexample. Let the first curve be a square of side 1, so $p_1 = 4$ and $A_1 = 1$. Let the second curve be a square with $p_2 = 40$ and $A_2 = 100$. Let the third curve be a rectangle with sides $11 + 2\sqrt{5}$ and $101/(11 + 2\sqrt{5})$ so that $p_3 = 44$ and $A_3 = 101$. These three curves fulfill the requirements of the problem, and yet $8\pi A_3 > p_3^2$.

Let us incorporate the additional requirement that $p_1^2 + p_2^2 = 2p_1p_2$. Then the required inequality can be proved as follows. The isoperimetric inequality applied to any of the curves is

$$A_i \leq \pi\left(\frac{p_i}{2\pi}\right)^2,$$

and thus $4\pi A_i \leq p_i^2$. Therefore

$$4\pi A_3 = 4\pi A_1 + 4\pi A_2 \leq p_1^2 + p_2^2.$$

With the newly-added condition we get

$$8\pi A_3 = 8\pi A_1 + 8\pi A_2 \leq 2p_1^2 + 2p_2^2 = p_1^2 + p_2^2 + 2p_1p_2 = (p_1+p_2)^2 = p_3^2.$$

Also solved by G. Apostolopoulos (Greece), R. Bagby, B. Burdick, R. Chapman (U.K.), W. J. Cowieson, P. P. Dályay (Hungary), J.-P. Grivaux (France), K. Hanes, J. H. Lindsey II, M. D. Meyerson, J. Minkus, J. Simons (U.K.), R. Stong, and the Microsoft Research Problems Group.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before August 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11564. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Prove that

$$\int_0^\infty \frac{e^{-x}(1-e^{-6x})}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})}\,dx = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

11565. *Proposed by Shai Covo, Kiryat-Ono, Israel.* Let $U_1, U_2, \ldots$ be independent random variables, each uniformly distributed on $[0,1]$.
(**a**) For $0 < x \le 1$, let N_x be the least n such that $\sum_{k=1}^n \sqrt{U_k} > x$. Find the expected value of N_x.
(**b**) For $0 < x \le 1$, let M_x be the least n such that $\prod_{k=1}^n U_k < x$. Find the expected value of M_x.

11566. *Proposed by Kent Holing, Statoil, Trondheim, Norway.* Let q be a monic quartic polynomial with rational coefficients, that is, $q(z) = z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$ with all coefficients rational. Let r be the *resolvent* of q, that is, $r(z) = z^3 - a_2z^2 + (a_3a_1 - 4a_0)z + (4a_2a_0 - a_3^2a_0 - a_1^2)$. Suppose r is irreducible over $\mathbb{Q}$ and q and r have a common zero.
(**a**) Determine the Galois group of q.
Suppose further that the coefficients of q are integers.
(**b**) Show that $a_3 \neq 0$.
(**c**) Show that only two quartics satisfy the hypotheses of part (b), and find them.

11567. *Proposed by David Callan, University of Wisconsin-Madison, Madison, WI.* How many arrangements $(a_1, \ldots, a_{2n})$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ satisfy the following two conditions: (i) All entries between the two occurrences of any given value i exceed i, and (ii) No three entries increase from left to right with the last two adjacent? (When $n = 3$, one such arrangement is 122133.)

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11568. *Proposed by Kurt Foster, Colorado Springs, CO.* For $n \geq 1$, let $f(n)$ be the least-significant nonzero decimal digit of $n!$. For $n \geq 2$, show that $f(625n) = f(n)$.

11569. *Proposed by M. H. Mehrabi, Nahavand, Iran.* Let a, b, and c be the lengths of the sides of a triangle, and let s, r, and R be the semi-perimeter, inradius, and circumradius, respectively, of that triangle. Show that

$$2 < \log\left(\frac{(a+b)(b+c)(c+a)}{abc}\right) < (a+b+c)\left(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}\right) - 6$$

and

$$8\left(\frac{r}{s}\right)^2 < \log\left(\frac{b+c}{a}\right)\log\left(\frac{c+a}{b}\right)\log\left(\frac{a+b}{c}\right) < \frac{2r}{R}.$$

11570. *Proposed by Kirk Bresniker, Hewlett-Packard, Granite Bay, CA, and Stan Wagon, Macalester College, St. Paul, MN.* Alice and Bob play a number game. Starting with a positive integer n, they take turns changing the number; Alice goes first. Each player in turn may change the current number k to either $k-1$ or $\lceil k/2 \rceil$. The person who changes 1 to 0 wins. For instance, when $n = 3$, the players have no choice, k proceeds from 3 to 2 to 1 to 0, and Alice wins. When $n = 4$, Alice wins if and only if her first move is to 2. For which initial n does Alice have a winning strategy?

11571. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let f be a nonnegative Lebesgue-measurable function on $[0, 1]$, with $\int_0^1 f(x)\,dx = 1$. Let $K(x, y) = (x-y)^2 f(x) f(y)$, $F(t) = \int_{[0,t]\times[0,t]} K(x, y)\,dy\,dx$, and $G(t) = \int_{[t,1]\times[t,1]} K(x, y)\,dy\,dx$. For $0 \leq t \leq 1$, prove that

$$\sqrt{F(t)} + \sqrt{G(t)} \leq \sqrt{F(1)}.$$

SOLUTIONS

An Occasional Congruence

11411 [2009, 179]. *Proposed by Alun Wyn-jones, KPMG, London, U.K.* For positive integers k and n, let $L_k(n) = \sum_{j=1}^{n-1}(-1)^j j^k$.
(**a**) Show that $L_1(n) \equiv L_5(n) \pmod{n}$ if and only if n is not a multiple of 4.
(**b**) Given distinct, odd, positive integers i and j with $\{i, j\} \neq \{1, 5\}$, show that the set of n such that $L_i(n) \equiv L_j(n) \pmod{n}$ is finite.

Solution to part **(a)** *by A. Kumar, Goleta, CA* . By induction on n, one can check that

$$L_5(n) - L_1(n) = (-1)^{n-1}(2n-1)\frac{(n-2)(n-1)n(n+1)}{4}.$$

The numerator contains the product of four consecutive integers. If n is not divisible by 4, then a factor other than n is divisible by 4, and n divides $L_5(n) - L_1(n)$. If n is divisible by 4, then $L_5(n) - L_1(n) = (-1)^{n-1}(2n-1)(n-2)(n-1)(n+1)(n/4) \equiv (-1)(-1)(-2)(-1)(1)(n/4) = n/2 \not\equiv 0 \pmod{n}$.

Solution to part **(b)** *by Richard Stong, Center for Communications Research, San Diego, CA* . We prove the restriction of the claim to odd n. The functions $L_k(n)$ are related to the Euler polynomials by $L_k(n) = \frac{1}{2}[E_k(0) + (-1)^{n-1}E_k(n)]$ (see 23.1.4 in

M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, 1972.) The sequence of Euler polynomials is usually defined by its exponential generating function:

$$\sum_{r=0}^{\infty} E_r(x)\frac{t^r}{r!} = \frac{2e^{xt}}{e^t+1}. \tag{1}$$

Since

$$\sum_{r=0}^{\infty} E_r(0)\frac{t^r}{r!} = \frac{2}{e^t+1} = 1 - \frac{e^t-1}{e^t+1} = 1 - \tanh(t/2),$$

and $\tanh(t/2)$ is an odd function, $E_0(0) = 1$ and $E_{2m}(0) = 0$ for $m \geq 1$. Also,

$$\left(\frac{2}{e^t+1}\right)' = \frac{-2e^t}{(e^t+1)^2} = \frac{-2}{e^t+1} + \frac{2}{(e^t+1)^2} = \frac{-2}{e^t+1} + \frac{1}{2}\left(\frac{2}{e^t+1}\right)^2,$$

so

$$\sum_{r=0}^{\infty} E_{r+1}(0)\frac{t^r}{r!} + \sum_{r=0}^{\infty} E_r(0)\frac{t^r}{r!} = \frac{1}{2}\left(\sum_{r=0}^{\infty} E_r(0)\frac{t^r}{r!}\right)^2.$$

Comparing the coefficients of t^r yields

$$E_{r+1}(0) + E_r(0) = \frac{1}{2}\sum_{k=0}^{r}\binom{r}{k}E_k(0)E_{r-k}(0). \tag{2}$$

With $r = 0$, we obtain $E_1(0) = -1/2$. Since the even terms vanish, setting $r = 2m$ in (**??**) yields a recurrence for the odd terms:

$$E_{2m+1}(0) = \frac{1}{2}\sum_{s=0}^{m-1}\binom{2m}{2s+1}E_{2s+1}(0)E_{2(m-s-1)+1}(0). \tag{3}$$

Induction on m yields $E_{2m+1}(0) \in \mathbb{Z}[\frac{1}{2}]$, where $\mathbb{Z}[\frac{1}{2}]$ is the set of rational numbers with powers of 2 as denominators.

Let $E'_{2m+1} = (-1)^{m+1}E_{2m+1}(0)$. Multiplying (**??**) by $(-1)^{m+1}$ yields $E'_{2m+1} = \frac{1}{2}\sum_{s=0}^{m-1} E'_{2s+1}E'_{2(m-s-1)+1}$. With $E_1(0) = -1/2$, inductively all E'_{2m+1} are positive. In particular, $|E_{2m+1}(0)| > \frac{m}{2}|E_{2m-1}(0)|$, so $|E_5(0)| < |E_7(0)| < \cdots$. Since $E_1(0) = E_5(0) = -\frac{1}{2}$ and $E_3(0) = \frac{1}{4}$, the values $E_{2m+1}(0)$ for $m \geq 0$ are distinct except for $E_1(0) = E_5(0)$.

Factoring the generating function (**??**) as $e^{xt}\cdot\frac{2}{e^t+1}$, we obtain

$$\sum_{r=0}^{\infty} E_r(x)\frac{t^r}{r!} = \sum_{r=0}^{\infty}\frac{x^r t^r}{r!}\cdot\sum_{r=0}^{\infty} E_r(0)\frac{t^r}{r!},$$

and hence $E_r(x) = \sum_{k=0}^{r}\binom{r}{k}E_{r-k}(0)x^k$. Thus E_r is a polynomial with coefficients in $\mathbb{Z}[\frac{1}{2}]$ and constant term $E_r(0)$. For odd i and j, we can now write

$$\begin{aligned} L_i(2m+1) - L_j(2m+1) &= \frac{1}{2}(E_i(0) + E_i(2m+1) - E_j(0) - E_j(2m+1)) \\ &= E_i(0) - E_j(0) + (2m+1)P(2m+1), \end{aligned}$$

where $xP(x) = \frac{1}{2}[E_i(x) - E_i(0) - E_j(x) + E_j(0)]$. Note that P is a polynomial with coefficients in $\mathbb{Z}[\frac{1}{2}]$. Thus $L_i(2m+1) \equiv L_j(2m+1) \pmod{2m+1}$ only if $2m+1$ divides the numerator of $E_i(0) - E_j(0)$, since its denominator is a power of 2. Since $E_i(0) \neq E_j(0)$ for $\{i, j\} \neq \{1, 5\}$, there are only finitely many such odd numbers.

Editorial comment. In part (b) the restriction to odd n was omitted in error. Several solvers pointed out that the claim is false when n is allowed to be even. Zoltán Vörös proved this as follows: For k odd and n even, $(n-i)^k \equiv (-i)^k \equiv -i^k \pmod{n}$, so the terms in the sum $L_k(n)$ cancel in pairs except for the middle term. Thus $L_k(n) \equiv (n/2)^k \pmod{n}$. Hence if i and j are odd and n is even but not divisible by 4, then $L_i(n) - L_j(n) \equiv (n/2)^i - (n/2)^j \equiv 0 \pmod{n}$. If both i and j are at least 3, then every even n will work.

Also solved by F. Alayont and by R. Chapman (U.K.). Part (a) only solved by D. Beckwith, Z. Vörös (Hungary), BSI Problems Group (Germany), GCHQ Problem Solving Group (United Kingdom), and the proposer. Falsity for even n also observed by D. Fleischman and J. H. Lindsey II.

Splitting Fields of Cubic Polynomials

11420 [2009, 277]. *Proposed by Kent Holing, StatoilHydro, Trondheim, Norway.* Let p be a monic cubic polynomial with integer coefficients and discriminant D. Show that if r is a zero of p and $d = \sqrt{D}$, then $\mathbb{Q}(r+d)$ is the splitting field of p.

Solution by Achava Nakhash, San Diego, CA. Let K be the splitting field of p. Let r, s, and t be the roots of p, so $d = \pm(r-s)(r-t)(s-t)$. If p splits into linear factors over $\mathbb{Q}$, then both r and d are rational, and it follows that $K = \mathbb{Q}(r+d) = \mathbb{Q}$. If p splits into a linear and an irreducible quadratic factor, and r is the rational root, then because $d = \sqrt{D}$ and $s, t, d \in \mathbb{Q}(\sqrt{D})$, $K = \mathbb{Q}(\sqrt{D}) = \mathbb{Q}(r+d)$. A similar proof works if r is one of the nonrational roots.

If p is irreducible, then the Galois group of K is either A_3 or S_3. In the former case, the discriminant is a rational square, so $\mathbb{Q}(r) = \mathbb{Q}(r+d) \neq \mathbb{Q}$; since K is of degree 3, it follows that $K = \mathbb{Q}(r) = \mathbb{Q}(r+d)$. Finally, if the Galois group of K is S_3, then it must include the permutation that fixes r and transposes s and t. This permutation does not fix d, so $r+d$ is not in $\mathbb{Q}(r)$, because the latter field is the fixed field of A_3. Thus, $\mathbb{Q}(r+d)$ properly contains $\mathbb{Q}(r)$ and is a subfield of the splitting field K. Since K is a degree-six extension of the rationals, it follows that $\mathbb{Q}(r+d) = K$, as claimed.

Also solved by R. Chapman (U.K.), P. Corn, P. P. Dályay (Hungary), S. M. Gagola Jr., J. Grivaux (France), J. H. Lindsey II, J. Simons (U.K.), J. H. Smith, R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U.K.), NSA Problem Solving Group, and the proposer.

Recurrences for k-Fibonacci Numbers

11421 [2009, 277]. *Proposed by Sergio Falcón and Ángel Plaza, University of las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.* Fix a positive integer k and define the sequence $\langle a \rangle$ by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = ka_n + a_{n-1}$ for $n \geq 1$.
(a) Show that if n, r, and h are nonnegative integers such that $r + h \leq n$, then $a_{n+r}a_{n+r+h} + (-1)^{h+1}a_{n-r-h}a_{n-r} = a_{2n}a_{2r+h}$.
(b) Show that if i and j are positive integers with $i \geq j$, then

$$k\sum_{r=0}^{j-1} a_{i-r}a_{j-r} = \begin{cases} a_i a_{j+1}, & \text{if } j \text{ is odd;} \\ a_i a_{j+1} - a_{i-j}, & \text{if } j \text{ is even.} \end{cases}$$

[Editor's note - In the original printing, "a_{n+r}" read "a_{n+1}"; we regret the error.]

Solution I by J. C. Linders, Eindhoven, The Netherlands.
(a) Set $\phi = (k + \sqrt{k^2+4})/2$ and $\omega = (k - \sqrt{k^2+4})/2$, which are the roots of $x^2 - kx - 1 = 0$. Since this is the characteristic equation of the recurrence, checking the initial conditions confirms that $a_n = (\phi^n - \omega^n)/(\phi - \omega)$ for $n \geq 0$. Thus

$$a_p a_q = \frac{\phi^p - \omega^p}{\phi - \omega} \cdot \frac{\phi^q - \omega^q}{\phi - \omega} = \frac{\phi^{p+q} - \phi^p\omega^q - \omega^p\phi^q + \omega^{p+q}}{(\phi - \omega)^2}$$

for $p, q \geq 0$. Since $\phi\omega = -1$, this yields

$$a_p a_q = \frac{\phi^{p+q} + \omega^{p+q} - (-1)^q\left(\phi^{p-q} - \omega^{p-q}\right)}{(\phi - \omega)^2}. \tag{4}$$

It follows that

$$a_{p+s}a_{q+s} - (-1)^s a_p a_q = \frac{\phi^{p+q+2s} + \omega^{p+q+2s} - (-1)^s\left(\phi^{p+q} + \omega^{p+q}\right)}{(\phi - \omega)^2}.$$

Applying (**??**) again yields, for $p, q, s \geq 0$,

$$a_{p+s}a_{q+s} - (-1)^s a_p a_q = a_{p+q+s}a_s \tag{5}$$

The claim follows by setting $p = n - r - h$, $q = n - r$, and $s = 2r + h$.
(b) Setting $q = 1$ in (**??**) yields $a_{p+s}a_{s+1} - (-1)^s a_p = a_{p+s+1}a_s$. Using this after applying the recurrence,

$$ka_{p+s}a_s = (a_{p+s+1} - a_{p+s-1})a_s = a_{p+s}a_{s+1} - (-1)^s a_p - a_{p+s-1}a_s,$$

assuming $p + s \geq 1$. For positive j and nonnegative p, we obtain

$$\begin{aligned} k\sum_{s=1}^{j} a_{p+s}a_s &= \sum_{s=1}^{j}\left(a_{p+s}a_{s+1} - a_{p+s-1}a_s\right) - \sum_{s=1}^{j}(-1)^s a_p \\ &= a_{p+j}a_{j+1} - a_p a_1 - a_p\sum_{s=1}^{j}(-1)^s = \begin{cases} a_{p+j}a_{j+1}, & \text{if } j \text{ is odd;} \\ a_{p+j}a_{j+1} - a_p, & \text{if } j \text{ is even.} \end{cases} \end{aligned}$$

The claim follows by setting $p = i - j$ and $s = j - r$.

Solution II to part (**a**) *by the editors.* The claim of (**a**) reduces to $a_{n+r}a_{2n} + 0 = a_{2n}a_{n+r}$ when $r + h = n$, so in what follows we assume $r + h < n$. The defining recurrence generates the k-Fibonacci numbers: a_n is the number of distinguishable tilings of a unit-width strip of length $n - 1$ using squares of k colors and uncolored dominos (see A. T. Benjamin and J. J. Quinn, Fibonacci and Lucas identities through colored tilings, *Utilitas Math.* **56** (1999) 137–142). To facilitate combinatorial proofs, let $b_n = a_{n+1}$ for $n \geq 0$ and $b_n = 0$ for $n < 0$.

Start the strip at point 0, so $b_r b_s$ tilings of length $r + s$ have a breakpoint at r. Similarly, $b_{r-1}b_{s+1}$ have a breakpoint at $r - 1$. Those with a square in position r are counted by both products. Thus $b_r b_{s-1} - b_{r-1}b_s = b_{r-2}b_{s-1} - b_{r-1}b_{s-2}$. For $r < s$, this yields $b_r b_{s-1} - b_{r-1}b_s = (-1)^r b_{s-r-1}$ by induction on r.

Now consider $b_{r+t}b_s - b_r b_{s+t}$. Among the tilings of length $r + s + t$ counted by the two products, those not counted by both are those in the first set that don't break at r and those in the second that don't break at $r + t$. For $r < s$, we have

$$\begin{aligned} b_{r+t}b_s - b_r b_{s+t} &= b_{r-1}b_{t-1}b_s - b_r b_{t-1}b_{s-1} \\ &= b_{t-1}[b_{r-1}b_s - b_r b_{s-1}] = (-1)^{r-1}b_{t-1}b_{s-r-1}. \end{aligned}$$

Setting $r = 2m + h - 1$, $s = n + m - 1$, and $t = n - m$, where $m + h < n$, converts this to the desired identity with m written in place of the original r.

Solution II to part (**b**) *by the proposers.* To give a combinatorial argument, we keep the notation from the previous solution to (**a**) and rewrite the desired identity (for $i \geq j$) as

$$k \sum_{r=1}^{j} b_{i-r} b_{j-r} = \begin{cases} b_{i-1} b_j, & \text{if } j \text{ is odd;} \\ b_{i-1} b_j - b_{i-j-1}, & \text{if } j \text{ is even.} \end{cases}$$

There are $b_{i-1} b_j$ pairs of tilings such that one has length $i - 1$ in positions 2 through i and the other has length j in positions 1 through j.

To count these another way, group the pairs by the first fault, where a *fault* is a position where both tilings have a breakpoint. The first fault immediately follows the leftmost square in the two tilings, say at position r, and there are k ways to color that square. This determines both tilings through position r, and there are $b_{i-r} b_{j-r}$ ways to complete the two tilings.

This counts all the pairs when j is odd, but when j is even there may be no fault. In this case both tilings begin with $j/2$ dominos, and there are b_{i-j-1} ways to complete the longer tiling.

Editorial comment. The special case of the identity in Solution II of **(a)** obtained by setting $r = n - 1$, $t = 1$, and $s = n + \ell$ was proved (for $k = 1$) and applied in Br. A. Brousseau, Summation of infinite Fibonacci series, *Fibonacci Quarterly* **7** (1969) 143–168.

Also solved by M. Bataille (France), D. Beckwith, R. Chapman (U.K.), P. Corn, C. Curtis, P. P. Dályay (Hungary), Y. Dumont (France), D. Fleischman, J. Grivaux (France), O. Kouba (Syria), H. Kwong, J. H. Lindsey II, O. P. Lossers (Netherlands), A. Nakhash, J. P. Robertson, J. Simons (U.K.), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), Microsoft Research Problems Group, and the proposer. Part (b) solved by G. C. Greubel and GCHQ Problem Solving Group (U.K.).

Equally Many Repetitions of Each Type

11424 [2009, 277]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Find the number of bit strings of length n in which the number of 00 substrings is equal to the number of 11 substrings. For example, when $n = 4$ we have 4 such bit strings: 0011, 0101, 1010, and 1100.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ. The answer is 2 when $n = 1$ and is $2\binom{n-2}{\lceil n/2 \rceil - 1}$ when $n > 1$.

Consider $n > 1$. Given a bit string of length n, let a_i be the number of 0s in the ith run of 0s, and let b_i be the number of 1s in the ith run of 1s. For example, the string 00011011 yields $a_1 = 3$, $b_1 = 2$, $a_2 = 1$, and $b_2 = 2$. The number of copies of 00 is $\sum_i (a_i - 1)$, and the number of copies of 11 is $\sum_i (b_i - 1)$.

Let $a = \sum_i a_i$ and $b = \sum_i b_i$. If the first and last bits differ, then the number of runs of each type is the same, and the desired condition holds if and only if $a = b$, which requires n to be even. If the first and last bits are 0, then there is an extra run of 0, and the desired condition holds if and only if $a = b + 1$, which requires n to be odd. Similarly, the condition is $a + 1 = b$ and n odd when the first and last bits are 1.

There are two possibilities for the first bit. Once this bit is chosen, the number of ways to satisfy the needed condition on a and b is $\binom{n-2}{(n-2)/2}$ when n is even, and it is $\binom{n-2}{(n-1)/2}$ when n is odd. Combining the computations yields $2\binom{n-2}{\lceil n/2 \rceil - 1}$ for the answer in both cases.

Editorial comment. Let $s(n,k)$ be the number of binary n-tuples beginning with 0 in which the number of copies of 00 exceeds the number of copies of 11 by exactly k. Several solvers showed that $s(n,k) = \binom{n-2}{\lceil (n-k)/2 \rceil - 1}$; setting $k = 0$ gives the desired result. Generalizing the problem in a different direction, Heckman asks how many binary n-tuples have the same number of copies of 00 and 01. Computational results suggest that this may be harder.

Also solved by T. Barcume, D. Beckwith, N. Caro (Colombia), R. Chapman (U.K.), P. Corn, C. Curtis, P. P. Dályay (Hungary), D. Grinberg, N. Grivaux (France), P. Landweber, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), D. Nacin, C. M. Nicolas, Á. Plaza & A. Saure & J. P. Pedro (Spain), C. R. Pranesachar (India), R. E. Prather, J. Riley, T. Rucker, B. Schmuland (Canada), J. Simons (U.K.), R. Staum, R. Stong, S. Sullivan, R. Tauraso (Italy), M. Tetiva (Romania), B. Tomper, H. Widmer (Switzerland), L. Zhou, CMC 328, GCHQ Problem Solving Group (U.K.), Hofstra University Problem Solvers, NSA Problems Group, Problem Solving Group Magdeburg (Germany), Texas State University Problem Solvers Group, and the proposer.

Sum of Two Squares in Every Congruence Class

11425 [2009, 366]. *Proposed by Erwin Just (emeritus), Bronx Community College, City University of New York, Bronx, NY.* For which positive integers m does every congruence class mod m contain the sum of two squares?

Solution by John H. Lindsey II, Cambridge, MA . Every congruence class modulo m contains a sum of two squares if and only if every prime whose square divides m is congruent to 1 modulo 4.

Necessity. Let p be a prime with $p^2 \mid m$. If every congruence class modulo m contains a sum of two squares, then the same holds modulo p^2. Modulo 4 we cannot get 3 as a sum of squares, so $p > 2$. If $a^2 + b^2 \equiv p \bmod p^2$, then a and b are not both divisible by p. Hence in the field $\mathbb{F}_p$ we have $a^2 + b^2 = 0$ and may assume a is nonzero; thus $(b/a)^2 = -1$, and b/a has order 4 in the multiplicative group of $p-1$ elements. This requires $4 \mid (p-1)$, as desired.

Sufficiency. Let p be a prime. We claim that every congruence class modulo p contains a sum of two squares. This is clearly true for $p = 2$, so suppose p is odd, and let h be the first quadratic nonresidue among $1, \dots, p-1$. Thus $h \equiv a^2 + 1 \bmod p$ for some integer a. The quadratic nonresidues are the $\frac{p-1}{2}$ numbers of the form $b^2 h$, where $b \neq 0$. Since $b^2 h = (ba)^2 + b^2$, the quadratic nonresidues are sums of two squares. The quadratic residues have the form $b^2 + 0^2$. Hence every congruence class modulo p contains a sum of two squares.

If also $p \equiv 1 \bmod 4$, then each class contains a sum of two squares that are not both divisible by p. This is obvious except for class 0; since -1 is a quadratic residue modulo p, we have $a^2 + 1 \equiv 0 \bmod p$ for some a with $p \nmid a$.

We show next that if all classes modulo p occur in the form $a^2 + b^2$ with p not dividing a, then for $i \geq 2$ all classes modulo p^i occur in the form $(a + pj)^2 + b^2$ for some j with $0 \leq j \leq p^{i-1} - 1$. For this, it suffices to show that for fixed a, the values $(a + pj)^2$ lie in p^{i-1} distinct classes modulo p^i, all of which are congruent to a^2 modulo p. If $(a + pj)^2$ and $(a + pk)^2$ lie in the same class modulo p^i, then p^i divides their difference, which equals $(2a + pj + pk)p(j - k)$. We have assumed that $p \nmid a$ and $p^{i-1} \nmid (j - k)$, so $(a + pj)^2 \not\equiv (a + pk)^2 \bmod p^i$. Also, $(a + pj)^2 \equiv a^2 \bmod p$.

Now suppose that m has prime factorization $\prod_{j=1}^{n} p_j^{i_j}$ such that $i_j \geq 2$ implies $p_j \equiv 1 \bmod 4$. Fix a congruence class $r \bmod m$. For each j with $1 \leq j \leq n$, there exist a_j and b_j such that $a_j^2 + b_j^2 \equiv r \bmod p_j^{i_j}$. By the Chinese Remainder Theorem, there exist a and b such that $a \equiv a_j \bmod p_j^{i_j}$ and $b \equiv b_j \bmod p_j^{i_j}$ for $1 \leq j \leq n$. Hence

$a^2+b^2 \equiv a_j^2+b_j^2 \equiv r \bmod p_j^{i_j}$ for $1 \le j \le n$. Since the various $p_j^{i_j}$ are relatively prime, $a^2+b^2 \equiv r \bmod m$.

Also solved by R. Chapman (U.K.), S. M. Gagola Jr., J. Grivaux (France), C. Lanski, O. P. Lossers (Netherlands), J. Moreira (Portugal), A. Nakhash, K. Schilling, J. Simons (U.K.), N. C. Singer, R. Stong, M. Tetiva (Romania), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

An Arccos Integral

11457 [2009, 747]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* For real numbers a and b with $0 \le a \le b$, find

$$\int_{x=a}^{b} \arccos\left(\frac{x}{\sqrt{(a+b)x-ab}}\right) dx.$$

Solution by FAU Problem Solving Group, Florida Atlantic University, Boca Raton, FL. We integrate by parts, using $x-\frac{ab}{a+b}$ as the antiderivative of 1. Notice that

$$\frac{d}{dx}\arccos\left(\frac{x}{\sqrt{(a+b)x-ab}}\right) = -\frac{b(x-a)-a(b-x)}{2(a+b)\left(x-\frac{ab}{a+b}\right)\sqrt{(b-x)(x-a)}},$$

and that $\arccos(x/\sqrt{(a+b)x-ab})$ vanishes for $x=a$ and $x=b$. Thus

$$\begin{aligned}\int_a^b \arccos\left(\frac{x}{\sqrt{(a+b)x-ab}}\right) dx &= \left(x-\frac{ab}{a+b}\right)\arccos\left(\frac{x}{\sqrt{(a+b)x-ab}}\right)\Bigg|_a^b \\ &\quad + \frac{1}{2(a+b)}\left[b\int_a^b \frac{\sqrt{x-a}}{\sqrt{b-x}}\,dx - a\int_a^b \frac{\sqrt{b-x}}{\sqrt{x-a}}\,dx\right] \\ &= \frac{1}{2(a+b)}\left[b\int_a^b \frac{\sqrt{x-a}}{\sqrt{b-x}}\,dx - a\int_a^b \frac{\sqrt{b-x}}{\sqrt{x-a}}\,dx\right].\end{aligned}$$

Writing B for Euler's beta function, we have

$$\int_a^b \frac{\sqrt{x-a}}{\sqrt{b-x}}dx = \int_a^b \frac{\sqrt{b-x}}{\sqrt{x-a}}dx = (b-a)B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{(b-a)\pi}{2}.$$

It follows that

$$\int_a^b \arccos\left(\frac{x}{\sqrt{(a+b)x-ab}}\right) dx = \frac{(b-a)^2\pi}{4(a+b)}.$$

Also solved by K. F. Andersen (Canada), G. Apostolopoulos (Greece), M. S. Ashbaugh, M. R. Avidon, R. Bagby, D. H. Bailey & J. M. Borwein (U.S.A. & Canada), D. Beckwith, K. N. Boyadzhiev, P. Bracken, R. Chapman (U.K.), H. Chen, C. Curtis, P. P. Dályay (Hungary), P. De (India), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), J. A. Grzesik, E. A. Herman, F. Holland (Ireland), M. E. H. Ismail, W. P. Johnson, P. Khalili, O. Kouba (Syria), C. Koutschan, G. Lamb, O. P. Lossers (Netherlands), K. McInturff, D. K. Nester, M. Omarjee (France), P. Perfetti (Italy), J. Posch, C. R. Pranesachar (India), O. G. Ruehr, F. Sami, B. Schmuland (Canada), J. Simons (U.K.), N. C. Singer, S. Singh, R. Stong, T. Tam, A. P. Taraporevala, R. Tauraso (Italy), M. Tetiva (Romania), T. P. Turiel, D. B. Tyler, E. I. Verriest, M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), GCHQ Problem Solving Group (U.K.), Hofstra University Problem Solvers, Missouri State University Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before September 30, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11572. *Proposed by Sam Sakmar, University of South Florida, Tampa, FL.* Given a circle C and two points A and B outside C, give a Euclidean construction to find a point P on C such that if Q and S are the second intersections with C of AP and BP respectively, then QS is perpendicular to AB. (Special configurations, including the case that A, B, and the center of C are collinear, are excluded.)

11573. *Proposed by Rob Pratt, SAS Institute, Cary, NC.* A *Sudoku permutation matrix* (SPM) of order n^2 is a permutation matrix of order n^2 with exactly one 1 in each of the n^2 submatrices of order n obtained by partitioning the original matrix into an n-by-n array of submatrices. Thus, for $n = 2$, the permutation 1324 yields an SPM, but the identity permutation 1234 does not. Find the number of SPMs of order n^2.

11574. *Proposed by M. Farrokhi D. G., Ferdowsi University of Mashhad, Mashhad, Iran.* Let F be a field with characteristic zero, let $p \in F[x]$ be a polynomial over F, and let D_p be the set of all polynomials q in $F[x]$ that divide $p \circ r$ for some r in $F[x]$. Prove that D_p is closed under multiplication.

11575. *Proposed by Tuan Le (student), Worcester Polytechnic Institute, Worcester, MA.* Prove that if a, b, and c are positive, then

$$\frac{16}{27}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3+\left(\frac{abc}{(a+b)(b+c)(c+a)}\right)^{1/3} \ge \frac{5}{2}.$$

11576. *Proposed by László Tóth, University of Pécs, Hungary.* Let $\omega(n)$ denote the number of distinct prime factors of n. Let $P(x, k)$ be the set of integers in $[1, x]$ that are relatively prime to k, and let $\phi(x, k) = |P(x, k)|$. Let

$$S(x, k) = \sum_{n \in P(x,k)} (-1)^{\omega(n)}.$$

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Show that for all real x in $[1, \infty)$,

$$\sum_{1\le n\le x} (-1)^{\omega(n)}\phi(x/n, n) = \sum_{1\le n\le x} S(x/n, n) = 1.$$

11577. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let n be a positive even integer and let p be prime. Show that the polynomial f given by $f(x) = p + \sum_{k=1}^{n} x^k$ is irreducible over $\mathbb{Q}$.

11578. *Proposed by Roger Cuculière, Clichy la Garenne, France.* Let E be a real normed vector space of dimension at least 2. Let f be a mapping from E to E, bounded on the unit sphere $\{x \in E\colon \|x\| = 1\}$, such that whenever x and y are in E, $f(x + f(y)) = f(x) + y$. Prove that f is a continuous, linear involution on E.

SOLUTIONS

Rounding Down

11428 [2009, 365]. *Proposed by Walter Blumberg, Coral Springs, FL.* Let p be a prime that is congruent to 3 mod 4, and let a and q be integers, with $p \nmid q$. Show that

$$\sum_{k=1}^{p} \lfloor (qk^2 + a)/p \rfloor = 2a + 1 + \sum_{k=1}^{p} \lfloor (qk^2 - a - 1)/p \rfloor.$$

Solution by Julien Grivaux (student), Université Pierre et Marie Curie, Paris, France. Let $N(x)$ be the number of congruence classes y such that $y^2 \equiv x \bmod p$. Since $p \equiv 3 \bmod 4$, the value -1 is not a quadratic residue modulo p, so $\{x, -x\}$ contains exactly one quadratic residue modulo p when $p \nmid x$. Thus always $N(x) + N(-x) = 2$.

Define

$$S(a) = \sum_{k=1}^{p} \lfloor (qk^2 + a)/p \rfloor \quad \text{and} \quad T(a) = \sum_{k=1}^{p} \lfloor (qk^2 - a - 1)/p \rfloor.$$

Let $[p] = \{1, \ldots, p\}$. Now

$$S(a+1) - S(a) = \left|\{k \in [p]\colon\ p \mid (qk^2 + a + 1)\}\right| = N(-q^{-1}(a+1)),$$
$$T(a) - T(a+1) = \left|\{k \in [p]\colon\ p \mid (qk^2 - a - 1)\}\right| = N(q^{-1}(a+1)).$$

Thus $\big[S(a+1) - T(a+1)\big] - \big[S(a) - T(a)\big] = 2$. We conclude that $S(a) - T(a) = 2a + [S(0) - T(0)]$. Furthermore, using $p \nmid q$, we have

$$S(0) - T(0) = \sum_{k=1}^{p} \big\lfloor qk^2/p \big\rfloor - \sum_{k=1}^{p} \big\lfloor (qk^2 - 1)/p \big\rfloor = \left|\{k \in [p]\colon\ p \mid qk^2\}\right| = 1$$

Thus $S(a) - T(a) = 2a + 1$.

Also solved by R. Chapman (U.K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), D. Gove, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), J. Simons (U.K.), N. C. Singer, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), K. S. Williams (Canada), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and NSA Problems Group.

A Triply-Determined Point on the Sphere

11433 [2009, 463]. *Proposed by Marius Cavachi, University "Ovidius," Constanţa, Romania.* Let n be a positive integer, and let $A_1, \dots, A_n$, $B_1, \dots, B_n$, and $C_1, \dots, C_n$ be points on the unit sphere S^2. Show that there exists P on S^2 such that

$$\sum_{k=1}^{n} |P - A_k|^2 = \sum_{k=1}^{n} |P - B_k|^2 = \sum_{k=1}^{n} |P - C_k|^2.$$

Solution I by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria. Let $U = \sum_{k=1}^{n}(B_k - A_k)$ and $V = \sum_{k=1}^{n}(C_k - A_k)$. Since $\dim \operatorname{span}(U, V) \le 2$, there is a point $Q \neq 0$ in $\mathbb{R}^3$ that is perpendicular to U and V. Let $P = Q/|Q|$ and $d = \sum_{k=1}^{n} |P - A_k|^2 - \sum_{k=1}^{n} |P - B_k|^2$. Expanding, $d = \sum_{k=1}^{n}(|P|^2 + |A_k|^2 - 2P \cdot A_k) - \sum_{k=1}^{n}(|P|^2 + |B_k|^2 - 2P \cdot B_k) = 2P \cdot U = 0$. Similarly, $\sum_{k=1}^{n} |P - A_k|^2 - \sum_{k=1}^{n} |P - C_k|^2 = 0$.

Solution II by Dan Jurca, California State University East Bay, Hayward, California. Define $f : S^2 \to \mathbb{R}^2$ by

$$f(P) = \left(\sum_{k=1}^{n} \left(|P - A_k|^2 - |P - B_k|^2\right), \sum_{k=1}^{n} \left(|P - A_k|^2 - |P - C_k|^2\right) \right).$$

Note that $|P - A_k|^2 - |P - B_k|^2 = -(|-P - A_k|^2 - |-P - B_k|^2)$ and similarly $|P - A_k|^2 - |P - C_k|^2 = -(|-P - A_k|^2 - |-P - C_k|^2)$. Thus $f(P) = -f(-P)$ for all $P \in S^2$. Since f is continuous on S^2, the Borsuk–Ulam theorem implies that there exists $P \in S^2$ such that $f(P) = f(-P) = (0, 0)$. The result follows.

Also solved by R. Bagby, M. Bataille (France), R. Chapman (U.K.), P. P. Dályay (Hungary), R. Garmanjani (Portugal), M. Goldenberg & M. Kaplan, D. Grinberg, E. A. Herman, B.-T. Iordache (Romania), Y. K. Jeon (Korea), H. Katsuura & E. Schmeichel, J. C. Linders (Netherlands), O. P. Lossers (Netherlands), K. McInturff, J. H. Nieto (Venezuela), J. Schaer (Canada), J. Simons (U.K.), N. C. Singer, R. Stong, M. Tetiva (Romania), CMC 328, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

Diagonal Intersections Not Collinear

11436 [2009, 463]. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* In a triangle ABC, let B' and C' be points on sides AC and AB, respectively. Let M be the intersection of BB' and CC'. Let distinct lines k and l intersecting inside triangle MBC meet segments $C'B$, MB, MC, and $B'C$ at K_1, K_2, K_3, K_4 and L_1, L_2, L_3, L_4, respectively. Show that the intersection points of the diagonals of $K_1K_2L_2L_1$, $K_2L_2K_3L_3$, and $K_3L_3L_4K_4$ are not collinear.

Solution I by Oliver Geupel, Brühl, Germany. If u and v are lines, let $u.v$ denote their intersection. Let $Q = K_1L_2.K_2L_1$, $O = k.l$, and $P = K_3L_4.K_4L_3$. Without loss of generality suppose that L_1 lies on the line segment $C'K_1$.

Let $U = K_1Q.K_4P$ and $V = QL_1.PL_4$. Then $U = K_1L_2.K_4L_3 = L_2Q.L_3P$ and $V = K_2L_1.K_3L_4 = QK_2.PK_3$. If the points at issue, Q, O, P, are collinear, then lines K_1K_4, L_1L_4, and QP are concurrent at O. Hence triangles K_1L_1Q and K_4L_4P are perspective from O, as are K_2L_2Q and K_3L_3P. By Desargues' theorem, $K_1L_1.K_4L_4$ (i.e., A), U, and V are collinear, as are $K_2L_2.K_3L_3$ (i.e., M), U, and V. Thus A, U, V, M are collinear, so U lies on AM. Triangle AK_1K_4 contains $K_1Q.AM$ in its interior but is disjoint from K_4P. This contradicts our assumption that Q, O, P are collinear.

Solution II by the GCHQ Problem Solving Group, Cheltenham, U.K. With the above notation, let $\mathbf{k}$ and $\mathbf{l}$ denote unit vectors in the directions OK_1 and OL_1, respectively. For $1 \le j \le 4$, let $OK_j = \alpha_j \mathbf{k}$ and $OL_j = \beta_j \mathbf{l}$. Now $P = K_3L_4.K_4L_3$, so $OP = \lambda\alpha_3\mathbf{k} + (1-\lambda)\beta_4\mathbf{l} = \mu\alpha_4\mathbf{k} + (1-\mu)\beta_3\mathbf{l}$. Thus $\lambda\alpha_3 = \mu\alpha_4$ and $(1-\lambda)\beta_4 = (1-\mu)\beta_3$. Therefore $(\alpha_3\beta_3 - \alpha_4\beta_4)OP = \alpha_3\alpha_4(\beta_3 - \beta_4)\mathbf{k} + \beta_3\beta_4(\alpha_3 - \alpha_4)\mathbf{l}$. Similarly, $(\alpha_1\beta_1 - \alpha_2\beta_2)OQ = \beta_1\beta_2(\alpha_1 - \alpha_2)\mathbf{k} + \beta_1\beta_2(\alpha_1 - \alpha_2)\mathbf{l}$, $(\alpha_1\beta_2 - \alpha_2\beta_1)OB = \alpha_1\alpha_2(\beta_2 - \beta_1)\mathbf{k} + \beta_1\beta_2(\alpha_1 - \alpha_2)\mathbf{l}$, and $(\alpha_3\beta_4 - \alpha_4\beta_3)OC = \alpha_3\alpha_4(\beta_4 - \beta_3)\mathbf{k} + \beta_3\beta_4(\alpha_3 - \alpha_4)\mathbf{l}$. Note that $OP \parallel ON$ if and only if $OB \parallel OC$, i.e., if and only if $\beta_1\beta_2\alpha_3\alpha_4(\alpha_1 - \alpha_2)(\beta_3 - \beta_4) = \alpha_1\alpha_2\beta_3\beta_4(\beta_1 - \beta_2)(\alpha_3 - \alpha_4)$. Since $O = k.l$ lies inside MBC, we conclude $OB \nparallel OC$, so $OP \nparallel OQ$, as desired.

Also solved by R. Chapman (U.K.), P. P. Dályay (Hungary), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), C. R. Pranesachar (India), J. Simons (U.K.), R. Stong, and the proposer.

Zeros of Polynomials with Unit Coefficients

11437 [2009, 464]. *Proposed by Tamás Erdélyi, Texas A&M University, College Station, TX.* Let $\mathcal{L}_k$ denote the set of all polynomials of degree k in x with each of their $k+1$ coefficients in $\{-1, 1\}$. Let M_k denote the largest multiplicity that a zero of a P in $\mathcal{L}_k$ can have at 1. Let $\langle C_k \rangle$ be a sequence of positive integers tending to infinity. Show that

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k\colon\ 1 \le k \le n \text{ and } M_k \ge C_k\}\right| = 0.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA . We claim that $M_k < 2^r$, where 2^r is the largest power of 2 dividing $k+1$. Let $P(x) = \sum_{n=0}^{k} e_n x^n$, where $e_n \in \{-1, 1\}$. If P has a zero of multiplicity at least 2^r at 1, then 1 also is a zero of the $(2^r - 1)$th derivative. Thus

$$0 = \frac{1}{(2^r-1)!}P^{(2^r-1)}(1) = \sum_{n=2^r-1}^{k} e_n \binom{n}{2^r - 1}.$$

Since $e_n = \pm 1 \equiv 1 \pmod 2$, we conclude that

$$0 = \sum_{n=2^r-1}^{k} e_k \binom{n}{2^r-1} \equiv \sum_{n=2^r-1}^{k} \binom{n}{2^r-1} = \binom{k+1}{2^r} \pmod 2.$$

By Lucas's theorem, $\binom{2^r s}{2^r}$ is odd when s is odd. Thus we have a contradiction and $M_k < 2^r$.

If $C_k \to \infty$, then for any r there exists N such that $C_k \ge 2^r$ for $k > N$. If $M_k \ge C_k$, then either $k \le N$ or 2^r divides $k+1$. Hence

$$\left|\{k\colon\ 1 \le k \le n \text{ and } M_k \ge C_k\}\right| \le N + \frac{n+1}{2^r}.$$

We conclude that

$$\limsup_{n\to\infty} \frac{1}{n}\left|\{k\colon\ 1 \le k \le n \text{ and } M_k \ge C_k\}\right| \le 2^{-r}.$$

Since the desired limit is nonnegative and r is arbitrary, the limit must be 0.

Editorial comment. The proposer noted that $(1-x)(1-x^2)\cdots(1-x^{2^{n-1}})$ is in $\mathcal{L}_{2^n-1}$, and its zero at 1 has multiplicity $n-1$. He also proved the much stronger result that for $\epsilon > 0$, and for almost every natural number k, every polynomial in $\mathcal{L}_k$ has at most $(1+\epsilon)\log k \log\log k$ zeros at 1.

Also solved by R. Chapman (U.K.), P. P. Dályay (Hungary), O. Geupel (Germany), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Simons (U.K.), GCHQ Problem Solving Group (U.K.), and the proposer.

A Triangle Inequality

11448 [2009, 647]. *Proposed by Wei-Dong Jiang, Weihai Vocational College, Weihai, China.* Let a, b, c be the side-lengths of a triangle, and let α, β, γ respectively denote half the measures of the angles opposite those sides. Show that

$$\frac{a}{b+c}\tan^2\beta\tan^2\gamma + \frac{b}{c+a}\tan^2\gamma\tan^2\alpha + \frac{c}{a+b}\tan^2\alpha\tan^2\beta \le \frac{1}{6}.$$

Solution by Cosmin Pohata, "Tudor Vianu" National College, Bucharest, Rumania. Let s be the semiperimeter of the triangle with side lengths a, b, c. Recall that $\tan\alpha = \sqrt{(s-b)(s-c)/s(s-a)}$, with similar formulas for $\tan\beta$ and $\tan\gamma$. The required inequality then becomes

$$\sum_{\text{cyc}} \frac{a}{b+c}\cdot(s-a)^2 \le \frac{s^2}{6}.$$

Let $s-a=x$, $s-b=y$, and $s-c=z$. Note that x, y, z are positive reals, with $y+z=a$, $z+x=b$, $x+y=c$. The inequality to be proved is now

$$\sum_{\text{cyc}} \frac{x^2(y+z)}{2x+y+z} \le \frac{(x+y+z)^2}{6}.$$

We show that this inequality holds for all positive real numbers x, y, z. This is equivalent to $A \le B$, where $A = 6\sum_{\text{cyc}}\left[x^2(y+z)(2y+z+x)(2x+y+z)\right]$ and $B = (x+y+z)^2\cdot\prod_{\text{cyc}}(2x+y+z)$.

Simply expanding $B-A$ yields

$$B-A = [5,0,0] + 5[4,1,0] + 6[3,1,1] - 5[2,2,1] - 7[3,2,0],$$

where $[p,q,r]$ is defined to be $\sum_{\text{sym}} x^p y^q z^r$.

By Schur's inequality, we have $[5,0,0]+[3,1,1] \ge 2[4,1,0]$. Hence

$$B-A \ge 7[4,1,0] + 5[3,1,1] - 5[2,2,1] - 7[3,2,0].$$

From Muirhead's inequality, we have $[4,1,0] \ge [3,2,0]$ and $[3,1,1] \ge [2,2,1]$. This proves that $B-A \ge 0$ and thus $A \le B$.

Editorial comment. Solvers Enkel Hysnelaj and Elton Bojaxiu point out that this problem by the same proposer appears at the RGMIA Problem Corner, at http://www.staff.vu.edu.au/RGMIA/pc.asp, with a solution by Miwa Lin.

Also solved by A. Alt, R. Bagby, D. Beckwith, E. Braune (Austria), R. Chapman (U.K.), C. Curtis, Y. Dumont (France), O. Faynshteyn (Germany), O. Geupel (Germany), J. Grivaux (France), E. Hysnelaj & E. Bojaxhiu (Australia & Germany), B.-T. Iordache (Romania), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, P. Nüesch (Switzerland), J. Oelschlager, C. R. Pranesachar (India), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), M. Vowe (Switzerland), S. Xiao (Canada), J. Zacharias, and GCHQ Problem Solving Group (U.K.).

Vandermonde Strikes Again

11459 [2009, 747]. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* Find all pairs (s, z) of complex numbers such that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!\,(n-k)!}\left(\prod_{j=1}^{k}(sj-z)\right)\left(\prod_{j=0}^{n-k-1}(sj+z)\right)$$

converges.

Solution by Oliver Geupel, Brühl, NRW, Germany. We claim that (s, z) has the desired property if and only if $|s| < 1$. Let

$$\alpha_n(s,z)=\sum_{k=0}^{n}\frac{1}{k!\,(n-k)!}\left(\prod_{j=1}^{k}(sj-z)\right)\left(\prod_{j=0}^{n-k-1}(sj+z)\right).$$

First consider $s = 0$. For $n \geq 1$ we have $\alpha_n(0, z) = (z^n/n!)\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$; hence $\sum_{n=0}^{\infty}\alpha_n(0, z)$ converges for each z. Now suppose $s \neq 0$. Applying Vandermonde's convolution formula

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n},$$

we obtain

$$\begin{aligned}\alpha_n(s,z)&=\sum_{k=0}^{n}\left(\prod_{j=1}^{k}\frac{z/s-j}{j}\right)\left(\prod_{j=0}^{n-k-1}\frac{-z/s-j}{j+1}\right)(-s)^n\\&=(-s)^n\sum_{k=0}^{n}\binom{z/s-1}{k}\binom{-z/s}{n-k}=(-s)^n\binom{-1}{n}=s^n.\end{aligned}$$

However, $\sum_{n=0}^{\infty}s^n$ converges if and only if $|s| < 1$. This completes the proof.

Also solved by M. S. Ashbaugh & F. Vial (U.S.A. & Chile), D. Beckwith, R. Chapman (U.K.), E. A. Herman, M. E. H. Ismail, O. Kouba (Syria), M. E. Larsen (Denmark), O. P. Lossers (Netherlands), D. K. Nester, J. Simons (U.K.), N. C. Singer, R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U.K.), and the proposer.

Asymptotics for an Elliptic Integral

11462 [2009, 844]. *Proposed by Nadezhda Alexandrova, Institute of Mining, Novosibirsk, Russia.* Find

$$\lim_{\alpha\to 0^+}\log\alpha+\int_{x=0}^{\pi}\frac{dx}{\sqrt{\sin^2 x+i\alpha}}.$$

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. We show that

$$\lim_{\alpha\to 0^+}\left(\log\alpha+\int_{x=0}^{\pi}\frac{dx}{\sqrt{\sin^2 x+i\alpha}}\right)=4\log 2-\frac{\pi}{2}i$$

by splitting the integrand up into two parts, one having an elementary antiderivative and the other remaining bounded as $\alpha \to 0^+$. In the calculation below, mutliple-valued functions should be interpreted as the versions which are analytic in the plane with the negative real-axis removed and that are positive on $(1, \infty)$.

By symmetry, we may write

$$\int_0^{\pi} \frac{dx}{\sqrt{\sin^2 x + i\alpha}} = 2\int_0^{\pi/2} \frac{dx}{\sqrt{\sin^2 x + i\alpha}}$$

$$= 2\int_0^{\pi/2} \frac{\cos x\, dx}{\sqrt{\sin^2 x + i\alpha}} + 2\int_0^{\pi/2} \frac{(1-\cos x)\, dx}{\sqrt{\sin^2 x + i\alpha}}.$$

On $[0, \pi/2]$ we have $|1-\cos x| \le \sin^2 x \le |\sqrt{\sin^2 x + i\alpha}|$, so by the dominated convergence theorem

$$\lim_{\alpha\to 0^+} 2\int_0^{\pi/2} \frac{(1-\cos x)\, dx}{\sqrt{\sin^2 x + i\alpha}} = 2\int_0^{\pi/2} \frac{(1-\cos x)\, dx}{\sin x} = 2\int_0^{\pi/2} \frac{\sin x\, dx}{1+\cos x}$$

$$= -2\log(1+\cos x)\big|_0^{\pi/2} = 2\log 2.$$

Recognizing the elementary antiderivative, we have

$$2\int_0^{\pi/2} \frac{\cos x\, dx}{\sqrt{\sin^2 x + i\alpha}} = 2\log\left(\sin x + \sqrt{\sin^2 x + i\alpha}\right)\Big|_0^{\pi/2}$$

$$= 2\log\left(1 + \sqrt{1+i\alpha}\right) - \log i - \log\alpha.$$

The stated result now follows immediately.

Editorial comment. Many solvers proceeded by recognizing the integral as $\frac{2}{\sqrt{i\alpha}} K\left(\frac{i}{\alpha}\right)$ and applying the known limiting behavior of the complete elliptic integrals.

Also solved by V. Adamchik, G. Apostolopoulos (Greece), K. N. Boyadzhiev, R. Chapman (U.K.), Y. Dumont (France), O. Kouba (Syria), G. Lamb, K. McInturff, M. Omarjee (France), O. G. Ruehr, J. Simons (U.K.), N. C. Singer, R. Stong, D. B. Tyler, Microsoft Research Problems Group, and the proposer.

A Generalization? Not!

11468 [2009, 940]. *Proposed by Cosmin Pohoata, Tudor Vianu National College of Informatics, Bucharest, Romania.* Let $A_1A_2A_3$ be a triangle, let $\mathcal{H}$ be a dilation mapping of the plane, and let $\mathcal{R}$ be a right angle rotation of the plane. Let P_1, P_2, and P_3 be the images under $\mathcal{H}\circ\mathcal{R}$ of A_1, A_2, and A_3, respectively, and suppose that P_1, P_2, and P_3 lie inside or on the boundary of $A_1A_2A_3$.

Let H_i for $i \in \{1,2,3\}$ be the foot of the perpendicular from P_i to the side of $A_1A_2A_3$ opposite A_i. Generalize the Erdős–Mordell inequality: show that

$$P_1A_1 + P_2A_2 + P_3A_3 \ge P_1H_2 + P_1H_3 + P_2H_3 + P_2H_1 + P_3H_1 + P_3H_2,$$

with equality if and only if $A_1A_2A_3$ is equilateral and each P_i is equal to the circumcenter of $A_1A_2A_3$.

Solution by GCHQ Problem Solving Group, Cheltenham, U.K. The result does not hold in general. Consider the isosceles right-angled triangle with $A_1 = (0,0)$, $A_2 = (0,4)$, and $A_3 = (4,0)$. Consider a dilation such that $P_1 = (0,2)$, $P_2 = (2,2)$, and $P_3 = (0,0)$. Now $H_1 = (1,3)$, $H_2 = (2,0)$, and $H_3 = (0,0)$, so $P_1A_1 + P_2A_2 + P_3A_3 = 2 + 2\sqrt{2} + 4 = 6 + 2\sqrt{2}$, but $P_1H_2 + P_1H_3 + P_2H_3 + P_2H_1 + P_3H_1 + P_3H_2 = 2\sqrt{2} + 2 + 2\sqrt{2} + \sqrt{2} + \sqrt{10} + 2 > 6 + 2\sqrt{2}$. The claim is false in this case.

Also solved by R. Stong.

A Series Estimate

11469 [2009, 940]. *Proposed by Slavko Simic, Mathematics Institute SANU, Belgrade, Serbia.* Let $\langle x_i \rangle$ be a sequence of positive numbers, and let $\langle p_i \rangle$ be a sequence of nonnegative numbers summing to 1. Let

$$A = \sum_{i=1}^{\infty} p_i x_i, \quad H = \left(\sum_{i=1}^{\infty} p_i / x_i \right)^{-1}.$$

Show that if s and t are nonnegative numbers such that $s \le \sqrt{x_i} \le s + t$ for all $i \ge 1$, then $H \le A \le t^2 + H$.

Solution by Allen Stenger, Alamogordo, NM. First note that it is enough to prove the result in the finite case; that is, when $p_i = 0$ for large i. Indeed: assume the finite case, and in the general case set

$$P_n = \sum_{i=1}^{n} p_i, \qquad A_n = \sum_{i=1}^{n} \frac{p_i}{P_i} x_i, \qquad H_n = \left(\sum_{i=1}^{n} \frac{p_i}{P_i} \frac{1}{x_i} \right)^{-1}.$$

By the finite case, $H_n \le A_n \le H_n + t^2$, and taking the limit as $n \to \infty$ yields the desired result. Also note that the limit argument applies when $\sum (p_i / x_i)$ diverges and we take $H = 0$.

Now assume $p_i = 0$ for $i > n$. Of course $H \le A$ is well known; this is a comparison of harmonic and arithmetic means. We must prove the other inequality. Write $m = \min_{i \le n} x_i$ and $M = \max_{i \le n} x_i$, so $m > 0$, $s \le \sqrt{m} \le \sqrt{M} \le s + t$, and $0 \le \sqrt{M} - \sqrt{m} \le t$.

The function $1/x$ is convex for $x > 0$, so its graph lies below the secant line passing through the points $(1/M, M)$ and $(1/m, m)$. The equation of this line is $y = L(x)$, where $L(x) = M + m - Mmx$, so $x \le L(1/x)$ for $m \le x \le M$. Then

$$A = \sum p_i x_i \le \sum p_i L(x_i^{-1}) = M + m - Mm \sum \frac{p_i}{x_i} = L\left(\sum \frac{p_i}{x_i} \right) = L\left(\frac{1}{H} \right).$$

From $m \le x_i \le M$ we have $m \le H \le M$, and therefore

$$A - H \le L\left(\frac{1}{H} \right) - H \le \max_{m \le x \le M} \left[L\left(\frac{1}{x} \right) - x \right].$$

This maximum occurs when $x = \sqrt{Mm}$, where the value is $M + m - 2\sqrt{Mm}$, which equals $(\sqrt{M} - \sqrt{m}\,)^2 \le t^2$.

Also solved by P. P. Dályay (Hungary), J. Grivaux (France), S. J. Herschkorn, O. P. Lossers (Netherlands), Á. Plaza (Spain), B. Schmuland (Canada), J. Simons (U.K.), N. C. Singer, R. Stong, GCHQ Problem Solving Group (U.K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before October 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11579. *Proposed by Hallard Croft, University of Cambridge, Cambridge, U. K., and Sateesh Mane, Convergent Computing, Shoreham, NY.* Let m and n be distinct integers, with $m, n \geq 3$. Let B be a fixed regular n-gon, and let A be the largest regular m-gon that does not extend beyond B. Let $d = \gcd(m, n)$, and assume $d > 1$. Show that:
(**a**) A and B are concentric.
(**b**) If $m \mid n$, then A and B have m points of contact, consisting of all the vertices of A.
(**c**) If $m \nmid n$ and $n \nmid m$, then A and B have $2d$ points of contact.
(**d**) A and B share exactly d common axes of symmetry.

11580. *Proposed by David Alfaya Sánchez, Universidad Autónoma de Madrid, Madrid, Spain, and José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.* For $n \geq 2$, let $a_1, \dots, a_n$ be positive numbers that sum to 1, let $E = \{1, \dots, n\}$, and let $F = \{(i, j) \in E \times E \colon i < j\}$. Prove that

$$\sum_{(i,j)\in F} \frac{(a_i - a_j)^2 + 2a_i a_j (1 - a_i)(1 - a_j)}{(1 - a_i)^2 (1 - a_j)^2} + \sum_{i \in E} \frac{(n+1)a_i^2 + n a_i}{(1 - a_i)^2} \geq \frac{n^2(n+2)}{(n-1)^2}.$$

11581. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous, nonconstant function from $[0, 1]$ to $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = 0$. Also, let $m = \min_{0\leq x\leq 1} f(x)$ and $M = \max_{0\leq x\leq 1} f(x)$. Prove that

$$\left| \int_0^1 x f(x)\,dx \right| \leq \frac{-mM}{2(M - m)}.$$

11582. *Proposed by Aleksandar Ilić, University of Niš, Serbia.* Let n be a positive integer, and consider the set S_n of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1} a_i$ with $a_1, \dots, a_k$ being positive integers that sum to n. Find S_n.

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11583. *Proposed by David Beckwith, Sag Harbor, NY.* The instructions for a magic trick are as follows: Pick a positive integer n. Next, list all partitions of n as nondecreasing strings—for instance, with $n = 3$, the list is $\{111, 12, 3\}$. Count 1 point for the string (n). For the string $\lambda_1 \cdots \lambda_k$ with $k > 1$, count $\prod_{j=1}^{k-1} \binom{\lambda_{j+1}}{\lambda_j}$ points. Add up your points, take the log base 2 of that, and add 1. Voilà! n. Explain.

11584. *Proposed by Raymond Mortini and Jérôme Noël, Université Paul Verlaine, Metz, France.* Let $\langle a_j \rangle$ be a sequence of nonzero complex numbers inside the unit circle such that $\prod_{k=1}^{\infty} |a_k|$ converges. Prove that

$$\left| \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{a_j} \right| \le \frac{1 - \prod_{j=1}^{\infty} |a_j|^2}{\prod_{j=1}^{\infty} |a_j|}.$$

11585. *Proposed by Bruce Burdick, Roger Williams University, Bristol, RI.* Show that

$$\sum_{k=3}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-2} \zeta(k-m)\zeta(m+1) - k \right) = 3 + \gamma^2 + 2\gamma_1 - \frac{\pi^2}{3}.$$

Here, ζ denotes the Riemann zeta function, γ is the Euler-Mascheroni constant, given by $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} 1/k - \log(n) \right)$, and γ_1 is the first Stieltjes constant, given by $\gamma_1 = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\log k}{k} - \frac{1}{2}(\log n)^2 \right)$.

SOLUTIONS

Extrema On the Edge

11449 [2009, 647]. *Proposed by Michel Bataille, Rouen, France.* (corrected) Find the maximum and minimum values of

$$\frac{(a^3 + b^3 + c^3)^2}{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}$$

given that $a + b \ge c > 0$, $b + c \ge a > 0$, and $c + a \ge b > 0$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO. Let F be the expression to be maximized. The maximum of F in the feasible region is 2, attained when $a = b = 1$ and $c = 2$, as well as at permutations and scalings of this.

Let $H = 2(b^2 + c^2)(c^2 + a^2)(a^2 + b^2) - (a^3 + b^3 + c^3)^2$. Since $F \le 2$ is equivalent to $H \ge 0$, we prove the latter. By symmetry, we may assume that $a \le b \le c$. By homogeneity, we may take $a = 1$. Hence, we can write $b = 1 + x$ and $c = 1 + x + y$ with $x, y \ge 0$. Since $a + b \ge c$, we have $y \le 1$. Expanding H as a polynomial in x with coefficients that are polynomials in y gives the following expansion:

$$\begin{aligned} H = {} & x^4[1 + 7(1+y)(1-y)] + 2x^3[1 + (1-y)(7y^2 + 21y + 13)] \\ & + x^2[1 + (1+y)(1-y)(13y^2 + 42y + 39)] \\ & + 2x(1+y)(1-y)(3y+7)(y^2 + 2y + 2) + (1+y)^2(1-y)(y^3 + 5y^2 + 7y + 7), \end{aligned}$$

which is evidently nonnegative. It is 0 if and only if $x = 0$ and $y = 1$. This corresponds to $(a, b, c) = (1, 1, 2)$.

Also solved by R. Agnew, A. Alt, M. Ashbaugh, R. Bagby, D. Beckwith, H. Caerols & R. Pellicer (Chile), R. Chapman (U. K.), H. Chen, C. Curtis,P. P. Dályay (Hungary), Y. Dumont (France), J. Fabrykowski and T. Smotzer, S. Falcón and Á. Plaza (Spain), D. Fleischman, J.-P. Grivaux (France), E. A. Herman, F. Holland (Ireland), T. Konstantopoulos (U. K.), O. Kouba (Syria), A. Lenskold, J. H. Lindsey II, B. Mulansky (Germany), P. Perfetti (Italy), C. R. Pranesachar (India), N. C. Singer, R. Stong, T. Tam, R. Tauraso (Italy), M. Tetiva (Romania), D. Tyler, E. I. Verriest, Z. Vörös (Hungary), S. Wagon, G. D. White, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Editorial comment. Two versions of this problem appeared; the first was not what the proposer intended. The treatment of the upper bound given in the March issue of this column (p. 278) fails as a solution to the corrected version. The maximum of F in the closure of the feasible region is attained not only at a corner, which is off-limits, but also at the other boundary points noted. The solver list here includes those who had supplied solutions under a new deadline. The editors regret the confusion.

Hexagon Inscribed in Circle

11470 [2009, 491]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let $ABCDEF$ be a hexagon inscribed in a circle. Let M, N, and P be the midpoints of the line segments BC, DE, and FA, respectively, and similarly let Q, R, and S be the midpoints of AD, BE, and CF. Show that if both MNP and QRS are equilateral, then the segments AB, CD, and EF have equal lengths.

Solution by Oliver Geupel, Brühl, NRW, Germany. Let the circle be the unit circle in the complex plane, and let $a, b, c, \ldots$ be the complex numbers corresponding to $A, B, C, \ldots$. Thus $2m = b + c$, $2n = d + e$, $2p = f + a$, $2q = a + d$, $2r = b + e$, and $2s = c + f$. Write $\epsilon = \exp(2\pi i/3)$. It is well known (for example: T. Andreescu and T. Andrica, *Complex Numbers from A to Z*, Birkhäuser, Boston, 2006, pp. 70*ff.*, Proposition (3.4)1) that a triangle UVW is equilateral if and only if $u + \epsilon v + \epsilon^2 w = 0$ or $u + \epsilon w + \epsilon^2 v = 0$, depending on the orientation of $\triangle UVW$. Without loss of generality, we may assume that $\triangle MNP$ is oriented so that $m + \epsilon n + \epsilon^2 p = 0$. Hence

$$(b + c) + \epsilon(d + e) + \epsilon^2(f + a) = 0. \tag{1}$$

We consider two cases, depending on the orientation of $\triangle QRS$.

Case 1: $\triangle MNP$ and $\triangle QRS$ have opposite orientation. In this case

$$(a + d) + \epsilon(c + f) + \epsilon^2(b + e) = 0. \tag{2}$$

Multiplying (1) by $\frac{-1-\epsilon+\epsilon^2}{2(\epsilon-1)}$, multiplying (2) by $\frac{-1+\epsilon+\epsilon^2}{2(\epsilon-1)}$, and adding, we obtain $a + \epsilon c + \epsilon^2 e = 0$. Multiplying (1) by $\frac{-1+\epsilon+\epsilon^2}{2(\epsilon-1)}$, multiplying (2) by $\frac{-1-\epsilon+\epsilon^2}{2(\epsilon-1)}$, and adding, we obtain $b + \epsilon d + \epsilon^2 f = 0$. Thus $\triangle ACE$ and $\triangle BDF$ are equilateral, which implies $AB = CD = EF$.

Case 2: $\triangle QRS$ has the same orientation as $\triangle MNP$. Now

$$(b + e) + \epsilon(c + f) + \epsilon^2(a + d) = 0. \tag{3}$$

Multiplying (1) by $\frac{1}{1-\epsilon}$, multiplying (3) by $-\frac{1}{1-\epsilon}$, and adding, we obtain $c - e = \epsilon(f - d)$. Therefore $CE = DF$, so $CD = EF$. Multiplying (1) by $\frac{\epsilon^2}{1-\epsilon}$, multiplying (3) by $-\frac{1}{1-\epsilon}$, and adding, we obtain $e - a = \epsilon(f - b)$. Therefore $EA = FB$, so $EF = AB$.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), M. Garner, M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), S. W. Kim (Korea), O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), R. Stong, S. Tonegawa & F. Vafa, and the proposer.

Product of Derivatives

11472 [2009, 941]. *Proposed by Mahdi Makhul, Shahrood University of Technology, Shahrood, Iran.* Let t be a nonnegative integer, and let f be a $(4t+3)$-times continuously differentiable function on $\mathbb{R}$. Show that there is a number a such that at $x = a$,

$$\prod_{k=0}^{4t+3} \frac{d^k f(x)}{dx^k} \geq 0.$$

Solution by Robin Chapman, University of Bristol, Bristol, England, U. K. We first claim that if g is a twice-differentiable function on $\mathbb{R}$, then there exists $b \in \mathbb{R}$ such that $g(b)g''(b) \geq 0$. To prove this, suppose that $g(x)g''(x) < 0$ for all $x \in \mathbb{R}$. Now $g(x) \neq 0$ for all $x \in \mathbb{R}$. Since g is continuous, g has constant sign. Hence, g'' has the opposite sign. Suppose that g is positive and g'' is negative (otherwise consider $-g$ in place of g). Hence g' is decreasing, and there exists $c \in \mathbb{R}$ with $g'(c) \neq 0$. By Taylor's theorem, for each $x \in \mathbb{R}$,

$$g(x) = g(c) + (x-c)g'(c) + \frac{(x-c)^2}{2} g''(\xi),$$

where ξ is between c and x. Since g'' is negative,

$$g(x) \leq g(c) + (x-c)g'(c).$$

Depending on the sign of $g'(c)$, this implies that $g(x) < 0$ for all large enough x or for all small enough x. Either way we have a contradiction. Hence there exists $b \in \mathbb{R}$ with $g(b)g''(b) \geq 0$.

Now let f be a $(4t+3)$-times continuously differentiable function on $\mathbb{R}$. Let $F(x) = \prod_{j=0}^{4t+3} f^{(j)}(x)$. If F is always negative, then F is always nonzero, so each $f^{(j)}$ with $0 \leq j \leq 4t+3$, since it is continuous, has constant sign. From the foregoing, $f^{(j)}$ and $f^{(j+2)}$ must have the same sign for $0 \leq j \leq 4t+1$. Therefore $\prod_{j=0}^{2t+1} f^{(2j)}$ and $\prod_{j=0}^{2t+1} f^{(2j+1)}$ are both positive, so F is positive, a contradiction.

Editorial comment. The special case $t = 0$ of this problem was problem A3 on the 1998 Putnam exam.

Also solved by G. Apostolopoulos (Greece), P. P. Dályay (Hungary), J.-P. Grivaux (France), O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), J. Simons (U. K.), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), X. Wang, GCHQ Problem Solving Group (U. K.), and the proposer.

A Series Equation

11473 [2009, 941]. *Proposed by Paolo Perfetti, Mathematics Dept., University "Tor Vergata Roma," Rome, Italy.* Let α and β be real numbers such that $-1 < \alpha + \beta < 1$ and such that, for all integers $k \geq 2$,

$$-(2k)\log(2k) \neq \alpha, \qquad (2k+1)\log(2k+1) \neq \alpha,$$
$$1 + (2k+1)\log(2k+1) \neq \beta, \qquad -1 - (2k+2)\log(2k+2) \neq \beta.$$

Let

$$T = \lim_{N\to\infty} \sum_{n=2}^{N} \prod_{k=2}^{n} \frac{\alpha + (-1)^k \cdot k\log(k)}{\beta + (-1)^{k+1}(1 + (k+1)\log(k+1))},$$

$$U = \lim_{N\to\infty} \sum_{n=2}^{N} ((n+1)\log(n+1)) \prod_{k=2}^{n} \frac{\alpha + (-1)^k \cdot k\log(k)}{\beta + (-1)^{k+1}(1 + (k+1)\log(k+1))}.$$

(**a**) Show that the limits defining T and U exist.
(**b**) Show that if, moreover, $|\alpha| < 1/2$ and $\beta = -\alpha$, then $T = -2U$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.
(**a**) The series for T and U are eventually alternating in sign, so for convergence it suffices to prove that the absolute value of the term decreases eventually and converges to zero. Since $(n+1)\log(n+1)$ is an increasing function of n, it suffices to prove this for U only. The negative of the quotient of two consecutive terms is

$$\frac{(n+1)\log(n+1)}{n\log n} \cdot \frac{(-1)^n\alpha + n\log n}{1 + (-1)^{n+1}\beta + (n+1)\log(n+1)}.$$

With the abbreviation $x_n = n\log n$, this expression can be written as

$$1 - \frac{1-(-1)^n(\alpha+\beta)}{x_n} + \left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right)(-1)^n(\alpha+\beta) + \mathrm{O}(x_n^{-2}).$$

Since $1/x_{n+1} - 1/x_n = \mathrm{O}(n^{-1}x_n^{-1})$ and $|\alpha+\beta| < 1$, this has the form $1 - c_n$ with $1 > c_n > \frac{1}{2}(1-|\alpha+\beta|)/x_n$ eventually. Therefore $\prod_{k=1}^n |1-c_k|$ is eventually decreasing. Also, since $\sum x_n^{-1}$ diverges, the product goes to zero. This proves that the limit for U, and hence also for T, exists.

(**b**) The equation $T = -2U$ is incorrect. Let $p_k = (-1)^k\alpha + x_k$ and $q_k = (-1)^{k+1}\beta + 1 + x_{k+1}$. If $\alpha+\beta = 0$, then the partial sums for $T + 2U$ can be written as

$$\sum_{n=2}^{N}(-1)^{n+1}(q_n + p_{n+1})\prod_{k=2}^{n}\frac{p_k}{q_k} = \sum_{n=2}^{N}(-1)^{n+1}\left(\frac{\prod_{k=2}^{n} p_k}{\prod_{k=2}^{n-1} q_k} + \frac{\prod_{k=2}^{n+1} p_k}{\prod_{k=2}^{n} q_k}\right).$$

This is a telescoping sum that simplifies to

$$-p_2 + (-1)^{N+1}p_{N+1}\prod_{k=2}^{N}\frac{p_k}{q_k}.$$

From the convergence of T and U, it follows that the second term goes to zero as N tends to infinity. Thus

$$T + 2U = -\alpha - 2\log 2.$$

Also solved by O. Kouba (Syria), R. Stong, and the GCHQ Problem Solving Group (U. K.).

An Inequality for Triangles

11476 [2010, 86]. *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy.* Let a, b, and c be the side-lengths of a triangle, and let r be its inradius. Show

$$\frac{a^2bc}{(b+c)(b+c-a)} + \frac{b^2ca}{(c+a)(c+a-b)} + \frac{c^2ab}{(a+b)(a+b-c)} \geq 18r^2.$$

Solution by P. Nüesch, Lausanne, Switzerland. Write s for the semiperimeter of the triangle. The left side of the inequality is (employing geometry's cyclic summation conventions)

$$\sum \frac{a^2bc}{(b+c)(b+c-a)} = \frac{abc}{2}\sum\frac{a}{(2s-a)(s-a)}.$$

The function f defined by

$$f(x) = \frac{x}{(2s-x)(s-x)}$$

is convex for $0 < x < s$. Setting $x_1 = a$, $x_2 = b$, $x_3 = c$ yields

$$\sum \frac{a}{(2s-a)(s-a)} = \sum f(x_i) \ge 3f\left(\frac{\sum x_i}{3}\right) = 3f\left(\frac{2s}{3}\right) = \frac{9}{2s}.$$

Together with $abc = 4Rrs$ and Euler's inequality $R \ge 2r$, we obtain

$$\frac{abc}{2}\sum \frac{a}{(2s-a)(s-a)} \ge \frac{abc}{2}\,\frac{9}{2s} = 9Rr \ge 18r^2.$$

Also solved by A. Alt, G. Apostolopoulos (Greece), R. Bagby, D. Beckwith, E. Bráune (Austria), R. Chapman (U. K.), P. P. Dályay (Hungary), J. Fabrykowski & T. Smotzer, H. Y. Far, O. Faynshteyn (Germany), V. V. Garcia (Spain), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, Á. Plaza & S. Falcón (Spain), C. Pohoata (Romania), C. R. Pranesachar (India), R. Stong, E. Suppa (Italy), M. Tetiva (Romania), M. Vowe (Switzerland), L. Wimmer (Germany), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

The Winding Density of a Non-Closing Poncelet Trajectory

11479 [2010, 87]. *Proposed by Vitaly Stakhovsky, National Center for Biotechnological Information, Bethesda, MD.* Two circles are given. The larger circle C has center O and radius R. The smaller circle c is contained in the interior of C and has center o and radius r. Given an initial point P on C, we construct a sequence $\langle P_k\rangle$ (the *Poncelet trajectory* for C and c starting at P) of points on C: Put $P_0 = P$, and for $j \ge 1$, let P_j be the point on C to the right of o as seen from P_{j-1} on a line through P_{j-1} and tangent to c. For $j \ge 1$, let ω_j be the radian measure of the angle counterclockwise along C from P_{j-1} to P_j. Let

$$\Omega(C, c, P) = \lim_{k\to\infty} \frac{1}{2\pi k}\sum_{j=1}^{k} \omega_j.$$

(**a**) Show that $\Omega(C, c, P)$ exists for all allowed choices of C, c, and P, and that it is independent of P.
(**b**) Find a formula for $\Omega(C, c, P)$ in terms of r, R, and the distance d from O to o.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We will show

$$\Omega(C, c, P) = \frac{F\left(\frac{1}{2}\arccos\frac{r-d}{R}\,\Big|\, m\right)}{K(m)}, \text{ where } m = \frac{4dR}{(R+d)^2 - r^2},$$

which is independent of P. We have used the incomplete elliptic integral of the first kind, defined by

$$F(\theta|m) = \int_0^{\theta} \frac{dt}{\sqrt{1 - m\sin^2 t}} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}\sqrt{1-my^2}},$$

and the corresponding complete integral $K(m) = F(\pi/2|m)$.

Use coordinates with c centered at the origin and C centered on the nonnegative x-axis. Parameterize c as $T(\theta) = (r\cos\theta, r\sin\theta)$ and C as $P(\phi) = (d +$

$R\cos\phi, R\sin\phi)$. Then $\|T(\theta)\|^2 = \|T'(\theta)\|^2 = r^2$ and $\langle T'(\theta), T(\theta)\rangle = 0$. The tangent line to c at $T(\theta)$ is given by $\langle X, T(\theta)\rangle = r^2$ and a point X on the tangent can be written as

$$X = T(\theta) \pm \frac{\sqrt{\|X\|^2 - r^2}}{r}\, T'(\theta),$$

using the $+$ sign if X is counterclockwise from $T(\theta)$ and the $-$ sign if X is clockwise from $T(\theta)$ as viewed from the origin.

For any two points $P(\phi_1)$ and $P(\phi_2)$ on C we have

$$P(\phi_1) - P(\phi_2) = 2\sin\left(\frac{\phi_1 - \phi_2}{2}\right)\left(-R\sin\left(\frac{\phi_1+\phi_2}{2}\right), R\cos\left(\frac{\phi_1+\phi_2}{2}\right)\right),$$

$$P'(\phi_1) + P'(\phi_2) = 2\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\left(-R\sin\left(\frac{\phi_1+\phi_2}{2}\right), R\cos\left(\frac{\phi_1+\phi_2}{2}\right)\right).$$

Hence these two vectors are parallel.

For a point $T(\theta)$ on the circle c, write $P(\phi_-)$ and $P(\phi_+)$ for the two points where the tangent to c at $T(\theta)$ meet C with ϕ_+ counterclockwise from $T(\theta)$ and $\phi_- < \phi_+ < \phi_- + 2\pi$. Then $\langle P(\phi_\pm), T(\theta)\rangle = r^2$ so $\langle P(\phi_+) - P(\phi_-), T(\theta)\rangle = 0$ and hence $\langle P'(\phi_+) + P'(\phi_-), T(\theta)\rangle = 0$. Now suppose we traverse the circle c so that

$$\frac{d\theta}{dt} = \langle P'(\phi_-), T(\theta)\rangle = -\langle P'(\phi_+), T(\theta)\rangle.$$

This makes $d\theta/dt > 0$, so we traverse c in counterclockwise order. Then from

$$0 = \frac{d}{dt}\langle P(\phi_\pm), T(\theta)\rangle = \langle P'(\phi_\pm), T(\theta)\rangle \frac{d\phi_\pm}{dt} + \langle P(\phi_\pm), T'(\theta)\rangle \frac{d\theta}{dt}$$

we see

$$\begin{aligned}\frac{d\phi_\pm}{dt} &= \pm\langle P(\phi_\pm), T'(\theta)\rangle \\ &= r\sqrt{\|P(\phi_\pm)\|^2 - r^2} = r\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi_\pm}.\end{aligned}$$

Thus the elliptic integral I given by

$$I = \int_{\phi_-}^{\phi_+} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}}$$

satisfies

$$\begin{aligned}\frac{dI}{dt} &= \frac{1}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi_+}}\frac{d\phi_+}{dt} \\ &\quad - \frac{1}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi_-}}\frac{d\phi_-}{dt} \\ &= r - r = 0\end{aligned}$$

and is a constant. One possible chord is the vertical one through the point $(r, 0)$ with $\theta = 0$, $\phi_\pm = \pm\arccos((r-d)/R)$, so we obtain

$$\begin{aligned}I &= 2\int_0^{\arccos((r-d)/R)} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}} \\ &= \frac{4}{\sqrt{(R+d)^2 - r^2}}\; F\left(\frac{1}{2}\arccos\frac{r-d}{R} \,\middle|\, \frac{4dR}{(R+d)^2 - r^2}\right).\end{aligned}$$

Let

$$J = \int_0^{2\pi} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}}$$
$$= \frac{4}{\sqrt{(R+d)^2 - r^2}}\, K\left(\frac{4dR}{(R+d)^2 - r^2}\right).$$

Now suppose $P_0 = (d + R\cos\phi_0, R\sin\phi_0)$ and let $\phi_k = \phi_0 + \sum_{j=1}^{k} \omega_j$. We have

$$\int_{\phi_0}^{\phi_k} \frac{d\phi}{\sqrt{R^2 + d^2 - r^2 + 2dR\cos\phi}} = kI.$$

This integral is over an interval of at least $\lfloor(\phi_k - \phi_0)/(2\pi)\rfloor$ complete periods and fewer than $\lceil(\phi_k - \phi_0)/(2\pi)\rceil$ complete periods. Hence

$$\left\lfloor \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rfloor J \le kI \le \left\lceil \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rceil J.$$

Thus

$$\frac{I}{J} - \frac{1}{k} \le \frac{1}{k}\left(\left\lceil \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rceil - 1\right) \le \frac{\sum_{j=1}^{k} \omega_j}{2\pi k} \le \frac{1}{k}\left(\left\lfloor \frac{\sum_{j=1}^{k} \omega_j}{2\pi} \right\rfloor + 1\right) \le \frac{I}{J} + \frac{1}{k}$$

and

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k} \omega_j}{2\pi k} = \frac{I}{J},$$

which is the quotient of elliptic integrals claimed.

Editorial comment.

In the classical case, when the trajectory closes—returns to its starting point after finitely many steps—this "winding density" is rational: the number of times the closed trajectory goes around the circle divided by the number of intervals in the trajectory. The use of elliptic integrals to compute it is known, and in many special cases it can be computed without elliptic integrals: see http://mathworld.wolfram.com/PonceletsPorism.html.

Also solved by J. A. Grzesik, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before December 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11586. *Proposed by Takis Konstantopoulos, Uppsala University, Uppsala, Sweden.* Let A_0, B_0, and C_0 be noncollinear points in the plane. Let p be a line that meets lines B_0C_0, C_0A_0, and A_0B_0 at A^*, B^*, and C^* respectively. For $n \geq 1$, let A_n be the intersection of B^*B_{n-1} with C^*C_{n-1}, and define B_n, C_n similarly. Show that all three sequences converge, and describe their respective limits.

11587. *Proposed by Andrei Ciupan, Harvard University, Cambridge, MA, and Bozgan Francisc, UCLA, Los Angeles, CA.* For which pairs (a, b) of positive integers do there exist infinitely many positive integers n such that n^2 divides $a^n + b^n$?

11588. *Proposed by Taras Banakh, Ivan Franko National University of Lviv, Lviv, Ukraine, and Igor Protasov, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine.* Show that $\mathbb{R} - \{0\}$ can be partitioned into countably many subsets, each of which is linearly independent over $\mathbb{Q}$, if and only if the continuum hypothesis holds.

11589. *Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.* Let P be a polynomial over $\mathbb{R}$ given by $P(x) = x^3 + a_2x^2 + a_1x + a_0$, with $a_1 > 0$. Show that P has a least one zero between $-a_0/a_1$ and $-a_2$.

11590. *Proposed by Khodakhast Bibak, University of Waterloo, Waterloo, Ontario, Canada.* Let m balls numbered 1 to m each be painted with one of n colors, with $n \geq 2$ and at least two balls of each color. For each positive integer k, let $P(k)$ be the number of ways to put these balls into urns numbered 1 through k so that no urn is empty and no urn gets two or more balls of the same color. Prove that

$$\sum_{k=1}^{m} \frac{(-1)^k}{k} P(k) = 0.$$

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11591. *Proposed by Dan White and Lenny Jones, Shippensburg University, Shippensburg, PA.* Let I_n be the set of all idempotent elements of $\mathbb{Z}/n\mathbb{Z}$. That is, $e \in I_n$ if and only if $e^2 \equiv e \pmod{n}$. Let $I_n^1 = I_n$, and for $k \geq 2$, let I_n^k be the set of all sums of the form $u + v$ where $u \in I_n$, $v \in I_n^{k-1}$, and the addition is done modulo n. Determine, in terms of n, the least k such that $I_n^k = \mathbb{Z}/n\mathbb{Z}$.

11592. *Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Find $\lim_{n\to\infty} \left(-\log(n) + \sum_{k=1}^n \arctan 1/k\right)$.

SOLUTIONS

A Telescoping Sum of Floors

11444 [2009, 548]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let k and s be positive integers with $s \leq k$. Let $f(n) = n - s\lfloor n/k \rfloor$. For $j \geq 0$, let f^j denote the j-fold composition of f, taking f^0 to be the identity function. Show that

$$\sum_{j=0}^{\infty} \left\lfloor \frac{f^j(n)}{k} \right\rfloor = -\left\lfloor \frac{q-n}{s} \right\rfloor,$$

where $q = \min\{k-1, n\}$.

Solution by Robin Chapman, University of Exeter, Exeter, U.K. We prove that the formula holds for nonnegative n. The formula as stated fails for negative n; we correct it. For $j \geq 0$, let $n_j = f^j(n)$, $a_j = \lfloor n_j/k \rfloor$, and

$$S(n) = \sum_{j=0}^{\infty} \left\lfloor \frac{f^j(n)}{k} \right\rfloor = \sum_{j=0}^{\infty} \left\lfloor \frac{n_j}{k} \right\rfloor = \sum_{j=0}^{\infty} a_j.$$

Consider first the case of nonnegative n. Clearly $f(n) = n$ if $0 \leq n \leq k-1$ and $f(n) < n$ if $n \geq k$. Also, $f(n) = n - s\lfloor n/k \rfloor \geq n - sn/k \geq 0$. Hence $\{n_j\}_{j\geq 0}$ is nonincreasing and reaches its integer limit. Since $a_j = (n_j - n_{j+1})/s$, the sum $S(n)$ thus has only finitely many nonzero terms. That is, there exists N such that

$$S(n) = \sum_{j=0}^{N} a_j = \frac{1}{s}\sum_{j=0}^{N} \left(n_j - n_{j+1}\right) = \frac{n_0 - n_{N+1}}{s}.$$

If $0 \leq n < k$, then $n_j = n$ for all j and $S(n) = 0$. If $n \geq k$, then $f(n) \geq (1 - s/k)n \geq (1 - s/k)k \geq k - s$, and $n_j \geq k - s$ for all j. Hence n_{N+1} is the unique integer m with $k - s \leq m < k$ that is congruent to n modulo s. That is, $m = n - s\lfloor (n-k+s)/s \rfloor$. Thus for $n \geq k$,

$$S(n) = \lfloor (n-k+s)/s \rfloor = -\lfloor (k-1-n)/s \rfloor = -\lfloor (q-n)/s \rfloor.$$

If $0 \leq n < k$, then $q = n$ and again $S(n) = 0 = -\lfloor (q-n)/s \rfloor$.

We now prove that $S(n) = \lfloor n/s \rfloor$ when n is a negative integer. In this case, $n < f(n) < n - s(n/k - 1) = n(1 - s/k) + s \leq s$. Thus $\{n_j\}_{j\geq 0}$ increases until it reaches a value m between 0 and $s-1$, after which it is stationary. Hence $m = n - s\lfloor n/s \rfloor$, and $S(n) = (n-m)/s = \lfloor n/s \rfloor$.

Also solved by D. Beckwith, P. P. Dályay (Hungary), D. Gove, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Simons (U. K.), R. Stong, S. Xiao (Canada), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

A Positive Sequence

11445 [2009, 548]. *Proposed by H. A. ShahAli, Tehran, Iran.* Given $a, b, c > 0$ with $b^2 > 4ac$, let $\langle \lambda_n \rangle$ be a sequence of real numbers, with $\lambda_0 > 0$ and $c\lambda_1 > b\lambda_0$. Let $u_0 = c\lambda_0$, $u_1 = c\lambda_1 - b\lambda_0$, and for $n \geq 2$ let $u_n = a\lambda_{n-2} - b\lambda_{n-1} + c\lambda_n$. Show that if $u_n > 0$ for all $n \geq 0$, then $\lambda_n > 0$ for all $n \geq 0$.

Solution I by J. C. Linders, Eindhoven, The Netherlands. Since $u_0 > 0$ and $u_1 > 0$, both λ_0 and λ_1 are positive. We show by induction on n that

$$c\lambda_n > \frac{n+1}{2n} b\lambda_{n-1} \quad \text{and} \quad \lambda_n > 0$$

for $n \geq 1$. Since $u_1 > 0$, this holds for $n = 1$. In general, $u_{n+1} > 0$ and the induction hypothesis imply for $n \geq 1$ that

$$\begin{aligned} c\lambda_{n+1} &> b\lambda_n - a\lambda_{n-1} > b\lambda_n - a\frac{2n}{n+1} \cdot \frac{c}{b}\lambda_n = \left(1 - \frac{2nac}{b^2(n+1)}\right) b\lambda_n \\ &> \left(1 - \frac{n}{2(n+1)}\right) b\lambda_n = \frac{n+2}{2(n+1)} b\lambda_n, \end{aligned}$$

where the last inequality follows from $b^2 > 4ac$ and $\lambda_n > 0$. This proves the two inequalities in the claim for $n + 1$.

Solution II by David Beckwith, Sag Harbor, NY. Define generating functions by letting $U(x) = \sum_{n=0}^{\infty} u_n x^n$ and $\Lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n$. The recursion yields $U(x) = (ax^2 - bx + c)\Lambda(x)$. The conditions on a, b, and c imply $ax^2 - bx + c = a(x - \rho_+)(x - \rho_-)$, where ρ_+ and ρ_- are the real and positive roots of $ax^2 - bx + c$. Thus

$$\begin{aligned} \Lambda(x) &= \frac{1}{ax^2 - bx + c} U(x) \;=\; \frac{1}{c(1 - \frac{x}{\rho_+})(1 - \frac{x}{\rho_-})} U(x) \\ &= \frac{1}{c}\left(\sum_{n=0}^{\infty} \frac{1}{\rho_+^n} x^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{\rho_-^n} x^n\right)\left(\sum_{n=0}^{\infty} u_n x^n\right). \end{aligned}$$

Since the product of three power series with all positive coefficients is a power series with all positive coefficients, it follows that $\lambda_n > 0$ for all n.

Editorial comment. From the proofs above, the claim also holds when $b^2 = 4ac$. O. P. Lossers showed also that the condition $a > 0$ is superfluous.

Also solved by R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), Y. Dumont (France), P. J. Fitzsimmons, D. Fleischman, M. Goldenberg & M. Kaplan, E. A. Herman, E. Hysnelaj & E. Bojaxhiu (Australia & Germany), T. Konstantopoulos (U. K.), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. C. Rhoades, C. R. & S. Selvaraj, J. Simons (U. K.), S. Song (Korea), R. Stong, M. Tetiva (Romania), E. I. Verriest, Z. Vörös (Hungary), L. Zhou, Fisher Problem Solving Group, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Matrices Whose Products Are All Different

11446 [2009, 647]. *Proposed by Christopher Hillar, Mathematical Research Sciences Institute, Berkeley, CA, and Lionel Levine, Massachusetts Institute of Technology,* *Cambridge, MA.* Prove or disprove: there exist 2×2 symmetric integer matrices A and B such that no element of the multiplicative semigroup generated by A and B can be written in two different ways. (Thus, A, B, AA, AB, BA, BB, AAA, AAB, ... are all different.)

Solution by Reiner Martin, Bad Soden-Neuenhain, Germany. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

No element of the multiplicative group generated by A and B can be written in two ways: When v is a column vector with two entries, both positive, the first entry of Av is larger than the second, and the first entry of Bv is smaller than the second. Therefore, when two products in A and B are equal, the first factor in the two products is the same. Since A and B are invertible, the products of the remaining factors must be the same. The claim follows by induction on the number of factors in the product.

Also solved by V. D. Blondel, R. Chapman (U. K.), C. Curtis, C. Delorme (France), O. Geupel (Germany), J. Grivaux (France), A. Ilić (Serbia), O. P. Lossers (Netherlands), V. S. Miller, R. Stong, J. V. Tejedor (Spain), A. Wyn-jones (U. K.), BSI Problems Group (Germany), Microsoft Research Problems Group, and the proposers.

A Sufficient Condition for a Division Ring

11451 [2009, 648]. *Proposed by Greg Oman, Otterbein College, Westerville, OH.* Let k and n be positive integers, with $k > 1$. Let R be a ring, not assumed to have an identity, with the following properties:

(i) There is an element of R that is not nilpotent.

(ii) If $x_1, \dots, x_k$ are nonzero elements of R, then $\sum_{j=1}^{k} x_j^n = 0$.

Show that R is a *division ring*, that is, the nonzero elements of R form a group under multiplication.

Solution by the NSA Problems Group, Fort Meade, MD. Take $a, x \in R$ with a non-nilpotent and x nonzero. With all x_j set to x in (ii), we obtain $kx^n = 0$. With $x_i = x$ for $i < k$ and $x_k = a$, we obtain $(k-1)x^n + a^n = 0$. Hence, $x^n = a^n$ for every nonzero $x \in R$. Let $e = a^n$; setting $x = a^2$ yields $e^2 = e$. Furthermore, $x^n = e \neq 0$ shows that R has no nonzero nilpotent elements.

We claim that e is the identity in R. First, $ex = a^n x = x^n x = x x^n = x a^n = xe$. Next, expand $(x - ex)^n$ by the binomial theorem, which applies since e commutes with every element of R. We obtain

$$\begin{aligned}(x - ex)^n &= \sum_{j=0}^{n} \binom{n}{j} x^j (-ex)^{n-j} = x^n + \sum_{j=0}^{n-1} \binom{n}{j} x^j e(-x)^{n-j} \\ &= x^n + e \sum_{j=0}^{n} \binom{n}{j} x^j (-x)^{n-j} - ex^n = x^n + e(x - x)^n - ex^n = 0.\end{aligned}$$

Hence $x - ex$ is nilpotent and must be 0, so $x = ex$ and e must be the identity. Finally, $x^n = e$ implies $x^{n-1} = x^{-1}$, and we conclude that R is a division ring.

Note that $x^{n+1} = x$ for $x \in R$, so a well-known theorem of Jacobson implies that R is commutative. Hence R is a field. Since $x^n = 1$ has at most n solutions in any field, R has at most n elements; thus it is a finite field whose characteristic divides k.

Editorial comment. Several other readers also showed that the conditions of the problem imply that R is a finite field. Jacobson's "$x^{n(x)} = x$" theorem appears in N. Jacobson, *Structure of Rings*, AMS Colloq. Pub., vol. 37, AMS, 1956, p. 217, as well as in Lam's *A First Course in Noncommutative Rings*, 2nd ed., and other graduate algebra texts.

Also solved by M. Angelelli (Italy), E. P. Armendariz, R. Bagby, A. J. Bevelacqua, W. D. Blair, P. Budney, N. Caro (Colombia), R. Chapman (U. K.), S. Dalton, P. P. Dályay (Hungary), A. Farrugia (Malta), D. Grinberg, J. Grivaux (France), T. Kezlan, D. Lenzi (Italy), O. P. Lossers (Netherlands), S. Markov & A. Alin, A. Nakhash, V. Ponomarenko, D. Ray, D. Saracino, K. Schilling, J. Simons (U. K.), J. H. Smith, R. Stong, J. V. Tejedor (Spain), M. Tetiva (Romania), G. P. Wene, S. Xiao (Canada), GCHQ Problem Solving Group (U. K.), Hofstra University Problem Solvers, Microsoft Research Problems Group, and the proposer.

Permutation Flipping

11452 [2009, 648]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Say that the permutations $a_1 \cdots a_k a_{k+1} \cdots a_n$ and $a_k \cdots a_1 a_{k+1} \cdots a_n$ are equivalent when $k = n$ or when a_{k+1} exceeds all of $a_1, \ldots, a_k$. Also say that two permutations are equivalent whenever they can be obtained from each other by a sequence of such flips. For example, $321 \equiv 123 \equiv 213 \equiv 312$ and $132 \equiv 231$. Show that the number of equivalence classes is equal to the Euler secant-and-tangent number for all n. (The nth secant-and-tangent number counts the "up-down" permutations of length n, namely the permutations like 25341 that alternately rise and fall beginning with a rise.)

Solution by Robin Chapman, University of Exeter, UK. We consider permutations of any totally ordered n-set; let E_n be the number of equivalence classes. We shall establish a recurrence for $\langle E_n \rangle_{n \geq 0}$. Set $E_0 = 1$. For a word a, let $\overline{a}$ denote its reversal.

If a is a permutation of a totally ordered n-set A with largest letter α, then $a = b\alpha c$, where b and c are permutations of complementary subsets B and C of $A - \{\alpha\}$. No flip can change the unordered pair $\{B, C\}$ (the sets can be exchanged and may be empty). Thus all permutations equivalent to a have the form $b'\alpha c'$ or $c'\alpha b'$, where $b' \equiv b$ and $c' \equiv c$. Conversely, any such permutation is equivalent to a: the presence of α allows transforming the part before α into anything in its equivalence class, and thus $b\alpha c \equiv b'\alpha c \equiv \overline{c}\alpha\overline{b'} \equiv c\alpha\overline{b'} \equiv \overline{c'}\alpha\overline{b'} \equiv b'\alpha c'$ and $b\alpha c \equiv \overline{b}\alpha c \equiv \overline{c}\alpha b \equiv c\alpha b \equiv c'\alpha b'$. Thus the equivalence class of $b\alpha c$ is $\{b'\alpha c', c'\alpha b' : b' \equiv b, c' \equiv c\}$.

To count the equivalence classes of permutations of A, we choose a partition of $A - \{\alpha\}$ into sets B and C of sizes k and $n - k - 1$ and populate the portions of the permutation before and after α with equivalence classes on those sets. Summing over k counts each equivalence class twice, since B and C can be switched. For $n \geq 2$,

$$2E_n = \sum_{k=0}^{n-1} \binom{n-1}{k} E_k E_{n-k-1}.$$

It is well known that the number of up-down permutations satisfies the same recurrence and initial condition; see, for example, the solution to Exercise 7.41 in Graham, Knuth, and Patashnik's *Concrete Mathematics*, Addison-Wesley, 1989. Thus, by induction, the two sequences are the same.

Editorial comment. The origin of the name for the numbers in this sequence is that its exponential generating function is $\sec x + \tan x$.

Also solved by D. Beckwith, P. P. Dályay (Hungary), A. Farrugia (Malta), D. Grinberg, Y. J. Ionin, P. Levande, O. P. Lossers (Netherlands), K. McInturff, J. Simons (U. K.), GCHQ Problem Solving Group (U. K.).

A Simplicial Complex Sum

11453 [2009, 746]. *Proposed by Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA.* Let Δ be a finite collection of sets such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. Fix $k \geq 0$. Suppose that every F in Δ (including $F = \varnothing$) with $|F| \leq k$ satisfies

$$\sum_{G\in\Delta, G\supseteq F} (-1)^{|G|} = 0.$$

Show that $|\Delta|$ is divisible by 2^{k+1}.

Solution I by Richard Bagby, New Mexico State University, Las Cruces, New Mexico. First we show that

$$|\Delta| = \sum_{\substack{F,G\in\Delta\\ F\subseteq G}} 2^{|F|}(-1)^{|G|-|F|}.$$

Indeed,

$$\begin{aligned}
\sum_{\substack{F,G\in\Delta\\ F\subseteq G}} 2^{|F|}(-1)^{|G|-|F|} &= \sum_{G\in\Delta}\left(\sum_{F\subseteq G} 2^{|F|}(-1)^{|G|-|F|}\right)\\
&= \sum_{G\in\Delta}\sum_{j=0}^{|G|}\binom{|G|}{j}2^j(-1)^{|G|-j}\\
&= \sum_{G\in\Delta}(2-1)^{|G|} = |\Delta|\,.
\end{aligned}$$

Interchanging the order of summation yields

$$|\Delta| = \sum_{F\in\Delta}(-2)^{|F|}\sum_{G\in\Delta, G\supseteq F}(-1)^{|G|}.$$

Now the contribution to the outer sum from each set F is either 0 (for $|F| \leq k$) or divisible by 2^{k+1} (for $|F| > k$).

Solution II by Richard Stong, Center for Communication Research, San Diego, CA. Let $P(x) = \sum_{G\in\Delta} x^{|G|}$; note that P is a polynomial with integer coefficients. For $m \leq k$,

$$\frac{(-1)^m}{m!}P^{(m)}(-1) = \sum_{G\in\Delta}\binom{|G|}{m}(-1)^{|G|} = \sum_{\substack{F\in\Delta\\ |F|=m}}\ \sum_{\substack{G\in\Delta\\ G\supseteq F}}(-1)^{|G|} \ = \ 0.$$

Hence -1 is a zero of P with multiplicity at least $k+1$, and we can write $P(x) = (x+1)^{k+1}Q(x)$ for some polynomial Q with integer coefficients. Setting $x = 1$ yields $|\Delta| = P(1) = 2^{k+1}Q(1)$; hence $|\Delta|$ is a multiple of 2^{k+1}.

Comment by the proposer. This result is the combinatorial analogue of a much deeper topological result of G. Kalai in *Computational Commutative Algebra and Combinatorics*, Adv. Stud. Pure Math., vol. 33, Math. Soc. Japan, 2002, 121–163 (Theorem 4.2), a special case of which can be stated as follows. Let Δ be a finite simplicial complex. Suppose that for any face F of dimension at most $k-1$ (including the empty face of dimension -1), the link of F (i.e., the set of all $G \in \Delta$ such that $F \cap G = \varnothing$ and

$F \cup G \in \Delta$) is acyclic (that is, has vanishing reduced homology). Letting f_i denote the number of i-dimensional faces of Δ, there exists a simplicial complex Γ, with g_i faces having dimension i, such that

$$\sum_{i \geq -1} f_i x^i = (1+x)^{k+1} \sum_{i \geq -1} g_i x^i.$$

The present problem does not follow from Kalai's result, since the hypotheses here concern only Euler characteristics, while Kalai's result concerns homology groups.

Also solved by D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), R. Ehrenborg, D. Grinberg, Y. J. Ionin, O. Kouba (Syria), J. H. Lindsey II, R. Martin (Germany), J. M. Sanders, K. Schilling, B. Schmuland (Canada), J. Simons (U. K.), M. Tetiva, and the proposer.

An Orientation Game

11454 [2009, 746]. *Proposed by Azer Kerimov, Bilkent University, Ankara, Turkey.* Alice and Bob play a game based on a 2-connected graph G with n vertices, where $n > 2$. Alice selects two vertices u and v. Bob then orients up to $2n-3$ of the edges. Alice then orients the remaining edges and selects some edge e, which may have been oriented by her or by Bob. If the oriented graph contains a path from u to v through e, then Bob wins; otherwise, Alice wins. Prove that Bob has a winning strategy, while if he is granted only $2n-4$ edges to orient, on some graphs he does not. (A graph is 2-connected if it has at least three vertices and each subgraph obtained by deleting one vertex is connected.)

Solution by Michelle Delcourt (student), Georgia Institute of Technology, Atlanta, GA. We show first that orienting $2n-4$ edges does not guarantee a win for Bob. Let G consist of two vertices adjacent to each other and to the remaining $n-2$ vertices; G has $2n-3$ edges. Alice chooses the high-degree vertices as u and v. Since Bob only orients $2n-4$ edges, some edge remains unoriented. Alice selects this edge as e and orients it into u and/or away from v. No path from u to v passes through e.

Now allow Bob to orient $2n-3$ edges. Bob produces a special vertex ordering and edge partition and uses them to orient at most $2n-3$ edges. Let u and v be the vertices chosen by Alice. Whitney's theorem for 2-connected graphs states that there are two paths from u to v with no shared internal vertices. Thus u and v lie on a cycle; let C be a shortest cycle containing them. Order its vertices by starting with u, then listing the internal vertices of one path from u to v along C, then listing the internal vertices of the other such path, then ending with v. Bob orients each edge of C from its earlier endpoint to its later endpoint, producing two oriented paths from u to v.

Bob now iteratively decomposes the rest of G into paths $P_1, \dots, P_r$. Let $G_0 = C$. Suppose that G_{i-1} has been defined, with a linear order on its vertices. If $G_{i-1} \neq G$, then there is a path joining distinct vertices of G_{i-1} whose edges and internal vertices are not in G_{i-1} (again by Whitney's theorem). Among all such paths, consider those whose earlier endpoint is earliest in the ordering of $V(G_{i-1})$; among these, consider those whose later endpoint is latest in the ordering; among these, let P_i be a shortest such path. Let $G_i = G_{i-1} \cup P_i$. Insert the internal vertices of P_i in the vertex ordering between its endpoints, ordered so that each new vertex has a neighbor occurring earlier and a neighbor occurring later in the ordering. Bob orients P_i if its length is at least 2, in that case orienting each edge from its earlier endpoint to its later endpoint. Bob leaves P_i unoriented if it has only one edge.

The number of edges of P_i oriented by Bob is at most twice the number of vertices added by P_i. The number of edges oriented in C is $|V(C)|$, and $|V(C)| \leq 2|V(C)| - 3$ since cycles have at least three vertices. Hence Bob orients at most $2n-3$ edges. The orientation produced by Bob explicitly has a path from u to v through each oriented edge (and a unique such path through each vertex outside $\{u, v\}$).

To prove that this partial orientation wins for Bob, it suffices to show that for any edge xy added as a path of length 1 (hence not oriented by Bob), there are disjoint paths oriented by Bob from u to x and from y to v. This is immediate when x precedes y in the ordering, so we may assume that x is later than y.

It suffices prove that these two paths exist in the first G_i that contains x and y, since they remain (oriented) as the rest of the decomposition is added. The claim holds when $i = 0$ since C was chosen to be a shortest cycle through u and v; thus x and y lie in distinct u, v-paths on C (and do not equal u or v).

For $i > 0$, vertices x and y cannot both be added by P_i, since then there would be a shorter path joining its endpoints that would be added instead. Let a and b be the first and last vertices of P_i in the ordering. If x is added by P_i, then it suffices to show that the u, x-path created then by Bob contains no vertex of the y, v-path in G_{i-1}. If it does, then y is earlier than a in the ordering, and the path that starts with yx and continues along P_i to b would be chosen in preference to P_i. Similarly, if y is added by P_i, then it suffices to show that the y, v-path created then by Bob contains no vertex of the u, x-path in G_{i-1}. If it does, then x is later than b in the ordering, and the path that follows P_i from a to y and finishes with yx would be chosen in preference to P_i.

Editorial comment. The list $C, P_1, \dots, P_r$ is an example of an *ear decomposition* of G. The vertex ordering is an example of an s, t-*numbering* with the source s being u and the terminus t being v; the condition is that each vertex outside $\{s, t\}$ has an earlier neighbor and a later neighbor in the ordering.

Also solved by D. Beckwith, J. Simons (U. K.), R. Stong, S. Xiao (Canada), and the proposer.

Our Gamma Inequality Flops

11474 [2010, 86]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Valentin Vornicu, Aops-MathLinks forum, San Diego, CA.* (Corrected) Show that when x, y, and z are greater than 1,

$$\Gamma(x)^{x^2+2yz}\Gamma(y)^{y^2+2zx}\Gamma(z)^{z^2+2xy} \geq (\Gamma(x)\Gamma(y)\Gamma(z))^{xy+yz+zx}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. When $x = y$, the inequality becomes $\Gamma(z)^{(z-x)^2} \geq 1$, which fails if $1 < z < 2$.

For $x, y, z > 2$, though, it follows from the fact that $\Gamma(x)$ is increasing on $[2, \infty)$ and $\Gamma(2) = 1$. Indeed: without loss we may assume $2 \leq x \leq y \leq z$. The desired inequality rearranges to

$$(y-x)(z-x)\log\Gamma(x) - (y-x)(z-y)\log\Gamma(y) + (z-x)(z-y)\log\Gamma(z) \geq 0.$$

The first term is nonnegative and the third term is greater than or equal to the second; hence this inequality holds.

Editorial comment. The corrected version of the problem, shown above, appeared in the April, 2010, issue of the MONTHLY.

Also solved by P. P. Dályay (Hungary), O. Kouba (Syria), O. P. Lossers (Netherlands), M. Muldoon (Canada), GCHQ Problem Solving Group (U. K.), and the Microsoft Research Problems Group.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before February 29, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11593. *Proposed by Peter McGrath, Brown University, Providence, RI.* For positive integers k and n, let $T(n,k)$ be the $n \times n$ matrix with (i,j)-entry $((i-1)n+j)^k$. Prove that for $n > k+1$, $\det(T(n,k)) = 0$.

11594. *Proposed by Harm Derksen and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.* Let

$$G_n = \prod_{k=1}^{n} \left(\prod_{j=1}^{k-1} \frac{j}{k} \right),$$

and let $\overline{G}_n = 1/G_n$.

(**a**) Show that if n is an integer greater than 1, then $\overline{G}_n$ is an integer.

(**b**) Show that for each prime p, there are infinitely many n greater than 1 such that p does not divide $\overline{G}_n$.

11595. *Proposed by Victor K. Ohanyan, Yerevan, Armenia.* Let $P_1, \ldots, P_n$ be the vertices of a convex n-gon in the plane. Let Q be a point in the interior of the n-gon, and let v be a vector in the plane. Let $\mathbf{r}_i$ denote the vector QP_i, with length r_i. Let Q_i be the (radian) measure of the angle between v and $\mathbf{r}_i$, and let F_i and Y_i be respectively the clockwise and counterclockwise angles into which the interior angle at P_i of the polygon is divided by QP_i. Show that

$$\sum_{i=1}^{n} \frac{1}{r_i} \sin(Q_i)(\cot F_i + \cot Y_i) = 0.$$

http://dx.doi.org/10.4169/amer.math.monthly.118.08.747

11596. *Proposed by Mehmet Sahin (student) Ankara University, Ankara, Turkey.* Let a, b, c be the side lengths of a triangle, and let r_a, r_b, r_c be the corresponding exradii. Prove that

$$\frac{a^2}{r_a^2} + \frac{b^2}{r_b^2} + \frac{c^2}{r_c^2} = 8\left(\frac{r_a + r_b + r_c}{a + b + c}\right)^2 - 2.$$

11597. *Proposed by Michel Bataille, Rouen, France.* Let $f(x) = x/\log(1-x)$. Prove that for $0 < x < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n(1-x)^n}{n!} f^{(n)}(x) = -\frac{1}{2} x f(x).$$

11598. *Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.* Let S be an additive semigroup of positive integers. Show that there is a finite subset T of S that generates S and that is contained in every generating set of S.

11599. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS, and Péter Komjáth, Eötvös Loránd University, Budapest, Hungary.* Prove that the following statement is equivalent to the axiom of choice: for any finite family $A_1, \dots, A_n$ of sets, there is a finite set F such that $|A_i \cap F| < |A_j \cap F|$ whenever $|A_i| < |A_j|$.

Here, equivalence is to be judged in the context of Zermelo-Fraenkel set theory, not assuming the axiom of choice, and to say that $|C| < |D|$ is to say that there is an injection from C to D, but none from D to C.

SOLUTIONS

A Generic Lower Bound for $a^2 + b^2 + c^2$ in a Triangle

11460 [2009, 844]. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College of Informatics, Bucharest, Romania.* Given a triangle of area S with sides of lengths a, b, and c, and positive numbers x, y, and z, show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S + \frac{2}{x+y+z}\left(a^2\frac{x^2 - yz}{x} + b^2\frac{y^2 - zx}{y} + c^2\frac{z^2 - xy}{z}\right).$$

Solution by Marian Dinca, Romania. Since the proposed inequality is homogeneous in x, y, z, we may assume without loss of generality that $x + y + z = 1$. The inequality may be written as

$$ma^2 + nb^2 + pc^2 \geq 4\sqrt{3}\,S,$$

where $m = 1 - 2(x^2 - yz)/x$, $n = 1 - 2(y^2 - zx)/y$, and $p = 1 - 2(z^2 - xy)/z$. A corollary of the Neuberg–Pedoe inequality (see comment below) tells us that

$$ma^2 + nb^2 + pc^2 \geq 4S\sqrt{mn + np + pm}.$$

It now suffices to show that $mn + np + pm = 3$, which may be done as follows: Let $t = 2(xy + yz + zx)$, so that $m = 1 - 2(1 - y - z) + 2yz/x = (2xy + 2yz + 2zx - x)/x = (t - x)/x$, $n = (t - y)/y$, and $p = (t - z)/z$. Then $mn = (t - x)(t - y)/(xy) = \big(zt^2 - (xz + yz)t + xyz\big)/(xyz)$, etc., so $mn + np + pm = \big((x + y + z)t^2 - t^2 + 3xyz\big)/(xyz) = 3$.

Editorial comment. Several solvers made note of the connection between this inequality and others already in the literature:

- The Weitzenbock inequality [8]: $a^2+b^2+c^2 \geq 4\sqrt{3}\,S$; in fact $a^2+b^2+c^2 \geq ab+bc+ca \geq a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \geq 3(a^2b^2c^2)^{1/3} \geq 4\sqrt{3}\,S$.
- The Hadwiger–Finsler inequality [4]: $a^2+b^2+c^2 \geq 4\sqrt{3}\,S+(a-b)^2+(b-c)^2+(c-a)^2$.
- The Neuberg–Pedoe inequality [5, 6]: for a second triangle of area T with sides of length x, y, z, we have

$$a^2(y^2+z^2-x^2)+b^2(z^2+x^2-y^2)+c^2(x^2+y^2-z^2) \geq 16ST,$$

  with equality if and only if the triangles are similar.
- The following corollary of the Neuberg–Pedoe inequality: Let m, n, p be any three positive numbers. A triangle exists with side lengths $x=\sqrt{n+p}$, $y=\sqrt{p+m}$, $z=\sqrt{m+n}$, since the triangle inequality holds for x, y, z. This triangle has area $T=\sqrt{mn+np+pm}\,/2$. Then noting $y^2+z^2-x^2=2m$, etc., by Neuberg–Pedoe we have $ma^2+nb^2+pc^2 \geq 4S\sqrt{mn+np+pm}$.

The proposed inequality becomes the Hadwiger–Finsler inequality when $x=a$, $y=b$, $z=c$. It becomes the Weitzenbock inequality when $x=y=z$.

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Also solved by G. Apostolopoulos (Greece), R. Chapman (U. K.), P. P. Dályay (Hungary), Á. Plaza & S. Falcón (Spain), M. A. Prasad (India), R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

Eigenvalues, Trace, and Determinant

11463 [2009, 844]. *Proposed by Xiang Qian Chang, Massachusetts College of Pharmacy and Health Sciences, Boston, MA.* Let A be a positive-definite $n \times n$ Hermitian matrix with minimum eigenvalue λ and maximum eigenvalue Λ. Show that

$$\left(\frac{n}{\operatorname{tr}((A+\lambda I)^{-1})}-\lambda\right)^n \leq \det(A) \leq \left(\frac{n}{\operatorname{tr}((A+\Lambda I)^{-1})}-\Lambda\right)^n.$$

Solution by BSI Problems Group, Bonn, Germany. An n-by-n positive-definite Hermitian matrix A has only positive eigenvalues, and it has eigenvectors forming a basis. Since $\operatorname{tr}((A+xI)^{-1})$ and $\det(A)$ are invariant under change of basis, we may assume that A is in diagonal form. We then must show, for positive $p_1, \dots, p_n$ with minimum λ and maximum Λ, that

$$\left(\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i+\lambda}} - \lambda\right)^n \le \prod_{i=1}^n p_i \le \left(\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i+\Lambda}} - \Lambda\right)^n.$$

Dividing the first inequality by λ^n and the second by Λ^n, it suffices to show

$$\left(\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i+1}} - 1\right)^n \le \prod_{i=1}^n x_i \qquad \text{and} \qquad \prod_{i=1}^n z_i \le \left(\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{z_i+1}} - 1\right)^n,$$

where $x_1, \dots, x_n \ge 1$ and $z_1, \dots, z_n \in (0, 1]$.

The function that maps y to $\log(1/y - 1)$ is convex on $(0, 1/2]$ and concave on $[1/2, 1)$. By Jensen's inequality,

$$n \log\left(\frac{1}{\frac{1}{n}\sum_{i=1}^n y_i} - 1\right) \le \sum_{i=1}^n \log\left(\frac{1}{y_i} - 1\right) \qquad \text{for } y_1, \dots, y_n \in (0, 1/2],$$

and

$$n \log\left(\frac{1}{\frac{1}{n}\sum_{i=1}^n y_i} - 1\right) \ge \sum_{i=1}^n \log\left(\frac{1}{y_i} - 1\right) \qquad \text{for } y_1, \dots, y_n \in [1/2, 1).$$

Exponentiating these inequalities and setting $y_i = 1/(1 + x_i)$ in the first and $y_i = 1/(1 + z_i)$ in the second yields the desired results.

Also solved by R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), N. C. Singer, R. Stong, M. Tetiva (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

More and More Balls in Urns

11464 [2009, 845]. *Proposed by David Beckwith, Sag Harbor, NY.* Let $a(n)$ be the number of ways to place n identical balls into a sequence of urns $U_1, U_2, \dots$ in such a way that U_1 receives at least one ball, and while any balls remain, each successive urn receives at least as many balls as in all the previous urns combined. Let $b(n)$ denote the number of partitions of n into powers of 2, with repeated powers allowed. (Thus, $a(6) = 6$ because the placements are 114, 123, 15, 24, 33, and 6, while $b(6) = 6$ because the partitions are 111111, 11112, 1122, 114, 222, and 24.) Prove that $a(n) = b(n)$ for all $n \in \mathbb{N}$.

Solution by Jerrold W. Grossman, Oakland University, Rochester, MI. Because $a(1) = b(1) = 1$, it suffices to show that both sequences satisfy the same recurrence.

When n is odd, the final urn in an acceptable distribution contains more than half of the balls, and removing one ball from it gives an acceptable distribution of $n - 1$ balls. Thus $a(2m + 1) = a(2m)$ for $m \ge 1$. When n is even, then either the final urn contains more than half the balls, and removing one from it gives an acceptable distribution of $n - 1$ balls, or the final urn contains exactly half the balls, and the others contain an acceptable distribution of $n/2$ balls. Thus $a(2m) = a(2m - 1) + a(m)$ for $m \ge 1$.

When n is odd, a 2-power partition of n contains a 1, and the rest is a 2-power partition of $n - 1$. Thus $b(2m + 1) = b(2m)$ for $m \ge 1$. When n is even, a 2-power partition of n contains a 1 plus a 2-power partition of $n - 1$, or all the parts are even and the partition is a doubling of a 2-power partition of $n/2$. Thus $b(2m) = b(2m - 1) + b(m)$ for $m \ge 1$.

Editorial comment. The sequence $(1, 2, 2, 4, 4, 6, 6, 10, 10, 14, 14, \ldots)$ and its recurrence are well known. See Sequence A018819 in *The On-Line Encyclopedia of Integer Sequences* (http://www2.research.att.com/~njas/sequences/) and its references.

Also solved by M. Andreoli, R. Bagby, C. K. Bailey & M. D. Meyerson, C. Bernardi (Italy), S. Binski, E. H. M. Brietzke (Brazil), D. Brown, S. M. Bryan, R. Chapman (U. K.), E. Cheng, W. J. Cowieson, P. P. Dályay (Hungary), C. Delorme (France), S. Eichhorn, D. Finley, J. P. Grivaux (France), J. Hook, Y. J. Ionin, D. E. Knuth, M. Kochanski, J. Lee (Canada), P. Levande, J. H. Lindsey II, O. P. Lossers (Netherlands), J. McKenna, D. Mitchell, J. H. Nieto (Venezuela), Á. Plaza (Spain), B. Popp, S. Post, M. A. Prasad (India), R. Pratt, B. Schmuland (Canada), J. Simons (U. K.), J. H. Smith, R. Staum, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), Z. Vörös (Hungary), S. Xiao (Canada), L. Zhou, Armstrong Problem Solvers, athenahealth Problem Solving Group, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

The Determinant of a Very Square Matrix

11467 [2009, 940]. *Proposed by Xiang Qian Chang, Massachusetts College of Pharmacy and Health Sciences, Boston, MA.* Find in closed form the determinant of the $n \times n$ matrix with entries $a_{i,j}$ given by

$$a_{i,j} = \begin{cases} \sum_{k=0}^{i-1} (j-k)^2 & \text{if } i \le j; \\ \sum_{k=1}^{j} k^2 + \sum_{k=0}^{i-j-1} (n-k)^2 & \text{if } i > j. \end{cases}$$

Solution by Jaime Vinuesa Tejedor, University of Cantabria, Santander, Spain. The answer is

$$(-1)^{n-1} \frac{n^{n-2}(n+1)(2n+1)[(n+2)^n - n^n]}{12}.$$

For i from $n-1$ to 1, subtracting row i from row $i+1$ does not change the determinant but transforms the matrix to a cyclic matrix with constant diagonals $1^2, \ldots, n^2$. The determinant of a cyclic matrix with elements $a_1, \ldots, a_n$ is $\prod_{k=0}^{n-1} \sum_{j=1}^{n} a_j \omega^{k(j-1)}$, where $\omega = e^{2\pi i/n}$ (see, for example, A. C. Aitken, *Determinants and Matrices*, U. M. T. Oliver and Boyd (1946)).

In our case, since $a_j = j^2$, the inner sum is $n(n+1)(2n+1)/6$ when $k = 0$. To evaluate the other factors, it follows when $\zeta^n = 1$ that

$$(1-\zeta)^2 \sum_{j=1}^{n} j^2 \zeta^{j-1} = n^2(\zeta - 1) - 2n$$

and hence

$$\sum_{j=1}^{n} j^2 \omega^{k(j-1)} = \frac{n^2(\omega^k - \frac{n+2}{n})}{(\omega^k - 1)^2}. \tag{1}$$

For $x \neq 1$, we have $\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1}(x - \omega^k)$; setting $x = \frac{n+2}{n}$ yields

$$\prod_{k=1}^{n-1} \left(\frac{n+2}{n} - \omega^k \right) = \frac{n}{2} \left[\left(\frac{n+2}{n} \right)^n - 1 \right].$$

To multiply the denominators in (1), note that $\prod_{k=1}^{n-1}(x - \omega^k) = \frac{x^n - 1}{x - 1} = \sum_{k=1}^{n} x^{k-1}$. Letting $x = 1$ in each polynomial yields $\prod_{k=1}^{n-1}(1 - \omega^k) = n$.

Now we can compute the determinant:

$$\prod_{k=0}^{n-1}\sum_{j=1}^{n} a_j\omega^{k(j-1)} = \frac{n(n+1)(2n+1)}{6}\prod_{k=1}^{n-1}\sum_{j=1}^{n} j^2\omega^{k(j-1)}$$

$$= \frac{n(n+1)(2n+1)}{6}\prod_{k=1}^{n-1}\frac{n^2(\omega^k - \frac{n+2}{n})}{(\omega^k-1)^2}$$

$$= \frac{n(n+1)(2n+1)}{12}(n^2)^{n-1}\frac{(-1)^{n-1}}{n^2}\frac{n}{2}\frac{(n+2)^n - n^n}{n^n}.$$

Editorial comment. Many of the solvers provided solution formulas not in closed form, but we have listed them anyway.

Also solved by D. Beckwith, R. Chapman (U. K.), C. Curtis, P. P. Dályay (Hungary), M. Goldenberg & M. Kaplan, E. A. Herman, Y. J. Ionin, O. Kouba (Syria), O. P. Lossers (Netherlands), K. McInturff, M. Omarjee (France), Á. Plaza (Spain), J. Simons (U. K.), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), P. Zwier, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, Missouri State University Problem Solving Group, and the proposer.

A Nonobtuse Altitude Inequality

11480 [2010, 87]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let a, b, and c be the lengths of the sides opposite vertices A, B, and C, respectively, in a nonobtuse triangle. Let h_a, h_b, and h_c be the corresponding lengths of the altitudes. Show that

$$\left(\frac{h_a}{a}\right)^2 + \left(\frac{h_b}{b}\right)^2 + \left(\frac{h_c}{c}\right)^2 \ge \frac{9}{4},$$

and determine the cases of equality.

Solution by Richard Bagby, New Mexico State University, LasCruces, NM. By scale invariance it suffices to restrict our attention to triangles of unit area. For such triangles we have

$$H := \left(\frac{h_a}{a}\right)^2 + \left(\frac{h_b}{b}\right)^2 + \left(\frac{h_c}{c}\right)^2 = \frac{4}{a^4} + \frac{4}{b^4} + \frac{4}{c^4}.$$

Let C be the largest angle, and consider varying a and b with C fixed.

Since the area is $\frac{1}{2}ab\sin C$, this amounts to fixing ab. We first compare H to the value H_0 obtained for the triangle with sides lengths $(\sqrt{ab}, \sqrt{ab}, c_0)$, where

$$c_0^2 = ab + ab - 2ab\cos C = 2ab(1-\cos C).$$

We observe that

$$c^2 - c_0^2 = (a^2 + b^2 - 2ab\cos C) - 2ab(1-\cos C) = (a-b)^2$$

and

$$c^2 + c_0^2 = (a+b)^2 - 4ab\cos C.$$

Consequently, we compute

$$\begin{aligned} H - H_0 &= \frac{4}{a^4} + \frac{4}{b^4} + \frac{4}{c^4} - \frac{4}{a^2b^2} - \frac{4}{a^2b^2} - \frac{4}{c_0^4} \\ &= \frac{4(a^2-b^2)^2}{a^4b^4} + \frac{4(c^4-c_0^4)}{c^4c_0^4} \\ &= 4(a-b)^2\left[\frac{(a+b)^2}{a^4b^4} - \frac{(a+b)^2}{c^4c_0^4} + \frac{4ab\cos C}{c^4c_0^4}\right]. \end{aligned}$$

Now $\pi/3 \le C \le \pi/2$. Since

$$c^4c_0^4 \ge c_0^8 = 16a^4b^4(1-\cos C)^4 \ge a^4b^4$$

for $\pi/3 \le C \le \pi/2$, we see that $H - H_0 \ge 0$ with equality if and only if $a = b$. Thus it suffices to look at isosceles triangles.

In this case, unit area gives $a^2 = b^2 = 2\csc C$ and $c^2 = c_0^2 = 4(1-\cos C)\csc C$ so we compute

$$\begin{aligned} H_0 - \frac{9}{4} &= 2\sin^2 C + \frac{\sin^2 C}{4(1-\cos C)^2} - \frac{9}{4} \\ &= 2(1-\cos^2 C) + \frac{1+\cos C}{4(1-\cos C)} - \frac{9}{4} \\ &= \frac{2(2\cos C - 1)^2\cos C}{4(1-\cos C)} \ge 0, \end{aligned}$$

with equality if and only if $C = \pi/3$ or $C = \pi/2$. Thus we get equality for an equilateral triangle or an isosceles right triangle.

Editorial comment. Many solvers noted that

$$\frac{h_a}{a} = \frac{1}{\cot B + \cot C}$$

and substituted $x = \cot A$, $y = \cot B$, and $z = \cot C$, to reduce the problem to

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \ge \frac{9}{4}$$

for $xy + yz + zx = 1$, which was a problem on the 1996 Iranian Mathematical Olympiad. However the earliest reference submitted was to J. F. Rigby, Sextic inequalities for the sides of a triangle, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* no. 498–541 (1975) 51–58.

Also solved by A. Alt, G. Apostolopoulos (Greece), M. Bataille (France), M. Can, C. Curtis, P. P. Dályay (Hungary), J. Fabrykowski & T. Smotzer, F. Holland (Ireland), J. H. Lindsey II, C. Pohoata (Romania), C. R. Pranesachar (India), R. Stong, E. Suppa (Syria), M. Tetiva (Romania), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), and the proposer.

Countable Dense Partition

11481 [2010, 182]. *Proposed by Ron Rietz, Gustavus Adolphus College, St. Peter, MN.* Let X be a countable dense subset of a separable metric space M with no isolated points. Show that there exists a countable partition $(X_1, X_2, \ldots)$ of X such that each X_n is dense in M.

Solution by Michael Barr, McGill University, Montréal, QC, Canada. We derive the conclusion under the weaker condition that M is first countable and Hausdorff. (A space M is first countable if for every x in M there is a sequence $U_1, U_2, \dots$ of neighborhoods of x such that for every neighborhood U of x, at least one of the U_i is a subset of U. It is Hausdorff if distinct points have disjoint neighborhoods.)

We begin with a few observations. First, since M is Hausdorff and has no isolated points, if N is a neighborhood of x, then $N - \{x\}$ is open and nonempty. Second, M is infinite. Third, any dense subset Y of M is infinite.

The plan now is to first show that we can split (partition into two components) any countable dense subset Y of M into dense subsets U and V. If we can do that, the problem is solved, because we can take $Y = X$, take $X_1 = U$, split V in the same fashion to get an X_2, then split the other part of V to get X_3, and so on. To split Y in the desired manner, we begin by showing that for every y in Y there are disjoint sequences of points in $Y - \{y\}$ that converge to y. (A sequence (y_j) converges to y if for every neighborhood N of y there is an index k such that for all $j \geq k$, $y_j \in N$.)

By hypothesis, there is a sequence $(N_1, N_2, \dots)$ of neighborhoods of y such that every neighborhood of y contains one of the N_j. It will be convenient to require that $N_j \subseteq N_k$ for $j \geq k$, and this can be achieved by replacing N_k with $\cap_{j=1}^{k} N_j$. Since M has no isolated points, each N_j contains a point other than y. Since Y is dense, for each j, $Y \cap (N_j - \{y\})$ is nonempty.

We construct a sequence $(s_1, s_2, \dots)$ by taking y_j from $Y \cap (N_j - \{y\})$, $j = 1, 2, \dots$. Every neighborhood of y contains some N_j as a subset, and hence all N_k with $k \geq j$ since the N_j are nested. Thus, (s_j) has the property that for every neighborhood N of y, there is an index k such that for $k \geq j$, $s_k \in N$. That is, (s_j) converges to y without hitting y.

By deleting duplicates and renumbering, we make (s_j) a sequence of distinct elements. The disjoint sequences (s_{2j}) and (s_{2j+1}) then provide our sequences converging to y from within Y.

Since Y is countable, there is a sequence $(y_1, y_2, \dots)$ of distinct elements of Y that exhaust Y. Using our construction above, for each $k \geq 1$ we build a disjoint pair $(u_{j,k})$ and $(v_{j,k})$ of sequences, free of repetion, drawn from $Y - \{y_k\}$ and converging to y_k. We put $U_k = \{u_{j,k} : j \geq 1\}$ and $V_k = \{v_{j,k} : j \geq 1\}$. We put $U_1' = U_1$, $V_1' = V_1$, and for $k \geq 2$ we put $U_k' = U_k - \cup_{j<k}(U_j \bigcup V_j)$, $V_k' = \cup_{j<k}(U_j \bigcup V_j)$. Since sequences converging to distinct limits share only finitely many entries, each U_k' and V_k' is infinite. Seen as sequences, U_k' and V_k' converge within $Y - \{y_k\}$ to y_k. Furthermore, any two distinct sets from the collection of U_k' and V_k' are disjoint.

Now take $U = \cup_{k=1}^{\infty} U_k'$, $V = \cup_{k=1}^{\infty} V_k'$. Then $Y \subseteq \overline{U}$ and $Y \subseteq \overline{V}$, so $\overline{U} = \overline{V} = M$. By construction, U and V are disjoint. Adding any unused elements of Y to U, we have our claimed partition of Y into disjoint subsets U and V, which completes the proof.

Also solved by J. Bryant, P. Budney, B. S. Burdick, R. Chapman (U. K.), J. Cobb, W. J. Cowieson, N. Eldredge, P. J. Fitzsimmons, J. Grivaux (France), J. W. Hagood, E. A. Herman, G. A. Heuer, O. P. Lossers (Netherlands), M. D. Meyerson, V. Pambuccian, D. Rose, K. A. Ross, B. Schmuland (Canada), J. Simons (U. K.), R. Stong, B. Tomper, J. Vinuesa (Spain), GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before March 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11600. *Proposed by Michael D. Hirschhorn, University of New South Wales, Sydney, Australia.* Suppose $a > 0$, $n \geq 1$, and $0 < r < a/n$. For given θ, let

$$\phi_k = \arctan\left(\frac{kr\sin\theta}{a - kr\cos\theta}\right), \qquad \rho_k = \sqrt{a^2 - 2kra\cos\theta + k^2r^2}.$$

Show that

$$\int_0^\infty \frac{\cos(\phi_1 + \cdots + \phi_n) - \theta\sin(\phi_1 + \cdots + \phi_n)}{\rho_1\cdots\rho_n}\,\frac{d\theta}{1+\theta^2} = \frac{\pi}{2a^n}.$$

11601. *Proposed by Harm Derksen and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.* The Farey series of order n is the set of reduced rational fractions j/k in the unit interval with denominator at most n. Let F_n be the product of these fractions, excluding $0/1$. That is,

$$F_n = \prod_{k=1}^{n}\prod_{\substack{j=1\\(j,k)=1}}^{k-1}\frac{j}{k}.$$

Let $\overline{F}_n = 1/F_n$. Show that $\overline{F}_n$ is an integer for only finitely many n.

11602. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Let p be a prime. Let F_n denote the nth Fibonacci number. Show that

$$\sum_{0<i<j<k<p}\frac{F_i}{ijk} \equiv 0 \pmod{p}.$$

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(A rational number is deemed congruent to 0 mod p if, when put in reduced form, the numerator is a multiple of p.)

11603. *Proposed by Alfonso Villani, Università di Catania, Catania, Italy.* Let I be the interval $[0, \infty)$. Let p and r be positive, with $r \geq 1$. Let f be a function on I that is absolutely continuous in every compact interval $[0, b]$ with $b > 0$. Assume that f is in $L^p(I)$ and that the (weak) derivative f' belongs to $L^r(I)$. (Weak derivatives are part of the theory of *distributions*.) Prove that $\lim_{x\to\infty} f(x) = 0$.

11604. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Given $0 \leq a \leq 2$, let $\langle a_n \rangle$ be the sequence defined by $a_1 = a$ and $a_{n+1} = 2^n - \sqrt{2^n(2^n - a_n)}$ for $n \geq 1$. Find $\sum_{n=1}^{\infty} a_n^2$.

11605. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let s, R, and r be the semiperimeter, circumradius, and inradius of a triangle with sides of length a, b, and c. Show that

$$\frac{R-2r}{2R} \geq \sum \frac{\sqrt{(s-a)(s-b)}}{c} - 2\sum \frac{(s-c)\sqrt{(s-a)(s-b)}}{ab},$$

and determine when equality occurs. Sums are cyclic.

11606. *Proposed by Kent Holing, Trondheim, Norway.* Let a, b, c, d be integers, the first two even and the other two odd. Let Q be the polynomial $x^4 + ax^3 + bx^2 + cx + d$, and assume that the Galois group of Q has order less than 24.

(a) Show that the Lagrange resolvent

$$x^3 + \left(2b - \frac{3}{4}a^2\right)x^2 + \left(b^2 - 4d + ac - a^2b + \frac{3}{16}a^4\right)x - \frac{1}{64}(a^3 - 4ab + 8c)^2$$

of Q has exactly one integer root; call it m.

(b) Show that $a^2 + 4(m - b)$ cannot be a nonzero square.

(c) Show that if $a = 0$, then the Galois group of Q is cyclic if and only if $(m-b)(m+b)^2 - 4c^2$ is square.

SOLUTIONS

An Equal Distance Sum Point

11482 [2010, 182]. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be a positive integer, and let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$, and $(c_1, \ldots, c_n)$ be n-tuples of points in $\mathbb{R}^2$ with noncollinear centroids. For $u \in \mathbb{R}^2$, let $\|u\|$ be the usual euclidean norm of u. Show that there is a point $p \in \mathbb{R}^2$ such that

$$\sum_{k=1}^{n} \|p - a_k\| = \sum_{k=1}^{n} \|p - b_k\| = \sum_{k=1}^{n} \|p - c_k\|.$$

Solution by Robin Chapman, University of Bristol (U.K.). Define maps f_a, f_b, f_c from $\mathbb{R}^2$ to $\mathbb{R}$ by

$$f_a(p) = \sum_{k=1}^{n} \|p - a_k\|, \quad f_b(p) = \sum_{k=1}^{n} \|p - b_k\|, \quad f_c(p) = \sum_{k=1}^{n} \|p - c_k\|.$$

Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $F(p) = \big(f_a(p) - f_c(p),\ f_b(p) - f_c(p)\big)$. The maps f_a, f_b, f_c, and F are continuous. We must show that F has a zero (a point p such that $F(p) = (0, 0)$.) Suppose instead that $F(p) \neq (0, 0)$ for all $p \in \mathbb{R}^2$. We use the concept of winding number. Let $\phi\colon [\alpha, \beta] \to \mathbb{R}^2$ be a loop; that is, a continuous map with $\phi(\alpha) = \phi(\beta)$. Since ϕ is a loop and F avoids $(0, 0)$, $F \circ \phi$ is a loop in $\mathbb{R}^2 \setminus \{(0, 0)\}$. As $\mathbb{R}^2$ is simply connected, ϕ can be deformed continuously into a path that is constant on $[\alpha, \beta]$; as it does, $F \circ \phi$ deforms continuously inside $\mathbb{R}^2 \setminus \{(0, 0)\}$ into another such path. It follows that the winding number of $F \circ \phi$ about $(0, 0)$ is 0. We obtain a contradiction by proving that there is a loop ϕ such that $F \circ \phi$ has nonzero winding number. Define $\phi\colon [0, 2\pi] \to \mathbb{R}^2$ by

$$\phi(\theta) = (R\cos\theta,\ R\sin\theta),$$

where R is a positive constant to be chosen later.

Let $\theta \in [0, 2\pi]$, and write $p = \phi(\theta)$. Note that $\|p\| = R$. We can take R as large as we like; we like $R > 2\max_k\big(\|a_k\|, \|b_k\|, \|c_k\|\big)$. Using the usual dot product in $\mathbb{R}^2$, for $1 \le k \le n$ we have

$$\begin{aligned}\|p - a_k\| &= \big(\|p\|^2 - 2p\cdot a_k + \|a_k\|^2\big)^{1/2} = R\left(1 - \frac{2p\cdot a_k}{R^2} + \frac{\|a_k\|^2}{R^2}\right)^{1/2} \\ &= R\left(1 - \frac{p\cdot a_k}{R^2} + O(1/R^2)\right) = R - \frac{p\cdot a_k}{R} + O(1/R)\end{aligned}$$

as $R \to \infty$, since $|p\cdot a_k| = O(R)$. Thus

$$f_a(p) = nR - \frac{n}{R}p\cdot a + O(1/R) \qquad \text{as } R \to \infty$$

where a is the centroid of the a_k. Similarly

$$f_b(p) = nR - \frac{n}{R}p\cdot b + O(1/R), \qquad f_c(p) = nR - \frac{n}{R}p\cdot c + O(1/R),$$

where b is the centroid of the b_k and c is the centroid of the c_k. Hence

$$F(p) = \frac{n}{R}\big((c - a)\cdot p,\ (c - b)\cdot p\big) + O(1/R) \qquad \text{as } R \to \infty.$$

We have assumed that a, b, c are noncollinear, so $c - a$ and $c - b$ are linearly independent. Thus

$$c - a = (s\cos\alpha, s\sin\alpha), \qquad c - b = (t\cos\beta, t\sin\beta)$$

where $s > 0$, $t > 0$, and $\alpha - \beta$ is not an integer multiple of π. (Note that s, t, α, β are constants and do not depend on R or θ.) Therefore

$$F(p) = n\big(s\cos(\theta - \alpha), t\cos(\theta - \beta)\big) + O(1/R) \qquad \text{as } R \to \infty.$$

Define $G\colon [0, 2\pi] \to \mathbb{R}^2$ by

$$G(\theta) := n\big(s\cos(\theta - \alpha), t\cos(\theta - \beta)\big).$$

We claim that G traces out an ellipse with center the origin exactly once, so that its winding number about $(0, 0)$ is ± 1. To see this, let $\gamma = \alpha - \beta$. Then

$$\cos(\theta - \beta) = \cos(\theta - \alpha + \gamma) = \cos\gamma\cos(\theta - \alpha) - \sin\gamma\sin(\theta - \alpha).$$

Now the nonsingular linear substitution $(x, y) = (nsu, ntu\cos\gamma - ntv\sin\gamma)$ converts the unit circle in the (u, v)-plane into the path G. Thus G traces out an ellipse centered at the origin, exactly once as θ increases from 0 to 2π. Hence the winding number of G is ± 1. Now $G(\theta) - F(\phi(\theta)) = O(1/R)$, so given $\varepsilon > 0$, for large enough R we have $\|G(\theta) - F(\phi(\theta))\| < \varepsilon$ for all θ. Provided ε is less than the distance from the origin to the path G, the line segment from $G(\theta)$ to $F(\phi(\theta))$ will not meet the origin. Therefore the paths G and $F \circ \phi$ are homotopic in $\mathbb{R}^2 \setminus \{(0,0)\}$, and so have the same winding number. This is the required contradiction.

Also solved by M. A. Prasad (India), K. Schilling, J. Simons (U. K.), R. Stong, and the proposer.

Triangle Tangent Inequality

11486 [2010, 183]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* Show that in an acute triangle with sides of lengths a_1, a_2, a_3 and opposite angles of radian measure A_1, A_2, A_3,

$$\prod_{k=1}^{3} \frac{(1-\cos A_k)}{\cos A_k} \geq \frac{8}{9} \frac{\sum_{k=1}^{3} a_k^2}{(\sum_{k=1}^{3} a_k)^2} \frac{(\sum_{k=1}^{3} \tan A_k)^3}{\prod_{k=1}^{3} (\tan A_k + \tan A_{k+1})}.$$

Solution by Pál Péter Dályay, Szeged, Hungary. Identifying $A_4 = A_1$, in an acute triangle we have $\sum_{k=1}^{3} \tan A_k > 0$ and $\prod_{k=1}^{3}(\tan A_k + \tan A_{k+1}) > 0$, so the inequality of the problem is equivalent to

$$\frac{\prod_{k=1}^{3}(\tan A_k + \tan A_{k+1})}{(\sum_{k=1}^{3} \tan A_k)^3} \frac{(\sum_{k=1}^{3} a_k)^2}{\sum_{k=1}^{3} a_k^2} \prod_{k=1}^{3} \frac{(1-\cos A_k)}{\cos A_k} \geq \frac{8}{9}. \tag{1}$$

Using trigonometric formulas and the relation $\sum_{k=1}^{3} \tan A_k = \prod_{k=1}^{3} \tan A_k$, we obtain

$$\frac{\prod_{k=1}^{3}(\tan A_k + \tan A_{k+1})}{(\sum_{k=1}^{3} \tan A_k)^3} = \left(\prod_{k=1}^{3} \frac{\sin(A_k + A_{k+1})}{\cos A_k \cos A_{k+1}}\right) \left(\prod_{k=1}^{3} \frac{\cos A_k}{\sin A_k}\right)^3 = \frac{\prod_{k=1}^{3} \cos A_k}{\prod_{k=1}^{3} \sin^2 A_k}. \tag{2}$$

Now use the relations $\sum_{k=1}^{3} \sin A_k = 4\prod_{k=1}^{3} \cos(A_k/2)$, $\sum_{k=1}^{3} \sin^2 A_k = \frac{1}{2}(3 - \sum_{k=1}^{3} \cos(2A_k)) = 2(1 + \prod_{k=1}^{3} \cos A_k)$, and the sine law to obtain

$$\frac{(\sum_{k=1}^{3} a_k)^2}{\sum_{k=1}^{3} a_k^2} = \frac{(\sum_{k=1}^{3} \sin A_k)^2}{\sum_{k=1}^{3} \sin^2 A_k} = \frac{16\prod_{k=1}^{3} \cos^2(A_k/2)}{2(1+\prod_{k=1}^{3} \cos A_k)} = \frac{\prod_{k=1}^{3}(1+\cos A_k)}{1+\prod_{k=1}^{3} \cos A_k}. \tag{3}$$

If L denotes the left-hand side of inequality (1), then using (2) and (3), we obtain

$$L = \left(\frac{\prod_{k=1}^{3} \cos A_k}{\prod_{k=1}^{3} \sin^2 A_k}\right) \left(\frac{\prod_{k=1}^{3}(1+\cos A_k)}{1+\prod_{k=1}^{3} \cos A_k}\right) \prod_{k=1}^{3} \frac{(1-\cos A_k)}{\cos A_k} = \frac{1}{1+\prod_{k=1}^{3} \cos A_k}.$$

Thus inequality (1) holds if and only if $1/(1+\prod_{k=1}^{3} \cos A_k) \geq 8/9$, that is, if and only if $\prod_{k=1}^{3} \cos A_k \leq 1/8$. This is a known inequality, but the proof is short: note that the function f given by $f(x) = \log(\cos x)$ is concave on $(0, \pi/2)$ because $f''(x) = -\cos^{-2} x < 0$. Thus we have

$$\sum_{k=1}^{3} \log(\cos A_k) \leq 3 \log\left(\cos\left(\frac{1}{3}\sum_{k=1}^{3} A_k\right)\right) = 3\log\left(\frac{1}{2}\right) = \log\left(\frac{1}{8}\right).$$

The required inequality follows. Equality holds when the triangle is equilateral.

Also solved by A. Alt, G. Apostolopoulos (Greece), P. De (India), M. Dincă (Romania), J. Fabrykowski & T. Smotzer, O. Faynshteyn (Germany), O. Kouba (Syria), O. P. Lossers (Netherlands), P. Nüesch (Switzerland), J. Rooin (Iran), C. R. & S. Selvaraj, R. Stong, M. Tetiva (Romania), M. Vowe (Switzerland), GCHQ Problem Solving Group (U.K.), and the proposer.

A Symmetric Inequality

11492 [2010, 278]. *Proposed by Tuan Le, student, Freemont High School, Anaheim, CA.* Show that for positive a, b, and c,

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}\geq\frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Solution by M. A. Prasad, India. First, note that $(\sqrt{a^3+b^3})/(a^2+b^2)\geq 1/\sqrt{a+b}$. Next, put

$$T_1=\left(\sum\sqrt{(b+c)(c+a)}\right)(a+b+c),\qquad T_2=6(ab+bc+ca),$$

where the sum is over all cyclic permutations of $\{a,b,c\}$. It suffices to show $T_1\geq T_2$. Let $x=\sqrt{a+b}$, $y=\sqrt{b+c}$, $z=\sqrt{c+a}$. Note $z^2<x^2+y^2$. Also

$$ab+bc+ca=\frac{(x^2+y^2+z^2)^2}{4}-\frac{x^4+y^4+z^4}{2}.$$

Let $D=4(T_1-T_2)$. Then

$$\begin{aligned}D&=12(x^4+y^4+z^4)+2(xy+yz+zx)(x^2+y^2+z^2)-6(x^2+y^2+z^2)^2\\&=3\sum(x^2-y^2)^2+2\sum xy(x-y)^2-\sum x^2(y-z)^2\\&=\sum(x-y)^2\big(3x^2+3y^2+8xy-z^2\big)\geq\sum(x-y)^2\big(2x^2+2y^2+8xy\big)\geq 0.\end{aligned}$$

Editorial comment. This problem can also be found at http://www.math.ust.hk/excalibur/v14_n2.pdf and http://ssmj.tamu.edu/problems/March-2010.pdf with two solutions at http://www.math.ust.hk/excalibur/v14_n3.pdf.

Also solved by D. Beckwith, P. P. Dályay (Hungary), P. De (India), O. Faynshteyn (Germany), G. C. Greubel, J. Grivaux (France), V. Krasniqi (Kosovo), J. H. Lindsey II, B. Mulansky (Germany), P. H. O. Pantoja (Brazil), P. Perfetti (Italy), J. Simons (U. K.), R. Stong, L. Zhou, GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, and the proposer.

Glaisher–Kinkelin

11494 [2010, 279]. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Let A be the Glaisher-Kinkelin constant, given by

$$A=\lim_{n\to\infty}n^{-n^2/2-n/2-1/12}e^{n^2/4}\prod_{k=1}^{n}k^k=1.2824\dots.$$

Show that

$$\prod_{n=1}^{\infty}\left(\frac{n!}{\sqrt{2\pi n}(n/e)^n}\right)^{(-1)^{n-1}}=\frac{A^3}{2^{7/12}\pi^{1/4}}.$$

Solution by Douglas B. Tyler, Raytheon, Torrance, CA. Let $p_n = \prod_{k=1}^{n} k^k$ and note that $p_n \sim An^{n^2/2+n/2+1/12}e^{-n^2/4}$. Let $a_n = n!/(\sqrt{2\pi n}(n/e)^n)$. By Stirling's formula, a_n tends to 1, and thus it suffices to evaluate the limit of the even-numbered partial products. Note that

$$\prod_{k=1}^{2n}(a_k)^{(-1)^{k-1}} = \prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}} = \prod_{k=1}^{n}\frac{(2k)^{2k}}{e\sqrt{2k(2k-1)}(2k-1)^{2k-1}}.$$

Rearranging gives

$$\frac{1}{e^n\sqrt{(2n)!}}\prod_{k=1}^{n}\frac{(2k)^{4k}}{(2k-1)^{2k-1}(2k)^{2k}} = \frac{2^{2n(n+1)}}{e^n\sqrt{(2n)!}}\cdot\frac{p_n^4}{p_{2n}}$$
$$\sim \frac{2^{2n(n+1)}}{(2n)^n\sqrt[4]{4\pi n}}\cdot\frac{A^4 n^{2n^2+2n+1/3}e^{-n^2}}{A(2n)^{2n^2+n+1/12}e^{-n^2}} = \frac{A^3}{\pi^{1/4}2^{7/12}}$$

as claimed.

Also solved by P. Bracken, B. S. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), J. Grivaux (France), E. A. Herman, O. Kouba (Syria), V. Krasniqi (Kosovo), O. P. Lossers (Netherlands), B. Mulansky (Germany), M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), P. .F. Refolio (Spain), J. Schlosberg, J. Simons (U. K.), S. D. Smith, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

Inequalities: One out of Two Ain't Bad

11497 [2010, 370]. *Proposed by Mihály Bencze, Brasov, Romania.* Given n real numbers $x_1, \ldots, x_n$ and a positive integer m, let $x_{n+1} = x_1$, and put

$$A = \sum_{k=1}^{n}\left(x_k^2 - x_k x_{k+1} + x_{k+1}^2\right)^m, \quad B = 3\sum_{k=1}^{n}x_k^{2m}.$$

Show that $A \le 3^m B$ and $A \le (3^m B/n)^n$.

Solution by M. A. Prasad, India. We first prove that

$$(x_1^2 - x_1x_2 + x_2^2)^m \le 3^m(x_1^{2m} + x_2^{2m})/2. \tag{1}$$

Now, $-x_1x_2 \le (x_1^2 + x_2^2)/2$, so $(x_1^2 - x_1x_2 + x_2^2)^m \le 3^m(x_1^2 + x_2^2)^m/2^m$. Next, $x_1^{2(m-r)}x_2^{2r} + x_1^{2r}x_2^{2(m-r)} \le x_1^{2m} + x_2^{2m}$, because

$$x_1^{2m} + x_2^{2m} - x_1^{2(m-r)}x_2^{2r} - x_1^{2r}x_2^{2(m-r)} = (x_2^{2r} - x_1^{2r})(x_2^{2(m-r)} - x_1^{2(m-r)}) \ge 0.$$

Now,

$$2(x_1^2 + x_2^2)^m = \sum_{r=0}^{m}\binom{m}{r}\left(x_1^{2(m-r)}x_2^{2r} + x_1^{2r}x_2^{2(m-r)}\right) \le \sum_{r=0}^{m}\binom{m}{r}\left(x_1^{2m} + x_2^{2m}\right),$$

which by the binomial theorem simplifies to $2^m\left(x_1^{2m} + x_2^{2m}\right)$, proving (1). Therefore,

$$A = \sum_{k=1}^{n}\left(x_k^2 - x_kx_{k+1} + x_{k+1}^2\right)^m \le \sum_{k=1}^{n}3^m\frac{x_k^{2m} + x_{k+1}^{2m}}{2} = 3^{m-1}B.$$

The second inequality, $A \le (3^m B/n)^n$, is incorrect: take $n = 10^{12}$, $m = 1$, and $x_1 = \cdots = x_n = 10^{-6}$. Now $A = 1$ and $B = 3$, so $(3^m B/n)^n = (9/10^{12})^{10^{12}} < A$.

Also solved by D. Beckwith, P. P. Dályay (Hungary), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), J. Simons (U. K.), R. Stong, Z. Vörös (Hungary), and GCHQ Problem Solving Group (U. K.).

A Power Series with Nonnegative Coefficients

11501 [2010, 834]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* (Corrected) Let

$$g(z) = 1 - \frac{3}{\frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of g about 0 are all nonnegative if and only if $a \geq \sqrt{3}$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Write

$$g(z) = \sum_{n=1}^{\infty} u_n z^n = \frac{az}{3} + \frac{2(a^2-3)z^2 + 8az^3}{3(3-2az+z^2)} = \frac{a}{3}z + \frac{2(a^2-3)}{9}z^2 + \cdots .$$

Thus, if $u_1, u_2 \geq 0$, then $a \geq \sqrt{3}$. If $a = \sqrt{3}$, then

$$g(z) = \frac{\sqrt{3}}{3}z + \frac{8(z/\sqrt{3})^3}{(1-z/\sqrt{3})^2},$$

so $u_n = 8(n-2)3^{-n/2} \geq 0$ for $n \geq 2$. Now suppose $a > \sqrt{3}$ and use partial fractions to expand in a power series as follows:

$$\begin{aligned}\frac{z}{3-2az+z^2} &= \frac{1}{2\sqrt{a^2-3}}\left(\frac{1}{1-\frac{a+\sqrt{a^2-3}}{3}z} - \frac{1}{1-\frac{a-\sqrt{a^2-3}}{3}z}\right) \\ &= \frac{1}{2\sqrt{a^2-3}}\sum_{m=1}^{\infty}\left[\left(\frac{a+\sqrt{a^2-3}}{3}\right)^m - \left(\frac{a-\sqrt{a^2-3}}{3}\right)^m\right]z^m.\end{aligned}$$

Hence all Taylor coefficients of $z/(3-2az+z^2)$ are nonnegative. Thus

$$\frac{2(a^2-3)z^2+8az^3}{3(3-2az+z^2)} = \frac{2(a^2-3)z+8az^2}{3}\cdot\frac{z}{3-2az+z^2}$$

also has all Taylor coefficients nonnegative, and therefore $g(z)$ does as well.

Also solved by M. Apagodu, G. Apostolopoulos (Greece), P. Bracken, N. Caro (Brazil), R. Chapman (U. K.), D. Constales (Belgium), P. P. Dályay (Hungary), Y. Dumont (France), O. Geupel (Germany), E. A. Herman, O. Kouba (Syria), K. McInturff, Á. Plaza & F. Perdomo (Spain), J. Simons (U. K.), M. Tetiva (Romania), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

A Twice Told Problem

11502, 11513 [2010, 458, 558]. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* For a triangle with area F, semiperimeter s, inradius r, circumradius R, and heights h_a, h_b, and h_c, show that

$$5(h_a+h_b+h_c) \geq \frac{2Fs}{Rr} + 18r \geq \frac{10r(5R-r)}{R}.$$

Solution by Vicente Vicario García, Huelva, Spain. Let a, b, and c be the side lengths of the triangle. We need the following result (Steinig's inequality or the first Gerretsen inequality; see, for example, Bottema et. al., *Geometric Inequalities,* Wolters-Noordhoff, Groningen, Netherlands, 1969, p. 50):

Lemma. *In any triangle,* $s^2 \geq 16Rr - 5r^2$.

This can be proved by rewriting the desired result in terms of x, y, and z, where $x = s - a$, $y = s - b$, and $z = s - c$, and noting that it rearranges to be Schur's inequality; it can also be proved by computing that $(s^2 - 16Rr + 5r^2)/9$ is the squared distance between the incenter and centroid.

Next, rewriting Heron's formula as

$$\begin{aligned} r^2 s^2 = F^2 &= s(s-a)(s-b)(s-c) \\ &= s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc \\ &= -s^3 + (ab+bc+ca)s - abc, \end{aligned}$$

and using Euler's formula $abc = 4Rrs$, we obtain the (well-known) fact that in any triangle, $ab + bc + ca = s^2 + 4Rr + r^2$. From this we compute

$$\begin{aligned} 5(h_a + h_b + h_c) - \frac{2Fs}{Rr} - 18r &= 10rs\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \frac{2s^2}{R} - 18r \\ &= \frac{10rs(ab+bc+ca)}{abc} - \frac{2s^2}{R} - 18r \\ &= \frac{5(s^2 + 4Rr + r^2)}{2R} - \frac{2s^2}{R} - 18r \\ &= \frac{s^2 - 16Rr + 5r^2}{2R} \geq 0. \end{aligned}$$

Also we have

$$\begin{aligned} \frac{2Fs}{Rr} + 18r - \frac{10r(5R - r)}{R} &= \frac{2s^2}{R} + 18r - \frac{10r(5R - r)}{R} \\ &= \frac{2(s^2 - 16Rr + 5r^2)}{R} \geq 0. \end{aligned}$$

Thus both of the desired inequalities follow from Gerretsen's inequality.

Editorial comment. The problem was accidentally republished as 11513.

Also solved by A. Alt, G. Apostolopoulos (Greece), M. Bataille (France), E. Braune (Austria), S. H. Brown, B. S. Burdick, M. Can, R. Chapman (U. K.), R. Cheplyaka, V. Lucic & L. Pebody, C. Curtis, J. Fabrykowski & T. Smotzer, O. Geupel (Germany), M. Goldenberg & M. Kaplan, B.-H. Gu (S. Korea), E. Hsynelaj (Australia) & E. Bojaxhiu (Germany), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, B. Mulansky (Germany), P. Nüesch (Switzerland), C. R. Pranesachar (India), R. Smith, R. Stong, M. Vowe (Switzerland), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before April 30, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11607. *Proposed by Jeffrey C. Lagarias and Andrey Mischenko, University of Michigan, Ann Arbor, MI.* Let C_0, C_1, C_2, C_3, with subscripts taken modulo 4, be circles in the Euclidean plane.

(a) Given for $k \in \mathbb{Z}_4$ that C_k and C_{k+1} intersect with orthogonal tangents, and the interiors of C_k and C_{k+2} are disjoint, show that the four circles have a common point.

(b)* Does the same conclusion hold in hyperbolic and spherical geometry?

11608. *Proposed by D. Aharonov and U. Elias, Technion—Israel Institute of Technology, Haifa, Israel.* Let f and g be functions on $\mathbb{R}$ that are differentiable $n+m$ times, where n and m are integers with $n \ge 1$ and $m \ge 0$. Let $A(x)$ be the $(n+m)\times(n+m)$ matrix given by

$$A_{j,k}(x) = \begin{cases} (f^{k-1}(x))^{(j-1)}, & \text{if } 1 \le j \le n; \\ (g^{k-1}(x))^{(j-1-n)}, & \text{if } n < j \le n+m, \end{cases}$$

Let $P = \prod_{r=1}^{n-1} r! \prod_{q=1}^{m-1} q!$. Prove that

$$\det A(x) = Pf(x)^n g(x)^m [g(x) - f(x)]^{mn} f'(x)^{n(n-1)/2} g'(x)^{m(m-1)/2}.$$

11609. *Proposed by M. N. Deshpande, Nagpur, India.* Let n be an integer greater than 1, and let $S_k(n)$ be the family of all subsets of $\{2, 3, \dots, n\}$ with k elements. Let $H(k) = \sum_{j=1}^{k} \frac{1}{j}$. Show that

$$\sum_{k=0}^{n-1} (2n+1-2k) \sum_{A \in S_k(n)} \prod_{j \in A} \frac{1}{j} = (n+1)((n+2) - H(n+1)).$$

http://dx.doi.org/10.4169/amer.math.monthly.118.10.936

11610. *Proposed by Richard P. Stanley, Massachussetts Institute of Technology, Cambridge, MA*. Let $f(n)$ be the number of binary words $a_1 \cdots a_n$ of length n that have the same number of pairs $a_i a_{i+1}$ equal to 00 as equal to 01. Show that

$$\sum_{n=0}^{\infty} f(n)t^n = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}}\right).$$

11611. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca Cluj, Romania.* Let f be a continuous function from [0, 1] into $[0, \infty)$. Find

$$\lim_{n\to\infty} n \int_{x=0}^{1} \left(\sum_{k=n}^{\infty} \frac{x^k}{k}\right)^2 f(x)\,dx.$$

11612. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Evaluate in closed form

$$\prod_{n=1}^{\infty} \left(\frac{n+z+1}{n+z}\right)^n e^{(2z-2n+1)/(2n)}.$$

11613. *Proposed by Stephen Morris, Newbury, U. K., and Stan Wagon, Macalester College, St. Paul, MN.* You are organizing a racing event with 25 horses on a track that can accommodate five horses per race. Each horse always runs the course in the same time, the 25 times are distinct, and you cannot use a stopwatch.

(a) Show how to arrange 7 races so that after all races are run, you will have enough information to determine which of the 25 horses present is fastest, which is next fastest, and which is third fastest. You may use the results of earlier races to schedule which horses compete in later races.

(b) Show that with just 6 races, it is not possible to be sure of knowing which are the top two horses.

(c) Give a procedure that uses 6 races and, with probability at least $3/10$, yields information sufficient to determine the fastest horse and the runner up. You have no a priori knowledge of the relative strengths of the 25 horses.

(d) Give a procedure that uses 6 races and, with probability at least $1/20$, yields information sufficient to determine which horse of the 25 is fastest, next fastest, and third fastest.

SOLUTIONS

More than Meets the Eye

11294 [2007,451]. *Proposed by John Zucker, King's College, London, U. K. and Ross McPhedran, University of Sydney, Sydney, Australia.* Show that

$$\sum_{m=-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^{m+n}}{(5m)^2+(5n+1)^2}\right\}$$
$$= \frac{\pi}{25}\left(\log(11+5\sqrt{5}) - \sqrt{5}\log\left(\sqrt{5}+1-\sqrt{5+2\sqrt{5}}\right)\right). \qquad (1)$$

Solution by Robin Chapman, University of Exeter, Exeter, U. K. The solution was too long to present in full detail; this is a condensed version. Let S be the value of the required sum. The double series in (1) does not converge absolutely, so care is needed, but the order of summation can be reversed. The Poisson summation formula is then applied to the new inner sum to yield (after simplification)

$$S = \frac{2\pi}{5} \sum_{r=0}^{\infty} \sum_{\substack{k \geq 1 \\ k \equiv \pm 1}} (-1)^{k-1} \frac{u^{(2r+1)k}}{k},$$

where $u = e^{-\pi/5}$ and the congruence in the inner sum is modulo 5. With $\zeta = e^{2\pi i/5}$, this can be rewritten as

$$S = \frac{\pi}{25} \log \prod_{r=0}^{\infty} \frac{\left(1+u^{2r+1}\right)^5}{1+u^{5(2r+1)}} + \frac{\pi\sqrt{5}}{25} \log \prod_{r=0}^{\infty} \frac{(1+\zeta u^{2r+1})(\zeta^{-1}u^{2r+1})}{(1+\zeta^2 u^{2r+1})(1+\zeta^{-2}u^{2r+1})}.$$

We define A and P by rewriting this as $S = \frac{\pi}{25}\left(\log A + \sqrt{5}\log P\right)$.

Now A can be related to the Dedekind eta function, defined for τ in the upper half-plane by

$$\eta(\tau) = \exp(\pi i \tau/12) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n \tau)).$$

The Weber function $\mathfrak{f}$ is now defined by

$$\begin{aligned} \mathfrak{f}(z) &= \exp(-\pi i/24)\eta((z+1)/2)/\eta(z) \\ &= \exp(-\pi i z/24) \prod_{n=1}^{\infty} (1 + \exp(\pi i (2n-1)z)); \end{aligned} \qquad (1)$$

it satisfies $\mathfrak{f}(z+2) = \exp(-\pi i/12)\mathfrak{f}(z)$ and $\mathfrak{f}(-1/z) = \mathfrak{f}(z)$. A straightforward computation now shows that $A = \mathfrak{f}(i/5)^5/\mathfrak{f}(i)$. The identities $\eta(-1/z) = \sqrt{z/i}\,\eta(z)$ and $f(z)^6 - f(z)^5 f(5z)^5 + 4f(z)f(5z) + f(5z)^6 = 0$ (see [5, Section 18.5] and [6, Section 7.14]) now allow for the evaluations $\mathfrak{f}(i) = 2^{1/4}$ and $\mathfrak{f}(i/5) = 2^{1/4}(1+\sqrt{5})/2$. For P, the Jacobi triple product identity ([6 Section 7.4], or `http://wikipedia.org/wiki/Triple_product_identity`) yields

$$\begin{aligned} P &= \prod_{r=0}^{\infty} \frac{(1+\zeta u^{2r+1})(\zeta^{-1}u^{2r+1})}{(1+\zeta^2 u^{2r+1})(1+\zeta^{-2}u^{2r+1})} \\ &= \prod_{r=0}^{\infty} \frac{(1+\zeta u^{2r+1})(\zeta^{-1}u^{2r+1})(1-u^{-2r})}{(1+\zeta^2 u^{2r+1})(1+\zeta^{-2}u^{2r+1})(1-u^{-2r})} \\ &= \frac{\sum_{m=-\infty}^{\infty} \zeta^m u^{m^2}}{\sum_{m=-\infty}^{\infty} \zeta^{2m} u^{m^2}} = \frac{\theta_1}{\theta_2}, \end{aligned}$$

where $\theta_j = \sum_{m=-\infty}^{\infty} \zeta^{jm} u^{m^2}$. Further calculations relate $\theta_1 + \theta_2$ and $\theta_1\theta_2$ to $\mathfrak{f}(i/5)$, $\mathfrak{f}(i)$, and $\eta(i/5)$. With this information in hand, θ_1 and θ_2 can be evaluated and the solution completed.

Editorial comment. Another full solution grew out of a partial solution provided by George Lamb. Lamb determined A and reduced the determination of P to the task of proving an intriguing identity. To state this identity, we introduce the standard notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

Lamb conjectured that if $q=e^{-\pi}$, then

$$\frac{1}{4}q^{1/6}\left\{q^{3/5}(-q;q^{10})_\infty^2(-q^9;q^{10})_\infty^2 + q^{-3/5}(-q^3;q^{10})_\infty^2(-q^7;q^{10})_\infty^2\right\} = 1. \qquad (3)$$

The editors then contacted Bruce Berndt. He and Mathew Rogers found that (3) could be reformulated in terms of Ramanujan's theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q|<1,$$

and the Rogers-Ramanujan continued fraction

$$R(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\ddots}}}.$$

With $q=e^{-\pi}$, the claimed identity, (3), can be written in the equivalent form

$$R^2(q^4)\left\{\frac{\varphi(q)}{\varphi(q^5)}+1\right\}^2 + R^{-2}(q^4)\left\{\frac{\varphi(q)}{\varphi(q^5)}-1\right\}^2 = 16q^{5/6}\frac{(q^{10};q^{10})_\infty^2}{\varphi^2(q^5)}. \qquad (4)$$

Ramanujan gave $128/(\sqrt{5}+1)^4$ for the right side of (4). As to the left side, Ramanujan's second notebook yields

$$\varphi\left(e^{-5\pi}\right) = \frac{\varphi(e^{-\pi})}{\sqrt{5\sqrt{5}-10}},$$

while in Ramanujan's first notebook one finds $R(e^{-4\pi}) = \sqrt{c^2+1}-c$, where $2c = 1+(5^{1/4}+1)\sqrt{5}/(5^{1/4}-1)$. Combining these gives (4); full details are available in [4]. The paper includes further results in the same vein. One additional special case of these is that

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+n}}{m^2+(3n+1)^2} = \frac{2\pi}{9}\log\left(2(\sqrt{3}-1)\right).$$

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5. Serge Lang, *Elliptic Functions* (2nd ed.), Springer-Verlag, New York, 1987.
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Also solved by the proposers. Partially solved George Lamb and by Albert Stadler (Switzerland).

Inequality via Equality

11504 and **11512** [2010, 458 and 558]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let N be a nonnegative integer. For $x \geq 0$, prove that

$$\sum_{m=0}^{N} \frac{1}{m!} \left(\sum_{k=1}^{N-m+1} \frac{x^k}{k} \right)^m \geq 1 + x + \cdots + x^N.$$

Solution by Nicholas C. Singer, Annandale, VA. The term corresponding to $m = 0$ is 1, so for $N = 0$ both sides are 1. Now assume $N \geq 1$. For nonnegative integer k and power series $f(x)$, let $[x^k]f(x)$ denote the coefficient of x^k in $f(x)$. All coefficients on the left side are nonnegative, so it suffices to prove that for $1 \leq s \leq N$,

$$[x^s] \sum_{m=1}^{N} \frac{1}{m!} \left(\sum_{k=1}^{N-m+1} \frac{x^k}{k} \right)^m = 1.$$

Let P be a polynomial of degree n, let Q be a polynomial or power series, and let m be a nonnegative integer. If $s < m$, then $[x^s](x^{n+2}Q(x) + xP(x))^m = 0 = [x^s](xP(x))^m$. If $m \leq s \leq n + m$, then $0 \leq s - m \leq n$ and

$$\begin{aligned} [x^s](x^{n+2}Q(x) + xP(x))^m &= [x^s](x^m(x^{n+1}Q(x) + P(x))^m) \\ &= [x^{s-m}](x^{n+1}Q(x) + P(x))^m \\ &= [x^{s-m}](P^m(x) + mP^{m-1}(x)x^{n+1}Q(x) + \cdots) \\ &= [x^{s-m}]P^m(x) \\ &= [x^s](xP(x))^m. \end{aligned}$$

We take $n = N - m$,

$$P(x) = \sum_{k=1}^{N-m+1} \frac{x^{k-1}}{k}, \qquad \text{and} \qquad Q(x) = \frac{\log \frac{1}{1-x} - xP(x)}{x^{N-m+2}}.$$

Since $\log(1/(1-x)) = \sum_{k=1}^{\infty} x^k/k$,

$$[x^s] \left(\log \frac{1}{1-x} \right)^m = [x^s] \left(\sum_{k=1}^{N-m+1} \frac{x^k}{k} \right)^m, \qquad 0 \leq s, m \leq N.$$

Let $\left[{s \atop m} \right]$ denote the coefficient of x^s in $\prod_{k=0}^{m-1}(x + k)$. (These coefficients are known as *unsigned Stirling numbers of the first kind*.) Then $\left[{n \atop 0} \right] = 0$ for $n > 0$ and $\left[{s \atop m} \right] = 0$ for $m > s$. By Graham, Knuth, and Patashnik, *Concrete Mathematics* (Addison-Wesley, Boston, 1988), 7.50 and 6.9, we have

$$[x^s] \ \log^m \frac{1}{1-x} = \frac{m!}{s!} \left[{s \atop m} \right], \qquad \sum_{m=0}^{n} \left[{n \atop m} \right] = n!.$$

The required result now follows:

$$[x^s]\sum_{m=1}^{N}\frac{1}{m!}\left(\sum_{k=1}^{N-m+1}\frac{x^k}{k}\right)^m=\sum_{m=1}^{N}\frac{1}{m!}\,[x^s]\,\log^m\frac{1}{1-x}=\sum_{m=1}^{N}\frac{1}{s!}\begin{bmatrix}s\\m\end{bmatrix}=1.$$

Editorial comment. The editors slipped up and posed the problem twice, having logged it into the system twice and gotten positive reviews both times, from different reviewers.

Also solved by G. Apostolopoulis (Greece), R. Bagby, D. Beckwith, R. Chapman (U. K.), R. Cheplyaka, V. Lucic & L. Pebody, P. P. Dályay (Hungary), E. Ehrenborg, M. Goldenberg & M. Kaplan, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. McDonald, P. Perfetti & R. Tauraso (Italy), B. Schmuland (Canada), J. Simons (U. K.), T. Starbird, A. Stenger, R. Stong, M. Tetiva (Romania), J. Vinuesa (Spain), H. Widmer (Switzerland), S. Xiao (Canada), S.-J. Yoon (Korea), BSI Problems Group (Germany), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

Equality of Integrals

11506 [2010, 459]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Show that for positive integers m and n with $m+n<mn$, and for positive a and b,

$$\sin\left(\frac{\pi}{n}\right)\int_{x=0}^{\infty}\frac{x^{1/n}}{x+a}\,\frac{b^{1/m}-x^{1/m}}{b-x}\,dx=\sin\left(\frac{\pi}{m}\right)\int_{x=0}^{\infty}\frac{x^{1/m}}{x+b}\,\frac{a^{1/n}-x^{1/n}}{a-x}\,dx.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. In fact, more generally, we will prove that

$$\sin(\alpha\pi)\int_0^{\infty}\frac{x^{\alpha}}{x+a}\,\frac{b^{\beta}-x^{\beta}}{b-x}\,dx=\sin(\beta\pi)\int_0^{\infty}\frac{x^{\beta}}{x+b}\,\frac{a^{\alpha}-x^{\alpha}}{a-x}\,dx$$

for every positive a, b, α, and β such that $\alpha+\beta<1$.

Define F by

$$F(a,\alpha,b,\beta)=\frac{\pi}{\sin(\alpha\pi)}\int_0^{\infty}\frac{x^{\beta}}{x+b}\,\frac{a^{\alpha}-x^{\alpha}}{a-x}\,dx.$$

For positive a,b,α,β such that $\alpha+\beta<1$, the defining integral converges. It remains to show that $F(a,\alpha,b,\beta)=F(b,\beta,a,\alpha)$.

Let $I(\alpha)=\int_0^{\infty}(t^{\alpha-1}\,dt)/(1+t)$. Putting $t=e^x$ gives $I(\alpha)=\int_{-\infty}^{\infty}(e^{\alpha x}\,dx)/(1+e^x)$. To compute this integral, we put $f(z)=e^{\alpha z}/(1-e^z)$, integrate f counterclockwise around the rectangle with vertices $\pm R\pm i\pi$, and then apply the residue theorem. The only singularity of $f(z)$ inside the rectangle is at the origin, where the residue is -1. The integrals along the vertical sides of the rectangle tend to 0 as $R\to\infty$ and the integrals along the horizontal sides sum to $-2i\sin(\alpha\pi)\int_{-R}^{R}(e^{\alpha x}\,dx)/(1+e^x)$. Hence $\sin(\alpha\pi)\int_{-\infty}^{\infty}(e^{\alpha x}\,dx)/(1+e^x)=\pi$, or $I(\alpha)=\pi/\sin(\alpha\pi)$.

Substituting t/λ for t in this equation, when λ is positive we get

$$\frac{\pi}{\sin(\alpha\pi)}\lambda^{\alpha}=\int_0^{\infty}\frac{\lambda}{t(\lambda+t)}\,t^{\alpha}\,dt.$$

It follows that, for $a>0$ and $x>0$,

$$\frac{\pi}{\sin(\alpha\pi)}(a^{\alpha}-x^{\alpha})=\int_0^{\infty}\left(\frac{a}{t(a+t)}-\frac{x}{t(x+t)}\right)t^{\alpha}\,dt=\int_0^{\infty}\frac{a-x}{(a+t)(x+t)}\,t^{\alpha}\,dt.$$

That is,

$$\frac{\pi}{\sin(\alpha\pi)}\frac{a^\alpha - x^\alpha}{a-x} = \int_0^\infty \frac{t^\alpha}{(a+t)(x+t)}\,dt.$$

Using this in the definition of F, and noting that the integrands are positive, we conclude that

$$F(a,\alpha,b,\beta) = \int_{x=0}^\infty \int_{t=0}^\infty \frac{x^\beta}{x+b}\cdot\frac{t^\alpha}{a+t}\cdot\frac{1}{x+t}\,dt\,dx.$$

Thus, $F(a,\alpha,b,\beta) = F(b,\beta,a,\alpha)$.

Editorial comment. Kouba remarked that

$$F(a,\alpha,b,\beta) = \frac{\pi^2}{a+b}\,\frac{b^{\alpha+\beta}\sin(\beta\pi) + a^{\alpha+\beta}\sin(\alpha\pi) - a^\alpha b^\beta \sin((\alpha+\beta)\pi)}{\sin(\alpha\pi)\sin(\beta\pi)\sin((\alpha+\beta)\pi)}.$$

Also solved by D. Beckwith, K. N. Boyadzhiev, R. Chapman (U. K.), H. Chen, J. A. Grzesik, G. Lamb, V. H. Moll, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Apply AM–GM

11514 [2010, 559]. *Proposed by Mihaly Bencze, Braşov, Romania.* Let k be a positive integer, and let $a_1,\dots,a_n$ be positive numbers such that $\sum_{i=1}^n a_i^k = 1$. Show that

$$\sum_{i=1}^n a_i + \frac{1}{\prod_{i=1}^n a_i} \ge n^{1-1/k} + n^{n/k}.$$

Solution by Pál Péter Dályay, Szeged, Hungary. By the AM–GM inequality, we have $1 = \sum_{i=1}^n a_i^k \ge n(\prod_{i=1}^n a_i)^{k/n}$, so $\prod_{i=1}^n a_i \le n^{-n/k}$. Using the AM–GM inequality again,

$$\sum_{i=1}^n a_i + \frac{1}{\prod_{i=1}^n a_i} \ge n\left(\prod_{i=1}^n a_i\right)^{1/n} + \frac{1}{\prod_{i=1}^n a_i}. \tag{1}$$

The function f given by $f(x) = nx^{1/n} + x^{-1}$ is differentiable on $\mathbb{R}^+$, and $f'(x) = x^{1/n-1} - x^{-2} = x^{-2}(x^{1+1/n} - 1)$. Note that $f'(x) < 0$ on $(0,1)$, so f is strictly decreasing on $(0,1]$. Since $0 < \prod_{i=1}^n a_i \le n^{-n/k} \le 1$, it follows that

$$n\left(\prod_{i=1}^n a_i\right)^{1/n} + \frac{1}{\prod_{i=1}^n a_i} \ge f\left(n^{-n/k}\right) = n^{1-1/k} + n^{n/k}. \tag{2}$$

The required inequality follows from (1) and (2).

Editorial comment. The stated result can be generalized. Five solvers observed that it is not necessary to restrict k to integer values. The most general result, obtained by Marian Tetiva (Romania), is that for $k > 0$,

$$\frac{\sum_{i=1}^n a_i}{\left(\sum_{i=1}^n a_i^k\right)^{1/k}} + \frac{\left(\sum_{i=1}^n a_i^k\right)^{n/k}}{\prod_{i=1}^n a_i} \ge n^{1-1/k} + n^{n/k}.$$

Also solved by R. Bagby, P. Bracken, M. Can, R. Chapman (U. K.), R. Cheplyaka & V. Lucic & L. Pebody, D. Fleischman, M. Goldenberg & M. Kaplan, S. Hazratpour (Iran), E. A. Herman, E. Hysnelaj (Australia) &

E. Bojaxhiu (Germany), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), P. Perfetti (Italy), K. Schilling, J. Simons (U. K.), S. Song (Korea), A. Stenger, R. Stong, M. Tetiva (Romania), E. I. Verriest, H. Widmer (Switzerland), Y. Xu (China), BSI Problems Group (Germany), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, and the proposer.

Hidden Telescope

11515 [2010, 559]. *Proposed by Estelle L. Basor, American Institute of Mathematics, Palo Alto, CA, Steven N. Evans, University of California, Berkeley, CA, and Kent E. Morrison, California Polytechnic State University, San Luis Obispo, CA.* Find a closed-form expression for

$$\sum_{n=1}^{\infty} 4^n \sin^4\left(2^{-n}\theta\right).$$

Solution by Nicolás Caro, Colombia. Since $\sin^4 x = \sin^2 x - \frac{1}{4}\sin^2(2x)$, we have $4^n \sin^4(2^{-n}\theta) = 4^n \sin^2(2^{-n}\theta) - 4^{n-1}\sin^2(2^{-(n-1)}\theta)$. So this is a telescoping series, with sum equal to

$$\begin{aligned}\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta) &= \lim_{n\to\infty} 4^n \sin^2(2^{-n}\theta) - \sin^2\theta \\ &= \lim_{n\to\infty}\left[\frac{\sin(2^{-n}\theta)}{2^{-n}}\right]^2 - \sin^2\theta = \theta^2 - \sin^2\theta.\end{aligned}$$

Editorial comment. Several solvers noted that the finite version of this sum appears as equation 1.362 in Gradshteyn and Ryzhik, *Table of Integrals, Series and Products*. The earliest reference provided (by Giorgio Malisani) was to item 47 on p. 33 of C.-A. Laisant, *Essai sur les fonctions hyperboliques*, Gauthier-Villars, 1874.

Also solved by U. Abel (Germany), T. Amdeberhan & V. H. Moll, R. Bagby, D. H. Bailey (U.S.A.) & J. M. Borwein (Canada), M. Bataille (France), D. Beckwith, M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), E. H. M. Brietzke (Brazil), M. A. Carlton, R. Chapman (U. K.), H. Chen, R. Cheplyaka & V. Lucic & L. Pebody, P. P. Dályay (Hungary), P. Deiermann, C. Delorme (France), P. J. Fitzsimmons, C. Georghiou (Greece), M. L. Glasser, M. Goldenberg & M. Kaplan, G. C. Greubel, J. M. Groah, J. W. Hagood, C. C. Heckman, E. A. Herman, C. Hill, E. Hysnelaj (Australia) & E. Bojaxhiu (Germany), W. P. Johnson, O. Kouba (Syria), G. Lamb, R. Lampe, W. C. Lang, O. P. Lossers (Netherlands), J. Magliano, G. Malisani (Italy), M. McMullen, N. Mecholsky & Y.-N. Yoon, B. Mulansky (Germany), S. Mutameni, R. Nandan, M. Omarjee (France), É. Pité (France), J. Posch, R. C. Rhoades, H. Riesel (Sweden), R. E. Rogers, O. G. Ruehr, B. Schmuland (Canada), C. R. & S. Selvaraj, J. Senadheera, B. Sim, R. A. Simón (Chile), J. Simons (U. K.), N. C. Singer, S. Singh, S. Song (Korea), A. Stenger, I. Sterling, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), N. Thornber, D. B. Tyler, E. I. Verriest, J. Vinuesa (Spain), M. Vowe (Switzerland), H. Widmer (Switzerland), J. B. Zacharias, S. M. Zemyan, BSI Problems Group (Germany), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, NSA Problems Group, and the proposers.

A Third-Derivative Integral Inequality

11517 [2010, 649]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f be a three-times differentiable real-valued function on $[a, b]$ with $f(a) = f(b)$. Prove that

$$\left|\int_a^{(a+b)/2} f(x)\,dx - \int_{(a+b)/2}^{b} f(x)\,dx\right| \le \frac{(b-a)^4}{192}\sup_{x\in[a,b]}|f'''(x)|.$$

Solution by John H. Smith, Boston College, Boston, MA. By simple changes of variables we may assume that $a = -1$, $b = 1$, and $f(-1) = f(1) = 0$. Since subtracting a multiple of $1 - x^2$ from $f(x)$ does not affect either side of the inequality, we may also assume that $f(0) = 0$. This suggests a comparison with $x^3 - x$, for which the inequality is equality. Multiplying by a suitable constant, we may assume that $\sup_{x\in[-1,1]} |f'''(x)| = 6$ and $\int_{-1}^{0} f(x)\,dx - \int_0^1 f(x)\,dx > 0$. Suppose that $\int_{-1}^{0} f(x)\,dx - \int_0^1 f(x)\,dx > \frac{1}{2}$; we show that this leads to a contradiction by finding a point h with $-1 < h < 1$ and $f'''(h) > 6$.

Let $g(x) = f(x) - (x^3 - x)$. Then $\int_{-1}^{0} g(x)\,dx - \int_0^1 g(x)\,dx > 0$, so either there exists c in $(-1, 0)$ at which g is positive, or c in $(0, 1)$ at which g is negative. We treat the first case; the second case is quite similar. Note that $g(-1) = g(0) = g(1) = 0$.

We are given c with $-1 < c < 0$ and $g(c) > 0$. Thus, there exist d_1, d_2, d_3 with $-1 < d_1 < c < d_2 < 0 < d_3 < 1$ such that g' is positive at d_1, negative at d_2, and 0 at d_3. Hence there are e_1 and e_2 with $d_1 < e_1 < d_2 < e_2 < d_3$ such that $g''(e_1) < 0$ and $g''(e_2) > 0$. We conclude that there exists h such that $e_1 < h < e_2$ and $g'''(h) > 0$. Equivalently, $f'''(h) > 6$.

Also solved by O. J. L. Alfonso (Colombia), K. F. Andersen (Canada), G. Apostolopoulos (Greece), R. Bagby, M. Benito & Ó. Ciaurri & E. Fernandéz & L. Roncal (Spain), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), N. Eldredge, M. Goldenberg & M. Kaplan, E. A. Herman, S. Hitotumatu (Japan), O. Kouba (Syria), E. Kouris (France), J. H. Lindsey II, B. Mulansky (Germany), W. Nuij (Netherlands), M. Omarjee (France), P. Perfetti (Italy), Á. Plaza (Spain), E. Poffald, K. Saha (India), J. Simons (U. K.), Z.-M. Song & L. Yin (China), A. Stenger, R. Stong, R. Tauraso (Italy), T. Trif (Romania), Barclays Capital Quantitative Analytics Group, GCHQ Problem Solving Group (U. K.), Matematicamente.It Forum Community (Italy), Mathramz Problem Solving Group, and the proposers.

A Zeta Inequality

11518 [2010, 649]. *Proposed by Mihaly Bencze, Brasov, Romania.* Suppose $n \ge 2$ and let $\lambda_1, \dots, \lambda_n$ be positive numbers such that $\sum_{k=1}^{n} 1/\lambda_k = 1$. Prove that

$$\frac{\zeta(\lambda_1)}{\lambda_1} + \sum_{k=2}^{n} \frac{1}{\lambda_k}\left(\zeta(\lambda_k) - \sum_{j=1}^{k-1} j^{-\lambda_k}\right) \ge \frac{1}{(n-1)(n-1)!}.$$

Solution by Barclays Capital Quantitative Analytics Group, London, U. K. Let L be the expression on the left side of the inequality. Since $\lambda_k > 1$, the usual series for the zeta function converges, so

$$L = \sum_{k=1}^{n} \frac{1}{\lambda_k} \sum_{j=k}^{\infty} j^{-\lambda_k} = \sum_{k=1}^{n} \frac{1}{\lambda_k} \sum_{j=0}^{\infty} (j+k)^{-\lambda_k}.$$

All of the terms are nonnegative, so we may rearrange them to obtain

$$L = \sum_{j=0}^{\infty} \sum_{k=1}^{n} \frac{(j+k)^{-\lambda_k}}{\lambda_k}.$$

The weighted arithmetic mean–geometric mean inequality gives

$$\sum_{k=1}^{n} \frac{(j+k)^{-\lambda_k}}{\lambda_k} \ge \prod_{k=1}^{n} \left((j+k)^{-\lambda_k}\right)^{1/\lambda_k} = \frac{1}{\prod_{k=1}^{n}(j+k)} = \frac{j!}{(j+n)!}.$$

Therefore

$$
\begin{aligned}
L \geq \sum_{j=0}^{\infty} \frac{j!}{(j+n)!} &= \sum_{j=0}^{\infty} \frac{1}{n-1} \left(\frac{(j+n)j!}{(j+n)!} - \frac{(j+1)j!}{(j+n)!} \right) \\
&= \frac{1}{n-1} \sum_{j=0}^{\infty} \left(\frac{j!}{(j+n-1)!} - \frac{(j+1)!}{(j+n)!} \right) \\
&= \frac{1}{n-1} \lim_{N\to\infty} \sum_{j=0}^{N} \left(\frac{j!}{(j+n-1)!} - \frac{(j+1)!}{(j+n)!} \right) \\
&= \frac{1}{n-1} \lim_{N\to\infty} \left(\frac{1}{(n-1)!} - \frac{(N+1)!}{(N+n)!} \right) \\
&= \frac{1}{(n-1)(n-1)!},
\end{aligned}
$$

as required.

Editorial comment. The GCHQ Problem Solving Group (U. K.) reported empirical evidence suggesting the stronger inequality

$$
\frac{\zeta(\lambda_1)}{\lambda_1} + \sum_{k=2}^{n} \frac{1}{\lambda_k} \left(\zeta(\lambda_k) - \sum_{j=1}^{k-1} j^{-\lambda_k} \right) > \frac{1}{(n-1)(n-1)!} + \frac{1}{(n+1)!}.
$$

Also solved by T. Amderberhan & V. De Angelis, R. Bagby, P. P. Dályay (Hungary), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Errata and End Notes for 2011.

Several readers noted a gap in the logic of the published solution (2010, 745) to 11384 (2008, 757). This solution relied on the alternating series test, which requires that the terms' absolute values eventually tend *monotonically* to zero. Some information about the distribution of primes is therefore required for a solution, contrary to the claim in the editorial comment.

Peter Nüesch wrote to inform us that problem 11552 (2011, 178) is a special case of problem 1320 by V. Komecny in *Mathematics Magazine* (1989, 137).

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before May 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11614. *Proposed by Moubinool Omarjee, Lycée Jean-Lurçat, Paris, France.* Let α be a real number with $\alpha > 1$, and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty}(u_n - u_{n+1})/u_n^\alpha$ exists and is nonzero. Prove that $\sum_{n=1}^\infty u_n$ converges if and only if $\alpha < 2$.

11615. *Proposed by Constantin Mateescu, Zinca Golescu High School, Pitesti, Romania.* Let A, B, and C be the vertices of a triangle, and let K be a point in the plane distinct from these vertices and the lines connecting them. Let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let D, E, and F be the intersections of the lines through MK and NP, NK and PM, and PK and MN, respectively. Prove that the parallels from D, E, and F to AK, BK, and CK, respectively, are concurrent.

11616. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Let $x_1, \dots, x_n$ be distinct points in $\mathbb{R}^3$, and let $k_1, \dots, k_n$ be positive real numbers. A test object at x is attracted to each of $x_1, \dots, x_n$ with a force along the line from x to x_j of magnitude $k_j\|x - x_j\|^2$, where $\|u\|$ denotes the usual euclidean norm of u. Show that when $n \geq 2$ there is a unique point x^* at which the net force on the test object is zero.

11617. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.* Let C be the ring of continuous functions on $\mathbb{R}$, equipped with pointwise addition and pointwise multiplication. Let D be the ring of differentiable functions on $\mathbb{R}$, equipped with the same addition and multiplication. The ring identity in both cases is the function f_1 on $\mathbb{R}$ that sends every real number to 1. Is there a subring E of D, containing f_1, that is isomorphic to C? (The ring isomorphism must carry f_1 to f_1.)

11618. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let a, b, c, and d be real numbers such that $a < c < d < b$ and $b - a = 2(d - c)$. Let S be the set of twice-differentiable functions from $[a, b]$ to $\mathbb{R}$ with continuous second derivative such that

http://dx.doi.org/10.4169/amer.math.monthly.119.01.068

$f(c) = f(d) = 0$ and $\int_{x=a}^{b} f(x)\,dx \neq 0$. Let p be a real number with $p > 1$. Show that the map ϕ from S to $\mathbb{R}$ given by

$$\phi(f) = \frac{\int_a^b |f''(x)|^p\,dx}{\left|\int_a^b f(x)\,dx\right|^p}$$

attains a minimum on S, and find that minimum in terms of p, a, b, c, d.

11619. *Proposed by Christopher Hillar, Mathematical Research Sciences Institute, Berkeley, CA.* Given an $n \times n$ complex matrix A, its *field of values* $\mathcal{F}(A)$ is given by

$$\mathcal{F}(A) = \{x^*Ax : \, x^*x = 1\}.$$

(Here, x^* is the conjugate transpose of x.) Call a matrix A *completely invertible* if 0 is not an element of $\mathcal{F}(A)$. Prove that if A is completely invertible then A^{-1} is also completely invertible.

11620. *Proposed by Mathew Rogers, Université de Montréal, Montreal, Canada.* Let H_k be the kth Hermite polynomial, given by $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$. Suppose

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho_1+\rho_1} & \frac{1}{\rho_1+\rho_2} & \cdots & \frac{1}{\rho_1+\rho_M} \\ \frac{1}{\rho_2+\rho_1} & \frac{1}{\rho_2+\rho_2} & \cdots & \frac{1}{\rho_2+\rho_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\rho_M+\rho_1} & \frac{1}{\rho_M+\rho_2} & \cdots & \frac{1}{\rho_M+\rho_M} \end{pmatrix} \begin{pmatrix} \frac{1}{\rho_1} \\ \frac{1}{\rho_2} \\ \vdots \\ \frac{1}{\rho_M} \end{pmatrix},$$

where $\rho_1, \dots, \rho_M$ are complex numbers for which $\sum_{k=1}^{M} 1/\rho_k > 0$. Prove that each ρ_k is a root of the equation

$$H_M(ix) - i\sqrt{2M}\,H_{M-1}(ix) = 0.$$

SOLUTIONS

Steiner–Lehmus Theorem

11511 [2010, 558]. *Proposed by Retkes Zoltan, Szeged, Hungary.* For a triangle ABC, let f_A denote the distance from A to the intersection of the line bisecting angle BAC with edge BC, and define f_B and f_C similarly. Prove that ABC is equilateral if and only if $f_A = f_B = f_C$.

Solution by H. T. Tang, Hayward, CA. The "only if" part is clear. The "if" part follows from the Steiner–Lehmus Theorem: *If the bisectors of the base angles of a triangle are equal, then the triangle is isosceles.* This problem was proposed in 1840 by D. C. Lehmus (1780–1863) to Jacob Steiner (1796–1863). For a proof of the Steiner–Lehmus Theorem, see for example N. Altschiller-Court, *College Geometry* (Johnson Pub. Co., Richmond, VA, 1925), p. 72–73; or L. S. Shiveley, *An Introduction to Modern Geometry* (Wiley & Sons, New York, 1884), p. 141.

Also solved by R. Bagby, M. Bataille (France), D. Beckwith, P. Budney, M. Can, R. Chapman (U. K.), R. Cheplyaka & V. Lucic & L. Pebody, J. E. Cooper III, C. Curtis, P. P. Dályay (Hungary), P. De (India), M. J. Englefield (Australia), D. Fleischman, V. V. García (Spain), J. Grivaux (France), E. A. Herman, L. Herot,

J. W. Kang (Korea), I. E. Leonard (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), R. E. Prather, J. Schaer (Canada), J. Simons (U. K.), S. Song (Korea), R. Stong, M. Tetiva (Romania), M. Vowe (Switzerland), S. V. Witt, Ellington Management Problem Solving Group, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, and the proposer.

Special Points on the Sphere

11521 [2010, 550]. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be a positive integer and let $A_1, \dots, A_n$, $B_1, \dots, B_n$, $C_1, \dots, C_n$ be points on the unit two-dimensional sphere $\mathbb{S}_2$. Let $d(X, Y)$ denote the geodesic distance on the sphere from X to Y, and let $e(X, Y)$ be the Euclidean distance across the chord from X to Y. Show that

(**a**) There exists $P \in \mathbb{S}_2$ such that $\sum_{i=1}^n d(P, A_i) = \sum_{i=1}^n d(P, B_i) = \sum_{i=1}^n d(P, C_i)$.

(**b**) There exists $Q \in \mathbb{S}_2$ such that $\sum_{i=1}^n e(Q, A_i) = \sum_{i=1}^n e(Q, B_i)$.

(**c**) There exist a positive integer n, and points $A_1, \dots, A_n, B_1, \dots, B_n, C_1, \dots, C_n$ on $\mathbb{S}_2$, such that for all $R \in \mathbb{S}_2$, $\sum_{i=1}^n e(R, A_i)$, $\sum_{i=1}^n e(R, B_i)$, and $\sum_{i=1}^n e(R, C_i)$ are not all equal. (That is, part (**b**) cannot be strengthened to read like part (**a**).)

Solution by Texas State Problem Solving Group, Texas State University, San Marcos, TX.

(**a**) If P and Q are any pair of points on $\mathbb{S}_2$, then Q lies on a great circle connecting P and its antipode $-P$, so $d(P, Q) + d(-P, Q) = d(P, -P) = \pi$. Let $f(P) = \sum_{i=1}^n d(P, A_i)$, $g(P) = \sum_{i=1}^n d(P, B_i)$, and $h(P) = \sum_{i=1}^n d(P, C_i)$. Note $F(P) = n\pi - f(-P)$, and the analogous equations hold for g and h. Let $r(P) = f(P) - g(P)$ and $s(P) = f(P) - h(P)$. Now

$$r(-P) = f(-P) - g(-P) = n\pi - f(P) - (n\pi - g(P)) = g(P) - f(P) = -r(P).$$

Similarly, $s(P) = -s(P)$. Thus $P \to (r(P), s(P))$ defines a continuous function from the sphere $\mathbb{S}_2$ into the plane R^2. The two-dimensional Borsuk-Ulam theorem says that for any continuous map from $\mathbb{S}_2$ to R^2, there exist antipodal points P and $-P$ in $\mathbb{S}_2$ that map to the same value. If $(r(P), s(P)) = (r(-P), s(-P)) = -(r(P), s(P))$, then $r(P) = s(P) = 0$ and $f(P) = g(P) = h(P)$ as required.

(**b**) For $P \in \mathbb{S}_2$ we can compute the average distance $\bar{e}_P = \int_{Q \in \mathbb{S}_2} e(Q, P) \frac{d\sigma}{4\pi}$ over the sphere from P. The group of isometries of the sphere is transitive, so this average does not depend on the point P, and we denote it by $\bar{e}$. Now let $w(Q) = \sum_{i=1}^n e(Q, A_i) - \sum_{i=1}^n e(Q, B_i)$. Computing $\bar{w} = \int_{Q \in \mathbb{S}_2} w(Q) \frac{d\sigma}{4\pi}$ term by term yields $\bar{w} = n\bar{e} - n\bar{e} = 0$. Pick any $Q_1 \in \mathbb{S}_2$ with $w(Q_1) \neq 0$. Without loss of generality, we may assume $w(Q_1) > 0$. Since $\bar{w} = 0$, there must exist some point $Q_2 \in \mathbb{S}_2$ such that $w(Q_2) < 0$. Now by the Intermediate Value Theorem, any continuous arc joining Q_1 and Q_2 must contain a point Q with $w(Q) = 0$ as required.

(**c**) Let $A_1 = A_2 = A_3 = (0, 0, 1)$. Let $B_1 = B_2 = B_3 = (0, 0, -1)$. Let C_1, C_2, C_3 be three points equally spaced on the great circle $x_3 = 0$. If a point Q contradicts the claim of (**c**), then $\sum_{i=1}^3 e(Q, A_i) = \sum_{i=1}^3 e(Q, B_i)$. Note that Q must lie on the horizontal great circle $(x_3 = 0)$ and we have $\sum_{i=1}^3 e(Q, A_i) = \sum_{i=1}^3 e(Q, B_i) = 3\sqrt{2}$. We may assume without loss of generality that Q lies on the arc of length $2\pi/3$ on the great circle connecting C_1 and C_2. Now $e(Q, C_1) + e(Q, C_2) < d(Q, C_1) + d(Q, C_2) = 2\pi/3$. Also $e(Q, C_3) \leq 2$, so $\sum_{i=1}^3 e(Q, C_i) < 2\pi/3 + 2$. Since $2\pi/3 + 2 < 3\sqrt{2}$, no point Q exists such that $e(Q, A_i) = \sum_{i=1}^3 e(Q, B_i) = \sum_{i=1}^3 e(Q, C_i)$.

Editorial comment. The result can easily be generalized to collections of points on $\mathbb{S}_m$ for $m > 2$. We may apply the general Borsuk-Ulam theorem for maps from $\mathbb{S}_m$ to $\mathbb{R}^m$ and the method of part (**a**) to obtain that if $A_{i,j}$ is a point on the sphere $\mathbb{S}_m$ for

$1 \le i \le n$ and $1 \le j \le m+1$, then there is some $P \in \mathbb{S}_m$ such that the $m+1$ sums $\sum_{i=1}^{n} d(P, A_{i,j})$ are equal. Also, the method of part (**b**) shows that there exists a point $P \in \mathbb{S}_m$ such that $\sum_{i=1}^{n} e(P, A_{i,1}) = \sum_{i=1}^{n} e(P, A_{i,2})$.

To see that in higher dimensions (**b**) cannot be strengthened to read like (**a**), let $A_{i,1} = E_{m+1}$ and let $A_{i,2} = -E_{m+1}$ for $1 \le i \le 2m$, where E_i is the vector with 1 in position i and other entries 0. Let $A_{i,3} = E_i$ and $A_{m+i,3} = -E_i$ for $1 \le i \le m$. If $P = (x_1, \ldots, x_{n+1})$ is a point such that $\sum_{i=1}^{n} e(P, A_{i,1}) = \sum_{i=1}^{n} e(P, A_{i,2})$, then $x_{n+1} = 0$ and $\sum_{i=1}^{n} e(P, A_{i,1}) = 2m\sqrt{2}$. However, for each antipodal pair E_i and $-E_i$, there exists a great circle of radius 1 containing P, E_i, and $-E_i$. Now $e(P, E_i) + e(P, -E_i)$ has maximum value $2\sqrt{2}$ on this circle, and this maximum is achieved at the two points exactly $\pi/2$ radians from E_i and $-E_i$, where $x_i = 0$. Hence, $\sum_{i=1}^{n} e(P, A_{i,3}) \le 2m\sqrt{2}$, with equality if and only if $x_1 = x_2 = \cdots = x_n = 0$. Since $x_{n+1} = 0$ and $P \in \mathbb{S}_{m+1}$, we have $x_i \ne 0$ for some i, and $\sum_{i=1}^{n} e(P, A_{i,3}) < 2m\sqrt{2}$. Hence no point P yields equal values of $\sum_{i=1}^{n} e(P, A_{i,j})$ for $1 \le j \le 3$.

Also solved by R. Chapman (U. K.), M. D. Meyerson, J. Simons (U. K.), R. Stong, Barclays Capital Quantitative Analytics Group (U. K.), and the proposer.

A 4-Volume

11522 [2010, 650]. *Proposed by Moubinool Omarjee, Lycée Jean Lurçat, Paris, France.* Let E be the set of all real 4-tuples (a, b, c, d) such that if $x, y \in \mathbb{R}$, then $(ax+by)^2 + (cx+dy)^2 \le x^2 + y^2$. Find the volume of E in $\mathbb{R}^4$.

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. The required volume is $2\pi^2/3$.

The condition on (a, b, c, d) is equivalent to the requirement that the matrix formed from the coefficients of the quadratic form q given by

$$q(x, y) = (1 - a^2 - c^2)x^2 - 2(ab + cd)xy + (1 - b^2 - d^2)y^2$$

be positive semidefinite. This is equivalent to

$$(ab + cd)^2 \le (1 - a^2 - c^2)(1 - b^2 - d^2)$$

with both of the factors on the right nonnegative. Multiplying this out and simplifying, we find that E is defined by the inequality,

$$a^2 + b^2 + c^2 + d^2 \le 1 + (ad - bc)^2, \tag{1}$$

along with the conditions $a^2 + c^2 \le 1$ and $b^2 + d^2 \le 1$. We may describe E in terms of a pair of polar coordinate systems by introducing $a = r\cos\theta$, $b = r\sin\theta$, $c = s\cos\phi$, $d = s\sin\phi$. Now E is parametrized by the conditions $r, s \in [0, 1]$ and $\theta, \phi \in [-\pi, \pi]$ with

$$r^2 + s^2 \le 1 + r^2 s^2 \sin^2(\theta - \phi),$$

which implies

$$0 \le r^2 \le \frac{1 - s^2}{1 - s^2 \sin^2(\theta - \phi)}.$$

Calling E' the set of all r, s, θ, ϕ that satisfy these conditions, the volume of E is given by the integral

$$
\begin{aligned}
|E| &= \int_{E'} rs\,drdsd\theta d\phi = \frac{1}{2}\int_0^1\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1-s^2}{1-s^2\sin^2(\theta-\phi)}\,s\,d\theta d\phi ds \\
&= 4\pi\int_0^1\int_0^{\pi/2}\frac{1-s^2}{1-s^2\sin^2\theta}\,s\,d\theta ds,
\end{aligned}
$$

by periodicity and symmetry. Note that $0 < 1-s^2 \le 1-s^2\sin^2(\theta-\phi)$ in the interior of E', so the integrand reflects the bound $r \le 1$. To perform the integration over θ, substitute $u = s\tan\theta$ with s held constant, so that $d\theta = s\,du/(s^2+u^2)$ and $\sin^2\theta = u^2/(s^2+u^2)$. For $0 < s < 1$, this yields

$$
\int_0^{\pi/2}\frac{1-s^2}{1-s^2\sin^2\theta}\,s\,d\theta = \int_0^{\infty}\frac{s^2(1-s^2)\,du}{s^2+(1-s^2)u^2} = \frac{\pi s}{2}\sqrt{1-s^2}.
$$

Therefore, the required volume is

$$
|E| = 2\pi^2\int_0^1 s\sqrt{1-s^2}\,ds = \frac{2}{3}\pi^2.
$$

Also solved by N. Caro (Brazil), R. Chapman (U. K.), W. J. Cowieson, E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, W. Nuij (Netherlands), P. Perfetti (Italy), J. Simons (U. K.), W. Song, R. Stong, M. Wildon (U. K.), L. Zhou, Barclays Capital Quantitative Analytics Group (U. K.), and the proposer.

The Short Vector Problem

11524 [2010, 741]. *Proposed by H. A. ShahAli, Tehran, Iran.* A vector v in $\mathbb{R}^n$ is *short* if $\|v\| \le 1$.
(**a**) Given six short vectors in $\mathbb{R}^2$ that sum to zero, show that some three of them have a short sum.
(**b**)* Let $f(n)$ be the least M such that, for any finite set T of short vectors in $\mathbb{R}^n$ that sum to 0, and any integer k with $1 \le k \le |T|$, there is a k-element subset S of T such that $\|\sum_{v\in S} v\| \le M$. The result of part (**a**) suggests $f(2) = 1$. Find $f(n)$ for $n \ge 2$.

Solution by the proposer. We need a preliminary result.
Lemma. *Given a collection of two or more short vectors in $\mathbb{R}^2$ that sum to zero, some two of them have a short sum.*

Proof. If one of the vectors is zero, then together with any other we have a short sum and are done. Now assume we have m vectors, all nonzero. We show that the angle θ between some two of them is at least $2\pi/3$, so the cosine of their angle is at most $-1/2$ and their sum is short. If the two vectors are u and w, with $\|u\| \le \|w\|$, then

$$
\begin{aligned}
\|u+w\|^2 &= \|u\|^2 + \|w\|^2 + 2\cos\theta\|u\|\|w\| \le \|u\|^2 + \|w\|^2 - \|u\|\|w\| \\
&= \|w\|^2 + \|u\|\big(\|u\| - \|w\|\big) \le \|w\|^2 \le 1.
\end{aligned}
$$

Write $v_1, \dots, v_m$ for the given vectors, with numbering to be determined. We may rotate coordinates so that one of the vectors, v_1, lies on the positive x-axis; let each vector v_j make angle θ_j with the positive x-axis. Thus $0 \le \theta_j < 2\pi$, with $\theta_1 = 0$. If all vectors lie on the x-axis, then (since their sum is 0) one of them (say v_2) lies on the negative x-axis, so $v_1 + v_2$ is short, and we are done. Now assume not all the vectors are on the x-axis. Because the sum is 0, at least one vector is in the upper half plane. Among these, let v_2 be one with the largest angle. Thus $0 < \theta_2 < \pi$. If $\theta_2 \ge 2\pi/3$, then $v_1 + v_2$ is short and we are done, so we may assume $0 < \theta_2 < 2\pi/3$.

The sum of all the vectors is zero, and one of them, v_1, is on one side of the line through 0 and v_2, so another of them, say v_3, is on the other side of that line, hence $\theta_2 < \theta_3 < \theta_2 + \pi$. Now $\theta_2 < \theta_3 < \pi$ is ruled out by the maximal choice for θ_2. If $2\pi/3 \le \theta_3 \le 4\pi/3$, then $v_1 + v_3$ is short and we are done. Thus we have $\theta_3 > 4\pi/3$, so $\theta_3 - \theta_2 < \pi$ and $\theta_3 - \theta_2 > 4\pi/3 - 2\pi/3 = 2\pi/3$, and so $v_2 + v_3$ is short. ■

(**a**). If one of the vectors is 0, then apply the Lemma to the remaining vectors to get two with short sum; with the zero vector we then have three with short sum. Now assume the given vectors are all nonzero. We write $v_1, \dots, v_6$ for the given vectors, with numbering to be determined. Apply the Lemma to $\{v_1, \dots, v_6\}$ to conclude that some two have short sum, say v_1 and v_2. Now apply the Lemma to $\{v_1 + v_2, v_3, v_4, v_5, v_6\}$ to conclude that some two have short sum. If $v_1 + v_2 + v_j$ is short for some j with $3 \le j \le 6$, we are done. Therefore, we may assume some two from $\{v_3, \dots, v_6\}$ have short sum, say v_3 and v_4. Now apply the Lemma to $\{v_1 + v_2, v_3 + v_4, v_5, v_6\}$ to conclude that some two have short sum. If that choice of two is one of $v_1 + v_2, v_3 + v_4$ and one of v_5, v_6, then we are done. If $v_1 + v_2 + v_3 + v_4$ is short, then so is $v_5 + v_6$. So we may assume $v_5 + v_6$ is short.

Now among the three short vectors u_{12}, u_{34}, and u_{56} given by $u_{12} = v_1 + v_2$, $u_{34} = v_3 + v_4$, and $u_{56} = v_5 + v_6$, there are two such that the angle θ between them satisfies $\theta \le 2\pi/3$, since the entire circle has circumference 2π. We consider two cases: First assume $\theta > \pi/3$. We rotate again to put the u_{jk}'s into the x-y plane, and, taking θ_{jk} to be the signed angle of u_{jk} with the positive x-axis,where $-\pi \le \theta_{jk} < \pi$, we spin them so that $\theta_{12} = -\theta/2$ and $\theta_{34} = \theta/2$. The sum is 0, and u_{12} and u_{34} are both in the right half-plane, while u_{56} is in the left half-plane. Thus at least one of v_5 and v_6 is in the left half-plane. Say v_5 is in the left half-plane. If v_5 is in the upper half-plane, then the angle between u_{12} and v_5 is greater than $2\pi - \theta/2 - \pi$, which is more than $2\pi/3$, and hence $u_{12} + v_5$ is short. Similarly, if v_5 is in the lower half-plane, then $u_{34} + v_5$ is short.

The other case is $\theta \le \pi/3$, so in particular $\theta < \pi/2$ and the dot product $u_{12} \cdot u_{34}$ is positive. Compute

$$\begin{aligned}
&\|v_5 + u_{12}\|^2 + \|v_5 + u_{34}\|^2 + \|v_6 + u_{12}\|^2 + \|v_6 + u_{34}\|^2 \\
&\quad = 2\big(\|v_5\|^2 + \|v_6\|^2 + \|u_{12}\|^2 + \|u_{34}\|^2 + (v_5 + v_6) \cdot (u_{12} + u_{34})\big) \\
&\quad = 2\big(\|v_5\|^2 + \|v_6\|^2 + \|u_{12}\|^2 + \|u_{34}\|^2 + u_{56} \cdot (u_{12} + u_{34})\big) \\
&\quad = 2\big(\|v_5\|^2 + \|v_6\|^2 + \|u_{12}\|^2 + \|u_{34}\|^2 - (u_{12} + u_{34}) \cdot (u_{12} + u_{34})\big) \\
&\quad = 2\big(\|v_5\|^2 + \|v_6\|^2 - 2\, u_{12} \cdot u_{34}\big) \le 4.
\end{aligned}$$

Therefore at least one vector in $\{v_5 + u_{12}, v_5 + u_{34}, v_6 + u_{12}, v_6 + u_{34}\}$ is short.

Editorial comment. The other solution for (**a**) also involves taking cases. No solution for (**b**) was received. The proposer conjectures that $f(n) = \sqrt{2 - \frac{2}{\max\{2,n\}}}$, which is achieved when $k = 2$ and the vectors are the vertices of the regular simplex centered at the origin.

Part (**a**) also solved by Barclays Capital Quantitative Analytics Group (U. K.).

Plane Geometric Arrangements

11525 [2010, 741]. *Proposed by Grigory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.*

(**a**) Prove that for each $n \geq 3$ there is a set of regular n-gons in the plane such that every line contains a side of exactly one polygon from this set.
(**b**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly one circle from the set?
(**c**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly two circles from the set?
(**d**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly three circles from the set?

Composite Solution by Jim Simons, Cheltenham, U. K., and Barclays Capital Quantitative Analytics Group, London, U. K.
(**a**) For odd n, consider any regular n-gon W. Choose a direction d that is not parallel to any side of W, and consider the set of all translations of W in the direction d. Clearly, if a line l is parallel to a side of W, then it is a side of exactly one of these n-gons.

Now take the n-gons described above and rotate each of them clockwise about the origin by all possible angles in the range $[0, \frac{2\pi}{n})$. Every line l can be made parallel to a side of W by rotating it counterclockwise around the origin by exactly one angle in this range. Thus, every line is the side of exactly one of these n-gons.

For even n, the construction above does not quite work. Since opposite sides of W are parallel, the construction above would produce a set of polygons with every line containing a side of two of them. We modify the construction by choosing the direction d more carefully and allowing only certain translates. Suppose the initial polygon has width w between two parallel sides and the direction d makes an angle α with these sides. The two polygons that differ by a translation in the d direction through a distance $w \csc \alpha$ have sides that lie on the same line. Suppose we choose a set $X \subset \mathbb{R}$ of translation amounts such that for all x exactly one of $\{x, x + w \csc \alpha\}$ is in X. Every line parallel to this pair of sides will contain a side of exactly one translate, as required.

Let $m = n/2$. The m pairs of parallel sides all have the same width w and make angles of $\alpha + 2\pi k/n$ with d, where $0 \leq k \leq m - 1$. Thus we need only show that there is a subset $X \subset \mathbb{R}$ such that for every x and every k exactly one of $\{x, x + w \csc(\alpha + 2\pi k/n)\}$ lies in X. For $0 \leq k \leq m - 1$, let $D_k = w \csc(\alpha + 2\pi k/n)$. If these D_k are linearly dependent over $\mathbb{Q}$, then there are integers r_k, not all zero, such that $\sum_{k=0}^{m-1} r_k \csc(\alpha + 2\pi k/n) = 0$. The left side of this equation written in terms of complex exponentials is a rational function of $e^{i\alpha}$. If $r_k \neq 0$, then this rational function has a pole at $e^{i\alpha} = e^{-2\pi i k/n}$ and hence is non-trivial. Hence the equation has only finitely many solutions. Thus there are only countably many α for which the D_k are linearly dependent over $\mathbb{Q}$.

Choose the direction d so that the distances D_k are linearly independent over $\mathbb{Q}$. Choose a Hamel basis of $\mathbb{R}$ containing these distances, and for $a \in \mathbb{R}$ let $p_k(a)$ be the coefficient of D_k in the expansion of a in this basis. Now one can take

$$X = \left\{ a : \sum_{k=0}^{m-1} \lfloor p_k(a) \rfloor \equiv 0 \pmod 2 \right\}$$

as the set of translations. This set has the required property.
(**b**) There is no such set of circles. If two circles are not nested, then there is a line tangent to both. Therefore in any such set the circles would have to be nested, totally ordered by radius. The intersection of the compact circular discs defined by these circles would be a non-empty closed set F and any line intersecting F would not be tangent to any of the circles.

(**c**, **d**). Using the Axiom of Choice, we will show that for any $k > 1$, there is a set of circles in the plane such that every line in the plane is tangent to exactly k circles from the set.

Let $\mathfrak{c}$ be the cardinality of the real numbers, and let $\omega_{\mathfrak{c}}$ be the first ordinal of this cardinality. There are $\mathfrak{c}$ lines in the plane, so they can be indexed as $\{l_\alpha : \alpha < \omega_{\mathfrak{c}}\}$. Using transfinite induction, we construct the set C of circles and simultaneously the set L of lines in the plane tangent to precisely k of the circles in C. Initially C and L are empty. Throughout the induction we will have no line in the plane tangent to more than k of the circles in C, and $|C| < \mathfrak{c}$. Note that for each pair of circles there are at most 4 lines tangent to both of them, and therefore $|L| \le 4\left|\binom{C}{2}\right| < \mathfrak{c}$.

For the inductive step, let α be the smallest ordinal for which $l_\alpha \notin L$. At any given point of l_α there are $\mathfrak{c}$ circles tangent to l_α, but fewer than $\mathfrak{c}$ of these are already in C or are tangent to a line in L. Hence we can choose a circle, not already in C, that is tangent to l_α, and whose addition does not produce a line tangent to more than k circles. Repeating this construction at most k times, we shall be adding l_α to L. At the point when we are up to step α in the induction, we have $|C| \le k|\alpha| < \mathfrak{c}$ as required for the induction step. The construction ends when we reach ordinal $\omega_{\mathfrak{c}}$, and at this point L is all lines in the plane.

Editorial comment. The proposers showed that (**c**) holds without requiring the Axiom of Choice. Simply take the set to consist of all circles whose radius is an odd integer and whose center is on the unit circle.

We did not count a single point as a degenerate polygon or circle.

Also solved by GCHQ Problem Solving Group (U. K.)(part b), and the proposers (parts a–c).

Expanders Increase Dimension

11526 [2010, 742]. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Prove that there is no function f from $\mathbb{R}^3$ to $\mathbb{R}^2$ with the property that $\|f(x) - f(y)\| \ge \|x - y\|$ for all $x, y \in \mathbb{R}^3$.

Solution by Ralph Howard, University of South Carolina, Columbia, SC. When (X, d) is a metric space and $k \in (0, \infty)$, write $\mathcal{H}^k_{(X,d)}$ for the k-dimensional Hausdorff outer measure defined on the subsets of X. The Hausdorff dimension of (X, d) is

$$\dim_{\mathcal{H}}(X, d) = \inf\{k \in (0, \infty) : \mathcal{H}^k_{(X,d)}(X) = 0\}.$$

Recall that, with the usual metric, $\dim_{\mathcal{H}}(\mathbb{R}^n) = n$.

If (X, d_X) and (Y, d_Y) are metric spaces, call a map $f : X \to Y$ an *expanding map* if $d_Y(f(x_1), f(x_2)) \ge d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. Such a map need not be continuous, but it is clearly injective. It suffices to prove the following: if there is an expanding map from (X, d_X) to (Y, d_Y), then $\dim_{\mathcal{H}}(X, d_X) \le \dim_{\mathcal{H}}(Y, d_Y)$.

Let $f : X \to Y$ be an expanding map. The image $f[X]$ is a subset of Y, so $\dim_{\mathcal{H}}(f[X]) \le \dim_{\mathcal{H}}(Y)$. Without loss of generality, we may replace Y by $f[X]$ and assume that f is surjective and thus bijective. Now f has an inverse $g : Y \to X$. As f is an expanding map, g is a contraction; that is, $d_X(g(y_1), g(y_2)) \le d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$. For $k > 0$ and $S \subseteq Y$, it follows directly from the definition of the Hausdorff outer measures that $\mathcal{H}^k_{(X,d_X)}(g[S]) \le \mathcal{H}^k_{(Y,d_Y)}(S)$. Thus, since g is surjective, we have $\dim_{\mathcal{H}}(X, d_X) \le \dim_{\mathcal{H}}(Y, d_Y)$.

Also solved by N. Eldredge, O. Geupel (Germany), J. Grivaux (France), E. A. Herman, O. P. Lossers (Netherlands), K. Schilling, J. Simons (U. K.), R. Stong, Barclays Capital Quantitative Analytics Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before April 30, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11621. *Proposed by Z. K. Silagadze, Budker Institute of Nuclear Physics and Novosibirsk State University, Novosibirsk, Russia.* Find

$$\int_{s_1=-\infty}^{\infty}\int_{s_2=-\infty}^{s_1}\int_{s_3=-\infty}^{s_2}\int_{s_4=-\infty}^{s_3}\cos(s_1^2-s_2^2)\cos(s_3^2-s_4^2)\,ds_4\,ds_3\,ds_2\,ds_1.$$

11622. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* In triangle ABC, let A_1, B_1, C_1 be the points opposite A, B, C at which symmedians of the triangle meet the opposite sides. Prove that

$$m_a(c\cos\alpha_1-b\cos\alpha_2)+m_b(a\cos\beta_1-c\cos\beta_2)+m_c(b\cos\gamma_1-a\cos\gamma_2)=0,$$

$$m_a(\sin\alpha_1-\sin\alpha_2)+m_b(\sin\beta_1-\sin\beta_2)+m_c(\sin\gamma_1-\sin\gamma_2)=0,\ \text{and}$$

$$m_a(\cos\alpha_1+\cos\alpha_2)+m_b(\cos\beta_1+\cos\beta_2)+m_c(\cos\gamma_1+\cos\gamma_2)=3s,$$

where a, b, c are the lengths of the sides, m_a, m_b, m_c are the lengths of the medians, s is the semiperimeter, $\alpha_1=\angle CAA_1$, $\alpha_2=\angle A_1AB$, and similarly with the β_j and γ_j.

11623. *Proposed by Aruna Gabhe, Pendharkar's College, Dombivali, India, and M. N. Deshpande, Nagpur, India.* A fair coin is tossed n times and the results recorded as a bit string. A *run* is a maximal subsequence of (possibly just one) identical tosses. Let the random variable X_n be the number of runs in the bit string not immediately followed by a longer run. (For instance, with bit string 1001101110, there are six runs, of lengths 1, 2, 2, 1, 3, and 1. Of these, the 2nd, 3rd, 5th, and 6th are not followed by a longer run, so $X_{10}=4$.) Find $E(X_n)$.

11624. *Proposed by David Callan, University of Wisconsin, Madison, WI, and Emeric Deutsch, Polytechnic Institute of NYU, Brooklyn, NY.* A Dyck n-path is a lattice path of

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n upsteps U (changing by $(1, 1)$) and n downsteps D (changing by $(1, -1)$) that starts at the origin and never goes below the x-axis. A peak is an occurrence of UD and the peak height is the y-coordinate of the vertex between its U and D.

The peak heights multiset of a Dyck path is the set of peak heights for that Dyck path, with multiplicity. For instance, the peak heights multiset of the Dyck 3-path $UUDUDD$ is $\{2, 2\}$. In terms of n, how many different multisets occur as the peak heights multiset of a Dyck n-path?

11625. *Proposed by Lane Bloome, Peter Johnson, and Nathan Saritzky (students) Auburn University Research Experience for Undergraduates in Algebra and Discrete Mathematics 2011.* Let $V(G)$, $E(G)$, and $\chi(G)$ denote respectively the vertex set, edge set, and chromatic number of a simple graph G. For each positive integer n, let $g(n)$ and $h(n)$ respectively denote the maximum and the minimum of $\chi(G) + \chi(H) - \chi(G \cup H)$ over all pairs of simple graphs G and H with $|V(G) \cup V(H)| \leq n$ and $E(G) \cap E(H) = \emptyset$. Find $g(n)$ and $\lim_{n\to\infty} \frac{h(n)}{n}$.

11626. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let x_1, x_2, and x_3 be positive numbers such that $x_1 + x_2 + x_3 = x_1x_2x_3$. Treating indices modulo 3, prove that

$$\sum_1^3 \frac{1}{\sqrt{x_k^2+1}} \leq \sum_1^3 \frac{1}{x_k^2+1} + \sum_1^3 \frac{1}{\sqrt{(x_k^2+1)(x_{k+1}^2+1)}} \leq \frac{3}{2}.$$

11627. *Proposed by Samuel Alexander, The Ohio State University, Columbus, Ohio.* Let $\mathbb{N}$ be the set of nonnegative integers. Let M be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. For a function f_0 from an interval $[0, m]$ in $\mathbb{N}$ to $\mathbb{N}$, say that f *extends* f_0 if $f(n) = f_0(n)$ for $0 \leq k \leq m$. Let $F(f_0)$ be the set of all extensions in M of f_0, and equip M with the topology in which the open sets of M are unions of sets of the form $F(f_0)$. Thus, $\{f \in M\colon f(0) = 7 \text{ and } f(1) = 11\}$ is an open set.

Let S be a proper subset of M that can be expressed both as $\bigcup_{i\in\mathbb{N}}\bigcap_{j\in\mathbb{N}} X_{i,j}$ and as $\bigcap_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} Y_{i,j}$, where each set $X_{i,j}$ or $Y_{i,j}$ is a subset of M that is both closed and open (clopen). Show that there is a family $Z_{i,j}$ of clopen sets such that $S = \bigcup_{i\in\mathbb{N}}\bigcap_{j\in\mathbb{N}} Z_{i,j}$ and $S = \bigcap_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} Z_{i,j}$.

SOLUTIONS

Eigenvalues of Sums and Differences of Idempotent Matrices

11466 [2009, 845]. *Proposed by Tian Yongge, Central University of Finance and Economics, Beijing, China.* For a real symmetric $n \times n$ matrix A, let $r(A)$, $i_+(A)$, and $i_-(A)$ denote the rank, the number of positive eigenvalues, and the number of negative eigenvalues of A, respectively. Let $s(A) = i_+(A) - i_-(A)$. Show that if P and Q are symmetric $n \times n$ matrices, $P^2 = P$, and $Q^2 = Q$, then $i_+(P - Q) = r(P + Q) - r(Q)$, $i_-(P - Q) = r(P + Q) - r(P)$, and $s(P - Q) = r(P) - r(Q)$.

Solution by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Dortmund University of Technology, Dortmund, Germany. We solve the more general problem in which idempotent P and Q are Hermitian with complex entries. We view them as $n \times n$ complex orthogonal projectors.

The solution is based on a joint decomposition of the projectors P and Q. Let P have rank ρ, where $0 < \rho \leq n$. By the Spectral Theorem, there is an $n \times n$ unitary matrix U such that

$$P = U \begin{pmatrix} I_\rho & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

where I_ρ is the identity matrix of order ρ and U^* denotes the conjugate transpose of U. We use this expression for P to partition the projector Q. Using the same matrix U, we write

$$Q = U \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} U^*,$$

where A and D are Hermitian matrices of orders ρ and $n - \rho$, respectively. Let $\bar{A} = I_\rho - A$. Since $Q^2 = Q$, we have $\bar{A} = \bar{A}^2 + BB^* = \bar{A}\bar{A}^* + BB^*$. Since $\bar{A}\bar{A}^*$ and BB^* are both nonnegative definite, $\bar{A}$ is also nonnegative definite. With $\mathcal{R}(\cdot)$ denoting the column space of a matrix argument, we obtain

$$\mathcal{R}(\bar{A}) = \mathcal{R}(\bar{A}\bar{A}^* + BB^*) = \mathcal{R}(\bar{A}\bar{A}^* + \mathcal{R}(BB^*) = \mathcal{R}(\bar{A}) + \mathcal{R}(B),$$

and hence $\mathcal{R}(B) \subseteq \mathcal{R}(\bar{A})$. Other relationships among A, B, and D are found in Lemmas 1–5 of [1]; we use two of these. The first expresses the orthogonal projector P_D onto the column space of D as $P_D = D + B^*\bar{A}B$, where $\bar{A}$ is the Moore–Penrose inverse of $\bar{A}$. The second expresses the rank of $\bar{A}$ as $r(\bar{A}) = \rho - r(A) + r(B)$. Furthermore, Theorem 1 of [1] gives $r(Q) = r(A) - r(B) + r(D)$, and Lemma 6 of [1] gives $r(P + Q) = \rho + r(D)$. Taking differences of these expressions yields $r(P + Q) - r(Q) = \rho - r(A) + r(B)$ and $r(P + Q) - r(P) = r(D)$. Since the third of the desired equations is just the difference of the first two, it suffices to show that $P - Q$ has $r(\bar{A})$ positive eigenvalues and $r(D)$ negative eigenvalues.

Theorem 5 in [1] expresses $P - Q$ as

$$P - Q = U \begin{pmatrix} \bar{A} & -B \\ -B^* & -D \end{pmatrix} U^*,$$

which can be rewritten as

$$P - Q = U \begin{pmatrix} I_\rho & 0 \\ -B^*\bar{A} & I_{n-\rho} \end{pmatrix} \begin{pmatrix} \bar{A} & 0 \\ 0 & -P_D \end{pmatrix} \begin{pmatrix} I_\rho & -\bar{A}B \\ 0 & I_{n-\rho} \end{pmatrix} U^*.$$

The matrices before and after the central matrix in the product on the right are nonsingular and are conjugate transposes of each other. By Sylvester's Law of Inertia (see [2, Section 1.3]), the numbers of positive and negative eigenvalues are unchanged by conjugation. Since $\bar{A}$ and P_D are nonnegative definite (the eigenvalues of an idempotent matrix lie in the interval [0, 1]), we conclude that $P - Q$ has $r(\bar{A})$ positive eigenvalues and $r(D)$ negative eigenvalues, as desired.

[1] O. M. Baksalary and G. Trenkler, Eigenvalues of functions of orthogonal projectors, *Linear Alg. Appl.* **431** (2009) 2172–2186.

[2] R. A. Horn, F. Zhang, Basic properties of the Schur complement, in *The Schur Complement and its Applications,* edited by F. Zhang, Springer Verlag, New York, 2005, 17–46.

Also solved by R. Chapman (U. K.), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, K. Schilling, J. Simons (U. K.), R. Stong, Z. Vörös (Hungary), S. Xiao (Canada), GCHQ Problem Solving Group (U. K.), and the proposer.

A Hankel Determinant Limit

11471 [2009, 941]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let A be an $r \times r$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. For $n \geq 0$, let $a(n)$ be the trace of A^n. Let $H(n)$ be the $r \times r$ Hankel matrix with (i, j) entry $a(i + j + n - 2)$. Show that

$$\lim_{n\to\infty} |\det H(n)|^{1/n} = \prod_{k=1}^{r} |\lambda_k|.$$

Solution by Jim Simons, Cheltenham, U. K. The eigenvalues of A^n are $\lambda_1^n, \ldots, \lambda_r^n$, so $a(n) = \sum_{k=1}^{r} \lambda_k^n$. Therefore, $H(n)_{i,j} = \sum_{k=1}^{r} \lambda_k^{n+i+j-2}$. It is well known that the Vandermonde matrix V, given by $V_{i,j} = \lambda_i^{j-1}$ for $i, j \in \{1, \ldots, n\}$, has determinant $\prod_{j<i}(\lambda_i - \lambda_j)$. Multiplying row i of V by λ_i^n yields a matrix $V(n)$ in which $V(n)_{i,j} = \lambda_i^{n+j-1}$, having determinant $(\prod_{k=1}^{r} \lambda_k)^n \prod_{j<i}(\lambda_i - \lambda_j)$. With V' denoting the transpose of V,

$$(V'V(n))_{i,j} = \sum_{k=1}^{r} \lambda_k^{i-1}\lambda_k^{n+j-1} = \sum_{k=1}^{r} \lambda_k^{n+i+j-2} = H(n)_{i,j}.$$

Therefore,

$$\det H(n) = \det(V'V(n)) = \left(\prod_{k=1}^{r} \lambda_k\right)^n \prod_{j<i}(\lambda_i - \lambda_j)^2.$$

The second factor is constant, so its nth root tends to 1.

Editorial comment. Simons observed that the computation of $\det H(n)$ is valid over any field.

Also solved by R. Chapman (U. K.), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Stong, E. I. Verriest, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Pretty Boxes All in a Row

11477 [2010, 86]. *Proposed by Antonio González, Universidad de Sevilla, Seville, Spain, and José Heber Nieto, Univesidad del Zulia, Maracaibo, Venezuela.* Several boxes sit in a row, numbered from 0 on the left to n on the right. A frog hops from box to box, starting at time 0 in box 0. If at time t, the frog is in box k, it hops one box to the left with probability k/n and one box to the right with probability $1 - k/n$. Let $p_t(k)$ be the probability that the frog launches its $(t + 1)$th hop from box k. Find $\lim_{i\to\infty} p_{2i}(k)$ and $\lim_{i\to\infty} p_{2i+1}(k)$.

Solution by Robin Chapman, University of Exeter, Exeter, U. K. We show that $\lim_{i\to\infty} p_{2i}(k)$ is $\binom{n}{k}/2^{n-1}$ when k is even and 0 when k is odd. Also, $\lim_{i\to\infty} p_{2i+1}$ is $\binom{n}{k}/2^{n-1}$ when k is odd and 0 when k is even.

In standard language, we have a Markov chain with states $0, \ldots, n$ and transition probabilities $p_{k,k-1} = k/n$ and $p_{k,k+1} = 1 - k/n$ (all others equal 0). This Markov chain is periodic with period 2, since the state switches parity on each move. Thus $p_j(k) = 0$ when j and k have opposite parity.

Taking two hops at once converts the Markov chain into two others, one on the odd states and one on the even states. Each is ergodic and thus has a unique stationary distribution. (For a chain to be ergodic it suffices that one can reach any state from any other and that it is possible to remain in the current state at any step.) The stationary distributions by definition are the limits to be computed, so the limits exist.

Let $a_k = \lim_{i\to\infty} p_{2i}(k)$ for even k and $a_k = \lim_{i\to\infty} p_{2i-1}(k)$ for odd k. In order for these to form stationary distributions, we must have $\sum_{k=0}^{n} a_k = 2$ and

$$a_k = \frac{n-k+1}{n} a_{k-1} + \frac{k+1}{n} a_{k+1}$$

for $0 \le k \le n$ (with $a_{-1} = a_{n+1} = 0$). These linear equations determine the $n+1$ values $\{a_k\}_{k=0}^{n}$. Therefore, it suffices to check that setting $a_k = \binom{n}{k} 2^{n-1}$ for all k satisfies the equations.

Editorial comment. Stephen J. Herschkorn wrote "The problem begs the question as to why, from a probabilistic point of view, the binomial should be the stationary distribution for this simple random walk." Herschkorn communicated the following intuition from Sheldon Ross: Flip n fair coins; the number of heads has a binomial distribution. Pick a random coin and turn it over. The new number of heads arises from the old by the same transition probability as in the random-walk model, but the new number of heads still has the binomial distribution, because each coin still has probability $1/2$ of being heads.

Daniel M. Rosenblum noted a similarity to Problem 11032 (2003, 637), in which the frog's probabilities of jumping to the right and left are reversed, yielding the same Markov chain as the Ehrenfest urn model (see, for example, Sections 4 and 5 of M. Kac, Random Walk and the Theory of Brownian Motion, *Amer. Math. Monthly* **54** (1947) 269–391.)

Some solvers used generating functions. It is also possible to avoid mentioning the theorem on stationary distributions and instead prove that the limits exist by direct methods particular to the problem.

Also solved by A. Agnew, M. Andreoli, D. Beckwith, K. David & P. Fricano, D. Fleischman, O. Geupel (Germany), C. González-alcón & Á. Plaza (Spain), S. J. Herschkorn, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), D. M. Rosenblum, R. K. Schwartz, J. Simons (U. K.), N. C. Singer, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), NSA Problems Group, Skidmore College Problem Group, and the proposer.

Separating the Degrees of Polynomials

11478 [2010, 87]. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let K be a field of characteristic 0, and let f and g be relatively prime polynomials in $K[x]$ with $\deg(g) < \deg(f)$. Suppose that for infinitely many λ in K there is a sublist of the roots of $f + \lambda g$ (counting multiplicity) that sums to 0. Show that $\deg(g) < \deg(f) - 1$ and that the sum of all the roots of f (again counting multiplicity) is 0.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. For a monic polynomial p of degree n with roots $\alpha_1, \dots, \alpha_n$ (taken with multiplicity) the product Q_k defined by

$$Q_k = \prod_{1\le i_1<i_2<\cdots<i_k\le n} (\alpha_{i_1} + \cdots + \alpha_{i_k})$$

is a symmetric function in the roots of p. Hence Q_k is given by a universal polynomial in the coefficients of p. When p is a constant multiple of $f + \lambda g$ (choosing the constant to make p monic), Q_k is a polynomial in λ. By hypothesis, there are infinitely many values of λ such that $\prod_{k=1}^{n} Q_k$ vanishes. Hence one of these polynomials, say Q_j, is the 0 polynomial. Thus Q_j vanishes for all λ, and the desired sublist exists for all λ.

The same conclusion holds even when we replace K by a larger field, specifically the field $K(t)$ of rational functions in a new indeterminate t. By Gauss's Lemma, if the polynomial $f(x) + tg(x)$ is reducible over $K(t)$, then it is reducible over the polynomial ring $K[t]$. However, since it is linear in t, one of the factors would be independent of t and would give a common factor of f and g. Thus $f(x) + tg(x)$ is irreducible over $K(t)$. Hence its Galois group G acts transitively on the roots $\alpha_1, \dots, \alpha_n$ of $f(x) + tg(x)$. Suppose without loss of generality that $\alpha_1 + \cdots + \alpha_k = 0$. Now

$$0 = \sum_{\phi \in G} \phi(\alpha_1 + \cdots + \alpha_k) = \frac{|G|k}{n}(\alpha_1 + \cdots + \alpha_n).$$

Thus $\alpha_1 + \cdots + \alpha_n = 0$, and hence the coefficient of x^{n-1} in $f + tg$ vanishes. Now $\deg(g) \le n - 2$, and the sum of the roots of f vanishes as desired.

Also solved by R. Chapman (U. K.), O. P. Lossers (Netherlands), and the proposer.

Orthogonality of Matrices under Additivity of Traces of Powers

11483 [2010, 182]. *Proposed by Éric Pité, Paris, France.* Let A and B be real $n \times n$ symmetric matrices such that $\operatorname{tr}(A + B)^k = \operatorname{tr} A^k + \operatorname{tr} B^k$ for every nonzero integer k. Show that $AB = 0$.

Composite solution by the editors. We prove the stronger statement that if A and B are $n \times n$ Hermitian matrices such that $\operatorname{tr}(A + B)^k = \operatorname{tr} A^k + \operatorname{tr} B^k$ for every integer k such that $1 \le k \le 3n$, then $AB = 0$.

We show first that if the sums of the kth powers of two lists of complex numbers, of length l and m respectively, are equal for $1 \le k \le l + m$, then the lists are the same (up to order of the entries). To see this, let the first list have distinct entries $\alpha_1, \dots, \alpha_r$ with multiplicities $a_1, \dots, a_r$, and let the second list have distinct entries $\beta_1, \dots, \beta_s$ with multiplicities $b_1, \dots, b_s$. The hypothesis is now that $\sum_{i=1}^{r} a_i \alpha_i^k - \sum_{j=1}^{s} b_j \beta_j^k = 0$ for $1 \le k \le \sum a_i + \sum b_j = l + m$. Since the Vandermonde matrix is invertible, the hypothesis requires the lists to have the same entries.

This immediately yields the following: If S_1, S_2, and S_3 are three lists of complex numbers, and the sum of the kth powers of the entries in S_1 and S_2 equals the sum of the kth powers of the entries in S_3 whenever k is at most the sum of the lengths of the three lists, then the entries of the concatenation of S_1 and S_2 are the same as the entries in S_3.

Now let A and B be Hermitian matrices, and let $C = A + B$. Let the lists of nonzero eigenvalues of these matrices be $\{\alpha_i\}_{i=1}^{r}$, $\{\beta_i\}_{i=1}^{s}$, and $\{\gamma_i\}_{i=1}^{t}$, respectively. The condition $\operatorname{tr}(A + B)^k = \operatorname{tr} A^k + \operatorname{tr} B^k$ is the same as $\sum_{i=1}^{t} \gamma_i^k = \sum_{i=1}^{r} \alpha_i^k + \sum_{i=1}^{s} \beta_i^k$, imposed for $1 \le k \le 3n$. Hence, the nonzero eigenvalues of C are exactly the nonzero eigenvalues of A and B, including multiplicities. Consequently, $\operatorname{rank}(C) = \operatorname{rank}(A) + \operatorname{rank}(B)$. On the other hand, the images satisfy $\operatorname{Im}(A + B) \subseteq \operatorname{Im}(A) + \operatorname{Im}(B)$. Thus, $\operatorname{Im}(A + B) = \operatorname{Im}(A) + \operatorname{Im}(B)$. Let $V = \operatorname{Im}(A + B)$. Viewed as a linear transformation on V, C is invertible.

Finally, we argue that $AB = 0$. By spectral factorization, since A and B are Hermitian, there are orthonormal vectors $\{u_i\}_{i=1}^{r}$ for A and $\{v_i\}_{i=1}^{s}$ for B such that $A = \sum_{i=1}^{r} \alpha_i u_i u_i^*$ and $B = \sum_{j=1}^{s} \beta_j v_j v_j^*$. Moreover, the space V is spanned by $\{u_i\}_{i=1}^{r}$ and $\{v_i\}_{i=1}^{s}$. Since $r + s = \dim V$, it follows that $\{u_i\}_{i=1}^{r} \cup \{v_i\}_{i=1}^{s}$ is a linearly independent set and forms a basis for V. It follows that

$$(A+B)u_i = \alpha_i u_i + \sum_{j=1}^{s} \beta_j v_j (v_j^* u_i) \qquad \text{for } 1 \le i \le r$$

and

$$(A+B)v_j = \sum_{i=1}^{r} \alpha_i u_i (u_i^* v_j) + \beta_j v_j \qquad \text{for } 1 \le j \le s.$$

Under the basis $\{u_1, \ldots, u_r, v_1, \ldots, v_s\}$, the matrix representation of $A+B$ is

$$\begin{pmatrix} D_\alpha & D\alpha E^* \\ D_\beta E & D_\beta \end{pmatrix} = \begin{pmatrix} D_\alpha & 0 \\ 0 & D_\beta \end{pmatrix} \begin{pmatrix} I & E \\ E^* & I \end{pmatrix}, \tag{1}$$

where $D_\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_r)$, $D_\beta = \operatorname{diag}(\beta_1, \ldots, \beta_s)$, and $E_{i,j} = u_i^* v_j$.

Since the nonzero eigenvalues of $A+B$ are $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^s$, the determinants of both sides of (1) equal $\prod_{i=1}^r \alpha_i \prod_{i=1}^s \beta_i$. This yields $\det \left(\begin{smallmatrix} I & E \\ E^* & I \end{smallmatrix}\right) = 1$. Also, $\left(\begin{smallmatrix} I & E \\ E^* & I \end{smallmatrix}\right)$ is just the Gram matrix of $u_1, \ldots, u_r, v_1, \ldots, v_s$. By the Hadamard determinant inequality, $E = 0$; that is, $u_i^* v_j = 0$ for all i and j. It follows that

$$AB = \left(\sum_{i=1}^{r} \alpha_i u_i u_i^*\right)\left(\sum_{j=1}^{s} \beta_j v_j v_j^*\right) = \sum_{i=1}^{r}\sum_{j=1}^{s} \alpha_i \beta_j u_i (u_i^* v_j) v_j^* = 0.$$

Editorial comment. It would be nice to extend the result to normal matrices. The problem is that $C = A+B$ is not normal when A and B are normal. Thus the rank of C is not necessarily the same as the number of nonzero eigenvalues of C. Other than this, everything works for normal matrices.

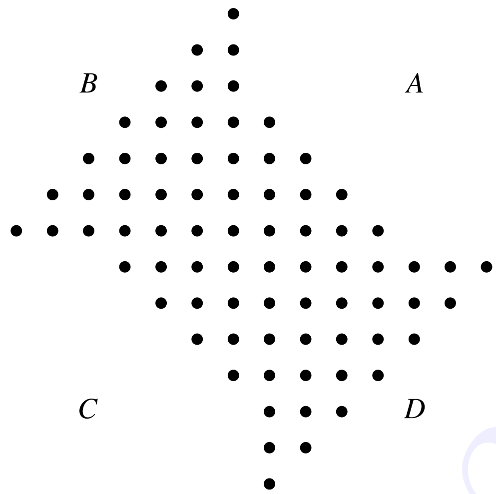
One may wonder whether the condition "$k \le 3n$" be replaced with "$k \le n$". This fails at least when $n = 1$, since $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$ for all numbers A and B, but $AB \ne 0$.

Also solved by J. Simons (U. K.), R. Stong, and the proposer.

Friendly Paths

11484 [2010, 182]. *Proposed by Giedrius Alkauskas, Vilnius University, Vilnius, Lithuania.* An *uphill* lattice path is the union of a (doubly infinite) sequence of directed line segments in $\mathbb{R}^2$, each connecting an integer pair (a, b) to an adjacent pair, either $(a, b+1)$ or $(a+1, b)$. A *downhill* lattice path is defined similarly, but with $b-1$ in place of $b+1$, and a *monotone* lattice is an uphill or downhill lattice path. Given a finite set P of points in $\mathbb{Z}^2$, a *friendly* path is a monotone lattice path for which there are as many points in P on one side of the path as on the other. (Points that lie on the path do not count.)
(a) Show that if $N = a^2 + b^2 + a + b$ for some positive integer pair (a, b) satisfying $a \le b \le a + \sqrt{2a}$, then for some set of N points there is no friendly path.
(b)* Is it true that for every odd-sized set of points there is a friendly path?

Solution to **(a)** *by the proposer.* Let P be the centrally symmetric configuration consisting of triangles of points in four quadrants as in the figure (where $a = 4$ and $b = 7$). The first and third quadrants contain triangles meeting a diagonals, comprising $a(a+1)/2$ points. The second and fourth quadrants contain triangles meeting b



diagonals, comprising $b(b+1)/2$ points. In total, $|P| = N$. Let A, B, C, D denote the subsets in the four quadrants.

We prove that there is no friendly path for P. If the first and last points of P on a monotone path Q lie in neighboring quadrants, then at least $N/2$ points lie on one side, and Q is not friendly. If the first and last points are in C and A, then Q hits one point in each of $2a+1$ diagonals. Since N is even, this leaves an odd number of points of P outside Q, and they cannot be split equally.

It remains to consider a downhill lattice path Q whose first and last points are in B and D. If Q hits a point of P at every step between these extremes, then Q hits $2b+1$ points of P, and again the remainder cannot be split equally. Hence, we may assume that an odd number of lattice points along Q between its ends are not in P. By symmetry, we may assume these points are in A. We claim that every such path has more points of P below it than above it.

Consider the point x just above the leftmost column of the triangle in A. The downhill path Q containing x that has the most points of P above and to its right goes directly rightward to x and then down. There are $\binom{b-a-1}{2}$ points of P above Q in B, $\binom{a}{2}$ points of P to the right of Q in A, and $\binom{b}{2}$ points of P to the right of Q in D. Meanwhile, on the other side of Q are $\binom{a+1}{2} + \binom{b+1}{2} - \binom{b-a+1}{2}$ points. An equal split requires

$$\binom{a}{2} + \binom{b}{2} + \binom{b-a-1}{2} \geq \binom{a+1}{2} + \binom{b+1}{2} - \binom{b-a+2}{2},$$

which simplifies to $a + b \leq (b-a)^2$. The left side is at least $2a$, and the right side is at most $2a$, so $b = a$ is necessary, but then $2a \leq 0$.

As we move from x to any other point in the first quadrant outside A as a point of Q outside P between points of Q in P, the number of points above Q decreases, while the number of points below Q increases. Hence the two sides can never have equal size.

Editorial comment. We do not know the answer to part **(b)***. Parity considerations made part **(a)** easy using a centrally symmetric configuration. However, a centrally symmetric configuration of odd size has a central point. Any symmetric path through that point is a friendly path. This makes it difficult to construct a counterexample.

No other solutions were received.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before July 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11628. *Proposed by Jeffrey C. Lagarias and Michael E. Zieve, University of Michigan, Ann Arbor, MI.* Define the *Lenstra constant* $L(R)$ of a commutative ring R to be the size of the largest subset A of R such that $a - b$ is a unit (invertible element) in R for any distinct elements $a, b \in A$. Show that for each positive integer N, the Lenstra constant of the ring $\mathbb{Z}[1/N]$ is the least prime that does not divide N.

11629. *Proposed by Olivier Oloa, University of Versailles, Rambouillet, France.* Let

$$f(\sigma) = \int_0^1 x^\sigma \left(\frac{1}{\log x} + \frac{1}{1-x} \right)^2 dx.$$

(**a**) Show that $f(0) = \log(2\pi) - 3/2$.

(**b**) Find a closed form expression for $f(\sigma)$ for $\sigma > 0$.

11630. *Proposed by Constantin Mateescu, High School 'Zinca Golescu', Pitesti, Romania.* For triangle ABC, let H be the orthocenter, I the incenter, O the circumcenter, and R the circumradius. Let b and c be the lengths of the sides opposite B and C, respectively, and let l be the length of the line segment from A to BC along the angle bisector at A. Let α be the radian measure of angle BAC. Prove that

$$\frac{bc}{l} + \max\{b, c\} \le 4R\cos(\alpha/4),$$

with equality if and only if rays AH, AI, and AO divide angle BAC into four equal angles.

11631. *Proposed by Pál Péter Dályay, Szeged, Hungary.* A *quasigroup* $(Q, *)$ is a set Q together with a binary operation $*$ such that for each $a, b \in Q$ there exist unique x and unique y (which may be equal) such that $ax = b$ and $ya = b$. The *Cayley table* of a finite quasigroup is its 'times table'. A quasigroup has *property P* if each row of the table is a rotation of the first row.

Find all positive integers n for which there exists a quasigroup $(\{1, \dots, n\}, *)$ with property P in which all elements are idempotent. (For instance, the Cayley table below defines a binary operation on $\{1, \dots, 5\}$ with property P in which each element is idempotent.)

$*$	1	2	3	4	5
1	1	5	4	3	2
2	3	2	1	5	4
3	5	4	3	2	1
4	2	1	5	4	3
5	4	3	2	1	5

11632. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Bucharest, Romania.* Let n be a positive integer, and write a vector $\mathbf{x} \in \mathbb{R}^n$ as $(x_1, \dots, x_n)$. For $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ let

$$[\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} = \sum_{1 \le i, j \le n} x_i y_j \min(a_i, b_j).$$

Show that for $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^n$ with nonnegative entries,

$$[\mathbf{x}, \mathbf{x}]_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{z}]^2_{\mathbf{b},\mathbf{c}} + [\mathbf{y}, \mathbf{y}]_{\mathbf{b},\mathbf{b}} \cdot [\mathbf{z}, \mathbf{x}]^2_{\mathbf{c},\mathbf{a}}$$
$$\le [\mathbf{x}, \mathbf{x}]^{1/2}_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{y}]^{1/2}_{\mathbf{b},\mathbf{b}} \cdot [\mathbf{z}, \mathbf{z}]_{\mathbf{c},\mathbf{c}} \cdot \left([\mathbf{x}, \mathbf{x}]^{1/2}_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{y}]^{1/2}_{\mathbf{b},\mathbf{b}} + [\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} \right).$$

11633. *Proposed by Anthony Sofo, Victoria University, Melbourne, Australia.* For real a, let $H_n^{(a)} = \sum_{j=1}^n j^{-a}$. Show that for integers a, b, and n with $a \ge 1$, $b \ge 0$, and $n \ge 1$,

$$\sum_{k=1}^n \frac{k(H_k^2 + H_k^{(2)}) + 2(k+b)^a H_k^{(1)} H_{k+b-1}^{(a)}}{k(k+b)^a} = H_{n+b}^{(a)} (H_n^2 + H_n^{(2)}).$$

11634. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade, Buzău, Romania.* Let $(x_1, \dots x_n)$ be an n-tuple of positive numbers, and let $X = \sum_{k=1}^n x_k$. Let a and m be nonnegative numbers, and let b, c, d be positive. Suppose $p \ge 1$ and $cX^p > d \max_{1 \le k \le n} x_k^p$. Show that

$$\sum_{k=1}^n \frac{aX + bx_k}{cX^p - dx_k^p} \ge \frac{(an+b)n^{mp}}{(cn^p - d)^m} X^{1-mp}.$$

SOLUTIONS

Another Hankel Determinant

11475 [2010, 86]. *Proposed by Ömer Eğecioğlu, University of California Santa Barbara, Santa Barbara, CA.* Let $h_k = \sum_{j=1}^k \frac{1}{j}$, and let D_n be the determinant of the $(n+1) \times (n+1)$ Hankel matrix with (i, j) entry h_{i+j+1} for $0 \le i, j \le n$. (Thus, $D_1 = -5/12$ and $D_2 = 1/216$.) Show that for $n \ge 1$,

$$D_n = \frac{\prod_{i=1}^{n} i!^4}{\prod_{i=1}^{2n+1} i!} \cdot \sum_{j=0}^{n} \frac{(-1)^j (n+j+1)!(n+1)h_{j+1}}{j!(j+1)!(n-j)!}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The desired value is the determinant of the matrix B that results from the given matrix by subtracting row $n-1$ from row n, row $n-2$ from row $n-1$, and so on to row 0 from row 1. This produces $b_{0,j} = h_{j+1}$, and $b_{i,j} = 1/(i+j+1)$ for $i \geq 1$. In particular, the bottom n rows of B agree with the bottom n rows of the $(n+1) \times (n+1)$ Hilbert matrix H, where $H_{i,j} = 1/(i+j+1)$ for $i, j \in \{0, \dots, n\}$. In the 19th century, Hilbert computed

$$\det H = \frac{\prod_{i=1}^{n} i!^4}{\prod_{i=1}^{2n+1} i!}$$

and

$$H_{i,j}^{-1} = (-1)^{i+j}(i+j+1)\binom{n+i+1}{n-j}\binom{n+j+1}{n-i}\binom{i+j}{i}.$$

Hence expanding $\det B$ along the top row yields

$$\begin{aligned} D_n = \det B &= (\det H) \sum_{j=0}^{n} b_{0,j} H_{j,0}^{-1} \\ &= \frac{\prod_{i=1}^{n} i!^4}{\prod_{i=1}^{2n+1} i!} \cdot \sum_{j=0}^{n} \frac{(-1)^j (n+j+1)!(n+1)h_{j+1}}{j!(j+1)!(n-j)!}. \end{aligned}$$

Also solved by R. Chapman (U. K.), O. Kouba (Syria), O. P. Lossers (Netherlands), K. McInturff GCHQ Problem Solving Group, and the proposer.

Short Runs from an Urn

11485 [2011, 182]. *Proposed by Neetu Badhoniya, K. S. Bhanu, and M. N. Deshpande, Institute of Science, Nagpur, India.* An urn contains a white balls and b black balls, and $a \geq 2b+3$. Balls are drawn at random from the urn and placed in a row as they are drawn. Drawing halts when three white balls are drawn in succession. Let X be the number of isolated pairs of white balls in the lineup produced during play, and let Y be the number of isolated white balls. Show that

$$E[X] = \frac{b}{a+1}, \quad \text{and} \quad E[Y] = \frac{b(a+b+1)}{(a+1)(a+2)}.$$

Solution by Bob Tomper, Mathematics Department, University of North Dakota. Let $E[X_{a,b}]$ and $E[Y_{a,b}]$ be the expected values of X and Y respectively when starting with an urn containing a white balls and b black balls, where $a \geq 2b+3$.

For $a \geq 3$, we have $E[X_{a,0}] = E[Y_{a,0}] = 0$, which agrees with the given formulae. We use induction on b, the number of black balls. Assume the formulae are correct up to $b-1$, where $b \geq 1$ (and all appropriate a values), and consider the case of a white balls and b black balls with $a \geq 2b+3$.

The lineup begins with B, WB, WWB, or WWW, where W and B indicate drawing a white or a black ball, respectively. The last also ends the lineup; after each of the others, the number of white balls remaining in the urn is at least three more than twice the number of black balls, so the conditions for the induction hypothesis apply. Accounting for the isolated W in WB and the paired WW in WWB,

$$E[X_{a,b}] = \Pr[B]E[X_{a,b-1}] + \Pr[WB]E[X_{a-1,b-1}] + \Pr[WWB]\left(1 + E[X_{a-2,b-1}]\right)$$

which gives this expression for $E[X_{a,b}]$:

$$\frac{b}{a+b}\frac{b-1}{a+1} + \frac{a}{a+b}\frac{b}{a+b-1}\frac{b-1}{a} + \frac{a}{a+b}\frac{a-1}{a+b-1}\frac{b}{a+b-2}\left(1 + \frac{b-1}{a-1}\right).$$

This simplifies to $\frac{b}{a+1}$. Similarly,

$$E[Y_{a,b}] = \Pr[B]E[Y_{a,b-1}] + \Pr[WB]\left(1 + E[Y_{a-1,b-1}]\right) + \Pr[WWB]E[Y_{a-2,b-1}]$$

whereby

$$E[Y_{a,b}] = \frac{b}{a+b}\frac{(b-1)}{(a+1)}\frac{(a+b)}{(a+2)} + \frac{a}{a+b}\frac{b}{a+b-1}\left(1 + \frac{(b-1)(a+b-1)}{a(a+1)}\right)$$
$$+\frac{a}{a+b}\frac{a-1}{a+b-1}\frac{b}{a+b-2}\frac{(b-1)(a+b-2)}{(a-1)a} = \frac{b(a+b+1)}{(a+1)(a+2)}.$$

Also solved by M. Andreoli, D. Beckwith, R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), C. González-Alcón & Á. Plaza (Spain), N. Grivaux (France), S. J. Herschkorn, O. Kouba (Syria), J. Lobo (Costa Rica), O. P. Lossers (Netherlands), R. Martin (Germany), K. McInturff, M. A. Prasad (India), R. Pratt, K. Schilling, D. Senft, J. Simons (U. K.), R. Stong, R. Tauraso (Italy), S. Xiao, CMC 328, GCHQ Problem Solving Group (U. K.), and the proposers.

A Weighted Fermat Triangle Problem

11491 [2010, 278]. *Proposed by Nicolae Anghel, University of North Texas, Denton, TX.* Let P be an interior point of a triangle having vertices A_0, A_1, and A_2, opposite sides of length a_0, a_1, and a_2, respectively, and circumradius R. For $j \in \{0, 1, 2\}$, let r_j be the distance from P to A_j. Show that

$$\frac{r_0}{a_0^2} + \frac{r_1}{a_1^2} + \frac{r_2}{a_2^2} \geq \frac{1}{R}.$$

Solution I by Marian Dinca, Romania . Let $E = r_0/a_0^2 + r_1/a_1^2 + r_2/a_2^2$. Let P_j be the projection of P to the side of length a_j, and let $d_j = PP_j$. Now P, P_1, A_0, and P_2 lie on the circle with diameter A_0P (of length r_0), so $\angle P_1PP_2 = A_1 + A_2$ and $P_1P_2 = r_0 \sin A_0$. Thus, $P_1P_2^2 = d_1^2 + d_2^2 - 2d_1d_2\cos(A_1 + A_2) = (d_1 \sin A_2 + d_2 \sin A_1)^2 + (d_1 \cos A_2 - d_2 \cos A_1)^2$ and $P_1P_2 = r_0 \sin A_0 \geq d_1 \sin A_2 + d_2 \sin A_1$. By the law of sines, $r_0a_0 \geq d_1a_2 + d_2a_1$. Similarly, $r_1a_1 \geq d_0a_2 + d_2a_0$ and $r_2a_2 \geq d_0a_1 + d_1a_0$. Thus

$$\begin{aligned}
E &\geq \frac{d_1a_2 + d_2a_1}{a_0^3} + \frac{d_2a_0 + d_0a_2}{a_1^3} + \frac{d_0a_1 + d_1a_0}{a_2^3} \\
&= \left(\frac{a_1}{a_2^3} + \frac{a_2}{a_1^3}\right)d_0 + \left(\frac{a_2}{a_0^3} + \frac{a_0}{a_2^3}\right)d_1 + \left(\frac{a_0}{a_1^3} + \frac{a_1}{a_0^3}\right)d_2 \\
&\geq \left(\frac{2}{a_1a_2}\right)d_0 + \left(\frac{2}{a_2a_0}\right)d_1 + \left(\frac{2}{a_0a_1}\right)d_2 = \frac{2(a_0d_0 + a_1d_1 + a_2d_2)}{a_1a_2a_3} = \frac{4K}{4KR},
\end{aligned}$$

where K is the area of $\Delta A_0A_1A_2$.

Solution II by Zoltán Vőrős, Tiszavasvári, Hungary . With r_i and d_i as above, the weighted Erdős–Mordell inequality states that $r_0x^2 + r_1y^2 + r_2z^2 \geq 2(d_0yz + d_1zx + d_2xy)$ for real x, y, z. Letting $x = 1/a_0$, $y = 1/a_1$, and $z = 1/a_2$, we get

$$\frac{r_0}{a_0^2} + \frac{r_1}{a_1^2} + \frac{r_2}{a_2^2} \geq 2\left(\frac{d_0}{a_1a_2} + \frac{d_1}{a_2a_0} + \frac{d_1}{a_0a_1}\right),$$

from which the result follows as above.

Editorial comment. This problem was on the 2000 US Olympiad Team Selection Test. It is a *weighted Fermat triangle problem* whose general solution is known (see Yujin Shen & Juan Tolosa, "The weighted Fermat triangle problem," *Int. J. Math. Math. Sci.* (2008), 16 pp., http://dx.doi.org/10.1155/2008/283846). The weighted Erdős–Mordell inequality can be found in *Am. Math. Monthly* **108** (2001) 165–168; available at http://dx.doi.org/10.2307/2695531. The notation of R_1 and r_1 as the distance from P to A and BC helps relate Erdős–Mordell's $R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3)$ to Euler's $R > 2r$.

Also solved by R. Bagby, P. P. Dályay (Hungary), J. Fabrykowski & T. Smotzer, J. Hamilton & T. Smotzer, P. Nüesch (Switzerland), M. Tetiva (Romania), L. Zhou, and the proposer.

A linear transformation and Hermite polynomials

11493 [2010, 279]. *Proposed by Johann Cigler, Universität Wien, Vienna, Austria.* Consider the Hermite polynomials H_n, defined by

$$H_n(x, s) = \sum_{0 \leq k \leq n/2} \binom{n}{2k} (2k-1)!!\,(-s)^k x^{n-2k},$$

where $m!! = \prod_{i<m/2}(m-2i)$ for positive m, with $(-1)!! = 1$. Let L be the linear transformation from $\mathbb{Q}[x, s]$ to $\mathbb{Q}[x]$ determined by $L1 = 1$, $Lx^ks^j = x^kLs^j$ for $j, k \geq 0$, and $LH_{2n}(x, s) = 0$ for $n > 0$. (Thus, for example, $0 = LH_2(x, s) = L(x^2 - s) = x^2 - Ls$, so $Ls = x^2$.) Define the *tangent numbers* T_{2n+1} by $\tan z = \sum_{n\geq 0} T_{2n+1}z^{2n+1}/(2n+1)!$, and the *Euler numbers* E_{2n} by $\sec z = \sum_{n\geq 0} E_{2n}\frac{z^{2n}}{(2n)!}$.

(**a**) Show that

$$LH_{2n+1}(x, s) = (-1)^n T_{2n+1} x^{2n+1}.$$

(**b**) Show that

$$Ls^n = \frac{E_{2n}}{(2n-1)!!} x^{2n}.$$

Solution by BSI Problems Group, Bonn, Germany. We work in the ring of formal power series in the indeterminate t with coefficients in the ring $\mathbb{Q}[x, s]$. We extend L to this ring coefficient-wise: if $F(t) = \sum_i f_i(x, s)t^i$, then $L\big(F(t)\big) = \sum_i L\big(f_i(x, s)\big)t^i$. It is then easy to see that, for $G(t) \in \mathbb{Q}[x][[t]]$,

$$L\big(G(t)F(t)\big) = G(t)L\big(F(t)\big).$$

Using $(2k-1)!! = (2k)!/2^kk!$, we find that

$$\sum_{n=0}^{\infty} H_n(x, s)\frac{t^n}{n!} = \sum_{k,n} \frac{(-s)^kt^{2k}}{2^kk!}\frac{(xt)^{n-2k}}{(n-2k)!} = e^{xt-st^2/2}.$$

Extracting even and odd powers of t gives

$$\sum_{n=0}^{\infty} H_{2n}(x,s)\frac{t^{2n}}{(2n)!} = \cosh(xt)e^{-st^2/2} \text{ and} \tag{1}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x,s)\frac{t^{2n+1}}{(2n+1)!} = \sinh(xt)e^{-st^2/2}. \tag{2}$$

Let $[t^n]F(t)$ denote the coefficient of t^n in $F(t)$. Applying L to (1) gives

$$1 = \cosh(xt)L\left(e^{-st^2/2}\right), \tag{3}$$

so

$$Ls^n = (-1)^n 2^n n!\,[t^{2n}]L\left(e^{-st^2/2}\right) = (-1)^n 2^n n!\,[t^{2n}]\operatorname{sech}(xt) = \frac{E_{2n}}{(2n-1)!!}x^{2n},$$

which proves (b).

Applying L to (2), and using (3), we have

$$\begin{aligned} LH_{2n+1}(x,s) &= (2n+1)!\,[t^{2n+1}]\sinh(xt)L\left(e^{-st^2/2}\right) \\ &= (2n+1)!\,[t^{2n+1}]\frac{\sinh(xt)}{\cosh(xt)} = (-1)^n T_{2n+1}x^{2n+1}, \end{aligned}$$

which proves (a).

Also solved by D. Beckwith, E. H. M. Brietzke (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), G. C. Greubel, O. P. Lossers (Netherlands), J. Matysiak and W. Matysiak (Poland), K. McInturff, R. Stong, S. Xiao, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Zeros of Symmetric Functions

11495 [2010, 370]. *Proposed by Marc Chamberland, Grinnell College, Grinnell IA.* Let a, b, and c be rational numbers such that exactly one of $a^2b + b^2c + c^2a$, $ab^2 + bc^2 + ca^2$, and $a^3 + b^3 + c^3 + 6abc$ is zero. Show that $a + b + c = 0$.

Composite solution by Jim Simons, Cheltenham, U. K., and Richard Stong, Center for Communications Research, San Diego, CA. We may assume that a, b, and c are not all 0. Since multiplication by a constant does not affect the statements, we may also assume that they are integers and have no common factor.

For each prime p, write $p^\alpha \| X$ to mean $p^\alpha \mid X$ and $p^{\alpha+1} \nmid X$. We use the fact that if $X + Y + Z = 0$, $p^\alpha \| X$, $p^\beta \| Y$, and $p^\gamma \| Z$, then the smallest two of α, β, and γ are equal.

Suppose that $a^2b + b^2c + c^2a = 0$. Let p be a prime divisor of abc, with $p^\alpha \| a$, $p^\beta \| b$, and $p^\gamma \| c$. Since $\gcd(a, b, c) = 1$, at least one of α, β, and γ is 0. If $\alpha = 0$, then $p^\beta \| a^2b$, $p^{2\beta+\gamma} \| b^2c$, and $p^{2\gamma} \| c^a$; it follows that $\beta = 2\gamma$. Similarly, $\beta = 0$ implies $\gamma = 2\alpha$, and $\gamma = 0$ implies $\alpha = 2\beta$. Thus there are pairwise relatively prime numbers X, Y, and Z such that $a = XY^2$, $b = YZ^2$, and $c = ZX^2$. In fact, X (respectively, Y and Z) is the product of all prime powers dividing $\gcd(c, a)$ (respectively, $\gcd(a, b)$ and $\gcd(b, c)$).

Substituting these values into $a^2b + b^2c + c^2a$ yields

$$0 = X^2Y^5Z^2 + Y^2Z^5X^2 + Z^2X^5Y^2 = (XYZ)^2(X^3 + Y^3 + Z^3).$$

If $XYZ \neq 0$, then $X^3 + Y^3 + Z^3 = 0$; this is the cubic case of Fermat's Last Theorem, which has no nontrivial solutions. Hence at least one of $\{X, Y, Z\}$ is 0, so two of $\{a, b, c\}$ equal 0, and hence $ab^2 + bc^2 + ca^2 = 0$. By symmetry, $ab^2 + bc^2 + ca^2 = 0$ implies $a^2b + b^2c + c^2a = 0$.

However, by hypothesis exactly one of the three given expressions equals 0, so it must be the third expression. Letting $\omega = e^{2\pi i/3}$, we may write

$$\begin{aligned} 0 &= a^3 + b^3 + c^3 + 6abc = 3(a^3 + b^3 + c^3 + 6abc) \\ &= (a+b+c)^3 + (a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3. \end{aligned}$$

This is a solution to Fermat's equation in the ring $\mathbb{Z}[\omega]$, but the usual proof of Fermat's Last Theorem in the cubic case shows that, in fact, there are no nontrivial solutions in $\mathbb{Z}[\omega]$. Thus, one of the three terms must be zero. If $a + b\omega + c\omega^2 = 0$, then all three of a, b, c must be zero and so $a + b + c = 0$; the same is true if $a + b\omega^2 + c\omega = 0$. What remains is that $a + b + c = 0$, as claimed.

Also solved by R. Chapman (U. K.), S. Chatadus (Poland), O. P. Lossers (Netherlands), C. R. Pranesachar (India), R. Prasad (India), R. E. Prather, M. Tetiva (Romania), the GCHQ Problem Solving Group (U. K.), and the proposer.

A Series with Harmonic Numbers

11499 [2010, 371]. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let H_n be the nth harmonic number, given by $H_n = \sum_{k=1}^{n} 1/k$. Let

$$S_k = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\log k - (H_{kn} - H_n)\right).$$

Prove that for $k \geq 2$,

$$S_k = \frac{k-1}{2k} \log 2 + \frac{1}{2} \log k - \frac{\pi}{2k^2} \sum_{l=1}^{\lfloor k/2 \rfloor} (k + 1 - 2l) \cot\left(\frac{(2l-1)\pi}{2k}\right).$$

Solution by Douglas B. Tyler, Raytheon, Torrance, CA . Fix $k \geq 2$. Let $a_n = \log k - (H_{kn} - H_n)$. Since $H_n = \log n + \gamma + o(1/n)$ as $n \to \infty$, we have $a_n = o(1/n)$, so $a_n \to 0$. Thus

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} (a_{2n-1} - a_{2n}) = \sum_{n=1}^{\infty} \sum_{m=1}^{k} \left(\frac{1}{(2n-1)k + m} - \frac{1}{2nk} \right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{k} \int_0^1 (x^{2nk-k+m-1} - x^{2nk-1})\, dx. \end{aligned}$$

The integrands are nonnegative, so we may sum first:

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{k} (x^{2nk-k+m-1} - x^{2nk-1}) &= \frac{x^k}{1+x^k} \cdot \frac{1}{1-x} + \frac{kx^{2k-1}}{x^{2k}-1} \\ &= \frac{kx^{2k-1}}{x^{2k}-1} + \frac{1}{2(x-1)} - \frac{x^k - 1}{2(x-1)(x^k+1)}. \end{aligned}$$

Let

$$I_k = \int_0^1 \frac{x^k - 1}{(x-1)(x^k+1)}\,dx.$$

Since

$$\int_0^1 \left(\frac{kx^{2k-1}}{x^{2k}-1} + \frac{1}{2(x-1)}\right) dx = \frac{1}{2}\log(2k),$$

it remains to prove that

$$I_k = \frac{1}{k}\log 2 + \frac{\pi}{k^2}\sum_{l=1}^{[k/2]}(k+1-2l)\cot\frac{(2l-1)\pi}{2k}.$$

Expanding the integrand yields $\frac{x^k-1}{(x-1)(x^k+1)} = \frac{2}{k}\sum_\zeta \frac{\zeta}{\zeta-1}\cdot\frac{1}{x-\zeta}$, where we sum over the kth roots of -1. Except for $\zeta = -1$ when k is odd, the summands occur in conjugate pairs $\frac{\zeta}{\zeta-1}\cdot\frac{1}{x-\zeta} + \frac{\bar\zeta}{\bar\zeta-1}\cdot\frac{1}{x-\bar\zeta} = \frac{x+1}{(x-\cos\theta)^2+\sin^2\theta}$, where $\zeta = e^{i\theta}$. With $J = \int_0^1 \frac{x+1}{(x-\cos\theta)^2+\sin^2\theta}\,dx$, we have

$$\begin{aligned} J &= \left[\frac{1}{2}\log(x^2 - 2x\cos\theta + 1) + \frac{1+\cos\theta}{\sin\theta}\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right)\right]_0^1 \\ &= \frac{1}{2}\log(2-2\cos\theta) + \cot\frac{\theta}{2}\left(\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) + \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)\right) \\ &= \log\left(2\sin\frac{\theta}{2}\right) + \cot\frac{\theta}{2}\left(\frac{\pi-\theta}{2}\right). \end{aligned}$$

Now if $\theta = (2l-1)\pi/k$, then

$$\begin{aligned} I_k &= \frac{2}{k}\sum_{l=1}^{\lfloor k/2\rfloor}\left(\log\left(2\sin\frac{(2l-1)\pi}{2k}\right) + \frac{(k+1-2l)\pi}{2k}\cot\frac{(2l-1)\pi}{2k}\right) \\ &= \frac{2}{k}\sum_{l=1}^{\lfloor k/2\rfloor}\log\left(2\sin\frac{(2l-1)\pi}{2k}\right) + \frac{\pi}{k^2}\sum_{l=1}^{\lfloor k/2\rfloor}(k+1-2l)\cot\frac{(2l-1)\pi}{2k}. \end{aligned}$$

If k is odd, then $l = \frac{k+1}{2}$ in the first sum corresponds to $\zeta = -1$ and $\frac{2}{k}\int_0^1 \frac{1/2}{x+1}\,dx = \frac{1}{k}\log(2\sin\frac{\pi}{2})$. The first sum thus equals $\frac{1}{k}\sum_{l=1}^{k}\log(2\sin\frac{(2l-1)\pi}{2k}) = \frac{1}{k}\log 2$, from the identity $\prod_{l=1}^{k}\sin\frac{(2l-1)\pi}{2k} = \frac{1}{2^{k-1}}$. This completes the proof.

Editorial comment. Some solvers derived the equivalent $S_k = \frac{k-1}{2k}\log 2 + \frac{1}{2}\log k - \frac{\pi}{4k}\sum_{j=1}^{k}\cot\frac{j\pi}{2k}$ by applying the digamma identity $\Psi(x) = \Psi(1-x) - \pi\cot\pi x$ to the sum $S_k = -\frac{1}{2k}\sum_{j=1}^{k-1}\Psi(\frac{1}{2}+\frac{j}{2k}) - \frac{k-1}{2k}\gamma$.

Also solved by P. Bracken, R. Chapman (U. K.), H. Chen, E. A. Herman, O. P. Lossers (Netherlands), A. Stenger, R. Stong, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before August 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11635. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bârlad, Romania, and Nicuşor Minculete, "Dimitrie Cantemir" University, Braşov, Romania.*
(**a**) Let α and β be distinct nonzero real numbers. Let a, b, c, x, y, z be real, with $0 < a < b < c$ and $a \le x < y < z \le c$. Prove that if

$$x^\alpha + y^\alpha + z^\alpha = a^\alpha + b^\alpha + c^\alpha \text{ and } x^\beta + y^\beta + z^\beta = a^\beta + b^\beta + c^\beta$$

then $x = a$, $y = b$, and $z = c$.
(**b**) Let $\alpha_1, \alpha_2, \alpha_3$ be distinct nonzero real numbers. Let $a_1, a_2, a_3, a_4, x_1, x_2, x_3, x_4$ be real, with $0 < a_1 < a_2 < a_3 < a_4$ and $a_1 \le x_1 < x_2 < x_3 < x_4 \le a_4$. If

$$\sum_{k=1}^{4} x_k^{\alpha_j} = \sum_{k=1}^{4} a_k^{\alpha_j}$$

for $1 \le j \le 3$, must a_k then equal x_k for $1 \le k \le 4$?

11636. *Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.* Let $ABCD$ be a convex quadrilateral, and suppose there is a point M on the diagonal BD with the property that the perimeters of ABM and CBM are equal and the perimeters of ADM and CDM are equal. Prove that $|AB| = |CB|$ and $|AD| = |CD|$.

11637. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj, Romania.* Let $m \ge 1$ be a nonnegative integer. Let $u = u - \lfloor u \rfloor$; the quantity u is called the *fractional part* of u. Prove that

$$\int_0^1 \left\{ \frac{1}{x} \right\}^m x^m \, dx = 1 - \frac{1}{m+1} \sum_{k=1}^{m} \zeta(k+1).$$

(Here ζ is denotes the Riemann zeta function.)

http://dx.doi.org/10.4169/amer.math.monthly.119.04.344

11638. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let a, b, c be positive real numbers. Prove that

$$a^3 + b^3 + c^3 + 3 \geq 3\left((a^2b+1)(b^2c+1)(c^2a+1)\right)^{1/3}.$$

11639. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Evaluate $\int_0^{\pi/2}(\log(2\sin x))^2\,dx$.

11640. *Proposed by Karl David, Milwaukee School of Engineering, Milwaukee, WI.* For $x > 0$ and $x \neq 1$, let $f(x) = x^{1/(x-1)}$, and let $f(1) = e$. Show that $f''(x) > 0$ for $x > 0$.

11641. *Proposed by Nicolae Bourbăcuţ, Sarmizegetusa, Romania.* Let f be a convex function from $\mathbb{R}$ into $\mathbb{R}$ and suppose that $f(x+y) + f(x-y) - 2f(x) \leq y^2$ for all real x and y.
(**a**) Show that f is differentiable.
(**b**) Show that for all real x and y,

$$|f'(x) - f'(y)| \leq |x-y|.$$

SOLUTIONS

Adjacency Matrices of Acyclic Digraphs

11487 [2011, 183]. *Proposed by Stephen Gagola Jr., Kent State University, Kent, OH.* Let A be a 0,1-matrix of order n with the property that $\mathrm{tr}(A^k) = 0$ for every positive integer k. Prove or disprove: A is similar by way of a permutation matrix to a strictly upper-triangular 0,1-matrix.

Solution by Victor S. Miller, CCR, Princeton, NJ. The statement is true. It is equivalent to the standard proposition in graph theory that every acyclic directed graph has a *ranking*, which is a vertex ordering such that every edge is directed from an earlier vertex to a later vertex. (In computer science, this ranking is called a 'topological sort' of the directed graph.)

A 0,1-matrix A is the adjacency matrix of a directed graph. Position (i, j) of A^k counts the directed walks from the ith vertex to the jth. If the traces of all powers equal 0, then the digraph has no closed walk and hence no cycle. Ordering the vertices by their distance from a source yields a ranking. Adjacency matrices with respect to reorderings of the vertices are obtained by conjugating by a permutation matrix. Some such matrix renumbers the vertices according to the ranking. The adjacency matrix with respect to this ordering is strictly upper triangular.

Also solved by G. Apostolopoulos (Greece), R. Chapman (U. K.), P. P. Dályay (Hungary), C. M. da Fonseca (Portugal), A. Gewirtz (France), O. Kouba (Syria), O. P. Lossers (Netherlands), R. Martin (Germany), M. Omarjee (France), J. Simons (U. K.), R. Stong, J. Stuart, M. Tetiva (Romania), GCHQ Problem Solving Group (U.K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Positive Semidefinite Combinations of Hermitian Matrices

11488 [2010, 278]. *Proposed by Dennis I. Merino, Southeastern Louisiana University, Hammond, LA, and Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL.*

(**a**) Show that if k is a positive odd integer, and A and B are Hermitian matrices of the same size such that $A^k + B^k = 2I$, then $2I - A - B$ is positive semidefinite.
(**b**) Find the largest positive integer p such that for all Hermitian matrices A and B of the same size, $2^{p-1}(A^p + B^p) - (A + B)^p$ is positive semidefinite.

Solution to Part (a) by Eugene A. Herman, Grinnell College, Grinnell, IA. More generally, let k be any positive integer such that $A^k + B^k = 2I$.

First consider odd k. Let L be an eigenspace of A with eigenvalue λ. Since k is odd and λ is real, L is also the eigenspace of A^k with eigenvalue λ^k. Hence L is the eigenspace of B^k with eigenvalue $2 - \lambda^k$, and so L is also the eigenspace of B with eigenvalue $(2 - \lambda^k)^{1/k}$. Having common eigenspaces, A and B are simultaneously unitarily diagonalizable. By applying this unitary similarity to both $A^k + B^k = 2I$ and $2I - A - B$, we may assume that A and B are diagonal. With $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ and $B = \operatorname{diag}(\beta_1, \ldots, \beta_n)$, we must show that $\alpha_i^k + \beta_i^k = 2$ implies $2 - \alpha_i - \beta_i \geq 0$. For this, it suffices to show that $x^k + y^k = 2$ implies $x + y \leq 2$.

Let $z = (x + y)/2$. If $z > 1$, then $2z^k > 2 = x^k + y^k$. Thus $z^k > (x^k + y^k)/2$, which contradicts the convexity of the kth-power function.

Next we show that if $2I - A^{2j} - B^{2k}$ is positive semidefinite (for $j \in \mathbb{N}$), then $2I - A^j - B^j$ is positive semidefinite. Since k is expressible as an odd number times a power of 2, this completes the proof for general k.

Since every eigenvalue of $A^{2j} + B^{2j}$ is real and at most 2, for $x \in \mathbb{C}^n$ we compute

$$\begin{aligned}\|A^k x\|^2 + \|B^k x\|^2 &= \langle A^k x, A^k x\rangle + \langle B^k x, B^k x\rangle \\ &= \langle (A^{2k} + B^{2k})x, x\rangle \leq \langle 2x, x\rangle = 2\|x\|^2,\end{aligned}$$

and hence

$$\left(\|A^k x\| + \|B^k x\|\right) \leq 2\left(\|A^k x\|^2 + \|B^k x\|^2\right) \leq 4\|x\|^2.$$

Solution to Part (b) by Richard Stong, Center for Communications Research, San Diego, CA. The largest such value of p is 2. For $p = 2$, we have $2(A^2 + B^2) - (A + B)^2 = (A - B)^2$; hence the expression is always positive semidefinite. For larger odd p, take $A = -I$ and $B = 0$. Now $2^{p-1}(A^p + B^p) - (A + B)^p = -(2^{p-1} - 1)I$, and the expression is not positive semidefinite.

For larger even p, let x be a small positive number, and take $A = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}0 & x\\ x & 0\end{smallmatrix}\right)$. Note that $A^p = A$ and $B^p = x^p I$. Let a_p be the $(2, 2)$-entry of $(A + B)^p$. Note that $a_0 = 1$, $a_1 = 0$, and $a_p = a_{p-1} + x^2 a_{p-2}$ for $p \geq 2$. Hence $a_p = x^2 + O(x^4)$. Thus the $(2, 2)$-entry of $2^{p-1}(A^p + B^p) - (A + B)^p$ is $-x^2 + O(x^4)$, and the matrix is not positive semidefinite.

Also solved by J. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, J. Simons (U. K.), and the proposers.

Twisted Semigroups

11490 [2010, 278]. *Proposed by Gábor Mészáros, Kemence, Hungary.* A semigroup S *agrees with* an ordered pair (i, j) of positive integers if $ab = b^j a^i$ whenever a and b are distinct elements of S. Find all ordered pairs (i, j) of positive integers such that if a semigroup S agrees with (i, j), then S has an idempotent element.

Solution by Nicolás Caro, Colombia. The required pairs are all pairs other than $(1, 1)$. Let (i, j) be a pair of positive integers, and let S be a semigroup agreeing with (i, j) that has no idempotent elements. For $k \geq 0$ and $x \in S$, we have $xx^2x^k = (x^2)^j x^i x^k$

(since $x \neq x^2$). Hence $x^{2j+i+k} = x^{3+k}$. Since x^{3+k} is not idempotent, $x^{2j+i+k} \neq x^{6+2k}$, and so $2j + i + k \neq 6 + 2k$ for $k \geq 0$. This implies $3 \leq 2j + i \leq 5$. If $2j + i = 4$, then $x^4 = x^3$ for all x implies $x^3 = x^4 = x^5 = x^6$ (contradicting $x^3 \neq x^6$). If $2j + i = 5$, then $x^5 = x^3$ for all x implies $x^4 = x^6 = x^8$ (contradicting $x^4 \neq x^8$). It follows that $2j + i = 3$, so $(i, j) = (1, 1)$.

Finally, the positive integers under addition form a semigroup agreeing with (1, 1) that has no idempotent elements. This concludes the proof.

Also solved by M. Angelelli (Italy), K. Benningfield, P. Budney, R. Chapman (U. K.), C. Curtis, P. P. Dályay (Hungary), J. Guerreiro (Portugal), Y. J. Ionin, S. C. Locke, J. Lockhart, B. Mulansky (Germany), V. Pambuccian, M. A. Prasad (India), K. Schilling, J. Simons (U. K.), R. Stong, B. Tomper, A. Wyn-Jones (U. K.), GCHQ Problem Solving Group (U. K.), Missouri State University University Problem Solving Group, NSA Problems Group, and the proposer.

A Matrix-Sum Inequality

11496 [2010, 370]. *Proposed by Benjamin Bogoşel (student), West University of Timisoara, Timisoara, Romania, and Cezar Lupu (student), University of Bucharest, Bucharest, Romania.* For a matrix X with real entries, let $s(X)$ be the sum of its entries. Prove that if A and B are $n \times n$ real matrices, then

$$\begin{aligned} &n\left(s(AA^T)s(BB^T) - s(AB^T)s(BA^T)\right) \geq \\ &\qquad s(AA^T)(s(B))^2 + s(BB^T)(s(A))^2 - s(A)s(B)\left(s(AB^T) + s(BA^T)\right). \end{aligned}$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The required inequality has been corrected above; there was an extra "+" within the first term, and $A^T B$ was printed instead of BA^T.

Let $\mathbf{1}$ be the column vector with each entry 1, so $s(X) = \mathbf{1}^T\mathbf{1}$. For $n \times n$ matrices A, B, and C, let

$$M = \begin{pmatrix} \mathbf{1}^T A \\ \mathbf{1}^T B \\ \mathbf{1}^T C \end{pmatrix};$$

note that M is a $3 \times n$ matrix. Also,

$$MM^T = \begin{pmatrix} s(AA^T) & s(AB^T) & s(AC^T) \\ s(BA^T) & s(BB^T) & s(BC^T) \\ s(CA^T) & s(CB^T) & s(CC^T) \end{pmatrix}.$$

Since $\det(MM^T) \geq 0$, expanding the determinant gives

$$\begin{aligned} s(CC^T)&\left(s(AA^T)s(BB^T) - s(AB^T)s(BA^T)\right) \geq \\ &\qquad s(AA^T)s(BC^T)s(CB^T) + s(BB^T)s(AC^T)s(CA^T) \\ &\qquad - s(AC^T)s(CB^T)s(BA^T) - s(BC^T)s(CA^T)s(AB^T). \end{aligned}$$

Specializing to $C = I$ yields the desired inequality.

Also solved by O. Kouba (Syria), O. P. Lossers (Netherlands), J. Simons (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Areas in a Subdivided Quadrilateral

11498 [2010, 371]. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Let $ABCD$ be a convex quadrilateral. A line through the intersection O of the diagonals AC and BD intersects the interior of edge BC at L and the interior of AD at N. Another line through O likewise meets AB at K and CD at M. This dissects $ABCD$ into eight triangles AKO, KBO, BLO, and so on. Prove that the arithmetic mean of the reciprocals of the areas of these triangles is greater than or equal to the sum of the arithmetic and quadratic means of the reciprocals of the areas of triangles ABO, BCO, CDO, and DAO. (The quadratic mean is also known as the root mean square; it is the square root of the mean of the squares of the given numbers.)

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The problem is invariant under affine transformations—which preserve ratios of areas—so we may assume $O = (0, 0)$, $A = (0, a)$, $B = (b, 0)$, $C = (-c, 0)$, and $D = (0, -d)$. Let MK have slope m, so

$$K = \left(\frac{ab}{b+ma}, \frac{mab}{b+ma}\right) \qquad \text{and} \qquad M = -\left(\frac{cd}{d+mc}, \frac{mcd}{d+mc}\right).$$

The areas of AOK, KOB, COM, and MOD are $ma^2b/(2(b+ma))$, $a^2b/(2(b+ma))$, $mc^2d/(2(d+mc))$, and $c^2d/(2(d+mc))$, respectively. The sum S of their reciprocals is given by $S = 4/ab + 4/cd + 2m(b^{-2} + d^{-2}) + 2m^{-1}(a^{-2} + c^{-2})$. Thus by the AM-GM inequality, $S \geq 4/ab + 4/cd + 4\sqrt{(b^{-2} + d^{-2})(a^{-2} + c^{-2})}$. Similarly the sum of the reciprocals of the other four areas is at least $4/bc + 4/da + 4\sqrt{(b^{-2} + d^{-2})(a^{-2} + c^{-2})}$. Adding these and dividing by 8 gives a lower bound of

$$\frac{1}{4}\left(\frac{2}{ab} + \frac{2}{bc} + \frac{2}{cd} + \frac{2}{da}\right) + \sqrt{\frac{1}{4}\left(\left(\frac{2}{ab}\right)^2 + \left(\frac{2}{bc}\right)^2 + \left(\frac{2}{cd}\right)^2 + \left(\frac{2}{da}\right)^2\right)}.$$

Now $ab/2, bc/2, cd/2, da/2$ are the areas of AOB, BOC, COD, DOA, so this is the desired result.

Also solved by P. P. Dályay (Hungary), O. Kouba (Syria), J. H. Lindsey II, GCHQ Problem Solving Group (U. K.), and the proposer.

Everything in Its Own Place, Almost

11500 [2010, 371]. *Proposed by Bhavana Deshpande, Poona College, Camp Pune, Maharashtra, India, and M. N. Deshpande, Institute of Science, Nagpur, India.* We have n balls, labeled 1 through n, and n urns, also labeled 1 through n. Ball 1 is put into a randomly chosen urn. Thereafter, as j increments from 2 to n, ball j is put into urn j if that urn is empty, otherwise, it is put into a randomly chosen empty urn. Let the random variable X be the number of balls that end up in the urn bearing their own number. Show that the expected value of X is $n - H_{n-1}$.

Solution I by Justin S. Dyer, Stanford University, Stanford, CA. Write X as X_n, and let $\mu_n = E(X_n)$. If the first ball is in the first urn, which happens with probability $1/n$, then every ball is in its own urn, contributing 1 to μ_n. If the first ball is in the kth urn, where $k \geq 2$, then the $k-2$ balls $2, \ldots, k-1$ are in their own urns, and $n-k+1$ urns are left when the next random placement occurs. They are labeled $1, k+1, k+2, \ldots, n$, and this is the same situation as before, except that if the next ball lands in urn 1, then it does not contribute. Hence the result of the remaining random placements is X_{n-k+1} if the next ball is not in urn 1, and $X_{n-k+1} - 1$ if it is, so its expectation is $\mu_{n-k+1} - \frac{1}{n-k+1}$. Summing over all locations for the first ball, we have

$$\mu_n = 1 + \frac{1}{n}\sum_{k=2}^{n}\left((k-2) + \mu_{n-k+1} - \frac{1}{n-k+1}\right).$$

We now compute

$$n\mu_n - (n-1)\mu_{n-1} = 1 + (n-2) + \mu_{n-1} - \frac{1}{n-1}.$$

Rearranging yields $\mu_n = \mu_{n-1} + 1 - \frac{1}{n-1}$, and hence $\mu_n = n - H_{n-1}$.

Solution II by Kenneth Schilling, University of Michigan-Flint, Flint, MI. Let $S = \{2, 3, \ldots, n\}$. We first prove that for $T \subseteq S$, there is exactly one placement of the n balls such that the subset of S corresponding to balls in their own urns is precisely $S - T$. Furthermore, the probability p_T of this placement is $\frac{1}{n}\prod_{k\in T}\frac{1}{n-k+1}$.

If $T = \varnothing$, then ball 1 is in urn 1. The rest fall into place, and $p_\varnothing = 1/n$. Otherwise, let $T = \{k_1, \ldots, k_t\}$, indexed in increasing order. Having ball i in urn i for $2 \le i < k_1$ but ball k_1 not in urn k_1 requires having ball 1 in urn k_1. Next, having ball i in urn i for $k_1 < i < k_2$ and ball k_2 not in urn k_2 requires having ball k_1 in urn k_2. Continuing, ball k_i must be in urn k_{i+1} for $1 \le i \le t-1$, and finally ball k_t must be in urn 1. For the probability claim, there are n urns in which ball 1 may be placed, for $k \notin T$ ball k is placed by rule, and the placement of ball k_i is chosen from $n - k_i + 1$ unfilled urns.

Now fix $j \in S$. We pair the subsets of S so that the sets T and T' in a pair differ only in whether they contain j. By the claim above, if $j \in T$ and $j \notin T'$, then $(n - j + 1)p_T = p_{T'}$. Summing over all pairs yields P(ball j is not in urn j) = $\frac{1}{n-j+2}$ and and P(ball j is in urn j) = $\frac{n-j+1}{n-j+2} = 1 - \frac{1}{n-j+2}$.

Summing over $j \in S$ and counting also $1/n$ for ball 1, we have $E(X) = \frac{1}{n} + \sum_{j=2}^{n}(1 - \frac{1}{n-j+2}) = n - H_{n-1}$.

Editorial comment. Justin S. Dyer also showed that X_n/n tends to 1 almost surely as $n \to \infty$.

Also solved by D. Beckwith, D. F. Behan, D. Brown & J. Zerger, N. Caro (Columbia), R. Chapman (U. K.), K. David & P. Fricano, P. J. Fitzsimmons, J. Freeman, D. Glass, N. Grivaux (France), S. J. Herschkorn, B.-T. Iordache (Romania), J. H. Lindsey II, O. P. Lossers (Netherlands), J. & W. Matysiak (Poland), K. McInturff, M. D. Meyerson, Á. Plaza & C. González-Alcón (Spain), R. Prasad (India), R. Pratt, B. Schmuland (Canada), J. Simons (U. K.), N. C. Singer, T. Starbird, J. H. Steelman, R. Stong, S. Xiao, CMC 328, GCHQ Problem Solving Group (U. K.), Szeged Problem Group "Fejéntaláltuka" (Hungary), and the proposer.

Runs of Heads and Covariance

11503 [2010, 458]. *Proposed by K. S. Bhanu, Institute of Science, Nagpur, India, and M. N. Deshpande, Nagpur, India.* We toss an unbiased coin to obtain a sequence of heads and tails, continuing until r heads have occurred. In this sequence, there will be some number R of runs (runs of heads or runs of tails) and some number X of isolated heads. (Thus, with $r = 4$, the sequence *HHTHTTH* yields $R = 5$ and $X = 2$.) Find the covariance of R and X in terms of r.

Solution by Jim Simons, Cheltenham, U. K. If $r = 1$, then $X = 1$ with probability 1, and the covariance is 0. For $r > 1$, we show that the answer is $r/2$.

For $1 \le i \le r$, let t_i be the random variable having value 1 if the ith head is immediately preceded by a tail and value 0 otherwise. Clearly $t_i = 1$ with probability $1/2$, and $t_1, \ldots, t_r$ are independent.

The first head is isolated if and only if $t_2 = 1$, and the last head is isolated if and only if $t_r = 1$. For $2 \le i \le r - 1$, the ith head is isolated if and only if $t_i = t_{i+1} = 1$. Thus $X = t_2 + t_r + \sum_{i=2}^{r-1} t_i t_{i+1}$, and

$$E(X) = 1/2 + 1/2 + (r-2)(1/4) = (r+2)/4.$$

If all t_i equal zero, then $R = 1$. Changing t_1 from 0 to 1 adds 1 to R. For $2 \le i \le r$, switching t_i to 1 adds 2 to R. Hence $R = 1 + t_1 + 2\sum_{j=2}^{r} t_j$, and

$$E(R) = 1 + 1/2 + 2(r-1)(1/2) = (2r+1)/2.$$

Since $t_i^2 = t_i$, expanding the product XR yields

$$\begin{aligned} XR &= \left(t_2 + t_r + \sum_{i=2}^{r-1} t_i t_{i+1}\right)\left(1 + t_1 + 2\sum_{j=2}^{r} t_j\right) \\ &= X(1+t_1) + 2t_2\left(1 + \sum_{j=3}^{r} t_j\right) + 2t_r\left(1 + \sum_{j=2}^{r-1} t_j\right) \\ &\quad + 2\sum_{i=2}^{r-1}\left(t_i t_{i+1}\left(\sum_{j=2}^{i-1} t_j + 2 + \sum_{j=i+2}^{r} t_j\right)\right). \end{aligned}$$

Since X is independent of t_1, we obtain

$$\begin{aligned} E(XR) &= \frac{3}{2}\left(\frac{r+2}{4}\right) + \frac{2}{2}\left(1 + \frac{r-2}{2}\right) + \frac{2}{2}\left(1 + \frac{r-2}{2}\right) \\ &\quad + 2\left(\frac{r-2}{4}\right)\left(2 + \frac{r-3}{2}\right), \end{aligned}$$

which simplifies to $(2r^2 + 9r + 2)/8$. We can now compute the required covariance:

$$E(XR) - E(X)E(R) = \frac{2r^2 + 9r + 2}{8} - \left(\frac{r+2}{4}\right)\left(\frac{2r+1}{2}\right) = \frac{r}{2}.$$

Also solved by D. Beckwith, N. Caro (Colombia), R. Chapman (U. K.), M. P. Cohen, P. J. Fitzsimmons, J. Gaisser, S. J. Herschkorn, J. H. Lindsey II, K. McInturff, M. Nyenhuis (Canada), M. A. Prasad (India), R. Pratt, K. Schilling, B. Schmuland (Canada), T. Starbird, R. Stong, and the proposers.

Computing Pi from Fibonacci Numbers

11505 [2010, 458]. *Proposed by Bruce Burdick, Roger Williams University, Bristol, RI.* Define $\{a_n\}$ to be the periodic sequence given by $a_1 = a_3 = 1$, $a_2 = 2$, $a_4 = a_6 = -1$, $a_5 = -2$, and $a_n = a_{n-6}$ for $n \ge 7$. Let $\{F_n\}$ be the Fibonacci sequence with $F_1 = F_2 = 1$. Show that

$$\sum_{k=1}^{\infty} \frac{a_k F_k F_{2k-1}}{2k-1} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1} F_{kn+3k-1}} = \frac{\pi}{4}.$$

Composite solution by Roberto Tauraso, Università di Roma "Tor Vergata", Roma, Italy; Rituraj Nandan, St. Peters, MO; and the proposer . By d'Ocagne's identity, (see http://mathworld.wolfram.com/dOcagnesIdentity.html), $F_N F_{M+1} - F_{N+1} F_M = (-1)^M F_{N-M}$. With $M = kn + 2k - 1$ and $N = k(n+1) + 2k - 1$, we have

$$\begin{aligned} \frac{(-1)^{kn}}{F_{kn+2k-1} F_{kn+3k-1}} &= \frac{-F_{k(n+1)+2k-1} F_{kn+2k} + F_{k(n+1)+2k} F_{kn+2k-1}}{F_k F_{kn+2k-1} F_{k(n+1)+2k-1}} \\ &= \frac{1}{F_k}\left(\frac{F_{k(n+1)+2k}}{F_{k(n+1)+2k-1}} - \frac{F_{kn+2k}}{F_{kn+2k-1}}\right). \end{aligned}$$

Hence we have the telescoping sum

$$\sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1}F_{kn+3k-1}} = \frac{1}{F_k} \sum_{n=0}^{\infty} \left(\frac{F_{k(n+1)+2k}}{F_{k(n+1)+2k-1}} - \frac{F_{kn+2k}}{F_{kn+2k-1}} \right)$$

$$= \frac{1}{F_k} \left(\phi - \frac{F_{2k}}{F_{2k-1}} \right),$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Thus

$$\sum_{k=1}^{\infty} \frac{a_k F_k F_{2k-1}}{2k-1} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1}F_{kn+3k-1}} = \sum_{k=1}^{\infty} \frac{a_k(\phi F_{2k-1} - F_{2k})}{2k-1}$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1},$$

since $F_{n+1} - \phi F_n = -\phi^{-n}$. The sequence $1, 2, 1, -1, -2, -1, \ldots$ is the sum of two sequences whose cycles are $1, -1, 1, -1, 1, -1$ and $0, 3, 0, 0, -3, 0$. Continuing the computation,

$$\sum_{k=1}^{\infty} \frac{a_k}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1} + \sum_{k=1}^{\infty} \frac{3(-1)^{k+1}}{6k-3} \left(\frac{1}{\phi} \right)^{6k-3}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\frac{1}{\phi^3} \right)^{2k-1}$$

$$= \arctan\left(\frac{1}{\phi} \right) + \arctan\left(\frac{1}{\phi^3} \right).$$

Using the arctangent addition formula and $\phi^2 = \phi + 1$, this becomes

$$\arctan\left(\frac{1}{\phi} \right) + \arctan\left(\frac{1}{\phi^3} \right) = \arctan\left(\frac{\frac{1}{\phi} + \frac{1}{\phi^3}}{1 - \frac{1}{\phi} \cdot \frac{1}{\phi^3}} \right) = \arctan\left(\frac{\phi^3 + \phi}{\phi^4 - 1} \right)$$

$$= \arctan\left(\frac{\phi^3 + \phi}{\phi^3 + \phi^2 - 1} \right) = \arctan\left(\frac{\phi^3 + \phi}{\phi^3 + \phi} \right) = \arctan(1) = \frac{\pi}{4}.$$

Editorial comment. Omran Kouba proved the general identity

$$\arctan\left(\frac{2x\cos\theta}{1-x^2} \right) = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}\cos((2k-1)\theta)}{2k-1} x^{2k-1}$$

for $\theta \in \mathbb{R}$ and $|x| < 1$, from which the second part of the proof above follows with $\theta = 2\pi/3$ and $x = 1/\phi$.

Also solved by R. Chapman (U. K.), O. Kouba (Syria), K. D. Lathrop, M. A. Prasad (India), R. Stong, S. Y. Xiao (Canada), and GCHQ Problem Solving Group (U. K.).

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before September 30, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11642. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let α, β, γ be positive real numbers, with $\gamma > 1$.
(**a**) Prove that

$$\lim_{x\to 1^-}(1-x)^\beta \sum_{n=1}^{\infty} \gamma^{n\alpha} x^{\gamma^n} = \begin{cases} 0 & \text{when } \beta > \alpha, \\ \infty & \text{when } \beta < \alpha. \end{cases}$$

(**b**) Does the limit exist if $\beta = \alpha$?

11643. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* Let r be a real number with $0 < r < 1$, and define a discrete probability measure P on $\mathbb{N}$ by $P(k) = (1-r)r^{k-1}$ for $k \geq 1$. Show that there are uncountably many triples (A_1, A_2, A_3) of subsets of $\mathbb{N}$ that are mutually independent, that is, $P(A_i \bigcap A_j) = P(A_i)P(A_j)$ for $i \neq j$ and $P(A_1 \bigcap A_2 \bigcap A_3) = P(A_1)P(A_2)P(A_3)$.

11644. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let n be a nonnegative integer, and let B_j be the jth Bernoulli number, defined for $j \geq 0$ by $x/(e^x - 1) = \sum_{k=0}^{\infty} B_k x^k/k!$. Let

$$I_n = \int_0^\infty \left(\frac{1}{x^n(e^x-1)} - \left(\frac{1}{x^n} + \sum_{k=0}^{n} B_k \frac{x^{k-n-1}}{k!} \right) e^{-x} \right) dx.$$

Prove that $I_0 = \gamma - 1$, that $I_1 = 1 - (1/2)\log(2\pi)$, and that for $n \geq 1$,

$$I_{2n} = (\log(2\pi) + \gamma)\frac{B_{2n}}{(2n)!} + (-1)^n \frac{2\zeta'(2n)}{(2\pi)^{2n}} + \frac{1}{2(2n-1)!}H_{2n-1} - \sum_{k=0}^{n-1} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n-2k}}{(2n-2k)!},$$

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and that for $n \geq 1$,

$$I_{2n+1} = (-1)^n \frac{\zeta(2n+1)}{2(2\pi)^{2n}} - \frac{1}{2(2n)!} H_{2n} + \sum_{k=0}^{n} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n+1-2k}}{(2n+1-2k)!}.$$

Here, H_n denotes $\sum_{k=1}^{n} 1/k$, ζ, the Riemann zeta function, and γ, Euler's constant.

11645. *Proposed by Christopher J. Hillar, University of California, Berkeley, CA, Lionel Levine, Cornell University, Ithaca, NY, and Darren Rhea, University of California San Francisco, San Francisco, CA.* Determine all positive integers n such that the polynomial g in two variables given by $g(x, y) = 1 + y^2 \sum_{k=1}^{n} x^{2k} + y^4 x^{2n+2}$ factors in $\mathbb{C}[x, y]$.

11646. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let ABC be an acute triangle, and let A_1, B_1, C_1 be the intersection points of the angle bisectors from A, B, C to the respective opposite sides. Let R and r be the circumradius and the inradius of ABC, and let R_A, R_B, R_C be the circumradii of the triangles AC_1B_1, BA_1C_1, and CA_1B_1, respectively. Let H be the orthocenter of ABC, and let d_a, d_b, d_c be the distances from H to sides BC, CA, and AB, respectively. Show that

$$2r(R_A + R_B + R_C) \geq R(d_a + d_b + d_c).$$

11647. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA and Tudorel Lupu, Decebal High School, Constanta, Romania.* For continuous Ψ on $[0, 1]$, let $V\Psi$ be the function on $[0, 1]$ given by $V\Psi(t) = \int_0^t \Psi(x)\,dx$. For ϕ a differentiable function from $[0, 1]$ to $\mathbb{R}$ satisfying $\phi'(x) \neq 0$ for $0 < x < 1$, let $V_\phi\Psi(t) = \int_0^t \phi(x)\Psi(x)\,dx$. Show that if f and g are continuous real-valued functions on $[0, 1]$ then there exists $x_0 \in (0, 1)$ such that

$$(V_\phi f)(x_0) \int_0^1 g(x)\,dx - (V_\phi g)(x_0) \int_0^1 f(x)\,dx$$
$$= \phi(0) \left(Vf(x_0) \int_0^1 g(x)\,dx - Vg(x_0) \int_0^1 f(x)\,dx \right).$$

11648. *Proposed by Moubinool Omarjee, Paris, France.* Let E be the set of all continuous, differentiable functions from $(0, 1]$ into $\mathbb{R}$ such that $\int_0^1 t^{1/2} f^2(t)\,dt$ converges. Let F be the set of all f in E such that $\int_0^1 t^{-3/2} f^2(t)\,dt$ and $\int_0^1 t^{1/2} f'(t)^2\,dt$ converge. Equip E with the distance

$$d(f, g) = \left(\int_0^1 t^{1/2} (f - g)^2(t)\,dt \right)^{1/2}$$

to make it a metric space. Is F a closed subset of E?

SOLUTIONS

Piercing Many Segments

11507 [2010, 459]. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be a positive integer and let R be a plane region of perimeter 1. Inside R there are a finite number of line segments, the sum of whose lengths is greater than n. Prove that there exists a line that intersects at least $2n + 1$ of the segments.

Solution by Jim Simons, Cheltenham, U. K. We may assume that R is convex, for otherwise we can take its convex hull, which will have perimeter less than 1, and then dilate about a point in its interior to create a convex region with perimeter 1 that still includes all the line segments. For a bounded convex set C, we define $W_\theta(C)$, the width of C in direction θ, to be the minimum width of a strip in the plane that includes C and that is bounded by two parallel lines making angle θ with the x-axis.

Represent a point z on the boundary of R as the pair (t, γ), where t is the arc length along the boundary from a fixed starting point and γ is the angle between the x-axis and the tangent to the boundary of R at z. It follows that

$$W_\theta(R) = \frac{1}{2} \oint \left|\sin(\theta - \gamma)\right| dt.$$

We compute the average value of $W_\theta(R)$ over all angles θ:

$$\begin{aligned}\frac{1}{\pi} \int_0^\pi W_\theta(R)\, d\theta &= \frac{1}{2\pi} \int_0^\pi \oint \left|\sin(\theta - \gamma)\right| dt\, d\theta \\ &= \frac{1}{2\pi} \oint \int_0^\pi \left|\sin(\theta - \gamma)\right| d\theta\, dt = \frac{1}{\pi} \oint dt = \frac{1}{\pi}.\end{aligned}$$

Similarly, if S is the given set of line segments, and if $s \in S$ has length $|s|$ and makes an angle $\gamma(s)$ with the x-axis, then

$$\sum_{s \in S} W_\theta(s) = \sum_{s \in S} |s| \left|\sin\bigl(\theta - \gamma(s)\bigr)\right|.$$

Averaging this quantity over all angles, we obtain

$$\begin{aligned}\frac{1}{\pi} \int_0^\pi \sum_{s \in S} W_\theta(s)\, d\theta &= \frac{1}{\pi} \int_0^\pi \sum_{s \in S} |s| \left|\sin\bigl(\theta - \gamma(s)\bigr)\right| d\theta \\ &= \frac{1}{\pi} \sum_{s \in S} |s| \int_0^\pi \left|\sin\bigl(\theta - \gamma(s)\bigr)\right| d\theta = \frac{2\sigma}{\pi},\end{aligned}$$

where $\sigma = \sum_{s \in S} |s| > n$. It follows that there is some angle θ for which $\sum_{s \in S} W_\theta(s) \geq 2\sigma W_\theta(R)$. Now consider the lines that make angle θ with the x-axis and that intersect R. They lie in the strip we have defined, and we can index them by the distance u from one edge of the strip. Let $n(u)$ be the number of segments from S that intersect such a line. Now

$$\sum_{s \in S} W_\theta(s) = \int_0^{W_\theta(R)} n(u)\, du.$$

We have $n(u) \geq 2\sigma$ for some u, and hence $n(u) \geq 2n + 1$.

Also solved by R. Chapman (U. K.), O. P. Lossers (Netherlands), T. Starbird, R. Stong, GCHQ Problems Group (U. K.), and the proposer.

Perfect Squares with Specified Differences

11508 [2010, 459]. *Proposed by Mih'aly Bencze, Brasov, Romania.* Prove that for all positive integers k there are infinitely many positive integers n such that $kn + 1$ and $(k + 1)n + 1$ are both perfect squares.

Solution by Yury J. Ionin, Champaign, IL. We prove a more general result: If a and b are positive integers such that ab is not a perfect square, then for all integers c and d there are infinitely many positive integers n such that $an + c^2$ and $bn + d^2$ are both perfect squares.

Letting $an + c^2 = x^2$ and $bn + d^2 = y^2$, we have

$$bx^2 - ay^2 = bc^2 - ad^2. \tag{1}$$

Conversely, if (x, y) is a solution to (1) and $x^2 \equiv c^2 \pmod{a}$, then, for $n = (x^2 - c^2)/a$, we have $an + c^2 = x^2$ and $bn + d^2 = y^2$.

Let (u, v) be a solution to the Pell equation

$$u^2 - abv^2 = 1, \tag{2}$$

and let $x = cu + adv$ and $y = du + bcv$. Now (x, y) is a solution to (1) and $x^2 \equiv c^2u^2 \equiv c^2 \pmod{a}$. To complete the proof, note that since ab is positive and is not a perfect square, the equation (2) has infinitely many solutions.

Also solved by G. Apostolopoulos (Greece), B. D. Beasley, D. Beckwith, R. Chapman (U. K.), and H. M. Choe & E. Jee & S. Kim (S. Korea).

A Combinatorial Identity

11509 [2010, 558]. *Proposed by William Stanford, University of Illinois-Chicago, Chicago, IL.* Let m be a positive integer. Prove that

$$\sum_{k=m}^{m^2-m+1} \frac{\binom{m^2-2m+1}{k-m}}{k\binom{m^2}{k}} = \frac{1}{m\binom{2m-1}{m}}.$$

Solution I by Kim McInturff, Santa Barbara, CA. Among the $\binom{m^2}{2m-1}$ subsets of size $2m-1$ in $\{1, \dots, m^2\}$, exactly $\binom{k-1}{m-1}\binom{m^2-k}{m-1}$ have k as the median element. Therefore

$$\sum_{k=m}^{m^2-m+1} \frac{\binom{m^2-2m+1}{k-m}}{k\binom{m^2}{k}} = \frac{(m^2-2m+1)!}{(m^2)!} \sum_{k=m}^{m^2-m+1} \frac{(k-1)!}{(k-m)!} \cdot \frac{(m^2-k)!}{(m^2-m+1-k)!}$$

$$= \frac{(m^2-2m+1)!}{(m^2)!} ((m-1)!)^2 \sum_{k=m}^{m^2-m+1} \binom{k-1}{k-m} \cdot \binom{m^2-k}{m-1}$$

$$= \frac{(m^2-2m+1)!}{(m^2)!} ((m-1)!)^2 \binom{m^2}{2m-1} = \frac{((m-1)!)^2}{(2m-1)!} = \frac{1}{m\binom{2m-1}{m}}.$$

Solution II by Takis Konstantopoulos, Uppsala University, Uppsala, Sweden. Let A, B, a, and b be integers such that $A \ge 1$, $B \ge 1$, $a \ge b$, and $B - A \ge a - b$. We prove the more general identity

$$\sum_{k=a}^{A+a} \frac{\binom{A}{k-a}}{\binom{B}{k-b}} = \frac{B+1}{(a-b+1)\binom{B-A+1}{a-b+1}}. \tag{$*$}$$

The desired result follows by setting $A = (m-1)^2$, $B = m^2 - 1$, $a = m$, and $b = 1$ and dividing by m^2. To prove $(*)$, recall the beta integral

$$\int_0^1 t^p(1-t)^q\, dt = \frac{p!\,q!}{(p+q+1)!}$$

for nonnegative integers p and q. Using this, we write

$$\frac{1}{\binom{B}{k-b}} = (B+1)\int_0^1 t^{k-b}(1-t)^{B+b-k}\,dt$$

when $b \le k \le B+b$ and in particular when $a \le k \le A+a$. The sum in $(*)$ then becomes

$$\begin{aligned}\sum_{k=a}^{A+a} \frac{\binom{A}{k-a}}{\binom{B}{k-b}} &= (B+1)\int_0^1 t^{-b}(1-t)^{B+b}\sum_k \binom{A}{k-a}\left(\frac{t}{1-t}\right)^k dt\\ &= (B+1)\int_0^1 t^{-b}(1-t)^{B+b}\left(\frac{t}{1-t}\right)^a\left(1+\frac{t}{1-t}\right)^A dt\\ &= (B+1)\int_0^1 t^{a-b}(1-t)^{B-A+b-a}\,dt \;=\; \frac{B+1}{(a-b+1)\binom{B-A+1}{a-b+1}}.\end{aligned}$$

Solution III by Stanley Xiao, University of Waterloo, Waterloo, Ontario, Canada. We generalize the problem: given positive integers R, B, and b with $b \le B$, we show

$$\sum_{r=0}^{R} \frac{b\binom{B}{b}\binom{R}{r}}{(b+r)\binom{B+R}{b+r}} = 1. \qquad (**)$$

The desired result follows by setting $B = 2m-1$, $R = (m-1)^2$, and $b = m$.

To prove $(**)$, consider a deck of B blue cards and R red cards. A game is played where the player pulls cards without replacement from a shuffled deck and wins as soon as he obtains b blue cards. The probability of winning is 1, since $b \le B$. We compute the probability that the player wins after drawing exactly r red cards, with $0 \le r \le R$. The probability that exactly b of the first $b+r$ cards are blue is $\binom{B}{b}\binom{R}{r}/\binom{B+R}{b+r}$. The probability that the last card is blue given that exactly b of the first $b+r$ cards are blue is $b/(b+r)$. Hence the probability of the player winning after drawing exactly r red cards is

$$\frac{b\binom{B}{b}\binom{R}{r}}{(b+r)\binom{B+R}{b+r}}.$$

Summing over r gives $(**)$.

Also solved by M. Anton & E. Niehaus & E. Shirley, M. Bataille (France), D. Beckwith, R. Chapman (U. K.), R. Cheplyaka & V. Lucic & L. Pebody, P. De (India), M. N. Deshpande (India), M. Goldenberg & M. Kaplan, O. Kouba (Syria) M. E. Larsen (Denmark), J.-Y. Lee (Korea), O. P. Lossers (Netherlands), Á. Plaza (Spain), O. G. Ruehr, J. Schlosberg, J. Simons (U. K.), N. C. Singer, S. Song (Korea), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), F. Vrabec (Austria), H. Widmer (Switzerland), BSI Problems Group (Germany), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, and the proposer.

The Inf of the Circumcenter-Centroid-Incenter Angle is $\pi/2$

11516 [2010, 649]. *Proposed by Elton Bojaxhiu, Albania, and Enkel Hysnelaj, Australia.* Let $\mathcal{T}$ be the set of all nonequilateral triangles. For T in $\mathcal{T}$, let O be the circumcenter, Q the incenter, and G the centroid. Show that $\inf_{T\in\mathcal{T}} \angle OGQ = \pi/2$.

Editorial comment. As pointed out by O. Geupel (Germany) and B. Mulansky (Germany), the solution to this problem was actually contained in: Andrew P. Guinand, "Euler Lines, Tri-tangent Centers, and Their Triangles" in *Amer. Math. Monthly* **91** (1984) 290–300. In that article, Guinand defines the "critical circle" of a triangle as that for which the segment between the centroid and the orthocenter is a diameter. (This is also known as the "Circle of Bellot-Rosada".)

Guinand's Theorem 1 (p. 291) states: "The incenter of a non-equilateral triangle lies inside the critical circle, and all the excenters lie outside it." Thus we have immediately that the inf desired in this problem cannot be less than $\pi/2$.

Guinand's Theorem 4 (p. 296) states, in part, that: "Every point inside the critical circle except the nine-point center is the incenter of some triangle." (He assumes G and O are fixed, hence so is the critical circle.) The nine-point center must be exempted because, in some sense, it corresponds to the equilateral triangle. This clause of Theorem 4 guarantees that $\angle OGQ$ can be made arbitrarily close to $\pi/2$ by making Q close to G and even closer to the boundary of the critical circle.

Two readers, J.-P. Grivaux (France) and J. Schlosberg, pointed out that the claim that $\angle OGQ$ is always obtuse follows from Problem 10955, *Amer. Math. Monthly* **111** (2004) 67–69.

Solved by R. Bagby, C. Curtis, P. P. Dályay (Hungary), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, L. R. King, O. Kouba (Syria), K. McInturff, B. Mulansky (Germany), C. R. Pranesachar (India), J. Schlosberg, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), J. B. Zacharias, Barclays Capital Quantitative Analytics Group (U. K.), and the proposers.

A Harmonious Sum

11519 [2010, 649]. *Proposed by Ovidiu Furdui, Câmpia Turzii, Cluj, Romania.* Find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m},$$

where H_n denotes the nth harmonic number.

Solution I by Wim Nuij, Eindhoven, The Netherlands. We show that the value is $\frac{\pi^2}{12}-\frac{\log 2}{2}-\frac{\log^2 2}{2}$. Since the sum is not absolutely convergent, instead we consider $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-x)^{n+m}H_{n+m}/(n+m)$. This series is absolutely convergent for $|x|<1$, so combining the terms where $n+m=k+1$ yields

$$\sum_{k=0}^{\infty}(-1)^{k+1}x^{k+1}H_{k+1}\frac{k}{k+1}, \tag{1}$$

which can be split into

$$\sum_{k=0}^{\infty}(-1)^{k+1}x^{k+1}H_{k+1}+\sum_{k=1}^{\infty}(-1)^{k}\frac{x^{k+1}}{k+1}H_k+\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{k+1}}{(k+1)^2}.$$

The power series of $\log(1+x)$ is $\sum_{k=0}^{\infty}(-1)^k x^{k+1}/(k+1)$, so

$$\frac{\log(1+x)}{1+x}=\sum_{k=0}^{\infty}(-1)^k x^{k+1}\sum_{i=0}^{k}\frac{1}{i+1}=\sum_{k=0}^{\infty}(-1)^k x^{k+1}H_{k+1}.$$

Integration leads to

$$\frac{\log^2(1+x)}{2}=\sum_{k=1}^{\infty}(-1)^{k-1}H_k\frac{x^{k+1}}{k+1},$$

so (1) equals

$$-\frac{\log(1+x)}{1+x}-\frac{\log^2(1+x)}{2}+\sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{(k+1)^2}.$$

For $x \to 1^-$ this tends to $-\frac{\log 2}{2}-\frac{\log^2 2}{2}+\frac{\pi^2}{12}$, but we need to justify that this limit is the sum of the original series.

Let $T_n(x)=\sum_{m=1}^{\infty}(-1)^{m-1}x^{n+m}H_{n+m}/(n+m)$. Since H_p/p is strictly decreasing and $\lim_{p\to\infty}H_p/p=0$, the alternating series $T_n(x)$ converges uniformly on [0, 1], so it is continuous on this interval. For $0<x\le 1$ we have

$$\frac{H_{p-1}}{p-1}-x\frac{H_p}{p}>x\frac{H_p}{p}-x^2\frac{H_{p+1}}{p+1}>0 \quad \text{for all } p>1,$$

where the first inequality follows from the discriminant

$$\left(\frac{H_p}{p}\right)^2-\frac{H_p-\frac{1}{p}}{p-1}\cdot\frac{H_p+\frac{1}{p}}{p+1}<0,$$

and the second inequality follows from H_p/p being strictly decreasing. Hence $T_n(x)>T_{n+1}(x)>0$ for $0<x\le 1$, implying that $T_n(x)\to 0$ uniformly. Thus the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}T_n(x)$ converges uniformly on [0, 1]. Its sum is continuous at $x=1$, justifying taking the limit.

Solution II by Richard Stong, San Diego, CA. Let S denote the desired sum. Since the inner series is alternating with terms decreasing in magnitude, we have

$$\left|\sum_{m=1}^{\infty}(-1)^{m+n}\frac{H_{n+m}}{n+m}\right|\le\frac{H_{n+1}}{n+1}\to 0$$

as $n\to\infty$. Thus the terms in the outer sum tend to 0. Hence it suffices to show that even-indexed partial sums of S converge, and we may add pairs of consecutive terms (say the $(2r-1)$-st and $(2r)$-th). Doing the same for the inner sum as well gives

$$\begin{aligned}S&=\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\left((-1)^{m+1}\frac{H_{2r+m-1}}{2r+m-1}+(-1)^m\frac{H_{2r+m}}{2r+m}\right)\\&=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left(\frac{H_{2r+2s-2}}{2r+2s-2}-\frac{2H_{2r+2s-1}}{2r+2s-1}+\frac{H_{2r+2s}}{2r+2s}\right)\\&=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{2H_{2r+2s}-3}{(2r+2s-2)(2r+2s-1)(2r+2s)}.\end{aligned}$$

This sum now converges absolutely. Rearrange it by letting $t=r+s$ and noting that each value of t occurs for $t-1$ pairs (r,s) (and include the vanishing term where $t=1$) to get $S=\sum_{t=1}^{\infty}(2H_{2t}-3)/(4t(2t-1))$. Applying the same regularization procedure to the well-known identities

$$\sum_{t=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}=\int_0^1\frac{-\log(1-x)}{1+x}\,dx=\frac{\pi^2}{12}-\frac{\log^2 2}{2}, \qquad \sum_{t=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\log 2$$

gives

$$\sum_{t=1}^{\infty} \frac{H_{2t}-1}{2t(2t-1)} = \frac{\pi^2}{12} - \frac{\log^2 2}{2}, \qquad \sum_{t=1}^{\infty} \frac{1}{2t(2t-1)} = \log 2.$$

Comparing these formulas yields $S = \frac{\pi^2}{12} - \frac{\log 2}{2} - \frac{\log^2 2}{2}$.

Editorial comment. A number of incomplete solutions were received. Most found $\lim_{x\to 1^-} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-x)^{n+m} H_{n+m}/(n+m)$ (as in Solution I) and tried to invoke Abel's theorem to argue that it is the sum of the series. However, this requires the rearrangement of the terms of the double series into a single series, so the use of the limit needs to be justified. For example, the series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m}$ with $a_{n,1} = 1$, $a_{n,2} = -1$, and $a_{n,m} = 0$ for $n \geq 1$ and $m \geq 3$ has sum 0, but $\lim_{x\to 1^-} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} x^{n+m} = 1$.

Mark Wildon showed that if $\langle a_n \rangle$ is a decreasing sequence of positive numbers with limit 0 as $n \to \infty$ such that $(a_n - a_{n+1})$ and a_n/a_{n+1} are also decreasing, then $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} a_{n+m} = \lim_{x\to 1^-} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-x)^{n+m} a_{n+m}$. Summing the original series over the diagonal $k = m + n$ yields a divergent series.

The solution of two similar problems appeared recently in *Mathematics Magazine*: Problem 1838, **84** (2011), 65–67, and Problem 1849, **84** (2011), 234–235.

Also solved by R. Bagby, D. Beckwith, B. S. Burdick, M. Chamberland & E. A. Herman, H. Chen, J. Grivaux (France), O. Kouba (Syria), G. Lamb, D. O'Brien & C. Rousseau, P. Perfetti (Italy), N. C. Singer, A. Stenger, R. Tauraso (Italy), D. B. Tyler, M. Wildon (U. K.), S. Zhao (China), Barclays Capital Quantitative Analytics Group (U. K.), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Short Proof that a Factor Ring of a PID Is Armendariz

A commutative ring R is said to be Armendariz if, $f = \sum_{i=0}^{m} a_i X^i$, $g = \sum_{j=0}^{n} b_j X^j \in R[X]$ with $fg = 0$ implies $a_i b_j = 0$ for all i, j.

The Gauss lemma shows A/I is Armendariz for a PID A and an ideal I.

Indeed, recall over a PID A, the ideal $c(f)$ generated by the coefficients of a polynomial $f \in A[X]$ is called the content ideal and, Gauss's lemma says that $c(fg) = c(f)c(g)$. Now, if $f = \sum_{i=0}^{m} a_i X^i$, $g = \sum_{j=0}^{n} b_j X^j \in A[X]$ with $fg \in I[X]$, then writing l, m for the GCD of the a_i's and the b_j's, respectively, we have the content ideals $c(f) = (l)$, $c(g) = (m)$. So, for each i, j,

$$a_i b_j \in (l)(m) = c(f)c(g) = c(fg) \subseteq I.$$

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before October 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11649. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Let p be real with $p > 1$. Let $(x_0, x_1, \ldots)$ be a sequence of nonnegative real numbers. Prove that

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right)^p < \infty \;\Rightarrow\; \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\sum_{k=0}^{j}x_k\right)^p < \infty.$$

11650. *Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.* Evaluate

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-(x-y)^2}\sin^2(x^2+y^2)\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx.$$

11651. *Proposed by Marcel Celaya and Frank Ruskey, University of Victoria, Victoria, BC, Canada.* Show that the equation

$$\left\lfloor\frac{n+1}{\phi}\right\rfloor = n - \left\lfloor\frac{n}{\phi}\right\rfloor + \left\lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor - \left\lfloor\frac{\left\lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor + \left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor - \cdots$$

holds for every nonnegative integer n if and only if $\phi = (1+\sqrt{5})/2$.

11652. *Proposed by Ajai Choudhry, Foreign Service Institute, New Delhi, India.* For $a, b, c, d \in \mathbb{R}$, and for nonnegative integers i, j, and n, let

$$t_{i,j} = \sum_{s=0}^{i}\binom{n-i}{j-s}\binom{i}{s}a^{n-i-j+s}b^{j-s}c^{i-s}d^s.$$

http://dx.doi.org/10.4169/amer.math.monthly.119.06.522

Let $T(a, b, c, d, n)$ be the $(n+1)$-by-$(n+1)$ matrix with (i, j)-entry given by $t_{i,j}$, for $i, j \in \{0, \ldots, n\}$. Show that $\det T(a, b, c, d, n) = (ad - bc)^{n(n+1)/2}$.

11653. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let n be a positive integer. Determine all entire functions f that satisfy, for all complex s and t, the functional equation

$$f(s+t) = \sum_{k=0}^{n-1} f^{(n-1-k)}(s) f^{(k)}(t).$$

Here, $f^{(m)}$ denotes the mth derivative of f.

11654. *Proposed by David Borwein, University of Western Ontario, Canada, and Jonathan M. Borwein and James Wan, CARMA, University of Newcastle, Australia.* Let Cl denote the *Clausen* function, given by $\mathrm{Cl}(\theta) = \sum_{n=1}^{\infty} \sin(n\theta)/n^2$. Let ζ denote the Riemann zeta function.

(**a**) Show that

$$\int_{y=0}^{2\pi} \int_{x=0}^{2\pi} \log(3 + 2\cos x + 2\cos y + 2\cos(x-y))\, dx\, dy = 8\pi \mathrm{Cl}(\pi/3).$$

(**b**) Show that

$$\int_{y=0}^{\pi} \int_{x=0}^{\pi} \log(3 + 2\cos x + 2\cos y + 2\cos(x-y))\, dx\, dy = \frac{28}{3}\zeta(3).$$

11655. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let $ABCD$ be a convex quadrilateral, and let α, β, γ, and δ be the radian measures of angles DAB, ABC, BCD, and CDA, respectively. Suppose $\alpha + \beta > \pi$ and $\alpha + \delta > \pi$, and let $\eta = \alpha + \beta - \pi$ and $\phi = \alpha + \delta - \pi$. Let a, b, c, d, e, f be real numbers with $ac = bd = ef$. Show that if $abe > 0$, then

$$a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta + e\cos\eta + f\cos\phi \le \frac{be}{2a} + \frac{cf}{2b} + \frac{de}{2c} + \frac{af}{2d},$$

while for $abe < 0$ the inequality is reversed.

SOLUTIONS

A Triangle Inequality

11527 [2010, 742]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* Prove that in an acute triangle with sides of length a, b, c, inradius r, and circumradius R,

$$\frac{a^2}{b^2+c^2-a^2} + \frac{b^2}{c^2+a^2-b^2} + \frac{c^2}{a^2+b^2-c^2} \ge \frac{3}{2} \cdot \frac{R}{r}.$$

Solution by Thomas Smotzer, Youngstown State University, Youngstown, OH. Let $\triangle ABC$ be acute, with side lengths a, b, c, area K, and semiperimeter p. Let $S = S(a, b, c)$ be the sum on the left in the required inequality. Note that $K = \frac{1}{2}bc\sin A$, and by the law of cosines $b^2 + c^2 - a^2 = 2bc\cos A$, and similar formulas hold for angles B and C. So

$$S = \frac{a^2}{2bc\cos A} + \frac{b^2}{2ca\cos B} + \frac{c^2}{2ab\cos C}$$
$$= \frac{a^2\tan A}{4K} + \frac{b^2\tan B}{4K} + \frac{c^2\tan C}{4K}.$$

It is enough to show $a^2\tan A + b^2\tan B + c^2\tan C \geq 6KR/r = 6pR$, since $K = rp$. By the law of sines $a = 2R\sin A$, etc., so we must show that

$$a\,2R\ \sin A\tan A + b\,2R\ \sin B\tan B + c\,2R\ \sin C\tan C \geq 3(a+b+c)R.$$

Equivalently, we must show that $a\sin A\tan A + b\sin B\tan B + c\sin C\tan C \geq \frac{3}{2}(a+b+c)$. Note that since the triangle is acute, $\sin A\tan A$, $\sin B\tan B$, and $\sin C\tan C$ occur in the same order after sorting as do the corresponding quantities a, b, and c.

Using Chebyshev's inequality, it suffices to show that $\frac{1}{3}(a+b+c)(\sin A\tan A + \sin B\tan B + \sin C\tan C) \geq \frac{3}{2}$. This simplifies to

$$\sin A\tan A + \sin B\tan B + \sin C\tan C \geq \tfrac{9}{2}.$$

Since $\sin x\tan x$ is a convex function of x on $[0, \pi/2)$, by Jensen's inequality we have $\sin A\tan A + \sin B\tan B + \sin C\tan C \geq 3\sin\frac{1}{3}(A+B+C)\tan\frac{1}{3}(A+B+C)$. The right side of this simplifies to $3\sin(\pi/6)\tan(\pi/6) = 9/2$.

Also solved by A. Alt, G. Apostolopoulos (Greece), R. Bagby, M. Bataille (France), D. Beckwith, M. Can, M. Caragiu, C. Curtis, P. P. Dályay (Hungary), H. Y. Far, O. Faynshteyn (Germany), M. Goldenberg & M. Kaplan, J. G. Heuver (Canada), E. Hysnelaj & E. Bojaxhiu (Australia, Germany), E. Jee & S. Kim (S. Korea), W.-D. Jiang (China), O. Kouba (Syria), K.-W. Lau (China), J. H. Lee (Korea), K. McInturff, N. Minculete (Romania), P. Nüesch (Switzerland), Á. Plaza (Spain), C. R. Pranesachar (India), E. A. Smith, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), S. Wagon, H. Wang & J. Wojydylo, J. B. Zacharias, Barclays Capital Problems Solving Group (U. K.), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

An Inequality for Three Circumradii

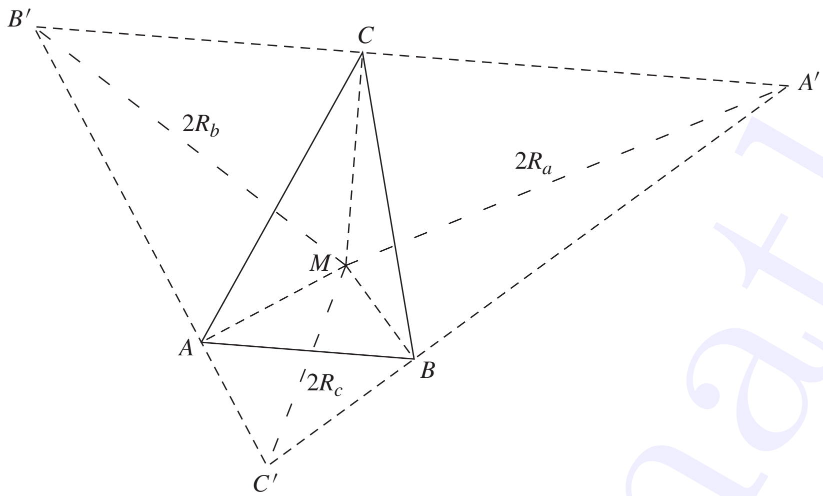
11531 [2010, 834]. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC and let $\lambda_1, \lambda_2, \lambda_3$ be positive real numbers. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Show that

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3\left(\frac{|MA|}{\lambda_1} + \frac{|MB|}{\lambda_2} + \frac{|MC|}{\lambda_3}\right).$$

(Here, for $V \in \{A, B, C\}$, $|MV|$ denotes the length of the line segment MV.)

Solution by George Apostolopoulis, Massolonghi, Greece. With points named as in the figure, we have that $B'C'$ is perpendicular to MA, $C'A'$ is perpendicular to MB, and $A'B'$ is perpendicular to MC. Using Pappus's Theorem, we have the inequality

$$|B'C'||MA'| \geq |C'A'||MC| + |A'B'||MB|. \qquad (1)$$



However, $|MA'|$ is the diameter of the circumcircle for $\triangle BAC$, that is, $|MA'| = 2R_a$. Therefore, using this in (1) gives

$$2R_a \geq \frac{|C'A'|}{|B'C'|}|MC| + \frac{|A'B'|}{|B'C'|}|MB|. \tag{2}$$

Similarly,

$$2R_b \geq \frac{|A'B'|}{|C'A'|}|MA| + \frac{|B'C'|}{|C'A'|}|MC|, \tag{3}$$

$$2R_c \geq \frac{|B'C'|}{|A'B'|}|MB| + \frac{|C'A'|}{|A'B'|}|MA|. \tag{4}$$

Multiplying (2) by λ_1^2, (3) by λ_2^2, and (4) by λ_3^2, then adding, we obtain

$$2\lambda_1^2 R_a + 2\lambda_2^2 R_b + 2\lambda_3^2 R_c \geq \left(\frac{\lambda_2^2|A'B'|}{|C'A'|} + \frac{\lambda_3^2|C'A'|}{|A'B'|}\right)|MA|$$
$$+ \left(\frac{\lambda_1^2|A'B'|}{|B'C'|} + \frac{\lambda_3^2|B'C'|}{|A'B'|}\right)|MB| + \left(\frac{\lambda_1^2|C'A'|}{|B'C'|} + \frac{\lambda_2^2|B'C'|}{|C'A'|}\right)|MC|. \tag{5}$$

For any real x and y, $x^2 + y^2 \geq 2xy$. Applying this to the coefficient of $|MA|$ on the right of (5), we get

$$\frac{\lambda_2^2|A'B'|^2 + \lambda_3^2|C'A'|^2}{|A'B'||C'A'|} \geq 2\lambda_2\lambda_3.$$

Similar inequalities follow for the other two terms in (5), and so (5) implies

$$2\lambda_1^2 R_a + 2\lambda_2^2 R_b + 2\lambda_3^2 R_c \geq 2\lambda_2\lambda_3|MA| + 2\lambda_1\lambda_3|MB| + 2\lambda_1\lambda_2|MC|.$$

This is the required inequality, namely

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3\left(\frac{|MA|}{\lambda_1} + \frac{|MB|}{\lambda_2} + \frac{|MC|}{\lambda_3}\right).$$

Equality holds when $\triangle ABC$ is equilateral, M is the circumcenter, and $\lambda_1 = \lambda_2 = \lambda_3$.

Also solved by M. Bataille (France), P. P. Dályay (Hungary), O. Faynshteyn (Germany), O. Geupel (Germany), O. Kouba (Syria), J. H. Smith, T. Smotzer, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), J. B. Zacharias & K. T. Greeson, and the proposer.

Zero–Nonzero Matrices

11534 [2010, 835]. *Proposed by Christopher Hillar, Mathematical Sciences Research Institute, Berkeley, CA.* Let k and n be positive integers with $k < n$. Characterize the $n \times n$ real matrices M with the property that for all $v \in \mathbb{R}^n$ with at most k nonzero entries, Mv also has at most k nonzero entries.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We show that either M has at most k nonzero rows or M has at most 1 nonzero entry in every column. Such a matrix has the desired property: in the first case Mv has at most k nonzero entries for any v, and in the second case Mv has at most as many nonzero entries as v.

Now suppose that some matrix M with the desired property is not of the form stated. Thus M has at least $k+1$ rows with nonzero entries and at least one column with at least two nonzero entries. Build a list of columns of M as follows: say a list of columns *represents* a row if and only if at least one of the columns in the set has a nonzero entry in that row. Start with a column w_1 with at least two nonzero entries. If $w_1, \dots, w_r$ have been chosen, and together they represent fewer than $k+1$ rows, then choose w_{r+1} to be any column that represents a new row and append it to the list. Stop when $w_1, \dots, w_r$ represent at least $k+1$ rows. Now we started with a column representing two rows, and each time we added a new column we got at least one new row. Hence $r \le k$. Thus any linear combination $\sum_{j=1}^r a_j w_j$ is of the form Mv, where v has at most k nonzero entries. Fix $k+1$ rows represented by the w_j. For the ith such row, let V_i be the set of all r-tuples $(a_1, \dots, a_r)$ such that $\sum_{j=1}^r a_j w_j$ has a nonzero entry in that ith row. Since $w_1, \dots, w_r$ do represent this row, V_i is the nullspace of a nontrivial linear equation on r-tuples and therefore is a codimension-1 subspace of $\mathbb{R}^r$. However, the required property of M says that for any r-tuple $(a_1, \dots, a_r)$, the linear combination $\sum_{j=1}^r a_j v_j$ has at most k nonzero entries; thus $(a_1, \dots, a_r)$ must lie in one of these $k+1$ subspaces. But of course $\mathbb{R}^r$ cannot be covered by finitely many codimension-1 subspaces. This contradiction shows that such an M cannot exist.

Editorial comment. Several solvers noted that the same result holds for any field of characteristic 0. John Smith (Needham, MA) noted that the result holds for rectangular matrices.

Also solved by P. Budney, N. Caro (Brazil), P. P. Dályay (Hungary), E. A. Herman, Y. J. Ionin, J. H. Lindsey II, O. P. Lossers (Netherlands), R. E. Prather, J. Simons (U. K.), J. H. Smith, M. Tetiva (Romania), E. I. Verriest, Barclays Capital Problems Solving Group (U. K.), NSA Problems Group, and the proposer.

How Closely Does This Sum Approximate the Integral?

11535 [2010, 835]. *Proposed by Marian Tetiva, Bîrlad, Romania.* Let f be a continuously differentiable function on $[0, 1]$. Let $A = f(1)$ and let $B = \int_0^1 x^{-1/2} f(x)\,dx$. Evaluate

$$\lim_{n\to\infty} n \left(\int_0^1 f(x)\,dx - \sum_{k=1}^n \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} \right) f\left(\frac{(k-1)^2}{n^2} \right) \right)$$

in terms of A and B.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. Answer: $A - B/2$.
Proposition. *If h is continuously differentiable on* $[0, 1]$, *then*

$$\lim_{n\to\infty} n \left(\int_0^1 h(x)\,dx - \frac{1}{n} \sum_{k=1}^n h\left(\frac{k-1}{n} \right) \right) = \frac{h(1) - h(0)}{2}.$$

Proof. Let H denote an antiderivative of h. By Taylor's theorem,

$$\int_0^1 h(y)\,dy - \frac{1}{n}\sum_{k=1}^n h\left(\frac{k-1}{n}\right) = \sum_{k=1}^n \left(\int_{(k-1)/n}^{k/n} h(y)\,dy - \frac{1}{n} h\left(\frac{k-1}{n}\right)\right)$$

$$= \sum_{k=1}^n \left(H\left(\frac{k}{n}\right) - H\left(\frac{k-1}{n}\right) - \frac{1}{n}H'\left(\frac{k-1}{n}\right)\right) = \frac{1}{2n^2}\sum_{k=1}^n H''(y_k),$$

for some list $(y_1, \ldots, y_n)$ with $y_k \in ((k-1)/n, k/n)$ for $1 \le k \le n$. Therefore,

$$\lim_{n\to\infty} n\left(\int_0^1 h(y)\,dy - \sum_{k=1}^n h\left(\frac{k-1}{n}\right)\frac{1}{n}\right) = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n H''(y_k)$$

$$= \frac{1}{2}\int_0^1 h'(y)\,dy = \frac{h(1)-h(0)}{2}. \qquad \blacksquare$$

For this problem, let $h(y) = 2yf(y^2)$. Now $(h(1)-h(0))/2 = (2f(1))/2 = A$. Using the substitution $y = \sqrt{x}$, we have

$$\int_0^1 f(x)\,dx = \int_0^1 h(y)dy, \quad B = \int_0^1 x^{-1/2} f(x)\,dx = 2\int_0^1 f(y^2)\,dy.$$

Therefore, by the proposition,

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)\,dx - \sum_{k=1}^n \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty} n\left(\int_0^1 h(y)\,dy - \sum_{k=1}^n \left(\frac{2(k-1)}{n^2} + \frac{1}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty}\left[n\left(\int_0^1 h(y)\,dy - \frac{1}{n}\sum_{k=1}^n h\left(\frac{k-1}{n}\right)\right) - \frac{1}{n}\sum_{k=1}^n f\left(\frac{(k-1)^2}{n^2}\right)\right]$$

$$= \frac{h(1)-h(0)}{2} - \int_0^1 f(y^2)\,dy = A - \frac{B}{2}.$$

Also solved by P. Bracken, N. Caro (Brazil), H. Chen, D. Constales (Belgium), P. P. Dályay (Hungary), Y. Dumont (France), P. J. Fitzsimmons, D. Fleischman, J.-P. Grivaux (France), F. Holland (Ireland), S. Kaczkowski, P. Khalili, O. Kouba (Syria), W. C. Lang, J. H. Lindsey II, R. Nandan, M. Omarjee (France), K. Schilling, J. Schlosberg, J. Simons (U. K.), N. C. Singer, Z. Song & L. Yin (China), A. Stenger, R. Stong, T. Tam, J. A. Van Casteren (Belgium), E. I. Verriest, P. Xi (China), J. B. Zacharias & K. T. Greeson, Barclays Capital Problems Solving Group (U. K.), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Use Hayashi's Inequality

11536 [2010, 835]. *Proposed by Mihaly Bencze, Brasov, Romania.* Let K, L, and M denote the respective midpoints of sides AB, BC, and CA in triangle ABC, and let P be a point in the plane of ABC other than K, L, or M. Show that

$$\frac{|AB|}{|PK|} + \frac{|BC|}{|PL|} + \frac{|CA|}{|PM|} \ge \frac{|AB|\cdot|BC|\cdot|CA|}{4|PK|\cdot|PL|\cdot|PM|},$$

where $|UV|$ denotes the length of segment UV.

Solution by D. Marinescu, Colegiul Naţional "Iancu de Hunedoara", Hunedoara, Romania, and M. Monea, Colegiul Naţional"Decebral", Deva, Romania. In 1913, T. Hayashi proved **Hayashi's Inequality**: *For any triangle ABC with opposite sides of lengths a, b, c respectively, and for and an arbitrary point M in its plane,*

$$a|MB| \cdot |MC| + b|MC| \cdot |MA| + c|MA| \cdot |MB| \geq abc.$$

See D. M. Mitronović, J. E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities,* (Kluwer, 1989), p. 297. We now apply Hayashi's Inequality with triangle KLM and point P to get

$$|KL| \cdot |PK| \cdot |PL| + |KM| \cdot |PK| \cdot |PM| + |ML| \cdot |PM| \cdot |PL| \geq |KL| \cdot |ML| \cdot |MK|.$$

Since $|KL| = |AC|/2$, $|KM| = |BC|/2$, and$|ML| = |AC|/2$,

$$|CA| \cdot |PK| \cdot |PL| + |BC| \cdot |PK| \cdot |PM| + |AB| \cdot |PM| \cdot |PL| \geq \frac{1}{4}|CA| \cdot |BC| \cdot |AB|,$$

which is equivalent to the inequality to be proved.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), D. Beckwith, M. Caragiu, P. P. Dályay (Hungary), O. Geupel (Germany), O. Kouba (Syria), N. Minculete (Romania), B. Mulansky (Germany), C. R. Pranesachar (India), J. Schlosberg, T. Smotzer, M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, and the proposer.

A Circumradius Inequality

11541 [2010, 929]. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Let $|MA|$, $|MB|$, and $|MC|$ be the distances from M to A, B, and C. Show that

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \leq \frac{3}{2}.$$

Solution by Oleh Faynshteyn, Leipzig, Germany. Let $\varphi_1 = \angle CAM$, $\varphi_2 = \angle MAB$, $\varphi_3 = \angle ABM$, $\varphi_4 = \angle MBC$, $\varphi_5 = \angle BCM$, and $\varphi_6 = \angle BCM$. Observe that $\sum_{i=1}^{6} \varphi_i = \pi$. From triangles AMC and ABM, it follows that

$$|MA| = 2R_b \sin\varphi_6 = 2R_c \sin\varphi_3,$$

hence

$$\frac{|MA|}{R_b + R_c} = \frac{2}{\csc\varphi_3 + \csc\varphi_6} \leq \frac{1}{2}(\sin\varphi_3 + \sin\varphi_6),$$

where the inequality is a consequence of the arithmetic-harmonic mean inequality. Similarly we get

$$\frac{|MB|}{R_c + R_a} \leq \frac{1}{2}(\sin\varphi_2 + \sin\varphi_5), \qquad \frac{|MC|}{R_a + R_b} \leq \frac{1}{2}(\sin\varphi_1 + \sin\varphi_4).$$

Adding these three inequalities, we obtain

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \leq \frac{1}{2}\sum_{i=1}^{6} \sin\varphi_i.$$

Since the sine function is concave down on $(0, \pi)$, Jensen's inequality gives

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \le 3 \sin \frac{\pi}{6} = \frac{3}{2}.$$

Equality holds if and only if all the φ_i are $\pi/6$, that is, if and only if ABC is equilateral and M is its center.

Editorial comment. Pál Péter Dálylay and Marian Dincă (independently) remarked that the problem and solution generalize as follows. Let M be a point in the interior of the convex n-gon$A_1 \cdots A_n$ (with all indices interpreted mod n). With R_k denoting the circumradius of triangle MA_kA_{k+1}, we have

$$\sum_{k=1}^{n} \frac{|MA_k|}{R_{k-1} + R_k} \le n \cos \frac{\pi}{n},$$

with equality if and only if the n-gon is regular and M is its center.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), M. Can, R. Chapman (U. K.), P. P. Dálylay (Hungary), M. Dincă (Romania), W. Jiang (China), O. Kouba (Syria), C. R. Pranesachar (India), J. Schlosberg, R. A. Simon (Chile), J. Simons (U. K.), R. Smith, T. Smotzer, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), J. B. Zacharias & K. T. Greeson, GCHQ Problem Solving Group (U. K.), and the proposer.

Gamma and Beta Inequalities

11542 [2010, 929]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania.* Show that for $x, y, z > 1$, and for positive α, β, γ,

$$(2x^2 + yz)\Gamma(x) + (2y^2 + zx)\Gamma(y) + (2z^2 + xy)\Gamma(z)$$
$$\ge (x + y + z)(x\Gamma(x) + y\Gamma(y) + z\Gamma(z)),$$

and

$$B(x, \alpha)^{x^2+2yz} B(y, \beta)^{y^2+2zx} B(z, \gamma)^{z^2+2xy}$$
$$\ge (B(x, \alpha)B(y, \beta)B(z, \gamma))^{xy+yz+zx}.$$

Here, $B(x, \alpha)$ is Euler's beta function, defined by $B(x, \alpha) = \int_0^1 t^{x-1}(1-t)^{\alpha-1}dt$.

Solution by M. A. Prasad, India. The first inequality is equivalent to

$$(x - y)(x - z)\Gamma(x) + (y - z)(y - x)\Gamma(x) + (z - x)(z - y)\Gamma(z) \ge 0.$$

It is symmetric in x, y, z, so we may assume $x \ge y \ge z$. The first and third terms are nonnegative, and the middle term is nonpositive. Note also that $\Gamma(x)$ is a convex function for $x > 0$, since $(d^2/dx^2)\Gamma(x) = (\log x)^2 \int_0^\infty e^{-t}t^{x-1}\,dx \ge 0$. Therefore, $\Gamma(y) \le \max\{\Gamma(x), \Gamma(z)\}$. Since $|(y - z)(y - x)| \le \min\{(x - y)(x - z), (z - x)(z - y)\}$, one of the nonnegative terms is at least as large as the nonpositive term in absolute value. This completes the proof for the first inequality. The second inequality is incorrect. For a counterexample, consider $x > y > z$, and α, γ very large, and $\beta = 1$. The inequality is equivalent to

$$B(x, \alpha)^{(x-y)(x-z)} B(y, \beta)^{(y-z)(y-x)} B(z, \gamma)^{(z-x)(z-y)} \ge 1.$$

Now as $\alpha, \gamma \to \infty$, we have $B(x, \alpha), B(z, \gamma) \to 0$, so the left side is less than 1.

Also solved by G. Apostopoulos (Greece), R. Bagby, R. Chapman (U. K.), P. P. Dálylay (Hungary), R. Stong, J. V. Tejedor (Spain), and GCHQ Problem Solving Group (U. K.)

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before December 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11656. *Proposed by Valerio De Angelis, Xavier University of Louisiana, New Orleans, LA.* The *sign chart* of a polynomial f with real coefficients is the list of successive pairs (ϵ, σ) of signs of (f', f) on the intervals separating real zeros of ff', together with the signs at the zeros of ff' themselves, read from left to right. Thus, for $f = x^3 - 3x^2$, the sign chart is $((1, -1), (0, 0), (-1, -1), (0, -1), (1, -1), (1, 0), (1, 1))$. As a function of n, how many distinct sign charts occur for polynomials of degree n?

11657. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Given a set V of n points in $\mathbb{R}^2$, no three of them collinear, let E be the set of $\binom{n}{2}$ line segments joining distinct elements of V.
(**a**) Prove that if $n \not\equiv 2 \pmod 3$, then E can be partitioned into triples in which the length of each segment is greater than the sum of the other two.
(**b**) Prove that if $n \equiv 2 \pmod 3$ and e is an element of E, then $E\backslash\{e\}$ can be so partitioned.

11658. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.* Let V be the vector space over $\mathbb{R}$ of all (countably infinite) sequences $(x_1, x_2, \ldots)$ of real numbers, equipped with the usual addition and scalar multiplication. For $v \in V$, say that v is *binary* if $v_k \in \{0, 1\}$ for $k \geq 1$, and let B be the set of all binary members of V. Prove that there exists a subset I of B with cardinality $2^{\aleph_0}$ that is linearly independent over $\mathbb{R}$. (An infinite subset of a vector space is linearly independent if all of its finite subsets are linearly independent.)

11659. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let x be real with $0 < x < 1$, and consider the sequence $\langle a_n \rangle$ given by $a_0 = 0$, $a_1 = 1$, and, for $n > 1$,

$$a_n = \frac{a_{n-1}^2}{xa_{n-2} + (1-x)a_{n-1}}.$$

http://dx.doi.org/10.4169/amer.math.monthly.119.07.608

Show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

11660. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Consider the following differential equation: $s''(t) = -s(t) - s(t)^2 \operatorname{sgn}(s'(t))$, where $\operatorname{sgn}(u)$ denotes the sign of u. Show that if $s(0) = a$ and $s'(0) = b$ with $ab \neq 0$, then (s, s') tends to $(0, 0)$ with $\sqrt{s^2 + s'^2} \leq C/t$ as $t \to \infty$, for some $C > 0$.

11661. *Proposed by Giedrius Alkauskas, Vilnius University, Vlinius, Lithuania.* Find every function f on $\mathbb{R}^+$ that satisfies the functional equation

$$(1 - z) f(x) = f\left(\frac{1 - z}{z} f(xz)\right)$$

for $x > 0$ and $0 < z < 1$.

11662. *Proposed by H. Stephen Morse, Fairfax, Va.* Let $ABCD$ be the vertices of a square, in that order. Insert P and Q on AB (in the order $APQB$) so that each of P and Q divides AB 'in extreme and mean ratio' (that is, $|AB|/|BQ| = |BQ|/|QA|$ and $|AB|/|AP| = |AP|/|PB|$.) Likewise, place R and S on CD so that $CRSD$ is divided in the same proportions as $APQB$. The four intersection points of AR, BS, CP, and DQ are called the *harmonious quartet* of the square on its *base pair* (AB, CD). They form a rhombus whose long diagonal has length $(\sqrt{5} + 1)/2$ times the length of its short diagonal.

Given a cube, create the harmonious quartet for each of its six faces, using each edge as part of a base pair exactly once, according to this scheme: label the vertices on one face of the cube $ABCD$ and the corresponding vertices of the opposite face $A'B'C'D'$. Pair AB with CD, AA' with BB', and BC with $B'C'$. The rest of the pairings are then forced: $A'B'$ with $C'D'$, AD with $A'D'$, and CC' with DD'. This generates 24 points.

(**a**) Show that these 24 points are a subset of the 32 vertices of a *rhombic triacontahedron* (a convex polyhedron bounded by 30 congruent rhombic faces, meeting three each across their obtuse angles at 20 vertices, and five each across their acute angles at 12 vertices), and find a construction for the remaining eight vertices.

(**b**) Show, moreover, that the 12 end points of the longer diagonals of the six constructed rhombi are the vertices of an icosahedron I, and these diagonals are edges of the icosahedron.

(**c**) Show that the 12 end points of the shorter diagonals of the constructed rhombi, together with the eight additional vertices of the triacontahedron, are the vertices of a dodecahedron. Show also that these shorter diagonals are edges of that dodecahedron.

SOLUTIONS

An Inequality in Three Variables

11543 [2010, 390]. *Proposed by Richard Stong, Center for Communications Research, San Diego, CA.* Let x, y, z be positive numbers with $xyz = 1$. Show that $(x^5 + y^5 + z^5)^2 \geq 3(x^7 + y^7 + z^7)$.

Solution by Erik I. Verriest, Georgia Institute of Technology, Atlanta, GA. The problem is equivalent to showing that the minimum of

$$\frac{\left(x^5+y^5+z^5\right)^2}{x^7+y^7+z^7},$$

subject to the constraints $xyz = 1$ and $x, y, z > 0$, is at least 3. Since the ratio equals 3 when $x = y = z = 1$, the bound can be achieved.

First suppose that $x \to 0$. In this case or y, z tends to ∞, and

$$\frac{\left(x^5+y^5+z^5\right)^2}{x^7+y^7+z^7} \geq \frac{\left(\max\{x, y, z\}\right)^{10}}{3\max\{x, y, z\}^7} \to \infty.$$

Similar reasoning holds for $y \to 0$ and $z \to 0$. Therefore the minimum value is achieved at an interior point where $x, y, z > 0$.

We apply the method of Lagrange multipliers, letting

$$L = \frac{\left(x^5+y^5+z^5\right)^2}{x^7+y^7+z^7} + \lambda(xyz-1).$$

Necessary conditions for a stationary point are

$$\frac{\partial}{L}\lambda = \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0.$$

Thus, at any stationary point, $xyz = 1$. Setting $U = x^5 + y^5 + z^5$ and $V = x^7 + y^7 + z^7$ and computing the other partials derivatives gives

$$\begin{aligned} 10x^5UV - 7x^7U^2 + \lambda V^2 &= 0, \\ 10y^5UV - 7y^7U^2 + \lambda V^2 &= 0, \\ 10z^5UV - 7z^7U^2 + \lambda V^2 &= 0. \end{aligned} \tag{1}$$

Adding and simplifying gives $U^2V + \lambda V^2 = 0$, and since $V \neq 0$, $\lambda = -U^2/V$. Substituting this into (1) and dividing (as we may) by UV^2 gives

$$\frac{7}{V}x^7 - \frac{10}{U}x^5 + 1 = 0. \tag{2}$$

and counterparts with y and z in place of x.

Consider now for $a > 0$ the function f defined by $f(a) = pa^7 - qa^5 + 1$, where both p and q are positive. Note $f(0) = 1$. This function is minimal at that a (call it a^*) such that $7pa^6 - 5qa^4 = 0$. Since $a^* > 0$, it follows that $a^* = \sqrt{5q}/\sqrt{7p}$. The minimum is

$$f(a^*) = \left(\left(\frac{5}{7}\right)^{7/2} - \left(\frac{5}{7}\right)^{5/2}\right)\frac{q^{7/2}}{p^{5/2}} + 1 = 1 - \frac{2}{7}\left(\frac{5}{7}\right)^{5/2}\frac{q^{7/2}}{p^{5/2}} < 1.$$

Thus f has two positive zeros, one (double) positive zero, or no positive zeros, when $f(a^*)$ is less than, equal to, or larger than 0, respectively. Now return to (2) by writing $p = 7/V$ and $q = 10/U$, which we can do because U and V are positive. If f has no positive zero, then (2) has no solution. If f has one positive zero, then the solution to (2) is $x = y = z$, and since $xyz = 1$ this is the first case $x = y = z = 1$. If f has two positive zeros, then in the solution to (2), two of x, y, z are equal, say $x = y$, and the other is $z = 1/x^2$. This leads to the minimization of

$$\frac{(2x^{15}+1)^2}{x^6(2x^{21}+1)}$$

over $x > 0$. The stationary value is given by

$$2(x^3)^{12} + 8(x^3)^5 - 9(x^3)^7 - 1 = 0,$$

which has a unique positive solution for $X = x^3 = 1$ and corresponds to a minimum. This again yields the first solution $x = y = z = 1$.

Editorial comment. Oliver Geupel and the proposer independently proved the inequality using the method found in V. Cîrtoaje, The equal variable method, *Journal of Inequalities in Pure and Applied Mathematics* **8** (2007) issue 1, article 15, corollary 1.7 (page 3), available at http://www.emis.de/journals/JIPAM/images/059_06_JIPAM/059_06.pdf.

Also solved by R. Bagby, P. Bracken, D. Constales (Belgium), A. Cooper, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), K. McInturff, D. J. Moore, P. Perfetti (Italy), C. R. Pranesachar (India), A. Stenger, C. Y. Yildirim (Turkey), Y. Yu, S. M. Zemyan, GCHQ Problem Solving Group (U. K.), Texas State University Problem Solvers Group, and the proposer.

A Rhombus From a Triangle

11547 [2011, 84]. *Proposed by Francisco Javier García Capitán, I.E.S Álvarez Cubero, Priego de Córdoba, Spain, and Juan Bosco Romero Márquez, University of Valladolid, Spain.* Let the altitude AD of triangle ABC be produced to meet the circumcircle again at E. Let K, L, M, and N be the projections of D onto the lines BA, AC, CE, and EB, and let P, Q, R, and S be the intersections of the diagonals of $DKAL$, $DLCM$, $DMEN$, and $DNBK$, respectively. Let $|XY|$ denote the distance from X to Y, and let α, β, γ be the radian measure of angles BAC, CBA, ACB, respectively. Show that $PQRS$ is a rhombus and that $|QS|^2/|PR|^2 = 1 + \cos(2\beta)\cos(2\gamma)/\sin^2\alpha$.

Solution by Robin Chapman, University of Exeter, Exeter, England. We have a cyclic quadrilateral $BACE$ whose diagonals are perpendicular and meet at D. Let $a = |DA|$, $b = |DB|$, $c = |DC|$, and $e = |DE|$. (Note that a, b and c are not the side-lengths of triangle ABC.) We use Cartesian coordinates with origin D. By aligning the coordinate axes appropriately, we get $A = (0, a)$, $B = (b, 0)$, $C = (-c, 0)$, and $E = (0, -e)$. By a standard property of intersecting chords of a circle, $ae = bc$.

The point K is the unique point on AB for which DK and AB are perpendicular. A routine calculation gives the ordered pair $K = \left(\frac{a^2b}{a^2+b^2}, \frac{ab^2}{a^2+b^2}\right)$, and, similarly, the pairs $L = \left(-\frac{a^2c}{a^2+c^2}, \frac{ac^2}{a^2+c^2}\right)$, $M = \left(-\frac{ce^2}{c^2+e^2}, -\frac{c^2e}{c^2+e^2}\right)$, and $N = \left(\frac{be^2}{b^2+e^2}, -\frac{b^2e}{b^2+e^2}\right)$.

The point P is the intersection of the line KL with the y-axis DA. Let $X = (x, y)$ and $U = (u, v)$ with $x \neq u$. The intersection of the line XU and the y-axis is $\big(0, (xv - yu)/(x - u)\big)$. Performing the calculation gives

$$P = \left(0, \frac{abc}{a^2 + bc}\right) = \left(0, \frac{ae}{a + e}\right),$$

since $ae = bc$. Similarly,

$$Q = \left(-\frac{bc}{b+c}, 0\right), \qquad R = \left(0, -\frac{ae}{a+e}\right), \qquad S = \left(\frac{bc}{b+c}, 0\right).$$

Thus $PQRS$ is a parallelogram, with centre D and with perpendicular diagonals. Hence $PQRS$ is a rhombus.

Since $e = bc/a$, we have that

$$\frac{|QS|^2}{|PR|^2} = \frac{(a+e)^2}{(b+c)^2} = \frac{(a^2+bc)^2}{a^2(b+c)^2}.$$

Considering the right-angled triangle BDA yields $\tan\beta = \frac{a}{b}$.

Therefore,

$$\cos(2\beta) = \frac{1-\tan^2\beta}{1+\tan^2\beta} = \frac{b^2-a^2}{a^2+b^2}.$$

Similarly,

$$\cos(2\gamma) = \frac{c^2-a^2}{a^2+c^2}.$$

Computing twice the area of triangle ABC one way yields

$$|AD|\,|BC| = a(b+c),$$

and another yields

$$|AB|\,|AC| \sin\alpha = \sqrt{(a^2+b^2)(a^2+c^2)}\ \sin\alpha.$$

Equating the right sides gives

$$\sin^2\alpha = \frac{a^2(b+c)^2}{(a^2+b^2)(a^2+c^2)}.$$

Therefore,

$$\begin{aligned} 1 + \frac{\cos(2\beta)\cos(2\gamma)}{\sin^2\alpha} &= 1 + \frac{(a^2-b^2)(a^2-c^2)}{a^2(b+c)^2} \\ &= \frac{a^4+2a^2bc+b^2c^2}{a^2(b+c)^2} = \frac{(a^2+bc)}{a^2(b+c)^2} = \frac{|QS|^2}{|PR|^2}. \end{aligned}$$

Also solved by M. Bataille (France), P. P. Dályay (Hungary), A. Ercan (Turkey), M. Goldenberg & M. Kaplan, D. Gove, J.-P. Grivaux (France), J. G Heuver (Canada), O. Kouba (Syria), P. Nüesch (Switzerland), C. R. Pranesachar (India), J. Schlosberg, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposers.

Derivative Cauchy–Schwarz

11548 [2011, 85]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* Let f be a twice-differentiable real-valued function with continuous second derivative, and suppose that $f(0) = 0$. Show that

$$\int_{-1}^{1} (f''(x))^2\,dx \geq 10\left(\int_{-1}^{1} f(x)\,dx\right)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China. Let $g(x) = f(x) + f(-x)$ so that $g(0) = g'(0)$ and g is twice continuously differentiable. Integrate by parts twice:

$$\int_0^1 g(x)\,dx = -\int_0^1 (x-1)g'(x)\,dx = \frac{1}{2}\int_0^1 (x-1)^2 g''(x)\,dx.$$

By Cauchy–Schwarz, and abbreviating $g(x)\,dx$ to g and so on, we have

$$\left(\int_0^1 g\right)^2 \le \frac{1}{4}\int_0^1 (x-1)^4 \int_0^1 \left(g''\right)^2 = \frac{1}{20}\int_0^1 \left(g''\right)^2. \tag{1}$$

Since $(g''(x))^2 = (f''(x) + f''(-x))^2 \le 2\big((f''(x))^2 + (f''(-x))^2\big)$, we have

$$\int_0^1 \left(g''\right)^2 \le 2\int_0^1 \left((f''(x))^2 + (f''(-x))^2\right) = 2\int_{-1}^1 \left(f''\right)^2. \tag{2}$$

Finally, from (1) and (2) we have

$$\int_{-1}^1 \left(f''\right) \ge 10\left(\int_0^1 g\right)^2 = 10\left(\int_0^1 \left(f(x) + f(-x)\right)\right)^2 = 10\left(\int_{-1}^1 f\right)^2.$$

Also solved by K. F. Andersen (Canada), R. Bagby, M. Bello-Hernández & M. Benito (Spain), G. E. Bilodeau, M. W. Botsko & L. Mismas, C. Burnette, R. Chapman (U. K.), H. Chen, D. Constales (Belgium), P. P. Dálylay (Hungary), P. J. Fitzsimmons, P. Gallegos (Chile), J.-P. Grivaux (France), L. Han (U.S.A.) & L. Yu (China), E. A. Herman, E. J. Ionascu, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Mortini & J. Noël (France), M. Omarjee (France), P. Perfetti (Italy), Á. Plaza (Spain), K. Schilling, J. Simons (U. K.), A. Stenger, R. Stong, R. Tauraso (Italy), T. Trif (Romania), E. I. Verriest, M. Vowe (Switzerland), H. Wang & Y. Xia, Y. Wang, T. Wiandt, L. Zhou, Barclays Capital Problems Solving Group, NSA Problems Group, and the proposer.

A Triple Functional Equation

11549 [2011, 85]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.* Determine all continuous functions f on $\mathbb{R}$ such that for all x,

$$f(f(f(x))) - 3f(x) + 2x = 0.$$

Solution by Barclays Capital Problems Solving Group, London (U. K.). We show that all such functions f have the form $f(x) = x + c$ or $f(x) = c - 2x$, for a real constant c. These are readily verified to be solutions, so we need only show that there are no others.

Fix a solution f. First note that f is injective. Indeed, if $f(x) = f(y)$, then $2x = 3f(x) - f(f(f(x))) = 3f(y) - f(f(f(y))) = 2y$, so $x = y$. It follows that f is strictly monotone, and hence $f(x)$ has a limit (finite or infinite) as $x \to \infty$. If f tends to a finite limit a as $x \to \infty$, then $2x = 3f(x) - f(f(f(x))) \to 3a - f(f(a))$, a contradiction. Similarly, we see that $f(x)$ tends to an infinite limit as $x \to -\infty$. Thus f is surjective.

Now write f^i for the ith iterate of f, which makes sense for all integers (positive, negative, and zero). For a given real number x, we have $f^{i+3}(x) - 3f^{i+1}(x) + 2f^i(x) = 0$ for all i. Solving this linear homogeneous recurrence yields real numbers A_x, B_x, C_x such that $f^i(x) = A_x + B_x i + C_x(-2)^i$ for all i.

Now we consider two cases: f increasing and f decreasing. First, suppose that f is increasing. For a given x, examine $f^i(x) = A_x + B_x i + C_x(-2)^i$. If $C_x > 0$, then $f^i(x)$ is large and positive when i is large and even, and large and negative when i is large and odd.

Thus there are arbitrarily large positive X such that $f(X)$ is large and negative, which contradicts $f(x) \to \infty$ as $x \to \infty$. A similar contradiction arises if $C_x < 0$. Thus $C_x = 0$, so $f^i(x) = A_x + B_x i$ for all i. Substituting $i = 0$, we have $f^i(x) = x + B_x i$ for all i.

If $f(x) = x$ for all x, then we are done. Suppose there is some x with $f(x) \neq x$. Now $B_x \neq 0$. Since f is increasing, for any integer n and real number y we have $x + B_x n \le y$ if and only if $x + B_x(n+1) \le f(y)$. Iterating yields $x + B_x n \le y$ if and only if $x + B_x(n+k) \le f^k(y) = y + B_y k$. Varying k thus does not affect whether $x + B_x(n+k) \le y + B_y k$. Sending k to ∞ and to $-\infty$, we conclude that $B_x = B_y$. Thus $f(x) = x + c$ for some c.

Second, suppose that f is decreasing. For a given x, examine $f^i(x) = A_x + B_x i + C_x(-2)^i$. If $B_x > 0$, then for i large and negative, $f^i(x)$ and $f^{i+1}(x)$ are both large and negative. Thus there are arbitrarily large negative X such that $f(X)$ is large and negative, which contradicts $f(x) \to \infty$ as $x \to -\infty$. Again, a similar contradiction arises if $B_x < 0$. Thus $B_x = 0$, and hence $f^i(x) = A_x + C_x(-2)^i$ for all i. Note that $f^i(x) \to A_x$ as $i \to -\infty$, so $f(A_x) = A_x$. A decreasing function can have at most one fixed point, so $A_x = A_y$ for all y.

Since $f(x) = A_x - 2C_x = 3A_x - 2(A_x + C_x) = 3A_x - 2x$, it follows that $f(x) = c - 2x$ for some c.

Also solved by M. Bataille (France), C. Burnette, R. Chapman (U. K.), P. P. Dályay (Hungary), C. Delorme (France), N. Grivaux (France), E. A. Herman, M. Huibregtse, O. P. Lossers (Netherlands), J. Simons (U. K.), R. Stong, T. Trif (Romania), E. I. Verriest, GCHQ Problem Solving Group (U. K.), and the proposer.

Angles at an Inside Point of a Triangle

11550 [2011, 85]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Let G be a point inside triangle ABC. Let α, β, γ be the radian measures of angles BGC, CGA, AGB, respectively. Let O, R, S be the triangle's circumcenter, circumradius, and area, respectively. Let $|XY|$ be the distance from X to Y. Prove that

$$|GA| \cdot |GB| \cdot |GC|(|GA| \sin\alpha + |GB| \sin\beta + |GC| \sin\gamma) = 2S(R^2 - |GO|^2).$$

Solution by Michael Vowe, Therwil, Switzerland. Writing $[ABC]$ for the area of triangle ABC, we have $2[BGC] = |GB| \cdot |GC| \sin\alpha$, $2[CGA] = |GC| \cdot |GA| \sin\beta$, and $2[AGB] = |GA| \cdot |GB| \sin\gamma$. Let (g_1, g_2, g_3) be the normalized barycentric coordinates of G, and for points P and Q, let $\overrightarrow{PQ}$ denote the vector from P to Q. Then

$$g_1 = \frac{[BGC]}{[ABC]}, \qquad g_2 = \frac{[CGA]}{[ABC]}, \qquad g_3 = \frac{[AGB]}{[ABC]}, \qquad g_1 + g_2 + g_3 = 1,$$

and

$$g_1\overrightarrow{GA} + g_2\overrightarrow{GB} + g_3\overrightarrow{GC} = 0.$$

The desired equality is equivalent to

$$R^2 = |GO|^2 + g_1|GA|^2 + g_2|GB|^2 + g_3|GC|^2.$$

Finally, we obtain

$$
\begin{aligned}
R^2 &= g_1 R^2 + g_2 R^2 + g_3 R^2 = g_1|OA|^2 + g_2|OB|^2 + g_3|OC|^2 \\
&= g_1(\overrightarrow{OG} + \overrightarrow{GA}) \cdot (\overrightarrow{OG} + \overrightarrow{GA}) + g_2(\overrightarrow{OG} + \overrightarrow{GB}) \cdot (\overrightarrow{OG} + \overrightarrow{GB}) \\
&\quad + g_3(\overrightarrow{OG} + \overrightarrow{GC}) \cdot (\overrightarrow{OG} + \overrightarrow{GC}) \\
&= |OG|^2 + g_1|GA|^2 + g_2|GB|^2 + g_3|GC|^2 + 2\overrightarrow{OG} \cdot (g_1\overrightarrow{GA} + g_2\overrightarrow{GB} + g_3\overrightarrow{GC}) \\
&= |OG|^2 + g_1|GA|^2 + g_2|GB|^2 + g_3|GC|^2.
\end{aligned}
$$

Also solved by M. Alexander & T. Smotzer, M. Bataille (France), R. Chapman (U. K.), P. P. Dályay (Hungary), P. De (India), A. Ercan (Turkey), O. Geupel (Germany), M. Goldenberg & M. Kaplan, S. Hitotumatu (Japan), O. Kouba (Syria), J. H. Lindsey II, C. R. Pranesachar (India), J. Schlosberg, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), J. B. Zacharias & K. T. Greeson, Barclays Capital Problems Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

Points in Figures

11551 [2011, 178]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Given a finite set S of closed bounded convex sets in $\mathbb{R}^n$ having positive volume, prove that there exists a finite set X of points in $\mathbb{R}^n$ such that each $A \in S$ contains at least one element of X and any $A, B \in S$ with the same volume contain the same number of elements of X.

Solution by Jim Simons, Cheltenham, U. K. Let the sets of S be $\{A_i\}_{i=1}^s$, and consider the $2^s - 1$ sets created by taking the intersection, as i ranges from 1 to s, of either A_i or its complement A_i^c, but excluding $A_1^c \cap A_2^c \cap \cdots \cap A_s^c$. These sets are not necessarily convex or even connected (indeed, they may have infinitely many components of positive volume), but they are measurable. Discard any that are empty or have zero measure, leaving $\{B_j\}_{j=1}^t$. Each A_i is a finite disjoint union of some of the B_j, together with a set of measure zero.

Consider giving real "weights" w_j to the B_j and adding these weights to define weights v_i for the A_i, so that $v_i = \sum w_j$, where the sum is over all j with $B_j \subseteq A_i$. Let V_i be the volume (measure) of A_i. The requirement that $V_i = V_{i'}$ imply $v_i = v_{i'}$ takes the form of a set L of linear equations in the w_j with rational coefficients (indeed, with coefficients $-1, 0, 1$ only).

Now (since the rank may be computed by evaluating certain determinants) a matrix with rational entries has the same rank over the reals as it has over the rationals. So the rational solution space for L has the same dimension as the real solution space, and therefore is dense in it. There is a positive real solution (namely, each w_j is the measure of B_j). So there is a nearby rational solution (we only need it near enough that all w_j are positive). Then we can multiply by a common denominator to get a solution in positive integers. For the set X, choose w_j points in each set B_j.

Note that this solution provides more information than requested. We can insure that $V_i < V_{i'}$ implies $v_i < v_{i'}$, $V_i = V_{i'}$ implies $v_i = v_{i'}$, and $V_i > V_{i'}$ implies $v_i > v_{i'}$.

Also solved by O. P. Lossers (Netherlands), R. Stong, and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before February 28, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11663. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* The unit interval is broken at two randomly chosen points along its length. Show that the probability that the lengths of the resulting three intervals are the heights of a triangle is equal to

$$\frac{12\sqrt{5}\log((3+\sqrt{5})/2)}{25} - \frac{4}{5}.$$

11664. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ, and Darij Grinberg, Massachusetts Institute of Technology, Cambridge, MA.* Let a, b, and c be the side lengths of a triangle. Let s denote the semiperimeter, r the inradius, and R the circumradius of that triangle. Let $a' = s - a$, $b' = s - b$, and $c' = s - c$.
(**a**) Prove that $\frac{ar}{R} \le \sqrt{b'c'}$.
(**b**) Prove that

$$\frac{r(a+b+c)}{R}\left(1 + \frac{R-2r}{4R+r}\right) \le 2\left(\frac{b'c'}{a} + \frac{c'a'}{b} + \frac{a'b'}{c}\right).$$

11665. *Proposed by Raitis Ozols, student, University of Latvia, Riga, Latvia.* Let $a = (a_1, \dots, a_n)$, where $n \ge 2$ and each a_j is a positive real number. Let $S(a) = a_1^{a_2} + \cdots + a_{n-1}^{a_n} + a_n^{a_1}$.
(**a**) Prove that $S(a) > 1$.
(**b**) Prove that for all $\epsilon > 0$ and $n \ge 2$ there exists a of length n with $S(a) < 1 + \epsilon$.

11666. *Proposed by Dmitry G. Fon-Der-Flaass (1962–2010), Institute of Mathematics, Novosibirsk, Russia, and Max. A. Alekseyev, University of South Carolina, Columbia,*

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SC. Let m be a positive integer, and let A and B be nonempty subsets of $\{0, 1\}^m$. Let n be the greatest integer such that $|A| + |B| > 2^n$. Prove that $|A + B| \geq 2^n$. (Here, $|X|$ denotes the number of elements in X, and $A + B$ denotes $\{a + b\colon\ a \in A, b \in B\}$, where addition of vectors is componentwise modulo 2.)

11667. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Softwin Co., Bucharest, Romania.* Let f, g, and h be elements of an inner product space over $\mathbb{R}$, with $\langle f, g\rangle = 0$.
(**a**) Show that

$$\langle f, f\rangle\langle g, g\rangle\langle h, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

(**b**) Show that

$$(\langle f, f\rangle\langle h, h\rangle)\langle h, f\rangle^2 + (\langle g, g\rangle\langle h, h\rangle)\langle g, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

11668. *Proposed by Dimitris Stathopoulos, Marousi, Greece.* For positive integer n and $i \in \{0, 1\}$, let $D_i(n)$ be the number of derangements on n elements whose number of cycles has the same parity as i. Prove that $D_1(n) - D_0(n) = n - 1$.

11669. *Proposed by Herman Roelants, Catholic University of Leuven, Louvain, Belgium.* Prove that for all $n \geq 4$ there exist integers $x_1, \dots, x_n$ such that

$$\frac{x_{n-1}^2 + 1}{x_n^2}\prod_{k=1}^{n-2}\frac{x_k^2 + 1}{x_k^2} = 1$$

satisfying the following conditions: $x_1 = 1$, $x_{k-1} < x_k < 3x_{k-1}$ for $2 \leq k \leq n - 2$, $x_{n-2} < x_{n-1} < 2x_{n-2}$, and $x_{n-1} < x_n < 2x_{n-1}$.

SOLUTIONS

An Equation Satisfied only by the Identity Matrix

11510 [2010, 558]. *Proposed by Vlad Matei (student), University of Bucharest, Bucharest, Romania.* Prove that if I is the n-by-n identity matrix, A is an n-by-n matrix with rational entries, $A \neq I$, p is prime with $p \equiv 3 \pmod 4$, and $p > n + 1$, then $A^p + A \neq 2I$.

Solution by C. T. Stretch, University of Ulster at Coleraine, Coleraine, Londonderry, Northern Ireland. The primality and congruence conditions on p are not needed; we require only $p > n + 1$. We prove more generally that $A^p + (q - 1)A = qI$ cannot hold for any prime q.

If $A^p + (q - 1) - qI = 0$, then the minimal polynomial $m(x)$ of A divides $\phi(x)$, where $\phi(x) = x^p + (q - 1)x - q$. Note that $\phi(x) = (x - 1)\psi(x)$, where

$$\psi(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + q.$$

Since $A \neq I$, we have $m(x) \neq x - 1$. Thus $m(x)$, which has degree at most n, is a factor of $\psi(x)$, which has degree greater than n. We obtain a contradiction and complete the proof by showing that $\psi(x)$ is irreducible over the rationals.

Since $\psi(1) \neq 0$ and $\psi(0) = q$, it suffices to show that $\psi(x)$ is irreducible over the integers. If α is a (complex root) of $\phi(x)$, then $\alpha^p = q - (q - 1)\alpha$. If $|\alpha| \leq 1$, then

$|q-(q-1)\alpha| = |\alpha|^p \le 1$. Since $|q-(q-1)\alpha| \ge q-(q-1)|\alpha| \ge 1$, we obtain $|q-(q-1)\alpha| = 1$, which occurs if and only if $\alpha = 1$. Thus every root α of $\psi(x)$ satisfies $|\alpha| > 1$.

Suppose that $\psi(x) = f(x)g(x)$, where both $f(x)$ and $g(x)$ have positive degree and integer coefficients. Consider the factorization $f(x) = \prod_{i=1}^{k}(x-\alpha_i)$ over the complex numbers. As shown above, $|\alpha_i| > 1$ for all i. Since $f(0) = (-1)^k \prod_{i=1}^{k} \alpha_i$, also $|f(0)| > 1$; similarly, $|g(0)| > 1$. Since $f(0)$ divides $\psi(0)$ and q is prime, we have $|f(0)| = q$; similarly, $|g(0)| = q$. But then $q = \psi(0) = f(0)g(0) = \pm q^2$, a contradiction. We conclude that $\psi(x)$ is irreducible.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), J. Simons (U. K.), N. C. Singer, R. Stong, M. Tetiva (Romania), Ellington Management Problem Solving Group, GCHQ Problem Solving Group, and the proposer.

An Expression for the kth Smallest Element of a Set

11520 [2010, 649]. *Proposed by Peter Ash, Cambridge Math Learning, Bedford, MA.* Let n and k be integers with $1 \le k \le n$, and let A be a set of n real numbers. For i with $1 \le i \le n$, let S_i be the set of all subsets of A with i elements, and let $\sigma_i = \sum_{s\in S_i} \max(s)$. Express the kth smallest element of A as a linear combination of $\sigma_0, \dots, \sigma_n$.

Solution by Mark Wildon, Royal Holloway, University of London, Egham, United Kingdom. Let $A = \{a_1, \dots, a_n\}$ with $a_1 < \cdots < a_n$. There are exactly $\binom{m-1}{k-1}$ k-subsets of A in which a_m is largest, so $\sigma_k = \sum_{m=1}^{n} \binom{m-1}{k-1} a_m$ for $1 \le k \le n$. The following computation expresses a_k as a linear combination of $\sigma_k, \dots, \sigma_n$, where the final step uses that $\sum_{r=1}^{n}(-1)^r\binom{m-k}{r-k} = 0$ except when $m = k$:

$$\begin{aligned}
\sum_{r=1}^{n}(-1)^{k+r}\binom{r-1}{k-1}\sigma_r &= (-1)^k \sum_{m=1}^{n} a_m \sum_{r=1}^{n}(-1)^r \binom{m-1}{r-1}\binom{r-1}{k-1} \\
&= (-1)^k \sum_{m=1}^{n} a_m \sum_{r=1}^{n}(-1)^r \binom{m-1}{k-1}\binom{m-k}{r-k} \\
&= (-1)^k \sum_{m=1}^{n} a_m \binom{m-1}{k-1} \sum_{r=1}^{n}(-1)^r \binom{m-k}{r-k} \\
&= a_k.
\end{aligned}$$

Editorial comment. The main step here can be viewed as inverting a matrix of binomial coefficients. Jayantha Senadheera cited Cramer's Rule. Li Zhou cited the book of Polya and Szegö, where the verification is a solved problem. Oliver Geupel cited a general binomial coefficient identity of which the verification is a special case (see R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics: a Foundation for Computer Science*, (Addison–Wesley, 1989), page 70).

Also solved by T. Amdeberhan & V. De Angelis, R. Bagby, D. Beckwith, M. Benedicty, N. Caro (Brazil), R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, S. J. Herschkorn, Y. J. Ionin, J.-W. Kang (Korea), O. Kouba (Syria), J. H. Lindsey II, J. H. Nieto (Venezuela), W. Nuij (Netherlands), É. Pité (France), Á. Plaza & S. Falcón (Spain), R. Pratt, J. Schlosberg, J. Senadheera, J. Simons (U. K.), J. H. Smith, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), S. Xiao (Canada), L. Zhou, Barclays Capital Quantitative Analytics Group, GCHQ

Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, Texas State University Problem Solving Group, and the proposer.

A Geometric Inequality

11552 [2011, 178]. *Proposed by Weidong Jiang, Weihai Vocational College, Weihai, China.* In triangle ABC, let A_1, B_1, C_1 be the points opposite A, B, C at which the angle bisectors of the triangle meet the opposite sides. Let R and r be the circumradius and inradius of ABC. Let a, b, c be the lengths of the sides opposite A, B, C, and let a_1, b_1, c_1 be the lengths of the line segments B_1C_1, C_1A_1, A_1B_1. Prove that

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq 1 + \frac{r}{R}.$$

Solution by Prithwijit De, HBCSE, Mumbai, India. If X and Y are the feet of the perpendiculars on BC from C_1 and B_1, respectively, then $a_1 = |B_1C_1| \geq |XY|$. However,

$$|XY| = a - (|BC_1| \cos B + |B_1C| \cos C) = 1 - \left(\frac{ac \cos B}{a+b} + \frac{ab \cos C}{a+c} \right).$$

Therefore,

$$\frac{a_1}{a} \geq 1 - \left(\frac{c}{a+b} \cos C + \frac{b}{a+c} \cos C \right). \tag{1}$$

Similarly,

$$\frac{b_1}{b} \geq 1 - \left(\frac{a}{b+c} \cos C + \frac{c}{b+a} \cos A \right), \tag{2}$$

$$\frac{c_1}{c} \geq 1 - \left(\frac{b}{c+a} \cos A + \frac{a}{b+c} \cos B \right). \tag{3}$$

Adding, we get

$$\begin{aligned} &\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq \qquad\qquad (4) \\ &3 - \left(\frac{a}{b+c}(\cos B + \cos C) + \frac{b}{c+a}(\cos C + \cos A) + \frac{c}{a+b}(\cos A + \cos B) \right). \end{aligned}$$

Now,

$$\begin{aligned} \frac{a(\cos B + \cos C)}{b+c} &= \frac{\sin A(\cos B + \cos C)}{\sin B + \sin C} \\ &= \frac{2\sin(A/2)\cos(A/2) \cdot 2\cos((B+C)/2)\cos((B-C)/2)}{2\sin((B+C)/2)\cos((B-C)/2)} \\ &= \frac{4\sin(A/2)\cos(A/2)\sin(A/2)\cos((B-C)/2)}{2\cos(A/2)\cos((B-C)/2)} \\ &= 2\sin^2\frac{A}{2} = 1 - \cos A. \end{aligned}$$

Similarly,

$$\frac{b}{c+a}(\cos C + \cos A) = 1 - \cos B,$$
$$\frac{c}{a+b}(\cos A + \cos B) = 1 - \cos C.$$

Putting these into (4), we have

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq \cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Editorial comment. Peter Nüesch (Switzerland) notes that this problem may be viewed as a special case of Problem 1320 in *Mathematics Magazine*, proposedby V. Kovner in vol. 62 (1989), p. 137, solved by J. Heuver and Richard E. Pfiefer in vol. 63(1990) pp. 130–131.

Also solved by P. P. Dályay (Hungary), P. Nüesch (Switzerland), J. Posch, R. Stong, and the proposer.

Triangle Center $X(79)$

11554 [2011, 178]. *Proposed by Zhang Yun, Xi'an Jiao Tong University Sunshine High School, Xi'an, China.* In triangle ABC, let I be the incenter, and let A', B', C' be the reflections of I through sides BC, CA, AB, respectively. Prove that the lines AA', BB', and CC' are concurrent.

Solution by Alin Bostan, INRIA, Rocquencourt, France. First we identify this problem as a particular case of two different classical theorems in Euclidean geometry: Jacobi's Theorem and Kariya's Theorem (which is itself a particular case of an older theorem of Lemoine's, see below). We then give two proofs of Problem 11554.

Jacobi's Theorem (sometimes called "the Isogonal Theorem"): *If ABC is a triangle, and A', B', and C' are points in its plane such that $\angle B'AC = \angle BAC'$, $\angle C'BA = \angle CBA'$, and $\angle A'CB = \angle ACB'$, then the lines AA', BB', and CC' are concurrent.* This is a generalization of the famous "Napoleon's Theorem", available at http://en.wikipedia.org/wiki/Napoleon's_theorem. It was seemingly discovered by Carl Friedrich Andreas Jacobi [not to be confused with Carl Gustav Jacob Jacobi], and published in 1825 in Latin: C. F. A. Jacobi, *De triangulorum rectilineorum proprietatibus quibusdam nondum satis cognitis,* Naumburg (1825).

Kariya's Theorem: *Let I be the incenter of a triangle ABC, and let X, Y, Z be the points where the incircle of $\triangle ABC$ touches the sides BC, CA, AB, respectively. If A', B', C' are three points on the half-lines IX, IY, IZ, respectively, such that $IA' = IB' = IC'$, then the lines AA', BB', and CC' are concurrent.* This theorem has a long history. It was discovered independently by Auguste Boutin and by V. Retali: A. Boutain, "Sur un groupe de quatre coniques remarquables,"*Journal de mathématiques spéciales* ser. 3, **4** (1890) 104–107, 124–127; A. Boutin, "Problèmes sur le triangle,"*Journal de mathématiques spéciales* ser. 3, **4** (1890) 265–269; V. Retali, *Periodico di Matematica* (Rome) **11** (1896) 71.

The result only became well known with Kariya's paper (which inspired many results appearing in *l'Enseignement* over the following years): J. Kariya, "Un probléme sur le triangle,"*L'Enseignement mathématique* **6** (1904) 130–132, 236, 406. Actually, a generalization of this result was obtained before Kariya by Emile Lemoine in Section 4 of: E. Lemoine, "Contributions à la géométrie du triangle,"*Congrès de l'AFAS,* Paris, 1889, p. 197–222.

Lemoine explicitly states and proves on page 202 the following: *Let ABC be a triangle, M a point in its plane, and X, Y, Z the projections of M on BC, CA, AB, respectively. If A', B', C' are points on the half-lines MX, MY, and MZ, respectively, such that $MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'$, then AA', BB', CC' are concurrent.* Auric gave in 1915 another generalization of Kariya's Theorem: A. Auric, "Généralisation du théorème de Kariya,"*Nouvelles annales de mathématiques* 4e série**15** (1915) 222–225. The statement is the same as Lemoine's Theorem except that the assumption $MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'$ is replaced by $MX/MA' = MY/MB' = MZ/MC'$.

Now we give the two solutions to Problem 11554, both based on Ceva's Theorem.

(1) This solution is possibly new (less elegant than the second one, but a bit shorter). Let P be the intersection of AA' and BC, and let Q be the intersection of AI and BC. Applying Menelaus' Theorem twice (once for $\triangle APQ$ and transversal IA', once for $\triangle AIA'$ and transversal BC), we find that $BP/PC = (a^2 + c^2 - b^2 + ca)/(b^2 + a^2 - c^2 + ab)$. Since the numerator is obtained from the denominator by the cyclic permutation $a \to b \to c \to a$, the conclusion follows from Ceva's Theorem.

(2) The second solution is much more elegant, and is possibly due to the Romanian geometer Gheorghe Titeica (it appears as Problem 1138 in his book *Problems of Geometry* (in Romanian)). Let the parallel to BC passing through A' intersect AB and AC in A_1 and A_2, respectively. Construct similarly the points B_1, B_2, C_1, and C_2. By symmetry, $A'A_1 = C'C_2$, $A'A_2 = B'B_1$, and $B'B_2 = C'C_1$. Let P be the intersection of AA' and BC, let Q be the intersection of BB' and AC, and let R be the intersection of CC' and AB. Thales' Theorem implies $BP/PC = A_1A'/A'A_2$, $CQ/QA = B_1B'/B'B_2$, and $AR/RB = C_1C'/C'C_2$. It follows that

$$\frac{BP}{PC}\,\frac{CQ}{QA}\,\frac{AR}{RB} = \frac{A_1A'}{A'A_2}\,\frac{B_1B'}{B'B_2}\,\frac{C_1C'}{C'C_2} = 1,$$

and the conclusion follows from Ceva's Theorem.

Final notes: (i) Nowadays the point J of concurrence in Problem 11554 is sometimes called "Gray's point" after Steve Gray who noted a seemingly new property, namely that the line IJ is parallel to the Euler line OH of $\triangle ABC$.

(ii) The point J is called $X(79)$ in Kimberling's *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Also solved by Y. An (China), G. Apostolopoulos (Greece), M. Bataille (France), R. B. Campos (Spain), C. Curtis, P. P. Dályay (Hungary), P. De (India), C. Delorme (France), A. Ercan (Turkey), O. Faynshteyn (Germany), R. Frank & H. Riede (Germany), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, S. Hitotumatu (Japan), Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), R. Mabry, R. Murgatroyd, C. R. Pranesachar (India), J. Schlosberg, T. Smith, R. Stong, M. Tetiva (Romania), R. S. Tiberio, Z. Vörös (Hungary), Z. Xintao (China), P. Yff, J. B. Zacharias, D. Zeilberger, GCHQ Problem Solving Group (U. K.), and the proposer.

Value Defined by an Integral

11555 [2011, 178]. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous real-valued function on $[0, 1]$ such that $\int_0^1 f(x)\,dx = 0$. Prove that there exists c in the interval $(0, 1)$ such that $c^2 f(c) = \int_0^c (x + x^2) f(x)\,dx$.

Solution I by Michael W. Botsko, Saint Vincent College, PA. First, let $F(x) = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$ on $[0, 1]$. By its construction, $F'(x) = \int_0^x f(t)\,dt$ and

$F'(0) = 0$. Since $\int_0^1 f(x)\,dx = 0$, also $F'(1) = 0$. Now by Flett's Mean Value Theorem (T. M. Flett, "A mean value theorem", *Math. Gazette*, **42**(1958), 38-39), there exists $a \in (0, 1)$ such that

$$\frac{F(a) - F(0)}{a - 0} = F'(a).$$

Therefore,

$$\int_0^a xf(x)\,dx = 0, \quad a \in (0, 1). \tag{1}$$

Next, let $G(x) = e^{-x} \int_0^x tf(t)\,dt$. From (1), $G(0) = G(a) = 0$. By Rolle's Theorem, there exists $b \in (0, a)$ such that

$$0 = G'(b) = -e^{-b} \int_0^b xf(x)\,dx + e^{-b}bf(b).$$

Therefore,

$$\int_0^b xf(x)\,dx = bf(b), \quad b \in (0, a). \tag{2}$$

Finally, let $H(x) = x\int_0^x tf(t)\,dt - \int_0^x (t + t^2)f(t)\,dt$. Then

$$H'(x) = \int_0^x tf(t)\,dt - xf(x).$$

Using (2), we have that $H'(0) = H'(b) = 0$. Once again using Flett's Mean Value Theorem, there exists $c \in (0, b)$ such that

$$\frac{H(c) - H(0)}{c - 0} = H'(c).$$

Therefore,

$$c\int_0^c xf(x)\,dx - \int_0^c (x + x^2)f(x)\,dx = c\int_0^c xf(x)\,dx - c^2 f(c).$$

This implies $\int_0^c (x + x^2)f(x)\,dx = c^2 f(c)$.

Solution II by Hongwei Chen, Christopher Newport University, Newport News, VA. If f is identically zero, there is nothing to prove, so assume that $f(x)$ is not identically zero. Since $\int_0^1 f(x)\,dx = 0$, there exist a and $b \in [0, 1]$ such that $a \neq b$ and

$$f(a) = \max_{x\in[0,1]} f(x) > 0, \qquad f(b) = \min_{x\in[0,1]} f(x) < 0.$$

Define $F(x) = x^2 f(x) - \int_0^x (t + t^2)f(t)\,dt$. Note that $F(x)$ is continuous on $[0, 1]$. Since $(t + t^2)f(a) \geq (t + t^2)f(t)$ for all $t \in [0, 1]$,

$$F(a) \geq a^2 f(a) - \int_0^a (t + t^2)f(a)\,dt = a^2\left(\frac{1}{2} - \frac{a}{3}\right)f(a) > 0.$$

Similarly, $(t + t^2)f(b) \leq (t + t^2)f(t)$ for all $t \in [0, 1]$, so

$$F(b) \leq b^2 f(b) - \int_0^b (t + t^2)f(b)\,dt = b^2\left(\frac{1}{2} - \frac{b}{3}\right)f(b) < 0.$$

The Intermediate Value Theorem implies that there exists a number $c \in (a, b) \subset (0, 1)$ such that $F(c) = 0$, so that $c^2 f(c) = \int_0^c (x + x^2) f(x)\,dx$.

Also solved by K. F. Andersen (Canada), M. W. Botsko, P. Bracken, H. Chen, D. Constales (Belgium), P. P. Dályay (Hungary), N. Grivaux (France), L. Han, E. Ionascu, K.-W. Lau (China), J. H. Lindsey II, M. Omarjee (France), S. Pauley, N. Weir & A. Welter, P. Perfetti (Italy), Á. Plaza (Spain), A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), H. Wang & Y. Xia, GCHQ Problem Solving Group (U. K.), and the proposer.

A Four-Number Summetric Inequality

11556 [2011, 179]. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* For positive real numbers a, b, c, d, show that

$$\frac{9}{a(b+c+d)} + \frac{9}{b(c+d+a)} + \frac{9}{c(d+a+b)} + \frac{9}{d(a+b+c)}$$
$$\geq \frac{16}{(a+b)(c+d)} + \frac{16}{(a+c)(b+d)} + \frac{16}{(a+d)(b+c)}.$$

Solution by Marian Dincă, Bucharest, Romania. Suppose f is a convex function on the interval $I \subset \mathbb{R}$. Given numbers x, y, z, let $x' = (y+z)/2$, $y' = (z+x)/2$, and $z' = (x+y)/2$. Combining Popoviciu's Inequality

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2\left[f(z') + f(y') + f(x')\right]$$

and Jensen's Inequality in the form

$$f(x) + f(y) + f(z) = \frac{f(x)+f(y)}{2} + \frac{f(x)+f(z)}{2} + \frac{f(y)+f(z)}{2}$$
$$\geq f(z') + f(y') + f(x')$$

(specifically adding the first to twice the second) gives

$$3f(x) + 3f(y) + 3f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 4\left[f(z') + f(y') + f(x')\right].$$

Applying this to the convex function $f(t) = \frac{1}{t}$, $t > 0$, for $x, y, z > 0$ we have

$$\frac{3}{x} + \frac{3}{y} + \frac{3}{z} + \frac{9}{x+y+z} \geq \frac{8}{x+y} + \frac{8}{x+z} + \frac{8}{y+z}.$$

Summing the four inequalities we get by taking x, y, z to be any three of a, b, c, d we obtain

$$\frac{9}{a} + \frac{9}{b} + \frac{9}{c} + \frac{9}{d} + \frac{9}{a+b+c} + \frac{9}{a+b+d} + \frac{9}{a+c+d} + \frac{9}{b+c+d}$$
$$\geq \frac{16}{a+b} + \frac{16}{a+c} + \frac{16}{a+d} + \frac{16}{b+c} + \frac{16}{b+d} + \frac{16}{c+d}.$$

Noting that

$$\frac{1}{a} + \frac{1}{b+c+d} = \frac{a+b+c+d}{a(b+c+d)} \quad \text{and} \quad \frac{1}{a+b} + \frac{1}{c+d} = \frac{a+b+c+d}{(a+b)(c+d)},$$

and symmetrically, we see that this is exactly the desired inequality multiplied by $a+b+c+d$.

Also solved by S. Hitotumatu (Japan), E. Hysnelaj & E. Bojaxhiu (Australia & Germany), O. Kouba (Syria), J. H. Lindsey II, P. H. O. Pantoja (Brazil), P. Perfetti (Italy), C. R. Pranesacher (India), A. Stenger, R. Stong, M. Tetiva (Romania), L. Zhou, Zhou X. (China), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before March 31, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11670. *Proposed by Miranda Bakke, Benson Wu, and Bogdan Suceavă, California State University, Fullerton, CA.* Prove that if $n \geq 3$ and $a_1, \ldots, a_n > 0$, then

$$\frac{(n-1)}{4}\sum_{k=1}^{n} a_k \geq \sum_{1 \leq j < k \leq n} \frac{a_j a_k}{a_j + a_k},$$

with equality if and only if all a_j are equal.

11671. *Proposed by Sam Northshield, SUNY-Plattsburgh, Plattsburgh, NY.* Show that if relatively prime integers a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2,$$

then $|a + b + c|$ can be written as $m^2 - mn + n^2$ for some integers m and n.

11672. *Proposed by José Luis Palacios, Universidad Simón Bolívar, Caracas, Venezuela.* A random walk starts at the origin and moves up-right or down-right with equal probability. What is the expected value of the first time that the walk is k steps below its then-current all time high? (Thus, for instance, with the walk UDDUUUUDDUDD$\cdots$, the walk is three steps below its maximum-so-far on step 12.)

11673. *Proposed by Kent Holing, Statoil, Trondheim, Norway.* Let Q and g be monic polynomials in $\mathbb{Z}[x]$, with Q an irreducible quartic. Let $f = Q \circ g$. Suppose that f is irreducible over Q and that the order of the Galois group of f is a power of 2. Which groups are possible as the Galois group of Q? If, moreover, Q has negative discriminant, determine the Galois group of Q.

11674. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let a and b be real numbers with $a < 0 < b$. Let S be the set of continuous functions f from $[0, 1]$ to $[a, b]$ with

http://dx.doi.org/10.4169/amer.math.monthly.119.09.800

$\int_0^1 f(x)\,dx = 0$. Let g be a strictly increasing function from $[0, 1]$ to $\mathbb{R}$. Define ϕ from S to $\mathbb{R}$ by $\phi(f) = \int_0^1 f(x)g(x)\,dx$.
(a) Find $\sup_{f\in S}\phi(f)$ in terms of a, b, and g.
(b) Prove that this supremum is not attained.

11675. *Proposed by Mircea Merca, Constantin Istrati Technical College, Campina, Romania.* Let p be the Euler partition function, i.e., $p(n)$ is the number of nondecreasing lists of positive integers that sum to n. Let $p(0) = 1$, and let $p(n) = 0$ for $n < 0$. Prove that for $n \geq 0$ with $n \neq 3$,

$$p(n) - 4p(n-3) + 4p(n-5) - p(n-8) > 0.$$

11676. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania.* For real t, find

$$\lim_{x\to\infty} x^{\sin^2 t}\left(\Gamma(x+2)^{(\cos^2 t)/(x+1)} - \Gamma(x+1)^{\cos^2 t/x}\right).$$

Here, Γ is the Euler gamma function.

SOLUTIONS

Fair Permutations and Random k-sets

11523 [2010, 741]. *Proposed by Timothy Chow, Princeton, NJ.* Given boxes 1 through n, put balls in k randomly chosen boxes. The *score* of a permutation π of $\{1, \ldots, n\}$ is the least i such that box $\pi(i)$ has a ball. Thus, if $\pi = (3, 4, 1, 5, 2)$ with $(n, k) = (5, 2)$, and boxes 1 and 4 have balls, then π has score 2.
(a) A permutation π is *fair* if, regardless of the value of k, the probability that π scores lower than the identity permutation equals the probability that it scores higher. Show that π is fair if and only if for each i in $[1, n]$, either $\pi(i) > i$ and $\pi^{-1}(i) > i$, or $\pi(i) \leq i$ and $\pi^{-1}(i) \leq i$.
(b) Let $f(n)$ be the number of fair permutations of $\{1, \ldots, n\}$, with the convention that $f(0) = 1$. Show that $\sum_{n=0}^{\infty} f(n)x^n/n! = e^x \sec(x)$.
(c) Assume now that $n = m^3$ with $m \geq 2$, and the boxes are arranged in m rows of length m^2. Alice scans the top row left to right, then the row below it, and so on, until she finds a box with a ball in it. Bob scans the leftmost column top to bottom, then the next column, and so on. They start simultaneously and both check one box per second. For which k are Alice and Bob equally likely to be the first to discover a ball?

Solution by Jim Simons, Cheltenham, U. K. Let $[n] = \{1, \ldots, n\}$. Define a *back-and-forth* permutation of $[n]$ to be a permutation whose nontrivial cycles alternate moving up and down. Thus every cycle in such a permutation is a fixed point or has even length. Since $\pi(i) = i$ if and only if $\pi^{-1}(i) = i$, and a nontrivial cycle has $\pi^{-1}(i), i, \pi(i)$ in succession, a permutation satisfies the condition specified in **(a)** if and only if it is a back-and-forth permutation.

(a) Fix n and π. For $A \subseteq [n]$, let s_A and s'_A denote the score of the identity and π on A, respectively, when A is the set of boxes with balls. Let $S_j = \{A \subseteq [n] \colon\ \min\{s_A, s'_A\} = j\}$, and let S_j^k be the family of k-sets in S_j. Let $\pi([l])$ be the image under π of $[l]$. Let $R_j = [j-1] \cup \pi([j-1])$. Note in particular that S_j consists of the subsets of $[n]$ that contain j or $\pi(j)$ but are disjoint from R_j.

Let π be a back-and-forth permutation. To prove sufficiency of the condition, it suffices to show for all k and j that $l_j^k = g_j^k$, where $l_j^k = |\{A \in S_j^k\colon\ \pi(j) \in A\}|$ and $g_j^k = |\{A \in S_j^k\colon\ j \in A\}|$, since $s'_A \le s_A$ for sets counted by l_j^k and $s'_A \ge s_A$ for sets counted by g_j^k, and $s'_A = s_A$ precisely for the sets counted by both. Note that $l_1^k = g_1^k$ for all k; each equals $\binom{n-1}{k-1}$.

For $j \ge 2$, the back-and-forth condition yields $\pi(j) \in [j-1]$ if and only if $j \in \pi([j-1])$. Consider $A \in S_j^k$. Since $s_A \ge j$, $[j-1] \cap A$ is empty. Since $s'_A \ge j$, $\pi([j-1] \cap A$ is also empty. The back-and-forth condition implies that neither or both of $\{j, \pi(j)\}$ are excluded from A. Therefore, there are the same number of k-subsets of the unexcluded elements that contain j (and are counted by g_j^k) as contain $\pi(j)$ (and are counted by l_j^k).

If π is not a back-and-forth permutation, then let j be the least index such that exactly one of $\pi(j) \in [j-1]$ and $j \in \pi([j-1])$ holds. Thus exactly one of j and $\pi(j)$ lies in R_j. Let $k = n - |R_j|$. Now the only set in S_j^k is the complement of R_j, and it contains exactly one of j and $\pi(j)$, so $l_j^k \ne g_j^k$. On the other hand, $l_i^k = g_i^k$ for $i > j$, while the reasoning of the previous paragraph yields $l_i^k = g_i^k$ for $i < j$. Summing over i, we find that the probabilities of π scoring higher or lower than the identity are different.

(b) As noted at the beginning, the back-and-forth permutations consist of fixed points and up/down alternating even cycles. Say that a back-and-forth permutation is *strict* if it has no fixed points. Let $g(n)$ be the number of strict back-and-forth permutations of $[n]$. Partitioning back-and-forth permutations by the number of fixed points, we have $f(n) = \sum_{i=0}^{n} \binom{n}{i} g(n-i)$, so

$$\sum_{n=0}^{\infty} f(n)x^n/n! = e^x \sum_{n=0}^{\infty} g(n)x^n/n!.$$

A permutation π is *alternating* if $\pi(1) < \pi(2) > \pi(3) < \cdots$. It is well known that $\sec(x) = \sum_{n=0}^{\infty} h(n)x^n/n!$, where $h(n)$ is the number of alternating permutations when n is even and is 0 when n is odd. It therefore suffices to show that, for even n, the number of strict back-and-forth permutations equals the number of alternating permutations.

We use the well-known bijection that maps a permutation π in word form to the permutation whose canonical cycle representation is that same word. That is, left-to-right minima in π begin cycles in π'. Writing the cycles of π' with least element first, in decreasing order of least elements, and erasing the parentheses yields π. For example, [46281537] corresponds to the permutation with cycle representation (46)(28)(1537). In an alternating permutation, left-to-right minima occur at odd positions, so each cycle in the image has even length and alternates.

(c) We show that the values of k where the probability is equal are $k = 1$ and $k \ge m^3 - 2m + 2$.

Label the boxes $0, 1, \ldots, m^3 - 1$ in the order in which Alice scans them, and write these labels as 3-digit numbers in base m. Define a permutation π of these 3-digit numbers by $\pi(abc) = cab$. The ith box that Bob scans is the one labelled $\pi(i)$. The question becomes "For which k is it equally likely that π scores lower or higher than the identity? Since π has m fixed points and $(m^3 - m)/3$ cycles of length 3, by part **(a)** equality does not hold for all k. Call a value of k where the probability is equal *fair*.

We have seen that the condition $\pi(j) \ge j$ if and only if $\pi^{-1}(j) \ge j$ implies $l_j^k = g_j^k$. We consider the relationship between l_j^k and g_j^k for all j. If j is a fixed point of π, or if j is the smallest member or the largest member of its 3-cycle, then $\pi(j) \geq j$ if and only if $\pi^{-1}(j) \geq j$, so so $l_j^k = g_j^k$ for all k.

Hence it remains only to consider the case where j is the middle member of its 3-cycle, and precisely one of $\pi(j) \geq j$ and $\pi^{-1}(j) \geq j$ is true. To analyze this case, we consider sets of 3-digit numbers in base m defined by

$$[ab(c:d)] = \{xyz : x = a, y = b, c \leq z \leq d\},$$

and with a similar definition for $[a(b:c)d]$ or even $[a(b:c)(d:e)]$. For $j \in [0(1:m-1)0]$, we have $\pi(j) < j < \pi^{-1}(j)$, so with $R'_j = R_j \cup \{0, \pi(0)\}$,

$$g_j^k = \binom{n-1-|R'_j|}{k-1} \text{ and } l_j^k = 0.$$

Similarly, for $j \in [10(1:m-1)]$, we have $\pi(j) > j > \pi^{-1}(j)$, so

$$l_j^k = \binom{n-1-|R'_j|}{k-1} \text{ and } g_j^k = 0.$$

For a given k, define the *unfairness* of k to be $\sum_{j=0}^{m^3-1}\left(g_j^k - l_j^k\right)$. Since the two blocks of numbers specified above have the same size, and since all the binomial coefficients that appear in the first block are at least as large as any in the second block, the total contribution to the unfairness from these two blocks is nonnegative.

All the other cases divide similarly into pairs of blocks of the same size that between them make a nonnegative contribution to the unfairness. These pairs of blocks are indexed by the first digit d of j, with $1 \leq d \leq m-2$, as follows:

$$\begin{array}{lll} \text{if} & j \in [d(d+1:m-1)(0:d)], & \text{then } \pi(j) < j < \pi^{-1}(j); \\ \text{if} & j \in [(d+1)(0:d)(d:m-1)], & \text{then } \pi(j) > j > \pi^{-1}(j). \end{array}$$

If $k = 1$, then all these binomial coefficients equal 1, and the unfairness is 0. If $k \geq m^3 - 2m + 2$, then all the binomial coefficients are 0, and again these values of k are fair. For $1 < k < m^3 - 2m + 2$, the first coefficient is strictly positive and strictly greater than the second, and so these values of k are unfair.

Also solved by R. Hutchinson, J. H. Lindsey II, Barclay's Capital Problem Solving Group (U.K.), and the proposer.

Convergence of an Averaging Expression

11528 [2010, 742]. *Proposed by Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let p, a, and b be positive integers with $a < b$. Consider a sequence $\langle x_n \rangle$ defined by the recurrence $nx_{n+1} = (n + 1/p)x_n$ and an initial condition $x_1 \neq 0$. Evaluate

$$\lim_{n\to\infty} \frac{x_{an} + x_{an+1} + \cdots + x_{bn}}{nx_{an}}.$$

Solution by Omran Kouba, H.I.A.S.T., Damascus, Syria. The answer is $\frac{b^{1+c}-a^{1+c}}{(1+c)a^c}$, where $c = 1/p$. Indeed, the solution is valid for any nonnegative real number c in place of $1/p$, not just reciprocals of integers.

Let $y_n = \frac{n-1}{1+c}x_n$. Since

$$y_{n+1} - y_n = \frac{nx_{n+1} - (n-1)x_n}{1+c} = \frac{(n+c)x_n - (n-1)x_n}{1+c} = x_n,$$

we obtain

$$\sum_{k=an}^{bn} x_k = \sum_{k=an}^{bn}(y_{k+1} - y_k) = y_{bn+1} - y_{an} = \frac{bnx_{bn+1} - (an-1)x_{an}}{1+c}.$$

Consequently,

$$\frac{1}{nx_{an}}\sum_{k=an}^{bn} x_k = \frac{b}{1+c}\cdot\frac{x_{bn+1}}{x_{an}} - \frac{an-1}{n(1+c)}. \tag{1}$$

Hence the problem reduces to finding $\lim z_n$, where $z_n = x_{bn+1}/x_{an}$. Writing

$$z_n = \prod_{k=an}^{bn}\frac{x_{k+1}}{x_k} = \prod_{k=an}^{bn}\left(1+\frac{c}{k}\right),$$

we have

$$z_n\left(\frac{an-1}{bn}\right)^c = z_n\prod_{k=an}^{bn}\left(\frac{k-1}{k}\right)^c = \prod_{k=an}^{bn}\left(1+\frac{c}{k}\right)\left(1-\frac{1}{k}\right)^c.$$

Since $(1+\frac{c}{k})(1-\frac{1}{k})^c = 1 + O(k^{-2})$, the infinite product $\prod_{k=1}^{\infty}(1+\frac{c}{k})(1-\frac{1}{k})^c$ converges. Thus

$$\lim_{n\to\infty} z_n\left(\frac{an-1}{bn}\right)^c = \lim_{n\to\infty}\prod_{k=an}^{bn}\left(1+\frac{c}{k}\right)\left(1-\frac{1}{k}\right)^c = 1.$$

Therefore, $\lim_{n\to\infty} z_n = (b/a)^c$. Finally, (1) yields

$$\lim_{n\to\infty}\frac{1}{nx_{an}}\sum_{k=an}^{bn} x_k = \frac{b}{1+c}\lim_{n\to\infty} z_n - \frac{a}{1+c} = \frac{b^{1+c} - a^{1+c}}{(1+c)\,a^c}.$$

Editorial comment. David Beckwith gave the solution of the recurrence as

$$x_n = x_1\prod_{k=1}^{n-1}\frac{k+1/p}{k} = \frac{x_1}{\Gamma(1+1/p)}\frac{\Gamma(n+1/p)}{\Gamma(n)}.$$

He then expressed the desired quantity asymptotically as a Riemann sum and evaluated the integral to obtain the answer.

Also solved by D. Beckwith, P. Bracken, P. P. Dályay (Hungary), S. J. De Luxán & Á. Plaza (Spain), O. Furdui (Romania), M. Goldenberg & M. Kaplan, E. A. Herman, R. Hutchinson, J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), Á. Plaza (Spain), K. Schilling, J. Simons (U. K.), Z. Song & Y. Lin (China), A. Stenger, R. Stong, Barclays Capital Problems Solving Group (U.K.), FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Greatest Integer Sum

11529 [2010, 742]. *Proposed by Walter Blumberg, Coral Springs, FL.* For $n \geq 1$, let $A_n = \left(3\sum_{k=1}^{n}\left\lfloor\frac{k^2}{n}\right\rfloor\right) - n^2$. Let p and q be distinct primes with $p \equiv q \pmod 4$. Show that $A_{pq} = A_p + A_q - 2$.

Solution by Robert Tauraso, Università di Roma "Tor Vergata", Rome, Italy. Let $r_n(m)$ be the remainder of m on division by n. Since $m = n\lfloor m/n \rfloor + r_n(m)$,

$$A_n = \frac{3}{n}\sum_{k=1}^{n}\left[k^2 - r_n(k^2)\right] - n^2 = \frac{3n+1}{2} - \frac{3}{n}\sum_{k=0}^{n-1} r_n(k^2).$$

Since p is an odd prime, the number of square roots of j modulo p is $\left(\frac{j}{p}\right)+1$, where $\left(\frac{j}{p}\right)$ is the *Legendre symbol*, which equals 1 when j is a nonzero square modulo p, -1 with j is a nonsquare modulo p, and 0 when j is divisible by p. Thus

$$\sum_{k=0}^{p-1} r_p(k^2) = \sum_{j=0}^{p-1}\left(\left(\frac{j}{p}\right)+1\right)j.$$

Using the Chinese Remainder Theorem,

$$\sum_{k=0}^{pq-1} r_{pq}(k^2) = \sum_{j=0}^{pq-1}\left(\left(\frac{j}{p}\right)+1\right)\left(\left(\frac{j}{q}\right)+1\right)j.$$

Thus

$$A_p = \frac{3p+1}{2} - \frac{3}{p}\sum_{j=0}^{p-1}\left(\left(\frac{j}{p}\right)+1\right)j = 2 - \frac{3}{p}\sum_{j=0}^{p-1}\left(\frac{j}{p}\right)j.$$

Similarly,

$$A_q = 2 - \frac{3}{q}\sum_{j=0}^{q-1}\left(\frac{j}{q}\right)j$$

and

$$A_{pq} = 2 - \frac{3}{pq}\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right)j - \frac{3}{pq}\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)j - \frac{3}{pq}\sum_{j=0}^{pq-1}\left(\frac{j}{q}\right)j.$$

Since $\sum_{j=0}^{p-1}\left(\frac{j}{p}\right) = 0$,

$$\begin{aligned}\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)j &= \sum_{a=0}^{q-1}\sum_{r=0}^{p-1}\left(\frac{ap+r}{p}\right)(ap+r)\\ &= p\sum_{a=0}^{q-1}a\sum_{r=0}^{p-1}\left(\frac{r}{p}\right) + q\sum_{r=0}^{p-1}\left(\frac{r}{p}\right)r = q\sum_{r=0}^{p-1}\left(\frac{r}{p}\right)r,\end{aligned}$$

and similarly with p and q switched. Thus

$$A_{pq} - A_p - A_q + 2 = \sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right)j,$$

and it suffices to prove that the value of this sum is 0 when $p \equiv q \pmod 4$. We have $p \equiv q \equiv 1$ or $p \equiv q \equiv -1 \pmod 4$. In either case,

$$\left(\frac{-1}{p}\right)\left(\frac{-1}{q}\right) = (-1)^{(p-1)/2+(q-1)/2} = 1.$$

Thus

$$\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right)j=\sum_{j=0}^{pq-1}\left(\frac{pq-j}{p}\right)\left(\frac{pq-j}{q}\right)(pq-j)$$
$$=\sum_{j=0}^{pq-1}\left(\frac{-j}{p}\right)\left(\frac{-j}{q}\right)(pq-j)=\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right)(pq-j),$$

so

$$2\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right)j=pq\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right).$$

Now

$$\sum_{j=0}^{pq-1}\left(\frac{j}{p}\right)\left(\frac{j}{q}\right)=\sum_{a=0}^{q-1}\sum_{r=0}^{p-1}\left(\frac{ap+r}{p}\right)\left(\frac{ap+r}{q}\right)=\sum_{r=0}^{p-1}\left(\frac{r}{p}\right)\sum_{a=0}^{q-1}\left(\frac{ap+r}{q}\right)$$
$$=\sum_{r=0}^{p-1}\left(\frac{r}{p}\right)\sum_{k=0}^{q-1}\left(\frac{k}{q}\right)=0,$$

since for each r, the set $\{ap+r\colon\ 0\le a\le q-1\}$ is a complete system of residues modulo q.

Also solved by O. P. Lossers (Netherlands), R. E. Prather, Barclays Capital Problems Solving Group (U.K.), and the proposer.

A Prime Multiple of the Identity Matrix

11532 [2010, 834]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicentiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Find all prime numbers p such that there exists a 2×2 matrix A with integer entries, other than the identity matrix I, for which $A^p+A^{p-1}+\cdots+A=pI$.

Solution by Stephen Pierce, San Diego State University, San Diego, CA. The only primes that qualify are 2 and 3. Let $f(x)=-px^0+\sum_{i=1}^{p}x^i$.

For $p=2$, let $A=\begin{pmatrix}1&0\\0&-2\end{pmatrix}$. For $p=3$, note that $f(x)=(x-1)(x^2+2x+3)$. Let A be the "companion matrix" of x^2+2x+3, that is, $A=\begin{pmatrix}0&-3\\1&-2\end{pmatrix}$. We obtain $A^2+2A+3I=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

For $p\ge 5$, we make some elementary observations about f.

(a) From the triangle inequality, $f(x)=0$ for x in the closed unit disk only when $x=1$.

(b) The root 1 is a simple root (by differentiation).

(c) If $f(A)=0$, then the minimal polynomial of A divides f.

Given a matrix A with $f(A)=0$, let λ and μ be the eigenvalues of A. If A is a multiple of the identity, then λ is an integer dividing p, and $f(\lambda)$ has the same sign as λ. The only such solution is $A=I$.

If A is not a multiple of the identity, then $\lambda\mu$ is an integer dividing p, by (c). Since p is prime, $|\lambda\mu|\in\{1,p\}$. If $|\lambda\mu|=1$, then $\lambda=\mu=1$ from (a), but this contradicts (b). If $|\lambda\mu|=p$, then then the product of the other roots of f is ± 1. Now the rest of the roots must all be 1, which contradicts (b) when $p>3$.

Also solved by A. Bostan (France), P. P. Dályay (Hungary), E. A. Herman, Y. J. Ionin, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), É. Pité (France), J. Simons (U. K.), N. C. Singer, A. Stenger, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), M. Wildon (U. K.), Barclay Capital Problems Solving Group (U.K.), NSA Problems Group, Skidmore College Problem Group, and the proposers.

Squares From Totients

11544 [2011, 84]. *Proposed by Max A. Alekseyev, University of South Carolina, Columbia, SC, and Frank Ruskey, University of Victoria, Victoria, BC, Canada.* Prove that if m is a positive integer, then

$$\sum_{k=0}^{m-1} \varphi(2k+1) \left\lfloor \frac{m+k}{2k+1} \right\rfloor = m^2.$$

Here φ denotes the Euler totient function.

Solution by Charles Burnette, Philadelphia, PA. From the well-known identity

$$\sum_{d|(2k+1)} \varphi(d) = 2k+1,$$

we obtain $\sum_{k=0}^{m-1} \sum_{d|(2k+1)} \varphi(d) = m^2$.

We interchange the order of summation. Let n_d be the number of occurrences of $\varphi(d)$ in the sum. Such terms occur when d is positive, odd, and at most $2m-1$, so

$$\sum_{k=0}^{m-1} \sum_{d|(2k+1)} \varphi(d) = \sum_{\substack{1 \le d \le 2m-1 \\ d \text{ is odd}}} \varphi(d) \cdot n_d.$$

Since $\varphi(d)$ appears once for each odd multiple of d not exceeding $2m-1$, we have $(2n_d - 1)d \le 2m-1$, and hence $n_d = \left\lfloor \frac{2m+d-1}{2d} \right\rfloor$. Since d is odd, we can rewrite the expression as

$$\begin{aligned} m^2 &= \sum_{\substack{1 \le d \le 2m-1 \\ d \text{ is odd}}} \varphi(d) \left\lfloor \frac{2m+d-1}{2d} \right\rfloor \\ &= \sum_{k=0}^{m-1} \varphi(2k+1) \left\lfloor \frac{2m+2k+1-1}{2(2k+1)} \right\rfloor = \sum_{k=0}^{m-1} \varphi(2k+1) \left\lfloor \frac{m+k}{2k+1} \right\rfloor. \end{aligned}$$

Editorial comment. Most solvers used induction. J. Vondra mentioned other identities that can be proved similarly: $\sum_{k=1}^{2m} \varphi(k) \left\lfloor \frac{2m+k}{2k} \right\rfloor = \frac{m(3m+1)}{2}$ (also conjectured by R. Daileda), $\sum_{k=1}^{m} \varphi(k) \left\lfloor \frac{m}{k} \right\rfloor = \frac{m(m+1)}{2}$ (also noted by L. Zhou), and $\sum_{k=1}^{m} \varphi(2k) \left\lfloor \frac{m+k}{2k} \right\rfloor = \frac{m(m+1)}{2}$.

Also solved by L. Bush (student) & R. Mabry, R. Chapman (U. K.), J. Christopher, F. Çiçek (Turkey), C. Curtis, R. C. Daileda, P. P. Dályay (Hungary), D. Fleischman, S. M. Gagola Jr., D. Gove, S. Graham, Y. J. Ionin, O. Kouba (Syria), O. P. Lossers (Netherlands), V. S. Miller, U. Milutinović (Slovenia), K. Schilling, J. Schlosberg, J. Simons (U. K.), N. C. Singer, J. H. Smith, J. H. Steelman, A. Stenger, R. Stong, P. Straffin, H. T. Tang, R. Tauraso (Italy), M. Tetiva (Romania), A. Velozo (Chile), J. Vondra (Australia), Z. Vörös (Hungary), M. Vowe (Switzerland), F. Vrabec (Austria), L. Zhou, Barclays Capital Problems Solving Group (U.K.), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before April 30, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11677. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Evaluate

$$\prod_{n=1}^{\infty}\left(1+2e^{-m\sqrt{3}}\cosh(mn/\sqrt{3})\right).$$

11678. *Proposed by Farrukh Ataev Rakhimjanovich, Westminster International University in Tashkent, Tashkent, Uzbekistan.* Let F_k be the kth Fibonacci number, where $F_0 = 0$ and $F_1 = 1$. For $n \geq 1$ let A_n be an $(n+1)\times(n+1)$ matrix with entries $a_{j,k}$ given by $a_{0,k} = a_{k,0} = F_k$ for $0 \leq k \leq n$ and by $a_{j,k} = a_{j-1,k} + a_{j,k-1}$ for $j, k \geq 1$. Compute the determinant of A_n.

11679. *Proposed by Tim Keller, Orangeville, CT.* Let n be an integer greater than 2, and let $a_2, \ldots, a_n$ be positive real numbers with product 1. Prove that

$$\prod_{k=2}^{n}(1+a_k)^k > \frac{2}{e}\left(\frac{n}{2}\right)^{2n-1}.$$

11680. *Proposed by Benjamin Bogoşel, University of Savoie, Savoie, France, and Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let $x_1, \ldots, x_n$ be nonnegative real numbers. Show that

$$\left(\sum_{i=1}^{n}\frac{x_i}{i}\right)^4 \leq 2\pi^2\sum_{i,j=1}^{n}\frac{x_i x_j}{i+j}\sum_{k,l=1}^{n}\frac{x_k x_l}{(k+l)^3}.$$

11681. *Proposed by Des MacHale, University College Cork, Cork, Ireland.* For any group G, let Aut$G$ denote the group of automorphisms of G.

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(**a**) Show that there is no finite group G with $|\mathrm{Aut}G| = |G| + 1$.
(**b**) Show that there are infinitely many finite groups G with $|\mathrm{Aut}G| = |G|$.
(**c**) Find all finite groups G with $|\mathrm{Aut}G| = |G| - 1$.

11682. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Compute

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2.$$

11683. *Proposed by Raimond Struble, Santa Monica, CA.* Given a triangle ABC, let F_C be the foot of the altitude from the incenter to AB. Define F_B and F_C similarly. Let C_A be the circle with center A that passes through F_B and F_C, and define C_B and C_C similarly. The *Gergonne point* of a triangle is the point at which segments AF_A, BF_B, and CF_C meet. Determine, up to similarity, all isosceles triangles such that the Gergonne point of the triangle lies on one of the circles C_A, C_B, or C_C.

SOLUTIONS

Matrices Whose Powers Have Bounded Entries

11530 [2010, 834]. *Proposed by Pál Peter Dályay, Szeged, Hungary* Let A be an $m \times m$ matrix with nonnegative entries $a_{i,j}$ and with the property that there exists a permutation σ of $\{1, \dots, m\}$ for which $\prod_{i=1}^{m} a_{i,\sigma(i)} \geq 1$. Show that the union over $n \geq 1$ of the set of entries of A^n is bounded if and only if some positive power of A is the identity matrix.

Solution by John H. Smith, Needham, MA. If some positive power of A is the identity, then the powers A^n run over a finite set and hence have bounded entries. We prove the converse.

Define A' to agree with A in the positions $(i, \sigma(i))$ and be 0 elsewhere. Let $A'' = A - A'$; note that A'' has nonnegative entries. Let k be the least common multiple of the lengths of the cycles in σ, so k is the order of σ. Let $B = A^k - A'^k$; since $A = A' + A''$, the entries of B are nonnegative.

The matrix A'^k is diagonal. Let b_j be the product of the entries $(i, \sigma(i))$ in A corresponding to the jth cycle in σ, and let l_j be its length. The entries on the diagonal of A'^k corresponding to this cycle are b_j^{k/l_j}.

If $b_j > 1$, then the entries of A'^{kn} are unbounded as n grows. Since $A^{kn} - A'^{kn}$ has nonnegative entries, the entries of A^{kn} would also be unbounded. Hence $b_j \leq 1$ for each j; since the product over all cycles is at least 1, we have $b_j = 1$ for each j.

Hence $A^k = I + C$, where C has nonnegative entries. If $C \neq 0$, then $I + nC$ has unbounded entries (over all n). The same also holds for A^{kn}, which equals $I + nC + D_n$, where D_n has nonnegative entries. Hence boundedness of the entries of A^n over all n implies $A^k = I$.

Also solved by P. Budney, R. Chapman (U. K.), D. Constales (Belgium), D. Fleischman, E. A. Herman, R. A. Horn & J. L. Stuart, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), S. Pierce, É. Pité (France), R. E. Prather, A. R. Schep, J. Simons (U. K.), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), Barclay Capital Problems Solving Group, Con Amore Problem Group (Denmark), NSA Problems Group, and the proposer.

A Sufficient Condition for a Boolean Ring

11533 [2010, 835]. *Proposed by Erwin Just (emeritus), Bronx Community College of the City College of New York, Bronx, NY.* Let t be a positive integer and let R be a ring, not necessarily having an identity element, such that $x + x^{2t+1} = x^{2t} + x^{10t+1}$ for each x in R. Prove that R is a Boolean ring, that is, $x = x^2$ for all $x \in R$.

Solution by Richard Stong, CCR, San Diego CA. Summing the given identity for x and $-x$ yields $2x^{2t} = 0$. The identity for $2x$ yields

$$2x = 2x^{2t}\left[2^{2t-1} - 2^{2t}x + 2^{10t}x^{8t+1}\right] = 0.$$

Thus, R has characteristic 2. Now fix x and define a linear map $\mathbb{Z}_2[X] \to R$ by $p(X) \mapsto xp(x)$. The kernel of this map is an ideal I in $\mathbb{Z}_2[X]$. Since $\mathbb{Z}_2$ is a field, I is a principal ideal; let $q(X)$ be a generator of I. Note that $X^{10t} - X^{2t} + X^{2t-1} - 1 \in I$, so $X \nmid q(X)$.

Suppose that $h(X)$ is an irreducible factor of $q(X)$ having degree m, and let $\alpha \in \mathbb{F}_{2^m}$ be a root of h. For each nonzero $\beta \in \mathbb{F}_{2^m}$, there is a polynomial $g(X)$ with $\alpha g(\alpha) = \beta$. Since the given identity applies to $xg(x)$,

$$X^{10t+1}g(X)^{10t+1} - X^{2t+1}g(X)^{2t+1} + X^{2t}g(X)^{2t} - Xg(X)$$

is in I and hence is a multiple of the irreducible polynomial $h(X)$. Setting $X = \alpha$, we have $\beta^{10t+1} - \beta^{2t+1} + \beta^{2t} - \beta = 0$. Hence every nonzero element of $\mathbb{F}_{2^m}$ is a root of $X^{10t+1} - X^{2t+1} + X^{2t} - X$. This means that $X^{2^m-1} - 1$ divides $X^{10t+1} - X^{2t+1} + X^{2t} - X$. When we reduce $X^{10t+1} - X^{2t+1} + X^{2t} - X$ modulo $X^{2^m-1} - 1$, we replace every exponent by its residue mod $2^m - 1$. Hence, since R has characteristic 2, the four exponents $10t + 1, 2t + 1, 2t$ and 1 must pair up when reduced mod $2^m - 1$. If they pair up as $(10t + 1, 2t + 1)$ and $(2t, 1)$, then $2^m - 1$ divides both $10t + 1 - (2t + 1)$ and $2t - 1$, which implies $m = 1$ since $\gcd(8t, 2t - 1) = 1$. Similarly, the other two pairings also lead to $m = 1$. Hence, $h(X) = X - 1$.

Now let $k(X) = X^{10t+1} - X^{2t+1} + X^{2t} - X$. Since $k'(1) = 1$, we cannot have 1 as a double root of $k(X)$. Thus $q(X)|(X - 1)$, and $X - 1 \in I$. In terms of x this yields $x^2 = x = 0$ or $x = x^2$, as desired.

Also solved by A. J. Bevelacqua, D. Constales (Belgium), P. P. Dályay (Hungary), O. P. Lossers (Netherlands), J. M. Sanders, R. Tauraso (Italy), and the proposer.

An Application of Fermat's Little Theorem

11537 [2010, 929]. *Proposed by Lang Withers, Jr., MITRE, McClean, VA.* Let p be a prime and a be a positive integer. Let X be a random variable having a Poisson distribution with mean a, and let M be the pth moment of X. Prove that $M \equiv 2a \bmod p$.

Solution by Alin Bostan and Bruno Salvy, INRIA, France. For $n \geq 0$, the nth moment $M_n(a)$ of a random variable having a Poisson distribution with mean a is given by

$$M_n(a) = e^{-a} \sum_{k \geq 0} \frac{k^n a^k}{k!}.$$

We compute its exponential generating function $\sum_{n \geq 0} M_n(a)t^n/n!$ as follows:

$$\sum_{n \geq 0} \frac{M_n(a)}{n!}t^n = e^{-a} \sum_{k \geq 0} \frac{a^k}{k!} \sum_{n \geq 0} \frac{(tk)^n}{n!} = e^{-a} \sum_{k \geq 0} \frac{a^k}{k!} e^{tk} = e^{a(e^t-1)}.$$

By coefficient extraction, it follows that $M_n(a)$ is a monic polynomial of degree n in a (it is known as the *Touchard polynomial*). Its coefficient $\left[a^k\right] M_n(a)$ on a^k, denoted $\left\{ n \atop k \right\}$, is the *Stirling number of the second kind*, given by

$$\left\{ n \atop k \right\} = \left[\frac{t^n}{n!}\right] \frac{(e^t - 1)^k}{k!} = \left[\frac{t^n}{n!}\right] \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} e^{tj} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

By Fermat's Little Theorem, if p is a prime number and $2 \le k \le p-1$, then mod p,

$$k! \left\{ p \atop k \right\} \equiv \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j = k \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} = k(1-1)^{k-1} = 0.$$

Therefore, p divides $\left\{ p \atop k \right\}$ when $2 \le k \le p-1$, yielding

$$M_p \equiv a^p + a \equiv 2a \pmod{p}.$$

Also solved by O. J. L. Alfonso (Colombia), T. Becker (Germany), L. Bogdan (Canada), J. R. Buchanan, M. Caragiu, M. A. Carlton, N. Caro (Brazil), R. Chapman (U. K.), D. Constales (Belgium), W. J. Cowieson, N. Grivaux (France), K. Kim, O. Kouba (Syria), J. Lobo (Costa Rica), O. P. Lossers (Netherlands), R. D. Nelson (U.K.), M. A. Prasad (India), D. Promislaw (Canada), C. M. Russell, J. Simons (U. K.), N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), Z. Vörös (Romania), M. Vowe (Switzerland), D. M. Warme, GCHQ Problem Solving Group (U. K.), Texas State University Problem Solvers Group, and the proposer.

A Condition for Multiplicative Identities in Commutative Rings

11538 [2010, 929]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.* Prove that a finite commutative ring in which every element can be written as a product of two (not necessarily distinct) elements has a multiplicative identity.

Solution by John H. Smith, Needham, MA. Let R be such a ring. For $x \in R$, let $xR = \{xz\colon z \in R\}$. We call xR *maximal* if it is not properly contained in yR for any y. By the hypothesis, every element of R is in some such set. Since R is finite, each yR is contained in a maximal such set.

When xR is maximal, we show that (i) $x \in xR$, (ii) xR contains an element e_x that acts as a multiplicative identity on xR, (iii) if yR is also maximal, then $xR = yR$, and (iv) $xR = R$. Together, (ii) and (iv) yield the desired multiplicative identity on R.

(i): We are given $x = yz$, which yields $xR \subseteq yR$. If $x \notin xR$, then the containment is proper, contradicting the maximality of xR.

(ii): Since $x \in xR$, we have $x = xe_x = e_x x$ (by commutativity) for some $e_x \in R$. Hence $xR \subseteq e_x R$, and these sets are equal by the maximality of xR. Thus also $e_x R$ is maximal, and by (i) we have $e_x \in e_x R = xR$. Since $e_x x = x$, it follows for $xy \in xR$ that $e_x xy = xy$, and hence e_x is a multipicative identity on xR.

(iii): Suppose that xR and yR are both maximal; let e_x and e_y be the corresponding identity elements, and let $f = e_x + e_y - e_x e_y$. We compute

$$fx = e_x x + e_y x - (e_x) e_y x = x + e_y x - e_y x = x.$$

Similarly, $fy = y$, so fR contains both xR and yR. By maximality, fR equals both xR and yR, and hence they equal each other.

(iv): Every element of R lies in some maximal set yR and hence in xR. Thus $R = xR$.

Also solved by G. Apostolopoulos (Greece), H. E. Bell (Canada), J. Bergen, A. J. Bevelacqua, W. D. Burgess, J. Cade, N. Caro, R. A. Caruthers, R. Chapman (U. K.), J. E. Cooper III, A. Habil (Syria), A. Hays, C. Lanski, O. P. Lossers (Netherlands), A. Nakhash, D. Opitz, D. Promislow (Canada), J. Simons (U. K.), T. Smith, R. Stong, R. Tauraso (Italy), X. Wang, J. W. Ward, the Microsoft Research Problems Group, the NSA Problems Group, the Texas State University Problem Solvers Group, and the proposer.

Quadratic form plus a cube

11539 [2010, 929]. *Proposed by William C. Jagy, MSRI, Berkeley, CA.* Let E be the set of all positive integers not divisible by 2 or 3 or by any prime q represented by the quadratic form $4u^2 + 2uv + 7v^2$. (Thus, the first few members of E are $1, 5, 11, 17, 23$, and 25.) Show that $4x^2 + 2xy + 7y^2 + z^3$ is not an element of $\{2n^3, -2n^3, 32n^3, -32n^3\}$ for $n \in E$ and $x, y, z \in \mathbb{Z}$.

Solution by Robin Chapman, Exeter, UK. Let $Q_1(x, y) = x^2 + 27y^2$ and $Q_2(x, y) = 4x^2 + 2xy + 7y^2$. Both Q_1 and Q_2 are primitive positive definite integral quadratic forms with discriminant -108. By special cases of quadratic and cubic reciprocity, we have the following (Theorems 2.13 and 4.15 of [1]):

(i) a prime p is represented by one of these forms if and only if $p \equiv 1 \pmod 3$;

(ii) p is represented by the quadratic form Q_1 if and only if $p \equiv 1 \pmod 3$ and 2 is a cubic residue modulo p.

Therefore, if p is represented by Q_2, then 2 is not a cubic residue modulo p.

Now suppose that $4x^2 + 2xy + 7y^2 + z^3 = kn^3$ with $k \in \{\pm 2, \pm 32\}$, for $x, y, z \in \mathbb{Z}$ and $n \in E$. We have $Q_2(x, y) = kn^3 - z^3$, but we cannot have $x = y = 0$, since k is not the cube of a rational. Now $Q_2(x, y) = \frac{1}{4}((4x + y)^2 + 27y^2) \ge \frac{27}{4} > 6$ if $y \ne 0$, and $4x^2 + 2xy + 7y^2 = 4x^2 \ge 4$ if $y = 0$.

Consider the possible values of $Q_2(x, y)$. If $Q_2(x, y)$ is even, then y is even; let $y = 2y'$. Now $Q_2(x, y) = 4(x^2 + xy' + 7(y')^2)$. For $x^2 + xy' + 7(y')^2$ to be even, both x and y' must be even. Repeating this argument, we see that the 2-adic valuation $v_2(Q_2(x, y))$ must be even. (Here $v_2(m)$ is defined as the highest power of 2 that divides m.) Since $n \in E$, n is odd, and therefore $v_2(kn^3)$ is 1 or 5. Since $z^3 = kn^3 - Q_2(x, y)$, we have $v_2(z^3) = \min(v_2(kn^3), v_2(Q_2(x, y))) \in \{0, 1, 2, 4, 5\}$. However, $v_2(z^3)$ is a multiple of 3, so $v_2(z^3) = 0$, and hence $Q_2(x, y)$ is odd.

Now $Q_2(x, y) \equiv x^2 + 2xy + y^2 = (x + y)^2 \pmod 3$. If $3 \mid Q_2(x, y)$, then $x \equiv -y \pmod 3$, so we can write $x = 3t - y$, where $t \in \mathbb{Z}$. Now $Q_2(x, y) = 36t^2 - 18t + 9y^2$ so it is a multiple of 9, so $z^3 \equiv kn^3 \pmod 9$. However, $n \in E$ yields $3 \nmid n$, so $n \equiv \pm 1 \pmod 3$. Therefore, $n^3 \equiv \pm 1 \pmod 9$, leading to $z^3 \equiv \pm 2$ or $\pm 5 \pmod 9$, which is impossible. We conclude that $Q_2(x, y)$ is coprime to 6.

With $m = \gcd(x, y)$ we get $Q_2(x, y) = m^2 Q_2(x', y')$, where $x = mx'$, $y = my'$, and $\gcd(x', y') = 1$. Now $Q_2(x', y') \ge 4$, and we claim that $Q_2(x', y')$ has a prime factor p represented by Q_2. Let p be any prime factor of $Q_2(x', y')$. We have $4Q_2(x', y') \equiv (4x' + y')^2 + 27(y')^2 \pmod p$, and hence $p \equiv 1 \pmod 3$ by (i). Now p is represented by Q_1 or Q_2; let us suppose by Q_1 so that $p = u^2 + 27v^2$.

Define $a = ux' - vx' - 7vy'$ and $b = uy' + 4vx' + vy'$ to get

$$4a^2 + 2ab + 7b^2 = (u^2 + 27v^2)\left(4(x')^2 + 2x'y' + 7(y')^2\right) = pQ_2(x', y').$$

Since $u^2 \equiv -27v^2 \pmod p$, we can write $u \equiv \xi v \pmod p$, where $\xi^2 \equiv -27 \pmod p$ (ξ exists, since -3 is a quadratic residue modulo p). Since $(4x' + y')^2 \equiv 27(y')^2 \pmod p$, we also have $4x' + y' \equiv \pm \xi y' \pmod p$. Replacing u by $-u$ and ξ by $-\xi$, if necessary, we may assume that $u \equiv \xi v$ and $4x' \equiv -(1 + \xi)y' \pmod p$. Now

$$4a \equiv \left(-\xi(1 + \xi) + (1 + \xi) - 28\right) vy' = -(\xi^2 + 27)vy' \equiv 0 \pmod p$$

and $b \equiv \left(\xi - (1+\xi) + 1\right) vy' \equiv 0 \pmod{p}$. Thus $p \mid a$ and $p \mid b$, so

$$\frac{Q_2(x', y')}{p} = \frac{Q_2(a, b)}{p^2} = Q_2\left(\frac{a}{p}, \frac{b}{p}\right)$$

and hence $Q_2(x', y')/p$ is represented by Q_2. Iterating this argument must eventually find a prime factor of $Q_2(x', y')$ represented by Q_2.

Let the prime p divide $Q_2(x', y')$ and be represented by Q_2, so 2 is not a cubic residue modulo p. We have $kn^3 \equiv z^3 \pmod{p}$. Since also $k = \pm 2$ or $k = \pm 2^5$, we conclude that k is not a cubic residue modulo p. Hence $p \mid n$, contradicting $n \in E$. This shows that $Q_2(x, y) + z^3 \notin \{\pm 2n^3, \pm 32n^3 : n \in E\}$.

References

[1] David A. Cox, *Primes of the form $x^2 + ny^2$*, John Wiley & Sons, 1989.

Also solved by the proposer.

A Stirling sum

11545 [2011, 84]. *Proposed by Manuel Kauers, Research Institute for Symbolic Computation, Linz, Austria, and Sheng-Lan Ko, National Taiwan University, Taipei, Taiwan.* Find a closed-form expression for

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} s(n+k, k),$$

where s refers to the (signed) Stirling numbers of the first kind.

Solution I by Jim Simons, Cheltenham, U. K. The answer is $\prod_{i=1}^{n}(2i-1)$. Let $c(n, k)$ denote the unsigned Stirling number of the first kind, the number of permutations of $[n]$ with k cycles. By definition, $s(n, k) = (-1)^{n-k} c(n, k)$. Subsituting this definition into the sum and then setting $k = n - i$ transforms the sum to

$$\sum_{i=0}^{n} (-1)^i \binom{2n}{i} c(2n-i, n-i).$$

Now $\binom{2n}{i} c(2n-i, n-i)$ is the number of ways to construct a permutation of $[2n]$ with n cycles by choosing i fixed points and constructing a permutation with $n - i$ cycles on the remaining $2n - i$ elements. By inclusion-exclusion, the sum is the number of permutations of $[2n]$ having n cycles and no fixed points. Each cycle in such a permutation must be a 2-cycle. It is well known that the number of pairings of $2n$ elements is $\prod_{i=1}^{n}(2i-1)$.

Solution II by Kim McInturff, Santa Barbara, CA. We obtain the answer in the form $(2n)!/2^n n!$. It is well known that

$$\sum_{n \geq k} s(n, k) \frac{t^n}{n!} u^k = (1+t)^u,$$

Now

$$\begin{aligned} e^{-ut}(1+t)^u &= \sum_{i=0}^{\infty} (-1)^i \frac{(ut)^i}{i!} \sum_{j,k} s(j+k, k) \frac{t^{j+k}}{(j+k)!} u^k \\ &= \sum_{i,j,k} (-1)^i \binom{i+j+k}{j+k} s(j+k, k) \frac{t^{i+j+k}}{(i+j+k)!} u^{i+k}. \end{aligned}$$

The coefficient of $t^{2n}u^n/n!$ in this sum is obtained by taking the terms in which $j = n$ and $i = n - k$, yielding

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n+k}s(n+k,k)$$

which is $(-1)^n$ times the desired sum. Computing the coefficient another way yields

$$e^{-ut}(1+t)^u = e^{-ut+u\log(1+t)} = \exp\bigl(u(-t^2/2+t^3/3-\cdots)\bigr)$$

so the coefficient of $u^n t^{2n}/(2n)!$ is $(-1)^n(2n)!/2^n n!$.

Also solved by D. Beckwith, C. Burnette, D. Callan, R. Chapman (U. K.), D. Constales, K. David, C. Fuerst (Austria), J.-P. Grivaux (France), O. Kouba (Syria), O. P. Lossers (Netherlands), Á. Plaza (Spain), J. Quaintance, N. C. Singer, R. Stong, R. Tauraso (Italy), M. Wildon, Barclays Capital Problems Solving Group (U.K.), CMC 328, GCHQ Problem Solving Group (U. K.), and the proposer.

2-adic Valuation of Bernoulli-style Sums

11546 [2011, 84]. *Proposed by Kieren MacMillan, Toronto, Canada, and Jonathan Sondow, New York, NY.* Let d, k, and q be positive integers, with k odd. Find the highest power of 2 that divides $\sum_{n=1}^{2^d k} n^q$.

Solution by Barclays Capital Problems Solving Group, London, (U.K.). If q is even or equals 1, then the answer is 2^{d-1}. Otherwise, the answer is $2^{2(d-1)}$.

In the case $q = 1$, the sum equals $2^{d-1}(k(2^d k - 1))$, which clearly is divisible by 2^{d-1} and not by 2^d. Now restrict to $q > 1$.

If $d = 1$, then the sum contains exactly k odd terms and hence is odd, so the answer is 2^0. We proceed by induction on d; consider $d > 1$. We pair high and low terms from the sum, using

$$\sum_{n=1}^{2^d k} n^q = (2^d k)^q - (2^{d-1}k)^q + \sum_{n=1}^{2^{d-1}k}\left(n^q + (2^d k - n)^q\right).$$

For even q, we take residues modulo 2^d. Since $q(d-1) \ge d$, both $(2^d k)^q$ and $(2^{d-1}k)^q$ are divisible by 2^d. Also, $(2^d k - n)^q \equiv n^q \bmod 2^d$ (since q is even). Thus

$$\sum_{n=1}^{2^d k} n^q \equiv 2\sum_{n=1}^{2^{d-1}k} n^q \pmod{2^d}.$$

By the induction hypothesis, $\sum_{n=1}^{2^{d-1}k} n^q$ is divisible by 2^{d-2} but not 2^{d-1}. Hence the sum on the left is divisible by 2^{d-1} but not 2^d.

For odd q, we instead work modulo 2^{2d-1}. Since $q(d-1) \ge 2d-1$ (using $q \ge 3$) both $(2^d k)^q$ and $(2^{d-1}k)^q$ are divisible by 2^{2d-1}. Since q is odd, expanding the binomial yields $(2^d k - n)^q = -n^q + 2^d kqn^{q-1} - 2^{2d}k^2\binom{q}{2}n^{q-2} + \cdots$; all terms after the second are divisible by 2^{2d-1}. Thus $n^q + (2^d k - n)^q \equiv 2^d qkn^{q-1} \bmod 2^{2d-1}$, so

$$\sum_{n=1}^{2^d k} n^q \equiv 2^d qk\sum_{n=1}^{2^{d-1}k} n^{q-1} \pmod{2^{2d-1}}.$$

By the induction hypothesis, $\sum_{n=1}^{2^{d-1}k} n^{q-1}$ is divisible by 2^{d-2} but not 2^{d-1}. Hence the sum on the left is divisible by 2^{2d-2} but not 2^{2d-1}.

Also solved by R. Chapman (U. K.), J. Christopher, P. P. Dályay (Hungary), Y. J. Ionin, O. P. Lossers (Netherlands), Á. Plaza (Spain), R. Stong, R. Tauraso (Italy), J. Vondra (Australia), Z. Vörös (Hungary), CMC 328, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

Largest odd divisors

11553 [2011, 178]. *Proposed by Mihály Bencze, Brasov, Romania.* For a positive integer k, let $\alpha(k)$ be the largest odd divisor of k. Prove that for each positive integer n,

$$\frac{n(n+1)}{3} \leq \sum_{k=1}^{n} \frac{n-k+1}{k}\alpha(k) \leq \frac{n(n+3)}{3}.$$

Solution by O. P. Lossers, Eindhoven, The Netherlands. Let $T(k) = \sum_{j=1}^{k} \frac{\alpha(j)}{j}$. Since $\alpha(2j) = \alpha(j)$ and $\alpha(2j-1) = 2j-1$,

$$T(k) = \sum_{j=1}^{\lceil k/2 \rceil} \frac{\alpha(2j-1)}{2j-1} + \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{\alpha(2j)}{2j} = \left\lceil \frac{k}{2} \right\rceil + \frac{T(\lfloor k/2 \rfloor)}{2}, \tag{1}$$

We prove by induction on k (with trivial basis $k = 1$) that

$$\frac{2k}{3} < T(k) < \frac{2(k+1)}{3}. \tag{2}$$

For $k > 1$, using (1) and the induction hypothesis yields

$$\frac{2k}{3} < \left\lceil \frac{k}{2} \right\rceil + \frac{\lfloor k/2 \rfloor}{3} < T(k) < \left\lceil \frac{k}{2} \right\rceil + \frac{\lfloor k/2 \rfloor + 1}{3} \leq \frac{2(k+1)}{3}.$$

Since $\sum_{k=1}^{n} T(k)$ counts each instance of $\alpha(j)/j$ exactly $n-j+1$ times, summing (2) over $1 \leq k \leq n$ yields

$$\frac{n(n+1)}{3} < \sum_{j=1}^{n} \frac{n-j+1}{j}\alpha(j) < \frac{n(n+3)}{3}.$$

Editorial comment. R. A. MacLeod (On the Largest Odd Divisor of n, *Amer. Math. Monthly* **75** (1968) 647–648) proved that $\frac{2k}{3} + \frac{1}{3k} \leq T(k) \leq \frac{2(k+1)}{3} - \frac{2}{3(k+1)}$, with equality for $k = 2^m$ in the first and for $k = 2^m - 1$ in the second inequality. Many solvers also showed that $\frac{n(n+7/4)}{3} \leq \sum_{k=1}^{n} \frac{n-k+1}{k}\alpha(k) \leq \frac{n(n+2)}{3}$ and that equality is achieved in the second inequality whenever $n = 2^m - 1$. O. Kouba (On Certain Sums Related to the Largest Odd Divisor, arXiv:1103.2295v1 [math.NT], Mar 2011, arxiv.org) gave a sharp lower bound for $\sum_{k=1}^{n} \frac{n-k+1}{k}\alpha(k)$ and described when equality holds.

Also solved by M. Bataille (France), D. Beckwith, C. Burnette, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), N. Grivaux (France), A. Habil (Syria), M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), J. H. Lindsey II, D. Nacin, Á. Plaza (Spain), C. R. Pranesachar (India), R. E. Prather, J. Simons (U. K.), N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, J. Vinuesa (Spain), Z. Xintao (China), C. Y. Yıldırım (Turkey), GCHQ Problem Solving Group (U. K.), and the proposer.

Errata and End Notes for 2012. In the credits for problem 10912 [2003,745], P. T. Krasopoulos' name was misspelled.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before May 31, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11684. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France, and Rudolf Rupp, Georg-Simon-Ohm Hochschule Nürnberg, Nuremberg, Germany.* For complex a and z, let $\phi_a(z) = (a - z)/(1 - \overline{a}z)$ and $\rho(a, z) = |a - z|/|1 - \overline{a}z|$.
(**a**) Show that whenever $-1 < a, b < 1$,

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\rho(a, b), \text{ and}$$

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2.$$

(**b**) For complex α, β with $|\alpha| = |\beta| = 1$, let

$$m(z) = m_{a,b,\alpha,\beta}(z) = |\alpha\phi_a(z) - \beta\phi_b(z)|.$$

Determine the maximum and minimum, taken over z with $|z| = 1$, of $m(z)$.

11685. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{2^{2^k} - 1}\right) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k-1}\left(2^{2^j} - 1\right)}.$$

In other words, prove that

$$(1 + 1)\left(1 + \tfrac{1}{3}\right)\left(1 + \tfrac{1}{15}\right)\left(1 + \tfrac{1}{255}\right)\cdots = \tfrac{1}{2} + 1 + 1 + \tfrac{1}{3} + \tfrac{1}{3\cdot 15} + \tfrac{1}{3\cdot 15\cdot 255} + \cdots .$$

11686. *Proposed by Michel Bataille, Rouen, France.* Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Prove that

$$\frac{\cot x + \cot y + \cot z}{\tan x + \tan y + \tan z} \ge 4(\sin^2 x + \sin^2 y + \sin^2 z).$$

http://dx.doi.org/10.4169/amer.math.monthly.120.01.076

11687. *Proposed by Steven Finch, Harvard University, Cambridge, MA.* Let T be a solid torus in $\mathbb{R}^3$ with center at the origin, tube radius 1 and spine radius r with $r \geq 1$ (so that T has volume $\pi \cdot 2\pi r$). Let P be a 'random' nearby plane. Find the conditional probability, given that P meets T, that the intersection is simply connected. For what value of r is this probability maximal? (The plane is chosen by first picking a distance from the origin uniformly between 0 and $1 + r$, and then picking a normal vector independently and uniformly on the unit sphere.)

11688. *Proposed by Samuel Alexander, The Ohio State University, Columbus, OH.* Consider $f\colon \mathbb{N}^3 \to \mathbb{N}$ such that $\lim_{a\to\infty} \inf_{b,c,d\in\mathbb{N},b<a} f(a,c,d) - f(b,c,d) = \infty$. Show that for $B \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$f(a,c,d) = k \Rightarrow \max\{c,d\} > B.$$

11689. *Proposed by Yagub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Two circles w_1 and w_2 intersect at distinct points B and C and are internally tangent to a third circle w at M and N, respectively. Line BC intersects w at A and D, with A nearer B than C. Let r_1 and r_2 be the radii of w_1 and w_2, respectively, with $r_1 \leq r_2$. Let $u = \sqrt{|AC| \cdot |BD|}$ and $v = \sqrt{|AB| \cdot |CD|}$. Prove that

$$\frac{u - v}{u + v} < \sqrt{\frac{r_1}{r_2}}.$$

11690. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let M be a point in the interior of a convex polygon with vertices $A_1, \ldots, A_n$ in order. For $1 \leq i \leq n$, let r_i be the distance from M to A_i, and let R_i be the radius of the circumcircle of triangle MA_iA_{i+1}, where $A_{n+1} = A_1$. Show that

$$\sum_{i=1}^{n} \frac{R_i}{r_i + r_{i+1}} \geq \frac{n}{4\cos(\pi/n)}.$$

SOLUTIONS

A Special Ratio of Cosines

11540 [2010, 929]. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be an integer greater than 1, other than 4. Let p and q be positive integers less than n and relatively prime to n. Let $a = \frac{\cos(2\pi p/n)}{\cos(2\pi q/n)}$. Show that if a^k is rational for some positive integer k, then a^k is either 1 or -1.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Let α be the primitive root $e^{2\pi i/n}$ of $z^n - 1 = 0$. We compute

$$(\alpha^q + \alpha^{-q})^k a^k = (\alpha^p + \alpha^{-p})^k.$$

All $\phi(n)$ primitive roots of $z^n - 1$ are conjugates of α and hence satisfy the same equation. If β runs through the set P of primitive roots, then so do β^p and β^q. Thus

$$\prod_{\beta\in P}(\beta + \beta^{-1})^k a^{k\phi(n)} = \prod_{\beta\in P}(\beta + \beta^{-1})^k,$$

so $a^{k\phi(n)} = 1$ and $a^k \in \{1, -1\}$.

Also solved by R. Chapman (U. K.), Y. J. Ionin, A. Nakhash, A. Stenger, R. Stong, M. Tetiva (Romania), and the proposer.

A Rectangle With Vertices On Prescribed Circles

11558 [2011, 275]. *Proposed by Andrew McFarland, Płock, Poland.* Given four concentric circles, find a necessary and sufficient condition that there be a rectangle with one corner on each circle.

Solution by J. C. Linders, Eindhoven, The Netherlands. Suppose there is a rectangle with one corner on each circle. Choose a rectangular coordinate system with origin at the common center of the circles and axes parallel to the sides of the rectangle. There are real numbers a, b, c, d such that the vertices of the rectangle, starting at the lower left corner and proceeding clockwise, are (a, b), (c, b), (c, d), (a, d).

Let the circles on which they lie have radii r_1, r_2, r_3, r_4, respectively. We have

$$a^2 + b^2 = r_1^2, \qquad c^2 + b^2 = r_2^2, \qquad c^2 + d^2 = r_3^2, \qquad a^2 + d^2 = r_4^2.$$

Thus $r_1^2 + r_3^2 = a^2 + b^2 + c^2 + d^2 = r_2^2 + r_4^2$. By rotating the picture, if necessary, we may assume that r_1 is the least radius and therefore r_3 is the greatest.

Conversely, suppose that $0 < r_1 < r_2 < r_4 < r_3$ satisfy $r_1^2 + r_3^2 = r_2^2 + r_4^2$. We may take $a = r_1$, $b = 0$, $c = r_2$, and $d = \sqrt{r_3^2 - r_2^2} = \sqrt{r_4^2 - r_1^2}$. The points (a, b), (c, b), (c, d), (a, d) lie on the circles, and they form a rectangle.

Editorial comment. Solvers found this problem in: C. Blattner & G. Wanner, "Note on rectangles with vertices on prescribed circles." *Elem. Math.* **62** (2007) 127–129; E. J. Ionascu & P. Stanica, "Extreme values for the area of rectangles with vertices on concentrical circles."*Elem. Math.* **62** (2007) 30–39.

Also solved by Y. N. Aliyev (Azerbaijan), Y. An (China), D. Beckwith, C. Burnette, N. Caro (Brazil), D. Chakerian, R. Chapman (U. K.), J. Christopher, P. P. Dályay (Hungary), C. Delorme (France), A. Ercan (Turkey), O. Geupel (Germany), J.-P. Grivaux (France), S. Habil, C. C. Heckman, E. A. Herman, E. J. Ionascu, Y. J. Ionin, J. E. Kettner, O. Kouba (Syria),L. Lipták, M. D. Meyerson, J. Minkus, H. W. Park (Korea), Á. Plaza & J. Sánchez-Reyes (Spain), R. E. Prather, J. Schlosberg, J. Simons (U. K.), J. H. Steelman, D. Stone & J. Hawkins, R. Stong, M. Tetiva (Romania), E. I. Verriest, M. Vowe (Switzerland), J. B. Zacharias, Armstrong Problem Solvers, Barclays Capital Problems Solving Group (U. K.), Ellington Management Problem Solving Group, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Rational Recurrence

11559 [2011, 275]. *Proposed by Michel Bataille, Rouen, France.* For positive p and $x \in (0, 1)$, define the sequence $\langle x_n \rangle$ by $x_0 = 1$, $x_1 = x$, and, for $n \geq 1$,

$$x_{n+1} = \frac{p x_{n-1} x_n + (1 - p) x_n^2}{(1 + p) x_{n-1} - p x_n}.$$

Find positive real numbers α, β such that $\lim_{n \to \infty} n^\alpha x_n = \beta$.

Solution by Douglas B. Tyler, Raytheon, Torrance, CA. We claim that the values are $\alpha = 1/p$ and $\beta = \Gamma(u)/\Gamma(ux)$, where $u = 1/(p(1 - x))$. The statement to be proved implies $x_{n+1}/x_n \to 1$, and the recursion is first order in this ratio. This suggests setting $y_n = 1 - x_{n+1}/x_n$. Now $1/y_n - 1/y_{n-1} = p$, which is linear in $1/y_n$. Thus $1/y_n = np + 1/(1 - x) = (n + u)p$, and

$$x_n = \prod_{k=0}^{n-1} \frac{x_{k+1}}{x_k} = \prod_{k=0}^{n-1} \frac{kp + up - 1}{(k + u)p} = \prod_{k=0}^{n-1} \frac{kp + uxp}{(k + u)p} = \frac{\Gamma(n + ux)}{\Gamma(n + u)} \cdot \frac{\Gamma(u)}{\Gamma(ux)}.$$

By Stirling's formula, $\Gamma(x+1) \sim (x/e)^x\sqrt{2\pi x}$, so $\Gamma(n+u)/\Gamma(n+ux) \sim n^{ux-u} = n^{-1/p}$, completing the proof.

Also solved by R. Agnew, R. Bagby, M. Benito, Ó. Ciaurri, E. Fernández & L. Roncal (Spain), N. Bouzar, P. Bracken, C. Burnette, R. Chapman (U. K.), P. P. Dálych (Hungary), S. de Luxán (Spain), D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. Grivaux (France), A. Habil (Syria), E. A. Herman, B. D. Hughes (Australia), O. Kouba (Syria), J. C. Linders (Netherlands), J. H. Lindsey II, L. Lipták, M. Omarjee (France), E. Omey (Belgium), N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), E. I. Verriest, Barclays Capital Problems Solving Group (U. K.), Ellington Management Problem Solving Group, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Rational-Area Triangles

11560 [2011, 275]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.*
(**a**) The diagonals of a convex pentagon $P_0P_1P_2P_3P_4$ divide it into 11 regions, of which 10 are triangular. Of these 10, five have two vertices on the diagonal P_0P_2. Prove that if each of these has rational area, then the other five triangles and the original pentagon all have rational areas.
(**b**) Let $P_0, P_1, \ldots, P_{n-1}$ with $n \geq 5$ be points in the plane. Suppose that no three are collinear, and, interpreting indices on P_k as periodic modulo n, suppose that for all k, $P_{k-1}P_{k+1}$ is not parallel to P_kP_{k+2}. Let Q_k be the intersection of $P_{k-1}P_{k+1}$ with P_kP_{k+2}. Let α_k be the area of triangle $P_kQ_kP_{k+1}$, and let β_k be the area of triangle $P_{k+1}Q_kQ_{k+1}$. For $0 \leq j \leq 2n-1$, let

$$\gamma_j = \begin{cases} \alpha_{j/2}, & \text{if } j \text{ is even,} \\ \beta_{(j-1)/2}, & \text{if } j \text{ is odd.} \end{cases}$$

Interpreting indices on γ_j as periodic modulo $2n$, find the least m such that if m consecutive γ_j are rational, then all are rational.

Solution by Richard Stong, Center for Communications Research,San Diego, CA. Call a point in the Cartesian plane *rational* if both its coordinates are rational. Call a line *rational* if it can be defined by an equation with rational coefficients. Note that: (i) Two rational points determine a rational line; (ii) The intersection of two rational lines is a rational point; (iii) A polygon all of whose vertices are rational points has rational area; and (iv) If A and B are rational points, then the locus of all C such that $\triangle ABC$ has a given rational area is two rational lines parallel to AB.

For part (**b**), we claim that the least m is $2n-4$ .We will begin by showing that rationality of any $2n-4$ consecutive values of γ_j implies that affine coordinates may be chosen so that all P_i and Q_j are rational points. Let $X_i = P_{i/2}$ for i even and $X_{(i-1)/2}$ for i odd. Now γ_j is the area of $\triangle X_jX_{j+1}X_{j+2}$ and $\gamma_j + \gamma_{j+1}$ is the area of $\triangle X_jX_{j+2}X_{j+3}$. Note that no three consecutive X_j can be collinear, since either case would imply that four consecutive P_j are collinear, contradicting the hypotheses.

Suppose $\gamma_j, \gamma_{j+1}, \ldots, \gamma_{j+2n-5}$ are all rational. By an affine transformation, we may arrange that X_j, X_{j+1}, and X_{j+2} are all rational. Before and after this transformation, $\triangle X_jX_{j+1}X_{j+2}$ has rational area, so the determinant of the associated matrix must be rational, and therefore this map sends rational areas to rational areas.

Now $\triangle X_{j+1}X_{j+2}X_{j+3}$ and $\triangle X_jX_{j+2}X_{j+3}$ both have rational area (γ_{j+1} and $\gamma_j + \gamma_{j+1}$,respectively), so from (iv) we may conclude that X_{j+3} lies on a certain rational line parallel to $X_{j+1}X_{j+2}$ and on another rational line parallel to X_jX_{j+2}. These two lines are not parallel, so (ii) implies that X_{j+3} is a rational point. Iterating this argument, we conclude that $X_j, \ldots, X_{j+2n-3}$ are all rational points. Thus, at most two of the X_j are irrational points.

On the other hand, each X_k lies at the intersection of two lines through four of the X_j (including X_k itself). If all but two of the X_j are rational, then by (i) these two lines must be rational and hence X_k itself is rational. Thus all the X_j are rational, and by (iii) all areas of polygons with vertices among the X_j are rational. Note that in the particular case $n = 5$, this implies part (**a**).

For the converse, fix rational points $P_0, P_1, \ldots, P_{n-2}, P'_{n-1}$, and then choose P_{n-1} to be an irrational point on the line $P_1 P'_{n-1}$. In this case all the P_i are rational except P_{n-1}, and all the Q_j are rational except possibly Q_{n-2} and Q_{n-3}. Thus all the areas except possibly $\alpha_{n-3} = \gamma_{2n-6}$, $\beta_{n-3} = \gamma_{2n-5}$, $\alpha_{n-2} = \gamma_{2n-4}$, $\beta_{n-2} = \gamma_{2n-3}$, and $\alpha_{n-1} = \gamma_{2n-2}$ are rational.

Thus there are $2n - 5$ consecutive rational γ_j. There cannot be more than this since P_{n-1} is irrational. If either γ_{2n-6} or γ_{2n-2} were rational, then the argument above would imply P_{n-1} is rational, a contradiction.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), Ellington Management Problem Solving Group, and the proposers; case (a) also solved by J.-P. Grivaux (France), M. Tetiva (Romania), and GCHQ Problem Solving Group (U. K.).

Three Inequalities for Orthogonal Functions

11561 [2011, 276]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania.* Let $f_1, \ldots, f_n$ be continuous real valued functions on $[0, 1]$, none identically zero, such that $\int_0^1 f_i(x) f_j(x)\,dx = 0$ if $i \neq j$. Prove that

$$\prod_{k=1}^n \int_0^1 f_k^2(x)\,dx \geq n^n \left(\prod_{k=1}^n \int_0^1 f_k(x)\,dx \right)^2,$$

$$\sum_{k=1}^n \int_0^1 f_k^2(x)\,dx \geq \left(\sum_{k=1}^n \int_0^1 f_k(x)\,dx \right)^2, \quad \text{and}$$

$$\sum_{k=1}^n \frac{\int_0^1 f_k^2(x)\,dx}{\left(\int_0^1 f_k(x)\,dx \right)^2} \geq n^2.$$

Solution by Allen Stenger, Alamogordo, NM. For the second inequality, we may apply the Cauchy–Schwarz inequality:

$$\left(\sum_{k=1}^n \int_0^1 f_k(x)\,dx \right)^2 = \left(\int_0^1 \left(\sum_{k=1}^n f_k(x) \right) \cdot 1\,dx \right)^2 \leq \int_0^1 \left(\sum_{k=1}^n f_k(x) \right)^2 dx \cdot 1$$
$$= \sum_{k=1}^n \sum_{m=1}^n \int_0^1 f_k(x) f_m(x)\,dx = \sum_{k=1}^n \int_0^1 f_k^2(x)\,dx,$$

where we have used orthogonality in the last step.In the first and third inequalities, we may assume that $\int_0^1 f_k(x)\,dx \neq 0$ for all k.These two inequalities are homogeneous in each f_k, so without loss of generality we may assume $\int_0^1 f_k^2(x)\,dx = 1$ for all k.

Furthermore, possibly replacing f_k by $-f_k$,we may assume $\int_0^1 f_k(x)\,dx > 0$. For the first inequality, we apply the arithmetic mean–geometric mean inequality and use our second inequality already proved. We find that

$$\left(\prod_{k=1}^{n}\int_0^1 f_k(x)\,dx\right)^{1/n} \le \frac{1}{n}\sum_{k=1}^{n}\int_0^1 f_k(x)\,dx \le \frac{1}{n}\left(\sum_{k=1}^{n}\int_0^1 f_k^2(x)\,dx\right)^{1/2}$$

$$= \frac{1}{n}\left(\sum_{k=1}^{n} 1\right)^{1/2} = n^{-1/2} = n^{-1/2}\left(\prod_{k=1}^{n}\int_0^1 f_k^2(x)\,dx\right)^{1/(2n)}.$$

Raising the extremes here to the power $2n$ and rearranging yields the first inequality. For the third inequality, we apply the arithmetic mean–geometric mean inequality and our first inequality to obtain (abbreviating $\int_0^1 f_k(x)\,dx$ to $\int_0^1 f_k$)

$$\frac{n}{\sum_{k=1}^{n}\left(\int_0^1 f_k\right)^{-2}} \le \left(\prod_{k=1}^{n}\left(\int_0^1 f_k\right)^2\right)^{1/n} \le \frac{1}{n}\left(\prod_{k=1}^{n}\int_0^1 f_k^2\right)^{1/n} = \frac{1}{n}.$$

Using $n = n\int_0^1 f_k^2$ in the numerator on the left, we see that the resulting inequality is equivalent to our third inequality.

Also solved by K. F. Andersen (Canada), P. P. Dályay (Hungary), P. J. Fitzsimmons, D. Fleischman, O. Geupel (Germany), J. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, M. Omarjee (France), I. Pinelis, Á. Plaza & K. Sadarangani (Spain), K. Schilling, R. Stong, E. I. Verriest, J. Vinuesa (Spain), Ellington Management Problem Solving Group, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer. Partial solutions by M. W. Botsko, P. Bracken, N. Caro (Brazil), E. Hysnelaj & E. Bojaxhiu (Australia & Germany), and NSA Problems Group.

A Definite Integral

11564 [2011, 371]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Prove that

$$\int_0^\infty \frac{e^{-x}(1-e^{-6x})}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})}\,dx = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

Solution by K. D. Lathrop, Ridgway, CO. The desired integral is $I(1)$, where

$$I(a) = \int_0^\infty \frac{e^{-ax}(1-e^{-6x})\,dx}{x(1+e^{-2x}++e^{-4x}+e^{-6x}+e^{-8x})}.$$

Now $\sum_{n=0}^{4} e^{-2nx} = (1-e^{-10x})/(1-e^{-2x})$, so

$$\frac{dI}{da} = -\int_0^\infty e^{-ax}(1-e^{-6x})(1-e^{-2x})\sum_{n=0}^{\infty} e^{-10nx}\,dx.$$

Summation and integration may be interchanged and the integration over x performed to yield

$$\frac{dI}{da} = -\sum_{n=0}^{\infty}\left(\frac{1}{a+10n} - \frac{1}{a+2+10n} - \frac{1}{a+6+10n} + \frac{1}{a+8+10n}\right).$$

Let $\psi(z) = d\big(\log\Gamma(z)\big)/dz$. Note that ψ satisfies the identity

$$\psi(\alpha) - \psi(\beta) = -\sum_{n=0}^{\infty}\left(\frac{1}{n+\alpha} - \frac{1}{n+\beta}\right).$$

Therefore we have (first in general, and then specializing to $a = 1$)

$$I(a) = \log \frac{\Gamma(\frac{a}{10})\Gamma(\frac{a+8}{10})}{\Gamma(\frac{a+2}{10})\Gamma(\frac{a+6}{10})}, \qquad I(1) = \log \frac{\Gamma(\frac{1}{10})\Gamma(\frac{9}{10})}{\Gamma(\frac{3}{10})\Gamma(\frac{3}{10})}.$$

Exactly this combination of Γ-functions was mentioned in the solution of Problem 11426 (this MONTHLY **117** (2010) 842); it can for example be evaluated using the identity $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$. Thus $I(1) = \log \frac{\sin(3\pi/10)}{\sin(\pi/10)} = \log \frac{3+\sqrt{5}}{2}$.

Editorial comment. Several solvers found substitutions that convert this integral into ones found in integral tables. Integrals of this type were considered by Euler.

Also solved by T. Amdeberhan & V. H. Moll, R. Bagby, M. Bataille (France), D. Beckwith, A. Bostan (France), P. Bracken, R. Chapman (U. K.), H. Chen, S. de Luxán (Spain), A. Ercan (Turkey), L. Gérard (France), O. Geupel (Germany), M. L. Glasser, J. Grivaux (France), J. A. Grzesik, F. Holland (Ireland), O. Kouba (Syria), G. Lamb, K.-W. Lau (China), O. P. Lossers (Netherlands), M. Omarjee (France), J. Rosenberg, R. Stong, R. Tauraso (Italy), T. Trif (Romania), M. Vowe (Switzerland), H. Wang, H. Widmer (Switzerland), C. Y. Yıldırım (Turkey), Barclays Capital Problems Solving Group (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Square-Roots and Products of Uniform Random Variables

11565 [2011, 371]. *Proposed by Shai Covo, Kiryat-Ono, Israel.* Let $U_1, U_2, \ldots$ be independent random variables, each uniformly distributed on [0, 1].
(**a**) For $0 < x \le 1$, let N_x be the least n such that $\sum_{k=1}^{n} \sqrt{U_k} > x$. Find the expected value of N_x.
(**b**) For $0 < x \le 1$, let M_x be the least n such that $\prod_{k=1}^{n} U_k < x$. Find the expected value of M_x.

Solution by N. Bouzar, University of Indianapolis, Indianapolis, IN. In part (**a**) the random variables $\sqrt{U_1}, \sqrt{U_2}, \ldots$ are independent with common probability density function f given by $f(s) = 2s$ for $0 < s < 1$ (and 0 otherwise.) Let $S_n = \sum_{k=1}^{n} \sqrt{U_k}$ for $n \ge 1$. By the definition of N_x and the fact that the summands of S_n are nonnegative, we have

$$[N_x > n] = [S_1 \le x, \ldots, S_n \le x] = [S_n \le x], \qquad n \ge 1.$$

Noting that $N_x \ge 1$, we now have

$$E(N_x) = 1 + \sum_{n=1}^{\infty} P(N_x > n) = 1 + \sum_{n=1}^{\infty} P(S_n \le x).$$

From independence, we have

$$P(S_{n+1} \le x) = \int_0^x P(S_n \le x - s) 2s\, ds, \qquad n \ge 1$$

(W. Feller, *Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed., John Wiley and Sons, p. 6).

By induction we see that $P(S_n \le x) = (x\sqrt{2})^{2n}/(2n)!$. Therefore, $E(N_x) = \cosh(x\sqrt{2})$.

(**b**) Let $Y_k = -\ln U_k$, $k \ge 1$, so that $Y_1, Y_2, \ldots$ are independent and exponentially distributed with mean 1. Also note that M_x is the least n such that $\sum_{k=1}^{n} Y_k > -\ln x$.

Now let $S_n = \sum_{k=1}^n Y_k$. Proceeding as in (**a**), we obtain

$$E(M_x) = 1 + \sum_{n=1}^{\infty} P(S_n \le -\ln x).$$

Now S_n has a gamma distribution with probability density function g_n given by $g_n(t) = \frac{1}{(n-1)!} t^{n-1} e^{-t}$ (W. Feller, Vol. II, 2nd ed., p. 11). Thus

$$E(M_x) = 1 + \sum_{n=1}^{\infty} \int_0^{-\ln x} g_n(t)\, dt = 1 + \int_0^{-\ln x} e^{-t} \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}\, dt = 1 - \ln x.$$

Also solved by R. A. Agnew, R. Bagby, D. Beckwith, L. Bogdan (Canada), M. A. Carlton, N. Caro (Brazil), R. Chapman (U. K.), O. Geupel (Germany), D. Gove, N. Grivaux (France), J. A. Grzesik, S. J. Herschkorn, T. Le & S. Singh, J. H. Lindsey II, E. Omey & S. Van Gulck (Belgium), K. Schilling, J. Simons (U. K.), N. C. Singer, R. Stong, D. B. Tyler, Barclays Capital Problems Solving Group (U. K.), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

An Integral Kernel Inequality

11571 [2011, 372]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let f be a nonnegative Lebesgue-measurable function on [0, 1], with $\int_0^1 f(x)\,dx = 1$. Let $K(x, y) = (x-y)^2 f(x) f(y)$, $F(t) = \int_{[0,t]\times[0,t]} K(x, y)\,dy\,dx$, and $G(t) = \int_{[t,1]\times[t,1]} K(x, y)\,dy\,dx$. For $0 \le t \le 1$, prove that

$$\sqrt{F(t)} + \sqrt{G(t)} \le \sqrt{F(1)}.$$

Solution by Kenneth Schilling, Mathematics Department, Universityof Michigan-Flint, Flint, MI. Let X be a random variable on [0, 1] with density function f, and let $A = \{X \le t\}$, $A' = \{X > t\}$, and $p = P(A)$. Now

$$\begin{aligned} F(t) &= 2\int_0^t x^2 f(x)dx \cdot \int_0^t f(x)dx - 2\left(\int_0^t x f(x)dx\right)^2 \\ &= 2\mathrm{Var}(X|A)\cdot p^2 = 2\sigma_{X|A}^2 p^2. \end{aligned}$$

Similarly, $G(t) = 2\sigma_{X|A'}^2(1-p)^2$. Now

$$\frac{1}{\sqrt{2}}\left(\sqrt{F(t)} + \sqrt{G(t)}\right) = p\sigma_{X|A} + (1-p)\sigma_{X|A'} \le \sqrt{p\sigma_{X|A}^2 + (1-p)\sigma_{X|A'}^2},$$

because the arithmetic mean is always at most the corresponding root-mean-square. Also, for any square-integrable random variable Y, the quantity $E((Y-a)^2)$ achieves its minimum value of σ_Y^2 at $a = \mu_Y$. Hence

$$\begin{aligned} p\sigma_{X|A}^2 + (1-p)\sigma_{X|A'}^2 &= p\cdot E((X-\mu_{X|A})^2|A) + (1-p)\cdot E((X-\mu_{X|A'})^2|A') \\ &\le p\cdot E((X-\mu_X)^2|A) + (1-p)\cdot E((X-\mu_X)^2|A') \\ &= E((X-\mu_X)^2) = \sigma_X^2 = \frac{1}{2}F(1). \end{aligned}$$

Combining this with the previous bound gives the desired result.

Also solved by R. Bagby, P. P. Dályay (Hungary), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, M. Omarjee (France), I. Pinelis, A. Sen (Canada), J. L. Shomberg, J. Simons (U. K.), R. Stong, Barclays Capital Problems Solving Group (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before June 30, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11691. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Show that the $2n$th moment $\int_0^\infty x^{2n} f(x)\,dx$ of the function f given by

$$f(x) = \frac{d}{dx} \arctan\left(\frac{\sinh x}{\cos x}\right)$$

is zero when n is an odd positive integer.

11692. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Ştefan Spătaru, International Computer High School of Bucharest, Bucharest, Romania.* Let a_1, a_2, a_3, a_4 be real numbers in $(0, 1)$, with $a_4 = a_1$. Show that

$$\frac{3}{1 - a_1a_2a_3} + \sum_{k=1}^{3} \frac{1}{1 - a_k^3} \geq \sum_{k=1}^{3} \frac{1}{1 - a_k^2 a_{k+1}} + \frac{1}{1 - a_k a_{k+1}^2}.$$

11693. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA, and Richard Stong, CCR, San Diego CA.* Let T be an equilateral triangle inscribed in the d-dimensional unit cube $[0, 1]^d$, with $d \geq 2$. As a function of d, what is the maximum possible side length of T?

11694. *Proposed by Kent Holing, Trondheim, Norway.* Let $g(x) = x^4 + ax^3 + bx^2 + ax + 1$, where a and b are rational. Suppose g is irreducible over $\mathbb{Q}$. Let G be the Galois group of g. Let $\mathbb{Z}_4$ denote the additive group of the integers mod 4, and let D_4 be the dihedral group of order 8. Let $\alpha = (b + 2)^2 - 4a^2$ and $\beta = a^2 - 4b + 8$.
(**a**) Show that G is isomorphic to one of $\mathbb{Z}_4$ or D_4 if and only if neither α nor β is the square of a rational number, and G is cyclic exactly when $\alpha\beta$ is the square of a rational number.

http://dx.doi.org/10.4169/amer.math.monthly.120.02.174

(**b**) Suppose neither α nor β is square, but $\alpha\beta$ is. Let r be one of the roots of g. (Trivially, $1/r$ is also a root.) Let $s = \sqrt{\alpha\beta}$, and let

$$t = ((s + (b-6)a)r^3 + (as + (b-8)a^2 + 4(b+2))r^2 + \\ ((b-1)s + (b^2 - b + 2)a - 2a^3)r + 2(b+2)b - 6a^2)/(2s).$$

Show that $t \in \mathbb{Q}[r]$ is one of the other two roots of g. Comment on the special case $a = b = 1$.

11695. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* The *Stirling numbers of the first kind*, denoted $s(n, k)$, can be defined by their generating function: $z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n, k)z^k$. Let m and p be nonnegative integers with $m > p - 4$. Prove that

$$\int_0^1 \int_0^1 \frac{\log x \cdot \log^m(xy) \cdot \log y}{(1-xy)^p}\, dx\, dy$$
$$= \begin{cases} (-1)^m \frac{1}{6}(m+3)!\zeta(m+4), & \text{if } p = 1; \\ (-1)^{m+p-1} \frac{(m+3)!}{6(p-1)!} \sum_{k=1}^{p-1} (-1)^k s(p-1, k)\zeta(m+4-k) & \text{if } p > 1. \end{cases}$$

11696. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia, and Elton Bojaxhiu, Kriftel, Germany.* Let T be a triangle with sides of length a, b, c, inradius r, circumradius R, and semiperimeter p. Show that

$$\frac{1}{2(r^2 + 4Rr)} + \frac{1}{9}\sum_{\text{cyc}} \frac{1}{c(p-c)} \geq \frac{4}{9}\sum_{\text{cyc}} \left(\frac{1}{9Rr - c(p-c)}\right).$$

11697. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let n and q be integers, with $2n > q \geq 1$. Let

$$f(t) = \int_{\mathbb{R}^q} \frac{e^{-t(x_1^{2n} + \cdot + x_q^{2n})}}{1 + x_1^{2n} + \cdots x_q^{2n}}\, dx_1 \cdots dx_q.$$

Prove that $\lim_{t\to\infty} t^{q/2n} f(t) = n^{-q}(\Gamma(1/2n))^q$.

SOLUTIONS

Some Inequalities for Triangles

11569 [2011, 372]. *Proposed by M. H. Mehrabi, Nahavand, Iran.* Let a, b, and c be the lengths of the sides of a triangle, and let s, r, and R be the semi-perimeter, inradius, and circumradius, respectively, of that triangle. Show that

$$2 < \log\left(\frac{(a+b)(b+c)(c+a)}{abc}\right) < (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 6$$

and

$$8\left(\frac{r}{p}\right) < \log\left(\frac{b+c}{a}\right)\log\left(\frac{c+a}{b}\right)\log\left(\frac{a+b}{c}\right) < \frac{2r}{R}.$$

Solution by Oliver Geupel, Brühl, NRW, Germany. In fact, the first two inequalities hold for any three positive numbers, even if they are not the sides of a triangle. The first inequality can be sharpened: for positive a, b, c we have

$$8 \le \frac{(a+b)(b+c)(c+a)}{abc}. \tag{1}$$

Indeed, by the Arithmetic Mean-Geometric Mean Inequality,

$$8 = \frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}}{abc} \le \frac{(a+b)(b+c)(c+a)}{abc}.$$

The second inequality can also be sharpened: for positive a, b, c we have

$$\log\left[\frac{(a+b)(b+c)(c+a)}{abc}\right] < (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \log 8 - 9.$$

In fact, let $u = (a+b)(b+c)(c+a)/(abc) = (a+b+c)(1/a+1/b+1/c) - 1$. By (1),

$$8 - \log 8 \le u - \log u = (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 1 - \log u.$$

For the second pair of inequalities, we will use three different formulas for the area F of the triangle: Heron's formula $F^2 = s(s-a)(s-b)(s-c)$, $F = rs$, and$F = abc/(4R)$. Note that for $0 < x < 1$ we have $2x < \log((1+x)/(1-x))$.

Write $x = (s-a)/s$ to get $2(s-a)/s < \log((b+c)/a)$. Let $a' = b+c$, $b' = c+a$, $c' = a+b$. Now

$$8\left(\frac{r}{s}\right)^2 = \frac{8F^2}{s^4} = \frac{8(s-a)(s-b)(s-c)}{s^3} < \log\left(\frac{a'}{a}\right)\log\left(\frac{b'}{b}\right)\log\left(\frac{c'}{c}\right).$$

Finally, for positive real x we have $\log(1+x) < x$.

Write $x = 2(s-a)/a$ to get $2(s-a)/a > \log((b+c)/a)$. Hence,

$$\log\left(\frac{a'}{a}\right)\log\left(\frac{b'}{b}\right)\log\left(\frac{c'}{c}\right) < \frac{8(s-a)(s-b)(s-c)}{abc} = \frac{2F}{s}\cdot\frac{4F}{abc} = \frac{2r}{R}.$$

Also solved by M. Bataille (France), P. P. Dályay (Hungary), D. Fleischman, E. Hysnelaj & E. Bojaxhiu (Australia & Germany), O. Kouba (Syria), J. Minkus, R. Stong, Z. Vörös (Hungary), M. Vowe (Switzerland), J. B. Zacharias, GCHQ Problem Solving Group (U.K.), and the proposer.

Euclidean Construction

11572 [2011, 463]. *Proposed by Sam Sakmar, University of South Florida, Tampa, FL.* Given a circle C and two points A and B outside C, give a Euclidean construction to find a point P on C such that if Q and S are the second intersections with C of AP and BP respectively, then QS is perpendicular to AB. (Special configurations, including the case that A, B, and the center of C are collinear, are excluded.)

Solution by Robert A. Russel, New York, NY. Construct the circle C' through A and B orthogonal to C. Let P_1 and P_2 be the two intersections of C and C'. We claim that P_1 and P_2 each satisfy the conditions of the problem. The circle C' is the circumcircle of $\triangle ABA'$, where A' is the inverse of A with respect to C. On the other hand, from the Inscribed Angle Theorem and the fact that an angle formed by a chord and a tangent is equal to the inscribed angle, we deduce that the circumcircle C' intersects C at the same angle as the angle between the lines AB and QS.

Editorial comment. Several solvers noted that A and B need not be outside C.

Also solved by R. Bagby, M. Bataille (France), C. T. R. Conley, P. P. Dályay (Hungary), O. Geupel (Hungary), M. Goldenberg & M. Kaplan, O. Kouba (Syria), J. H. Lindsey II, J. McHugh, R. Murgatroyd, R. Stong, and the proposer.

A Three-Variable Inequality

11575 [2011, 463]. *Proposed by Tuan Le (student), Worcester Polytechnic Institute, Worcester, MA.* Prove that if a, b, and c are positive, then

$$\frac{16}{27}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3+\left(\frac{abc}{(a+b)(b+c)(c+a)}\right)^{1/3}\geq\frac{5}{2}.$$

Solution by Michel Bataille, Rouen, France. The required inequality has the form $\frac{16}{27}A^3+B^{1/3}\geq\frac{5}{2}$. Note that $(a+b)(b+c)(c+a)\geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}=8abc>0$. Thus, $0<B^{1/3}\leq 1/2$. Also,

$$A=\frac{a^3+b^3+c^3}{(a+b)(b+c)(c+a)}+1+B\geq 2(1-2B),$$

where we have used $a^3+b^3+c^3\geq(a+b)(b+c)(c+a)-5abc$, a rewriting of Schur's inequality $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)\geq 0$. As a result, it suffices to prove that $\phi(x)\geq 5/2$ for any $x\in(0,1/2]$, where ϕ is given by $\phi(x)=\frac{128}{27}(1-2x^3)^3+x$. Compute

$$\phi'(x)=1-\frac{256x^2}{3}(1-2x^3)^2,\qquad \phi''(x)=-\frac{512x}{3}(1-2x^3)(1-8x^3).$$

Since $\phi''(x)<0$ for $x\in(0,1/2)$, the function ϕ' decreases from $\phi'(0)=1$ to $\phi'(1/2)<0$. Thus there exists $\alpha\in(0,1/2)$ such that ϕ is increasing on $[0,\alpha]$ and decreasing on $[\alpha,1/2]$. Therefore $\min\{\phi(x)\colon x\in[0,1/2]\}=\min\big(\phi(0),\phi(1/2)\big)=5/2$, as required.

Also solved by G. Apostolopoulos (Greece), D. Beckwith, E. Braune (Austria), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), E. A. Herman, F. Holland (Ireland), E. Hysnelaj & E. Bojaxhiu (Australia & Germany), S. Kaczkowski, K.-W. Lau (China), J. H. Lee (Korea), J. H. Lindsey II, J. Loverde, P. Perfetti (Italy), E. A. Smith, R. Stong, E. I. Verriest, M. Vowe (Switzerland), H. Wang & J. Wojdylo, T. R. Wilkerson, J. Zacharias, GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, and the proposer.

An unusual functional equation

11578 [2011, 464]. *Proposed by Roger Cuculière, Clichy la Garenne, France.* Let E be a real normed vector space of dimension at least 2. Let f be a mapping from E to E, bounded on the unit sphere $\{x\in E\colon\ \|x\|=1\}$, such that whenever x and y are in E, $f(x+f(y))=f(x)+y$. Prove that f is a continuous, linear involution on E.

Solution by Nicholás Caro, Universidade Federal dePernambuco, Recife, Brazil. We have $f(f(z))=f(0)+z$ for all z, so $f(z)=0$ implies $z=0$. Taking $z=-f(0)$, we obtain $f(f(z))=0$, and hence $z=0$. Thus $f(0)=0$ and $f(f(z))=z$ for all z. Hence for all x and z we have $f(x+z)=f(x+f(f(z)))=f(x)+f(z)$. Therefore f is additive and hence is a $\mathbb{Q}$-linear involution on E.

Let $M > 0$ be such that $f(z) \le M$ for all $z \in E$ with$\|z\| = 1$. By $\mathbb{Q}$-linearity, for any $x \ne 0$ in E with $\|x\| \in \mathbb{Q}$, we have $\|f(x)\| \le M\|x\|$. For each $x \ne 0$in E, let $z \in E$ be a vector with x and z being $\mathbb{R}$-linearly independent. For every rational r with $0 < r < \|x\|$, consider the map $P_r : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}$ given by

$$P_r(a, b) = \left\| x + r\frac{ax + bz}{\|ax + bz\|} \right\|.$$

Since P_r is continuous, $\mathrm{Im}(P_r)$ is an interval in $\mathbb{R}$ containing the points $\|x\| \pm r$ (corresponding to $b = 0$ and $a = \pm 1$). Hence there exists $y_r = x + r(a_r x + b_r z)/\|a_r x + b_r z\| \in E$ such that $\|x\| - r < \|y_r\| < \|x\| + r$ and $\|y_r\|$ is rational.

Since we also have $\|x - y_r\| = r$ rational, it follows that $\|f(x)\| \le \|f(y_r)\| + \|f(x - y_r)\| \le M(\|y_r\| + r) \le M(\|x\| + 2r)$. Letting r tend to zero, we have $\|f(x)\| \le M\|x\|$ for all $x \in E$.

Thus f is Lipschitz continuous, which implies $\mathbb{R}$-linearity. Indeed, if $r_n \to r$ with r_n rational, then $f(r_n x) = r_n f(x) \to f(rx)$, so $f(rx) = rf(x)$.

Also solved by R. Bagby, J. Boersema, C. Burnette, T. Castro & M. Velasquez (Colombia), R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), P. J. Fitzsimmons, R. Ger (Poland), N. Grivaux (France), E. A. Herman, J. C. Kieffer, O. Kouba (Syria), O. P. Lossers (Netherlands), R. Mortini (France), M. Omarjee (France), K. Schilling, J. Simons (U. K.), R. Stong, M. Tetiva (Romania), and the proposer.

Optimally Nested Regular Polygons

11579 [2011, 557]. *Proposed by Hallard Croft, University of Cambridge, Cambridge, U. K., and Sateesh Mane, Convergent Computing, Shoreham, NY.* Let m and n be integers, with $m, n \ge 3$. Let B be a fixed regular n-gon, and let A be the largest regular m-gon that does not extend beyond B. Let $d = \gcd(m, n)$, and assume $d > 1$. Show the following:

(**a**) A and B are concentric;

(**b**) If $m \mid n$, then A and B have m points of contact, these being the vertices of A;

(**c**) If $m \nmid n$ and $n \nmid m$, then A and B have $2d$ points of contact;

(**d**) A and B share exactly d common axes of symmetry.

Editorial comment. The proofs can be found in the paper: S. J. Dilworth & S. R. Mane, "On a problem of Croft on optimally nested regular polygons." *Journal of Geometry* **99** (2010) 43–66. Claim (**a**) is Proposition 4.2 in the paper; statements (**b**) and (**c**) follow from Corollary 4.7; part (**d**) is Corollary 4.6.

Also solved by O. Geupel (Germany), R. Simon (Chile), J. Simons (U. K.), R. Stong, Ellington Management Problem Solving Group, and the proposers.

That's Sum Minimum!

11580 [2011, 557]. *Proposed by David Alfaya Sánchez, Universidad Autónoma de Madrid, Madrid, Spain, and José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.* For $n \ge 2$, let $a_1, \ldots, a_n$ be positive numbers that sum to 1, let $E = \{1, \ldots, n\}$, and let $F = \{(i, j) \in E \times E\colon\ i < j\}$. Prove that

$$\sum_{(i,j)\in F} \frac{(a_i - a_j)^2 + 2a_i a_j(1 - a_i)(1 - a_j)}{(1 - a_i)^2(1 - a_j)^2} + \sum_{i\in E} \frac{(n + 1)a_i^2 + na_i}{(1 - a_i)^2} \ge \frac{n^2(n + 2)}{(n - 1)^2}.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. A reshuffling of the terms on the left side leads to

$$\sum_{i<j} \frac{(a_i - a_j)^2}{(1 - a_i)^2(1 - a_j)^2} + 2\sum_{i<j} \frac{a_i a_j}{(1 - a_i)(1 - a_j)} + \sum \frac{a_i^2}{(1 - a_i)^2} + n\sum \frac{a_i^2 + a_i}{(1 - a_i)^2}.$$

The first sum is nonnegative; we leave it out completely. The rest can be simplified to

$$\left(\sum \frac{a_i}{1-a_i}\right)^2 + n \sum \frac{a_i + a_i^2}{(1-a_i)^2}.$$

The functions $x/(1-x)$ and $(x+x^2)/(1-x)^2$ are products of positive, increasing, convex functions, so they are convex. Under the assumption that $\sum a_i = 1$, these sums attain their minimum for a_i all equal, that is $a_i = 1/n$. It follows that the minimum is $\frac{n^2}{(n-1)^2} + n^2 \frac{n+1}{(n-1)^2}$.

Also solved by M. Bataille (France), M. A. Carlton, M. Cipu (Romania), P. P. Dályay (Hungary), D. Fleischman, M. Goldenberg & M. Kaplan, E. Hysnelaj & E. Bojaxhiu (Australia & Germany), D.-H. Kim (Korea), O. Kouba (Syria), J. C. Linders (Netherlands), R. Stong, Z. Vörös (Hungary), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

Inequality for an Integral

11581 [2011, 557]. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous, nonconstant function from [0, 1] to $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = 0$. Also, let $m = \min_{0\le x\le 1} f(x)$ and $M = \max_{0\le x\le 1} f(x)$. Prove that

$$\left|\int_0^1 x f(x)\,dx\right| \le \frac{-mM}{2(M-m)}.$$

Solution by Katie Elliott (student), Westmont College, Santa Barbara, CA. Since f is nonconstant and integrates to zero, $m < 0 < M$. Multiplying f by a scalar, we may assume $M - m = 1$ and $\int_0^1 x f(x)\,dx > 0$. Consider the function F defined by

$$F(x) = \begin{cases} m & 0 \le x \le M \\ M & M < x \le 1 \end{cases}$$

and note that $\int_0^1 F(x)\,dx = 0$. Using $-m = 1 - M$, we get

$$\int_0^1 x F(x)\,dx = \int_0^M mx\,dx + \int_M^1 Mx\,dx = \frac{-mM}{2(M-m)}.$$

Now we show that $\int_0^1 x F(x)\,dx$ is an upper bound for $\int_0^1 x f(x)\,dx$; equivalently, that $\int_0^1 x(f(x) - F(x))dx \le 0$. Since $f(x) \ge F(x)$ on $[0, M]$ and $F(x) \ge f(x)$ on $(M, 1]$, it follows that

$$\begin{aligned}\int_0^1 x(f(x)-F(x))dx &= \int_0^M x(f(x)-F(x))dx + \int_M^1 x(f(x)-F(x))dx \\ &\le M\int_0^M (f(x)-F(x))dx + M\int_M^1 (f(x)-F(x))dx \\ &= M\int_0^1 (f(x)-F(x))dx = 0.\end{aligned}$$

Therefore

$$\left|\int_0^1 x f(x)\,dx\right| \le \int_0^1 x F(x)\,dx = \frac{-mM}{2(M-m)}.$$

Also solved by U. Abel (Germany), R. Bagby, R. Chapman (U. K.), D. Constales (Belgium), W. J. Cowieson, P. P. Dályay (Hungary), N. Eldredge, P. J. Fitzsimmons, O. Geupel (Germany), W. R. Green, E. A. Herman, J. C. Kieffer, O. Kouba (Syria), K.-W. Lau (China), J. C. Linders (Netherlands), J. H. Lindsey II, O. P. Lossers (Netherlands), P. R. Mercer, J. Noël (France), J. M. Pacheco, Á. Plaza & K. Sadarangani (Spain), P. Perfetti (Italy), I. Pinelis, P. Rodríguez-Chavez (Mexico), K. Schilling, J. Simons (U. K.), A. Stenger, R. Stong, M. Tetiva (Romania), M. Wildon (U. K.), W. C. Yuan (Singapore), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Line and a Triangle Generate Three Convergent Point Sequences

11586 [2011, 653]. *Proposed by Takis Konstantopoulos, Uppsala University, Uppsala, Sweden.* Let A_0, B_0, and C_0 be noncollinear points in the plane. Let p be a line that meets lines B_0C_0, C_0A_0, and A_0B_0 at A^*, B^*, and C^* respectively. For $n \geq 1$, let A_n be the intersection of B^*B_{n-1} with C^*C_{n-1}, and define B_n and C_n similarly. Show that all three sequences converge, and describe their respective limits.

Editorial comment. As pointed out by several readers, we must assume that p does not pass through any of the three initial points. Also, this must be viewed in the projective plane, since it may happen that a point in one of the sequences is defined by the intersection of two parallel lines.

Solution I by George Apostopoulos, Messolonghi, Greece. Let ϕ be a projective transformation taking line p to the line at infinity. Since points A_0, B_0, and C_0 are noncollinear, so are their images under ϕ. Since A^*, B_0, C_0 are collinear, so are their images, which is to say that line $\phi(B_0)\phi(C_0)$ is parallel to the direction of $\phi(A^*)$, a point at infinity. Similarly for the other two cases. The image of line A^*A_0 is the line through $\phi(A_0)$ parallel to the direction of $\phi(A^*)$, that is, parallel to $\phi(B_0)\phi(C_0)$, and similarly for the other two cases. Therefore, triangle $\phi(A_0)\phi(B_0)\phi(C_0)$ is the median triangle of triangle $\phi(A_1)\phi(B_1)\phi(C_1)$. By induction, for every n, triangle $\phi(A_n)\phi(B_n)\phi(C_n)$ is the median triangle of triangle $\phi(A_{n+1})\phi(B_{n+1})\phi(C_{n+1})$.

Thus all $\phi(A_n)$ are collinear, and—since the triangles increase regularly in size—$\lim_{n\to\infty} \phi(A_n)$ is the infinite point of line $\phi(A_0)\phi(A_1)$, which is the intersection of line $\phi(A_0)\phi(A_1)$ and the line at infinity $\phi(A^*)\phi(B^*)\phi(C^*)$. Removing the transformation ϕ, we see that $\lim_{n\to\infty} A_n$ is the intersection of line A_0A_1 and line $A^*B^*C^* = p$. Similarly for $\lim_{n\to\infty} B_n$ and $\lim_{n\to\infty} C_n$.

Solution II by Á. Montesdeoca, Univ. de la Luna, Spain, and Á. Plaza, Univ. de Las Palmas de Gran Canaria, Spain. Corresponding sides of triangles $A_0B_0C_0$ and $A_1B_1C_1$ meet at points on line p, so by Desargues's Theorem the two triangles are perspective from a point, say P. By induction, all subsequent triangles $A_nB_nC_n$ are also mutually perspective from this same point P. Thus all the points A_n are collinear, all B_n are collinear, and all C_n are collinear. We claim that the limits of the respective sequences are the intersection points with p of the lines through the sequences. We show this for sequence $\{A_n\}$.

We use homogeneous coordinates $(x : y : z)$ with $A_0B_0C_0$ as reference triangle, so that $A_0 = (1 : 0 : 0)$, $B_0 = (0 : 1 : 0)$, and $C_0 = (0 : 0 : 1)$. We write UV for the line through U and V. Let the equation of p be $qx + ry + sz = 0$ for some choice of coefficients q, r, s. Then p intersects B_0C_0 $(x = 0)$ at $A^* = (0 : -s : r)$. It intersects C_0A_0 $(y = 0)$ at $B^* = (s : 0 : -q)$. It intersects A_0B_0 $(z = 0)$ at $C^* = (-r : q : 0)$. Since point P is the intersection of A_0A_1 $(ry = sz)$ and C_0C_1 $(qx = ry)$, we have $P = (1/q : 1/r : 1/s)$. Therefore $A_1 = (1/q : -1/r : -1/s) = (rs : -sq : -qr)$, $B_1 = (-rs : sq : -qr)$, and $C_1 = (-rs : -sq : qr)$.

Since A_2 is the intersection of B^*B_1 $(qx + 2ry + sz = 0)$ with C^*C_1 $(qx + ry + 2sz = 0)$, it follows that $A_2 = (3rs : -sq : -qr)$, and similarly for B_2 and C_2. The next points are $A_3 = (5rs : -3sq : -3qr)$, $A_4 = (11rs : -5sq : -5qr)$, and in general $A_n = (a_n rs : -a_{n-1}sq : -a_{n-1}qr) = ((a_n/a_{n-1})rs : -sq : -qr)$, where $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_n = 2a_{n-1} + (-1)^n$, the Jacobsthal sequence. Take the limit as $n \to \infty$ to find (since $a_n/a_{n-1} \to 2$) that $A_n \to A^* = (2rs : -sq : -qr)$, so A^* is the intersection of p with the line A_0P ($ry = sz$, same as A_0A_1), since its coordinates satisfy both equations.

Also solved by R. Chapman (U. K.), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), A. Habil (Syria), M. E. Kidwell & M. D. Meyerson, L. R. King, O. Kouba (Syria), J. C. Linders (Netherlands), O. P. Lossers (Netherlands), J. Minkus, R. Stong, GCHQ Problem Solving Group (U. K.), University of Louisiana at Lafayette Math Club, and the proposer.

Where Are the Zeros?

11589 [2011, 653]. *Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.* Let P be a polynomial over $\mathbb{R}$ given by $P(x) = x^3 + a_2x^2 + a_1x + a_0$, with $a_1 > 0$. Show that P has a least one zero between $-a_0/a_1$ and $-a_2$.

Solution by William J. Cowieson, Fullerton College, Fullerton, CA. Note that $P(-a_0/a_1) = (a_0^2/a_1^3)(a_1a_2 - a_0)$ and $P(-a_2) = a_0 - a_1a_0$, so that

$$P(-a_0/a_1) = -(a_0^2/a_1^3)P(-a_2). \tag{1}$$

There are three cases.

(1) If $a_0 - a_1a_2 = 0$, then the interval reduces to a single point, and that point is a zero of P.

(2) If $a_0 = 0$, then $P(x) = x(x^2 + a_2x + a_0)$ has zeros at 0 and at $(-a_2 \pm \sqrt{a_2^2 - 4a_1})/2$. If $a_2^2 - 4a_1 < 0$, then 0 is the only real zero of P. Otherwise, $(-a_2 \pm \sqrt{a_2^2 - 4a_1})/2$ are both strictly between $-a_0/a_1 = 0$ and $-a_2$, since $a_1 > 0$.

(3) Both $a_0 - a_1a_2 \neq 0$ and $a_0 \neq 0$. In this case, from (1) we see that $P(-a_0/a_1)$ and $P(-a_2)$ are nonzero and of opposite sign when $a_1 > 0$. Hence the Intermediate Value Theorem implies that there is a zero between $-a_0/a_1$ and $-a_2$.

Also solved by B. K. Agarwal (India), G. Apostolopoulos (Greece), S. J. Baek & D.-H. Kim (Korea), B. D. Beasley, M. W. Botsko, D. Brown & J. Zerger, V. Bucaj, P. Budney, H. Caerols (Chile), E. M. Campbell & D. T. Bailey, M. Can, M. A. Carlton, T. Castro, J. Montero & A. Murcia (Colombia), R. Chapman (U. K.), H. Chen, W. ChengYuan (Singapore), J. Christopher, D. Constales (Belgium), W. J. Cowieson, C. Curtis, P. P. Dályay (Hungary), C. Degenkolb, C. R. Diminnie, K. Farwell, J. Ferdinands, D. Fleischman, V. V. García (Spain), O. Geupel (Germany), W. R. Green & T. D. Lesaulnier, J.-P. Grivaux (France), M. Hajja (Jordan), E. A. Herman, G. A. Heuer, S. Kaczkowski, B. Kalantari, B. Karaivanov, T. Keller, L. Kennedy, J. C. Kieffer, O. Kouba (Syria), P. T. Krasopoulos (Greece), R. Lampe, K.-W. Lau (China), J. C. Linders (Netherlands), J. H. Lindsey II, O. López (Colombia), O.P. Lossers (Netherlands), J. Loverde, Y.-H. McDowell & F. Mawyer, F. B. Miles, S. Mosiman, K. Muthuvel, M. Omarjee (France), Á. Plaza & K. Sadarangani (Spain), P. Pongsriiam & T. Pongsriiam (U. S. A. & Thailand), V. Ponomarenko, C. R. Pranesachar (India), R. E. Prather, R. Pratt, D. Ritter, A. J. Rosenthal, U. Schneider (Switzerland), C. R. Selvaraj & S. Selvaraj, A. K. Shafie & S. Gholami (Iran), J. Simons (U. K.), N. C. Singer, E. A. Smith, N. Stanciu & T. Zvonaru (Romania), J. H. Steelman, A. Stenger, R. Stong, M. Tetiva (Romania), N. Thornber, V. Tuck & A. Stancu, D. B. Tyler, D. Vacaru (Romania), E. I. Verriest, J. Vinuesa (Spain), T. Viteam (Germany), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), R. Wieler, S. V. Witt, N. Youngberg, J. Zacharias, Z. Zhang, Fejéntaláltuka Szeged Problem Solving Group (Hungary), GCHQ Problem Solving Group (U. K.), University of Louisiana at Lafayette Math Club, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before September 30, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11712. *Proposed by Daniel W. Cranston, Virginia Commonwealth University, Richmond, Va, and Douglas B. West, Zhejiang Normal University, Jinhua, China, and University of Illinois, Urbana, IL.* In the game of *Bulgarian solitaire*, n identical coins are distributed into two piles, and a *move* takes one coin from each existing pile to form a new pile. Beginning with a single pile of size n, how many moves are needed to reach a position on a cycle (a position that will eventually repeat)? For example, $5 \to 41 \to 32 \to 221 \to 311 \to 32$, so the answer is 2 when $n = 5$.

11713. *Proposed by Mihaly Bencze, Brasov, Romania.* Let $x_1, \dots, x_n$ be nonnegative real numbers. Let $S = \sum_{k=1}^{n} x_k$. Prove that

$$\prod_{k=1}^{n}(1 + x_k) \le 1 + \sum_{k=1}^{n} \left(1 - \frac{k}{2n}\right)^{k-1} \frac{S^k}{k!}.$$

11714. *Proposed by Nicuşor Minculete, "Dimitrie Cantenemir" University, Braşov, Romania, and Cătălin Barbu, "Vasile Alecsandri" National College, Bacău, Romania.* Let $ABCD$ be a cyclic quadrilateral (the four vertices lie on a circle). Let $e = |AC|$ and $f = |BD|$. Let r_a be the inradius of BCD, and define r_b, r_c, and r_d similarly. Prove that $er_ar_c = fr_br_d$.

11715. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.* Prove that

$$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^5} = \frac{1}{2}\left(\frac{2^5-1}{2^5} \cdot \frac{3^5-1}{3^5}\zeta(5) + \frac{11}{8}\left(\frac{\pi}{3}\right)^5 \cdot \frac{1}{\sqrt{3}}\right).$$

http://dx.doi.org/10.4169/amer.math.monthly.120.06.569

11716. *Proposed by Oliver Knill, Harvard University, Cambridge, MA.* Let $\alpha = (\sqrt{5}-1)/2$. Let p_n and q_n be the numerator and denominator of the nth continued fraction convergent to α. (Thus, p_n is the nth Fibonacci number and $q_n = p_{n+1}$). Show that

$$\sqrt{5}\left(\alpha - \frac{p_n}{q_n}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{q_n^{2k+2} 5^k},$$

where C_k denotes the kth *Catalan number*, given by $C_k = \frac{(2k)!}{(k!(k+1)!)}$.

11717. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Given a circle c and line segment AB tangent to c at a point E that lies strictly between A and B, provide a compass and straightedge construction of the circle through A and B to which c is internally tangent.

11718. *Proposed by Arkady Alt, San Jose, CA.* Given positive real numbers $a_1, \ldots, a_n$ with $n \ge 2$, minimize $\sum_{i=1}^{n} x_i$ subject to the conditions that $x_1, \ldots, x_n$ are positive and that $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i x_i$.

SOLUTIONS

A Polygon Equation

11595 [2011, 747]. *Proposed by Victor K. Ohanyan, Yerevan, Armenia.* Let $P_1, \ldots, P_n$ be the vertices of a convex n-gon in the plane. Let Q be a point in the interior of the n-gon, and let $\mathbf{v}$ be a vector in the plane. Let $\mathbf{r}_i$ denote the vector QP_i, with length r_i. Let Q_i be the (radian) measure of the angle between $\mathbf{v}$ and $\mathbf{r}_i$, and let F_i and Y_i be, respectively, the clockwise and counterclockwise angles into which the interior angle at P_i of the polygon is divided by QP_i. Show that

$$\sum_{i=1}^{n} \frac{1}{r_i} \sin(Q_i)(\cot F_i + \cot Y_i) = 0.$$

Solution by O. P. Lossers, The Netherlands. We assume without loss of generality that $\mathbf{v}$ is a unit vector. Let $\mathbf{k}$ be a unit vector in three-space orthogonal to the plane of the polygon. Note that $\sin(Q_i)\,\mathbf{k} = \frac{1}{r_i}(\mathbf{r}_i \times \mathbf{v})$. We have

$$\cot F_i = \frac{\mathbf{r}_i \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times (\mathbf{r}_{i+1} - \mathbf{r}_i)\|} \quad \text{and} \quad \cot Y_i = \frac{\mathbf{r}_i \cdot (\mathbf{r}_{i-1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times (\mathbf{r}_{i-1} - \mathbf{r}_i)\|}$$

(subscripts are taken modulo n). Since $\mathbf{v}$ is arbitrary and $\mathbf{r}_i \times \mathbf{r}_i = \mathbf{0}$, we must prove that

$$\sum_{i=1}^{n} \left(\frac{\mathbf{r}_i \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times \mathbf{r}_{i+1}\|} + \frac{\mathbf{r}_i \cdot (\mathbf{r}_{i-1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times \mathbf{r}_{i-1}\|} \right) \frac{\mathbf{r}_i}{r_i^2} = \mathbf{0}.$$

For geometric reasons, the vector s_i defined by

$$\mathbf{s}_i = \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\|\mathbf{r}_i \times \mathbf{r}_{i+1}\|} + \frac{\mathbf{r}_{i-1} - \mathbf{r}_i}{\|\mathbf{r}_i \times \mathbf{r}_{i-1}\|}$$

is a multiple of $\mathbf{r}_i$; this may also be seen by computing $\mathbf{s}_i \times \mathbf{r}_i = \mathbf{k} - \mathbf{k} = \mathbf{0}$. It follows that $(\mathbf{r}_i \cdot \mathbf{s}_i)\mathbf{r}_i / r_i^2 = \mathbf{s}_i$, so our sum simplifies and telescopes to $\mathbf{0}$:

$$\sum_i \mathbf{s}_i = \sum_i \left(\frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\|\mathbf{r}_{i+1} \times \mathbf{r}_i\|} - \frac{\mathbf{r}_i - \mathbf{r}_{i-1}}{\|\mathbf{r}_i \times \mathbf{r}_{i-1}\|} \right) = \mathbf{0}.$$

Also solved by E. A. Herman, B. Karaivanov, Á. Plaza & J. Sánchez-Reyes (Spain), R. Stong, J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

An Isosceles Condition

11605 [2011, 847]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let s, R, and r be the semiperimeter, circumradius, and inradius of a triangle with sides of length a, b, and c. Show that

$$\frac{R-2r}{2R} \geq \sum \frac{\sqrt{(s-a)(s-b)}}{c} - 2\sum \frac{(s-c)\sqrt{(s-a)(s-b)}}{ab},$$

and determine when equality occurs. The sums are cyclic.

Solution by Borislav Karaivanov, University of South Carolina, Columbia, SC. Using formulas

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}},$$

we get

$$\begin{aligned}\frac{2r}{R} &= \frac{2\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} \cdot \frac{4\sqrt{s(s-a)(s-b)(s-c)}}{abc} \\ &= \frac{2\sqrt{(s-a)(s-b)}}{c} \cdot \frac{2\sqrt{(s-b)(s-c)}}{a} \cdot \frac{2\sqrt{(s-c)(s-a)}}{b}.\end{aligned}$$

Hence,

$$\begin{aligned}&2\left(\frac{R-2r}{2R} - \sum \frac{\sqrt{(s-a)(s-b)}}{c} + 2\sum \frac{(s-c)\sqrt{(s-a)(s-b)}}{ab}\right) \\ &= 1 - \frac{2\sqrt{(s-a)(s-b)}}{c} \cdot \frac{2\sqrt{(s-b)(s-c)}}{a} \cdot \frac{2\sqrt{(s-c)(s-a)}}{b} \\ &\qquad - \sum \frac{2\sqrt{(s-a)(s-b)}}{c} + \sum \frac{2\sqrt{(s-c)(s-a)}}{b} \cdot \frac{2\sqrt{(s-b)(s-c)}}{a} \\ &= 1 - ABC - (A+B+C) + (AB+BC+CA) = (1-A)(1-B)(1-C),\end{aligned}$$

where

$$A = \frac{2\sqrt{(s-b)(s-c)}}{a}, \quad B = \frac{2\sqrt{(s-c)(s-a)}}{b}, \quad C = \frac{2\sqrt{(s-a)(s-b)}}{c}.$$

We compute

$$A = 2\sqrt{\frac{(a-(b-c))(a+(b-c))}{4a^2}} = \sqrt{1 - \left(\frac{b-c}{a}\right)^2} \leq 1. \tag{1}$$

Similarly, $B \leq 1$ and $C \leq 1$. Therefore $(1 - A)(1 - B)(1 - C) \geq 0$, which proves the required inequality.

We claim that equality holds if and only if the triangle is isosceles. Of course, $(1 - A)(1 - B)(1 - C) = 0$ if and only if one of the three factors is zero. By (1) we have $1 - A = 0$ if and only if $b = c$. The other two cases are the same.

Also solved by G. Apostolopoulos (Greece), E. Braune (Austria), R. Chapman (U. K.), P. P. Dályay (Hungary), O. Faynshteyn (Germany), D. Fleischman, V. V. García (Spain), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. G. Heuver (Canada), I. J. Hwang (Korea), K.-W. Lau (China), J. Loverde, P. Nüesch (Switzerland), C. R. Pranesachar (India), R. Stong, Z. Vörös (Hungary), C. Y. Wu (Singapore), Z. Zhang, T. Zvonaru & N. Stanciu (Romania), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer..

Must Four Circles Really Meet That Way?

11607 [2011, 936]. *Proposed by Jeffrey C. Lagarias and Andrey Mischenko, University of Michigan, Ann Arbor, MI.* Let C_0, C_1, C_2, C_3, with subscripts taken modulo 4, be circles in the Euclidean plane.
(**a**) Given for $k \in \mathbb{Z}_4$ that C_k and C_{k+1} intersect with orthogonal tangents, and the interiors of C_k and C_{k+2} are disjoint, show that the four circles have a common point.
(**b**)* Does the same conclusion hold in hyperbolic and spherical geometry?

Composite solution by the editors. In plane geometry, the *power* of a point P with respect to a circle with center O and radius r is defined as $OP^2 - r^2$. If T is a point on the circle, and the tangent there passes through P, then—by the Pythagorean Theorem—the power of P reduces to PT^2. The *radical axis* of circles C_1 and C_2 with non-intersecting interiors is the set of points P in the plane for which the powers of P with respect to the two circles are equal, i.e., for which all four tangent segments are congruent. This set is a line, and it is perpendicular to the line through the centers of C_1 and C_2. A circle is orthogonal to both C_1 and C_2 if and only if its center lies on the radical axis of C_1 and C_2. If C_1 and C_2 are also disjoint, then all such mutually orthogonal circles intersect the segment joining C_1 and C_2 and do so at the same two distinct points. (For proofs see: H. W. Guggenheimer, *Plane Geometry and Its Groups;* D. Pedoe, *Circles: A Mathematical View;* H. S. M. Coxeter & S. L. Greitzer, *Geometry Revisited;* or R. A. Johnson, *Advanced Euclidean Geometry*.)

(**a**) We cannot have C_1 and C_3 disjoint, because then C_0 and C_2 would intersect twice, and so their interiors would intersect. But by hypothesis they do not.

If C_1 and C_3 touch at point P on the segment joining their centers, then the radical axis of C_1 and C_3 passes through the point P, as well as through the centers of C_0 and C_2. By an analogous argument, C_0 and C_2 must touch at the same point P, which lies therefore at the intersection of the segment joining the centers of C_0 and C_2 with that joining the centers of C_1 and C_3. This point P is common to all four circles.

(**b**) On the sphere, there is ambiguity with respect to the "interior" of a great circle. First, suppose that great circles are considered to be "circles" in the statement of the problem. If one is permitted to define either hemisphere defined by a great circle as its "interior", then the theorem of concern here is false. As a counterexample, let C_0 and C_2 be the Arctic and Antarctic Circles; let C_1 and C_3 be the great circle that defines 0° longitude, with the interior of C_1 the Eastern Hemisphere and the interior of C_3 the Western. Then there is no point in common to all four circles, but the conditions on meeting orthogonally and disjoint interiors are met.

Now suppose that great circles are not considered to be "circles" in spherical geometry, which may be reasonable as they are the "straight lines". If "circle" is taken to be a non-great circle, with its interior being the smaller of the two possibilities, then the theorem is true. This may be proved by performing a stereographic projection onto the Euclidean plane from any point Q not on or interior to any of the four circles. Such a projection takes circles to circles and is isogonal, so the spherical version of the theorem follows from the Euclidean version. (Why is there such a point Q? Since C_1 and C_3 are small circles with disjoint interiors, there is a great circle C with C_1 strictly on one side and C_3 strictly on the other. Interiors of each of C_0 and C_2 cover less than half of C. Hence there are points Q on C not on or interior to any of the four circles C_k.)

The theorem is also true in hyperbolic geometry. This is shown by considering the Poincaré disk model.

Also solved by R. Chapman (U. K.), C. Delorme (France), D. Gove, E. J. Ionascu, B. Karaivanov, M. E. Kidwell & M. D. Meyerson & D. Ruth & M. Wakefield, J. Minkus, J. Schaer (Canada), R. Stong, and D. B. Tyler; part (a) only by J-P. Grivaux (France), H. W. Guggenheimer, J. H. Lindsey II, H. S. Morse, H. Widmer, GCHQ Problem Solving Group (U. K.), and the proposers.

A Determinant of Derivatives and Powers

11608 [2011, 936]. *Proposed by D. Aharonov and U. Elias, Technion-Israel Institute of Technology, Haifa, Israel.* Let f and g be functions on $\mathbb{R}$ that are differentiable $n+m$ times, where n and m are integers with $n \geq 1$ and $m \geq 0$. Let $A(x)$ be the $(n+m) \times (n+m)$ matrix given by

$$A_{j,k}(x) = \begin{cases} (f^{k-1}(x))^{(j-1)}, & \text{if } 1 \leq j \leq n; \\ (g^{k-1}(x))^{(j-1-n)}, & \text{if } n < j \leq n+m. \end{cases}$$

Let $P = \prod_{r=1}^{n-1} r! \prod_{q=1}^{m-1} q!$. Prove that

$$\det A(x) = P f(x)^n g(x)^m [g(x) - f(x)]^{mn} f'(x)^{n(n-1)/2} g'(x)^{m(m-1)/2}.$$

Solution by Robin Chapman, University of Exeter, Exeter, U. K. There is a minor error in the statement of the problem. Either the factor $f(x)^n g(x)^m$ should be removed from the formula for $\det A(x)$, or $f^{k-1}(x)$ and $g^{k-1}(x)$ should be $f^k(x)$ and $g^k(x)$ in the definition of $A(x)$. Here the original definition of $A(x)$ is taken, and the corrected formula for $\det A(x)$ is proved.

Define a matrix $V(x,t)$ by

$$V_{j,k}(x,t) = \begin{cases} f(x + (j-1)t)^{k-1} & \text{if } 1 \leq j \leq n, \\ g(x + (j-n-1)t)^{k-1} & \text{if } n < j \leq n+m. \end{cases}$$

Since $V(x,t)$ is a Vandermonde matrix,

$$\det V(x,t) = \prod_{0 \leq j < k < n} (f(x+kt) - f(x+jt))$$

$$\cdot \prod_{0 \leq j < k < m} (g(x+kt) - g(x+jt)) \cdot \prod_{j=0}^{n-1} \prod_{k=0}^{m-1} (g(x+kt) - f(x+jt)).$$

For each positive integer r, define an $r \times r$ matrix $T^{(r)}$ by

$$T^{(r)}_{j,k} = \begin{cases} (-1)^{k-j} \binom{j-1}{k-1}, & \text{if } j \geq k \\ 0, & \text{if } j < k. \end{cases}$$

Now $T^{(r)}$ is lower triangular and $\det T^{(r)} = 1$. Let T be the diagonal sum of T_n and T_m. Note that $\det V(x,t) = \det B(x,t)$, where $B(x,t)$ is given by

$$B_{j,k}(x,t) = \begin{cases} \sum_{i=0}^{j-1}(-1)^{j-i-1}\binom{j-1}{i} f(x+it)^{k-1} & \text{if } 1 \le j \le n, \\ \sum_{i=0}^{j-n-1}(-1)^{j-n-1-i}\binom{j-n-1}{i} g(x+it)^{k-1} & \text{if } n < j \le n+m. \end{cases}$$

If the function h is differentiable s times, then

$$\lim_{t\to 0} t^{-s} \sum_{i=0}^{s} (-1)^{s-i}\binom{s}{i} h(x+it) = f^{(s)}(x)$$

so

$$\lim_{t\to 0} t^{-u} B(x,t) = A(x),$$

where $u = \sum_{s=0}^{n-1} s + \sum_{s=0}^{m-1} s = n(n-1)/2 + m(m-1)/2$. Therefore,

$$\begin{aligned}\det A(x) &= \lim_{t\to 0} t^{-u} \det B(x,t) = \lim_{t\to 0} t^{-u} \det V(x,t) \\ &= \prod_{0\le j<k<n} \lim_{t\to 0} \frac{f(x+kt)-f(x+jt)}{t} \\ &\quad \cdot \prod_{0\le j<k<n} \lim_{t\to 0} \frac{g(x+kt)-g(x+jt)}{t} \cdot (g(x)-f(x))^{mn} \\ &= \prod_{0\le j<k<n} (k-j) f'(x) \cdot \prod_{0\le j<k<m} (k-j) g'(x) \cdot (g(x)-f(x))^{mn} \\ &= P\, f'(x)^{n(n-1)/2} g'(x)^{m(m-1)/2} (g(x)-f(x))^{mn}.\end{aligned}$$

Editorial comment. All solvers noted the inaccuracy in the statement.

Also solved by P. P. Dályay (Hungary), J. Grivaux (France), B. Karaivanov, J. H. Smith, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposers.

An Infinite Product

11612 [2011, 937]. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Evaluate in closed form

$$\prod_{n=1}^{\infty} \left(\frac{n+z+1}{n+z}\right)^n e^{(2z-2n+1)/(2n)}.$$

Solution I by Charles Martin. The answer is $z\Gamma(z)e^{\gamma z+z+1}\sqrt{e^{\gamma}/(2\pi)}$, where γ is Euler's constant. Let z be a complex number other than a negative integer. Let

$$p_n = \prod_{k=1}^{n} \left(\frac{k+z+1}{k+z}\right)^n e^{(2z-2k+1)/(2k)}.$$

Now

$$\begin{aligned}p_n &= \left[\left(\frac{2+z}{1+z}\right)\left(\frac{3+z}{2+z}\right)^2\left(\frac{4+z}{3+z}\right)^3 \cdots \left(\frac{n+z+1}{n+z}\right)^n\right] \exp\left[\sum_{k=1}^{n} \frac{2z-2k+1}{2k}\right] \\ &= \left[\frac{(n+z+1)^n}{(1+z)(2+z)(3+z)\cdots(n+z)}\right] \exp\left[\left(z+\frac{1}{2}\right)H_n - n\right],\end{aligned}$$

where H_n is the nth harmonic number, $\sum_{k=1}^{n} 1/k$. On the other hand, we also have $H_n = \log n + \gamma + \epsilon_n$ with $\epsilon_n \to 0$, so

$$p_n = \prod_{k=1}^{n} \frac{n+z+1}{k+z} \cdot \exp\left[\log(n^z\sqrt{n}) + \gamma z + \frac{\gamma}{2} - n + \left(z+\frac{1}{2}\right)\epsilon_n\right]$$

$$= n^z \prod_{k=1}^{n} \frac{k}{k+z} \cdot \left[\frac{n^n\sqrt{2\pi n}}{n!e^n}\right] \cdot \left(1+\frac{z+1}{n}\right)^n \frac{e^{\gamma(z+1/2)}}{\sqrt{2\pi}} e^{\epsilon_n(z+1/2)}.$$

As $n \to \infty$ in the last line here, the product approches $z\Gamma(z)$ (this is Euler's original definition of the Gamma function); the bracketed expression approaches 1 by Stirling's approximation of $n!$, the factor $(1+(z+1)/n)^n$ approaches e^{z+1}, and the final exponential approaches 1. The claimed result follows.

Solution II by M. L. Glasser, Clarkson University, Potsdam, NY. We evaluate a variant of this product: if $0 < b < a$, then let

$$Q = \prod_{n=1}^{\infty} \left(\frac{n+a}{n+b}\right)^n \exp\left\{\frac{(a-b)}{2}\,\frac{(a-n)+(b-n)}{n}\right\}$$

$$= \frac{\Gamma(1+a)^a}{\Gamma(1+b)^b} e^{\gamma(a^2-b^2)/2} \exp\left[-\int_{1+b}^{1+a} \log\Gamma(t)\,dt\right], \tag{1}$$

where γ is Euler's constant. Setting $a = z+1$ and $b = z$ yields the original problem. The integral is (2.2.3(2)) in A. P. Prudnikov et. al., *Tables of Integrals and Series,* Vol. 2;

$$\int_{z+1}^{z+2} \log\Gamma(t)\,dt = (z+1)\log(z+1) - (z+1) + \log\sqrt{2\pi}.$$

Thus $Q = \Gamma(z+1)e^{(z+1/2)\gamma+z+1}/\sqrt{2\pi}$. When a and b are integers, the integral is *ibid.* (2.2.3(3)) , hence

$$\int_{b+1}^{a+1} \log\Gamma(t)\,dt = \sum_{k=b+1}^{a} k\log k + \frac{a-b}{2}\log(2\pi) - \frac{a(a+1)-b(b+1)}{2},$$

and

$$Q = \frac{\Gamma(1+a)^q\Gamma(1+b)^{-b}}{\sqrt{(2\pi)^{a-b}}\prod_{k=b+1}^{a} k^k} \exp\left\{\frac{1}{2}[(\gamma+1)(a^2-b^2)+(a-b)]\right\}.$$

Now we prove (1). Write

$$\log Q = \lim_{\eta\to 1^+} \sum_{n=1}^{\infty} \left\{n^\eta\left[\log\left(1+\frac{a}{n^\eta}\right) - \log\left(1+\frac{b}{n^\eta}\right)\right] - \frac{a-b}{n^{\eta-1}} + \frac{a^2-b^2}{2n^\eta}\right\}.$$

Replace the logarithms by their Taylor series, and note that the first two terms cancel the last two terms in the curly brackets, at which point the limit can be taken. Therefore,

$$Q = \sum_{n=1}^{\infty}\sum_{l=3}^{\infty} \frac{(-1)^{l+1}}{l\,n^{l-1}}(a^l-b^l) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k+1}\zeta(k)(a^{k+1}-b^{k+1}).$$

Here, the sum over n has been expressed as a Riemann zeta function and l has been replaced by $k+1$.

Now we apply the generating function *ibid.* (5.3.1(4))

$$\sum_{k=2}^{\infty} \zeta(k) t^k = -t\big[\gamma + \psi(1-t)\big],$$

where ψ is the logarithmic derivative of the Gamma function. Finally, integrate and put $z = -t$:

$$\sum_{k=2}^{\infty} \frac{(-1)^k z^{k+1}}{k+1} \zeta(k) = \frac{1}{2}\gamma z^2 + z \log\Gamma(1+z) - \int_0^{1+z} \log\Gamma(t)\, dt.$$

The value (1) follows.

Also solved by R. Bagby, B. S. Burdick, R. Chapman (U. K.), H. Chen, D. Constales (Belgium), D. Fleischman, O. Geupel (Germany), J. Grivaux (France), O. Kouba (Syria), O. P. Lossers (Netherlands), J. Magliano, M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), P. F. Refolio (Spain), N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), T. Trif (Romania), D. B. Tyler, J. Vinuesa (Spain), Z. Vörös (Hungary), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

Convergence of a Series

11614 [2012, 68]. *Proposed by Moubinool Omarjee, Lycée Jean-Lurçat, Paris, France.* Let α be a real number with $\alpha > 1$, and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty}(u_n - u_{n+1})/u_n^\alpha$ exists and is nonzero. Prove that $\sum_{n=1}^{\infty} u_n$ converges if and only if $\alpha < 2$.

Solution by Nicole Grivaux, Lycée Lavoisier, Paris, France. Let $l = \lim_{n\to\infty}(u_n - u_{n+1})/u_n^\alpha$. Since $l \neq 0$, for n large enough (say, for $n \geq N$) the sequence $\{u_n\}$ is strictly monotone. However, u_n is positive and $\lim_{n\to\infty} u_n = 0$, so $\{u_n\}_{n\geq N}$ strictly decreases. For $n \geq N$, if $u_{n+1} \leq t \leq u_n$, then $u_{n+1}^{\alpha-1} \leq t^{\alpha-1} \leq u_n^{\alpha-1}$, since $\alpha > 1$. Thus

$$\frac{u_n - u_{n+1}}{u_n^{\alpha-1}} \leq \int_{u_{n+1}}^{u_n} \frac{dt}{t^{\alpha-1}} \leq \frac{u_n - u_{n+1}}{u_{n+1}^{\alpha-1}}. \tag{1}$$

If $\alpha < 2$, then the function $1/t^{\alpha-1}$ is integrable on $]0, 1]$, so by (1) the series $\sum(u_n - u_{n+1})/u_n^{\alpha-1}$ converges. Since $lu_n \sim (u_n - u_{n+1})/u_n^{\alpha-1}$, the series $\sum u_n$ also converges.

On the other hand, if $\alpha \geq 2$, then the function $1/t^{\alpha-1}$ is not integrable on $]0, 1]$, so by (1) the series $\sum(u_n - u_{n+1})/u_{n+1}^{\alpha-1}$ diverges. The hypothesis $(u_n - u_{n+1})/u_n \sim lu_n^{\alpha-1}$ implies $\lim_{n\to\infty}(1 - u_n/u_{n+1}) = 0$. Hence $u_n \sim u_{n+1}$ and $u_n \sim (u_n - u_{n+1})/u_{n+1}^{\alpha-1}$, so $\sum u_n$ diverges.

Also solved by R. Bagby, M. Bataille (France), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, A. Habil (Syria), E. J. Ionascu, S. James (Canada), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), Á. Plaza & K. Sadarangani (Spain), K. Schilling, B. Schmuland (Canada), N. C. Singer, A. Stenger R. Stong, D. B. Tyler, J. Vinuesa (Spain), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before November 30, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11719. *Proposed by Nicolae Anghel, University of North Texas, Denton, TX.* Let f be a twice-differentiable function from $[0,\infty)$ into $(0,\infty)$ such that

$$\lim_{x\to\infty}\frac{f''(x)}{f(x)(1+f'(x)^2)^2}=\infty.$$

Show that

$$\lim_{x\to\infty}\int_{t=0}^{x}\frac{\sqrt{1+f'(t)^2}}{f(t)}\,dt\int_{t=x}^{\infty}\sqrt{1+f'(t)^2}\,f(t)\,dt=0.$$

11720. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* Let $E_n(t)$ be the Eulerian polynomial defined by

$$\sum_{k=0}^{\infty}(k+1)^n t^k=\frac{E_n(t)}{(1-t)^{n+1}},$$

and let B_n be the nth Bernoulli number. Show that $(E_{n+1}(t)-(1-t)^n)B_n$ is a polynomial with integer coefficients.

11721. *Proposed by Roberto Tauraso, Universitá di Roma "Tor Vergata", Rome, Italy.* Let p be a prime greater than 3, and let q be a complex number other than 1 such that $q^p=1$. Evaluate

$$\sum_{k=1}^{p-1}\frac{(1-q^k)^5}{(1-q^{2k})^3(1-q^{3k})^2}.$$

http://dx.doi.org/10.4169/amer.math.monthly.120.07.660

11722. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Let ABC be an acute triangle in the plane, and let M be a point inside ABC. Let O_1, O_2, and O_3 be the circumcenters of BCM, CAM, and ABM, respectively. Let c be the circumcircle of ABC. Let D, E, and F be the points opposite A, B, and C, respectively, at which AM, BM, and CM meet c. Prove that O_1D, O_2E, and O_3F are concurrent at a point P that lies on c.

11723. *Proposed by L. R. King, Davidson, NC.* Let A, B, and C be three points in the plane, and let D, E, and F be points lying on BC, CA, and AB, respectively. Show that there exists a conic tangent to BC, CA, and AB at D, E, and F, respectively, if and only if AD, BE, and CF are concurrent.

11724. *Proposed by Andrew Cusumano, Great Neck, NY.* Let $f(n) = \sum_{k=1}^{n} k^k$ and let $g(n) = \sum_{k=1}^{n} f(k)$. Find

$$\lim_{n\to\infty} \frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)}.$$

11725. *Proposed by Mher Safaryan, Yerevan State University, Yerevan, Armenia.* Let m be a positive integer. Show that, as $n \to \infty$,

$$\left| \log 2 - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \right| = \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots + \frac{C_m}{n^m} + o\left(\frac{1}{n^m}\right),$$

where

$$C_k = (-1)^k \sum_{i=1}^{k} \frac{1}{2^i} \sum_{j=1}^{i} (-1)^j \binom{i-1}{j-1} j^{k-1}$$

for $1 \le k \le m$.

SOLUTIONS

A Sum and Product Inequality

11584 [2011, 558]. *Proposed by Raymond Mortini and Jérôme Noël, Université Paul Verlaine, Metz, France.* Let $\langle a_j \rangle$ be a sequence of nonzero complex numbers inside the unit circle, such that $\prod_{k=1}^{\infty} |a_k|$ converges. Prove that

$$\left| \sum_{j=1}^{\infty} \frac{1-|a_j|^2}{a_j} \right| \le \frac{1-\prod_{j=1}^{\infty} |a_j|^2}{\prod_{j=1}^{\infty} |a_j|}.$$

Solution I by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Assume that $\prod_{j=1}^{\infty} |a_j|$ converges to a positive number P, and let $f(x) = \frac{1}{x} - x$. For $a, b \in (0, 1)$, we have

$$f(ab) - f(a) - f(b) = \frac{1}{ab}(1-a)(1-b)(1-ab) > 0,$$

so that $f(a) + f(b) < f(ab)$. By induction, $\sum_{j=1}^{n} f(|a_j|) < f(|a_1 a_2 \cdots a_n|)$. Taking the limit as $n \to \infty$, we obtain $\sum_{j=1}^{\infty} f(|a_j|) \le f(P)$. Thus, $\sum_{j=1}^{\infty} \frac{1}{a_j}(1-|a_j|^2)$ converges absolutely, and the desired result follows.

Solution II by Douglas B. Tyler, Raytheon, Torrence, California. Assume as in the first solution that $\prod_{j=1}^{\infty}|a_j|$ converges to P, and let $|a_j| = e^{-x_j}$. Thus, $x_j > 0$ and $\sum_{j=1}^{\infty} x_j = -\log P$. For $x, y > 0$, $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x \geq \sinh x + \sinh y$. We have

$$\left|\sum_{j=1}^{\infty} \frac{1-|a_j|^2}{|a_j|}\right| \leq \sum_{j=1}^{\infty} \frac{1}{|a_j|} - |a_j| = 2\sum_{j=1}^{\infty} \sinh x_j \leq 2\sinh\left(\sum_{j=1}^{\infty} x_j\right) = \frac{1}{P} - P.$$

Editorial comment. The proposers' solution used the Schwartz–Pick inequality: $|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$ for any analytic automorphism of the unit disk. Taking $f(z)$ to be $\prod_{j=1}^{\infty} \frac{a_j}{|a_j|} \frac{a_j - z}{1-\bar{a}_j z}$ leads to the required conclusion.

We of course follow the usual convention: To say that an infinite product with nonzero factors "converges", means that the sequence of partial products converges to a nonzero value.

Also solved by R. Chapman (U. K.), D. Constales (Belgium), W. J. Cowieson, P. P. Dályay (Hungary), O. Geupel (Germany), M. Goldenberg & M. Kaplan, E. A. Herman, K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Simons (U. K.), N. C. Singer, A. Stenger, R. Stong, M. Tetiva (Romania), E. I. Verriest, J. Vinuesa (Spain), H. Widmer (Switzerland), Ellington Management Problem Solving Group, and GCHQ Problem Solving Group (U. K.).

An Equivalent of CH

11588 [2011, 653]. *Proposed by Taras Banakh, Ivan Franko National University of Lviv, Lviv, Ukraine, and Igor Protasov, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine.* Show that $\mathbb{R} - \{0\}$ can be partitioned into countably many subsets, each of which is linearly independent over $\mathbb{Q}$, if and only if the continuum hypothesis holds.

Editorial comment. Most solvers noted that the result is a theorem of Paul Erdős and Shizuo Kakutani, which can be found in "On Non-denumerable Graphs," *Bull. Amer. Math. Soc.* **49** (1943) 457–461. O. Guepel and R. Mabry observed that the proposers published the result with proof in their article "Partitions of groups and matroids into independent subsets," *Algebra Discrete Math.* **10** (2010) 1–7, also available at `http://arxiv.org/abs/1010.1359`.

Solved by N. Caro (Brazil), C. Degenkolb, O. Geupel (Germany), R. Mabry, K. Muthuvel, V. Pambuccian, S. Scheinberg, R. Stong, and the proposers.

Arggh! Eye Factorial . . . Arg(i!)

11592 [2011, 654]. *Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Find $\lim_{n\to\infty}\left(-\log(n) + \sum_{k=1}^{n} \arctan\frac{1}{k}\right)$.

Solution I by Nora Thornber, Raritan Valley Community College, Somerville, New Jersey. Let L be the desired limit. Since $\lim_{n\to\infty}(-\log n + \sum_{k=1}^{n}\frac{1}{k}) = \gamma$, Euler's constant, we have

$$L = \gamma - \lim_{n\to\infty} \sum_{k=1}^{n}\left(\frac{1}{k} - \arctan\frac{1}{k}\right). \tag{1}$$

Note that $\operatorname{Im}\log(1+i/k) = \operatorname{Arg}(1+i/k) = \arctan(1/k)$. Since

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{z/k}, \tag{2}$$

we have

$$\operatorname{Im}\log\Gamma(i) = -\gamma - \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{1}{k} - \arctan\left(\frac{1}{k}\right),$$

and thus $L = -\pi/2 - \operatorname{Arg}\Gamma(i) = -\operatorname{Arg}\Gamma(1+i)$. Since both sides are between 0 and 2π, the branch of the logarithm was appropriate.

Solution II by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. As before, the limit is given by (1). Since $e^{i\arctan(1/k)} = (k+i)/\sqrt{1+k^2}$, we have

$$e^{iL} = e^{i\gamma}\prod_{k=1}^{\infty} e^{i\arctan(1/k)-i/k} = \left(\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)\right)^{-1/2} \cdot e^{i\gamma}\prod_{k=1}^{\infty}\left(1+\frac{i}{k}\right)e^{-i/k}.$$

Thus, $iL = -(1/2)\log C - \log\Gamma(1+i)$, where $C = \prod_{k=1}^{\infty}(1+1/k^2)$. Note that C is real. Taking imaginary parts and using (2), we have $L = -\operatorname{Arg}\Gamma(1+i)$.

Solution III by Denis Constales, Ghent University, Ghent, Belgium. Consider the digamma function, defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Since $\psi(x+1) = \psi(x) + 1/x$ and $\psi(n) = \log(n) + O(1/n)$ as $n \to \infty$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k}{x^2+k^2} &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k+ix} + \frac{1}{k-ix}\right) \\
&= \frac{1}{2}\left(\psi(n+1+ix) - \psi(1+ix) + \psi(n+1-ix) - \psi(1-ix)\right) \\
&= -\frac{1}{2}\left(\psi(1+ix) + \psi(1-ix)\right) + \log(n) + O(1/n).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^1 \sum_{k=1}^{n} \frac{k}{x^2+k^2}\,dx &= \sum_{k=1}^{n} \arctan\frac{1}{k} \\
&= \frac{i}{2}\left[\log\Gamma(1+ix) - \log\Gamma(1-ix)\right]_{x=0}^{1} + \log(n) + O(1/n).
\end{aligned}$$

Thus, the desired limit is

$$\lim_{n\to\infty}\left(-\log(n) + \sum_{k=1}^{n}\arctan\frac{1}{k}\right) = \frac{i}{2}\log\frac{\Gamma(1+i)}{\overline{\Gamma(1+i)}} = -\operatorname{Arg}\Gamma(1+i).$$

Editorial comment. Many partial solutions were submitted, for example $\gamma - \zeta(3)/3 + \zeta(5)/5 - \zeta(7)/7 + \cdots$. D. Beckwith and others noted formula 6.1.27 in Abramowitz and Stegun's *Handbook of Mathematical Functions*, which states $\operatorname{Arg}\Gamma(x+iy) = y\psi(x) + \sum_{k=0}^{\infty}(\frac{y}{x+k} - \arctan\frac{y}{x+k})$, where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. The result follows by taking $x = 1$ and $y = -1$, since $\psi(1) = -\gamma$. D. Bailey, D. Borwein, and J. Borwein observed that $\lim_{n\to\infty}(-\log n + \sum_{k=2}^{n}\operatorname{arctanh}(1/k))$ is an easier problem, in which the limit is $-\log\sqrt{2}$.

Also solved by T. Amdeberhan & V. H. Moll, D. H. Bailey & D. Borwein & J. M. Borwein (Canada & Canada & Australia), D. Beckwith, R. Chapman (U. K.), P. J. Fitzsimmons, O. Furdui (Romania), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. Grivaux (France), E. A. Herman, K.-W. Lau (China), O. P. Lossers (Netherlands), N. C. Singer, A. Stenger, I. Sterling, R. Stong, M. Vowe (Switzerland), S. Wagon & M. Trott, T. Wiandt, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

A Limit of an Integral

11611 [2011, 937]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca Cluj, Romania.* Let f be a continuous function from $[0, 1]$ into $[0, \infty)$. Find

$$\lim_{n\to\infty} n \int_{x=0}^{1} \left(\sum_{k=n}^{\infty} \frac{x^k}{k} \right)^2 f(x)\, dx.$$

Solution by Byron Schmuland, University of Alberta, Edmonton, AB, Canada. Begin by noting that f may be replaced by a constant function. Indeed, let $g_n(x) = (\sum_{k=n}^{\infty} x^k/k)^2$. Suppose that $n\int_0^1 g_n(x)\,dx$ converges to a finite value c. For any continuous f and positive ε, let ρ be less than 1 but near enough to 1 that $\sup_{\rho\le x\le 1} |f(x) - f(1)| < \varepsilon$. For $0 \le x \le \rho$, we have the crude bound

$$g_n(x) \le \left(\sum_{k=n}^{\infty} \rho^k \right)^2 = \frac{\rho^{2n}}{(1-\rho)^2}.$$

Thus

$$\begin{aligned} \left| n\int_0^1 g_n f(x)\,dx - n\int_0^1 g_n f(1)\,dx \right| &\le 2\|f\|_\infty \cdot n\int_0^\rho g_n(x)\,dx + \varepsilon n \int_\rho^1 g_n(x)\,dx \\ &\le 2\|f\|_\infty \frac{n\rho^{2n}}{(1-\rho)^2} + \varepsilon n\int_0^1 g_n(x)\,dx. \end{aligned}$$

Take lim sup in n and then let $\varepsilon \to 0$ to conclude

$$\lim_{n\to\infty} n\int_0^1 g_n(x) f(x)\,dx = \lim_{n\to\infty} n\int_0^1 g_n(x) f(1)\,dx = cf(1),$$

as claimed.

Now we must compute c. For $m \ge 1$, let $A_m = [m,\infty) \times [m,\infty)$ and write

$$\int_0^1 g_n(x)\,dx = \int_0^1 \sum_{j=n}^{\infty}\sum_{k=0}^{\infty} \frac{x^{j+k}}{jk}\,dx = \sum_{j=n}^{\infty}\sum_{k=n}^{\infty} \frac{1}{jk(j+k+1)}.$$

Now let $n > 1$ and take $j, k \ge n$. On the square $(x, y) \in [j, j+1] \times [k, k+1]$,

$$\left(\frac{j+k}{j+k+1}\right) \frac{1}{xy(x+y)} \le \frac{1}{jk(j+k+1)} \le \frac{1}{(x-1)(y-1)(x+y-2)}.$$

Summing over all these squares yields

$$\left(\frac{2n}{2n+1}\right) \int_{A_n} \frac{dx\,dy}{xy(x+y)} \le \sum_{j=n}^{\infty}\sum_{k=n}^{\infty} \frac{1}{jk(j+k+1)} \le \int_{A_{n-1}} \frac{dx\,dy}{xy(x+y)}.$$

However,

$$\int_n^\infty \frac{dx}{xy(x+y)} = \left[\frac{1}{y^2}\log\frac{x}{x+y}\right]_n^\infty = \frac{1}{y^2}\log\frac{n+y}{y},$$

so

$$\int_{A_n} \frac{dx\,dy}{xy(x+y)} = \left[\frac{1}{n}\log\frac{y}{n+y} + \frac{1}{y}\log\frac{n}{n+y}\right]_n^\infty = \frac{2\log 2}{n},$$

and we conclude that $c = 2\log 2$.

Editorial comment. Several solvers noted that f need not have nonnegative values; in fact, the result holds for all bounded integrable f that are continuous at $x = 1$.

Also solved by N. Caro (Brazil), R. Chapman (U. K.), D. Constales (Belgium), P. J. Fitzsimmons, J. Grivaux (France), E. J. Ionascu, B. Karaivanov, J. C. Kieffer, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), U. Milutinović (Slovenia), M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), N. C. Singer, A. Stenger, R. Stong, D. B. Tyler, J. Vinuesa (Spain), T. Viteam (Uruguay), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

Concurrent Lines Defined by a Triangle

11615 [2012, 68]. *Proposed by Constantin Mateescu, Zinca Golescu High School, Pitesti, Romania.* Let A, B, and C be the vertices of a triangle, and let K be a point in the plane distinct from these vertices and the lines connecting them. Let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let D, E, and F be the intersections of the lines through MK and NP, NK and PM, and PK and MN, respectively. Prove that the parallels from D, E, and F to AK, BK, and CK, respectively, are concurrent.

Solution by C. R. Pranesechar, Indian Institute of Science, Bangalore, India. We use vectors. Let K be the origin, and use the same letters $A, B, C, M, N, P, D, E, F$ for the position vectors of the corresponding points. We describe lines by a point, a direction, and a parameter 't', so that the line through, say, Q and R is given by the set of all points of the form $Q + t(R - Q)$, or, for short, by the equation $\mathbf{r} = Q + t(R - Q)$.

Since K is not on any of the lines BC, CA, AB, we see that any two of the vectors A, B, C are linearly independent. Write $\mathbf{0} = lA + mB + nC$ where l, m, and n are all nonzero real numbers. Thus $A = -(m/l)B - (n/l)C$. The equation of line NP is

$$\mathbf{r} = P + t_1(P - N) = \frac{1}{2}(A + B) + \frac{t_1}{2}(B - C),$$

and the equation of line MK is

$$\mathbf{r} = t_2 D = \frac{t_2}{2}(B + C).$$

Next we find the point D, the intersection of these two lines. Proceeding from these two equations, we write A in terms of B and C, equate the coefficients of B and C, solve for t_1 and t_2 in terms of l, m, n, and substitute back for t_1 to get

$$D = \frac{l - m - n}{4l}(B + C).$$

Similarly,

$$E = \frac{m - n - l}{4m}(C + A).$$

The line L_1 through D and parallel to AK has equation

$$\mathbf{r} = D + t_3 A = \frac{l-m-n}{4l}(B+C) + t_3 A.$$

The line L_2 through E and parallel to BK is

$$\mathbf{r} = E + t_4 B = \frac{m-n-l}{4m}(C+A) + t_4 B.$$

Let T be the point of intersection of L_1 and L_2. As before, substitute for A in terms of B and C, equate coefficients of B and C in the two equations, solve for t_2 and t_4 in terms of l, m, n, and substitute back for t_3. This yields

$$T = \frac{(m-l)(m-n+l)}{4nl}B + \frac{(n-l)(n-m+l)}{4lm}C.$$

This may be rearranged as

$$\frac{A+B+C}{4} + \frac{l^3 A + m^3 B + n^3 C}{4lmn},$$

so it is symmetric with respect to the pairs (A, l), (B, m), (C, n). Thus T also lies on the line L_3 through F and parallel to CK. This proves the concurrency of the lines L_1, L_2, L_3, as desired.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), R. Chapman (U. K.), C. Delorme (France), O. Geupel (Germany), J.-P. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, R. Stong, D. Stout, T. Viteam (Germany), Z. Zhang, GCHQ Problem Solving Group (U. K.), and the proposer.

A Point of Zero Net Force

11616 [2012, 68]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Let $x_1, \dots, x_n$ be distinct points in $\mathbb{R}^3$, and let $k_1, \dots, k_n$ be positive real numbers. A test object at x is attracted to each of $x_1, \dots, x_n$ with a force along the line from x to x_j of magnitude $k_j \|x - x_j\|^2$, where $\|u\|$ denotes the usual Euclidean norm of u. Show that when $n \geq 2$, there is a unique point x^* at which the net force on the test object is zero.

Solution by Jeff Boersema, Seattle University, Seattle, WA. We will prove a stronger statement: *Let $x_1, \dots, x_n$ be distinct points in $\mathbb{R}^m$, with $n \geq 2$ and $m \geq 2$. Let $g_1, \dots, g_n$ be continuous strictly increasing functions from $[0, \infty)$ to $[0, \infty)$ with $g_j(0) = 0$. If a test object at x is attracted to each of $x_1, \dots, x_n$ with a force along the line from x to x_j of magnitude $g_j(\|x - x_j\|)$, then there is a unique point at which the net force is zero.*

Let $F(x)$ be the vector field on $\mathbb{R}^m$ defined by

$$F(x) = \sum_{j=1}^{n} g_j\left(\|x_j - x\|\right) \frac{x_j - x}{\|x_j - x\|},$$

where the jth term is zero if $x = x_j$. We must show that there is a unique point x^* such that $F(x^*) = 0$. To prove existence, let K be the convex hull of the points $x_1, \dots, x_n$. For each $x \in K$, there is a positive number ε such that the segment from x to $x + \varepsilon F(x)$ is contained in K. Indeed, at a point on the boundary of K, the direction of $F(x)$ is toward the interior of K. Since F is continuous and K is compact, there is a positive number ε^* such that $x + \varepsilon^* F(x) \in K$ for all $x \in K$. Now K is homeomorphic to a simplex (of some dimension less than or equal to m), so by the Brouwer Fixed Point Theorem, the function $x + \varepsilon^* F(x)$ has a fixed point x^*. Thus $F(x^*) = 0$.

For uniqueness, suppose that F has distinct zeros u and v. This leads to a contradiction as follows: Write $z = v - u$. Define $G\colon \mathbb{R} \to \mathbb{R}$ by $G(t) = F(u + tz) \cdot z$, where '$\cdot$' denotes the dot product. Note that G is the sum of the n strictly decreasing functions of t given by

$$g_j\left(\|x_j - u - tz\|\right) \frac{x_j - u - tz}{\|x_j - u - tz\|} \cdot z.$$

However, F has distinct zeros u and v, so $G(0) = G(1) = 0$, a contradiction.

Also solved by R. Chapman (U. K.), D. Constales (Belgium), P. J. Fitzsimmons, E. J. Ionascu, J. H. Lindsey II, O. P. Lossers (Netherlands), Á. Plaza & J. Sánchez-Reyes (Spain), C. R. Pranesachar (India), M. B. Y. Ranorovelonalohotsy (South Africa), J. G. Simmonds, N. C. Singer, R. Stong, R. Tauraso (Italy), E. I. Verriest, and the proposer.

A Quadruple Integral

11621 [2012, 161]. *Proposed by Z. K. Silagadze, Budker Institute of Nuclear Physics and Novosibirsk State University, Novosibirsk, Russia.* Find

$$\int_{s_1=-\infty}^{\infty} \int_{s_2=-\infty}^{s_1} \int_{s_3=-\infty}^{s_2} \int_{s_4=-\infty}^{s_3} \cos(s_1^2 - s_2^2)\cos(s_3^2 - s_4^2)\, ds_4\, ds_3\, ds_2\, ds_1.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA . Let I denote this value. We will show that $I = \pi^2/16$.

Let $F_c(x) = \int_{-\infty}^{x} \cos(t^2)\, dt$ and $F_s(x) = \int_{-\infty}^{x} \sin(t^2)\, dt$. These are essentially the Fresnel integrals; in particular

$$\lim_{x\to\infty} F_c(x) = \lim_{x\to\infty} F_s(x) = \sqrt{\frac{\pi}{2}}.$$

Integrating by parts, we see that in fact

$$F_s(x) = \sqrt{\frac{\pi}{2}} - \frac{\cos(x^2)}{2x} + O\left(\frac{1}{x^3}\right) \quad \text{as } x \to +\infty,$$

and

$$F_c(x) = \sqrt{\frac{\pi}{2}} + \frac{\sin(x^2)}{2x} + O\left(\frac{1}{x^3}\right), \quad \text{as } x \to +\infty.$$

Noting that $F_c(x) + F_c(-x) = \sqrt{\pi/2}$ and $F_s(x) + F_s(-x) = \sqrt{\pi/2}$, we have

$$F_c(x) = O\left(\frac{1}{x}\right) \quad \text{as } x \to -\infty,$$

and

$$F_s(x) = O\left(\frac{1}{x}\right) \quad \text{as } x \to -\infty.$$

Now apply the formula for the cosine of a difference to compute

$$\int_{-\infty}^{s_2} \int_{-\infty}^{s_3} \cos(s_3^2 - s_4^2)\, ds_4\, ds_3 = \int_{-\infty}^{s_2} \left(\cos(s_3^2) F_c(s_3) + \sin(s_3^2) F_s(s_3)\right) ds_3$$
$$= \frac{F_c(s_2)^2 + F_s(s_2)^2}{2}.$$

Now use integration by parts. (Here we write $F^2(u)$ for $(F(u))^2$.)

$$\begin{aligned}
I &= \frac{1}{2}\lim_{R\to\infty}\int_{-\infty}^{R}\int_{-\infty}^{s_1}\left(\cos(s_1^2)\cos(s_2^2)+\sin(s_1^2)\sin(s_2^2)\right)\left(F_c^2(s_2)+F_s^2(s_2)\right)ds_2\,ds_1\\
&= \frac{1}{2}\lim_{R\to\infty}\Bigg[F_c(R)\int_{-\infty}^{R}\cos(x^2)\big(F_c^2(x)+F_s^2(x)\big)\,dx\\
&\qquad + F_s(R)\int_{-\infty}^{R}\sin(x^2))\big(F_c^2(x)+F_s^2(x)\big)\,dx\\
&\qquad - \int_{-\infty}^{R}\big(\cos(x^2)F_c(R)+\sin(x^2)F_s(x)\big)\big(F_c^2(x)+F_s^2(x)\big)\,dx\Bigg]\\
&= \sqrt{\frac{\pi}{8}}\int_{-\infty}^{\infty}\big(\cos(x^2)+\sin(x^2)\big)\big(F_c^2(x)+F_s^2(x)\big)\,dx - \frac{1}{8}\big(F_c^2(x)+F_s^2(x)\big)^2\Big|_{-\infty}^{\infty}\\
&= \sqrt{\frac{\pi}{8}}\int_{-\infty}^{\infty}\big(\cos(x^2)+\sin(x^2)\big)\big(F_c^2(x)+F_s^2(x)\big)\,dx - \frac{\pi^2}{8}.
\end{aligned}$$

Here, in going from the second line to the third, we have used the asymptotics above to conclude that

$$\cos(x^2)\big(F_c^2(x)+F_s^2(x)\big) = \pi\cos(x^2) + \sqrt{\frac{\pi}{2}}\,\frac{\sin(2x^2)-1-\cos(2x^2)}{2x} + O\left(\frac{1}{x^3}\right)$$

as $x \to +\infty$. The oscillatory terms in this sum and the error term are integrable; hence from this formula and the analogous one for the sine, we get

$$\int_{-\infty}^{R}\cos(x^2)\big(F_c^2(x)+F_s^2(x)\big)\,dx = -\frac{\sqrt{\pi}}{2\sqrt{2}}\log R + O(1)$$

and

$$\int_{-\infty}^{R}\sin(x^2)\big(F_c^2(x)+F_s^2(x)\big)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}\log R + O(1)$$

as $R \to +\infty$. All other terms converge, and these $\log R$ terms cancel in the limit, so in particular the integral in the third line exists.

To evaluate this integral, note that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\big(\cos(x^2)+\sin(x^2)\big)\big(F_c^2(x)+F_s^2(x)\big)\,dx\\
&\qquad = \mathrm{Re}\left[(1+i)\int_{-\infty}^{\infty}e^{-ix^2}\big(F_c^2(x)+F_s^2(x)\big)\,dx\right]\\
&\qquad = \lim_{a\to i}\mathrm{Re}\left[\sqrt{2a}\int_{-\infty}^{\infty}e^{-ax^2}\big(F_c^2(x)+F_s^2(x)\big)\,dx\right],
\end{aligned}$$

where the final limit is taken over a with $\mathrm{Re}(a) > 0$. To justify exchanging the integral and the limit, note that by explicit computation

$$\lim_{a\to i}\int_0^{\infty}e^{-ax^2}dx = \int_0^{\infty}e^{-ix^2}dx;$$

$$\lim_{a\to i}\int_1^{\infty}\frac{e^{-ax^2+ibx^2}}{x}\,dx = \int_1^{\infty}\frac{e^{-ix^2+ibx^2}}{x}\,dx$$

for $b = 0$, -2, or 2. (For the first limit, this is the standard argument for evaluating $F_c(\infty) = F_s(\infty) = \sqrt{\pi/2}$. The second is similar.)

These limits correspond to the leading terms of the asymptotics of the integrand. The error terms tend to zero fast enough (like x^{-3} at $+\infty$ and x^{-2} at $-\infty$) that the Dominated Convergence Theorem applies. Thus we can exchange the limit and the integral. Noting that

$$F_c^2(x) + F_s^2(x) = \int_0^\infty e^{i(x-s)^2} ds \int_0^\infty e^{-i(x-t)^2} dt,$$

we see that the required integral is a standard sort of Gaussian integral and

$$\begin{aligned}
&\sqrt{2a} \int_{-\infty}^\infty e^{-ax^2} \left(F_c(x)^2 + F_s(x)^2\right) dx \\
&\quad = \sqrt{2a} \int_{-\infty}^\infty \int_0^\infty \int_0^\infty e^{-ax^2 + i(x-s)^2 - i(x-t)^2} ds\, dt\, dx \\
&\quad = \frac{\sqrt{2\pi}}{4} \left[\pi - i \log\left(\frac{ai-1}{ai+1}\right)\right].
\end{aligned}$$

As $\operatorname{Re}(a)$ tends to zero, the argument of $\frac{ai-1}{ai+1}$ tends to $\pi/2$ (though the magnitude grows without bound). Hence the real part of the expression tends to

$$\int_{-\infty}^\infty \left(\cos(x^2) + \sin(x^2)\right)\left(F_c^2(x) + F_s^2(x)\right) dx = \frac{3\pi^{3/2}}{4\sqrt{2}},$$

and we get

$$I = \frac{1}{2}\sqrt{\frac{\pi}{2}} \cdot \frac{3\pi^{3/2}}{4\sqrt{2}} - \frac{\pi^2}{8} = \frac{\pi^2}{16}.$$

Editorial comment. The (relatively long) solution published here avoids the hand-waving arguments found in some of the other solutions. The proposer notes that this integral is connected to the quantum-mechanical Landau–Zener problem. He conjectures that if we do this with $2n$ nested integrals rather than 4, we get the value $2(\pi/4)^n/n!$.

The proposer's solution to this problem unfortunately appeared on the web in an essay `http://arxiv.org/abs/1201.1975`, in time for would-be solvers to have read it.

Also solved by T. Amdeberhan & A. Straub, M. L. Glasser, J. A. Grzesik, M. Omarjee (France), J. G. Simmonds, and the proposer.

Limits and Derivatives (Correction)

11603 [2011, 847]. *Proposed by Alfonso Villani, Università di Catania, Catania, Italy.* In the May 2013 issue solution by Iosif Pinelis, on p. 476 line 2, one step of the solution we published was a garbled version of what we received. It should have read

$$|f(x)| \geq |f(x_k)| - \left|\int_{x_k}^x f'(u)\, du\right| > 2\varepsilon - \varepsilon = \varepsilon.$$

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before September 30, 2014. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11775. *Proposed by Isaac Sofair, Fredericksburg, VA.* Let $A_1, \ldots, A_k$ be finite sets. For $J \subseteq \{1, \ldots, k\}$, let $N_J = \left|\bigcup_{j\in J} A_j\right|$, and let $S_m = \sum_{J\,:\,|J|=m} N_J$.
(**a**) Express in terms of $S_1, \ldots, S_k$ the number of elements that belong to exactly m of the sets $A_1, \ldots, A_k$.
(**b**) Same question as in (**a**), except that we now require the number of elements belonging to at least m of the sets $A_1, \ldots, A_k$.

11776. *Proposed by David Beckwith, Sag Harbor, NY.* Given urns $U_1, \ldots, U_n$ in a line, and plenty of identical blue and identical red balls, let a_n be the number of ways to put balls into the urns subject to the conditions that
(i) each urn contains at most one ball,
(ii) any urn containing a red ball is next to exactly one urn containing a blue ball, and
(iii) no two urns containing a blue ball are adjacent.
(**a**) Show that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1+t+2t^2}{1-t-t^2-3t^3}.$$

(**b**) Show that

$$a_n = \sum_{j\ge 0}\sum_{m\ge 0} 4^j \left[\binom{n-2m}{j}\binom{m}{j} + \binom{n-2m-1}{j}\binom{m}{j} + 2\binom{n-2m}{j}\binom{m-1}{j}\right].$$

Here, $\binom{k}{l} = 0$ if $k < l$.

http://dx.doi.org/10.4169/amer.math.monthly.121.05.455

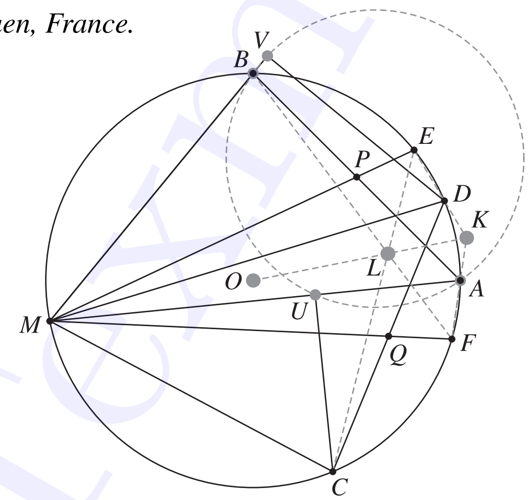
11777. *Proposed by Marian Dincă, Bucharest, Romania.* Let $x_1, \ldots, x_n$ be real numbers such that $\prod_{k=1}^{n} x_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{x_k^2}{x_k^2 - 2x_k \cos(2\pi/n) + 1} \geq 1.$$

11778. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.* Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Let $f(x, y, z) = 1/(\tan^2 x + 4\tan^2 y + 9\tan^2 z)$. Prove that

$$f(x, y, z) + f(y, z, x) + f(z, x, y) \leq \frac{9}{14}\left(\tan^2 x + \tan^2 y + \tan^2 z\right).$$

11779. *Proposed by Michel Bataille, Rouen, France.* Let M, A, B, C, and D be distinct points (in any order) on a circle Γ with center O. Let the medians through M of triangles MAB and MCD cross lines AB and CD at P and Q, respectively, and meet Γ again at E and F, respectively. Let K be the intersection of AF with DE, and let L be the intersection of BF with CE. Let U and V be the orthogonal projections of C onto MA and D onto MB, respectively, and assume $U \neq A$ and $V \neq B$. Prove that A, B, U, and V are concyclic if and only if O, K, and L are collinear.



11780. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Tudorel Lupu, Decebal High School, Constanţa, Romania.* Let f be a positive-valued, concave function on [0, 1]. Prove that

$$\frac{3}{4}\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{8} + \int_0^1 f^3(x)\,dx.$$

11781. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata", Rome, Italy.* For $n \geq 2$, call a positive integer *n-smooth* if none of its prime factors is larger than n. Let S_n be the set of all n-smooth positive integers. Let C be a finite, nonempty set of nonnegative integers, and let a and d be positive integers. Let M be the set of all positive integers of the form $m = \sum_{k=1}^{d} c_k s_k$, where $c_k \in C$ and $s_k \in S_n$ for $k = 1, \ldots, d$. Prove that there are infinitely many primes p such that $p^a \notin M$.

SOLUTIONS

Integrals with Bernoulli Numbers

11644 [2012, 426]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let n be a nonnegative integer, and let B_j be the jth Bernoulli number, defined for $j \geq 0$ by

$x/(e^x-1)=\sum_{k=0}^{\infty} B_k x^k/k!$. Let

$$I_n=\int_0^\infty\left(\frac{1}{x^n(e^x-1)}-\left(\frac{1}{x^n}+\sum_{k=0}^{n}B_k\frac{x^{k-n-1}}{k!}\right)e^{-x}\right)dx.$$

Prove that $I_0=\gamma-1$, that $I_1=1-(1/2)\log(2\pi)$, and that for $n\geq 1$,

$$I_{2n}=(\log(2\pi)+\gamma)\frac{B_{2n}}{(2n)!}+(-1)^n\frac{2\zeta'(2n)}{(2\pi)^{2n}}$$
$$+\frac{1}{2(2n-1)!}H_{2n-1}-\sum_{k=0}^{n-1}\frac{B_{2k}}{(2k)!}\cdot\frac{H_{2n-2k}}{(2n-2k)!},$$

and that for $n\geq 1$,

$$I_{2n+1}=(-1)^n\frac{\zeta(2n+1)}{2(2\pi)^{2n}}-\frac{1}{2(2n)!}H_{2n}+\sum_{k=0}^{n}\frac{B_{2k}}{(2k)!}\cdot\frac{H_{2n+1-2k}}{(2n+1-2k)!}.$$

Here, H_n denotes $\sum_{k=1}^{n}1/k$, ζ denotes the Riemann zeta function, and γ is Euler's constant.

Solution by the proposer. Note that

$$\frac{e^x}{x^n(e^x-1)}=\frac{1}{x^n}+\sum_{k=0}^{n}B_k\frac{x^{k-n-1}}{k!}+O(1)$$

in a neighborhood of $x=0$. Define

$$f_n(s)=\int_0^\infty x^{s-1}\left(\frac{1}{x^n(e^x-1)}-\left(\frac{1}{x^n}+\sum_{k=0}^{n}B_k\frac{x^{k-n-1}}{k!}\right)e^{-x}\right)dx.$$

The integral converges absolutely for $\operatorname{Re} s>0$ and uniformly in every compact subset contained in $\operatorname{Re} s\geq\varepsilon>0$. Therefore, $f_n(s)$ is analytic in $\operatorname{Re} s>0$. Thus $I_n=f_n(1)$, and we compute $f_n(1)$.

If $\operatorname{Re} s>n$, then

$$f_n(s)=\Gamma(s-n)\zeta(s-n)-\Gamma(s-n)-\sum_{k=0}^{n}\frac{B_k}{k!}\Gamma(s+k-n-1).\tag{1}$$

Note that (1) represents the analytic continuation of $f_n(s)$ as a meromorphic function in the whole complex plane. Also, the residues of f_n at $s\in\{1,\ldots,n\}$ all vanish.

We now take note of some well-known facts about the gamma and zeta functions. If m is a nonnegative integer, then in a neighborhood of $s=1$ we have

$$\Gamma(s-m)=\frac{\Gamma(s)}{(s-1)(s-2)\cdots(s-m)}$$
$$=\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{1}{s-1}-\gamma+H_{m-1}+O(s-1)\right).\tag{2}$$

By considering the residue of f_n at 1, we have

$$0=\frac{(-1)^{n-1}}{(n-1)!}\zeta(1-n)-\frac{(-1)^{n-1}}{(n-1)!}-\sum_{k=0}^{n}\frac{B_k}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!}.$$

For the last equality we used

$$xe^{-x} = \left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!}\right)\left(1-e^{-x}\right) = \left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!}\right)\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k!}\right)$$

and compared coefficients of x^n. Therefore,

$$\zeta(1-n) = (-1)^{n-1}\frac{B_n}{n}. \tag{3}$$

Using the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s), \tag{4}$$

we get

$$2^{1-n}\pi^{-n}\cos\frac{\pi n}{2}\Gamma(n)\zeta(n) = (-1)^{n-1}\frac{B_n}{n}$$

and

$$\zeta(2n) = (-1)^{n-1}\frac{B_{2n}}{2(2n)!}(2\pi)^{2n}.$$

Now

$$\begin{aligned}
I_0 = f_0(1) &= \lim_{s\to 1}\left(\Gamma(s)\zeta(s)-\Gamma(s)-\Gamma(s-1)\right) = -1+\lim_{s\to 1}\frac{\Gamma(s)(s-1)\zeta(s)-\Gamma(s)}{s-1} \\
&= -1+\lim_{s\to 1}\frac{1}{s-1}\Big(\left[1-\gamma(s-1)+O\left((s-1)^2\right)\right]\left[1+\gamma(s-1)+O\left((s-1)^2\right)\right] \\
&\qquad -\left[1-\gamma(s-1)+O\left((s-1)^2\right)\right]\Big) = \gamma - 1.
\end{aligned}$$

For $n \ge 1$,

$$\begin{aligned}
I_n &= \int_0^{\infty}\left(\frac{1}{x^n(e^x-1)} - \left(\frac{1}{x^n}+\sum_{k=0}^{n} B_k\frac{x^{k-n-1}}{k!}\right)e^{-x}\right)dx = f_n(1) \\
&= \lim_{s\to 1}\left(\Gamma(s-n)\zeta(s-n)-\Gamma(s-n)-\sum_{k=0}^{n}\frac{B_k}{k!}\Gamma(s+k-n-1)\right) \\
&= \frac{(-1)^{n-1}}{(n-1)!}\left(-\gamma+H_{n-1}\right)\zeta(1-n) + \frac{(-1)^{n-1}}{(n-1)!}\zeta'(1-n) \\
&\qquad - \frac{(-1)^{n-1}}{(n-1)!}\left(-\gamma+H_{n-1}\right) - \sum_{k=0}^{n}\frac{B_k}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!}\left(-\gamma+H_{n-k}\right) \\
&= \frac{(-1)^{n-1}}{(n-1)!}\left(\zeta(1-n)-1\right) + \frac{(-1)^{n-1}}{(n-1)!}\zeta'(1-n) - \sum_{k=0}^{n-1}\frac{B_k}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!}H_{n-k} \\
&= \frac{B_n}{n!}H_{n-1} - \frac{(-1)^{n-1}}{(n-1)!}H_{n-1} + \frac{(-1)^{n-1}}{(n-1)!}\zeta'(1-n) - \sum_{k=0}^{n-1}\frac{B_k}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!},
\end{aligned}$$

where we have used (2) and the fact that the residue at 1 of f_n is zero. To get from here to the required formulas, we will need to relate the values of ζ' at negative integers to values of ζ and ζ' at positive integers.

We have $\zeta(0) = -1/2$. From (4) we deduce

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\log(2\pi) - \frac{\pi}{2}\tan\frac{\pi s}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}.$$

In a neighborhood of $s = 1$,

$$\frac{\pi}{2}\tan\frac{\pi s}{2} = \frac{-1}{s-1} + O(s-1), \qquad \frac{\Gamma'(s)}{\Gamma(s)} = -\gamma + O(s-1),$$

and

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \gamma + O(s-1),$$

so

$$-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi), \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi).$$

We have

$$\frac{\Gamma'(s+n)}{\Gamma(s+n)} = \frac{1}{s+n-1} + \frac{1}{s+n-2} + \cdots + \frac{1}{s+1} + \frac{\Gamma'(s+1)}{\Gamma(s+1)}.$$

Thus,

$$\frac{\Gamma'(n)}{\Gamma(n)} = H_{n-1} - \gamma, \qquad \Gamma'(n) = (n-1)!\,(H_{n-1} - \gamma)\,.$$

From (4), we deduce

$$-\zeta'(1-s) = -2\log(2\pi)(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s) - 2(2\pi)^{-s}\frac{\pi}{2}\sin\frac{\pi s}{2}\Gamma(s)\zeta(s)$$
$$+ 2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma'(s)\zeta(s) + 2(2\pi)^{-s}\cos\frac{\pi 2}{2}\Gamma(s)\zeta'(s).$$

For $n \geq 1$, let $Z_n = \zeta'(1-n)(2\pi)^n/2(n-1)!$. We then have

$$Z_n = \log(2\pi)\cos\frac{\pi n}{2}\zeta(n) + \frac{\pi}{2}\sin\frac{\pi n}{2}\zeta(n)$$
$$-\cos\frac{\pi n}{2}\,(H_{n-1} - \gamma)\,\zeta(n) - \cos\frac{\pi n}{2}\zeta'(n).$$

Thus for odd n, $Z_n = \frac{\pi}{2}(-1)^{(n-1)/2}\zeta(n)$, while for even n,

$$Z_n = (-1)^{n/2}\left[(\log(2\pi) - H_{n-1} + \gamma)\,\zeta(n) - \zeta'(n)\right]$$
$$= (-\log(2\pi) + H_{n-1} - \gamma)\,\frac{B_n(2\pi)^n}{2(n!)} - (-1)^{n/2}\zeta'(n).$$

We thus conclude:

$$I_0 = \gamma - 1, \qquad I_1 = f_1(1) = \zeta'(0) + B_0 H_1 = 1 - \frac{1}{2}\log(2\pi),$$

and for $n \geq 1$, using the fact that $B_1 = -1/2$ while $B_{2k+1} = 0$ for $k > 0$,

$$I_{2n} = \frac{1}{2(2n-1)!} H_{2n-1} + (\log(2\pi) + \gamma) \frac{B_{2n}}{(2n)!} + (-1)^n \frac{2\zeta'(2n)}{(2\pi)^{2n}} - \sum_{k=0}^{n-1} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n-2k}}{(2n-2k)!},$$

$$I_{2n+1} = -\frac{1}{2(2n)!} H_{2n} + (-1)^n \frac{\zeta(2n+1)}{2(2\pi)^{2n}} + \sum_{k=0}^{n} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n+1-2k}}{(2n+1-2k)!}.$$

Also solved by B. Burdick.

An l^p Inequality

11649 [2012, 522]. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Let p be real with $p > 1$. Let $(x_0, x_1, \ldots)$ be a sequence of nonnegative real numbers. Prove that

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{x_k}{j+k+1} \right)^p < \infty \;\Rightarrow\; \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} x_k \right)^p < \infty.$$

Solution by Oliver Geupel, Brühl, NRW, Germany. For every nonnegative integer j, since $x_j > 0$, we have

$$\frac{1}{j+1} \sum_{k=0}^{j} x_k \leq \frac{2j+1}{j+1} \sum_{k=0}^{j} \frac{x_k}{j+k+1} \leq 2 \sum_{k=0}^{\infty} \frac{x_k}{j+k+1}.$$

If $p > 0$, then x^p strictly increases with x on the interval $[0, \infty)$. Thus, raising both sides of this inequality to the pth power and summing both sides over j yields

$$\sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} x_k \right)^p \leq 2^p \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{x_k}{j+k+1} \right)^p.$$

The proof also shows that the restriction on p can be relaxed to $p > 0$.

Editorial comment. Kenneth F. Anderson remarked that, conversely, since $(a+b)^p \leq 2^p(a^p + b^p)$ for $a, b \geq 0$, it follows that

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{x_k}{j+k+1} \right)^p \leq 2^p \left[\sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} x_k \right)^p + \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \frac{x_k}{k+1} \right)^p \right].$$

The convergence of the two series on the right-hand side implies convergence of $\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} x_k/(j+k+1) \right)^p$. See Hardy's discussion of Hilbert's Double Series Theorem (Hardy–Littlewood–Pólya, *Inequalities,* Cambridge University Press, 1967, Ch. 9).

Also solved by K. F. Andersen (Canada), R. Bagby, P. P. Dályay (Hungary), E. A. Herman, F. Holland (Ireland), B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), A. Stenger, R. Stong, R. Tauraso (Italy), T. Viteam (Chile), and the proposer.

A Double Integral

11650 [2012, 522]. *Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.* Evaluate

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-(x-y)^2}\sin^2(x^2+y^2)\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx.$$

Solution by Jan A. Van Casteren, University of Antwerp, Antwerp, Belgium. As preparation, we evaluate the following integral for $\sigma > 0$:

$$\begin{aligned}\int_0^\infty e^{-2\sigma\rho}\frac{\sin^2\rho}{\rho}\,d\rho &= \int_0^\infty e^{-\sigma\rho}\frac{\sin^2(\rho/2)}{\rho}\,d\rho = \frac{1}{2}\int_0^\infty\int_\sigma^\infty e^{-\tau\rho}\,d\tau(1-\cos\rho)d\rho\\ &= \frac{1}{2}\int_\sigma^\infty\int_0^\infty e^{-\tau\rho}(1-\cos\rho)d\rho\,d\tau = \frac{1}{2}\int_\sigma^\infty\left(\frac{1}{\tau}-\frac{\tau}{1+\tau^2}\right)d\tau\\ &= \frac{1}{2}\log\frac{(1+\sigma^2)^{1/2}}{\sigma}.\end{aligned}$$

Now for the integral J of the problem: passing first to polar coordinates via $x = r\cos\varphi$, $y = r\sin\varphi$, we compute

$$\begin{aligned}J &= \int_0^\infty\int_x^\infty e^{-(x-y)^2}\sin^2(x^2+y^2)\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx\\ &= \int_0^\infty\int_{\pi/4}^{\pi/2} e^{-r^2+2r^2\sin\varphi\cos\varphi}\sin^2(r^2)\frac{\cos^2\varphi-\sin^2\varphi}{r}\,d\varphi\,dr\\ &= \int_0^\infty\int_{\pi/4}^{\pi/2} e^{-r^2+r^2\sin 2\varphi}\cos 2\varphi\,d\varphi\frac{\sin^2(r^2)}{r}\,dr\\ &= -\frac{1}{2}\int_0^\infty\frac{1-e^{-r^2}}{r^2}\,\frac{\sin^2(r^2)}{r}\,dr \qquad (\text{substitute } \rho = r^2)\\ &= -\frac{1}{4}\int_0^\infty\frac{1-e^{-\rho}}{\rho}\,\frac{\sin^2\rho}{\rho}\,d\rho = -\frac{1}{2}\int_0^{1/2}\int_0^\infty e^{-2\sigma\rho}\,\frac{\sin^2\rho}{\rho}\,d\rho\,d\sigma\\ &= -\frac{1}{4}\int_0^{1/2}\log\frac{(1+\sigma^2)^{1/2}}{\sigma}\,d\sigma \qquad (\text{integrate by parts})\\ &= -\frac{1}{4}\left(\frac{1}{2}\log\frac{(1+(1/2)^2)^{1/2}}{1/2}+\arctan\frac{1}{2}\right) = -\frac{1}{16}\log 5-\frac{1}{4}\arctan\frac{1}{2}.\end{aligned}$$

Also solved by K. F. Andersen (Canada), D. Anderson (Ireland), R. Bagby, D. H. Bailey (U.S.) & J. M. Borwein (Australia), M. Benito, Ó. Ciaurri, E. Fernández & L. Roncal (Spain), K. N. Boyadzhiev, M. A. Carlton, R. Chapman (U. K.), H. Chen, B. E. Davis, S. de Luxán (Spain), E. S. Eyeson, C. Georghiou (Greece), O. Geupel (Germany), M. L. Glasser, J. A. Grzesik, A. Guetter & I. Roussos, E. A. Herman, F. Holland (Ireland), B. Karaivanov, O. Kouba (Syria), K. D. Lathrop, K.-W. Lau (China), O. P. Lossers (Netherlands), J. Magliano, T. L. McCoy, M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), I. Rusodimos, R. Stong, R. Tauraso (Italy), T. Trif (Romania), D. B. Tyler, E. I. Verriest, J. Vinuesa (Spain), M. Vowe (Switzerland), J. Wan (Australia), H. Wang & J. Wojdylo, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Binomial Determinant

11652 [2012, 522–523]. *Proposed by Ajai Choudhry, Foreign Service Institute, New Delhi, India.* For $a, b, c, d \in \mathbb{R}$, and for nonnegative integers i, j, and n, let

$$t_{i,j} = \sum_{s=0}^{i} \binom{n-i}{j-s}\binom{i}{s} a^{n-i-j+s} b^{j-s} c^{i-s} d^s.$$

Let $T(a, b, c, d, n)$ be the $(n+1)$-by-$(n+1)$ matrix with (i, j)-entry given by $t_{i,j}$, for $i, j \in \{0, \dots, n\}$. Show that $\det T(a, b, c, d, n) = (ad - bc)^{n(n+1)/2}$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let E denote the vector space $\mathbb{R}_n[x]$ of real polynomials with degree at most n, and let B denote the canonical basis $\{1, x, x^2, \dots\}$ of E. Consider the linear transformations V and $T_{\lambda,\mu}$ from E to E defined by $V(P(x)) = x^n P(1/x)$ and $T_{\lambda,\mu}(P(x)) = P(\lambda x + \mu)$, where $(\lambda, \mu) \in \mathbb{R}^2$.

For a linear transformation T from E to E, let $\det(T)$ denote the determinant of the matrix of T with respect to B. Since the matrices of V and $T_{\lambda,\mu}$ with respect to B are

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \mu & * & \cdots & * \\ 0 & \lambda & * & \cdots & * \\ 0 & 0 & \lambda^2 & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda^n \end{pmatrix},$$

we obtain $\det(V) = (-1)^{n(n+1)/2}$ and $\det(T_{\lambda,\mu}) = \lambda^{n(n+1)/2}$.

Now consider $(a, b, c, d) \in \mathbb{R}^4$ with $b \neq 0$, and let U be the linear transformation defined by $U = T_{b,a} \circ V \circ T_{c-ad/b,d/b}$. We have

$$\det(U) = \det(T_{b,a}) \det(V) \det(T_{c-ad/b,d/b}) = (ad - bc)^{n(n+1)/2}. \qquad (*)$$

On the other hand, for $0 \leq i \leq n$,

$$\begin{aligned} U(x^i) &= (a + bx)^{n-i}(c + dx)^i \\ &= \sum_{j=0}^{n} \left(\sum_{s\geq 0} \binom{n-i}{j-s}\binom{i}{s} a^{n-i-j+s} b^{j-s} c^{i-s} d^s \right) x^j = \sum_{j=0}^{n} t_{i,j} x^j. \end{aligned}$$

Thus, the matrix of U with respect to B is the transpose of the matrix $T(a, b, c, d, n)$. Using $(*)$, we obtain

$$\det(T(a, b, c, d, n)) = \det(U) = (ad - bc)^{n(n+1)/2}.$$

The case $b = 0$ follows by continuity.

Also solved by D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), B. Karaivanov, P. Lima-Filho, M. Omarjee (France), M. A. Prasad (India), J. H. Smith, J. H. Steelman, R. Stong, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before October 31, 2014. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11782. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* A *signed binary representation* of an integer m is a finite list $a_0, a_1, \ldots$ of elements of $\{-1, 0, 1\}$ such that $\sum a_i 2^i = m$. A signed binary representation is *sparse* if no two consecutive entries in the list are nonzero.
(**a**) Prove that every integer has a unique sparse representation.
(**b**) Prove that for all $m \in \mathbb{Z}$, every non-sparse signed binary representation of m has at least as many nonzero terms as the sparse representation.

11783. *Proposed by Zhang Yun, Xi'an City, Shaanxi, China.* Given a tetrahedron, let r denote the radius of its inscribed sphere. For $1 \le k \le 4$, let h_k denote the distance from the kth vertex to the plane of the opposite face. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \ge \frac{12}{5}.$$

11784. *Proposed by Abdurrahim Yilmaz, Middle East Technical University, Ankara, Turkey.* Let ABC be an equilateral triangle with center O and circumradius r. Given $R > r$, let ρ be a circle about O of radius R. All points named 'P' are on ρ.
(**a**) Prove that $|PA|^2 + |PB|^2 + |PC|^2 = 3(R^2 + r^2)$.
(**b**) Prove that $\min_{P\in\rho} |PA|\,|PB|\,|PC| = R^3 - r^3$ and that $\max_{P\in\rho} |PA|\,|PB|\,|PC| = R^3 + r^3$.
(**c**) Prove that the area of a triangle with sides of length $|PA|$, $|PB|$, and $|PC|$ is $\frac{\sqrt{3}}{4}(R^2 - r^2)$.
(**d**) Prove that if H, K, and L are the respective projections of P onto AB, AC, and BC, then the area of triangle HKL is $\frac{3\sqrt{3}}{116}(R^2 - r^2)$.
(**e**) With the same notation, prove that $|HK|^2 + |KL|^2 + |HL|^2 = \frac{9}{4}(R^2 + r^2)$.

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11785. *Proposed by Bhaskar Bagchi, India Statistics Institute, Bangalore, India.* Let $[n] = \{1, \dots, n\}$. For a subset A of $[n]$, a *run* of A is a maximal subset of A consisting of consecutive integers. Let $O(A)$ denote the number of runs of A with an odd number of elements, and let $\mu(A) = \frac{1}{2}(|A| + O(A))$. (For instance, if $n = 9$ and $A = \{1, 3, 4, 5, 8, 9\}$, then A has three runs, $O(A) = 2$, and $\mu(A) = 4$.)
(**a**) Show that if $0 \le k \le n$ and $k/2 \le i \le k$, then the number $N_{i,k}$ of subsets A of $[n]$ such that $\mu(A) = i$ and $|A| = k$ is given by

$$N_{i,k} = \binom{n-i}{k-i}\binom{n-k+1}{2i-k}.$$

(**b**)* Prove or disprove that if m is a positive integer and $m + 1 \le k \le 2m$, then the number of subsets A of $[3m + 1]$ such that $|A| = k$ and $\mu(A) \le m$ is equal to the number of subsets B of $[3m + 1]$ such that $|B| = 3m + 1 - k$ and $\mu(B) > m$.

11786. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let $x_1, x_2, \dots$ be a sequence of positive numbers such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \frac{\log x_n}{x_1+\cdots+x_n}$ is a negative number. Prove that $\lim_{n\to\infty} \frac{\log x_n}{\log n} = -1$.

11787. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Prove that

$$\sum_{k=1}^{\infty}(-1)^{k-1}kp_k\left(n-\frac{1}{2}k(k+1)\right) = \sum_{k=-\infty}^{\infty}(-1)^k\tau\left(n-\frac{1}{2}k(3k-1)\right).$$

Here, $p_k(n)$ denotes the number of partitions of n in which the greatest part is less than or equal to k (with $p_k(0) = 1$ and $p_k(n) = 0$ for $n < 0$), and $\tau(n)$ is the number of divisors of n (with $\tau(n) = 0$ for $n \le 0$).

11788. *Proposed by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece.* Let n be a positive integer, and suppose that $0 < y_i \le x_i < 1$ for $1 \le i \le n$. Prove that

$$\frac{\log x_1 + \cdots + \log x_n}{\log y_1 + \cdots + \log y_n} \le \sqrt{\frac{1-x_1}{1-y_1} + \cdots + \frac{1-x_n}{1-y_n}}.$$

SOLUTIONS

Another Property of Only the Golden Ratio

11651 [522]. *Proposed by Marcel Celaya and Frank Ruskey, University of Victoria, Victoria, BC, Canada.* Show that the equation

$$\left\lfloor\frac{n+1}{\phi}\right\rfloor = n - \left\lfloor\frac{n}{\phi}\right\rfloor + \left\lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor - \left\lfloor\frac{\left\lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor + \left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor - \cdots$$

holds for every nonnegative integer n if and only if $\phi = (1 + \sqrt{5})/2$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Requiring equality for $n = 0$ restricts our attention to positive ϕ, and thus for each n there are only finitely many nonzero terms in the sum.

Replacing n with $\lfloor n/\phi \rfloor$ and adding to the given equation yields

$$\left\lfloor \frac{n+1}{\phi} \right\rfloor + \left\lfloor \frac{\lfloor n/\phi \rfloor + 1}{\phi} \right\rfloor = n. \tag{1}$$

Conversely, if (1) holds for each nonnegative integer n, then so does the given equation. Hence, it is sufficient to show that (1) holds for each nonnegative integer n if and only if $\phi = (1+\sqrt{5})/2$.

First, suppose that (1) holds for all nonnegative integers. Dividing both sides by n and taking the limit as $n \to \infty$ gives $1/\phi + 1/\phi^2 = 1$, which yields $\phi = (1+\sqrt{5})/2$ since $\phi > 0$.

Conversely, suppose that $\phi = (1+\sqrt{5})/2$. We show first that (1) will follow by proving the following for all $n \geq 0$:

$$\left\lfloor \frac{\lfloor n/\phi \rfloor + 1}{\phi} \right\rfloor = \left\lfloor \frac{n+1}{\phi^2} \right\rfloor. \tag{2}$$

Given (2), we have

$$\begin{aligned} \left\lfloor \frac{n+1}{\phi} \right\rfloor + \left\lfloor \frac{\lfloor n/\phi \rfloor + 1}{\phi} \right\rfloor &= \left\lfloor \frac{n+1}{\phi} \right\rfloor + \left\lfloor \frac{n+1}{\phi^2} \right\rfloor \\ &= \left\lfloor \frac{n+1}{\phi} \right\rfloor + \left\lfloor \left(1 - \frac{1}{\phi}\right)(n+1) \right\rfloor = (n+1) + \left\lfloor \frac{n+1}{\phi} \right\rfloor + \left\lfloor -\frac{n+1}{\phi} \right\rfloor = n, \end{aligned}$$

since $\lfloor z \rfloor + \lfloor -z \rfloor = -1$ when z is not an integer.

Finally, we prove (2). Letting $n/\phi = \lfloor n/\phi \rfloor + \epsilon$, we have

$$\frac{n+1}{\phi^2} = n - \frac{n}{\phi} + \frac{1}{\phi^2} = n - \left\lfloor \frac{n}{\phi} \right\rfloor - \epsilon + \frac{1}{\phi^2} \tag{3}$$

and

$$\frac{\lfloor n/\phi \rfloor + 1}{\phi} = \frac{n}{\phi^2} + \frac{1-\epsilon}{\phi} = n\left(1 - \frac{1}{\phi}\right) + \frac{1-\epsilon}{\phi} = n - \left\lfloor \frac{n}{\phi} \right\rfloor - \epsilon + \frac{1-\epsilon}{\phi}. \tag{4}$$

The floors of (7) and (8) will be equal if $-\epsilon + 1/\phi^2$ and $-\epsilon + (1-\epsilon)/\phi$ are either both nonnegative or both negative. Since

$$-\epsilon + \frac{1-\epsilon}{\phi} = -\epsilon\left(1 + \frac{1}{\phi}\right) + \frac{1}{\phi} = -\epsilon\phi + \frac{1}{\phi} = \left(-\epsilon + \frac{1}{\phi^2}\right)\phi,$$

the result follows.

Editorial comment. The proposers note that the formula $b(n) + b(b(n-1)) = n$, where $b(n) = \lfloor (n+1)/\phi \rfloor$, is proved in V. Granville and J. P. Rasson, *A strange recursive relation*, J. Number Theory 30 (1988) 238–241.

L. Carlitz (*Fibonacci representations*, Fibonacci Quarterly 6 (1968) 193–220) studied the function e defined by $e(F_{k_1} + \cdots + F_{k_r}) = F_{k_1 - 1} + \cdots + F_{k_r - 1}$, where $k_1 > \cdots > k_r \geq 2$. He proved formulas closely related to (1), which in his notation becomes $e(n) + e(e(n-1)) = n$. For example, if $k_r > 2$, then $e(n) + e(e(n)) = n$.

Also solved by R. Bagby, N. Caro (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, K. Schilling, A. Stenger, R. Stong, T. Viteam (Chile), GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposers.

An Addition Formula

11653 [2012, 523]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let n be a positive integer. Determine all entire functions f that satisfy, for all complex s and t, the functional equation

$$f(s+t) = \sum_{k=0}^{n-1} f^{(n-1-k)}(s) f^{(k)}(t).$$

Here, $f^{(m)}$ denotes the mth derivative of f.

Solution by Denis Constales, Ghent University, Ghent, Belgium. We first note that for such f,

$$\begin{aligned} 0 &= \frac{\partial f(s+t)}{\partial s} - \frac{\partial f(s+t)}{\partial t} = \sum_{k=0}^{n-1} f^{(n-k)}(s) f^{(k)}(t) - \sum_{k=1}^{n} f^{(n-k)}(s) f^{(k)}(t) \\ &= f^{(n)}(s) f(t) - f(s) f^{(n)}(t). \end{aligned}$$

The identically zero function clearly is a solution of the stated functional equation, so henceforth we assume that f is not identically zero. Defining $\lambda = f^{(n)}(t)/f(t)$ for some t with $f(t) \neq 0$ shows that f must satisfy the ordinary differential equation $f^{(n)}(z) - \lambda f(z) = 0$.

Case 1: $\lambda = 0$ (and thus, f must be a polynomial of degree at most $n-1$). Let m be the degree of f and C its leading coefficient. The coefficient of s^m on the right side of the functional equation is $f^{(n-1)}(t)C$, which is nonzero only if $m = n-1$. The polynomials $f^{(k)}(t)$ for $0 \le k \le n-1$ thus form a basis for the vector space of all polynomials of degree at most $n-1$ in t. Comparison of the Taylor expansion $f(s+t) = \sum_{k=0}^{n-1} (f^{(k)}(t)/k!) s^k$ with the functional equation shows that we have a solution if and only if $f^{(n-1-k)}(s) = s^k/k!$ for $0 \le k \le n-1$. Hence, $f(z) = z^{n-1}/(n-1)!$ is the only nonzero polynomial solution.

Case 2: $\lambda \neq 0$. Write $\omega = \exp(2\pi i/n)$ for the nth root of unity and μ for any nth root of λ. The general solution to the ordinary differential equation is $f(z) = \sum_{k=0}^{n-1} a_k \exp(\mu\omega^k z)$. The right side (call it R) of the functional equation is given by

$$\begin{aligned} R &= \sum_{k=0}^{n-1} \sum_{p=0}^{n-1} a_p \mu^{n-1-k} \omega^{p(n-1-k)} \exp(\mu\omega^p s) \sum_{q=0}^{n-1} a_q \mu^k \omega^{qk} \exp(\mu\omega^q t) \\ &= \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} a_p a_q \mu^{n-1} \omega^{p(n-1)} \exp(\mu\omega^p s + \mu\omega^q t) \sum_{k=0}^{n-1} \omega^{k(q-p)}. \end{aligned}$$

The inner sum vanishes unless $p = q$ (in which case it is n), so

$$R = \sum_{p=0}^{n-1} n a_p^2 \mu^{n-1} \omega^{-p} \exp(\mu\omega^p (s+t)).$$

Matching this with the left side, we obtain a solution if and only if each a_p independently equals either zero or $\omega^p/(n\mu^{n-1})$.

Also solved by K. F. Andersen (Canada), R. Bagby, D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), E. Herman, R. Howard, O. Kouba (Syria), O. P. Lossers (Netherlands), T. L. McCoy, J. Stewart, R. Stong, E. I. Verriest, TCDmath Problem Group (Ireland), and the proposer.

Integrals Arising from a Plane Random Walk

11654 [2012, 523]. *Proposed by David Borwein, University of Western Ontario, Canada, and Jonathan M. Borwein and James Wan, CARMA, University of Newcastle, Australia.* Let Cl denote the *Clausen* function, given by $\mathrm{Cl}(\theta) = \sum_{n=1}^{\infty} \sin(n\theta)/n^2$. Let ζ denote the Riemann zeta function.

(**a**) Show that

$$\int_{y=0}^{2\pi} \int_{x=0}^{2\pi} \log(3 + 2\cos x + 2\cos y + 2\cos(x - y))\, dx\, dy = 8\pi \mathrm{Cl}(\pi/3).$$

(**b**) Show that

$$\int_{y=0}^{\pi} \int_{x=0}^{\pi} \log(3 + 2\cos x + 2\cos y + 2\cos(x - y))\, dx\, dy = \frac{28}{3}\zeta(3).$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Note that

$$2\log\left|1 + e^{ix} + e^{iy}\right| = \log\left(3 + 2\cos x + 2\cos y + 2\cos(x - y)\right)$$

and recall that

$$\int_0^{2\pi} \log\left|a + e^{ix}\right|\, dx = \begin{cases} 2\pi \log|a|, & \text{if } |a| > 1 \\ 0, & \text{if } |a| \le 1 \end{cases}.$$

Also recall that the Clausen function satisfies

$$\mathrm{Cl}(\theta) = \sum_{n=0}^{\infty} \frac{1}{2n} \int_{-\theta}^{\theta} e^{inx} dx = -\frac{1}{2} \int_{-\theta}^{\theta} \log\left(1 - e^{x}\right) dx.$$

We will use some polylogarithms. These are given by $\mathrm{Li}_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a}$ for $|z| < 1$ and elsewhere by analytic continuation. With this notation, for $a = 2$ we have the *dilogarithm* and for $a = 3$ the *trilogarithm* (see L. Lewin, *Polylogarithms and Associated Functions*, North Holland, 1981).

For part (**a**), the desired integral is

$$I = 2\int_0^{2\pi} \int_0^{2\pi} \log\left|1 + e^{ix} + e^{iy}\right|\, dx\, dy.$$

The inner integral in x vanishes for $2\pi/3 \le y \le 4\pi/3$, since in this range $|1 + e^{iy}| \le 1$. Outside this range, the inner integral is $2\pi \log|1 + e^{iy}|$. The integral in y of this over all of $[0, 2\pi]$ would evaluate to 0, so we obtain (substituting $y = z + \pi$):

$$I = -4\pi \int_{2\pi/3}^{4\pi/3} \log\left|1 + e^{iy}\right|\, dy = -4\pi \int_{-\pi/3}^{\pi/3} \log\left|1 - e^{iz}\right|\, dz = 8\pi \mathrm{Cl}(\pi/3).$$

For part (**b**), we integrate by parts in the variable x, then switch the order of integration to do the integral in y and substitute $z = e^{ix}$ to obtain expressions equal to the original integral J:

$$
\begin{aligned}
&2\int_0^{\pi}\int_0^{\pi} \log\left|1+e^{ix}+e^{iy}\right| \, dx\, dy \\
&\quad = 2\int_0^{\pi}\left[x\log|1+e^{ix}+e^{iy}|\Big|_{x=0}^{x=\pi} - \int_0^{\pi} x \operatorname{Re}\left(\frac{ie^{ix}}{1+e^{ix}+e^{iy}}\right) dx\right] dy \\
&\quad = -2\operatorname{Re}\int_0^{\pi}\int_0^{\pi} \frac{ixe^{ix}}{1+e^{ix}+e^{iy}}\, dy\, dx \\
&\quad = 2\operatorname{Re}\int_0^{\pi} \frac{xe^{ix}}{1+e^{ix}}\left(-iy+\log(1+e^{ix}+e^{iy})\right)\Big|_{y=0}^{y=\pi} dx \\
&\quad = 2\operatorname{Re}\int_0^{\pi} \frac{xe^{ix}\left(ix-i\pi-\log(2+e^{ix})\right)}{1+e^{ix}}\, dx \\
&\quad = 2\operatorname{Re}\int_{-1}^{1} \frac{\log z\,(\log z - i\pi - \log(2+z))}{1+z}\, dz \\
&\quad = 2\int_0^1 \frac{\log z\ \ (\log z - \log(2+z))}{1+z}\, dz + 2\int_0^1 \frac{\log z\ \ (\log z - \log(2-z))}{1-z}\, dz \\
&\quad = 4\int_0^1 \frac{(\log z)^2}{1-z^2}\, dz - 2\int_0^1 \frac{\log z\ \log(2-z)}{1-z}\, dz - 2\int_0^1 \frac{\log z\ \log(2+z)}{1+z}\, dz.
\end{aligned}
$$

Here in the penultimate step, we have shifted the contour from a semicircular arc onto the real axis and replaced z with $-z$ for the part along the negative real axis.

Now we evaluate the three integrals. The first integral is standard:

$$
\int_0^1 \frac{(\log z)^2}{1-z^2}\, dz = \sum_{n=0}^{\infty}\int_0^1 z^{2n}(\log z)^2 dz = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = \frac{7}{4}\zeta(3).
$$

The second integral reduces to an alternating Euler sum,

$$
\begin{aligned}
\int_0^1 \frac{\log z\ \log(2-z)}{1-z}\, dz &= \int_0^1 \frac{\log(1-z)\log(1+z)}{z}\, dz \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 z^{n-1}\log(1-z)\, dz = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n = -\frac{5}{8}\zeta(3).
\end{aligned}
$$

The third integral can be evaluated in terms of polylogarithms (as could the first two integrals). Using integral tables or computer algebra systems, we get

$$
\begin{aligned}
&\int_0^1 \frac{\log z\ \log(2+z)}{1+z}\, dz = -\frac{13}{24}\zeta(3) \\
&\quad + (i\pi + \log 3)\left[\operatorname{Li}_2\left(\frac{-1}{3}\right) - 2\operatorname{Li}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{6} - \frac{(\log 3)^2}{2}\right] \\
&\quad + \left[\operatorname{Li}_3\left(\frac{-1}{3}\right) - 2\operatorname{Li}_3\left(\frac{1}{3}\right) + \frac{13}{6}\zeta(3) - \frac{\pi^2\log 3}{6} + \frac{(\log 3)^3}{6}\right] = -\frac{13}{24}\zeta(3).
\end{aligned}
$$

We see that the first square-bracket expression vanishes either by noting that the integral is real or by combining the dilogarithm identities

$$
\operatorname{Li}_2(x) + \operatorname{Li}_2(-x) = \frac{1}{2}\operatorname{Li}_2(x^2) \quad \text{and} \quad \operatorname{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\operatorname{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} - \frac{(\log 3)^2}{6}.
$$

The second square-bracket expression vanishes, using the trilogarithm identity

$$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right) - \mathrm{Li}_3\left(\frac{z-1}{z+1}\right) = 2\,\mathrm{Li}_3(1-z) + 2\,\mathrm{Li}_3\left(\frac{1}{z+1}\right) - \frac{1}{2}\,\mathrm{Li}_3(1-z^2)$$
$$- \frac{7}{4}\,\zeta(3) - \frac{1}{3}\,(\log(1+z))^3 + \frac{\pi^2}{6}\log(1+z)$$

at $z = 2$, and the identity

$$\mathrm{Li}_3(z) = \mathrm{Li}_3\left(\frac{1}{z}\right) - \frac{1}{6}\,(\log(-z))^3 - \frac{\pi^2}{6}\log(-z)$$

at $z = -3$ (this last identity holds for $z \notin (0, 1)$).

Plugging in the values for the three integrals gives

$$J = 4 \cdot \frac{7}{4}\,\zeta(3) + 2 \cdot \frac{5}{8}\,\zeta(3) + 2 \cdot \frac{13}{24}\,\zeta(3) = \frac{28}{3}\,\zeta(3).$$

Editorial comment. Kouba and Stong solved (**b**) from scratch using polylogarithm identities. Van Casteren evaluated (**b**) in terms of a double series; together with the solution given here, it establishes

$$\frac{7}{3}\,\zeta(3) = 4\sum_{m=0}^{\infty}\sum_{k=2m+1}^{\infty}\frac{1}{k2^k(2m+1)^2} = 2\int_0^1\left(\log\frac{1}{s}\right)\frac{\log(1+2s)}{s(1+s)}\,ds.$$

Glasser and the proposers derived (**b**) from integrals evaluated elsewhere, but which are themselves very interesting. Glasser cited

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\log\left|1 + e^{ix} + e^{iy} + e^{iz}\right|\,dx\,dy\,dz = 28\pi\,\zeta(3),$$

found in D. W. Boyd, Speculations concerning the range of the Mahler measure, *Canad. Math. Bull.* **24** (1981) 453–469. The proposers cited

$$W_3(s) = \frac{1}{2^{2s+1}}\tan\frac{\pi s}{2}\binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\left.\begin{matrix}\frac12, \frac12, \frac12\\ \frac{s+3}{2}, \frac{s+3}{2}\end{matrix}\right|\frac14\right) + \binom{s}{\frac{s}{2}}{}_3F_2\left(\left.\begin{matrix}-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}\\ 1, -\frac{s-1}{2}\end{matrix}\right|\frac14\right),$$

$$W_4(s) = \frac{1}{2^{2s}}\tan\frac{\pi s}{2}\binom{s}{\frac{s-1}{2}}^2 {}_4F_3\left(\left.\begin{matrix}\frac12, \frac12, \frac12, \frac{s}{2}+1\\ \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2}\end{matrix}\right|1\right) + \binom{s}{\frac{s}{2}}{}_4F_3\left(\left.\begin{matrix}\frac12, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}\\ 1, 1, -\frac{s-1}{2}\end{matrix}\right|1\right),$$

under certain restrictions on s, where

$$W_n(s) = \int_{[0,1]^{n-1}}\left|1 + \sum_{k=1}^{n-1} e^{2\pi i x_k}\right|^s d\mathbf{x},$$

which expresses the sth moment of the distance to the origin after n steps for a random walk in the plane where each step is a unit step taken in a random direction. The two integrals in this problem are essentially $W_3'(0)$ and $W_4'(0)$. The proposers challenge our readers to similarly evaluate $W_5'(0) = 5\int_0^\infty\left(\log\frac{2}{t} - \gamma\right)J_0^4(t)J_1(t)\,dt \approx 0.54441256$. Here J is the Bessel function and γ is Euler's constant.

Also solved by M. L. Glasser, O. Kouba (Syria), and the proposers; part (a) only K. F. Andersen (Canada), O. Geupel (Germany), J. Van Casteren (Belgium), and GCHQ Problem Solving Group (U. K.).

A Functional Equation

11661 [2012, 609]. *Proposed by Giedrius Alkauskas, Vilnius University, Vilnius, Lithuania.* Find every function f on $\mathbb{R}^+$ that satisfies the functional equation

$$(1-z)f(x) = f\left(\frac{1-z}{z}f(xz)\right)$$

for $x > 0$ and $0 < z < 1$.

Solution by Michel Bataille, Rouen, France. We show that the solutions to the given functional equation are the functions $f_a\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_a(x) = \frac{x}{1+ax}$, where a is a nonnegative real number.

Direct computation shows that f_a is a solution when $a \geq 0$. Conversely, let f be a solution. Since f^+ is defined only on $\mathbb{R}^+$, in order for the functional equation to make sense, the image of f must be contained in $\mathbb{R}^+$.

Replacing x with $x+y$ and z with $x/(x+y)$ in the functional equation gives $y\frac{f(x+y)}{x+y} = f(yf(x)/x)$. Defining g by $g(x) = f(x)/x$ gives

$$g(x+y) = g(x)g\big(yg(x)\big) \tag{5}$$

for $x, y > 0$.

Next we show that $g(x) \leq 1$ for $x > 0$. If $g(z) > 1$ for some positive z, then taking $x = z$ and $y = z/\big(g(z)-1\big)$ in (5) gives $g\big(yg(z)\big) = g(z)g\big(yg(z)\big)$. Since $g(yg(z)) > 0$, this implies $g(z) = 1$, a contradiction.

It follows that $g(x+y) \leq g(x)$ for all $x, y > 0$, so g is nonincreasing.

First suppose $g(t) = 1$ for some positive t. Setting $x = t$ in (5) gives $g(t+y) = g(y)$, so $g(nt+y) = g(y)$ for every nonnegative integer n. For $x > 0$, there exists n such that $nt > x$, so $1 = g(nt) \leq g(x)$. Thus $g(x) = 1$ for all $x > 0$, and $f = f_0$.

Hence we may assume $g(x) < 1$ for $x > 0$. In this case (5) implies that g is strictly decreasing and hence injective. Let $u = g(1)$. From (5), we obtain

$$g(y+1) = ug(uy). \tag{6}$$

Setting $y = xg(x)$ in (6) and applying (5) gives

$$g\big(xg(x)+1\big) = ug\big(uxg(x)\big) = u\frac{g(x+ux)}{g(x)}. \tag{7}$$

On the other hand, applying (5) and then (6) gives

$$g\big((x/u+1)g(x)\big) = \frac{g(x/u+1+x)}{g(x)} = u\frac{g(x+ux)}{g(x)}. \tag{8}$$

Now (7) and (8) yield $g\big(xg(x)+1\big) = g\big((x/u+1)g(x)\big)$. Since g is injective, $xg(x)+1 = (x/u+1)g(x)$. Thus $g(x) = 1/(1+ax)$, where $a = (1-u)/u) > 0$. We conclude that $f = f_a$.

Also solved by Y.Q. Chen, P. P. Dályay (Hungary), M. Kim (Korea), O. P. Lossers (Netherlands), M. A. Prasad (India), N. C. Singer, GCHQ Problem Solving Group (U. K.), and the proposer. Three other submissions solved the problem under the assumption that solutions are differentiable.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before December 31, 2014. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11789. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let a and k be positive integers. Prove that for every positive integer d there exists a positive integer n such that d divides $ka^n + n$.

11790. *Proposed by Arkady Alt, San Jose, CA and Konstantin Knop, St. Petersburg, Russia.* Given a triangle with semiperimeter s, inradius r, and medians of length m_a, m_b, and m_c, prove that $m_a + m_b + m_c \le 2s - 3(2\sqrt{3} - 3)r$.

11791. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.* Show that for $r \ge 1$,

$$\sum_{s=1}^{r} \binom{6r+1}{6s-2} B_{6s-2} = -\frac{6r+1}{6},$$

where B_n denotes the nth Bernoulli number.

11792. *Proposed by Stephen Scheinberg, Corona del Mar, CA.* Show that every infinite dimensional Banach space contains a closed subspace of infinite dimension and infinite codimension.

11793. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Prove that

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} = -\zeta'(2) + \sum_{n=3}^{\infty} (-1)^{n+1} \frac{\zeta(n)}{n-2},$$

where ζ denotes the Riemann zeta function and ζ' denotes its derivative.

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11794. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Find every twice differentiable function f on $\mathbb{R}$ such that for all nonzero x and y, $xf(f(y)/x) = yf(f(x)/y)$.

11795. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let p be the partition counting function on the set $\mathbb{Z}^+$ of positive integers, and let g be the function on $\mathbb{Z}^+$ given by $g(n) = \frac{1}{2}\lceil n/2\rceil \lceil(3n+1)/2\rceil$. Let $A(n)$ be the set of positive integer triples (i, j, k) such that $g(i) + j + k = n$. Prove for $n \geq 1$ that

$$p(n) = \frac{1}{n} \sum_{(i,j,k)\in A(n)} (-1)^{\lceil i/2\rceil - 1} g(i) p(j) p(k).$$

SOLUTIONS

Inequality for a Convex Quadrilateral

11655 [2012, 523]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let $ABCD$ be a convex quadrilateral, and let α, β, γ, and δ be the radian measures of angles DAB, ABC, BCD, and CDA, respectively. Suppose $\alpha + \beta > \pi$ and $\alpha + \delta > \pi$, and let $\eta = \alpha + \beta - \pi$ and $\phi = \alpha + \delta - \pi$. Let a, b, c, d, e, f be real numbers with $ac = bd = ef$. Show that if $abe > 0$, then

$$a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta + e\cos\eta + f\cos\phi \leq \frac{be}{2a} + \frac{cf}{2b} + \frac{de}{2c} + \frac{af}{2d},$$

while for $abe < 0$ the inequality is reversed.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. It suffices to consider the case $abe > 0$; the other case follows by negating a, b, c, d, e, f. Write ν_{AB}, ν_{BC}, ν_{CD}, and ν_{DA} for outward unit normal vectors to the sides AB, BC, CD, and DA, respectively. We have

$$\begin{aligned}
\nu_{DA}\cdot\nu_{AB} &= -\cos\alpha, & \nu_{AB}\cdot\nu_{BC} &= -\cos\beta, & \nu_{BC}\cdot\nu_{CD} &= -\cos\gamma,\\
\nu_{CD}\cdot\nu_{DA} &= -\cos\delta, & \nu_{AB}\cdot\nu_{CD} &= -\cos\phi, & \nu_{BC}\cdot\nu_{DA} &= -\cos\eta.
\end{aligned}$$

Let

$$r = \frac{\sqrt{2abe}}{2e}, \qquad s = \frac{\sqrt{2abe}}{2a}, \qquad t = \frac{ac}{\sqrt{2abe}}, \qquad u = \frac{\sqrt{2abe}}{2b}.$$

We compute

$$a = 2ru, \qquad b = 2rs, \qquad c = 2st, \qquad d = 2tu, \qquad e = 2su, \qquad f = 2rt,$$

$$r^2 + s^2 + t^2 + u^2 = \frac{ab}{2e} + \frac{be}{2a} + \frac{ac^2}{2be} + \frac{ae}{2b} = \frac{af}{2d} + \frac{be}{2a} + \frac{cf}{2b} + \frac{de}{2c}.$$

Thus the right side of the desired inequality is $r^2 + t^2 + s^2 + u^2$, and the left side is the negation of

$$\begin{aligned}
&2ru\ \nu_{DA}\cdot\nu_{AB} + 2rs\ \nu_{AB}\cdot\nu_{BC} + 2st\ \nu_{BC}\cdot\nu_{CD}\\
&+ 2tu\ \nu_{CD}\cdot\nu_{DA} + 2su\ \nu_{BC}\cdot\nu_{DA} + 2rt\ \nu_{AB}\cdot\nu_{CD}.
\end{aligned}$$

Thus the desired inequality is equivalent to $\left\| r\ v_{AB} + s\ v_{BC} + t\ v_{CD} + u\ v_{DA} \right\|^2 \geq 0$, which holds trivially.

Also solved by GCHQ Problem Solving Group (U. K.), and the proposer.

How Many Polynomial Shapes Are There?

11656 [2012, 608]. *Proposed by Valerio De Angelis, Xavier University of Louisiana, New Orleans, LA.* The *sign chart* of a polynomial f with real coefficients is the list of successive pairs (ϵ, σ) of signs of (f', f) on the intervals separating real zeros of ff', together with the signs at the zeros of ff' themselves, read from left to right. Thus, for $x^3 - 3x^2$, the sign chart is $((1, -1), (0, 0), (-1, -1), (0, -1), (1, -1), (1, 0), (1, 1))$. As a function of n, how many distinct sign charts occur for polynomials of degree n?

Solution by Ronald E. Prather, Oakland CA. We count the sign charts, but do not prove that they can all be achieved by polynomials of the required degree.

Let $n \geq 1$. The polynomials f of degree n satisfying $\lim_{x\to-\infty} f(x) = +\infty$ produce half of the sign charts, so it suffices to count the sign charts for them.

Instead of sign charts, we first consider a related enumeration of a set $D(n)$ of *shapes* of polynomials f of degree n. A shape will be a finite sequence chosen from a set of twelve symbols. We write m when, scanning from the left, we encounter a local minimum of f (that is, f' has a zero of odd order, $f' < 0$ to the left, and $f' > 0$ to the right). We write M on meeting a local maximum (that is, a point at which f' has a zero of odd order, $f' > 0$ to the left, and $f' < 0$ to the right). We write I for a decreasing stationary point (that is, f' has a zero of even order and $f' < 0$ on both sides). We write J for an increasing stationary point (that is, f' has a zero of even order and $f' > 0$ on both sides). Each of these symbols will have subscript $+$, 0, or $-$, according as $f > 0$, $f = 0$, or $f < 0$ at the point. This yields twelve symbols: three ways to subscript each of m, M, I, and J.

The *shape* of a polynomial f is the sequence, from left to right, of the symbols corresponding to the zeros of f'. With the restriction that f is positive and f' is negative far to the left, we will be able to recover the sign chart from the shape of a polynomial and vice versa. In particular, from the data in the shape, we can discover the intervals where there is a point with $f = 0$ but $f' \neq 0$.

The polynomial $-x^3 + 3x^2$, the negative of the example in the problem statement, has shape $m_0 M_+$. Setting $f(x) = -x^3 + 2x^2$, let us recover the corresponding sign chart. Left of the point with symbol m_0 (or just 'm_0') f is positive and decreasing. Between m_0 and M_+, f is positive and increasing. Just right of M_+, f is positive and decreasing. Since $f \to -\infty$ at the far right, there is a zero of f where it changes sign. This yields $\big((-1, +1), (0, 0), (+1, +1), (0, +1), (-1, +1), (-1, 0), (-1, -1)\big)$.

Next we define *weight*. The weight of a subscripted m or M is 1, the weight of a subscripted I or J is 2, and the weight of a shape is the sum of the weights of its components. The case of weight 0 with no components is allowed. The weight of the shape of a polynomial f is equal to the degree of f', possibly reduced by an even number, so the weight is the degree of f minus an odd positive number. Write $S(w)$ for the set of all shapes of polynomials f with weight w for which $\lim_{x\to-\infty} f(x) = +\infty$. We have $|D(n)| = |S(n-1)| + |S(n-3)| + |S(n-5)| + \dots$, where we end at $|S(0)|$ or $|S(1)|$.

We must now compute $|S(w)|$. The difficulty is that not every sequence of the twelve symbols can occur. For example, following a zero of f' of type m_+ the function is positive and increasing, so the next symbol must be either M_+ or J_+. Another example: m_- can only be followed by M_+, M_0, M_-, J_+, J_0, or J_-.

We next provide a *code* for our shapes, symbol by symbol, according to the convention

$$\begin{array}{llllllllllll} m_+ & \mathbf{a} & m_0 & \mathbf{b} & m_- & \mathbf{c} & M_+ & \mathbf{c} & M_0 & \mathbf{a} & M_- & \mathbf{b} \\ I_+ & \mathbf{bb} & I_0 & \mathbf{ba} & I_- & \mathbf{aa} & J_+ & \mathbf{aa} & J_0 & \mathbf{ba} & J & \mathbf{bb}. \end{array}$$

The coding of a shape of weight w is a string of w symbols from the alphabet $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ such that the substring **ab** never appears. All such w-strings occur as encodings.

Given a string (with no **ab**), here is the method to obtain the corresponding shape. A string of w letters (**a**'s, **b**'s, and **c**'s) arising in this fashion from a polynomial is the concatenation of some k batches of strings $\mathbf{b}^u\mathbf{a}^v\mathbf{c}$, with the final **c** perhaps absent: $\mathbf{b}^{u_1}\mathbf{a}^{v_1}\mathbf{c}\mathbf{b}^{u_2}\mathbf{a}^{v_2}\mathbf{c}\ldots\mathbf{b}^{u_k}\mathbf{a}^{v_k}\mathbf{c}^{\epsilon}$ where $k \geq 0$, the u_j and v_j are nonnegative integers, and $\epsilon \in \{0, 1\}$.

Suppose we are turning a string into a shape, moving left to right, and we come to a batch $\mathbf{a}^u\mathbf{b}^v\mathbf{c}$ of letters knowing that any corresponding underlying function f is positive and decreasing. (The case of negative and increasing will be similar. These are the only two possibilities following a **c**, and we have enough information about the shape to know the local sign of any f that might underlie the shape we are building.) There are four cases for the upcoming batch, depending on the parities of the upcoming u and v.

If the batch has the form $\mathbf{b}^{2i+1}\mathbf{a}^{2j+1}\mathbf{c}$, then the corresponding next part of the shape is $I_+^i I_0 I_-{}^j m_-$. If it has the form $\mathbf{b}^{2i+1}\mathbf{a}^{2j}\mathbf{c}$, then the next part is $I_+{}^i m_0 J_+{}^j M_+$. If it has the form $\mathbf{b}^{2i}\mathbf{a}^{2j}\mathbf{c}$, then the next part is $I_+{}^i I_-{}^j m_-$. If it has the form $\mathbf{b}^{2i}\mathbf{a}^{2j+1}\mathbf{c}$, then the next part is $I_+{}^i m_+ J_+{}^j M_+$. If there is no final **c** (which can only happen if the current batch is the last), then drop the final shape item (an m or M). Note that $i = 0$ and $j = 0$ are allowed in all cases.

Let $s(w) = |S(w)|$. Because the coding is a bijection, $s(w)$ is the number of strings of length w with no **ab**. It can be computed recursively: $s(0) = 1$, $s(1) = 3$, $s(w) = 3s(w-1) - s(w-2)$. The solution is

$$s(w) = \frac{\phi^{2w+2} - \phi^{-2w-2}}{\sqrt{5}} = F_{2w+2},$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio and F_n is the nth Fibonacci number.

Finally, the solution to the problem itself is

$$\begin{aligned} 2\big(s(n-1) + s(n-3) + s(n-5) + \ldots\big) &= \frac{2\big(\phi^{2n+2} + \phi^{-2n-2}\big) + (-1)^{n+1} - 5}{5} \\ &= \frac{2L_{2n+2} + (-1)^{n+1} - 5}{5}, \end{aligned}$$

in terms of Lucas numbers L_n.

Editorial comment. As noted, this solution does not show that every shape of weight w can be realized by a polynomial of degree $w + 1$, another of degree $w + 3$, and so on. The proposer also omits this consideration, but to be fair the determination *by degree* was added by the editors. Stong found that all of the sign charts enumerated can be achieved by polynomials of the required degree, but his proof would have been too long to present with all details filled in.

Also (partially) solved by R. Stong and the proposer.

Partitioning Segments Into Triples

11657 [2012, 608]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Given a set V of n points in $\mathbb{R}^2$, no three of them collinear, let E be the set of $\binom{n}{2}$ line segments joining distinct elements of V.

(**a**) Prove that if $n \not\equiv 2 \pmod 3$, then E can be partitioned into triples in which the length of each segment is smaller than the sum of the other two.

(**b**) Prove that if $n \equiv 2 \pmod 3$ and e is an element of E, then $E \setminus \{e\}$ can be so partitioned.

Solution by the proposers. Part **(a)** is immediate for $n = 3$; we prove **(a)** for $n = 4$ and **(b)** for $n = 5$ and proceed by induction on n. For $x, y \in V$, let $|xy|$ denote the length of the segment xy. Since E is the edge set of the complete graph with vertex set V, for $x, y, z, w \in V$ we evoke terminology from graph theory by letting $\curlywedge wxyz$ denote the graph with vertex set $\{w, x, y, z\}$ and edge set $\{wx, wy, wz\}$ (a *claw*), letting $\sqcup xyzw$ denote the graph with vertex set $\{x, y, z, w\}$ and edge set $\{xy, yz, zw\}$ (a *path*), and letting $\triangle xyz$ denote the graph with vertex set $\{x, y, z\}$ and edge set $\{xy, xz, yz\}$ (a *triangle*).

When three segments satisfy the condition that the length of each is smaller than the sum of the other two, we say that the triple is *triangular*; this holds for any triangle $\triangle xyz$ and also for other triples of edges, including some paths and claws.

For $n = 4$, let $V = \{x, y, z, w\}$. Without loss of generality, let xy be a shortest segment in E, and label z and w so that $|xz| + |yz| \le |xw| + |yw|$. Using this inequality and the triangles containing zw, we have $2(|xw| + |yw|) \ge (|xw| + |yw|) + (|xz| + |yz|) = (|xw| + |xz|) + (|yw| + |yz|) > 2|zw|$, so $|xw| + |yw| > |zw|$. Also, $|xw| + |zw| \ge |xw| + |xy| > |yw|$ and $|yw| + |zw| \ge |yw| + |xy| > |xw|$. Hence the triangular triples $\triangle xyz$ and $\curlywedge wxyz$ suffice.

For $n = 5$, let $V = \{x_1, x_2, y_1, y_2, y_3\}$ and $E' = E \setminus \{x_1x_2\}$. We partition E' into three triangular triples. If $\sqcup x_1y_iy_jx_2$ is triangular, where $\{i, j, k\} = \{1, 2, 3\}$, then we use $\{\sqcup x_1y_iy_jx_2, \triangle x_1y_jy_k, \triangle x_2y_iy_k\}$. If $\curlywedge x_iy_1y_2y_3$ is triangular, where $\{i, j\} = \{1, 2\}$, then we apply the case $n = 4$ to partition the segments formed by $\{x_j, y_1, y_2, y_3\}$ into two triangular triples and use $\curlywedge x_iy_1y_2y_3$ as the third.

In the remaining case for $n = 5$, no path of the form $\sqcup x_1y_iy_jx_2$ and no claw of the form $\curlywedge x_iy_1y_2y_3$ is triangular. If y_iy_j is a longest segment in E', then $|y_iy_j| \ge |x_1y_i| + |x_2y_j|$ and $|y_iy_j| \ge |x_1y_j| + |x_2y_i|$; otherwise, $\sqcup x_1y_iy_jx_2$ or $\sqcup x_1y_jy_ix_2$ is triangular. This yields the contradiction

$$2|y_iy_j| \ge (|x_1y_i| + |x_1y_j|) + (|x_2y_j| + |x_2y_i|) > 2|y_iy_j|.$$

Therefore, any longest segment in E' has the form x_iy_j. Index the points so x_1y_1 is a longest segment and $|x_1y_1| \ge |x_1y_2| \ge |x_1y_3|$. Now $|x_1y_1| \ge |x_1y_2| + |x_1y_3|$ (otherwise $\curlywedge x_1y_1y_2y_3$ is triangular), and $|x_1y_3| + |y_3y_1| > |x_1y_1|$ (since $\triangle x_1y_1y_3$ is triangular), so $|y_1y_3| > |x_1y_2|$. Since $\sqcup x_1y_1y_3x_2$ is not triangular and x_1y_1 is a longest segment, $|x_1y_1| \ge |y_1y_3| + |y_3x_2|$. If y_1y_3 is a longest segment in $\sqcup x_1y_3y_1x_2$, then $|y_1y_3| \ge |x_1y_3| + |y_1x_2|$, yielding the contradiction $|x_1y_1| \ge |x_1y_3| + |y_3x_2| + |x_2y_1|$ (combining two triangles). Since $|y_1y_3| > |x_1y_2| \ge |x_1y_3|$, we conclude that x_2y_1 is longest in $\sqcup x_1y_3y_1x_2$. Since $\sqcup x_1y_3y_1x_2$ is not triangular, we obtain the contradiction $|x_2y_1| \ge |y_1y_3| + |y_3x_1| > |x_1y_1|$, finishing the case $n = 5$.

Now consider $n \ge 6$. In the case $n \not\equiv 2 \pmod 3$, let xy be a shortest segment in E, and let z be a point in V such that $|xy| + |yz| = \min_{w \notin \{x,y\}}\{|xw| + |yw|\}$. The proof given for $n = 4$ shows that $\curlywedge wxyz$ is triangular for all $w \notin \{x, y, z\}$. Hence $\triangle xyz$ and the claws $\curlywedge wxyz$ for $w \notin \{x, y, z\}$ can be combined with a partition of the pairs in $V \setminus \{x, y, z\}$ (obtained by the induction hypothesis) to complete the desired partition.

Finally, in the case $n \not\equiv 2 \pmod 3$, let $e = uv$. Let xy be a shortest segment determined by the points in $V \setminus \{u, v\}$, and let z be a point in $V \setminus \{x, y, u, v\}$ such that $|xy| + |yz|$ is the minimum of $|xw + yw|$ over all $w \notin \{x, y, u, v\}$ such that $xw, yw \in E$. Again, every claw $\curlywedge wxyz$ is triangular when $w \notin \{x, y, z, u, v\}$. Apply the case $n = 5$ for the pairs in $\{x, y, z, u, v\}$, and apply the induction hypothesis to the $n - 3$ points in $V \setminus \{x, y, z\}$, in both cases choosing $e = uv$ as the deleted line segment. Combine the resulting partitions with the claws $\curlywedge wxyz$ for $w \notin \{x, y, z, u, v\}$.

Editorial comment. The result applies to any metric space, since the proof given needs only the property that every actual triangle is triangular. The problem was printed with an unfortunate typo in the rewording of the triangle inequality, with the word "greater" appearing instead of "smaller."

Also solved by R. Stong.

Pentagonal Series as Limit

11659 [2012, 608]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let x be real with $0 < x < 1$, and consider the sequence $\langle a_n \rangle$ given by $a_0 = 0$, $a_1 = 1$, and, for $n > 1$,

$$a_n = \frac{a_{n-1}^2}{xa_{n-2} + (1 - x)a_{n-1}}.$$

Show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

Solution by Greg Martin, University of British Columbia, Vancouver, BC, Canada. Take the reciprocal of the recurrence relation and multiply both sides by a_{n-1} to get

$$\frac{a_{n-1}}{a_n} = x\,\frac{a_{n-2}}{a_{n-1}} + (1 - x), \qquad (n \geq 2).$$

Let $r_n = a_n/a_{n+1}$. We now have $r_0 = 0$ and the recurrence $r_n = xr_n + 1 - x$ for $n \geq 1$. The solution to this is $r_n = 1 - x^n$. Therefore

$$\lim_{n\to\infty} \frac{1}{a_n} = \lim_{n\to\infty} \prod_{k=1}^{n-1} \frac{a_k}{a_{k+1}} = \lim_{n\to\infty} \prod_{k=1}^{n-1} r_k = \lim_{n\to\infty} \prod_{k=1}^{n-1} (1 - x^k) = \prod_{k=1}^{\infty} (1 - x^k).$$

By Euler's pentagonal number theorem, for $|x| < 1$ this infinite product is equal to $\sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}$.

Also solved by M. Bataille (France), D. Beckwith, P. Bracken, B. Bradie, N. Caro (Brazil), R. Chapman (U. K.), H. Chen, J. E. Cooper III, P. P. Dályay (Hungary), E. S. Eyeson, S. M. Gagola Jr., C. Georghiou (Greece), O. Geupel (Germany), A. Habil (Syria), E. A. Herman, R. Howard, M. Kim (Korea), O. Kouba (Syria), W. C. Lang, J. Li, J. C. Linders (Netherlands), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), T. L. McCoy, R. Nandan, M. Omarjee (France), P. Perfetti (Italy), C. R. Pranesachar (India), M. A. Prasad (India), D. N. Sanyasi (India), R. K. Schwartz, C. R. Selvaraj & S. Selvaraj, N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, E. I. Verriest, J. Vinuesa (Spain), T. Viteam (Chile), M. Vowe (Switzerland), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

An Unusual Differential Equation

11660 [2012, 609]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Consider the following differential equation: $s''(t) = -s(t) - s(t)^2 \operatorname{sgn}(s'(t))$, where $\operatorname{sgn}(u)$ denotes the sign of u. Show that if $s(0) = a$ and $s'(0) = b$ with $ab \neq 0$, then (s, s') tends to $(0, 0)$ with $\sqrt{s^2 + s'^2} \leq C/t$ as $t \to \infty$, for some $C > 0$.

Solution by GCHQ Problem Solving Group, Cheltenham, UK. The claim does not always hold. Suppose that $s(t)$ is the position of a particle at time t, and denote $s'(t)$ by $v(t)$. If we let $s_1(t)$ denote s when $s(0) = a$ and $s'(0) = b$ and $s_2(t)$ denote s when $s(0) = -a$ and $s'(0) = -b$, then $s_2(t) = -s_1(t)$ for all t and the claim holds for s_1 if and only if it holds for s_2. Without loss of generality, we may assume $a \geq 0$. Whenever $v > 0$, the equation is $v\frac{dv}{ds} = -s - s^2$. Hence integration yields

$$\frac{1}{2}(v^2 - V^2) = \frac{1}{2}(S^2 - s^2) + \frac{1}{3}(S^3 - s^3),$$

where $(s, v) = (S, V)$ at some point of the motion. Whenever $v < 0$, the equation is $v\frac{dv}{ds} = -s + s^2$, so

$$\frac{1}{2}(v^2 - U^2) = \frac{1}{2}(R^2 - s^2) + \frac{1}{3}(s^3 - R^3),$$

where $(s, v) = (R, U)$ at some point of the motion. Thus, if $a, b > 0$ such that $3a^2 + 3b^2 + 2a^3 \geq 5$, then $v^2 = a^2 + b^2 - s^2 + \frac{2}{3}(a^3 - s^3)$, so the particle moves to the right until $v = 0$ or $s^2 + \frac{2}{3}s^3 = a^2 + b^2 + \frac{2}{3}a^3 \geq \frac{5}{3}$.

Letting α denote the smallest solution, we have $\alpha \geq 1$ and $\alpha^2 > \alpha$. When s reaches α, the particle stops instantaneously and reverses its direction. As the particle attempts to go left, $\operatorname{sgn}(s'(t))$ flips; since $\alpha^2 \geq \alpha$, the net acceleration then acts to the right and motion is instantaneously reversed again. Therefore $(s, v) = (\alpha, 0)$ for all future time.

It is not true that $s^2 + s'^2 \leq C/t$ as $t \to \infty$, but it is trivially true that $(s - \alpha)^2 + s'^2 \leq C/t$. Note also $s''(t) \not\to 0$ as $t \to \infty$, even when $\alpha = 1$. Similarly, if $a > 0$ and $b < 0$ with $3a^2 + 3b^2 - 2a^3 \geq 0$, then $v^2 = a^2 + b^2 - s^2 + \frac{2}{3}(s^3 - a^3)$. The particle moves to the left until $s^2 - \frac{2}{3}s^3 = a^2 + b^2 - \frac{2}{3}a^3 \geq \frac{5}{3}$. Letting β denote the largest negative solution, we have $\beta \leq -1$. Therefore, once s reaches β, we have $(s, v) = (\beta, 0)$ for all future time.

Now we show that the claim holds for the opposite inequality, and that $C = 10$ will suffice. Consider the case in which $a \geq 0$, $b > 0$, and $3a^2 + 2a^3 + 3b^2 < 5$. The particle moves to the right until $s = \alpha$, where $\alpha^2 + \frac{2}{3}\alpha^3 = a^2 + b^2 + \frac{2}{3}a^3$. Since $\alpha < 1$, the particle then accelerates left with motion governed by $v^2 = \alpha^2 - s^2 + \frac{2}{3}(s^3 - \alpha^3)$, stopping instantaneously when $0 = 3(\alpha^2 - s^2) + 2(s^3 - \alpha^3) = (\alpha - s)(3(\alpha + s) - 2(\alpha^2 + \alpha s + s^2)) = (\alpha - s)f(s)$, say. With $f(s)$ defined this way, we have $f(0) = \alpha(3 - 2\alpha) > 0$ and $f(\alpha) = 6\alpha(1 - \alpha) > 0$, so $f(s)$ has one negative root and one root larger than α. The particle will stop when s is the smaller of these roots, which we denote by β, where

$$\beta = \frac{1}{4}\left[3 - 2\alpha - \sqrt{(3 - 2\alpha)(3 + 6\alpha)}\,\right] = -\frac{1}{4}\sqrt{3 - 2\alpha}\left[\sqrt{3 + 6\alpha} - \sqrt{3 - 2\alpha}\,\right]. \tag{1}$$

Since $f(-\alpha) = -2\alpha^2 < 0$, we have $\beta > -\alpha$ or $|\beta| < |\alpha|$. The particle now accelerates to the right, obeying $v^2 = \beta^2 - s^2 + \frac{2}{3}(\beta^3 - s^3)$, and stops when $0 = 3(\beta^2 - s^2) + 2(\beta^3 - s^3) = (\beta - s)(3(\beta + s) + 2(\beta^2 + \beta s + s^2)) = (\beta - s)g(s)$, say. With $g(s)$ defined this way, we have $g(\beta) = 6\beta(1 + \beta) < 0$ and $g(0) = \beta(3 + 2\beta) < 0$.

Thus $g(s)$ has one positive root and one root smaller than β. The particle will stop when s is the larger of these roots, which we denote γ. We have

$$\gamma = \frac{1}{4}\Big[-(3+2\beta)+\sqrt{(3+2\beta)(3-6\beta)}\,\Big] = \frac{1}{4}\sqrt{3+2\beta}\Big[\sqrt{3-6\beta}-\sqrt{3+2\beta}\,\Big].$$

Since $g(-s) = 2\beta^2 > 0$, we know that $s_{++} < -\beta$ or $|s_{++}| < |\beta|$. The sequence of distances from the origin to the stationary points is decreasing and bounded below by 0 and must converge. If it converges to a positive limit P, then by (1) it would have to satisfy $-4P = 3-2P-\sqrt{(3-2P)(3+6P)}$, which yields $P = 0$. Hence $s \to 0$ and also $v \to 0$. Define T_0 to be the time that the particle comes to rest following its first rightward motion. Define $T_1, T_2, T_3, \ldots$ to be the times for subsequent rests, with odd (even) indices following leftward (rightward) motion. Let $s = S_j$ at time T_j and let $U_j = |S_j|$. Consider the motion of the particle during the cycle from $s = S_{2j}$ through $s = S_{2j+1}$ to $s = S_{2j+2}$. We have the following bounds: $v < 0$ implies $v^2 = S_{2j}^2 - s^2 + \frac{2}{3}(s^3 - S_{2j}^3)$, which implies $\sqrt{s^2+s'^2} \le U_{2j}$; if $v > 0$ then $v^2 = S_{2j+1}^2 - s^2 + \frac{2}{3}(S_{2j+1}^3 - s^3)$, which implies $\sqrt{s^2+s'^2} \le U_{2j+1}$.

We now show that $\sqrt{s^2+s'^2} \le C/t$ in four steps (**a**)–(**d**).

(**a**) $T_n < T_0 + n\pi$. If $v < 0$, then $s \ge S_{2j+1}$. Hence $v^2 = S_{2j}^2 - \frac{2}{3}S_{2j}^3 - s^2 + \frac{2}{3}s^3 \ge S_{2j}^2 - \frac{2}{3}S_{2j}^3 - s^2(1-\frac{2}{3}S_{2j+1})$ so the velocity is larger and travel time less than when travelling under $\frac{ds}{dt} = \sqrt{S_{2j}^2 - \frac{2}{3}S_{2j}^3 - s^2(1-\frac{2}{3}S_{2j+1})}$. This implies

$$t = \int \frac{ds}{\sqrt{S_{2j}^2 - \frac{2}{3}S_{2j}^3 - s^2(1-\frac{2}{3}S_{2j+1})}} = \frac{1}{\sqrt{1-\frac{2}{3}S_{2j+1}}}\arcsin\left(s\sqrt{\frac{1-\frac{2}{3}S_{2j+1}}{S_{2j}^2-\frac{2}{3}S_{2j}^3}}\right).$$

Therefore,

$$T_{2j+1} - T_{2j} \le \frac{1}{\sqrt{1-\frac{2}{3}S_{2j+1}}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{\sqrt{1+\frac{2}{3}U_{2j+1}}} < \pi.$$

When $v > 0$, we have $s \le S_{2j+2}$, so $v^2 = S_{2j+1}^2 + \frac{2}{3}S_{2j+1}^2 - s^2 - \frac{2}{3}s^3 \ge S_{2j+1}^2 + \frac{2}{3}S_{2j+1}^3 - s^2(1+\frac{2}{3}S_{2j+1})$, which yields $T_{2j+2} - T_{2j+1} < \pi$.

(**b**) $U_{j+1} < U_j - \frac{1}{3}U_j^2$. From above, we have $0 < U_j \le 3/2$ and $U_{j+1} = \frac{1}{4}\sqrt{3-2U_j}(\sqrt{3+6U_j}-\sqrt{3-2U_j})$. Now $0 < 9 - 6U_j + 2U_j^2$, so $0 < 8u_j^2 - \frac{16}{3}U_j^3 + \frac{16}{9}U_j^4$. Together with our range for U_j, this implies $0 < 9 + 12U_j - 12U_j^2 < (3 + 2U_j - \frac{4}{3}U_j^2)^2$. Taking the square root now yields $\sqrt{(3+6U_j)(3-2U_j)} < 3 + 2U_j - \frac{4}{3}U_j^2 = (3-2U_j) + (4U_j - \frac{4}{3}U_j^2)$. This implies (**b**).

(**c**) If $n \ge 0$, then $1/U_n \ge 1/U_0 + \frac{n}{3}$. Clearly this holds for $n = 0$, so assume it holds up to some $n = N$. By factoring, we have

$$\left(N+\frac{3}{U_0}\right)^2 - 1 < \left(N+\frac{3}{U_0}\right)^2 \Longrightarrow \frac{N+3/U_0-1}{(N+3/U_0)^2} < \frac{1}{N+3/U_0+1}.$$

The expression $x - \frac{1}{3}x^2$ is increasing on the range of interest, so

$$U_{N+1} < U_N - \frac{1}{3}U_N^2 < \frac{3}{N+3/U_0} - \frac{3}{(N+3/U_0)^2}$$
$$= \frac{3(N+3/U_0-1)}{(N+3/U_0)^2} < \frac{1}{N+3/U_0+1}.$$

The result holds for $n = N + 1$ and the induction is complete. Equality holds only when $n = 0$.

(**d**) $\sqrt{s^2 + s'^2} \le C/t$. For $T_n \le t \le T_{n+1}$, we have

$$\frac{1}{\sqrt{s^2+s'^2}} \ge \frac{1}{U_n} \ge \frac{1}{U_0} + \frac{n}{3} > \frac{1}{U_0} + \frac{T_n - T_0}{3\pi} \ge \frac{1}{U_0} + \frac{(t-\pi) - T_0}{3\pi} > \frac{t}{C}$$

for $C = 10$ and for sufficiently large t. Any choice of C with $C > 3\pi$ will work.

If $a \ge 0$ and $b \le 0$, then we obtain the same asymptotic result, and $C = 10$ again suffices.

Also solved by E. A. Herman, O. P. Lossers (Netherlands), R. Stong, D. B. Tyler, E. I. Verriest, TCDmath Problem Group (Ireland), and the proposer.

More Triangle Inequalities

11664 [2012, 699]. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ, and Darij Grinberg, Massachusetts Institute of Technology, Cambridge, MA.* Let a, b, and c be the side lengths of a triangle. Let s denote the semiperimeter, r the inradius, and R the circumradius of that triangle. Let $a' = s - a$, $b' = s - b$, and $c' = s - c$.
(**a**) Prove that $\frac{ar}{R} \le \sqrt{b'c'}$.
(**b**) Prove that

$$\frac{r(a+b+c)}{R}\left(1 + \frac{R-2r}{4R+r}\right) \le 2\left(\frac{b'c'}{a} + \frac{c'a'}{b} + \frac{a'b'}{c}\right).$$

Solution for (a) by Alper Ercan, Istanbul, Turkey. Recall these formulas, involving the area Δ of the triangle:

$$\Delta = rs = \frac{abc}{4R}, \qquad \Delta^2 = sa'b'c'.$$

The inequality to be proved becomes $4a'\sqrt{b'c'} \le (a'+b')(a'+c')$, because $bc = (a'+b')(a'+c')$. By the AM–GM inequality, $2\sqrt{a'b'} \le a' + b'$ and $2\sqrt{a'c'} \le a' + c'$. The desired inequality follows.

Solution for (b) by Paolo Perfetti, Università degli Studi di Roma "Tor Vergata", Rome, Italy. Recall

$$R = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}},$$

$$r = \frac{abc}{4Rs} = \frac{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}}{s}.$$

For convenience, write $x = a'$, $y = b'$, $z = c'$, and $D = (x+y)(y+z)(z+x)$. The inequality to be proved becomes

$$\frac{8(x+y+z)xyz}{D}\,\frac{5D - 4xyz}{4D + 4xyz} \le 2\left[\frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}\right].$$

Clearing denominators, we see that this is equivalent to

$$(xy)^3 + (yz)^3 + (zx)^3 + 3(xyz)^2 \ge x^3y^2z + y^3z^2x + z^3x^2y,$$

which, with $\alpha = xy$, $\beta = yz$, and $\gamma = zx$, becomes Schur's third-degree inequality $\alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta\gamma \ge \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$.

Editorial comment. Some solvers noted that part (a) is related to Problem 11306, this **Monthly 116** (2009) 88–89. Part (b) was also solved using various other geometric inequalities, such as Kooi's inequality.

Also solved by R. Boukharfane (Canada), M. Can, R. Chapman (U. K.), P. P. Dályay (Hungary), J. Fabrykowski & T. Smotzer, O. Geupel (Germany), M. Goldenberg & M. Kaplan, W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), P. Nüesch (Switzerland), V. Pambuccian, N. Stanciu & Z. Zvonaru (Romania), R. Stong, M. Vowe (Switzerland), J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

Inequalities for Inner Product Space

11667 [2012, 700]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Softwin Co., Bucharest, Romania.* Let f, g, and h be elements of an inner product space over $\mathbb{R}$, with $\langle f, g\rangle = 0$.
(**a**) Show that

$$\langle f, f\rangle\langle g, g\rangle\langle h, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

(**b**) Show that

$$(\langle f, f\rangle\langle h, h\rangle)\langle h, f\rangle^2 + (\langle g, g\rangle\langle h, h\rangle)\langle g, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

Solution I by Pál Péter Dályay, Szeged, Hungary. If f, g, or h is zero, then the inequalities clearly hold. Since $\langle f, g\rangle = 0$, note that

$$e = \frac{\langle h, f\rangle}{\langle f, f\rangle} f + \frac{\langle h, g\rangle}{\langle g, g\rangle} g$$

is the orthogonal projection of h onto the space spanned by $\{f, g\}$, and therefore $\|h\|^2 = \|e\|^2 + \|h - e\|^2$. Thus, $\|h\|^2 \geq \|e\|^2 = \langle h, f\rangle^2/\|f\|^2 + \langle h, g\rangle^2/\|g\|^2 \geq 2\langle h, f\rangle\langle h, g\rangle/(\|f\| \cdot \|g\|)$. Squaring both sides gives (a). By the AM–GM inequality,

$$\begin{aligned}\langle f,f\rangle\langle h,h\rangle\langle h,f\rangle^2+\langle g,g\rangle\langle h,h\rangle\langle g,h\rangle^2 &\geq 2\sqrt{\langle f,f\rangle\langle g,g\rangle(\langle h,h\rangle)^2(\langle h,f\rangle)^2(\langle g,h\rangle)^2} \\ &= 2\cdot\left[\sqrt{\langle f,f\rangle\langle g,g\rangle}\langle h,h\rangle\right]\cdot\langle h,f\rangle\langle g,h\rangle.\end{aligned}$$

By (a), the bracketed term is at least $2|\langle g, h\rangle\langle h, f\rangle|$, so (b) follows.

Solution II by Paolo Perfetti, Dipartimento di Matematica, Università Degli Studi di Roma, Rome, Italy. If f, g, or h is zero, then the result clearly holds, so we may define $F = f/\|f\|$, $G = g/\|g\|$, and $H = h/\|h\|$. Now (a) reads $\langle h, h\rangle^2 \geq 4\langle G, h\rangle^2\langle F, h\rangle^2$. By Bessel's inequality, $\|h\|^2 \geq \langle G, h\rangle^2 + \langle F, h\rangle^2 \geq 2\langle G, h\rangle\langle F, h\rangle$, and (a) follows by squaring this result. Similarly, part (b) reads $\|f\|^2\langle H, f\rangle^2 + \|g\|^2\langle H, g\rangle^2 \geq 4\langle H, f\rangle^2\langle H, g\rangle^2$. Again by AM–GM, $\|f\|^2\langle H, f\rangle^2+\|g\|^2\langle H, g\rangle^2 \geq 2\|f\| \cdot \|g\| \cdot |\langle H, f\rangle\langle H, g\rangle|$, and it remains to show $\|f\| \cdot \|g\| \geq 2|\langle H, f\rangle\langle H, g\rangle|$. When we multiply both sides by $\frac{\|h\|}{\|f\|\cdot\|g\|}$, this becomes $\|h\| \geq 2|\langle h, F\rangle\langle h, G\rangle|$, which is essentially (a).

Also solved by K. Andersen (Canada), G. Apostolopoulos (Greece), R. Boukharfane (Canada), P. Bracken, R. Chapman (U. K.), A. Ercan (Turkey), D. Fleischman, C. Georghiou (Greece), O. Geupel (Germany), K. Hanes, E. A. Herman, F. Holland, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), M. A. Prasad (India), N. C. Singer, R. Stong, R. Tauraso (Italy), T. Trif (Romania), D. B. Tyler, E. I. Verriest, J. Vinuesa (Spain), R. Wyant & T. Smotzer, GCHQ Problem Solving Group (U. K.), and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before February 28, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11796. *Proposed by Gleb Glebov, Simon Fraser University, Burnaby, Canada.* Find

$$\int_0^\infty \frac{\sin((2n+1)x)}{\sin x} e^{-\alpha x} x^{m-1}\, dx$$

in terms of α, m, and n, when $\alpha > 0$, $m \geq 1$, and n is a nonnegative integer.

11797. *Proposed by Zhang Yun, Xi'an, Shaanxi Province, China.* Let A_1, A_2, A_3, and A_4 be the vertices of a tetrahedron. Let h_k be the length of the altitude from A_k to the plane of the opposite face, and let r be the radius of the inscribed sphere. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \geq \frac{12}{5}.$$

11798. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* For positive integers n, let f_n be the polynomial given by

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r} x^{\lfloor r/2 \rfloor}.$$

(**a**) Prove that if $n+1$ is prime, then f_n is irreducible over $\mathbb{Q}$.
(**b**) Prove that for all n (whether $n+1$ is prime or not),

$$f_n(1+x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k} x^k.$$

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11799. *Proposed by Vicenţiu Rădulescu, King Abdulaziz University, Jeddah, Saudi Arabia.* Let a, b, and c be positive.

(**a**) Prove that there is a unique continuously differentiable function f from $[0,\infty)$ into $\mathbb{R}$ such that $f(0)=0$ and, for all $x \ge 0$,

$$f'(x)\left(1+a|f(x)|^b\right)^c = 1.$$

(**b**) Find, in terms of a, b, and c, the largest θ such that $f(x)=O(x^\theta)$ as $x\to\infty$.

11800. *Proposed by Oleksiy Klurman, University of Montreal, Montreal, Canada.* Let f be a continuous function from $[0,1]$ into $\mathbb{R}^+$. Prove that

$$\int_0^1 f(x)\,dx - \exp\left[\int_0^1 \log f(x)\,dx\right] \le \max_{0\le x,y\le 1}\left(\sqrt{f(x)}-\sqrt{f(y)}\right)^2.$$

11801. *Proposed by David Carter, Nahant, MA.* Let f be a polynomial in one variable with rational coefficients that has no nonnegative real root. Show that there is a nonzero polynomial g with rational coefficients such that the coefficients of fg are positive.

11802. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let $H_{n,2}=\sum_{k=1}^n k^{-2}$, and let $D_n = n!\sum_{k=0}^n(-1)^k/k!$. (This is the *derangement number* of n, that is, the number of permutations of $\{1,\dots,n\}$ that fix no element.) Prove that

$$\sum_{n=1}^\infty H_{n,2}\frac{(-1)^n}{n!} = \frac{\pi^2}{6e} - \sum_{n=0}^\infty \frac{D_n}{n!(n+1)^2}.$$

SOLUTIONS

An Uncountable Linearly Independent Set of Binary Sequences

11658 [2012, 608]. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.* Let V be the vector space over $\mathbb{R}$ of all (countably infinite) sequences $(x_1,x_2,\dots)$ of real numbers, equipped with the usual addition and scalar multiplication. For $v\in V$, say that v is *binary* if $v_k\in\{0,1\}$ for $k\ge 1$, and let B be the set of all binary members of V. Prove that there exists a subset I of B with cardinality $2^{\aleph_0}$ that is linearly independent over $\mathbb{R}$. (An infinite subset of a vector space is linearly independent if all of its finite subsets are linearly independent.)

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI. Given a bijection $\phi\colon \mathbb{N}\to\mathbb{Q}$, for each $r\in\mathbb{R}$, define $v(r)\in B$ by

$$v(r)_k = \begin{cases} 1 & \text{if } \phi(k)\le r,\\ 0 & \text{if } \phi(k) > r.\end{cases}$$

Let $I=\{v(r)\colon r\in\mathbb{R}\}$. We claim first that v is injective. Given $r,r'\in\mathbb{R}$ with $r<r'$, let q be a rational number between r and r'. Let $k=\phi^{-1}(q)$. Since $v(r)_k=0$ and $v(r')_k=1$, we have $v(r)\ne v(r')$. Thus I, $\mathbb{R}$, and B have the same cardinality, $2^{\aleph_0}$.

We show also that I is a linearly independent subset of B. Given $\sum_{i=1}^n a_i v(r_i)=\vec{0}$ for distinct real numbers $r_1,\dots,r_n$, we may assign indices so that $r_1<\dots<r_n$. Let

q be a rational number between r_{n-1} and r_n. Let $k = \phi^{-1}(q)$. We have $v(r_n)_k = 1$, but $v(r_i)_k = 0$ for $i < n$. Thus a_n must be 0. Dropping the nth term from the sum and repeating the argument eventually shows that all coefficients equal 0. Thus I is linearly independent.

Also solved by O. Antolin-Camarena, P. Budney, C. Burnette, N. Caro (Brazil), R. Chapman (U. K.), S. M. Gagola Jr., K. P. Hart (Netherlands), E. A. Herman, S. J. Herschkorn, R. Howard, Y. J. Ionin, O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), E. Ordman, V. Pambuccian, S. K. Patel & A. K. Desai (India), P. Perfetti (Italy), M. Rajeswari (India), C. P. Rupert, S. Scheinberg, R. Stong, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

The Harmonious Quartets of the Faces of a Cube

11662 [2012, 609]. *Proposed by H. Stephen Morse, Fairfax, Va.* Let $ABCD$ be the vertices of a square, in that order. Insert P and Q on AB (in the order $AQPB$) so that each of P and Q divides AB 'in extreme and mean ratio' (that is, $|AB|/|BQ| = |BQ|/|QA|$ and $|AB|/|AP| = |AP|/|PB|$.) The four intersection points of AP, BR, CQ, and DP are called the *harmonious quartet* of the square on its *base pair* (AB,CD). They form a rhombus whose long diagonal has length $(\sqrt{5}+1)/2$ times the length of its short diagonal.

Given a cube, create the harmonious quartet for each of its six faces, using each edge as part of a base pair exactly once, according to this scheme: label the vertices on one face of the cube $ABCD$ and the corresponding vertices of the bottom face $A'B'C'D'$. Pair AB with CD, AA' with BB', and BC with $B'C'$. The rest of the pairings are then forced: $A'B'$ with $C'D'$, AD with $A'D'$, and CC' with DD'. This generates 24 points.

(**a**) Show that these 24 points are a subset of the 32 vertices of a *rhombic triacontahedron* (a convex polyhedron bounded by 30 congruent rhombic faces, meeting three each across their obtuse angles at 20 vertices, and five each across their acute angles at 12 vertices), and find a construction for the remaining eight vertices.

(**b**) Show, moreover, that the 12 end points of the longer diagonals of the six constructed rhombi are the vertices of an icosahedron I, and that these diagonals are edges of the icosahedron.

(**c**) Show that the 12 end points of the shorter diagonals of the constructed rhombi, together with the eight additional vertices of the triacontahedron, are the vertices of a dodecahedron. Show also that these shorter diagonals are edges of that dodecahedron.

Solution by Robin Chapman, University of Exeter, Exeter, England, U.K. There is an error in the statement of the question: the roles of P and Q in the definition of extreme and mean ratio need to be reversed, yielding $|AB|/|BP| = |BP|/|PA|$ and $|AB|/|AQ| = |AQ|/|QB|$.

Start with just a line segment AB on the number line, A and B being the points -1 and 1, respectively. If P is at point x, then $\frac{2}{1-x} = \frac{1-x}{1+x}$. This gives a quadratic equation $x^2 - 4x - 1 = 0$ with solutions $x = 2 \pm \sqrt{5}$. Since $-1 < x < 1$, we have $x = 2 - \sqrt{5}$. By symmetry $Q = -2 + \sqrt{5}$. (Hence $APQB$ occur in this order; the original statement would have given the order $AQPB$.)

Consider the square $ABCD$ in the Cartesian plane, with A, B, C, D at $(-1, 1)$, $(1, 1)$, $(1, -1)$, and $(-1, -1)$, respectively. Now P, Q, R, and S are at $(2-\sqrt{5}, 1)$, $(\sqrt{5}-2, 1)$, $(\sqrt{5}-2, -1)$, and $(2-\sqrt{5}, -1)$, respectively. Letting $\tau = (1+\sqrt{5})/2$, we compute that lines AR and DQ meet at the point $(-\tau^{-2}, 0)$, lines BS and CP meet at $(\tau^{-2}, 0)$, lines AR and BS meet at $(0, -\tau^{-1})$, and lines CP and DQ meet at $(0, \tau^{-1})$.

Consider the cube $ABCDA'B'C'D'$ in $\mathbb{R}^3$ with A, B, C, D, A', B', C', and D' at points $(-1, 1, 1)$, $(1, 1, 1)$, $(1, -1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, 1, -1)$, $(1, -1, -1)$, and $(-1, -1, -1)$, respectively. In this coordinate system, the set of base pairs is invariant under the rotation $(x, y, z) \mapsto (y, z, x)$, so the 24 points of the six harmonious quartets are $(\pm\tau^{-2}, 0, \pm 1)$, $(0, \pm\tau^{-1}, \pm 1)$ and their images under cyclic permutations of the coordinates. Let $\mathcal{L}$ be the 12-element set of cyclic permutations of $(0, \pm\tau^{-1}, \pm 1)$ (the vertices of the longer diagonals of the six harmonious quartets) and let $\mathcal{S}$ be the 12-element set of cyclic permutations of $(\pm\tau^{-2}, 0, \pm 1)$ (the vertices of the shorter diagonals of the six harmonious quartets).

Now let $\mathcal{E}$ be the 8-element set with elements $(\pm\tau^{-1}, \pm\tau^{-1}, \pm\tau^{-1})$. We will show that $\mathcal{L} \cup \mathcal{S} \cup \mathcal{E}$, $\mathcal{L}$, and $\mathcal{S} \cup \mathcal{E}$ are the vertex sets of the required rhombic triacontahedron, regular icosahedron, and regular dodecahedron, respectively. The points of $\mathcal{E}$ can be geometrically constructed in various ways. One way is to divide the line segments from the centre of the original cube to its vertices in golden section.

We describe the faces of the rhombic triacontahedron in some detail. The six harmonious quartets account for six of the faces. Consider the four points $W = (0, \tau^{-1}, 1)$, $X = (\tau^{-2}, 0, 1)$, $Y = (1, 0, \tau^{-1})$ and $Z = (\tau^{-1}, \tau^{-1}, \tau^{-1})$ of $\mathcal{L} \cup \mathcal{S} \cup \mathcal{E}$. The midpoints of WY and of XZ are both $(1/2, \tau^{-1}/2, \tau/2)$. Therefore W, X, Y, and Z are coplanar and the vertices of a rhombus. Points W and Y lie in $\mathcal{L}$ and are $2\tau^{-1}$ apart; points X and Z lie in $\mathcal{S} \cup \mathcal{E}$ and are at distance $2\tau^{-2}$ apart. Thus the rhombus $WXYZ$ is congruent to each harmonious quartet. Taking images of $WXYZ$ under combinations of cyclic permutations of coordinates and under reflections perpendicular to the coordinate axes give 24 rhombi. These account for the remaining faces of the rhombic triacontahedron. To see how these fit together, we look at the faces meeting each vertex. By symmetry it suffices to consider one vertex from each of the sets $\mathcal{L}$, $\mathcal{S}$, and $\mathcal{E}$.

The vertex $(0, \tau^{-1}, 1) \in \mathcal{L}$ is adjacent to the vertices $(\tau^{-2}, 0, 1)$, $(\tau^{-1}, \tau^{-1}, \tau^{-1})$, $(0, 1, \tau^{-2})$, $(-\tau^{-1}, \tau^{-1}, \tau^{-1})$, and $(-\tau^{-2}, 0, 1)$ in cyclic order. The five vertices $(1, 0, \tau^{-1})$, $(\tau^{-1}, 1, 0)$, $(-\tau^{-1}, 1, 0)$, $(-1, 0, \tau^{-1})$, and $(0, -\tau^{-1}, 1)$ in $\mathcal{L}$ complete the five rhombi surrounding $(0, \tau^{-1}, 1)$.

The vertex $(\tau^{-2}, 0, 1) \in \mathcal{S}$ is adjacent to the vertices $(0, \tau^{-1}, 1)$, $(1, 0, \tau^{-1})$, and $(0, -\tau^{-1}, 1)$. The three vertices $(\tau^{-1}, \tau^{-1}, \tau^{-1})$, $(\tau^{-1}, -\tau^{-1}, \tau^{-1})$, and $(-\tau^{-2}, 0, 1)$ in $\mathcal{S} \cup \mathcal{E}$ complete the three rhombi surrounding $(\tau^{-2}, 0, 1)$.

The vertex $(\tau^{-1}, \tau^{-1}, \tau^{-1}) \in \mathcal{E}$ is adjacent to the vertices $(1, 0, \tau^{-1})$, $(0, \tau^{-1}, 1)$, and $(\tau^{-1}, 1, 0)$. The three vertices $(\tau^{-2}, 0, 1)$, $(0, 1, \tau^{-2})$, and $(1, \tau^{-2}, 0)$ in $\mathcal{S}$ complete the three rhombi surrounding $(\tau^{-1}, \tau^{-1}, \tau^{-1})$.

It is a well-known property of the rhombic triacontahedron that the long diagonals of the faces are the edges of a regular icosahedron. The endpoints of these diagonals form the set $\mathcal{L}$ and these diagonals include the long diagonals of the six harmonious quartets of the vertices of the cube. Likewise the short diagonals of the rhombic triacontahedron are the edges of a regular dodecahedron. The endpoints of these diagonals include the short diagonals of the six harmonious quartets of the vertices of the cube.

Also solved by O. Geupel (Germany), O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), R. A. Simon (Chile), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Are Random Breaks the Altitudes of a Triangle?

11663 [2012, 699]. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* The unit interval is broken at two randomly chosen points along its length.

Show that the probability that the lengths of the resulting three intervals are the heights of a triangle is equal to

$$\frac{12\sqrt{5}\log((3+\sqrt{5})/2)}{25} - \frac{4}{5}.$$

Solution by David Farnsworth and James Marento, Rochester Institute of Technology, Rochester, NY. Consider $\triangle ABC$ with side lengths a, b, c, vertex angles α, β, γ, and heights x, y, z as in Figure 1. We have

$$\frac{z}{b} = \sin\alpha = \frac{y}{c} \tag{1}$$

$$\frac{z}{a} = \sin\beta = \frac{x}{c} \tag{2}$$

$$\frac{y}{a} = \sin\gamma = \frac{x}{b}. \tag{3}$$

If $\triangle ABC$ is not acute, then these relations still hold, even though two of the altitudes lie outside the triangle.

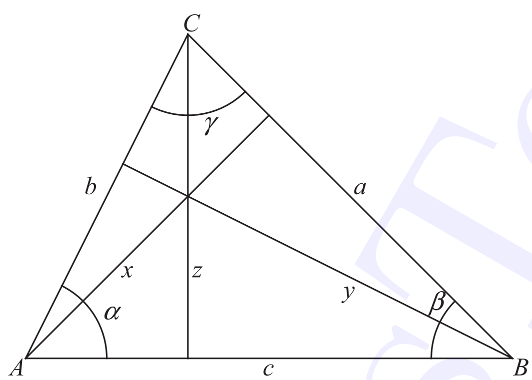


Figure 1.

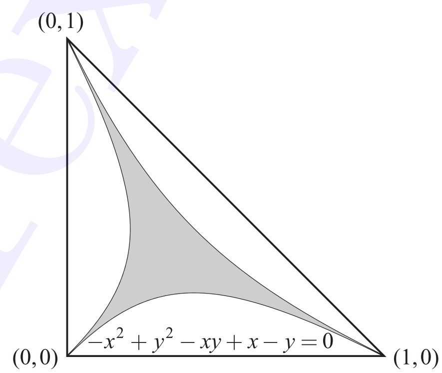


Figure 2.

From (2) and (3) we deduce

$$c < a+b \iff \frac{c}{a} < 1+\frac{b}{a} \iff \frac{x}{z} < 1+\frac{x}{y} \iff \frac{1}{z} < \frac{1}{x}+\frac{1}{y}. \tag{1'}$$

Similarly

$$b < a+c \iff \frac{1}{y} < \frac{1}{x}+\frac{1}{z} \tag{2'}$$

$$a < b+c \iff \frac{1}{x} < \frac{1}{y}+\frac{1}{z}. \tag{3'}$$

Therefore, by the triangle inequality, the three positive numbers x, y, z are the heights of a triangle if and only if the statements on the right of (1′), (2′), and (3′) all hold.

Now let x, y, z be the (random) lengths of the three intervals that are obtained by dividing the interval (0, 1) with two points that are independently chosen according to a uniform distribution on this interval. Since $z = 1 - (x + y)$, the inequalities on the right of (1′), (2′), and (3′) boil down to

$$x^2 + 3xy + y^2 - x - y < 0 \tag{1''}$$

$$-x^2 + y^2 - xy + x - y < 0 \tag{2''}$$

$$x^2 - y^2 - xy - x + y < 0. \tag{3''}$$

If "<" is replaced by "=" in these three inequalities, they become the equations of the three hyperbolas shown in Figure 2. Now (x, y) is uniformly distributed on the triangular region with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$. The probability that x, y, z are the heights of a triangle is therefore twice the shaded area in Figure 2. If E_k denotes the event that inequality (k'') holds for $k \in \{1, 2, 3\}$, then the probability of the shaded region is $1 - P(\overline{E_1} \cup \overline{E_2} \cup \overline{E_3})$.

Observe that if p, q, and r are any positive numbers, then at most one of the three inequalities $r < p + q$, $q < p + r$, $p < q + r$ is false. (If p, q, and r are the sides of a triangle, then of course none of the three inequalities is false.) So the three events $\overline{E_1}$, $\overline{E_2}$, and $\overline{E_3}$ are pairwise disjoint. By symmetry, these three events have the same probability. The desired probability P is therefore given by

$$\begin{aligned}
P = 1 - 3P(\overline{E_1}) &= 1 - 6\int_0^1 1 - x - \frac{1 - 3x + \sqrt{(3x-1)^2 - 4(x^2 - x)}}{2}\,dx \\
&= \frac{-7}{2} + 3\sqrt{5}\int_0^1 \sqrt{\left(x - \frac{1}{5}\right)^2 + \frac{4}{25}}\,dx = \frac{-7}{2} + \frac{12\sqrt{5}}{25}\int_{-1/2}^{2} \sqrt{u^2+1}\,du \\
&= \frac{-7}{2} + \frac{12\sqrt{5}}{25}\left[\frac{u\sqrt{u^2+1} + \log\left(u + \sqrt{u^2+1}\,\right)}{2}\right]_{u=-1/2}^{u=2} \\
&= \frac{12\sqrt{5}}{25}\log\left(\frac{3+\sqrt{5}}{2}\right) - \frac{4}{5} = \frac{24\sqrt{5}}{25}\log\phi - \frac{4}{5},
\end{aligned}$$

where ϕ is the golden ratio.

Also solved by G. Apostolopoulos (Greece), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (Canada), M. A. Carlton, R. Chapman (U. K.), C. Curtis, P. P. Dályay (Hungary), P. De (India), A. Ercan (Turkey), E. A. Herman, S. J. Herschkorn, B. Karaivanov, O. Kouba (Syria), J. Li, J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, M. Omarjee (France), O. Pavlyk, C. R. Pranesachar (India), M. A. Prasad (India), J. G. Simmonds, T. Smotzer, R. Stong, R. Tauraso (Italy), T. Trif (Romania), T. Viteam (Chile), M. Vowe (Switzerland), T. Wiandt, J. Zacharias, L. Zhang, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.

A Parity Problem for Derangements

11668 [2012, 700]. *Proposed by Dimitris Stathopoulos, Marousi, Greece.* For positive integer n and $i \in \{0, 1\}$, let $D_i(n)$ be the number of derangements on n elements whose number of cycles has the same parity as i. Prove that $D_1(n) - D_0(n) = n - 1$.

Solution I by Ronald E. Prather, Oakland, CA. We use induction on n. The traditional recurrence $D(n) = (n-1)[D(n-1) + D(n-2)]$ is proved by considering whether element n lies in a cycle of length more than 2 (following some element in a derangement of $[n-1]$) or in a cycle with just one other element (the rest forming a derangement of $n-2$ elements). In the first case the smaller permutation has the same number of cycles; in the second it has one less cycle. Thus

$$D_1(n) = (n-1)[D_1(n-1) + D_0(n-2)]$$
$$D_0(n) = (n-1)[D_0(n-1) + D_1(n-2)].$$

Since $D_0(0) = 1$ and $D_1(0) = D_1(1) = D_0(1) = 0$ validate the claim for $n \le 1$, for $n \ge 2$ we use the induction hypothesis to compute

$$\begin{aligned} D_1(n) - D_0(n) &= (n-1)[(D_1(n-1) - D_0(n-1)) - (D_1(n-2) - D_0(n-2))] \\ &= (n-1)[(n-2) - (n-3)] = n-1. \end{aligned}$$

Solution II by Richard Ehrenborg, University of Kentucky, Lexington, KY. Let c_k be the number of permutations of $[k]$ that are cycles, so $c_k = (k-1)!$. The exponential generating function for nontrivial cycles, indexed by length, is given by $C(x) = \sum_{k\ge 2} c_k x^k/k! = -\ln(1-x) - x$. By the Exponential Formula, the EGF for derangements is $e^{C(x)}$, obtained as $\sum_{m\ge 0}(C(x))^m/m!$; the term for m enumerates derangements with m cycles.

To incorporate parity of the number of cycles, let $E(x) = \sum_{m\ge 0}(-C(x))^m/m!$. The coefficient of x^n in $E(x)$ is the number of derangements having an even number of cycles minus the number having an odd number of cycles. Hence the desired value is the coefficient of x^n in $-E(x)$. We compute that $-E(x)$ is equal to

$$-e^{-C(x)} = -(1-x)e^x = xe^x - e^x = \sum_{n\ge 0} \frac{x^{n+1}}{n!} - \sum_{n\ge 0} \frac{x^n}{n!} = \sum_{n\ge 0} (n-1)\frac{x^n}{n!}.$$

Editorial comment. Robin Chapman observed that this problem is a trivial variation of Problem E907 (this *Monthly* **57** (1950), 184). That problem requested the numbers of even and odd derangements of n; the parity of a permutation of $[n]$ with k cycles is the parity of $n-k$. In addition to the approaches printed above, proofs are known using the determinant of the complement of the identity matrix and using signed involutions (see R. Chapman, An involution on derangements, *Discrete Math.* **231** (2001), 121–122).

Also solved by M. Andreoli, D. Beckwith, R. Boukharfane (Canada), R. Chapman (U.K.), C. Curtis, P. P. Dályay (Hungary), R. Ehrenborg, S. M. Gagola Jr., F. Galvin, O. Geupel (Germany), Y. J. Ionin, B. Karaivanov, J. H. Lindsey II, J. H. Nieto (Venezuela), C. R. Pranesachar (India), M. A. Prasad (India), R. E. Prather, R. Pratt, J. H. Steelman, R. Stong, R. Tauraso (Italy), T. Viteam (Chile), M. Wildon (U.K.), GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

A Surprise Visit from Fibonacci

11669 [2012,]. *Proposed by Herman Roelants, Catholic University of Leuven, Louvain, Belgium.* Prove that for $n \ge 4$ there exist integers $x_1, \dots, x_n$ such that

$$\frac{x_{n-1}^2 + 1}{x_n^2} \prod_{k=1}^{n-2} \frac{x_k^2+1}{x_k^2} = 1$$

satisfying the following conditions: $x_1 = 1$, $x_{k-1} < x_k < 3x_{k-1}$ for $2 \le k \le n-2$, $x_{n-2} < x_{n-1} < 2_{n-2}$, and $x_{n-1} < x_n < 2x_{n-1}$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO. Let $\{F_n\}$ be the Fibonacci numbers; that is, $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_k + F_{k+1}$ for $k \ge 0$.

Let $x_n = F_{2n-3}$, let $x_{n-1} = F_{2n-4}$, and let $x_k = F_{2k-1}$ for $1 \le k \le n-2$. We use the identities

$$F_{2i-1}F_{2i+3} = F_{2i+1}^2 + 1 \quad \text{and} \quad F_{2i-1}F_{2i+1} = F_{2i}^2 + 1,$$

which are instances of Catalan's identity $F_k^2 - F_{k+r}F_{n-r} = (-1)^{k-r}F_r^2$, first with $k = 2i+1$ and $r = 2$, and then with $k = 2i$ and $r = 1$. We have

$$\begin{aligned}
\frac{x_{n-1}^2+1}{x_n^2}\prod_{k=1}^{n-2}\frac{x_k^2+1}{x_k^2} &= \frac{F_{2n-4}^2+1}{F_{2n-3}^2}\cdot\frac{F_1^2+1}{F_1^2}\cdot\frac{F_3^2+1}{F_3^2}\prod_{k=3}^{n-2}\frac{F_{2k-1}^2+1}{F_{2k-1}^2} \\
&= \frac{5}{2}\cdot\frac{F_{2n-5}F_{2n-3}}{F_{2n-3}^2}\prod_{k=3}^{n-2}\frac{F_{2k-3}F_{2k+1}}{F_{2k-1}^2} \\
&= \frac{5}{2}\cdot\frac{F_{2n-5}F_{2n-3}}{F_{2n-3}^2}\,\frac{\prod_{k=2}^{n-3}F_{2k-1}}{\prod_{k=3}^{n-2}F_{2k-1}}\,\frac{\prod_{k=4}^{n-1}F_{2k-1}}{\prod_{k=3}^{n-2}F_{2k-1}} \\
&= \frac{5}{2}\cdot\frac{F_{2n-5}F_{2n-3}}{F_{2n-3}^2}\cdot\frac{F_3}{F_{2n-5}}\cdot\frac{F_{2n-3}}{F_5} = 1.
\end{aligned}$$

For $2 \le k \le n-2$, the condition $x_{k-1} < x_k < 3x_{k-1}$ is equivalent to $F_{2k-3} < F_{2k-1} = F_{2k-3} + F_{2k-2} < 3F_{2k-3}$, or $0 < F_{2k-2} < 2F_{2k-3}$. The condition $x_{n-2} < x_{n-1} < 2x_{n-2}$ becomes $F_{2n-5} < F_{2n-4} < 2F_{2n-5}$, which is true for $n \ge 4$. Finally, $x_{n-1} < x_n < 2x_{n-1}$ becomes $F_{2n-4} < F_{2n-3} < 2F_{2n-4}$, which holds similarly.

Also solved by D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, S. M. Gagola Jr., O. Geupel (Germany), Y. J. Ionin, S. Jo (Korea), B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), C. R. Pranesachar (India), M. A. Prasad (India), E. Schmeichel, N. C. Singer, T. Smotzer, R. Stong, D. B. Tyler, J. Van Hamme (Belgium), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

An Inequality

11670 [2012, 800]. *Proposed by Miranda Bakke, Benson Wu, and Bogdan Suceavă, California State University, Fullerton, CA.* Prove that if $n \ge 3$ and $a_1, \dots, a_n > 0$, then

$$\frac{(n-1)}{4}\sum_{k=1}^{n} a_k \ge \sum_{1\le j<k\le n}\frac{a_j a_k}{a_j + a_k},$$

with equality if and only if all a_j are equal.

Solution by Robert A. Agnew, Buffalo Grove, IL. By the Arithmetic–Harmonic Mean Inequality, we have $\frac{a_j+a_k}{2} \ge \frac{2a_j a_k}{a_j+a_k}$ with equality if and only if $a_j = a_k$. Thus

$$\frac{1}{4}\sum_{1\le j<k\le n}(a_j + a_k) \ge \sum_{1\le j<k\le n}\frac{a_j a_k}{a_j + a_k}$$

with equality if and only if all a_j are equal. Since each a_j occurs $n-1$ times in the first sum, the original inequality holds for $n \ge 2$.

Editorial comment. Charles Delorme (France) notes generalizations such as:

$$\frac{(n-1)(n-2)}{54}\sum_{k=1}^{n} a_k \ge \sum_{1\le i<j<k\le n}\frac{a_i a_j a_k}{(a_i + a_j + a_k)^2}.$$

Also solved by 75 others, including the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before April 30, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11803. *Proposed by Sam Speed, Germantown, PA.* Let $a_1(k,n) = (9^k(24n+5)-5)/8$, $a_2(k,n) = (9^k(24n+13)-5)/8$, $a_3(k,n) = (3\cdot 9^k(24n+7)-5)/8$, and $a_4(k,n) = (3\cdot 9^k(24n+23)-5)/8$. Show that for each nonnegative integer m there is a unique integer triple (j,k,n) with $j \in \{1,2,3,4\}$ and $k,n \geq 0$ such that $m = a_j(k,n)$.

11804. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Prove that $10|x^3+y^3+z^3-1| \leq 9|x^5+y^5+z^5-1|$ for real numbers x, y, and z with $x+y+z=1$. When does equality hold?

11805. *Proposed by Gleb Glebov, Simon Fraser University, Burnaby, Canada.*
(**a**) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}+\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{5\pi^3\sqrt{3}}{243}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}-\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{13}{18}\zeta(3).$$

(**b**) Prove that

$$\zeta(3)=\frac{9}{13}\int_0^1\frac{(\log x)^2}{x^3+1}\,dx-\frac{18}{13}\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}.$$

Here, ζ denotes the Riemann zeta function.

http://dx.doi.org/10.4169/amer.math.monthly.121.10.946

11806. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Prove that

$$\int_0^{2\pi} \log\Gamma\left(\frac{x}{2\pi}\right) e^{\cos x}\sin(x+\sin x)\,dx = (e-1)(\log(2\pi)+\gamma) + \sum_{n=2}^{\infty}\frac{\log n}{n!}.$$

Here Γ denotes the gamma function and γ denotes the Euler–Mascheroni constant.

11807. *Proposed by Robin Oakapple, Albany, OR.* Given a quadrilateral $ABCD$ inscribed in a circle K, and a point Z inside K, the rays AZ, BZ, CZ, and DZ meet K again at points E, F, G, and H, respectively, to yield another quadrilateral also inscribed in K. Develop a construction that takes as input A, B, C, and D and returns a point Z such that this second quadrilateral has (at least) three of its sides of equal length.

11808. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let Γ be the gamma function. Compute

$$\lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{(n!)^{-1/n}} \Gamma(nx)\,dx.$$

11809. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let $\langle a_n\rangle$ be a sequence of real numbers.
(**a**) Suppose that $\langle a_n\rangle$ consists of nonnegative numbers and is nonincreasing, and $\sum_{n=1}^{\infty} a_n/\sqrt{n}$ converges. Prove that $\sum_{n=1}^{\infty}(-1)^{\lfloor\sqrt{n}\rfloor}a_n$ converges.
(**b**) Find a nonincreasing sequence $\langle a_n\rangle$ of positive numbers such that $\lim_{n\to\infty}\sqrt{n}a_n = 0$ and $\sum_{n=1}^{\infty}(-1)^{\lfloor\sqrt{n}\rfloor}a_n$ diverges.

11795. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let p be the partition counting function on the set $\mathbb{Z}^+$ of positive integers, and let g be the function on $\mathbb{N}$ given by $g(n) = \frac{1}{2}\lceil n/2\rceil\,,\lceil(3n+1)/2\rceil$. Let $A(n)$ be the set of *nonnegative* integer triples (i, j, k) such that $g(i)+j+k = n$. Prove for $n \ge 1$ that

$$p(n) = \frac{1}{n}\sum_{(i,j,k)\in A(n)}(-1)^{\lceil i/2\rceil - 1} g(i)p(j)p(k).$$

SOLUTIONS

Large Sum of Sizes Implies Large Size of Sum

11666 [2012, 699–700]. *Proposed by Dmitry G. Fon-Der-Flaass (1962–2010), Institute of Mathematics, Novosibirsk, Russia, and Max A. Alekseyev, University of South Carolina, Columbia, SC.* Let m be a positive integer, and let A and B be nonempty subsets of $\{0,1\}^m$. Let n be the greatest integer such that $|A|+|B| > 2^n$. Prove that $|A+B| \ge 2^n$. (Here, $|X|$ denotes the number of elements in X, and $A+B$ denotes $\{a+b\colon\ a\in A, b\in B\}$, where addition of vectors is componentwise modulo 2.)

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove by induction on n that $|A|+|B| > 2^n$ implies $|A+B| \ge 2^n$ (for any n). The case $n = 0$ is trivial, since the sum of sets that are not both empty is nonempty.

Consider $n \geq 1$. If $|B| = 1$, then $|A + B| = |A| > 2^n - 1$, which suffices. By symmetry, we may therefore assume $|A|, |B| \geq 2$. Choose $w, x \in A$ and $y, z \in B$. Given nonzero elements u and v in $\{0, 1\}^m$ as an additive group, there is a homomorphism $\phi\colon \{0, 1\}^m \to \{0, 1\}$ such that $\phi(u) = \phi(v) = 1$. With $u = x - w$ and $v = z - y$, we obtain $\phi(x) \neq \phi(w)$ and $\phi(y) \neq \phi(z)$. For $i \in \{0, 1\}$, let $A_i = \{v \in A\colon \phi(v) = i\}$, and similarly for B_i. By construction, the four sets are nonempty, and $A + B$ is the disjoint union of $(A_0 + B_0) \cup (A_1 + B_1)$ (mapping to 0) and $(A_0 + B_1) \cup (A_1 + B_0)$ (mapping to 1).

Since $|A_0| + |A_1| + |B_0| + |B_1| > 2^n$, at least one of $|A_0| + |B_0|$ and $|A_1| + |B_1|$ exceeds 2^{n-1}, and similarly at least one of $|A_0| + |B_1|$ and $|A_1| + |B_0|$ exceeds 2^{n-1}. By the induction hypothesis, both sets in our decomposition of $A + B$ have size at least 2^{n-1}, so $|A + B| \geq 2^n$.

Editorial comment. Most solvers used induction. Traian Viteam and Robin Chapman used the combinatorial nullstellensatz. O.P. Lossers and Pál Peter Dályáy used Kneser's theorem.

Also solved by G. Apostolopoulos (Greece), R. Chapman (U. K.), P. P. Dályáy (Hungary), A. Habil (Syria), Y. J. Ionin, O. P. Lossers (Netherlands), R. Tauraso (Italy), T. Viteam (Chile), and the proposers.

The Gambler's Ruin in Disguise

11672 [2012, 800]. *Proposed by José Luis Palacios, Universidad Simón Bolívar, Caracas, Venezuela.* A random walk starts at the origin and moves up-right or down-right with equal probability. What is the expected value of the first time that the walk is k steps below its then-current all-time high? (Thus, for instance, with the walk UDDUUUUDDUDD$\cdots$, the walk is three steps below its maximum-so-far on step 12.)

Solution I by Padraig Condon, Trinity College, Dublin, Ireland. The answer, which we denote by p_k, is $k(k+1)$. Since p_1 is the expected time of the first D, we have $p_1 = 2$. Let q_k be the expected number of steps to reach k steps below the all-time high given a starting point $k - 1$ steps below the all-time high. Note that $q_1 = p_1 = 2$. To reach k steps below the all-time high, we must first reach $k - 1$ steps below the all-time high. Hence, $p_k = p_{k-1} + q_k$ for $k \geq 2$.

From a point $k - 1$ steps below the all-time high, after the next step with equal probability we are k or $k - 2$ steps below the all-time high. In the first case, we have arrived, while in the second case we must first return to $k - 1$ steps below the all-time high. Thus, the expected number of steps in the second case is $q_{k-1} + q_k$. Hence, $q_k = 1 + \frac{1}{2}(q_{k-1} + q_k)$, which simplifies to $q_k = 2 + q_{k-1}$. With $q_1 = 2$, we obtain $q_k = 2k$. Thus, $p_k = p_{k-1} + 2k$, and $p_1 = 2$ yields $p_k = \sum_{i=1}^{k} 2i = k(k+1)$.

Solution II by Richard Stong, San Diego, CA. We prove that the answer is $k(k+1)$ by expressing the problem in terms of the stopping time of the classical Gambler's Ruin problem in which one gambler starts with \$$k$ dollars and the other with \$$(k+1)$, and each step transfers \$1 from one gambler to the other, each direction having equal probability. Interpret upward and downward moves as wins and losses by the gambler currently having less money, respectively. (The total amount of money is odd so there can never be a tie.) At every step, each outcome has probability $1/2$.

At each time, the gambler with less money has $k - m$ dollars exactly when we are m steps below the current all-time high. This is true at the start and is easily checked to be preserved by each move. The only interesting case is when we move to a new all-time high. Before the move, the money is split $k + 1$ to k. The gambler with less money wins, the path reaches a new high and the money is again split $k+1$ to k, but the two gamblers interchange roles.

In the classical problem starting with \$$a$ and \$$b$, it is well known that the expected number of steps until one gambler is ruined is ab, in this case $k(k+1)$. We have shown that this also is the expected number of steps until the path is first k steps below its all-time high.

Also solved by M. Andreoli, W. Barta, R. Chapman (U. K.), C. Delorme (France), S. J. Herschkorn, G. Lau (U. K.), O. P. Lossers (Netherlands), H. M. Mahmoud, R. Martin (Germany), I. Pinelis, M. A. Prasad (India), R. Pratt, R. Tauraso (Italy), M. Wildon (U. K.), and the proposer.

Polynomials with Galois Groups of 2-power Order

11673 [2012, 800]. *Proposed by Kent Holing, Statoil, Trondheim, Norway.* Let Q and g be monic polynomials in $\mathbb{Z}[x]$, with Q an irreducible quartic, and let $f = Q \circ g$. Suppose that f is irreducible over $\mathbb{Q}$ and that the order of the Galois group of F is a power of 2. Which groups are possible as the Galois group of Q? If, moreover, Q has negative discriminant, determine the Galois group of A.

Solution by Robin Chapman, University of Exeter, Exeter, England, UK. The possible Galois groups of Q are the eight-element dihedral group D_4, the cyclic group Z_4, and the Klein four-group V_4. If the discriminant of Q is negative, then the group is D_4.

Let K and L be the splitting fields of Q and f, respectively, over $\mathbb{Q}$. The zeroes of f are the solutions β of $g(\beta) = \alpha$ such that α is a root of Q. Hence, $K \subseteq L$. Thus, if the Galois group of f has order a power of 2, then the index $[L : \mathbb{Q}]$ is a power of 2. Since $[K : \mathbb{Q}]$ is a factor of $[L : \mathbb{Q}]$, it follows that $[K : \mathbb{Q}]$ is also a power of 2. Since Q is an irreducible quartic, its Galois group is a transitive subgroup of S_4, and the only such subgroups whose order is a power of 2 are Z_4, V_4, and D_4.

Each of these groups can occur as the Galois groups of an irreducible quartic Q, such as when Q is $x^4 - 4x^2 + 2$, $x^4 + 1$, or $x^4 - 2$, respectively. In each case, one can take $g(x) = x$ or, less trivially, $g(x) = x^{2^k}$ for any positive integer k.

If Q has negative discriminant, then it has two real and two nonreal zeroes. Complex conjugation thus induces an element with order 2 in the Galois group that fixes one of the zeroes of Q, that is, a transposition. Of the three possibilities previously mentioned, only D_4 has such an element. This possibility does occur when Q is $x^4 - 2$.

Also solved by P. P. Dályay (Hungary), J. H. Lindsey II, R. Stong, M. Wildon (U. K.), and the proposer.

Norm of a Linear Functional

11674 [2012, 800]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let a and b be real numbers with $a < 0 < b$. Let S be the set of continuous functions f from $[0, 1]$ to $[a, b]$ with $\int_0^1 f(x)\,dx = 0$. Let g be a strictly increasing function from $[0, 1]$ to $\mathbb{R}$. Define ϕ from S to $\mathbb{R}$ by $\phi(f) = \int_0^1 f(x)g(x)\,dx$.
(a) Find $\sup_{f \in S} \phi(f)$ in terms of a, b, and g.
(b) Prove that this supremum is not attained.

Solution by Earl R. Barnes, Morgan State University, Baltimore, MD. Let $\xi = \frac{b}{b-a}$. Since $a < 0 < b$, we have $0 < \xi < 1$. Note that $\int_0^\xi a\,dx + \int_\xi^1 b\,dx = 0$. Let $\lambda = g(\xi)$. For any f satisfying the conditions of the problem, we have $\int_0^1 f(x)\,dx = 0$ and $a \le f(x) \le b$, so

$$\phi(f) = \int_0^1 (g(x)-\lambda)\, f(x)\, dx = \int_0^\xi (g(x)-\lambda) f(x)\, dx + \int_\xi^1 (g(x)-\lambda) f(x)\, dx$$

$$\leq \int_0^\xi (g(x)-\lambda)a\, dx + \int_\xi^1 (g(x)-\lambda)b\, dx = a\int_0^\xi g(x)\, dx + b\int_\xi^1 g(x)\, dx.$$

Equality can hold if and only if $f(x) = a$ a.e. on $[0, \xi]$ and $f(x) = b$ a.e. on $[\xi, 1]$. This can happen only if f is discontinuous at ξ, so the inequality is strict for all $f \in S$. On the other hand, this upper bound can be approached as closely as we like by choosing ε small and positive and taking $f(x) = a$ for $0 \leq x \leq \xi - \varepsilon$, $f(x) = b$ for $[\xi + \varepsilon, 1]$, and f linear on the interval $[\xi - \varepsilon, \xi + \varepsilon]$.

Also solved by. K. F. Andersen (Canada), R. Bagby, P. Bracken, R. Chapman (U. K.), S. J. Herschkorn, B. Karaivanov, O. Kouba (Syria), J. C. Linders (Netherlands), J. H. Lindsey II, O. P. Lossers (Netherlands), I. Pinelis, Á. Plaza (Spain), A. Stenger, R. Stong, E. I. Verriest, S. V. Witt, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

An Inequality for the Partition Function

11675 [2012, 801]. *Proposed by Mircea Merca, Constantin Istrati Technical College, Campina, Romania.* Let p be the Euler partition function, i.e., $p(n)$ is the number of nondecreasing lists of positive integers that sum to n. Let $p(0) = 1$, and let $p(n) = 0$ for $n < 0$. Prove that for $n \geq 0$ with $n \neq 3$,

$$p(n) - 4p(n-3) + 4p(n-5) - p(n-8) > 0.$$

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. Let $P(x)$ be the generating function for integer partitions,

$$P(x) = \sum_{n=0}^\infty p(n)x^n = \prod_{n=1}^\infty \frac{1}{1-x^n}.$$

We have to prove that for $n \neq 3$, the coefficient of x^n in $(1 - 4x^3 + 4x^5 - x^8)P(x)$ is positive. Since

$$1-4x^3+4x^5-x^8 = (x^2-x^3)(1-x)(1-x^2)+(1+x+x^2)(1-x)(1-x^2)(1-x^3),$$

we must prove positivity of the coefficient of x^n for $n \neq 3$ in

$$(x^2 - x^3)\prod_{n=3}^\infty \frac{1}{1-x^n} + (1 + x + x^2)\prod_{n=4}^\infty \frac{1}{1-x^n}.$$

It is clear that the coefficient of x^n in the second term is positive for $n \neq 3$, so it is sufficient to show that the coefficient of x^n in the first term is nonnegative for $n \neq 3$. It suffices to show that for $n \geq 4$, among partitions with all parts at least 3, there are at least as many with sum $n - 2$ as with sum $n - 3$. This follows from the injection that adds 1 to the largest part.

Also solved by G. Apostolopoulos (Greece), D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), C. Delorme (France), I. Gessel, J.-P. Grivaux (France), Y. J. Ionin, O. P. Lossers (Netherlands), M. A. Prasad (India), R. Stong, R. Tauraso (Italy), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

A Powered Gamma Limit

11676 [2012, 801]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania.* For real t, find

$$\lim_{x\to\infty} x^{\sin^2 t}\left(\Gamma(x+2)^{(\cos^2 t)/(x+1)} - \Gamma(x+1)^{\cos^2 t/x}\right).$$

Here, Γ is the gamma function.

Solution by Santiago de Luxán, Fraunhofer Heinrich-Hertz-Institute, Berlin. We will prove a generalization: with $f(t) = \cos^2 t e^{-\cos^2 t}$, and for $a, b \in \mathbb{R}$,

$$L(a,b) = \lim_{x\to\infty} x^{\sin^2 t}\left[\Gamma(x+a)^{\cos^2 t/(x+a-1)} - \Gamma(x+b)^{\cos^2 t/(x+b-1)}\right] = (a-b)f(t).$$

Assume that the limit exists (applying L'Hopital's rule at the end of the calculation verifies that it does). Apply Stirling's formula to both $\Gamma(x+a)$ and $\Gamma(x+b)$ to obtain

$$\begin{aligned} L(a,b) &= \lim_{x\to\infty} x^{\sin^2 t}\left(\left(\frac{x+a-1}{e}\right)^{\cos^2 t} - \left(\frac{x+b-1}{e}\right)^{\cos^2 t}\right) \\ &= \lim_{x\to\infty} e^{-\cos^2 t} x^{1-\cos^2 t}\left((x+a-1)^{\cos^2 t} - (x+b-1)^{\cos^2 t}\right) \\ &= e^{-\cos^2 t} \lim_{x\to\infty} \frac{\left(\frac{x+a-1}{e}\right)^{\cos^2 t} - \left(\frac{x+b-1}{e}\right)^{\cos^2 t}}{1/x}. \end{aligned}$$

This is an indeterminate limit that can be evaluated using l'Hopital's rule:

$$\begin{aligned} L(a,b) &= f(t)\lim_{x\to\infty}\left((a-1)\left(1+\frac{a-1}{x}\right)^{-\sin^2 t} - (b-1)\left(1+\frac{b-1}{x}\right)^{-\sin^2 t}\right) \\ &= f(t)\big((a-1)-(b-1)\big) = (a-b)\cos^2 t\, e^{-\cos^2 t}. \end{aligned}$$

For the case in the problem as stated, $a = 2$ and $b = 1$ so $L = e^{-\cos^2 t}\cos^2 t$.

Also solved by K. F. Andersen (Canada), R. Boukharfane (Canada), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), A. Ercan (Turkey), D. Fleischman, C. Georghiou (Greece), O. Geupel (Germany), M. L. Glasser, J.-P. Grivaux (France), O. Kouba (Syria), K.-W. Lau (China), J. Li, O. P. Lossers (Netherlands), H. M. Mahmoud, G. Martin (Canada)R. Nandan, M. Omarjee (France), P. Perfetti (Italy), I. Pinelis, R. Stong, D. B. Tyler, GCHQ Problem Solving Group (U. K.), and the proposers.

Dedekind η Function Disguised

11677 [2012, 880]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Evaluate

$$\prod_{n=1}^{\infty}\left(1 + 2e^{-n\pi\sqrt{3}}\cosh(n\pi/\sqrt{3})\right).$$

Solution by Radouan Boukharfane, Polytechnique Montréal, Montreal, Canada. The answer is $e^{\pi\sqrt{3}/18}/\sqrt[4]{3}$. We use the Dedekind η function defined for a complex number t with positive imaginary part by

$$\eta(t) = e^{\frac{\pi i t}{12}}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n t}\right).$$

It is well known that this function satisfies the functional equation $\eta(-1/t) = \sqrt{-it}\,\eta(t)$. Put $t = i/\sqrt{3}$, and use $-1/t = 3t$ to derive

$$\begin{aligned}
&\prod_{n=1}^{\infty}\left(1+2e^{-n\pi\sqrt{3}}\cosh(n\pi/\sqrt{3})\right)\\
&\quad=\prod_{n=1}^{\infty}\left(1+e^{-n\pi\sqrt{3}}\left(e^{\frac{n\pi}{\sqrt{3}}}+e^{-\frac{n\pi}{\sqrt{3}}}\right)\right)\\
&\quad=\prod_{n=1}^{\infty}\left(1+e^{2\pi int}+e^{4\pi int}\right)=\frac{\prod_{n=1}^{\infty}\left(1-e^{6\pi int}\right)}{\prod_{n=1}^{\infty}\left(1-e^{2\pi int}\right)}=\frac{e^{-\frac{\pi it}{4}}\eta(3t)}{e^{-\frac{\pi it}{12}}\eta(t)}\\
&\quad=e^{-\frac{\pi it}{6}}\frac{\eta(-1/t)}{\eta(t)}=e^{\frac{\pi}{6\sqrt{3}}}\sqrt{-it}=\frac{e^{\frac{\pi\sqrt{3}}{18}}}{\sqrt[4]{3}}.
\end{aligned}$$

Editorial comment. Unfortunately the problem appeared with typos, making the product divergent. Solutions showing divergence were also accepted.

Also solved by G. Apostolopoulos (Greece), R. Chapman (U. K.), D. Fleischman, O. Geupel (Germany), M. Omarjee (France), R. Stong, R. Tauraso (Italy), and the proposer.

The Determinant of the Fibonacci Matrix

11678 [2012, 880]. *Proposed by Farrukh Ataev Rakhimjanovich, Westminster International University in Tashkent, Tashkent, Uzbekistan.* Let F_k be the kth Fibonacci number, where $F_0 = 0$ and $F_1 = 1$. For $n \geq 1$, let A_n be an $(n+1)\times(n+1)$ matrix with entries $a_{j,k}$ given by $a_{0,k} = a_{k,0} = F_k$ for $a \leq k \leq n$ and by $a_{j,k} = a_{j-1,k} + a_{j,k-1}$ for $j, k \geq 1$. Compute the determinant of A_n.

Solution by Yuri Ionin, Central Michigan University, Mount Pleasant, MI. We show that the determinant is -2^{n-1}.

Let us call an m-by-m matrix an *NW-matrix* if each entry not in the the first row or column equals the sum of its northern and western neighbors; furthermore, it is a *unit NW-matrix* if all entries in the first column equal 1. Index the rows and columns from 1 to m. We claim that every unit NW-matrix has determinant 1. We use induction on m; the claim is immediate for $m = 1$.

For $m \geq 2$, let X be a unit NW-matrix of order m. Obtain Y from X by subtracting each row from the row immediately below it, leaving row 1 unchanged. Column 1 of Y is all 0 except for 1 in the first row. For $i \geq 2$, we have $y_{i,2} = x_{i,2} - x_{i-1,2} = x_{i,1} = 1$. Also, for $i \geq 3$ and $j \geq 2$,

$$y_{i,j} = x_{i.j} - x_{i-1,j} = (x_{i-1.j} + x_{i,j-1}) - (x_{i-2,j} + x_{i-1,j-1}) = y_{i-1,j} + y_{i,j-1},$$

so Y is an NW-matrix. The matrix Z obtained from Y by deleting the first row and column is a unit NW-matrix; by the induction hypothesis, $\det Z = 1$. Expanding the determinant of Y along the first column yields $\det(X) = \det(Y) = \det(Z) = 1$.

Clearly $\det A_1 = -1$, so choose $n \geq 2$. Obtain B from A_n by leaving the first two rows unchanged and subtracting from each subsequent row the two rows immediately above it; note that $\det B = \det A_n$. For $i \geq 3$ and $j \geq 1$,

$$\begin{aligned}
b_{i,j} - b_{i,j-1} &= (a_{i,j} - a_{i-1,j} - a_{i-2,j}) - (a_{i,j-1} - a_{i-1,j-1} - a_{i-2,j-1})\\
&= a_{i-1,j} - a_{i-2,j} - a_{i-3,j} = b_{i-1,j}.
\end{aligned}$$

The matrix B takes the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 2 & 3 & \cdots \\ 0 & 0 & 2 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 2 & \cdots \end{bmatrix}.$$

Expanding $\det B$ along the first two columns yields $\det B = -\det(2W)$, where W is a unit NW-matrix of order $n-1$. By the claim, $\det A_n = -2^{n-1}\det(W) = -2^{n-1}$.

Editorial comment. Sergio Falcón and Ángel Plaza observed that the problem and its solution appear as an example in A. R. Moghaddamfar, S. M. H. Pooya, Generalized Pascal triangles and Toeplitz matrices, *Electron. J. Lin. Alg.* **18** (2009), 564–588. Ionin's matrix W, with i, j-entry $\binom{i+j}{i}$, is shown to have determinant 1 via four proofs in A. Edelman and G. Strang, Pascal Matrices, this MONTHLY **111** (2004), 189–197; an earlier such proof appears in C. A. Rupp, Problem 3468, this MONTHLY **37** (1930), 552 (solution by H.T.R. Aude, **38** (1931), 355).

Also solved by D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), C. Delorme (France), S. Falcón & Á. Plaza (Spain), O. Geupel (Germany), J. P. Grivaux (France), E. A. Herman, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), R. E. Prather, C. P. Rupert, R. Stong, R. Tauraso (Italy), J. van Hamme (Belgium), Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.

Lower Bound on a Product

11679 [2012, 000]. *Proposed by Tim Keller, Orangeville, CT.* Let n be an integer greater than 2, and let $a_2, \dots, a_n$ be positive real numbers with product 1. Prove that

$$\prod_{k=2}^{n}(1+a_k)^k > \frac{2}{e}\left(\frac{n}{2}\right)^{2n-1}.$$

Solution by Traian Viteam, Punta Arenas, Chile. For $n = 2$ the inequality reduces to $4 > 2/e$, which is trivial. For $2 < k \le n$, the AM–GM inequality implies that

$$(1+a_k)^k = \left(\frac{1}{k-2} + \cdots + \frac{1}{k-2} + \frac{a_k}{2} + \frac{a_k}{2}\right)^k \ge \frac{k^k}{(k-2)^{k-2}}\frac{a_k^2}{4}.$$

Multiplying $(1+a_2)^2 > a_2^2$ together with these inequalities for $k \in \{3, \dots, n\}$ yields

$$\prod_{k=2}^{n}(1+a_k)^k > \frac{(n-1)^{n-1}n^n}{2^2}\frac{a_2^2\cdots a_n^2}{4^{n-2}} = 2\left(1-\frac{1}{n}\right)^{n-1}\left(\frac{n}{2}\right)^{2n-1},$$

since $\prod_{i=2}^{n} a_i = 1$ is assumed. Using $e^x \ge 1 + x$ with $x = \frac{1}{n-1}$, it follows that

$$e \ge \left(1+\frac{1}{n-1}\right)^{n-1} = \left(1-\frac{1}{n}\right)^{-(n-1)}$$

and hence $(1-\frac{1}{n})^{n-1} \ge e^{-1}$. Substituting this inequality into the product inequality above yields the stated result.

Also solved by G. Apostolopoulos (Greece), R. Boukharfane (Canada), E. Eyeson, D. Fleischman, N. Grivaux (France), S. Kaczkowski, O. Kouba (Syria), O. P. Lossers (Netherlands), R. E. Prather, D. B. Tyler, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted by the proposer to the internet before the due date for solutions. Submitted solutions should arrive before May 31, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11810. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $H_n = \sum_{k=1}^{n} 1/k$, and let ζ be the Riemann zeta function. Find

$$\sum_{n=1}^{\infty} H_n \left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3} \right).$$

11811. *Proposed by Vazgen Mikayelyan, Yerevan State University, Yerevan, Armenia.* Let $\langle a \rangle$ and $\langle b \rangle$ be infinite sequences of positive numbers. Let $\langle x \rangle$ be the infinite sequence given for $n \geq 1$ by

$$x_n = \frac{a_1^{b_1} \cdots a_n^{b_n}}{\left(\dfrac{a_1 b_1 + \cdots + a_n b_n}{b_1 + \cdots + b_n} \right)^{b_1 + \cdots + b_n}}.$$

(**a**) Prove that $\lim_{n\to\infty} x_n$ exists.
(**b**) Find the set of all c that can occur as that limit, for suitably chosen $\langle a \rangle$ and $\langle b \rangle$.

11812. *Proposed by Cristian Chiser, Craiova, Romania.* Let f be a twice continuously differentiable function from $[0, 1]$ into $\mathbb{R}$. Let p be an integer greater than 1. Given that $\sum_{k=1}^{p-1} f(k/p) = -\frac{1}{2}(f(0) + f(1))$, prove that

$$\left(\int_0^1 f(x)\, dx \right)^2 \leq \frac{1}{5!\, p^4} \int_0^1 (f''(x))^2\, dx.$$

http://dx.doi.org/10.4169/amer.math.monthly.122.01.75

11813. *Proposed by Greg Oman, University of Colorado-Colorado Springs, Colorado Springs, CO.* Let X be a set, and let $*$ be a binary operation on X (that is, a function from $X \times X$ to X). Prove or disprove: there exists an uncountable set X and a binary operation $*$ on X such that for any subsets Y and Z of X that are closed under $*$, either $Y \subseteq Z$ or $Z \subseteq Y$.

11814. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let ϕ be a continuously differentiable function from $[0, 1]$ into $\mathbb{R}$, with $\phi(0) = 0$ and $\phi(1) = 1$, and suppose that $\phi'(x) \neq 0$ for $0 \leq x \leq 1$. Let f be a continuous function from $[0, 1]$ into $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = \int_0^1 \phi(x) f(x)\,dx$. Show that there exists t with $0 < t < 1$ such that $\int_0^t \phi(x) f(x)\,dx = 0$.

11815. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let x, y, and z be positive numbers such that $x + y + z = 3$. Prove that

$$\frac{x^4 + x^2 + 1}{x^2 + x + 1} + \frac{y^4 + y^2 + 1}{y^2 + y + 1} + \frac{z^4 + z^2 + 1}{z^2 + z + 1} \geq 3xyz.$$

11816. *Proposed by Sabin Tabirca, University College Cork, Cork, Ireland.* Let ABC be an acute triangle, and let B_1 and C_1 be the points where the altitudes from B and C intersect the circumcircle. Let X be a point on arc BC, and let B_2 and C_2 denote the intersections of XB_1 with AC and XC_1 with AB, respectively. Prove that the line B_2C_2 contains the orthocenter of ABC.

SOLUTIONS

If the Sum of the Squares is the Square of the Sum, . . .

11671 [2012, 800]. *Proposed by Sam Northshield, SUNY-Plattsburgh, Plattsburgh, NY.* Show that if relatively prime integers a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2,$$

then $|a + b + c|$ can be written as $m^2 - mn + n^2$ for some integers m and n.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity. Note that $m^2 - mn + n^2$ is the norm of $m + n\omega$ in the number ring $\mathbb{Z}[\omega]$. This ring is a unique factorization domain. The primes that split in this number ring are 3 and all primes congruent to 1 modulo 3. Thus a positive integer can be written in the form $m^2 - mn + n^2$ if and only if every prime congruent to 2 modulo 3 divides it an even number of times.

Let $g = \gcd(a + b + c, a + b + d, a + c + d, b + c + d)$. Now $(a + b + d) + (a + c + d) + (b + c + d) - 2(a + b + c) = 3d$ and symmetrically, and since $\gcd(a, b, c, d) = 1$, g is a divisor of 3.

Thus for any prime p congruent to 2 modulo 3 that divides $a + b + c$, we can choose one of $a + b + d$, $a + c + d$, and $b + c + d$ that is not divisible by p. Rewriting the given equality as

$$(a + b + d)(a + b + c) = a^2 - a(-b) + (-b)^2,$$

we see that p divides the right side with even multiplicity and hence divides $a + b + c$ with even multiplicity. By the remarks above, $a + b + c$ can be written in the form $m^2 - mn + n^2$ for some integers m and n.

Also solved by G. Apostolopoulos (Greece), R. Chapman (U. K.), P. P. Dálylay (Hungary), Y. J. Ionin, O. P. Lossers (Netherlands), C. R. Pranesachar (India), M. A. Prasad (India), J. P. Robertson, T. Viteam (Chile), GCHQ Problem Solving Group (U. K.), and the proposer.

Carlson's Inequality

11680 [2012, 880]. *Proposed by Benjamin Bogoşel, University of Savoie, Savoie, France, and Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let $x_1, \dots, x_n$ be nonnegative real numbers. Show that

$$\left(\sum_{i=1}^{n} \frac{x_i}{i}\right)^4 \le 2\pi^2 \sum_{i,j=1}^{n} \frac{x_i x_j}{i+j} \sum_{k,l=1}^{n} \frac{x_k x_l}{(k+l)^3}.$$

Solution by Boukharfane Radouan, Quebec, Canada. This inequality is a direct application of the integral version of Carlson's inequality. Recall that this equality states that if f is a nonnegative function defined on $[0, \infty)$ such that $f(t)$ and $tf(t)$ are square-integrable, then

$$\left(\int_0^\infty f(t)\,dt\right)^4 \le \pi^2 \left(\int_0^\infty (f(t))^2\,dt\right)\left(\int_0^\infty t^2 (f(t))^2\,dt\right).$$

For the current problem we apply Carlson's inequality to the function $f(t) = \sum_{k=1}^{n} x_k e^{-kt}$. Then we compute

$$\int_0^\infty f(t)\,dt = \sum_{k=1}^{n} x_k \int_0^\infty e^{-kt}\,dt = \sum_{k=1}^{n} \frac{x_k}{k},$$

$$\int_0^\infty (f(t))^2\,dt = \sum_{k,j=1}^{n} x_k x_j \int_0^\infty e^{-(k+j)t}\,dt = \sum_{k,j=1}^{n} \frac{x_k x_j}{k+j},$$

and
$$\int_0^\infty t^2 (f(t))^2\,dt = \sum_{k,j=1}^{n} x_k x_j \int_0^\infty t^2 e^{-(k+j)t}\,dt = 2\sum_{k,j=1}^{n} \frac{x_k x_j}{(k+j)^3}.$$

Putting these pieces together gives the desired inequality.

Editorial comment. Reference: F. Carlson, Une inégalité, *Ark. Mat. Astron. Fys.* **25B** (1934) 1–5. Some solvers provided Hardy's proof for Carlson's inequality.

Also solved by G. Apostolopoulos (Greece), P. Bracken, R. Chapman (U. K.), P. P. Dálylay (Hungary), M. Omarjee (France), R. Stong, R. Tauraso (Italy), and the proposer.

Automorphisms Cannot One-Up their Group

11681 [2012, 880–881]. *Proposed by Des MacHale, University College Cork, Cork Ireland.* For any group G, let $\mathrm{Aut}(G)$ denote the group of automorphisms of G.
(**a**) Show that there is no finite group G with $|\mathrm{Aut}(G)| = |G| + 1$.
(**b**) Show that there are infinitely many finite groups G with $|\mathrm{Aut}(G)| = |G|$.
(**c**) Find all finite groups G with $|\mathrm{Aut}(G)| = |G| - 1$.

Solution by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO. For (**b**), it is well known that $\mathrm{Aut}(S_n) \cong S_n$ when $n \notin \{2, 6\}$, so $\{S_n : n \notin \{2, 6\}\}$ is such an infinite family.

Now consider **(a)** and **(c)**. Let $\mathrm{Inn}(G)$ denote the group of inner automorphisms of G, that is, the group of mappings τ_b defined by $\tau_b(x) = bxb^{-1}$. Let $Z(G)$ be the center of G. An elementary group-theoretic argument shows that $\mathrm{Inn}(G) \cong G/Z(G)$, so $|\mathrm{Inn}(G)|$ divides $|G|$. Since $\mathrm{Inn}(G)$ is a (normal) subgroup of $\mathrm{Aut}(G)$, the size of $\mathrm{Inn}(G)$ divides $|\mathrm{Aut}(G)|$. In **(a)** and **(c)**, $|\mathrm{Inn}(G)|$ divides two relatively prime integers, so $|\mathrm{Inn}(G)| = 1$. Hence G is Abelian.

We claim that also G is cyclic. If not, then $G \cong \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus H$ with p a prime and $1 \le \alpha \le \beta$. Define $f\colon\ G \to G$ by $f(x, y, z) = (x + y, y, z)$. This is an automorphism of order p^α, so p^α divides both $|G|$ and $|\mathrm{Aut}(G)|$. From the contradiction $p^\alpha \nmid 1$, we conclude that G is cyclic.

Since G is cyclic, $|\mathrm{Aut}(G)| = \varphi(G) < |G|$, so **(a)** cannot occur.

We claim that case **(c)** can occur if and only if G is cyclic of prime order. If $G \cong \mathbb{Z}_p$ with p a prime, then $|\mathrm{Aut}(G)| = \varphi(p) = p - 1 = |G| - 1$, as claimed. Otherwise, $|G| = n = pk$ with p a prime and $k > 1$. Now both p and $2p$ do not exceed n and are not relatively prime to n; hence $|\mathrm{Aut}(G)| = \varphi(|G|) < |G| - 1$. Thus if G is not cyclic of prime order, then $|\mathrm{Aut}(G)| < |G| - 1$.

Editorial comment. Bruce Burdick used similar ideas to prove that $|\mathrm{Aut}(G)| = |G| + 2$ if and only if $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $|\mathrm{Aut}(G)| = |G| - 2$ if and only if $G \cong \mathbb{Z}_4$.

Also solved by A. J. Bevelacqua, R. Black & A. Lizzi & N. Monson, P. Budney, B. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, S. M. Gagola Jr., O. Geupel (Germany) (part (b) only), N. Grivaux (France), Y. J. Ionin, J Konieczny, C. Lanski, C. Leuridan (France), J. H. Lindsey II, O. P. Lossers (Netherlands), C. P. Rupert, J. H. Smith, R. Stong, D. Tyler, the GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), NSA Problems Group, and the proposer.

An Alternating Sum of Squares of Alternating Sums

11682 [2012, 881]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Compute

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA. The sum equals $\pi^2/24$.

Since

$$\int_0^1 \frac{x^n}{1+x}\,dx = \int_0^1 x^n\left(\sum_{k=1}^{\infty}(-1)^{k-1}x^{k-1}\right)dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k},$$

we have

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2 = \sum_{n=0}^{\infty}(-1)^n\left(\int_0^1\frac{x^n}{1+x}\,dx\right)^2$$

$$= \sum_{n=0}^{\infty}(-1)^n\int_0^1\frac{x^n}{1+x}\,dx\int_0^1\frac{y^n}{1+y}\,dy = \sum_{n=0}^{\infty}(-1)^n\int_0^1\int_0^1\frac{x^ny^n}{(1+x)(1+y)}\,dy\,dx$$

$$= \int_0^1\int_0^1\frac{1}{(1+x)(1+y)(1+xy)}\,dy\,dx.$$

Now

$$\int_0^1\frac{1}{(1+y)(1+xy)}\,dy = \frac{\ln(1+y)-\ln(1+xy)}{1-x}\bigg|_{y=0}^{y=1} = -\frac{\ln((1+x)/2)}{1-x},$$

so

$$\int_0^1\int_0^1 \frac{1}{(1+x)(1+y)(1+xy)}\,dy\,dx = -\int_0^1 \frac{\ln((1+x)/2)}{(1-x)(1+x)}\,dx$$
$$= -\frac{1}{2}\int_0^1 \frac{\ln((1+x)/2)}{1+x}\,dx - \frac{1}{2}\int_0^1 \frac{\ln((1+x)/2)}{1-x}\,dx. \qquad (*)$$

For the first term in $(*)$, we compute

$$-\frac{1}{2}\int_0^1 \frac{\ln((1+x)/2)}{1+x}\,dx = -\frac{1}{4}\ln^2\left(\frac{1+x}{2}\right)\Bigg|_0^1 = \frac{1}{4}\ln^2 2.$$

For the second term in $(*)$, the substitution $u = (1-x)/2$ yields

$$-\frac{1}{2}\int_0^1 \frac{\ln((1+x)/2)}{1-x}\,dx = \frac{1}{2}\int_{1/2}^0 \frac{\ln(1-u)}{u}\,du = \frac{1}{2}\left(\frac{\pi^2}{12} - \frac{\ln^2 2}{2}\right).$$

(For the last step, if h is the integrand, think about the integral of h over $(0,1)$ and $(1/2,1)$, and use integration by parts.) Therefore

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2 = \frac{1}{4}\ln^2 2 + \frac{1}{2}\left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right) = \frac{\pi^2}{24}.$$

Also solved by U. Abel (Germany), D. Beckwith, M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), R. Boukharfane (Canada), K. N. Boyadzhiev, B. Burdick, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), O. Geupel (Germany), M. L. Glasser, J. P. Grivaux (France), O. Kouba (Syria), O. P. Lossers (Netherlands), O. Oloa (France), M. Omarjee (France), P. Perfetti (Italy), A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, J. Van Hamme (Belgium), M. Vowe (Switzerland), GWstat Problem Solving Group, and the proposer.

Special Gergonne Points

11683 [2012, 881]. *Proposed by Raimond Struble, Santa Monica, CA.* Given a triangle ABC, let F_C be the foot of the altitude from the incenter to AB. Define F_B and F_C similarly. Let C_A be the circle with center A that passes through F_B and F_C, and define C_B and C_C similarly. The *Gergonne point* of a triangle is the point at which segments AF_A, BF_B, and CF_C meet. Determine, up to similarity, all isosceles triangles such that the Gergonne point of the triangle lies on one of the circles C_A, C_B, or C_C.

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI. The triangle is isosceles. Assume that $\angle B$ and $\angle C$ are congruent. Then $\overline{AF_A}$ is a line of symmetry of the triangle, so it is perpendicular to BC, and it is a common tangent to C_B and C_C. Thus the Gergonne point, call it G, can lie on either C_B or C_C only if it coincides with F_A, but that implies that ABC is a degenerate triangle. Thus G is on C_A.

Let the lengths of the sides opposite A, B, C be a, b, c, respectively. Let $s = \frac{1}{2}(a + b + c)$. The distance from A to F_B is $s - a$, so the radius of C_A is $s - a$, and the distance from A to G is $s - a$. Also, $\overline{AF_A}$ is the bisector of $\angle A$. The observation that

$$\text{Area}(ABF_B) = \text{Area}(ABG) + \text{Area}(AGF_B)$$

allows us to write

$$\frac{1}{2}c(s-a)\sin A = \frac{1}{2}c(s-a)\sin\frac{A}{2} + \frac{1}{2}(s-a)^2\sin\frac{A}{2},$$
$$c\sin\frac{A}{2}\cos\frac{A}{2} = \frac{1}{2}(s-a+c)\sin\frac{A}{2},$$
$$c\cos\frac{A}{2} = \frac{1}{4}(b+c-a+2c).$$

Since $\triangle ABF_A$ is a right triangle and $b = c$, we have

$$\sqrt{c^2 - \frac{1}{4}a^2} = c - \frac{1}{4}a, \quad c^2 - \frac{1}{4}a^2 = c^2 - \frac{1}{2}ac + \frac{1}{16}a^2,$$
$$0 = \frac{5}{16}a^2 - \frac{1}{2}ac, \quad a = \frac{8}{5}c.$$

It follows that if the Gergonne point of an isosceles triangle lies on one of the circles C_A, C_B, C_C, then the three sides, in some order, are in the ratio $5:5:8$.

Also solved by R. Boukharfane (Canada), P. P. Dályay (Hungary), C. Delorme (France), A. Ercan (Turkey), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), K. Hanes, A. Johnston, N. Komanda, J. H. Lindsey II, O. P. Lossers (Netherlands), J. Minkus, C. P. Pranesachar (India), R. Stong, M. Vowe (Switzerland), H. Widmer (Switzerland), J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

Möbius Estimates

11684 [2013, 76]. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France, and Rudolf Rupp, Georg-Simon-Ohm Hochschule Nürnberg, Nuremberg, Germany.* For complex a and z, let

$$\phi_a(z) = \frac{a-z}{1-\overline{a}z}, \quad \rho(a,z) = \frac{|a-z|}{|1-\overline{a}z|}.$$

(**a**) Show that whenever $-1 < a, b < 1$,

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\rho(a,b)$$
$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2.$$

(**b**) For complex α, β with $|\alpha| = |\beta| = 1$, let

$$m(z) = m_{a,b,\alpha,\beta}(z) = |\alpha\phi_a(z) - \beta\phi_b(z)|.$$

Determine the maximum and minimum, taken over z with $|z| = 1$, of $m(z)$.

Solution by the proposers.
(**b**) Observe that ϕ_a is its own inverse. Let $c = (b-a)/(1-a\overline{b})$ and let

$$\lambda = -\frac{1-a\overline{b}}{1-\overline{a}b}.$$

Since ϕ_b is a bijection of the unit circle onto itself,

$$\max_{|z|=1}\left|\alpha\phi_a(z) - \beta\phi_b(z)\right| = \max_{|z|=1}\left|\alpha\overline{\beta}\phi_a\big(\phi_b(z)\big) - z\right| = \max_{|z|=1}\left|\alpha\overline{\beta}\lambda\phi_c(z) - z\right|.$$

The same identities hold when the maximum is replaced by the minimum. Put $\gamma = \alpha\overline{\beta}\lambda$, and let $-\pi < \arg\gamma \le \pi$. For $|z| = 1$, let $H(z) = |\gamma\phi_c(z) - z|$. We have

$$H(z) = \left|\gamma \frac{z(c\overline{z} - 1)}{1 - \overline{c}z} - z\right| = \left|\gamma \frac{1 - c\overline{z}}{1 - \overline{c}z} - 1\right| = \left|\gamma \frac{w}{\overline{w}} + 1\right|,$$

where $w = 1 - c\overline{z} = 1 - c/z$. As z moves around the unit circle, w moves around the circle $|w - 1| = |c|$. Write $w = |w|e^{i\theta}$. Note that θ varies on the interval $[-\theta_m, \theta_m]$, where $|\theta_m| < \pi/2$ and $\sin\theta_m = |c| = \rho(a, b)$. Now

$$H(z) = \left|\gamma e^{2i\theta} + 1\right| = 2\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right|.$$

Hence

$$\max_{|z|=1} H(z) = 2\max\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a, b)\right\} \qquad (*)$$

and

$$\min_{|z|=1} H(z) = 2\min\left\{\left|\cos\left(\frac{\arg\gamma}{2} + \theta\right)\right| : |\theta| \le \arcsin\rho(a, b)\right\}.$$

(**a**) Specialize $(*)$ by taking $a, b \in (-1, 1)$ and $\alpha = \beta = 1$, so that $\gamma = -1$. By the maximum principle, the maximum on the disk is achieved on the boundary, so

$$\max_{|z|\le 1} |\phi_a(z) - \phi_b(z)| = 2\max\left\{|\sin\theta| : |\theta| \le \arcsin\rho(a, b)\right\} = 2\rho(a, b).$$

For the other part of (**a**), instead specialize $(*)$ by taking $a, b \in (-1, 1)$ and $\alpha = 1$, $\beta = -1$, so that $\gamma = 1$. This gives

$$\max_{|z|\le 1} |\phi_a(z) + \phi_b(z)| = 2\max\left\{|\cos\theta| : |\theta| \le \arcsin\rho(a, b)\right\} = 2.$$

Also solved by P. P. Dályay (Hungary) and R. Stong. Part (a) only by A. Alt, D. Beckwith, D. Fleischman, O. P. Lossers (Netherlands), and T. Smotzer.

The Reciprocal of the Thue-Morse Constant

11685 [2013, 76]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that

$$\prod_{n=0}^{\infty}\left(1 + \frac{1}{2^{2^k} - 1}\right) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k-1}\left(2^{2^j} - 1\right)}.$$

In other words, prove that

$$(1 + 1)\left(1 + \tfrac{1}{3}\right)\left(1 + \tfrac{1}{15}\right)\left(1 + \tfrac{1}{255}\right)\cdots = \tfrac{1}{2} + 1 + 1 + \tfrac{1}{3} + \tfrac{1}{3\cdot 15} + \tfrac{1}{3\cdot 15\cdot 255} + \cdots.$$

Solution by Traian Viteam, Punta Arenas, Chile. For $n \ge 0$,

$$\prod_{k=0}^{n}\left(1 + \frac{1}{2^{2^k} - 1}\right) - \prod_{k=0}^{n-1}\left(1 + \frac{1}{2^{2^k} - 1}\right) = \prod_{k=0}^{n-1} 2^{2^k} \Big/ \left(\prod_{k=0}^{n} 2^{2^k} - 1\right)$$
$$= 2^{2^n - 1} \Big/ \left(\prod_{k=0}^{n} 2^{2^k} - 1\right) = \frac{1}{2}\left(\frac{1}{\prod_{j=0}^{n-1}(2^{2^j} - 1)} + \frac{1}{\prod_{j=0}^{n}(2^{2^j} - 1)}.\right)$$

Summing from $n = 0$ to $n = N$ yields

$$\prod_{k=0}^{N}\left(1+\frac{1}{2^{2^k}-1}\right)-1=\frac{1}{2}+\sum_{k=1}^{N}\frac{1}{\prod_{j=0}^{k-1}(2^{2^j}-1)}+\frac{1}{2}\frac{1}{\prod_{j=0}^{N}(2^{2^j}-1)}$$

for all N. Letting N tend to infinity yields the desired result.

Editorial comment. The proposer noted that this is the special case $x = 1/2$ of

$$\frac{1}{\prod_{k=0}^{\infty}\left(1-x^{2^k}\right)}=1-x+2\sum_{k=0}^{\infty}\frac{x^{2^k}}{\prod_{j=0}^{k-1}\left(1-x^{2^j}\right)}.$$

The left side is the reciprocal of the generating function $\mu(x)$ of the Thue-Morse sequence, and $\mu(1/2)$ is the Thue-Morse constant, which is the subject of Section 6.8 in *Mathematical Constants* by Steven R. Finch, Cambridge University Press (2003), pp. 436–441.

Also solved by R. Barnes, D. Beckwith, R. Boukharfane (Canada), B. Burdick, R. Chapman (U. K.), J. Fabrykowski & T. Smotzer, O. Geupel (Germany) C. Georghiou (Greece), Y. J. Ionin, O. Kouba (Syria), K. Kyun (Korea), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), J. Martinez (Spain), M. Omarjee (France), H. Roelants (Belgium), R. Sachdev (India), J. Schlosberg, R. Tauraso (Italy), M. Wildon (U. K.), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

A Fast-Growing Function

11688 [2013, 77]. *Proposed by Samuel Alexander, The Ohio State University, Columbus, OH.* Consider $f\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{a\to\infty} \inf_{b,c,d\in\mathbb{N},b<a} f(a,c,d) - f(b,c,d) = \infty$. Show that for $B \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$f(a,c,d) = k \quad\Rightarrow\quad \max\{c,d\} > B.$$

Solution by Iosif Pinelis, Michigan Technological University, Houghton, MI. We say that a subset S of $\mathbb{N}$ has *density zero* if

$$\lim_{n\to\infty}\frac{1}{n}|S\cap[n]| = 0,$$

where $[n] = \{1,\dots,n\}$.

First we show that if $h : \mathbb{N} \to \mathbb{N}$ is a function satisfying $\lim_{a\to\infty} h(a) - h(a-1) = \infty$, then $h(\mathbb{N})$ has density zero. For a positive integer m, there exists $a_m \in \mathbb{N}$ such that $h(a) - h(a-1) \ge m$ for $a > a_m$. Hence for all $n \in \mathbb{N}$,

$$|h(\mathbb{N})\cap[n]| \le |h([a_m])| + \frac{n}{m} + 1,$$

and thus $\limsup_{n\to\infty} \frac{1}{n}|h(\mathbb{N})\cap[n]| \le 1/m$. Since this holds for all m, it follows that $h(\mathbb{N})$ has density zero.

Now the given hypothesis implies for fixed $c, d \in \mathbb{N}$ that $\lim_{a\to\infty} f(a,c,d) - f(a-1,c,d) = \infty$, and thus the set $S_{c,d} = \{f(a,c,d)\colon\ a \in \mathbb{N}\}$ has density 0. Since the union of finitely many sets of density zero has density zero, for any $B \in \mathbb{N}$ the set $\bigcup_{c,d\le B} S_{c,d}$ has density zero. Therefore some $k \in \mathbb{N}$ is not in this set, so $f(a,c,d) = k$ implies $\max\{c,d\} > B$.

Also solved by R. Chapman (U. K.), O. Geupel (Germany), B. Karaivanov, O. P. Lossers (Netherlands), R. Martin (Germany), R. Stong, H. Takeda (Japan), GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before July 31, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11817. *Proposed by Mohammad Jahaveri, Siena College, Loudonville, NY.* A *cycle double cover* of a graph is a collection of cycles that, counting multiplicity, includes every edge exactly twice. Let X be an infinite set and let K_X be the complete graph on X. Construct a cycle double cover for X.

11818. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* Let ABC be a triangle and let A_1, B_1, and C_1 be the points on sides opposite A, B, and C, respectively, at which the ecircles of the triangle are tangent to those sides. Let R and r be the circumradius and inradius of the triangle. Let the name of a vertex of ABC or of $A_1B_1C_1$ also stand for the radian measure of the corresponding angle. Prove that

$$\frac{\cot A_1 + \cot(A/2)}{\cot A} + \frac{\cot B_1 + \cot(B/2)}{\cot B} + \frac{\cot C_1 + \cot(C/2)}{\cot C} = \frac{6R}{r}$$

wherever the expression is defined.

11819. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let f be a continuous, nonnegative function on [0, 1]. Show that

$$\int_0^1 f^3(x)\,dx \ge 4\left(\int_0^1 x^2 f(x)\,dx\right)\left(\int_0^1 x f^2(x)\,dx\right).$$

11820. *Proposed by Alborz Azarang, Shahid Chamran University of Ahvaz, Ahvaz, Iran.* Let K be a field and let R be a subring of $K[X]$ that contains K. Prove that R is noetherian, that is, that every ascending chain of ideals in R terminates.

http://dx.doi.org/10.4169/amer.math.monthly.122.02.175

11821. *Proposed by Finbarr Holland and Claus Koester, University College Cork, Cork, Ireland.* Let p be a positive integer. Prove that

$$\lim_{n\to\infty} \frac{1}{2^n n^p} \sum_{k=0}^{n} (n-2k)^{2p} \binom{n}{k} = \prod_{j=1}^{p} (2j-1).$$

11822. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Call a polynomial *real* if all its coefficients are real. Let P and Q be polynomials with complex coefficients such that the composition $P \circ Q$ is real. Show that if the leading coefficient of Q and its constant term are both real, then P and Q are real.

11823. *Proposed by Sabin Tabirca, University College Cork, Cork, Ireland.* Let P be a point inside a circle C.

(**a**) Prove that there exists a point P' outside C such that, for all chords XY of C through P, $(|XP'| + |YP'|)/|XY|$ is the same. (Here, $|UV|$ denotes the distance from U to V.)
(**b**) Is P' unique?

SOLUTIONS

A Consequence of Blundon's Inequality

11686 [2013, 76]. *Proposed by Michel Bataille, Rouen, France.* Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Prove that

$$\frac{\cot x + \cot y + \cot z}{\tan x + \tan y + \tan z} \geq 4(\sin^2 x + \sin^2 y + \sin^2 z).$$

Solution by John Zacharias, Arlington, VA. Let $A = 2x$, $B = 2y$, and $C = 2z$, and note that A, B, and C are the angles of a triangle. Let R, r, and s be the circumradius, inradius, and semiperimeter of this triangle, respectively. Then we have the standard identities

$$\cot(A/2) + \cot(B/2) + \cot(C/2) = \frac{s}{r},$$

$$\tan(A/2) + \tan(B/2) + \tan(C/2) = \frac{r+4R}{s},$$

and

$$\begin{aligned} 4(\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)) &= 2(3 - \cos A - \cos B - \cos C) \\ &= \frac{2(2R-r)}{R}. \end{aligned}$$

Thus, the desired inequality rearranges to

$$s^2 \geq \frac{2r(r+4R)(2R-r)}{R}.$$

This can be proved from Blundon's inequality (W. J. Blundon, "Inequalities Associated with the Triangle," *Canadian Mathematics Bulletin,* 1965, 615–626), which (is sharp and) states that

$$s^2 \geq 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)}.$$

Indeed, for $x \geq 2$,

$$\left(2x^2 - 6x + 3 + \frac{2}{x}\right)^2 = 4x(x-2)^3 + \frac{(x-2)^2(4x+1)}{x^2} \geq 4x(x-2)^3.$$

Therefore, after taking a square root,

$$2x^2 - 6x + 3 + \frac{2}{x} \geq 2(x-2)\sqrt{x^2 - 2x}.$$

We apply this with $x = R/r$. By Euler's inequality, ($R \geq 2r$ when R is the circumradius of a circle and r the inradius), $x \geq 2$. Rearranging gives

$$s^2 \geq 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)} \geq \frac{2r(r+4R)(2R-r)}{R},$$

as required.

Editorial comment. Mitrinovic, Pecaric, and Volenec, *Recent Advances in Geometric Inequalities,* (Kluwer Academic Publishers Group, Dordrecht, 1989), pp. 56–60, note that the inequality that bears Blundon's name has, in fact, been known since at least 1851.

Slicing a Torus

11687 [2013, 77]. *Proposed by Steven Finch, Harvard University, Cambridge, MA.* Let T be a solid torus in $\mathbb{R}^3$ with center at the origin, tube radius 1, and spine radius r with $r \geq 1$ (so that T has volume $\pi \cdot 2\pi r$.) Let P be a 'random' nearby plane. Find the conditional probability, given that P meets T, that the intersection is simply connected. For what value of r is this probability maximal? (The plane is chosen by first picking a distance from the origin uniformly between 0 and $1 + r$ and then picking a normal vector independently and uniformly on the unit sphere.)

Solution by Radouan Boukharfane, Polytechnique de Montreal, Montreal, Canada. We may assume that the axis of the torus T is the z-axis, and the center is the origin. Then T has equation

$$x^2 + y^2 = \left(r \pm \sqrt{1 - z^2}\right)^2.$$

We may assume the normal to the plane P is in the first quadrant of the xz-plane. Then P has equation

$$x\cos(\alpha) + z\sin(\alpha) = m,$$

where $0 \leq \alpha \leq \pi/2$ and m is the distance from the origin. If $0 < m \leq 1$, then P meets T for all α; if $1 < m \leq 1 + d$, then P meets T for $\alpha \leq \arccos((m-1)/r)$. So we may compute the denominator of our conditional probability as

$$D = \int_0^1 \frac{\pi}{2}\, dm + \int_1^{1+r} \arccos\left(\frac{m-1}{r}\right) dm = \frac{\pi}{2} + r.$$

Now we compute the numerator. Let C_+ be the circle $(r + \cos t, 0, \sin t)$ and C_- be the circle $(-r + \cos t, 0, \sin t)$. If $T \cap P \neq \varnothing$, then (aside from probability zero events) there are three possibilities: (a) P meets C_+ but not C_-, so $T \cap P$ is homeomorphic to a disk; (b) P meets both C_+ and C_-, so $T \cap P$ is homeomorphic to an annulus; and (c) P meets neither circle (it passes between them), so $T \cap P$ is homeomorphic to a disjoint union of two disks. We consider (a) simply connected, and the other two not simply connected.

Consider the case $r \geq 2$. Then the numerator is $N_1 = I_1 + J_1 + K_1$, where

$$I_1 = \int_0^1 \left[\arccos\left(\frac{1-m}{r}\right) - \arccos\left(\frac{1+m}{r}\right)\right] dm,$$

$$J_1 = \int_1^{r-1} \left[\arccos\left(\frac{m-1}{r}\right) - \arccos\left(\frac{m+1}{r}\right)\right] dm,$$

$$K_1 = \int_{r-1}^{r+1} \arccos\left(\frac{m-1}{r}\right) dm.$$

Now consider the case $1 < r \leq 2$. Then the numerator is $N_2 = I_2 + J_2 + K_2$, where

$$I_2 = \int_0^{r-1} \left[\arccos\left(\frac{1-m}{r}\right) - \arccos\left(\frac{1+m}{r}\right)\right] dm,$$

$$J_2 = \int_{r-1}^{1} \arccos\left(\frac{1-m}{r}\right) dm,$$

$$K_2 = \int_1^{r+1} \arccos\left(\frac{m-1}{r}\right) dm.$$

Computation yields $N = N_1 = N_2$, and in both cases, the conditional probability is

$$P(r) = \frac{N}{D} = \frac{2r + 2\arccos(1/r) - 2\sqrt{r^2-1}}{\frac{\pi}{2} + r}.$$

Numerically, we find that the maximal value for $P(r)$ is ≈ 0.810777, reached at $r \approx 1.24376$, a solution of the equation

$$\sin\left(\frac{\pi}{2}\sqrt{1-\left(\frac{r}{2}\right)^2}\right) = \frac{1}{r}.$$

Editorial comment. In order to be "simply connected," must a space in particular be connected? Some solvers made one choice; some made the other.

Also solved by C. Curtis, J. H. Lindsey II, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Inscribe an Equilateral Triangle in a Hypercube

11693 [2013, 174]. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA, and Richard Stong, CCR, San Diego CA.* Let T be an equilateral triangle inscribed in the d-dimensional unit cube $[0, 1]^d$, with $d \geq 2$. As a function of d, what is the maximum possible side length of T?

Solution by Yury J. Ionin, Champaign, IL. We will show that the maximum side length of T is $q(d)$, where

$$q(d)^2 = \begin{cases} \frac{2d}{3}, & \text{if } d \equiv 0 \pmod 3; \\ \frac{2d+16}{3} - 4\sqrt{2}, & \text{if } d \equiv 1 \pmod 3; \\ \frac{2d+20}{3} - 4\sqrt{3}, & \text{if } d \equiv 2 \pmod 3. \end{cases} \tag{1}$$

Since $[0, 1]^d$ is a compact set, there exists an equilateral triangle of the maximum side length contained in $[0, 1]^d$. We will call such a triangle *maximal*. Write m_d for the side length of a maximal triangle.

We begin with examples of triangles exhibiting the claimed side lengths $q(d)$. In these examples, $X, Y, Z \in [0,1]^d$, X is the origin, $Y = (y_1, \ldots, y_d)$, and $Z = (z_1, \ldots, z_d)$.

If $d \equiv 0 \pmod 3$, let $y_i = 1$ for $1 \le i \le 2d/3$, $y_i = 0$ otherwise, $z_i = 0$ for $1 \le i \le d/3$, and $z_i = 1$ otherwise.

If $d \equiv 1 \pmod 3$, let $y_1 = z_2 = 1$, $y_2 = z_1 = 2 - \sqrt{2}$, $y_1 = 1$ for $3 \le i \le (2d+1)/3$, $y_1 = 0$ otherwise, $z_i = 0$ for $3 \le i \le (d+5)/3$, and $z_i = 1$ otherwise.

If $d \equiv 2 \pmod 3$, let $y_1 = z_2 = 1$, $y_2 = z_1 = 2 - \sqrt{3}$, $y_i = 1$ for $3 \le i \le (2d+2)/3$, $y_i = 0$ otherwise, $z_i = 0$ for $3 \le i \le (d+4)/3$, and $z_i = 1$ otherwise.

These examples show $m_d \ge q(d)$. It remains to show that $m_d \le q(d)$. Let $X, Y, Z \in [0,1]^d$, $X = (x_1, \ldots x_d)$, $Y = (y_1, \ldots, y_d)$, $Z = (z_1, \ldots, z_d)$. With these points, we associate a $3 \times d$ matrix

$$\mathbf{A} = \begin{bmatrix} x_1 & \cdots & x_d \\ y_1 & \cdots & y_d \\ z_1 & \cdots & z_d \end{bmatrix}.$$

Note that

$$|XY|^2 + |XZ|^2 + |YZ|^2 = \sum_{i=1}^{d} \big((x_i - y_i)^2 + (x_i - z_i)^2 + (y_i - z_i)^2\big).$$

For a fixed i, the three numbers x_i, y_i, z_i have some order, say $x_i \ge y_i \ge z_i$; let $u_i = x_i - y_i$ and $v_i = x_i - z_i$. Then $0 \le u_i \le v_i \le 1$, and

$$(x_i - y_i)^2 + (x_i - z_i)^2 + (y_1 - z_i)^2 = 2\big(v_i^2 - u_i(v_i - u_i)\big) \le 2v_i^2 \le 2.$$

We have equality $(x_i - y_i)^2 + (x_i - z_i)^2 + (y_1 - z_i)^2 = 2$ if and only if the three numbers x_i, y_i, z_i consist of two 0s and a 1 or two 1s and a 0. Adding, we get

$$|XY|^2 + |XZ|^2 + |YZ|^2 \le 2d, \tag{2}$$

with equality if and only if each column of $\mathbf{A}$ consists of two 0s and a 1 or two 1s and a 0. We now have $q(d) \le m_d \le \sqrt{2d/3}$. If $d \equiv 0 \pmod 3$, then $q(d) = m_d = \sqrt{2d/3}$.

For $0 \le k \le d$, a *k-face* of the cube is a k-dimensional face. Thus, 0-faces are vertices, 1-faces are edges of the cube, and the d-face is the entire cube. If $P \in [0,1]^d$ is not a vertex of the cube, then there exists a unique $k = k(P)$ such that P lies in the interior of a k-face of the cube.

Claim 1. If an equilateral $\triangle XYZ$ is maximal, then its vertices lie on the boundary of the cube. Suppose, for example, that Z is in the interior of the cube. Since $m_d^2 < d$, there exists j such that $|x_j - y_j| < 1$. We replace points X and Y by $X' = (x'_1, \ldots, x'_d)$ and $Y' = (y'_1, \ldots, y'_d)$ with $x'_i = x_i$ and $y'_i = y_i$ for $i \ne j$ so that X' and Y' are in the cube and $|x'_j - y'_j| > |x_j - y_j|$. Moreover, we choose X' and Y' so close to X and Y, respectively, that a third vertex Z' of an equilateral $\triangle X'Y'Z'$ then can be chosen sufficiently close to Z and therefore in the interior of the cube. Since the side length of this triangle is greater than m_d, we have a contradiction.

For $d = 2$, this claim implies that one of the vertices of a maximal equilateral $\triangle XYZ$ is a vertex of the square $[0,1]^2$; otherwise, we would have had two vertices of the triangle inside parallel sides of the square and we could have shifted the triangle along these sides to obtain a maximal equilateral triangle with the third vertex inside the square. So we may assume $X = (0,0)$, $Y = (y,1)$, and $Z = (1,z)$, with $y^2 + 1 = 1 + z^2 = (1-y)^2 + (1-z)^2$, and so $y = z = 2 - \sqrt{3}$. This shows $m_2 = |XY| = q(2)$.

We proceed by induction on d. Suppose $d \geq 3$ and $m_{d-1} = q(d-1)$. Of course, we may also assume $d \not\equiv 0 \pmod 3$. Observe that $m_d \geq q(d) > q(d-1)$, and therefore, a maximal equilateral triangle inscribed in $[0, 1]^d$ is not contained in a $(d-1)$-face of $[0, 1]^d$. In other words, if an equilateral $\triangle XYZ$ is maximal, then no column of $\mathbf{A}$ consists of three 0s or of three 1s.

Claim 2. If an equilateral $\triangle XYZ$ is maximal, then its vertices lie on edges of the cube. Suppose there is a maximal equilateral $\triangle XYZ$ with $k(Z) \geq 2$; choose such a $\triangle XYZ$ such that $k = k(Z)$ is as large as possible. Let F be the k-face of $[0, 1]^d$ containing Z in its interior. By Claim 1, $k \leq d - 1$. Let M be the midpoint of segment $\overline{XY}$, and let hyperplane π through M be perpendicular to that segment. If $M \in F$, then M is on the boundary of the cube, so the entire segment $\overline{XY}$ is contained in F and thus $\triangle XYZ$ lies on a k-face of the cube, a contradiction. Therefore, $M \notin F$, so $F \cap \pi$ is a convex polyhedron η in the k-flat containing F (that is, the intersection of finitely many half-$(k-1)$-flats). (Note that $k \geq 2$.) Each vertex of η lies on the boundary of F, and at least one of them, say W, is further from M than Z. Then segment $\overline{MW}$ contains a point Z' such that $|MZ'| = |MZ|$. Since $Z' \in \pi$, the perpendicular bisector of $\overline{XY}$, and $Z' \notin F$, we obtain a maximal equilateral $\triangle XYZ'$ with $k(Z') > k(Z)$. This contradicts the choice of Z and proves the claim.

From Claim 2 we see: If an equilateral $\triangle XYZ$ is maximal, then each row of $\mathbf{A}$ contains at most one entry that equals neither 0 nor 1 (that is, at most one entry that is not an integer). We split $\mathbf{A}$ into a $3 \times d_1$ submatrix $\mathbf{A}_1$ with a noninteger entry in every column and a $3 \times d_2$ submatrix $\mathbf{A}_2$ with no noninteger entries, so $d_1 + d_2 = d$ and $0 \leq d_1 \leq 3$. These submatrices correspond to triangles $X_1Y_1Z_1$ in $[0, 1]^{d_1}$ and $X_2Y_2Z_2$ in $[0, 1]^{d_2}$. The triangles $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ lie in orthogonal subspaces, so $m_d^2 = |XY|^2 = |X_1Y_1|^2 + |X_2Y_2|^2$ and similarly for the other edges. The differences

$$\left||X_2Y_2|^2 - |X_2Z_2|^2\right|, \quad \left||X_2Y_2|^2 - |Y_2Z_2|^2\right|, \quad \left||X_2Z_2|^2 - |Y_2Z_2|^2\right|$$

are three nonnegative integers, of which the largest is equal to the sum of the other two. Thus, the three differences

$$\left||X_1Y_1|^2 - |X_1Z_1|^2\right|, \quad \left||X_1Y_1|^2 - |Y_1Z_1|^2\right|, \quad \left||X_1Z_1|^2 - |Y_1Z_1|^2\right|$$

are also three nonnegative integers, of which the largest is equal to the sum of the other two. But these differences are strictly less than d_1, the maximum square of the distance between two points of $[0, 1]^{d_1}$. This rules out $d_1 = 1$ and yields the following possible values for these differences: **(a)** 0, 0, 0 and **(b)** 0, 1, 1 for $d_1 = 2$ and 3, and **(c)** 1, 1, 2 and **(d)** 0, 2, 2 for $d_1 = 3$ only. The differences are the same for $\triangle X_2Y_2Z_2$. We will say that matrix $\mathbf{A}$ is of type **(a)**, **(b)**, **(c)**, or **(d)**. Note that if $d_1 \geq 2$, then each column of $\mathbf{A}_1$ has an entry equal to 0 and an entry equal to 1. Indeed, if $\mathbf{A}_1$ has a column with no entry equal 0 (equal 1), then we subtract (add) the same small positive number from (to) each entry of the column to obtain a maximal equilateral triangle having a vertex with two noninteger coordinates. But then for $d_3 = 3$, we have $1 < |X_1Y_1|^2 < 3$, and similarly for the other sides, so $0 \leq \left||X_1Y_1|^2 - |X_1Z_1|^2\right| < 2$, and similarly for the other differences. Thus, types **(c)** and **(d)** are impossible.

Without loss of generality, assume that all entries in the third row of $\mathbf{A}_2$ are 0s. The entries of the first and second row of $\mathbf{A}_2$ form, say, α pairs $(1, 0)$, β pairs $(0, 1)$, and γ pairs $(1, 1)$, so $\alpha + \beta + \gamma = d_2$. If $\mathbf{A}$ is of type **(a)**, then $\alpha + \beta = \alpha + \gamma = \beta + \gamma$, and then $d_2 \equiv 0 \bmod 3$ and $\alpha = \beta = \gamma = d_2/3$. If $\mathbf{A}$ is of type **(b)**, then (up to a permutation of α, β, γ), there are two possibilities: **(b1)** $\alpha + \beta = \alpha + \gamma = \beta + \gamma + 1$ and **(b2)** $\alpha + \beta = \alpha + \gamma = \beta + \gamma - 1$. In case **(b1)**, we obtain $d_2 \equiv 1 \pmod 3$, $\alpha = (d_2 + 2)/3$, $\beta = \gamma = (d_2 - 1)/3$; in case **(b2)**, $d_2 \equiv 2 \pmod 3$, $\alpha = (d_2 - 2)/3$, $\beta = \gamma = (d_1 + 1)/3$.

If $d_1 = 0$, then $\triangle X_2Y_2Z_2$ is equilateral. Then **A** is of type **(a)**, and we have $d = d_2 \equiv 0 \pmod 3$, so this case is finished.

As noted above, $d_1 = 1$ is impossible.

Let $d_1 = 2$. Then **A** is of type **(a)** or **(b)**, and we assume without loss of generality that the vertices of $\triangle X_1Y_1Z_1$ have coordinates $(0, 0)$, $(a, 1)$, and $(1, b)$. If $d \equiv 1 \pmod 3$, then $d_2 \equiv 2 \pmod 3$, and therefore, **A** is of type **(b2)**. If $(0, 0)$ is the apex of $\triangle X_1Y_1Z_1$, we obtain that $1 + a^2 = 1 + b^2 = (1 - a)^2 + (1 - b)^2 + 1$. Then $a = b = 2 - \sqrt{2}$, and the side length of $\triangle XYZ$ is $q(d)$. If $(1, b)$ is the apex of $\triangle X_1Y_1Z_1$, then $a^2 + 1 = b^2 > 1$, so this case is impossible. On the other hand, if $d \equiv 2 \pmod 3$, then $d_2 \equiv 0 \pmod 3$, and then **A** is of type **(a)**. Therefore, $\triangle X_1Y_1Z_1$ is equilateral. We have $1 + a^2 = 1 + b^2 = (1 - a)^2 + (1 - b)^2$, and then $a = b = 2 - \sqrt{3}$, and again the side length of $\triangle XYZ$ is $q(d)$.

Let $d_1 = 3$. Since every column of $\mathbf{A}_1$ has entries 0 and 1, each side of $\triangle X_1Y_1Z_1$ is greater than 1 and less than $\sqrt{3}$. Since $d_2 \equiv d \not\equiv 0 \pmod 3$, **A** is of type **(b)**. Let $t = |X_1Y_1|^2 = |X_1Z_1|^2 = |Y_1Z_1|^2 \pm 1$. Let $\theta = \angle X_1Y_1Z_1$. If $|Y_1Z_1|^2 = t - 1$, then $t > 2$ and $\cos^2\theta = (t - 1)/(2t) > 1/4$. Therefore, $\theta < 60^\circ$, $\angle Y_1X_1Z_1 > 60^\circ$, and then $\sqrt{t} = |X_1Z_1| < |Y_1Z_1| = \sqrt{t - 1}$, a contradiction. Thus, the case $d_1 = 3$ is impossible.

This completes the proof. Note that one and only one vertex of any maximal equilateral triangle is a vertex of the cube.

Also solved by C. Blatter (Switzerland), M. A. Prasad (India), and the proposers.

A Complex Three-Number Problem

11700 [2013, 365]. *Proposed by Evan O'Dorney (student), Harvard University, Cambridge, MA* . Let n be an integer greater than 1. Let a, b, and c be complex numbers with $a + b + c = a^n + b^n + c^n = 0$. Prove that the absolute values of a, b, and c cannot be distinct.

Solution by Allen Stenger, Alamogordo, NM. If $a = 0$, then $|b| = |c|$, and we are done. If $a \neq 0$, set $z = b/a$. Then $a^n + b^n + c^n = 0$ becomes $p(z) = 0$, where $p(z) = 1 + z^n + (-1 - z)^n$. We must show that all zeros z of $p(z)$ satisfy at least one of: $|z| = 1$, $|z + 1| = 1$, or $|z| = |z + 1|$. The first two are unit circles, and the last is the line $\operatorname{Re} z = -1/2$. These curves intersect only in the two nonreal cube roots of unity, $\omega = e^{2\pi i/3}$ and $\omega^2 = \overline{\omega}$. We will establish lower bounds for the number of zeros (counted with multiplicity) on these curves and show that the total lower bound is the degree of the polynomial, so there can be no zeros off the curves. If n is even, the degree of p is n, but if n is odd, the high-order terms cancel and the degree is $n - 1$.

There are six cases, according to the value of n mod 6, the three classes of zeros: real, cube root of unity, and "other." We will prove the lower bounds listed in this table:

n	real zeros	ω, ω^2	other zeros	total zeros	degree of p
$6m$	0	0	$6m$	$6m$	$6m$
$6m + 1$	2	4	$6(m - 1)$	$6m$	$6m$
$6m + 2$	0	2	$6m$	$6m + 2$	$6m + 2$
$6m + 3$	2	0	$6m$	$6m + 2$	$6m + 2$
$6m + 4$	0	4	$6m$	$6m + 4$	$6m + 4$
$6m + 5$	2	2	$6m$	$6m + 4$	$6m + 4$.

If n is even, there are no real zeros because $p(z) > 0$ for real arguments. If n is odd, there are real zeros at $z = 0$ and $z = -1$, so a lower bound is 2.

Because $p(\omega) = 1 + \omega^n + \omega^{2n}$, we have $p(\omega) = 0$ if n is not a multiple of 3 and $p(\omega) = 3$ if n is a multiple of 3. Also, $p'(\omega) = n(\omega^{n-1} - \omega^{2(n-1)}) = 0$ if

$n \equiv 1 \pmod 3$, so there is (at least) a double zero at ω if $n = 6m+1$ or $n = 6m+4$. Calculation for ω^2 is the same. So we have at least 0, 4, 2 zeros (counting multiplicity) according as n mod 3 is 0, 1, 2, respectively.

Now we consider the zeros on the arc Γ of the unit circle given parametrically by $z = e^{2\pi it}$ where $1/3 < t < 1/2$. None of these zeros has been counted yet. First we will show the number of uncounted zeros is at least 6 times the number in Γ. This comes from complex conjugation and from the fractional linear transformation $w(z) = -1/(z+1)$. This transformation sends the unit circle $|z| = 1$ to the line $\operatorname{Re} z = -1/2$, sends that line to the circle $|z+1| = 1$, and sends that circle back to the first circle. Also, it sends zeros of p to zeros of p (except for the zero $z = -1$ that we have already counted) because $p(w(z)) = p(z)/(-z-1)^n$. Further, w maps the real line to itself, and the cube roots ω, ω^2 to themselves, so all the other zeros produced by complex conjugation and by w from those on the arc Γ are distinct and have not been counted yet. Therefore, the "other" category contains at least six times the number of roots in the arc Γ.

To count the zeros in Γ we look at zero-crossings. Define

$$\begin{aligned} g(t) &:= e^{-\pi int} p(e^{2\pi it}) = e^{-\pi int}\left(1 + e^{2\pi int} + (-1)^n (e^{2\pi it} + 1)^n\right) \\ &= (e^{\pi int} + e^{-\pi int}) + (-1)^n (e^{\pi it} + e^{-\pi it})^n \\ &= 2\cos \pi nt + (-1)^n (2\cos \pi t)^n. \end{aligned}$$

Thus, $g(t)$ is real-valued and has the same zeros as $p(e^{2\pi it})$. Consider the values $t = k/n$ where k is an integer and $1/3 \le k/n \le 1/2$. For these t, we have $2\cos \pi nt = 2(-1)^k$. Because the cosine function is decreasing on this interval, we have $|(2\cos \pi t)^n| \le |(2\cos(\pi/3))^n| = 1$. Therefore, the sign of $g(k/n)$ is $(-1)^k$. So there is a zero in each interval between consecutive values k/n. The number of such intervals is $\lfloor n/2 \rfloor - \lceil n/3 \rceil$. Considering the six possible congruence classes $n = 6m + r$, we see that this number is m except for $n = 6m+1$, and in that case it is $m - 1$. This completes the table and the proof.

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), E. A. Herman, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), J. Ritter, R. Stong, R. Tauraso (Italy), E. I. Verriest, and GCHQ Problem Solving Group (U. K.).

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal nor posted to the Internet before the due date for solutions. Submitted solutions should arrive before July 31, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11824. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI and Yusheng Luo, Harvard University, Cambridge, MA.* A set X of points in the plane is said to be in *circular general position* if it has the property that every circle or straight line in the plane misses at least two points of X. (Such sets must have at least five elements, and most five-element sets qualify.)
(**a**) Show that if X is a set in circular general position and contains at least seven points, then it has a five-element subset that is in circular general position.
(**b**) Show that there exist sets X in circular general position containing exactly six points for which there is no five-element subset in circular general position.

11825. *Proposed by Marian Dincă, Vahalia University of Târgoviste, Bucharest, Romania, and Sorin Radulescu, Institute of Mathematical Statistic and Applied Mathematics, Bucharest, Romania.* Let E be a normed linear space. Given $x_1, \ldots, x_n \in E$ (with $n \geq 2$) such that $\|x_k\| = 1$ for $1 \leq k \leq n$ and the origin of E is in the convex hull of $\{x_1, \ldots, x_n\}$, prove that $\|x_1 + \cdots + x_n\| \leq n - 2$.

11826. *Proposed by Michel Bataille, Rouen, France.* Let m and n be positive integers with $m \leq n$. Prove that

$$\sum_{k=m}^{n} 4^{n+1-k} \binom{m+k-1}{m-1}^2 \geq \sum_{k=m}^{n} \binom{m+n}{k}^2.$$

11827. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Show that there are infinitely many rational triples (a, b, c) such that $a + b + c = abc = 6$.

http://dx.doi.org/10.4169/amer.math.monthly.122.03.284

11828. *Proposed by Roberto Tauraso, Universita di Roma "Tor Vergata," Rome, Italy.* Let n be a positive integer, and let z be a complex number that is not a kth root of unity for any k with $1 \le k \le n$. Let S be the set of all lists $(a_1, \dots, a_n)$ of n nonnegative integers such that $\sum_{k=1}^{n} k a_k = n$. Prove that

$$\sum_{a \in S} \prod_{k=1}^{n} \frac{1}{a_k! k^{a_k} (1 - z^k)^{a_k}} = \prod_{k=1}^{n} \frac{1}{1 - z^k}.$$

11829. *Proposed by Paul Bracken, University of Texas-Pan American, Edinburg, TX.* Let $\langle a \rangle$ be a monotone decreasing sequence of real numbers that converges to 0. Prove that

$$\sum_{n=1}^{\infty} a_n < \infty$$

if and only if $a_n = O(1/\log n)$ and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log n < \infty$.

11830. *Proposed by Leo Giugiuc, Drobeta-Turnu Severin, Romania, and Oai Thanh Dao, Quang Trung village, Kien Xuong district, Thai Binh Province, Vietnam.* Let A, B, C be the vertices of a triangle. Let P be a parabola tangent to the line BC at A_1, to CA at B_1, and to AB at C_1. Let A_2, B_2, and C_2 be the circumcenters of triangles AB_1C_1, BC_1A_1, and CA_1B_1, respectively.

(**a**) Show that there is a circle through A_2, B_2, C_2, and the focus of P.

(**b**) Show that the triangles ABC and $A_2B_2C_2$ are similar.

SOLUTIONS

A Tale of Three Circles

11689 [2013, 77]. *Proposed by Yagub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Two circles w_1 and w_2 intersect at distinct points B and C and are internally tangent to a third circle w at M and N, respectively. Line BC intersects w at A and D, with A nearer B than C. Let r_1 and r_2 be the radii of w_1 and w_2, respectively, with $r_1 \le r_2$. Let $u = \sqrt{|AC| \cdot |BD|}$ and $v = \sqrt{|AB| \cdot |CD|}$. Prove that

$$\frac{u - v}{u + v} < \sqrt{\frac{r_1}{r_2}}.$$

Solution by Richard Stong, Center for Communication Research, San Diego, CA. Lay down coordinates with $A = (0, 0)$ and $D = (2, 0)$. Define $s, t > 0$ by

$$C = \left(\frac{2s}{1 + s}, 0\right), \quad B = \left(\frac{2st}{1 + st}, 0\right),$$

so that

$$\frac{|AB| \cdot |CD|}{|AC| \cdot |BD|} = \frac{v^2}{u^2} = t < 1.$$

Define θ with $0 \le \theta < \pi/2$ by requiring that w have center $(1, \tan\theta)$, and hence, radius $r = \sec\theta$. For $i = 1, 2$, let the center of w_i be

$$\left(\frac{s(1 + t + 2st)}{(1 + s)(1 + st)}, y_i\right).$$

The fact that w_i passes through B and C yields

$$r_i^2 = y_i^2 + \frac{s^2(1-t)^2}{(1+s)^2(1+st)^2}, \tag{1}$$

and the fact that w_i is tangent to w yields

$$(\sec\theta - r_i)^2 = (\tan\theta - y_i)^2 + \frac{(1-s^2t)^2}{(1+s)^2(1+st)^2}.$$

Subtracting these, we obtain

$$y_i = r_i \csc\theta - \frac{s(1+t)\cot\theta}{(1+s)(1+st)},$$

and plugging this back into (1) gives

$$r_i^2 - \frac{2s(1+t)\sec\theta}{(1+s)(1+st)} r_i + \frac{s^2\sec^2\theta\big(1+2t\cos(2\theta)+t^2\big)}{(1+s)^2(1+st)^2} = 0. \tag{2}$$

Hence,

$$r_i = \frac{s\sec\theta}{(1+s)(1+st)}(1 \pm 2\sqrt{t}\sin\theta + t)$$

are the two roots of the quadratic equation (2). Therefore, the ratio

$$\frac{r_1}{r_2} = \frac{1-2\sqrt{t}\sin\theta+t}{1+2\sqrt{t}\sin\theta+t} = \frac{2(1+t)}{1+2\sqrt{t}\sin\theta+t} - 1$$

is a decreasing function of θ for $0 \le \theta < \pi/2$, and hence,

$$\frac{r_1}{r_2} > \left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right)^2 = \left(\frac{u-v}{u+v}\right)^2,$$

as desired.

Editorial comment. Two readers—L. R. King and C. R. Pranesachar—solved a slightly different problem, with "w_1 and w_2 internally tangent to w" replaced by "w internally tangent to w_1 and w_2." King proved that the ratio of v to u (whose square is the so-called *cross-ratio* $[B, C, A, D]$) is independent of the locations of A and D (for B and C fixed). Thus, it suffices to prove the claim for the case that the center of w is on the segment between the centers of w_1 and w_2 and the configuration has an axis of symmetry.

Also solved by R. Boukharfane (Canada), J. Chun (Korea), P. P. Dályay (Hungary), L. R. King, O. Kouba (Syria), and C. R. Pranesachar (India).

A Polygon Inequality

11690 [2013, 77]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let M be a point in the interior of a convex polygon with vertices $A_1, \ldots, A_n$ in order. For $1 \le i \le n$, let r_i be the distance from M to A_i, and let R_i be the radius of the circumcircle of triangle MA_iA_{i+1}, where $A_{n+1} = A_1$. Show that

$$\sum_{i=1}^{n} \frac{R_i}{r_i + r_{i+1}} \ge \frac{n}{4\cos(\pi/n)}.$$

Solution by Borislav Karaivanov, University of South Carolina, Columbia, SC.

For $1 \le i \le n$, let α_i denote the internal angle of the polygon at vertex A_i, and let $\alpha_i' = \angle M A_i A_{i+1}$ and $\alpha_i'' = \angle M A_i A_{i-1}$ be the two angles into which $M A_i$ splits α_i. By means of the sine rule in each triangle $M A_i A_{i+1}$, we have $\sin\alpha_{i+1}''/r_i = \sin\alpha_i'/r_{i+1} = 1/2R_i$. Using $\alpha_{n+1}'' = \alpha_1'$, we can write

$$\begin{aligned}\sum_{i=1}^n \frac{r_i + r_{i+1}}{R_i} &= \sum_{i=1}^n (2\sin\alpha_i' + 2\sin\alpha_{i+1}'') = 2\sum_{i=1}^n (\sin\alpha_i' + \sin\alpha_i'') \\ &\le 4\sum_{i=1}^n \sin\left(\frac{\alpha_i' + \alpha_i''}{2}\right) = 4\sum_{i=1}^n \sin\frac{\alpha_i}{2}. \qquad (1)\end{aligned}$$

The inequality is justified by concavity of the sine function on $[0, \pi]$ and convexity of the polygon. Convexity of the polygon guarantees $\alpha_i', \alpha_i'' \in [0, \pi]$ for $1 \le i \le n$. Applying the Jensen inequality to the last sum in (1) yields

$$\frac{1}{n}\sum_{i=1}^n \sin\frac{\alpha_i}{2} \le \sin\left(\frac{\sum_{i=1}^n \alpha_i}{2n}\right) = \sin\left(\frac{\pi(n-2)}{2n}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = \cos\frac{\pi}{n}. \qquad (2)$$

Combining (1) and (2), we obtain

$$\frac{1}{n}\sum_{i=1}^n \frac{r_i + r_{i+1}}{R_i} \le 4\cos\frac{\pi}{n}.$$

By the harmonic-arithmetic mean inequality, we have

$$\frac{n}{\sum_{i=1}^n \frac{R_i}{r_i + r_{i+1}}} \le \frac{1}{n}\sum_{i=1}^n \frac{r_i + r_{i+1}}{R_i}.$$

These last two inequalities imply

$$\frac{n}{\sum_{i=1}^n \frac{R_i}{r_i + r_{i+1}}} \le 4\cos\frac{\pi}{n}.$$

Inverting this inequality yields the required result. Equality holds if and only if the polygon is regular and point M is its center.

Also solved by A. Alt, G. Apostolopoulos (Greece), M. Bataille (France), D. Beckwith, R. Boukharfane & R. Tauraso (Canada & Italy), M. Can, R. Chapman (U.K.), P. De (India), M. Dincă (Romania), O. Geupel (Germany), K. Hanes, O. Kouba (Syria), P. T. Krasopoulos (Greece), O. P. Lossers (Netherlands), C. R. Pranesachar (India), S. Y. Ri (Korea), T. Smotzer, R. Stong, M. Vowe (Switzerland), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U.K.), and the proposer.

Some Moments That Vanish

11691 [2013, 174]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Show that the $2n$th moment $\int_0^\infty x^{2n} f(x)\,dx$ of the function f given by

$$f(x) = \frac{d}{dx}\arctan\left(\frac{\sinh x}{\cos x}\right)$$

is zero when n is an odd positive integer.

Solution by J. G. Simmonds, Charlottesville, VA. Because

$$f(x) = \frac{\cos x \cosh x + \sin x \sinh x}{\cos^2 x + \sinh^2 x} = \frac{1+i}{2\cosh(1+i)x} + \frac{1-i}{2\cosh(1-i)x},$$

the given integral may be expressed as $I_n + J_n$, where

$$I_n = \frac{1+i}{2}\int_0^\infty \frac{x^{2n}\,dx}{\cosh(1+i)x}, \quad J_n = \frac{1-i}{2}\int_0^\infty \frac{x^{2n}\,dx}{\cosh(1-i)x}.$$

Consider a sector S_+ of radius R and angle $\pi/4$ in the upper complex plane with one ray along the positive x-axis. The integrand of J_n has no poles inside this sector, and the integral along the one-eighth circle goes to 0 as $R \to \infty$, so J_n may be computed by integration along the diagonal ray instead of the positive real axis. Set $x = (1+i)r$ for $r > 0$ to obtain $J_n = 2^n i^n \int_0^\infty r^{2n}\operatorname{sech} 2r\,dr$.

Similarly, consider a sector S_- of radius R and angle $\pi/2$ in the lower complex plane with one ray along the positive x-axis. As before, I_n may be computed by integration along the diagonal ray instead of the positive real axis. Set $x = (1-i)r$ to obtain $I_n = (-1)^n 2^n i^n \int_0^\infty r^{2n}\operatorname{sech} 2r\,dr$. Hence, since n is an odd positive integer, $I_n + J_n = \big[(-1)^n + 1\big]J_n = 0$, as required.

Also solved by K. Andersen (Canada), D. Beckwith, R. Chapman (U.K.), F. Holland (Ireland), O. Kouba (Syria), K. D. Lathrop, R. Stong, GCHQ Problem Solving Group (U.K.), and the proposer.

An Inequality, Schurly

11692 [2013, 174]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Ştefan Spătaru, International Computer High School of Bucharest, Bucharest, Romania.* Let a_1, a_2, a_3, a_4 be real numbers in $(0, 1)$, with $a_4 = a_1$. Show that

$$\frac{3}{1-a_1a_2a_3} + \sum_{k=1}^{3}\frac{1}{1-a_k^3} \ge \sum_{k=1}^{3}\frac{1}{1-a_k^2a_{k+1}} + \frac{1}{1-a_ka_{k+1}^2}.$$

Solution by Traian Viteam, Punta Arenas, Chile. We use Schur's inequality $3xyz + x^3 + y^3 + z^3 \ge x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2$, (valid for $x, y, z \ge 0$), and $1/(1-x) = \sum_{k=0}^{\infty} x^k$ for $-1 < x < 1$. Thus,

$$\begin{aligned}
\frac{3}{1-a_1a_2a_3} + \sum_{k=1}^{3}\frac{1}{1-a_k^3} &= 3\sum_{j\ge 0}(a_1a_2a_3)^j + \sum_{k=1}^{3}\sum_{j\ge 0}(a_k^3)^j \\
&= \sum_{j\ge 0}\left(3a_1^ja_2^ja_3^j + (a_1^j)^3 + (a_2^j)^3 + (a_3^j)^3\right) \\
&\ge \sum_{j\ge 0}\left(\sum_{k=1}^{3}(a_k^j)^2a_{k+1}^j + \sum_{k=1}^{3}a_k^j(a_{k+1}^j)^2\right) \\
&= \sum_{k=1}^{3}\sum_{j\ge 0}(a_k^2a_{k+1})^j + \sum_{k=1}^{3}\sum_{j\ge 0}(a_ka_{k+1}^2)^j \\
&= \sum_{k=1}^{3}\frac{1}{1-a_k^2a_{k+1}} + \sum_{k=1}^{3}\frac{1}{1-a_ka_{k+1}^2}
\end{aligned}$$

as claimed. Reference: Cristinel Mortici, "A power series approach to some inequalities," this **Monthly 119** (2012), 147–151.

Also solved by G. Apostolopoulos (Greece), R. Boukharfane (Canada), P. Bracken, R. Chapman (U.K.), P. P. Dályay (Hungary), M. Dincă (Romania), O. Geupel (Germany), N. Grivaux (France), E. A. Herman, B. Karaivanov, H. Katsuura & E. Schmeichel, S. J. Kim (Korea), Y. Kim & S. Yi (Korea), O. Kouba (Syria), T. Koupelis, J. H. Lindsey II, O. P. Lossers (Netherlands), P. Perfetti (Italy), Á. Plaza (Spain), C. R. Pranesachar (India), M. A. Prasad (India), R. Stong, R. Tauraso (Italy), D. B. Tyler, H. Widmer (Switzerland), L. Zhou, CMC 328, GCHQ Problem Solving Group (U.K.), TCDmath Problem Group (Ireland), and the proposers.

A Rational Polynomial

11694 [2013, 174–175]. *Proposed by Proposed by Kent Holing, Trondheim, Norway.* Let $g(x) = x^4 + ax^3 + bx^2 + ax + 1$, where a and b are rational. Suppose g is irreducible over $\mathbb{Q}$. Let G be the Galois group of g. Let $\mathbb{Z}_4$ denote the additive group of the integers mod 4, and let D_4 be the dihedral group of order 8. Let $\alpha = (b+2)^2 - 4a^2$ and $\beta = a^2 - 4b + 8$.
(**a**) Show that G is isomorphic to $\mathbb{Z}_4$ or D_4 if and only if neither α nor β is the square of a rational number, and that G is cyclic exactly when $\alpha\beta$ is the square of a rational number.
(**b**) Suppose neither α nor β is a square, but $\alpha\beta$ is. Let r be one of the roots of g. (Trivially, $1/r$ is also a root.) Let $s = \sqrt{\alpha\beta}$, and let

$$t = \frac{1}{2s}\big((s + (b-6)a)r^3 + (as + (b-8)a^2 + 4(b+2))r^2 \\ + ((b-1)s + (b^2 - b + 2)a - 2a^3)r + 2(b+2)b - 6a^2\big).$$

Show that $t \in \mathbb{Q}[r]$ is one of the other two roots of g. Comment on the special case $a = b = 1$.

Solution by Richard Stong, Center for Communications Research, San Diego CA.
(a) Though it is not stated in this way, the hypothesis that g is irreducible implies that β is not a square. Indeed, write

$$\frac{g(x)}{x^2} = \left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) + b - 2.$$

Recognizing this as a quadratic in $x + 1/x$ with roots $(-a \pm \sqrt{\beta})/2$, we obtain

$$g(x) = \left(x^2 + \frac{a - \sqrt{\beta}}{2}x + 1\right)\left(x^2 + \frac{a + \sqrt{\beta}}{2}x + 1\right).$$

The splitting field of g is built from $\mathbb{Q}[\sqrt{\beta}]$ by adjoining roots of two quadratics. Hence, G is a 2-group and is one of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, and the eight-element dihedral group D_4. The discriminants of these quadratics are $\Delta_\pm = (a^2 - 2b - 4 \mp a\sqrt{\beta})/2$. Since these expressions are conjugate in $\mathbb{Q}[\sqrt{\beta}]$, if either is a square, then both are squares, and then $|G| = 2$, a contradiction. Therefore, neither is a square in $\mathbb{Q}[\sqrt{\beta}]$; it follows that G has order 8 (and is D_4) unless these discriminants generate the same quadratic extension of $\mathbb{Q}[\sqrt{\beta}]$. This requires that α is a square in $\mathbb{Q}[\sqrt{\beta}]$, with

$$\Delta_+\Delta_- = \frac{(a^2 - 2b - 4)^2 - a^2\beta}{4} = \alpha.$$

Writing α as $(p + q\sqrt{\beta})^2$ for rational p and q and looking at the coefficient of $\sqrt{\beta}$, we have $q = 0$ or $p = 0$. If $q = 0$, then α is a rational square. Since the discriminant

Δ of g is $\alpha\beta^2$, this implies that the Galois group G is in A_4, and hence, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. If $p = 0$, then $\alpha\beta$ is a rational square $(q\beta)^2$, and the Galois group is not in A_4. In this case, $G = \mathbb{Z}_4$.

(b) From the factorization above, we have four roots:

$$\left\{r, \frac{1}{r}\right\} = \frac{-a + \sqrt{\beta} \pm 2\sqrt{\Delta_+}}{4} \quad \text{and} \quad \left\{t, \frac{1}{t}\right\} = \frac{-a - \sqrt{\beta} \pm 2\sqrt{\Delta_-}}{4}.$$

Hence, $(r - 1/r)(t - 1/t) = \sqrt{\Delta_+\Delta_-} = \sqrt{\alpha}$, and so

$$t - \frac{1}{t} = \frac{r\sqrt{\alpha}}{r^2 - 1} \quad \text{and} \quad t + \frac{1}{t} = \frac{-\alpha - \sqrt{\beta}}{2} = -a - r - \frac{1}{r}.$$

With the notation just introduced, the root that matches the problem statement is actually $1/t$, not t. Subtracting the first from the second, multiplying by s, and using $s\sqrt{\alpha} = \alpha\sqrt{\beta} = \alpha(2r + 2/r + a)$, we have

$$\frac{2s}{t} = -\frac{\alpha(2r^2 + ar + 2)}{r^2 - 1} - as - sr - \frac{s}{r}.$$

Since $g(r) = 0$, we compute the above using the Euclidean algorithm and find

$$\begin{aligned}
\frac{1}{r} &= -r^3 - ar^2 - br - a, \\
\frac{\alpha}{r^2 - 1} &= 2ar^3 + (2a^2 - b - 2)r^2 + abr + 4a^2 - b^2 - 3b - 2, \\
\frac{\alpha(2r^2 + ar + 2)}{r^2 - 1} &= (b - 6)ar^3 + ((b - 8)a^2 + 4(b + 2)r^2 \\
&\quad + ((b^2 - b - 2)a - 2a^3)r + 2(b + 2)b - 6a^2.
\end{aligned}$$

Substituting these in and collecting terms gives the desired formula

$$\begin{aligned}
\frac{2s}{t} &= (s + (b - 6)a)r^3 + (as + (b - 8)a^2 + 4(b + 2))r^2 \\
&\quad + ((b - 1)s + (b^2 - b + 2)a - 2a^3)r + 2(b + 2)b - 6a^2.
\end{aligned}$$

If $a = b = 1$, then $\alpha = \beta = s = 5$, and this formula simplifies to $1/t = r^2$; since the roots of g are the fifth roots of unity in this special case, $1/t = r^2$ is a reasonable answer.

Also solved by R. Chapman (UK), C. P. Rupert, the TCDmath Problem Group (Ireland), and the proposer.

A Stirling Integral

11695 [2013, 175]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* The *Stirling numbers of the first kind*, denoted $s(n, k)$, can be defined by their generating function: $z(z - 1) \cdots (z - n + 1) = \sum_{k=0}^{n} s(n, k)z^k$. Let m and p be nonnegative integers with $m > p - 4$. Prove that

$$\begin{aligned}
&\int_0^1 \int_0^1 \frac{\log x \cdot \log^m(xy) \cdot \log y}{(1 - xy)^p} \, dx \, dy \\
&\quad = \begin{cases} (-1)^m \frac{1}{6}(m + 3)!\zeta(m + 4), & \text{if } p = 1; \\ (-1)^{m+p-1} \frac{(m+3)!}{6(p-1)!} \sum_{k=1}^{p-1} (-1)^k s(p - 1, k)\zeta(m + 4 - k) & \text{if } p > 1. \end{cases}
\end{aligned}$$

Solution by David Beckwith, Sag Harbor, NY. Let I denote the given integral. Making the changes of variables $x = e^{-u}$ and $y = e^{-v}$ and then $u + v = \alpha$ and $u - v = \beta$, we have

$$\begin{aligned}
I &= (-1)^m \int_0^\infty \int_0^\infty \frac{(u+v)^m e^{-(u+v)}}{(1-e^{-(u+v)})^p} uv\,du\,dv \\
&= \frac{(-1)^m}{8} \int_{\alpha=0}^\infty \int_{\beta=-\alpha}^{\alpha} \frac{\alpha^m e^{-\alpha}}{(1-e^{-\alpha})^p} (\alpha^2 - \beta^2) d\beta\,d\alpha \\
&= \frac{(-1)^m}{6} \int_0^\infty \frac{\alpha^{m+3} e^{-\alpha}}{(1-e^{-\alpha})^p} d\alpha.
\end{aligned}$$

Setting $t = e^{-\alpha}$ in the Taylor series expansion

$$\frac{1}{(1-t)^p} = \sum_{j=0}^\infty \binom{p-1+j}{p-1} t^j \quad (|t| < 1),$$

and integrating term-by-term (justified by positivity of all terms), we have

$$\begin{aligned}
I &= \frac{(-1)^m}{6} \sum_{j=0}^\infty \binom{p-1+j}{p-1} \int_0^\infty \alpha^{m+3} e^{-(j+1)\alpha}\,d\alpha \\
&= \frac{(-1)^m (m+3)!}{6} \sum_{j=0}^\infty \binom{p-1+j}{p-1} j^{-m-4}.
\end{aligned}$$

Taking $n = p - 1$ and $z = -j$ in the generating function for the Stirling numbers of the first kind and using the convention that $s(n, 0) = \delta_{n,0}$, for $p \geq 1$ we have

$$\begin{aligned}
\binom{p-1+j}{p-1} &= \frac{(-1)^{p-1}}{(p-1)!} j(j+1)\cdots(j+p-2) \\
&= \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{p-1} (-1)^k s(p-1, k) j^k.
\end{aligned}$$

Substituting this in the previous expression and reversing the order of summation, we have

$$I = (-1)^{m+p-1} \frac{(m+3)!}{6(p-1)!} \sum_{k=0}^{p-1} (-1)^k s(p-1, k) \zeta(m+4-k)$$

for $m > p - 4$, which (because of the convention for $s(p-1, 0)$) agrees with the requested formula for $p \geq 1$.

Also solved by K. Andersen (Canada), R. Boukharfane (Canada), P. Bracken, R. Chapman (U.K.), P. P. Dályay (Hungary), D. Fleischman, O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before August 31, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11831. *Proposed by Raitis Ozols, University of Latvia, Riga, Latvia.* Prove that for $\varepsilon > 0$ there exists an integer n such that the greatest prime divisor of $n^2 + 1$ is less than εn.

11832. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let $C(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$ (thus $C(z)$ is the generating function of the Catalan numbers). Prove that

$$\log(C(z))^2 = \sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n}.$$

Here, $H_k = \sum_{j=1}^{k} 1/j$; that is, H_k is the kth *harmonic number*.

11833. *Proposed by Mher Safaryan, Yerevan State University, Yerevan, Armenia, and Vahagn Aslanyan, University of Oxford, Oxford, U. K.* Let f be a real-valued function on an open interval (a, b) such that the one-sided limits $\lim_{t\to x^-} f(t)$ and $\lim_{t\to x^+} f(t)$ exist and are finite for all x in (a, b). Can the set of discontinuities of f be uncountable?

11834. *Proposed by Arkady Alt, San Jose, CA.* For nonnegative real numbers u, v, w, let $\Delta(u, v, w) = 2(uv + vw + wu) - (u^2 + v^2 + w^2)$. Say that two lists (a, b, c) and (x, y, z) *agree in order* if $(a - b)(x - y) \ge 0$, $(b - c)(y - z) \ge 0$, and $(c - a)(z - x) \ge 0$. Prove that if (x, y, z) and (a, b, c) agree in order, then $\Delta(a, b, c)\Delta(x, y, z) \ge 3\Delta(ax, by, cz)$.

11835. *Proposed by George Stoica, University of New Brunswick, St John, Canada.* Find all functions f from $[0, \infty)$ to $[0, \infty)$ such that whenever $x, y \ge 0$,

$$\sqrt{3} f(2x) + 5 f(2y) \le 2 f(\sqrt{3}x + 5y).$$

http://dx.doi.org/10.4169/amer.math.monthly.122.04.390

11836. *Proposed by Traian Viteam, Montevideo, Uruguay.* Let ABC be a triangle with sides of lengths a, b, and c, circumradius R, and inradius r. For $p, q, r > 0$, let $f(p, q, r) = pqr/(p+q)(r^2 - (p-q)^2)$. Prove that

$$\frac{R}{2r} \geq \frac{2}{3}\left(f(a,b,c) + f(b,c,a) + f(c,a,b)\right).$$

11837. *Proposed by Iosif Pinelis, Michigan Technological University, Houghton, MI.* Let $a_0 = 1$, and for $n \geq 0$ let $a_{n+1} = a_n + e^{-a_n}$. Let $b_n = a_n - \log n$. For $n \geq 0$, show that $0 < b_{n+1} < b_n$; also show that $\lim_{n\to\infty} b_n = 0$. (The proposer notes that the content of Problem B4 of the 73rd William Lowell Putnam Mathematical Competition—see, e.g., this Monthly, Volume 120, no. 8, pages 682–686—was the question of whether b_n has a finite limit as $n \to \infty$.)

SOLUTIONS

A Triangle Inequality

11696 [2013, 175]. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia, and Elton Bojaxhiu, Kriftel, Germany.* Let T be a triangle with sides of length a, b, c, inradius r, circumradius R, and semiperimeter p. Show that

$$\frac{1}{2(r^2+4Rr)} + \frac{1}{9}\sum_{\text{cyc}} \frac{1}{c(p-c)} \geq \frac{4}{9}\sum_{\text{cyc}}\left(\frac{1}{9Rr - c(p-c)}\right).$$

Solution by Theo Koupelis, Edison State College, Fort Myers, FL. Let A, B, C denote the angles opposite sides a, b, c, respectively. Now $p - c = r\cot(C/2)$ and $c = 2R\sin C$, so $c(p-c) = 2rR(1+\cos C)$, and thus $9rR - c(p-c) = rR(7 - 2\cos C)$. Therefore,

$$\frac{4}{9}\sum_{\text{cyc}}\left(\frac{1}{9Rr - c(p-c)}\right) - \frac{1}{9}\sum_{\text{cyc}}\frac{1}{c(p-c)} = \frac{1}{18rR}\sum_{\text{cyc}}\frac{1+10\cos A}{(1+\cos A)(7-2\cos A)}.$$

Using $r = R(\cos A + \cos B + \cos C - 1)$, we get $2(r^2+4rR) = 2rR(\cos A + \cos B + \cos C + 3)$. Thus the given inequality is equivalent to

$$\frac{9}{\cos A + \cos B + \cos C + 3} \geq \sum_{\text{cyc}}\frac{1+10\cos A}{(1+\cos A)(7-2\cos A)}.$$

On the other hand,

$$\frac{1+10\cos A}{(1+\cos A)(7-2\cos A)} \leq \frac{8\cos A + 2}{9},$$

because this inequality is equivalent to $\left(\cos A - \frac{1}{2}\right)^2(16\cos A - 20) \leq 0$. Therefore,

$$\sum_{\text{cyc}}\frac{1+10\cos A}{(1+\cos A)(7-2\cos A)} \leq \frac{8(\cos A + \cos B + \cos C) + 6}{9},$$

with equality when the triangle is equilateral.

Now $\cos A + \cos B + \cos C \le \frac{3}{2}$, with equality holding when $A = B = C = \pi/3$. Thus,

$$\frac{9}{\cos A + \cos B + \cos C + 3} \ge 2 \quad \text{while} \quad \frac{8(\cos A + \cos B + \cos C) + 6}{9} \le 2.$$

Therefore the given inequality is true, with equality for an equilateral triangle.

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), P. P. Dályay (Hungary), B. Karaivanov, O. Kouba (Syria), K.-W. Lau (China), P. Nüesch (Switzerland), P. Perfetti (Italy), C. R. Pranesachar (India), R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), and the proposers.

A Limit with Gamma

11697 [2013, 175]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let n and q be integers, with $2n > q \ge 1$. Let

$$f(t) = \int_{\mathbb{R}^q} \frac{e^{-t(x_1^{2n}+\cdots+x_q^{2n})}}{1 + x_1^{2n} + \cdots x_q^{2n}}\, dx_1 \cdots dx_q.$$

Prove that $\lim_{t\to\infty} t^{q/2n} f(t) = n^{-q}(\Gamma(1/2n))^q$.

Solution by Roberto Tauraso, Università di Roma, "Tor Vergata," Roma, Italy. Let $x_i^{2n} = u_i$ for $i = 1, \ldots, q$ so that $dx_i = \frac{1}{2n}(u_i)^{1/(2n)-1}\, du_i$ and

$$\begin{aligned}
f(t) &= 2^q \int_{[0,\infty)^q} \frac{e^{-t(x_1^{2n}+\cdots+x_q^{2n})}}{1 + x_1^{2n} + \cdots + x_q^{2n}}\, dx_1 \cdots dx_q \\
&= n^{-q} \int_{[0,\infty)^q} \frac{\prod_{i=1}^q u_i^{1/(2n)-1} e^{-tu_i}}{1 + u_1 + \cdots + u_q}\, du_1 \cdots du_q \\
&= n^{-q} \int_{[0,\infty)^q} \prod_{i=1}^q u_i^{1/(2n)-1} e^{-tu_i} \left(\int_0^\infty e^{-s(1+u_1+\cdots+u_q)}\, ds \right) du_1 \cdots du_q \\
&= n^{-q} \int_0^\infty e^{-s} \left(\int_{[0,\infty)^q} \prod_{i=1}^q u_i^{1/(2n)-1} e^{-(s+t)u_i}\, du_1 \cdots du_q \right) ds \\
&= n^{-q} \int_0^\infty e^{-s} \left(\int_0^\infty u^{1/(2n)-1} e^{-(s+t)u}\, du \right)^q ds \\
&= n^{-q} \int_0^\infty e^{-s} \left(\frac{1}{(s+t)^{1/(2n)}} \int_0^\infty r^{1/(2n)-1} e^{-r}\, dr \right)^q ds \\
&= n^{-q} \Gamma\left(\frac{1}{2n}\right)^q \int_0^\infty \frac{e^{-s}}{(s+t)^{q/(2n)}}\, ds.
\end{aligned}$$

The integral form of the Gamma function has been used in completing the last line. Now the required limit can be evaluated as follows:

$$\lim_{t\to\infty} t^{q/2n} f(t) = n^{-q} \Gamma\left(\frac{1}{2n}\right)^q \lim_{t\to\infty} \int_0^\infty \left(\frac{t}{s+t}\right)^{q/2n} e^{-s} ds = n^{-q} \Gamma\left(\frac{1}{2n}\right)^q.$$

The computation first uses $0 \le \left(\frac{t}{s+t}\right)^{q/2n} \le 1$ and $\lim_{t\to\infty} \left(\frac{t}{s+t}\right)^{q/2n} = 1$, and then the limit of the remaining integral over s is equal to $\Gamma(1) = 1$. The condition $2n > q$ has not been used and seems to be unnecessary.

Also solved by K. Andersen (Canada), D. Beckwith, M. Benito, Ó. Ciaurri, E. Fernández & L. Roncal (Spain), R. Boukharfane (Canada), P. Bracken, R. Chapman (U. K.), O. Furdui (Romania), J.-P. Grivaux (France), J. A. Grzesik, E. A. Herman, B. D. Hughes (Australia), O. Kouba (Syria), O. P. Lossers (Netherlands), P. Perfetti (Italy), M. A. Prasad (India), C. M. Russell, R. Stong, N. Thornber, BSI Problems Group (Germany), NSA PRoblems Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Rational Function that Simplifies at a Special Point

11698 [2013, 365]. *Proposed by Timothy Hall, Cambridge, MA.* Provide an algorithm that takes as input a positive integer n and a nonzero constant k and returns polynomials F and G in variables u and v such that when x^n is substituted for u, and $x + k/x$ is substituted for v, $F(u, v)/G(u, v)$ simplifies (disregarding removable singularities) to x. (For example, when $k = 1$ and $n = 3$, $F = u + v$ and $G = v^2 - 1$ will do.)

Solution by David Beckwith, Sag Harbor, NY. Powers of v expand as

$$\left(x + \frac{k}{x}\right)^n = x^n + \cdots + \binom{n}{j} x^{n-j} \frac{k^j}{x^j} + \cdots + \frac{k^n}{x^n}.$$

Combining the first and last terms, the second and next-to-last, and so on yields

$$\left(x + \frac{k}{x}\right)^n = \left(x^n + \frac{k^n}{x^n}\right) + \binom{n}{1} k \left(x^{n-2} + \frac{k^{n-2}}{x^{n-2}}\right) + \cdots$$
$$+ \binom{n}{2} k^2 \left(x^{n-4} + \frac{k^{n-4}}{x^{n-4}}\right) + \cdots + \begin{cases} \binom{n}{\lfloor n/2 \rfloor} k^{\lfloor n/2 \rfloor} (x + \frac{k}{x}) & \text{for } n \text{ odd,} \\ \binom{n}{n/2} k^{n/2} & \text{for } n \text{ even.} \end{cases} \tag{1}$$

With $v = x + k/x$, we claim for $n \geq 1$ that $x^n + k^n/x^n$ can be written as $\phi_n(v)$ with ϕ_n being a polynomial of degree n. The proof is by induction on n. Note that $\phi_1(v) = v$ and $\phi_2(v) = v^2 - 2k$. For $n \geq 3$, (1) gives

$$x^n + \frac{k^n}{x^n} = v^n - \binom{n}{1} k \phi_{n-2}(v) - \binom{n}{2} k^2 \phi_{n-4}(v) - \binom{n}{3} k^3 \phi_{n-6}(v) - \cdots,$$

proving the claim.

The formulas for the desired polynomials depend on n, so we write them as $F_n(u, v)$ and $G_n(u, v)$. When $n = 1$, we have $u = x$, so we may let $F_1(u, v) = u$ and $G_1(u, v) = 1$. When $n = 2$, we have $\frac{uv}{u+k} = \frac{x^2(x+2/x)}{x^2+2} = x$, so we set $F_2(u, v) = uv$ and $G_2(u, v) = u + k$.

For $n \geq 3$ with n odd, note that

$$\frac{\phi_{(n-1)/2}(v)}{\phi_{(n+1)/2}(v)} = \frac{\left(x^{(n-1)/2} + \frac{k^{(n-1)/2}}{x^{(n-1)/2}}\right) x^{(n+1)/2}}{\left(x^{(n+1)/2} + \frac{k^{(n+1)/2}}{x^{(n+1)/2}}\right) x^{(n+1)/2}} = \frac{u + k^{(n-1)/2} x}{ux + k^{(n+1)/2}}.$$

Solving for x gives

$$x = \frac{u\phi_{(n+1)/2}(v) - k^{(n+1)/2} \phi_{(n-1)/2}(v)}{u\phi_{(n-1)/2}(v) - k^{(n-1)/2} \phi_{(n+1)/2}(v)}.$$

Hence we may set $F_n(u, v) = u\phi_{(n+1)/2}(v) - k^{(n+1)/2} \phi_{(n-1)/2}(v)$ and $G_n(u, v) = u\phi_{(n-1)/2}(v) - k^{(n-1)/2} \phi_{(n+1)/2}(v)$.

Finally, for $n \geq 4$ with n even, note that $x^2 = vx - k$. Now

$$\frac{\phi_{(n-2)/2}(v)}{\phi_{n/2}(v)} = \frac{x^{(n+2)/2}\left(x^{(n-2)/2} + \frac{k^{(n-2)/2}}{x^{(n-2)/2}}\right)}{x^{(n+2)/2}\left(x^{n/2} + \frac{k^{n/2}}{x^{n/2}}\right)} = \frac{u - k^{n/2} + k^{(n-2)/2}vx}{(u + k^{n/2})x}.$$

Solving for x gives

$$x = \frac{(u - k^{n/2})\phi_{n/2}(v)}{(u + k^{n/2})\phi_{(n-2)/2}(v) - k^{(n-2)/2}v\phi_{n/2}(v)}.$$

Hence in this case we may set $F_n(u, v) = (u - k^{n/2})\phi_{n/2}(v)$ and $G_n(u, v) = (u + k^{n/2})\phi_{(n-2)/2}(v) - k^{(n-2)/2}v\phi_{n/2}(v)$.

Also solved by B. Radouan (Canada), R. Chapman (U. K.), E. A. Herman, O. P. Lossers (Netherlands), C. R. Pranesachar (India), R. E. Prather, C. P. Rupert, B. Schmuland (Canada), N. C. Singer, J. H. Steelman, R. Stong, GCHQ Problem Solving Group (U. K.), TCD Problem Group (Ireland), and the proposer.

A New Divisor Every Time?

11699 [2013, 635]. *Proposed by Bakir Farhi, University of Bejaia, Bejaia, Algeria.* Let $\langle a_k \rangle$ be a strictly increasing sequence of positive integers such that $\sum_{k=2}^{\infty} \frac{1}{a_k \log a_k}$ diverges. Prove that $\operatorname{lcm}(a_1, \ldots, a_k) = \operatorname{lcm}(a_1, \ldots, a_{k+1})$ for infinitely many k in $\mathbb{N}$.

Solution by TCDmath Problems Group, Trinity College, Dublin, Ireland. Suppose to the contrary that $\operatorname{lcm}(a_1, \ldots, a_k) < \operatorname{lcm}(a_1, \ldots, a_{k+1})$ for $k \geq N$. For $k > N$, there must then be a prime power q which divides a_k but does not divide a_j for any $j < k$. Let us choose one such prime power q_k for each $k > N$. By construction the prime powers q_k are distinct. Hence

$$\sum_{k=N+1}^{\infty} \frac{1}{a_k \log a_k} \leq \sum_{k=N+1}^{\infty} \frac{1}{q_k \log q_k} \leq \sum_{q} \frac{1}{q \log q},$$

where the last sum is taken over all prime powers q. However,

$$\sum_{q} \frac{1}{q \log q} = \sum_{p} \frac{1}{p \log p} \sum_{j=0}^{\infty} \frac{1}{(j+1)p^j} \leq \sum_{p} \frac{1}{p \log p} \sum_{j=0}^{\infty} \frac{1}{p^j} \leq \sum_{j=0}^{\infty} \frac{2}{p \log p}.$$

By the prime number theorem, $p_n \sim n \log n$ (where p_n is the nth prime). Hence $p_n \log p_n \sim n \log^2 n$. Since the sum $\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$ is easily seen to converge by comparison with $\int \frac{dx}{x \log^2 x}$, it follows that $\sum \frac{1}{a_k \log a_k}$ converges, contrary to the hypothesis.

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), O. P. Lossers (Netherlands), R. Martin (Germany), H. C. Morris, P. Pongsriiam (Thailand), M. A. Prasad (India), R. Stong, R. Tauraso (Italy), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Harmonic Sum Asymptotics

11701 [2013, 000]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania.*
(**a**) Let $\langle x_n \rangle$ be the sequence defined by $\sum_{k=1}^{mn} 1/k = \gamma + \log(mn + x_n)$, where γ is the Euler–Mascheroni constant. Find $\lim_{n\to\infty} x_n$.
(**b**) Let $\langle y_n \rangle$ be the sequence defined by $\sum_{k=1}^{mn} 1/k = \gamma + \log(m(n + y_n))$. Find $\lim_{n\to\infty} y_n$.

Solution by Paul Bracken, University of Texas, Edinburg, TX. (**a**) Define a sequence by $\gamma_n = -\log(n) + \sum_{k=1}^{n} 1/k$. Note that

$$\gamma_n - \gamma = \sum_{k=1}^{n} \frac{1}{k} - \log(n) - \gamma = \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right). \tag{1}$$

The equation that defines the sequence $\{x_n\}$ can be put in the form

$$-\log(mn + x_n) + \log(mn) - \log(mn) + \sum_{k=1}^{mn} 1/k = \gamma.$$

Hence, $\log\left(1 + \frac{x_n}{mn}\right) = \gamma_{mn} - \gamma$. Solving this equation for x_n yields

$$x_n = mn(e^{\gamma_{mn}-\gamma} - 1) = \left(\frac{e^{\gamma_{mn}-\gamma} - 1}{\gamma_{mn} - \gamma}\right) \cdot mn(\gamma_{mn} - \gamma).$$

The limit of the ratio on the right side exists since $\gamma_{mn} \to \gamma$ when $n \to \infty$ so

$$\lim_{n\to\infty} \frac{e^{\gamma_{mn}-\gamma} - 1}{\gamma_{mn} - \gamma} = \lim_{u\to 0} \frac{e^u - 1}{u} = 1. \tag{2}$$

The required limit is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} mn(\gamma_{mn} - \gamma) = \lim_{M\to\infty} M \cdot (\gamma_M - \gamma) = \frac{1}{2}.$$

(b) The equation that determines the sequence y_n is equivalent to

$$-\log(m(n + y_n)) + \log(mn) - \log(mn) + \sum_{k=1}^{mn} 1/k = \gamma.$$

Thus $\log\left(1 + \frac{y_n}{n}\right) = \gamma_{mn} - \gamma$, and

$$y_n = (e^{\gamma_{mn}-\gamma} - 1) \cdot n = \frac{e^{\gamma_{mn}-\gamma} - 1}{\gamma_{mn} - \gamma} \cdot n(\gamma_{mn} - \gamma).$$

Using (1) and (2) again,

$$\lim_{n\to\infty} y_n = \frac{1}{m} \lim_{n\to\infty} mn\,(\gamma_{mn} - \gamma) = \frac{1}{m} \lim_{M\to\infty} M \cdot (\gamma_M - \gamma) = \frac{1}{2m}.$$

This is the required limit for (b).

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), M. W. Coffey, D. Fleischman, O. Furdui (Romania), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), M. R. Modak (U. K.), M. Omarjee (France), P. Perfetti (Italy), P. Pongsriiam (Thailand), C. P. Rupert, J. Schlosberg, N. C. Singer, A. Stenger, R. Stong, D. B. Tyler, E. I. Verriest, M. Vowe (Switzerland), S. Wagon, L. Zhou, GHCQ Problem Solving Group (U. K.), GWstat Problem Solving Group, Missouri State University Problem Solving Group, and the proposers.

One Ring Rings True

11702 [2013]. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.* Find all nonzero rings R (not assumed to be commutative or to contain a multiplicative identity) with these properties:

(a) There exists $x \in R$ that is neither a left nor a right zero divisor, and

(b) Every map φ from R to R that satisfies $\varphi(x + y) = \varphi(x) + \varphi(y)$ also satisfies $\varphi(xy) = \varphi(x)\varphi(y)$. (Every additive homomorphism on R is a ring homomorphism.)

Solution by Bill Abrams. The only such ring is $\mathbb{Z}_2$, which clearly satisfies both properties. Conversely, let R be a ring satisfying (a) and (b), and let $x \in R$ be an element establishing (a). If φ is the additive homomorphism on R given by left multiplication by x, then for all $a, b \in R$ we use (b) to compute $xab = \varphi(ab) = \varphi(a)\varphi(b) = xaxb$. Since x is not a zero divisor, canceling x yields $ab = axb$. Putting $x = a$ yields $xb = xxb$, so $b = xb$; putting $x = b$ yields $ax = axx$, so $a = ax$. Thus, x is a two-sided multiplicative identity; call it 1.

Now let ψ be the ring homomorphism defined by $\psi(a) = a + a$. Since $\psi(a) = \psi(a)\psi(1)$ for all $a \in R$, it follows that $a + a = a + a + a + a$, so $a + a = 0$. Identify the subring $\{0, 1\}$ of R with $\mathbb{Z}_2$, so R is in fact a $\mathbb{Z}_2$-algebra. Let B be a basis for R over $\mathbb{Z}_2$ that contains 1, and choose $c \in B$ with $c \neq 1$. The mapping f that switches c and 1 and sends the rest of B to 0 is linear, so

$$1 = f(c) = f(1 \cdot c) = f(1)f(c) = c \cdot 1 = c,$$

which is a contradiction. Hence $B = \{1\}$ and $R = \mathbb{Z}_2$.

Editorial comment. Several solvers showed that R is a boolean ring.

Also solved by P. Budney, R. Chapman (U. K.), S. M. Gagola Jr., C. Lanski, O. P. Lossers (Netherlands), P. S. Peck, F. Perdomo & A. Francisco (Spain), C. P. Rupert, R. Stong, R. Trachtman, D. Tyler, the Missouri State University Problem Solving Group, the NSA Problems Group, the TCDmath Problem Group (Ireland), and the proposer.

A Focus of an Ellipse in a Cone

11703 [2013, 366]. *Proposed by Richard Bagby, New Mexico State University, Las Cruces, NM.* For $\lambda > 0$, let $\Gamma(\lambda) = \{(x, y, z) \in \mathbb{R}^3 \colon z \geq \lambda\sqrt{x^2 + y^2}\}$, and let $C(\lambda)$ be the (half-cone) boundary of $\Gamma(\lambda)$. Prove that every point in the interior of $\Gamma(\lambda)$ is the focus of at least one ellipse in $C(\lambda)$, and find the largest μ such that every ellipse in $C(\lambda)$ has at least one focus in $\Gamma(\mu)$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We may assume without loss of generality that a prescribed focus is given in the xz plane. At this focus, the plane of the ellipse is tangent to a sphere (the Dandelin sphere) that has its center on the axis of the cone and touches the cone. This center has equal distances to the focus and the intersection of the sphere with $C(\lambda)$; so, to find this center, intersect the axis with a parabola whose focus is the given focus and whose directrix is $z = \pm\lambda x$. There are two solutions: a larger sphere and a smaller sphere. The intersection of the tangent plane with $C(\lambda)$ is an ellipse only if the angle the plane makes with the axis is larger than the vertex angle of the cone. For the larger sphere, this is no problem, so there is at least one ellipse with the given point as focus. For the smaller sphere, however, the limiting case is a plane parallel to a half-line in $C(\lambda)$. (The intersection is then a parabola.) At the point of the sphere opposite the focus the sphere touches $C(\lambda)$. Suppose this point has coordinates $(-a, 0, \lambda a)$. The center of the sphere is then $(0, 0, \lambda + 1/\lambda)$, and the focus is $(a, 0, (\lambda + 2/\lambda)a)$. The largest μ is $\lambda + 2/\lambda$, since if $(a, 0, b)$ is a point with $b > |a|(\lambda + 2/\lambda)$, then it is the focus of an ellipse in $C(\lambda)$.

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), J.-P. Grivaux (France), M. E. Kidwell & M. D. Meyerson & D. Ruth & M. Wakefield, J. Martínez (Spain), P. Perfetti (Italy), K. A. Roper, R. Stong, E. I. Verriest, L. Zhou, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, TCDmath Problem Group (Ireland), and the proposer.

Cycles in Products of Involutions

11704 [2013, 366]. *Proposed by Olivier Bernardi, Brandeis University, Waltham, MA, Thaynara Arielly de Lima, Universidade de Brasilia, Brasilia, Brazil, and Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA.* Let S_{2n} denote the symmetric group of all permutations of $\{1, \ldots, 2n\}$ and let T_{2n} denote the set of all fixed-point-free involutions in S_{2n}. Choose u and v from T_{2n} at random (any element of T_{2n} being as likely as any other) and independently. What is the probability that 1 and 2 are in the same cycle of the permutation uv? (For example, when $n = 2$, $T_{2n} = \{2143, 3412, 4321\}$, (u, v) can be $(3412, 4321)$ or $(4321, 3412)$, and the required probability is $2/9$.)

Solution by Reiner Martin, Bad Soden-Neuenhain, Germany. We will show by induction on n that the probability is $\frac{2n-2}{6n-3}$. Let $[m] = \{1, \ldots, m\}$. For $n = 1$, the probability is 0, since u and v must both transpose 1 and 2.

Now consider $n > 1$. Begin with 1 and apply uv repeatedly to obtain the distinct elements in the cycle of uv containing 1. The transpositions in v can be viewed as a red matching on $[2n]$, and similarly u can be viewed as a blue matching on $[2n]$. Following the cycle of uv containing 1 is following red and blue alternately, completing an even cycle of elements. Because each element is in only one edge of each color, elements reached after an odd number of steps cannot also be reached after an even number of steps. Therefore, if $v(1) = 2$, then 1 and 2 cannot be in the same cycle under uv.

Assume now that $v(1) \neq 2$, which occurs with probability $\frac{2n-2}{2n-1}$. If $uv(1) = 2$, which occurs with probability $\frac{2n-2}{(2n-1)^2}$, then 1 and 2 are in the same cycle of uv.

The remaining case is $v(1) \neq 2$ and $uv(1) \neq 2$, with probability $\frac{(2n-2)(2n-3)}{(2n-1)^2}$. Now we construct fixed-point-free involutions u' and v' on $[2n] - \{v(1), uv(1)\}$. Obtain u' by restricting u to $[2n] - \{v(1), uv(1)\}$. Let v' agree with v on $[2n] - \{1, v(1), uv(1), vuv(1)\}$, but let $v'(vuv(1)) = 1$ and $v'(v(1)) = uv(1)$.

Note that 1 and 2 are in the same cycle of uv if and only if 1 and 2 are in the same cycle of $u'v'$. For each fixed choice of the distinct elements $v(1)$ and $uv(1)$, neither equal to 2, the involution u' is a random fixed-point-free involution on $2n - 2$ elements. Also, once the distinct elements $v(1)$ and $uv(1)$ are specified (neither in $\{1, 2\}$), the choice of $vuv(1)$ is random among the remaining elements other than 1 (it cannot equal 1 since $v(1) \neq uv(1)$). Therefore, v' is also a random fixed-point-free involution on the same $2n - 2$ elements as u'.

By the induction hypothesis, 1 and 2 are in the same cycle of $u'v'$ with probability $\frac{2n-4}{6n-9}$. To obtain the probability that 1 and 2 are in the same cycle of uv, we compute

$$\frac{2n-2}{(2n-1)^2} + \frac{(2n-2)(2n-3)}{(2n-1)^2} \cdot \frac{2n-4}{6n-9} = \frac{2n-2}{6n-3}.$$

Editorial comment. The proposers noted that the probability approaches $1/3$ as $n \to \infty$, while 1 and 2 are in the same cycle of a random permutation of S_n with probability exactly $1/2$ for $n \geq 2$. The proposers showed that the probability that $1, \ldots, k$ are all in the same cycle of uv is $\frac{(k-1)!}{(2k-1)!!} \cdot 2^k \binom{n-1}{k} / \binom{2n-1}{k}$.

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), C. Delorme (France), O. Geupel (Germany), D. Gove, Y. J. Ionin, D. Knuth, J. H. Lindsey II, O. P. Lossers (Netherlands), M. A. Prasad (India), R. E. Prather, C. P. Rupert, B. Schmuland (Canada), R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before September 30, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11838. *Proposed by Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA.* Let n be a positive integer. Find the least integer $f(n)$ with the following property: if M is an $n \times n$ matrix of nonnegative integers with every row and column sum equal to $f(n)$, then M contains n entries, all greater than 1, with no two of these n entries in the same row or column.

11839. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let R be the circumradius, r the inradius, and s the semiperimeter of a triangle. Prove that

$$16R^3 + 20R^2r + 15Rr^2 + 5r^3 \geq s^2(4R + r),$$

with equality if and only if the triangle is equilateral.

11840. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let $z_1, \ldots, z_n$ be complex numbers. Prove that

$$\left(\sum_{k=1}^{n} |z_k|\right)^2 - \left|\sum_{k=1}^{n} z_k\right|^2 \geq \left(\sum_{k=1}^{n} |\mathrm{Re} z_k| - \left|\sum_{k=1}^{n} \mathrm{Re} z_k\right|\right)^2.$$

(Here Re z denotes the real part of z.)

11841. *Proposed by Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let $ABCD$ be a convex quadrilateral. Let E be the midpoint of AC, and let F be the midpoint of BD. Show that

$$|AB| + |BC| + |CD| + |DA| \geq |AC| + |BD| + 2|EF|.$$

(Here $|XY|$ denotes the distance from X to Y.)

http://dx.doi.org/10.4169/amer.math.monthly.122.5.500

11842. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let ψ be the *Digamma function*, that is, $\psi(x) = (\log \Gamma(x))'$. Let $\phi = (1+\sqrt{5})/2$. Prove that

$$\sum_{n=1}^{\infty} \frac{\psi(n+\phi) - \psi(n-1/\phi)}{n^2+n-1} = \frac{\pi^2}{2\sqrt{5}} + \frac{\pi^2 \tan^2(\sqrt{5}\pi/2)}{\sqrt{5}} + \frac{4}{5}\pi \tan(\sqrt{5}\pi/2).$$

11843. *Proposed by Mihali Bencze, Bucharest, Romania.* Let n and k be positive integers, and let $x_j \geq 1$ for $1 \leq j \leq n$. Let $y = \prod_{j=1}^{n} x_j$. Show that

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \geq \sum_{j=1}^{n} \frac{1}{1+(x_j^{k-1} y)^{1/(n+k-1)}}.$$

11844. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For nonnegative integers m and n, prove

$$\sum_{k=0}^{n} (m-2k) \binom{m}{k}^3 = (m-n)\binom{m}{n} \sum_{j=0}^{m-1} \binom{j}{n} \binom{j}{m-n-1}.$$

(Here $\binom{u}{v}$ is zero if $u < v$, and a sum is zero if its range of summation is empty.)

SOLUTIONS

Expressions as a Sum of Primes

11705 [2013, 469]. *Proposed by John Loase, Concordia College, Westchester County, NY.* Let $C(n)$ be the number of distinct multisets of two or more primes that sum to n. Prove that $C(n+1) \geq C(n)$ for all n. (For instance, $C(4) = 1$, $C(5) = 1$, and $C(6) = 2$.)

Solution by Victor Pambuccian, Arizona State University - West Campus, Glendale, AZ. For $n \geq 2$, let S_n denote the set of multisets of primes summing to n. Define $\varphi\colon\ S_n \to S_{n+1}$ by letting $\varphi(A)$ be the multiset obtained from A by (a) replacing one 2 with 3 if $2 \in A$, (b) replacing the smallest odd prime p in A with $(p+1)/2$ copies of 2 if $2, 3 \notin A$, or (c) replacing all k copies of 3 in A with $(3k+1)/p$ copies of the smallest prime p dividing $3k+1$ if $2 \notin A$ and $3 \in A$.

Consider a resulting multiset A'. If A' arises in case (a), then $3 \in A'$, but in case (b) or (c) $3 \notin A'$. If A' arises in case (b) or (c), then $2 \in A'$; let q be the sum of the copies of 2. In case (c), $q-1$ is divisible by 3, but in case (b) $q-1$ is a prime greater than 3. Hence A' cannot arise from both case (b) and case (c).

We conclude that φ is injective, so $C(n+1) > C(n)$.

Editorial comment. Several solvers showed that $C(n+1) > C(n)$ for $n \geq 11$. Traian Viteam points out that this problem is related to (but not the same as) the partitions into primes in an article by P. T. Bateman and P. Erdős, *Publ. Math. Debrecen* **4** (1956) 198–200.

Also solved by R. Boukharfane (Canada), P. P. Dályay (Hungary), C. Hurlburt, Y. J. Ionin, O. P. Lossers (Netherlands), R. Martin (Germany), R. E. Prather, J. P. Robertson, C. P. Rupert, J. M. Sanders, R. Tauraso (Italy), and T. Viteam (Chile).

Strongly Uncorrelated Sequences of Projections

11708 [2013, 469]. *Proposed by James W. Moeller, University of Illinois at Chicago, Chicago, IL.* Let $\langle E_n\rangle$ and $\langle P_n\rangle$ be two sequences of distinct orthogonal projections on an infinite-dimensional Hilbert space H whose ranges are finite-dimensional and satisfy the *intersection property*

$$\operatorname{Ran} E_n \cap (\operatorname{Ran} P_n)^\perp = \{0\} = \operatorname{Ran} P_n \cap (\operatorname{Ran} E_n)^\perp.$$

Such sequences are *strongly uncorrelated* if $\langle E_n\rangle$ converges strongly to 0 while $\langle P_n\rangle$ converges strongly to I. (A sequence $\langle L_n\rangle$ of bounded linear operators on a Hilbert space H *converges strongly* to L if $L_n x \to Lx$ for all $x \in H$.)

Show that the set of strongly uncorrelated sequences of projections is nonempty.

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. Let H be the Hilbert space (real or complex) spanned by the orthonormal basis $\{e_k\}_{k=1}^\infty$ and let H_n denote the n-dimensional subspace of H spanned by $\{e_k\}_{k=1}^n$. The orthogonal projection P_n of H onto H_n is given by

$$P_n x = \sum_{k=1}^n (x\cdot e_k)\, e_k,$$

where $x \cdot y$ is the inner product of $x, y \in H$. Since $x = \sum_{k=1}^\infty (x\cdot e_k)\, e_k$ with $|x|^2 = \sum_{k=1}^\infty |x\cdot e_k|^2$,

$$|x - P_n x|^2 = \left|\sum_{k=n+1}^\infty (x\cdot e_k)e_k\right|^2 = \sum_{k=n+1}^\infty |x\cdot e_k|^2 \to 0, \quad n \to \infty.$$

Consequently, P_n converges strongly to the identity operator I in H.

For each positive integer n, let

$$a_n = \sin\frac{\pi}{2n}, \quad b_n = \cos\frac{\pi}{2n}.$$

Note that $|a_n|^2 + |b_n|^2 = 1$, so $\{f_k\colon\ 1 \le k \le n\}$ with $f_k = a_n e_k + b_n e_{n+k}$ is an orthonormal set in H_{2n}. As such, it spans an n-dimensional subspace G_n of H, and the orthogonal projection E_n of H onto G_n is given by

$$E_n x = \sum_{k=1}^n (x\cdot f_k)\, f_k.$$

By orthonormality,

$$\begin{aligned}|E_n x|^2 &= \sum_{k=1}^n |x\cdot f_k|^2 = \sum_{k=1}^n |a_n\, x\cdot e_k + b_n\, x\cdot e_{n+k}|^2\\ &= |a_n|^2 \sum_{k=1}^n |x\cdot e_k|^2 + 2\operatorname{Re}\left(a_n b_n \sum_{k=1}^n (x\cdot e_k)(x\cdot e_{n+k})\right) + |b_n|^2 \sum_{k=1}^n |x\cdot e_{n+k}|^2\\ &\le 2|a_n|^2 \sum_{k=1}^n |x\cdot e_k|^2 + 2|b_n|^2 \sum_{k=1}^n |x\cdot e_{n+k}|^2,\end{aligned}$$

where the last step uses Schwartz's inequality. Thus

$$\mid E_n x \mid^2 \le 2a_n^2 |x|^2 + 2b_n^2 \mid x - P_n x \mid^2 .$$

Since $a_n \to 0$ and $|b_n| \le 1$ while $|x - P_n x| \to 0$, we have $E_n x \to 0$ strongly in H.

Finally, we have $H_n \cap G_n^\perp = G_n \cap H_n^\perp = \{0\}$. Indeed, $G_n^\perp$ consists of all elements $x \in H$ with $x \cdot f_k = 0$ for $1 \le k \le n$, while for $x \in H_n$ we have

$$x \cdot f_k = a_n\, x \cdot e_k + b_n\, x \cdot e_{n+k} = a_n\, x \cdot e_k.$$

Thus $x = P_n x$ for all $x \in H_n$ and $P_n x = 0$ for all $x \in H_n \cap G_n^\perp$, which together imply $x = 0$. On the other hand, $H_n^\perp$ consists of all elements $x \in H$ with $x \cdot e_j = 0$ for $1 \le j \le n$, while $x = E_n x$ for all $x \in G_n$. However, it follows that

$$(E_n x) \cdot e_j = \left(\sum_{k=1}^{n} (x \cdot f_k)(a_n e_k + b_n e_{n+k}) \right) \cdot e_j = a_n (x \cdot f_j), \quad 1 \le j \le n.$$

Thus, for all $x \in H_n^\perp$, $E_n x = 0$.

Also solved by J. Boersema, K. P. Hart (Netherlands), O. P. Lossers (Netherlands), J. Martínez (Spain), and the proposer.

Fubini and Riemann–Lebesgue

11709 [2013, 470]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Find

$$\int_{x=0}^{\infty} \frac{1}{x} \int_{y=0}^{x} \frac{\cos(x-y) - \cos(x)}{y}\, dy\, dx.$$

Solution by Kenneth F. Andersen, Edmonton, Alberta, Canada. The value of the integral is $\pi^2/6$. Let $f(x, y) = \frac{1}{y}(\cos(x - y) - \cos(x))$. For $x > 0$, we have

$$\int_0^x f(x, y) dy = \int_0^1 \frac{\cos(1-t)x - \cos x}{t} dt = x \int_0^1 \frac{1}{t} \int_{1-t}^1 \sin ux\, du\, dt$$

and thus, for $R > 0$,

$$\int_0^R \frac{1}{x} \int_0^x f(x, y) dy\, dx = \int_0^R \int_0^1 \frac{1}{t} \int_{1-t}^1 \sin ux\, du\, dt\, dx.$$

Since $|\sin ux| \le 1$, the triple integral is absolutely convergent. Therefore Fubini's theorem justifies an interchange in the order of integration to yield

$$\begin{aligned}
\int_0^R \frac{1}{x} \int_0^x f(x, y) dy\, dx &= \int_0^1 \int_{1-t}^1 \frac{1}{t} \int_0^R \sin ux\, dx\, du\, dt \\
&= \int_0^1 \frac{1 - \cos Ru}{u} \int_{1-u}^1 \frac{1}{t}\, dt\, du \\
&= -\int_0^1 \frac{\log(1-u)}{u} du + \int_0^1 \frac{\log(1-u)}{u} \cos Ru\, du.
\end{aligned}$$

Since $|\log(1-u)/u| \in L^1([0, 1])$, the Riemann–Lebesgue lemma shows that

$$\lim_{R \to \infty} \int_0^1 \frac{\log(1-u)}{u} \cos Ru\, du = 0.$$

Thus we have

$$\lim_{R\to\infty}\int_0^R \frac{1}{x}\int_0^x f(x,y)dy\,dx = -\int_0^1 \frac{\log(1-u)}{u}du$$
$$= \int_0^1 \sum_{n=1}^{\infty}\frac{t^{n-1}}{n}dt = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Editorial comment. O. Furdui also noted that this integral appeared as an open problem (Problem 12-002) proposed by Z. K. Silagadze in *SIAM Problems and Solutions* (http://www.siam.org/journals/categories/12-002.php).

Also solved by R. Bagby, R. Chapman (U. K.), O. Furdui (Romania), A. Heinis (Netherlands), F. Holland (Ireland), O. Kouba (Syria), K. D. Lathrop, and S. Wagon.

A Pesky Fractional Identity

11711 [2013, 470]. *Proposed by J. A. Grzesik, Allwave Corporation, Torrance, CA.* Show, for integers n and k with $n \geq 2$ and $1 \leq k \leq n$, that

$$(-1)^{n-k}\binom{n}{k}k\sum_{j\in[n]-\{k\}}\frac{1}{k-j} = -\sum_{j\in[n]-\{k\}}(-1)^{n-j}\binom{n}{j}\frac{j}{k-j}.$$

Solution by Traian Viteam, Punta Arenas, Chile. We prove a more general identity from which the claim follows.

For distinct numbers $a_1,\ldots,a_n$, consider the partial fraction expansion

$$\frac{1}{\prod_{j=1}^{n}(x-a_j)} = \sum_{j=1}^{n}\frac{A_j}{x-a_j}. \tag{1}$$

The "Heaviside method" computes the coefficients by multiplying (1) by $\prod_{i=1}^{n}(x-a_i)$ and then setting $x=a_j$. This yields

$$A_j = \prod_{i\in S_j}\frac{1}{a_j-a_i}, \tag{2}$$

where $S_j = [n]-\{j\}$.

We need a different expression for the particular coefficient A_k. Subtracting $A_k/(x-a_k)$ from both sides of (1) yields

$$\frac{1-A_k\prod_{j\in S_k}(x-a_j)}{\prod_{j=1}^{n}(x-a_j)} = \sum_{j\in S_k}\frac{A_j}{x-a_j}. \tag{3}$$

With $L(x)$ denoting the left side, let x tend to a_k in (3). From (1), the numerator of $L(x)$ tends to 0. Hence we can evaluate $\lim_{x\to a_k}L(x)$ by l'Hospital's rule. Letting $S_{j,k}=[n]-\{j,k\}$, we obtain

$$\lim_{x\to a_k}L(x) = \lim_{x\to a_k}\frac{-A_k\sum_{j\in S_k}\prod_{i\in S_{j,k}}(x-a_i)}{\sum_{j\in[n]}\prod_{i\in S_j}(x-a_i)} = \frac{-A_k\sum_{j\in S_k}\prod_{i\in S_{j,k}}(a_k-a_i)}{\sum_{j\in[n]}\prod_{i\in S_j}(a_k-a_i)}$$
$$= \frac{-A_k\sum_{j\in S_k}\prod_{i\in S_{j,k}}(a_k-a_i)}{\prod_{i\in S_k}(a_k-a_i)} = -A_k\sum_{j\in S_k}\frac{1}{a_k-a_j}.$$

Thus as $x \to a_k$, (3) becomes the general identity

$$-A_k \sum_{j \in S_k} \frac{1}{a_k - a_j} = \sum_{j \in S_k} \frac{A_j}{a_k - a_j}. \tag{4}$$

Specializing (2) to $a_j = j$ for $1 \le j \le n$ yields

$$A_j = \prod_{i \in S_j} \frac{1}{j-i} = \frac{1}{(n-1)!}(-1)^{n-j}\binom{n-1}{j-1} = \frac{1}{n!}(-1)^{n-j}\binom{n}{j} j. \tag{5}$$

Substituting (5) into the specialization of (4) on both sides yields the desired identity.

Editorial comment. Several solvers used the partial fraction approach with less direct expansion formulas. Others treated the problem as an identity of functions in two integer variables and used properties of binomial coefficients, harmonic numbers, and weighted sums to give essentially inductive proofs.

Also solved by U. Abel (Germany), D. Beckwith, R. Boukharfane (Canada), K. N. Boyadzhiev, P. Bracken, R. Chapman (U. K.), C. T. R. Conley, P. P. Dályay (Hungary), I. Gessel, O. Geupel (Germany), F. Henderson, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), C. R. Pranesachar (India), R. Tauraso (Italy), D. B. Tyler, J. van Hamme (Belgium), GCHQ Problem Solving Group (U. K.), and the proposer.

Bulgarian Solitaire from One Pile

11712 [2013, 569]. *Proposed by Daniel W. Cranston, Virginia Commonwealth University, Richmond, VA, and Douglas B. West, Zhejiang Normal University, Jinhua, China, and University of Illinois, Urbana, IL.* In the game of *Bulgarian solitaire*, n identical coins are distributed into piles, and a *move* takes one coin from each existing pile to form a new pile. Beginning with a single pile of size n, how many moves are needed to reach a position on a cycle (a position that will eventually repeat)? For example, $5 \to 41 \to 32 \to 221 \to 311 \to 32$, so that answer is 2 when $n = 5$.

Solution by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI. Fixing n, the answer is $n - t$, where $t = \min\{k \in \mathbb{N}\colon \frac{k(k+1)}{2} \ge n\} = \lceil \frac{\sqrt{8n+1}-1}{2} \rceil$. This is easily checked for $n \le 4$, so consider $n \ge 5$.

Positions in the game correspond to partitions of the integer n. Hence for a position π we let π_1 denote the largest pile size, π_2 the largest size among the others (if any), and $|\pi|$ the number of piles. Let π' denote the position obtained from π by one move. If no two piles in π have the same size, then $|\pi| \le \pi_1$. Say that π is *stretched* if no two piles have the same size and $|\pi| \ge \pi_2$.

If π has no two piles of equal size, then at most one pile has size 1, and hence $|\pi| \le |\pi'| \le |\pi| + 1$. Therefore, if π is stretched and $\pi_1 \ge |\pi| + 2$, then π' is also stretched and $\pi'_1 = \pi_1 - 1$. The first move yields the stretched position $\pi = (n-1, 1)$ with $\pi_1 \ge |\pi| + 2$, and after some such positions we reach for the first time a stretched position ρ with $|\rho| \le \rho_1 \le |\rho| + 1$. We show that $\rho_1 = t$, that ρ is in a cycle, and that no preceding position is in the cycle. Since each move during this process reduces the size of the largest pile by 1, the number of moves to reach ρ is $n - t$.

If $|\rho| = \rho_1$, then $n = 1 + 2 + \cdots + |\rho|$ and $\rho_1 = |\rho| = t$. Since $\rho' = \rho$, the position ρ forms a cycle of length 1. No preceding position belongs to this cycle.

If $|\rho| + 1 = \rho_1$, then ρ consists of piles with sizes 1 through ρ_1 except for one missing size s, where $1 \le s < \rho_1$. Thus $n = \rho_1(\rho_1 + 1)/2 - s$, and $\rho_1 = t$. Let $T_{i,j}$ be the position consisting of piles with distinct sizes 1 through $t - 1$ except for one missing pile of size i and one extra pile of size j (here $j = t$ is allowed). Let $\hat{T}_{i,j}$ consist of $T_{i,j}$ plus one pile of size t. Set $i = 0$ to mean that there is no missing size and $j = 0$ to mean that there is no repeated size. Note that $\rho = \hat{T}_{s,0} = T_{s,t}$ and $\rho' = T_{s-1,t-1}$. In general, $T'_{i,j} = T_{i-1,j-1}$ and $\hat{T}'_{i,j} = \hat{T}_{i-1,j-1}$ for $i, j \geq 1$, while $T'_{0,j} = T_{t-1,j-1}$. Thus the game continues from ρ as

$$\rho, T_{s-1,t-1}, \ldots, T_{0,t-s}, \hat{T}_{t-1,t-s-1}, \ldots, \hat{T}_{s,0}.$$

The last listed position is ρ, the same as the first. No position in this cycle has a pile bigger than t, while every position preceding ρ in the process has such a pile. Thus no position preceding ρ is in the cycle.

Editorial comment. Several solvers noted that, starting with a single pile of size n, the first repetition of a position occurs after exactly n moves. This follows from the argument above and the fact that the cycle has t distinct positions.

Previous articles about Bulgarian solitaire include J. Brandt, Cycles of partitions, *Proc. AMS* **85**, no. 3 (1982) 483–486; M. Gardner, Bulgarian solitaire and other seemingly endless tasks, *Sci. Amer.* **249** (1983) 12–21; E. Akin and M. Davis, Bulgarian solitaire, *Amer. Math. Monthly* **92** no. 4 (1985) 237–250; I. Kiyoshi, Solution of the Bulgarian solitaire conjecture, *Math. Mag.* **58**, no. 5 (1985) 259–271; and B. Hopkins and M. A. Jones, Shift-induced dynamical systems on partitions and compositions, *Electr. J. Comb.* **13** (2006) 1.

Also solved by R. Boukharfane (Canada), R. Chapman (U. K.), P. P. Dályay (Hungary), K. David, O. Geupel (Germany), D. Gove, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), J. Olson, H. K. Pillai (India), R. E. Prather, C. P. Rupert, M. Safaryan (Armenia) I. Sterling, R. Stong, R. Tauraso (Italy), Armstrong Problem Solvers, DIMACS REU 2013 Bridge Workshop, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposers.

A Limit with Integrals

11719 [2013, 600]. *Proposed by Nicolae Anghel, University of North Texas, Denton, TX.* Let f be a twice-differentiable function from $[0, \infty)$ into $(0, \infty)$ such that

$$\lim_{x\to\infty} \frac{f''(x)}{f(x)(1 + f'(x)^2)^2} = \infty.$$

Show that

$$\lim_{x\to\infty} \int_{t=0}^{x} \frac{\sqrt{1 + f'(t)^2}}{f(t)}\,dt \int_{t=x}^{\infty} \sqrt{1 + f'(t)^2}\, f(t)\,dt = 0.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We will prove:

(a) The function f is integrable, i.e., $\int_0^\infty f(x)\,dx < +\infty$.
(b) There is a real $a > 0$ such that $f'(x) < 0$ for $x \geq a$ and $\lim_{x\to\infty} f'(x) = 0$.
(c) The function f is strictly decreasing on $[a, +\infty)$ and $\lim_{x\to\infty} f(x) = 0$.
(d) $\lim_{x\to\infty} f'(x)/f(x) = -\infty$.

Indeed, there exists a real number $a > 0$ such that

$$\frac{f''(x)}{f(x)(1 + f'(x)^2)^2} \geq \frac{1}{2} \quad \text{for } x \geq a, \tag{1}$$

or equivalently

$$f(x) \le \frac{2f''(x)}{(1+f'(x)^2)^2} = \left(\arctan\left(f'(x)\right) + \frac{f'(x)}{1+f'(x)^2}\right)'.$$

Thus, for $x \ge a$ we have

$$\int_0^x f(t)\,dt \le \arctan\left(f'(x)\right) + \frac{f'(x)}{1+f'(x)^2} + c \le \frac{\pi}{2} + \frac{1}{2} + c,$$

where $c = \int_0^a f(t)\,dt - \arctan(f'(a)) - f'(a)/(1+f'(a)^2)$. This proves (a).

From (1) we know that $f''(x) > 0$ for $x \ge a$, so f' is strictly increasing on $[a, +\infty)$. Suppose there exists some $x_0 \in [a, +\infty)$ such that $f'(x_0) > 0$ for $x > x_0$. This would imply that f is increasing on $[x_0, +\infty)$ and would contradict (a). Thus, $f'(x) < 0$ for $x \in [a, +\infty)$, and consequently $\lim_{x\to\infty} f'(x) = k \le 0$. If $k < 0$, then there would exist x_1 such that $f(x) < k/2$ for $x \ge x_1$. This would imply that $0 < f(x) \le (k/2)x + b$ for $x \ge x_1$, which leads to a contradiction when $x \to \infty$. This shows that $k = 0$ and completes the proof of (b).

From (b) we conclude that f is decreasing on $[a, +\infty)$, so $\lim_{x\to\infty} f(x)$ exists. By (a) this limit must be zero. This proves (c).

Now $\lim_{x\to\infty}(1+f'(x)^2)^2 = 1$, so we have $\lim_{x\to\infty} f''(x)/f(x) = +\infty$. Using l'Hospital's rule, we conclude that $\lim_{x\to\infty} f'(x)^2/f(x)^2 = +\infty$. However, $f'(x)/f(x) < 0$ for $x \ge a$, so $\lim_{x\to\infty} f'(x)/f(x) = -\infty$, which is (d).

Now define g on $[a, +\infty)$ by the formula

$$g(x) = -\sup_{t\ge x} \frac{f'(t)}{f(t)},$$

so that $-f'(t) \ge g(x)f(t)$ for $t \ge x$. According to (d), $\lim_{x\to\infty} g(x) = +\infty$.

For $x \ge a$ we have

$$\begin{aligned}\int_x^\infty \sqrt{1+f'(t)^2}\, f(t)\,dt &\le \sqrt{1+f'(a)^2}\int_x^\infty f(t)\,dt \\ &\le \sqrt{1+f'(a)^2}\int_x^\infty \frac{-f'(t)}{g(x)}\,dt \le \sqrt{1+f'(a)^2}\cdot\frac{f(x)}{g(x)}.\end{aligned}$$

Also, for $x \ge a$ we have

$$\begin{aligned}\int_a^x \frac{\sqrt{1+f'(t)^2}}{f(t)}\,dt &\le \sqrt{1+f'(a)^2}\int_a^x \frac{1}{f(t)}\,dt \le \sqrt{1+f'(a)^2}\int_a^x \frac{-f'(t)}{f(t)^2 g(x)}\,dt \\ &\le \sqrt{1+f'(a)^2}\cdot\frac{1}{g(x)}\left(\frac{1}{f(x)} - \frac{1}{f(a)}\right) \le \frac{\sqrt{1+f'(a)^2}}{g(x)f(x)}.\end{aligned}$$

Combining these, we obtain for $x \ge a$ that

$$\begin{aligned}&\int_0^x \frac{\sqrt{1+f'(t)^2}}{f(t)}\,dt \int_x^\infty \sqrt{1+f'(t)^2}\,dt \le \\ &\sqrt{1+f'(a)^2}\left(\int_0^a \frac{\sqrt{1+f'(t)^2}}{f(t)}\,dt\right)\frac{f(a)}{g(x)} + \frac{1+f'(a)^2}{g(x)^2}.\end{aligned}$$

Finally, since $\lim_{x\to\infty} g(x) = +\infty$, we obtain the desired conclusion by letting x tend to infinity.

Also solved by R. Bagby, P. Bracken, M. Omarjee (France), R. Stong, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal nor posted to the Internet before the due date for solutions. Submitted solutions should arrive before October 31, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11845. *Proposed by Gregory Galperin, Eastern Illinois University, and Yury Ionin, Central Michigan University.*
(**a**) Let P be a convex polyhedron inside a sphere S, and let $e_1, \dots, e_n$ be the edges of P. Let c_i be the chord of S containing edge e_i. Note that $c_i \backslash e_i$ is the union of two disjoint segments; we term these a_i and b_i.

Prove that if all the edges of P have the same length, then the $2n$-element set consisting of the a_i and the b_i can be partitioned into two subsets such that the sum of the lengths of the elements in each part is the same.
(**b**) Let $A_0, A_1, \dots, A_{n-1}$ be a regular n-gon inscribed in a circle γ. Let γ' be a circle containing γ, and let the tangent line to γ at A_i meet γ' at points X_i and Y_i. Prove that the $2n$-element set consisting of the segments A_iX_i and A_iY_i can be partitioned into two subsets such that the sum of the lengths of the elements in each part is the same.

11846. *Proposed by Kent Holing, Trondheim, Norway.* Let $f = \sum_{j=0}^{n} a_j x^j$ be an irreducible monic polynomial of odd degree with integer coefficients. Writing the terms $a_j x^j$ of f in order of increasing j, assume that either there is at least one term of f missing between two (nonmissing) terms of the same sign or there is more than one term missing between two (nonmissing) terms of opposite sign. (A term $a_j x^j$ is *missing* if $a_j = 0$. The sign of the term $a_j x^j$ is the sign of a_j.) Prove that the Galois group G of f is not abelian. Also, prove that G is not a dihedral group if, in addition, $f(x) = 0$ has at least two real roots.

11847. *Proposed by Mihaly Bencze, Brasov, Romania.* Prove that for $n \ge 1$,

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\log^2(1+1/k)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}.$$

http://dx.doi.org/10.4169/amer.math.monthly.122.6.604

11848. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Prove that

$$\frac{1}{2\pi}\mathrm{Li}_2\left(e^{-2\pi}\right) = \log(2\pi) - 1 - \frac{5\pi}{12} - \sum_{m=1}^{\infty}\frac{(-1)^m\zeta(2m)}{m(2m+1)}.$$

Here, ζ is the Riemann zeta function, and $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$.

11849. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Define numbers $a_0, a_1, \ldots$ by

$$\exp\left(\sum_{k=0}^{\infty} x^{2^k}\right) = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that $\liminf_{n\to\infty}\frac{\log a_n}{\log n} \le \frac{1}{\log 2} - 1 \le \limsup_{n\to\infty}\frac{\log a_n}{\log n}$.

11850. *Proposed by Zafar Ahmed, Bhabha Atomic Research Centre, Mumbai, India.* Let A_n be the polynomial given by

$$A_n(x) = \sqrt{\frac{2}{\pi}}\frac{1}{n!}(1+x^2)^{n/2}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Prove that $\int_{-\infty}^{\infty} A_m(x)A_n(x)\,dx = \delta(m,n)$ for nonnegative integers m and n. Here, $\delta(m,n) = 1$ if $m = n$, and otherwise $\delta(m,n) = 0$.

11851. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* For real a and b and integer $n \ge 1$, let $\gamma_n(a,b) = -\log(n+a) + \sum_{k=1}^{n} 1/(k+b)$.
(**a**) Prove that $\gamma(a,b) = \lim_{n\to\infty}\gamma_n(a,b)$ exists and is finite.
(**b**) Find

$$\lim_{n\to\infty}\left(\log\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma(a,b)\right)^n.$$

11829. *Proposed by Paul Bracken, University of Texas-Pan American, Edinburg, TX.* (Correction.) Let $\langle a\rangle$ be a monotone decreasing sequence of real numbers that converges to 0. Prove that $\sum_{n=1}^{\infty} a_n/n < \infty$ if and only if $a_n = O(1/\log n)$ and $\sum_{n=1}^{\infty}(a_n - a_{n+1})\log n < \infty$.

SOLUTIONS

Will a Mutation Take Over the Population?

11710 [2013, 470]. *Proposed by B. Voorhees, Athabasca University, Athabasca, Alberta, Canada.* Let n, k, and r be positive numbers such that $n \ge k+1$ and $r \ge 1$. Show that

$$r^{n+k} - 1 \ge \frac{(kr+n)(nr+k)}{(n-k)^2}\left(1 - \left(\frac{kr+n}{nr+k}\right)^{n-k}\right).$$

Solution by Josep Martínez, Universitat de València, Valencia, Spain. Letting $r \to \infty$ shows that the inequality is false when $n + k < 2$. We will prove the inequality for $n + k \geq 2$. We first claim

$$\frac{r^{n+k} - 1}{r - 1} \geq kr + n, \qquad r > 1, n \geq k + 1, n + k \geq 2. \tag{1}$$

This follows from the fact that $r^{n+k} - 1 - (kr + n)(r - 1)$ is a strictly increasing function of r for $r > 1$ and vanishes at $r = 1$. Next we claim

$$(n - k)\left(1 - \frac{kr + n}{nr + k}\right) \geq 1 - \left(\frac{kr + n}{nr + k}\right)^{n-k}. \tag{2}$$

This follows from $1 - s^a \leq a(1 - s)$ for $0 < s < 1$ and $a > 1$, which in turn follows from the mean value theorem applied to $\psi(x) = x^a$ on the interval $[s, 1]$.

Now apply (1) and (2) to get

$$\begin{aligned}\frac{(kr + n)(nr + k)}{(n - k)^2}\left(1 - \left(\frac{kr + n}{nr + k}\right)^{n-k}\right) &\leq \frac{(kr + n)(nr + k)}{(n - k)}\left(1 - \frac{kr + n}{nr + k}\right) \\ &= (kr + n)(r - 1) \leq r^{n+k} - 1.\end{aligned}$$

Editorial comment. Solvers provided alternate hypotheses under which to prove the inequality: $k > 1/2$ or $n \geq 2$ or n, k positive integers.

The proposer provided a context for the problem: In evolutionary graph theory, certain simple birth-and-death processes within a fixed population of size N can be represented by a weighted graph G with N vertices, each with weight 1 or r, where $r > 1$. Competition occurs randomly along edges, and when an r and a 1 compete, the r wins with probability $r/(r + 1)$. The winner resets the other vertex to its own weight. When the graph is the complete graph on N vertices, the probability that a lone vertex with weight r will sweep the graph is known to be $r^{N-1}/\sum_0^{N-1} r^k$. For a complete bipartite graph, there is another formula, and the claim of the problem can be seen as saying that mutations stand a better chance of sweeping in such graphs.

Also solved by P. P. Dályay (Hungary), D. Fleischman, D. Fritze (Germany), O. Geupel (Germany), O. P. Lossers (Netherlands), P. Perfetti (Italy), and GCHQ Problem Solving Group (U. K.).

A Product Inequality

11713 [2013, 569]. *Proposed by Mihaly Bencze, Brasov, Romania.* Let $x_1, \ldots, x_n$ be nonnegative real numbers. Let $S = \sum_{k=1}^{n} x_k$. Prove that

$$\prod_{k=1}^{n}(1 + x_k) \leq 1 + \sum_{k=1}^{n}\left(1 - \frac{k}{2n}\right)^{k-1}\frac{S^k}{k!}.$$

Solution by Robert A Agnew, Buffalo Grove, IL. Equality holds if $n = 1$. Assume now $n \geq 2$. From the AM–GM inequality, we have

$$\begin{aligned}\prod_{k=1}^{n}(1 + x_k) &\leq \left(\frac{1}{n}\sum_{k=1}^{n}(1 + x_k)\right)^n = \left(1 + \frac{S}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{S}{n}\right)^k \\ &= 1 + \sum_{k=1}^{n}\left(\frac{n!}{(n - k)!n^k}\right)\left(\frac{S^k}{k!}\right).\end{aligned}$$

For $k = 1$,

$$\frac{n!}{(n-1)!n^1} = 1 = \left(1 - \frac{1}{2n}\right)^0.$$

For $k \geq 2$,

$$\frac{n!}{(n-k)!} = n\prod_{i=1}^{k-1}(n-i),$$

and from the AM–GM inequality

$$\left(\prod_{k=1}^{k-1}(n-i)\right)^{1/(k-1)} \leq \frac{1}{k-1}\sum_{k=1}^{k-1}(n-i) = \frac{n(k-1)-k(k-1)/2}{k-1} = n - \frac{k}{2},$$

so that

$$\frac{n!}{(n-k)!n^k} \leq \frac{n}{n^k}\left(n - \frac{k}{2}\right)^{k-1} = \left(1 - \frac{k}{2n}\right)^{k-1}.$$

The assertion follows.

Also solved by D. Anderson (Ireland), M. Bataille (France), D. Beckwith, R. Boukharfane (Canada), M. A. Carlton, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, G. Funchess, O. Geupel (Germany), E. A. Herman, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Martínez (Spain), H. F. Mattson Jr., M. Omarjee (France), P. Perfetti (Italy), M. Safaryan (Armenia), B. Schmuland (Canada), A. Stenger, R. Stong, R. Tauraso (Italy), Z. Vörös (Hungary), J. Zacharias, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Inradii and Diagonals

11714 [2013, 569]. *Proposed by Nicuşor Minculete, "Dimitrie Cantenemir" University, Braşov, Romania, and Cătălin Barbu, "Vasile Alecsandri" National College, Bacău, Romania.* Let $ABCD$ be a cyclic quadrilateral (the four vertices lie on a circle). Let $e = |AC|$ and $f = |BD|$. Let r_a be the inradius of BCD, and define r_b, r_c, and r_d similarly. Prove that $er_ar_c = fr_br_d$.

Solution by Donald Jay Moore, Wichita, KS. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, and $d = |DA|$. Triangles BCD, CDA, DAB, and ABC all have the same circumradius R, hence

$$R = \frac{bcf}{4r_as_a} = \frac{cde}{4r_bs_b} = \frac{adf}{4r_cs_c} = \frac{abe}{4r_ds_d},$$

where $s_a = (b+c+f)/2$, $s_b = (c+d+e)/2$, $s_c = (a+d+f)/2$, and $s_d = (a+b+e)/2$. Thus, we have

$$R^2 = \frac{abcde^2}{16r_br_ds_bs_d} = \frac{abcdf^2}{16r_ar_cs_as_c}$$

or $e^2r_ar_cs_as_c = f^2r_br_ds_bs_d$. Thus, it suffices to show $es_as_c = fs_bs_d$. Expanding this equation, we get

$$abe + ace + bde + cde - acf - bcf - adf - bdf + (f-e)ef = 0.$$

Substituting for ef using Ptolemy's theorem ($ef = ac + bd$) and cancelling, the required equation becomes $abe + cde = bcf + adf$. Since abe, cde, bcf, and adf

are $4R$ times the areas of triangles ABC, BCD, CDA, and DAB, respectively, both $abe + cde$ and $bcf + adf$ are equal to $4R$ times the area of quadrilateral $ABCD$.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), R. Boukharfane (Canada), R. B. Campos (Spain), P. P. Dályay (Hungary), P. De (India), C. Delorme (France), O. Geupel (Germany), J. G. Heuver (Canada), S. Ibragimov (Uzbekistan), S. Jo (Korea), B. Karaivanov, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), J. Minkus, C. R. Pranesachar (India), J. Schlosberg, J. Song (Korea), R. Stong, T. Viteam (Chile), Z. Vörös (Hungary), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposers.

An Infinite Sum Introduces a Zeta

11715 [2013, 569]. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.* Prove that

$$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^5} = \frac{1}{2}\left(\frac{2^5-1}{2^5} \cdot \frac{3^5-1}{3^5}\zeta(5) + \frac{11}{8}\left(\frac{\pi}{3}\right)^5 \cdot \frac{1}{\sqrt{3}}\right).$$

Solution by Michael Hoffman, U.S. Naval Academy, Annapolis, MD. It is an elementary exercise in the use of Euler products to show that

$$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^{n+1}} + \sum_{k=0}^{\infty} \frac{1}{(6k+5)^{n+1}} = \frac{(3^{n+1}-1)(2^{n+1}-1)}{6^{n+1}}\zeta(n+1).$$

The result will follow from a (somewhat less elementary) identity for the difference of the sums on the left. For positive even n, define P_n by

$$\frac{d^n}{dx^n}\tan x = P_n(\tan x).$$

We shall show that

$$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^{n+1}} - \sum_{k=0}^{\infty} \frac{1}{(6k+5)^{n+1}} = \frac{\pi^{n+1}}{6^{n+1}n!}\, P_n(\sqrt{3}).$$

Adding this difference to the above sum yields a general result on reciprocal $(n+1)$th powers. Since $P_4(\sqrt{3}) = 352\sqrt{3}$, setting $n = 4$ solves the given problem.

We use Theorem 4.4 of Michael E. Hoffman, derivative polynomials for tangents and secants, this MONTHLY **102** (1995), 23–30. Let ψ be the function of period 6 with $\psi(1) = 1$, $\psi(-1) = -1$ and $\psi(0) = \psi(2) = \psi(3) = \psi(4) = 0$.

For even n, this gives

$$\sum_{j=1}^{\infty} \frac{\psi(j)}{j^{n+1}} = \frac{\pi^{n+1}}{2 \cdot 6^{n+1}n!}\sum_{p=1}^{5}\psi(p)P_n\left(\cot\frac{p\pi}{6}\right)$$

which for positive even n is the desired formula (note that P_n is an odd function when n is even).

Editorial comment. Richard Stong showed that the identity is an easy consequence of formula (23.1.18) of Abramowitz and Stegun. Stan Wagon typed the sum on the left into Mathematica, applied the command Simplify to the right side, and obtained the same expression. M. Ram Murty and Akshaa Vatwani observed that the crux of the matter is that the Dirichlet L-function $L(s, \chi)$ can be computed explicitly when s and

χ have the same parity ($\chi(-1) = 1$ and $\chi(-1) = -1$ correspond to even and odd, respectively). In this case $s = 5$, and χ and ψ are equal and odd. Related considerations allowed them to express the Hurwitz zeta function $\zeta(2k+1, a/q)$ in terms of π and $\psi(2k+1)$ for $q \in \{3, 4, 6\}$.

Also solved by K. F. Andersen (Canada), R. Bagby, M. Bataille (France), R. Boukharfane (Canada), P. Bracken, B. S. Burdick, M. A. Carlton, R. Chapman (U. K.), H. Chen, M. W. Coffey, M. L. Glasser, G. C. Greubel, J.-P. Grivaux (France), E. A Herman, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands) G. Malisani (Italy), J. Martínez (Spain), M. R. Murty & A. Vatwani (Canada), M. Omarjee (France), H. Riesel (Sweden), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), S. Wagon, J. Zacharias, J. Zucker (U. K.), GCHQ Problem Solving Group (U. K.), NSA Problems Group, TCDmath Problem Group (Ireland), and the proposer.

Catalan and Fibonacci Continued

11716 [2013, 570]. *Proposed by Oliver Knill, Harvard University, Cambridge, MA.* Let $\alpha = (\sqrt{5} - 1)/2$. Let p_n and q_n be the numerator and denominator of the nth continued fraction convergent to α. (Thus, p_n is the nth Fibonacci number and $q_n = p_{n+1}$.) Show that

$$\sqrt{5}\left(\alpha - \frac{p_n}{q_n}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{{q_n}^{2k+2} 5^k},$$

where C_k denotes the kth *Catalan number*, given by $C_k = \frac{(2k)!}{k!(k+1)!}$.

Solution by Borislav Karaivanov, University of South Carolina, Columbia, SC. We prove the desired identity assuming $p_n = F_{n-1}$ and $q_n = p_{n+1} = F_n$, where the Fibonacci numbers are defined by $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$.

We use the usual form of the generating function for the Catalan numbers:

$$\sum_{k=0}^{\infty} C_k x^k = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

The power series has radius of convergence $1/4$, as is easily verified with the ratio test. Setting $x = (-1)^{n+1}/(5F_n^2)$ in the generating function yields

$$\sum_{k=0}^{\infty} \frac{(-1)^{(n+1)k} C_k}{5^k F_n^{2k}} = \frac{\sqrt{5} F_n}{(-1)^{n+1}} \cdot \frac{\sqrt{5} F_n - \sqrt{5{F_n}^2 - 4(-1)^{n+1}}}{2}.$$

Hence

$$\sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{5^k q_n^{2k+2}} = \frac{(-1)^{n+1}}{F_n^2} \sum_{k=0}^{\infty} \frac{(-1)^{(n+1)k} C_k}{5^k F_n^{2k}}$$
$$= \frac{\sqrt{5}}{F_n} \cdot \frac{\sqrt{5} F_n - \sqrt{5F_n^2 + 4(-1)^n}}{2}.$$

Using Cassini's identity $F_n^2 + (-1)^n = F_{n+1} F_{n-1}$, we rewrite the terms under the root via

$$5F_n^2 + 4(-1)^n = F_n^2 + 4(F_n^2 + (-1)^n) = (F_{n+1} - F_{n-1})^2 + 4F_{n+1}F_{n-1}$$
$$= (F_{n+1} + F_{n-1})^2 = (F_n + 2F_{n-1})^2,$$

and obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{5^k q_n^{2k+2}} = \frac{\sqrt{5}}{F_n} \cdot \frac{\sqrt{5} F_n - (F_n + 2F_{n-1})}{2} = \sqrt{5} \left(\frac{\sqrt{5} - 1}{2} - \frac{F_{n-1}}{F_n} \right).$$

Editorial comment. Many solvers proved a slightly different formula based on a different interpretation of "the nth Fibonacci number."

Also solved by D. Beckwith, R. Boukharfane (Canada), B. S. Burdick, R. Chapman (U. K.), M. W. Coffey, O. Geupel (Germany), O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), A. Meyer, M. Omarjee (France), M. Somos, R. Stong, R. Tauraso (Italy), C. Vignat (France) & V. H. Moll, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

A Special (Degenerate) Case of the Problem of Apollonius

11717 [2013, 570]. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Given a circle c and line segment AB tangent to c at a point E that lies strictly between A and B, provide a compass and straightedge construction of the circle through A and B to which c is internally tangent.

Solution by Michel Bataille, Rouen, France. Consider the inversion $\mathbf{I}$ in the circle with center A and radius $|AE|$; note that $\mathbf{I}(c) = c$. If γ is any circle passing through A and B and tangent to c, then $\mathbf{I}(\gamma) = t$ is a line tangent to c passing through $\mathbf{I}(B)$. But $\mathbf{I}(B)$ is on line AB, so there is only one such line and $\gamma = \mathbf{I}(t)$. The constructions of $\mathbf{I}(B)$ and t are classical. Let t touch c at T, so AT intersects c again at T', and γ is the circumcircle of $\triangle ABT'$.

Editorial comment. Solver J. Schaer (University of Calgary, Canada) notes that this is a special case of the **Problem of Apollonius:** *To contruct all circles tangent to three given circles.* Here, two of the given circles, points A and B, have zero radius.

Also solved by R. Bagby, R. Bouharfane (Canada), J. Cade, P. P. Dályay (Hungary), C. Delorme (France), K. Farwell, O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), A. Habil (Syria), B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Nicholson, C. R. Pranesachar (India), R. A. Russell, J. Schaer (Canada), R. Stong, V. Tran, E. I. Verriest, T. Viteam (Chile), A. L. Yandl & C. Swenson, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

An Integral Sum

11721 [2013, 660]. *Proposed by Roberto Tauraso, Universitá di Roma "Tor Vergata", Rome, Italy.* Let p be a prime greater than 3, and let q be a complex number other than 1 such that $q^p = 1$. Evaluate

$$\sum_{k=1}^{p-1} \frac{(1-q^k)^5}{(1-q^{2k})^3(1-q^{3k})^2}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The value is $-(11p-1)(5p-1)/72$ when $p \equiv 5 \pmod 6$ and $-(55p-1)(p-1)/72$ when $p \equiv 1 \pmod 6$.

Let $\omega = q^k$. As k varies from 1 to $p-1$, the value of ω varies over all primitive pth roots of unity, and $\omega^p = 1$. For $p = 6m-1$, rewrite the summand (S_ω, say) as

$$
\begin{aligned}
S_\omega &= \frac{(1-\omega)^5}{(1-\omega^2)^3(1-\omega^3)^2} = \frac{(1-\omega)^2(1-\omega^{6m})^3}{(1-\omega^2)^3(1-\omega^3)^2} \\
&= \frac{(1-\omega)^2(1-\omega^6)^3}{(1-\omega^2)^3(1-\omega^3)^2}\,\frac{(1-\omega^{6m})^3}{(1-\omega^6)^3} = \frac{(1-\omega)^2(1-\omega^3)^3(1+\omega^3)^3}{(1-\omega^2)^3(1-\omega^3)^2}\,\frac{(1-\omega^{6m})^3}{(1-\omega^6)^3} \\
&= \frac{(1-\omega)^3(1+\omega+\omega^2)(1+\omega)^3(1-\omega+\omega^2)^3}{(1-\omega^2)^3}\,\frac{(1-\omega^{6m})^3}{(1-\omega^6)^3} \\
&= (1+\omega+\omega^2)(1-\omega+\omega^2)^3\left(\frac{1-\omega^{6m}}{1-\omega^6}\right)^3 \\
&= 1-2\omega+4\omega^2-4\omega^3+5\omega^4-4\omega^5+4\omega^6-2\omega^7+\omega^8)\times\left(\sum_{j=0}^{m-1}\omega^{6j}\right)^3.
\end{aligned}
$$

Let $P_m(\omega)$ denote the final expression. Note that P_m is a polynomial of degree $18m-10$ with $P_m(1)=3m^3$. Thus, the desired sum is $-3m^3+\sum_{\omega^p=1}P_m(\omega)$. The only monomials in P_m whose contribution does not cancel in this sum are those whose exponent is a multiple of p; hence, they are ω^0, ω^{6m-1}, and ω^{12m-2}.

The coefficient of ω^0 is 1. The coefficient of ω^{6m-1} is -4 times the coefficient of ω^{6m-6} in $(1+\omega^6+\omega^{12}+\cdots+\omega^{6m-6})^3$. This coefficient is the same as the coefficient of x^{m-1} in the Taylor series of $(1+x+x^2+\cdots)^3=\frac{1}{(1-x)^3}$, namely $\binom{m+1}{2}$. The coefficient of ω^{12m-2} is 5 times the coefficient of ω^{12m-6} in$(1+\omega^6+\omega^{12}+\cdots+\omega^{6m-6})^3$. By symmetry, this is the same as the coefficient of ω^{6m-12}, which by the above is $\binom{m}{2}$.

Hence, for the sum we compute

$$
\begin{aligned}
-3m^3+(6m-1)\left(1-4\binom{m+1}{2}+5\binom{m}{2}\right) &= -\frac{(11m-2)(5m-1)}{2} \\
&= -\frac{(11p-1)(5p-1)}{72}.
\end{aligned}
$$

The calculation is similar for $p=6m+1$. Using $\omega^p=1$ we obtain

$$
\frac{(1-\omega)^5}{(1-\omega^2)^3(1-\omega^3)^2} = \frac{(1-\omega)^2(\omega^{6m+1}-\omega)^3}{(1-\omega^2)^3(1-\omega^3)^2} = -\omega^3P_m(\omega),
$$

and the desired sum is $3m^3-\sum_{\omega^p=1}\omega^3P_m(\omega)$. In this case, the coefficient of ω^0 is 0, and the coefficients of ω^{6m+1} and ω^{12m+2} in $\omega^3P_m(\omega)$ are $5\binom{m+1}{2}$ and $-4\binom{m}{2}$, respectively. Hence, for the sum we compute

$$
\begin{aligned}
3m^3-(6m+1)\left(5\binom{m+1}{2}-4\binom{m}{2}\right) &= -\frac{m(55m+9)}{2} \\
&= -\frac{(55p-1)(p-1)}{72}.
\end{aligned}
$$

Editorial comment. Most solvers used either partial fraction decomposition of the summand and Laurent series of the terms or complex integrals and the residue theorem applied to the summand multiplied by either $p/(z(z^p-1))$ or $pz^{p-1}/(z^p-1)$, where $z=q^k$. Several solvers noted that the result is always an integer and that it holds for every p that is coprime to 6 as long as q is a primitive pth root of unity.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), I. M. Isaacs, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), N. C. Singer, A. Stadler (Switzerland), A. Stenger, J. Van Hamme (Belgium), GCHQ Problem Solving Group (U. K.), and the proposer.

A Concurrence on a Circumcircle

11722 [2013, 661]. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Let ABC be an acute triangle in the plane, and let M be a point inside ABC. Let O_1, O_2, and O_3 be the circumcenters of BCM, CAM, and ABM, respectively. Let c be the circumcircle of ABC. Let D, E, and F be the points opposite A, B, and C, respectively, at which AM, BM, and CM meet c. Prove that O_1D, O_2E, and O_3F are concurrent at a point P that lies on c.

Solution by Michel Bataille, Rouen, France. Use the complex plane as coordinates so that the origin lies at the circumcenter of ABC and the circumradius is 1. Thus, c is the unit circle. We will write the lower-case letter for the complex number corresponding to a point designated by the corresponding upper-case letter. To avoid confusion, we will write Δ for the unit circle.

Let r be a complex number on the unit circle, and let s be any other complex number. The second intersection of Δ and the line through R and S has coordinate $(r-s)/(r\overline{s}-1)$. From this result, we have in particular that $d = (a-m)/(a\overline{m}-1)$. On the other hand, since O_1 is on the perpendicular bisectors of BC, CM, and MB, we have $|o_1-m|^2 = |o_1-b|^2 = |o_1-c|^2$ so that $o_1(\overline{b}-\overline{c}) + \overline{o_1}(b-c) = 0$ and $o_1(\overline{b}-\overline{m}) + \overline{o_1}(b-m) = 1-m\overline{m}$. Now, $\overline{b} = 1/b$ and $\overline{c} = 1/c$, so we deduce that

$$o_1 = \frac{(b-c)(m\overline{m}-1)}{(\overline{m}-\overline{b})(b-c)-(m-b)(\overline{b}-\overline{c})} = \frac{bc(m\overline{m}-1)}{m-b-c+bc\overline{m}},$$

$$\overline{o_1} = \frac{m\overline{m}-1}{m-b-c+bc\overline{m}}.$$

Now, let $\beta = (a\overline{m}-1)(m-b-c+bc\overline{m})$. We compute

$$d - o_1 = \frac{m(a+b+c)-(ab+bc+ca)+abc\overline{m}(2-m\overline{m})-m^2}{\beta},$$

$$d\overline{o_1} - 1 = \frac{(ab+bc+ca)\overline{m}-(a+b+c)+m(2-m\overline{m})-abc\overline{m}^2}{\beta}.$$

From this, we see that $(d-o_1)/(d\overline{o_1}-1)$ is symmetric in a, b, c. This implies that the second point of intersection of O_1D with Δ is also the second point of intersection of O_2E and O_3F with Δ. Thus, lines DO_1, EO_2, and FO_3 are concurrent at a point that lies on Δ.

Editorial comment. Several readers pointed out that there is no need for the triangle to be acute or that M lie inside ABC. Of course, the construction will make sense only if M is neither on the circumcircle nor on the lines containing the sides of the triangle. This same problem (by the same proposer) appeared in *Crux Mathematicorum* vol. 37, no. 8 as Problem 3692.

Also solved by J.-P. Grivaux (France), O. Kouba (Syria), D. Lee (Korea), C. R. Pranesachar (India) R. Stong, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before December 31, 2015. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11852. *Proposed by Sam Northshield, SUNY Plattsburgh, Plattsburgh, NY.* For $n \in \mathbb{Z}^+$, let $\nu_n = k$ if 3^k divides n but 3^{k+1} does not. Let $X_1 = 2$, and for $n \geq 2$ let

$$X_n = 4\nu_n + 2 - \frac{2}{X_{n-1}},$$

so that $\langle X_n \rangle$ begins with $2, 1, 4, \frac{3}{2}, \frac{2}{3}, 3, \ldots$. Show that every positive rational number appears exactly once in the list $(X_1, X_2, \ldots)$.

11853. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Find

$$\sum_{n=1}^{\infty} \frac{1}{\sinh 2^n}.$$

11854. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* In the Euclidean plane, given distinct points $P_1, \ldots, P_n$ and distinct lines $l_1, \ldots, l_m$, prove that there is a half-line h such that for any point Q on h, any $k \in \{1, \ldots, m\}$, and any $j \in \{1, \ldots, n\}$, Q is nearer than P_j to l_k.

11855. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* For a continuous and nonnegative function f on $[0, 1]$, let $\mu_n = \int_0^1 x^n f(x)\,dx$. Show that $\mu_{n+1}\mu_0 \geq \mu_n\mu_1$ for $n \in \mathbb{N}$.

11856. *Proposed by Keith Kearnes, University of Colorado, Boulder, CO.* Let G be a finite group. Show that the number of Sylow subgroups of G is at most $\frac{2}{3}|G|$.

11857. *Proposed by Mehmet Şahin, Ankara University, Ankara, Turkey.* Let ABC be a triangle with corresponding sides of lengths a, b, and c, inradius r, and corresponding

http://dx.doi.org/10.4169/amer.math.monthly.122.7.700

exradii r_a, r_b, and r_c. Let $A'B'C'$ be another triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. Show that $A'B'C'$ has area given by

$$\frac{1}{2}\sqrt{r(r_a + r_b + r_c)}.$$

SOLUTIONS

When One Triangle Circumscribes Another

11706 [2013, 469]. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Let ABC and DEF be triangles in a plane.
(**a**) Provide a compass and straightedge construction, which may use ABC and DEF, of a triangle $A'B'C'$ that is similar to ABC and circumscribes DEF.
(**b**) Among all triangles $A'B'C'$ of the sort described in part (a), determine which one has the greatest area and which one has the greatest perimeter.

Editorial comment. Partial solution by the editors (using material submitted by the solvers). No complete solutions were received to either (**a**) or (**b**) that dealt with all possible shapes that the triangles ABC and DEF might have. Part (**b**) turns out to be quite complex. The largest circumscribing triangle cannot be identified until all possible circumscribing triangles can be surveyed, including different ways of matching up angles of ABC with sides/angles of DEF, plus possibly several anomalous cases. Fortunately, since all triangles under consideration are similar, area and perimeter will both increase and decrease with the length of any side, so we need concern ourselves only with maximizing the length of any one side.

There are several possible definitions of what it means to say "triangle PQR circumscribes triangle STU." We take the following definition: *PQR circumscribes STU when each closed side of PQR contains at least one of the vertices of STU.* There are stricter and looser definitions, but this one guarantees that the desired maximum circumscribing triangle actually exists.

When in fact PQR circumscribes STU and each open side of PQR contains at least one of the vertices of STU, we will say that *PQR strictly circumscribes STU*. Otherwise we will say that *PQR marginally circumscribes STU*.

We will show that when the angles of ABC are paired up in any of the six possible ways with the sides/angles of DEF, there exists $A'B'C'$ that strictly circumscribes DEF so that each angle of $A'B'C'$ faces the side of DEF (equivalently, is opposite the angle) with which it is paired. Such a triangle $A'B'C'$ can be turned clockwise through a continuum of circumscribing triangles until a case of marginal circumscription occurs. Similarly, it can be turned counterclockwise until a case of marginal circumscription occurs. In Figure 1, $\angle A$ is associated with side EF (opposite $\angle D$), $\angle B$ with ED (opposite $\angle F$), and $\angle C$ with DF (opposite $\angle E$). Note that DE is a chord of a circle whose center is at O_B and such that $\angle DO_BE = 2\angle B$. By the vantage-point theorem, also known as the inscribed-angle theorem, all angles (from now on, when we speak of an "angle," we mean also its measure) intercepted by chord DE from any point on the outer arc of this circle are equal to $\angle B$.

Point O_B may be constructed by first creating the perpendicular bisector of DE and a perpendicular to DE from endpoint D. Then $\angle B$ is laid off from the outward perpendicular ray at D in the direction of the angle bisector. Center O_B is the intersection of the angle bisector with the last-constructed ray.

The remainder of Figure 1 is constructed similarly. Two positions are shown. The thinly dashed one is a marginal circumscription—it is "at its clockwise limit." The

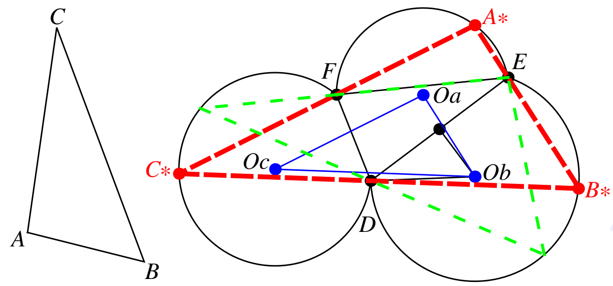


Figure 1. General and marginal positions

thickly dashed one is in the continuum of "general position" of its kind, and is optimal. All possible strict and marginal circumscribing triangles are obtained in this manner except for possibly a few anomalous cases of marginal circumscription (0, 2, 4, or 6, depending on the relationship of the angles of the two triangles). These anomalous cases involve a side of DEF lying along but interior to an edge of $A'B'C'$ with the opposite vertices of both triangles coinciding.

Next we will examine what all marginal circumscriptions look like. One vertex of DEF and one of $A'B'C'$ must coincide; let us say for definiteness that D and A' coincide. Now D counts as a vertex of DEF lying on both $A'B'$ and $A'C'$. One or both of E and F must lie on $B'C'$. One possibility is that E lies at B' and F at C' (or vice versa), in which case the triangles are similar. We call this case Shape A.

Another possibility is that E lies at B' but F is in the interior of $B'C'$ (or $A'C'$). This we call Shape B. The next case is that E lies in the interior of $A'B'$ and F in the interior of $B'C'$; this we call Shape C. Finally, E and F may both lie in the interior of $B'C'$. We refer to this as Shape D.

Figure 2 illustrates these four cases. Note that Shape A is a special case of Shape B (when lower right angles are equal), and Shape B is a special case of Shape C (when lower left angles are also equal). Shape D is anomalous.

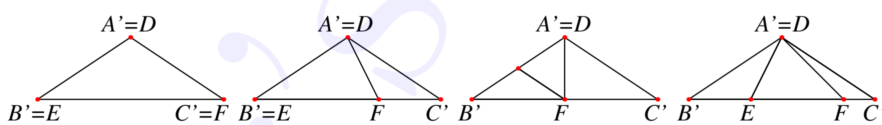


Figure 2. Shapes A–D

The Shape D configurations cannot be achieved by the process of turning through a continuum of strict configurations. If either of the ends of the side of $A'B'C'$ that wholly contains a side of DEF in its interior ($B'C'$ in Figure 2 Shape D) is moved, then some vertex of DEF is in the interior or the exterior of $A'B'C'$ and DEF is no longer circumscribed at all.

Assume that both ABC and DEF are nondegenerate. Let the angles of DEF be denoted X_1, X_2, X_3 and those of ABC be denoted Y_1, Y_2, Y_3, both listings from least to greatest. Much depends on the ordering of these six angles. We consider the following five cases based on which angles are largest and smallest.

(1) The two triangles are similar, so $X_1 = Y_1$, $X_2 = Y_2$, and $X_3 = Y_3$. This case is subsumed by each of the following, so it need not be dealt with explicitly.

(2) $X_1 \leq \cdots \leq X_3$, so the largest and smallest angles of the six are in DEF. The order of the other four angles is only a matter of where X_2 fits among Y_1, Y_2, and Y_3.

(3) $Y_1 \le \cdots \le Y_3$, so the largest and smallest angles of the six are in ABC. Again, the order of the remaining four angles is only a matter of how Y_2 fits among X_1, X_2, and X_3.

(4) $X_1 \le \cdots \le Y_3$. In this case, the other angles must satisfy $X_1 \le Y_1 \le Y_2 < X_2 \le X_3 \le Y_3$, as any other choice (except similarity) will contradict the fact that the angles of any triangle sum to the same value. If $X_2 < Y_2$, then the sum of the X angles is less than the sum of the Y angles, a contradiction. If $X_2 = Y_2$, then the triangles are similar.

(5) $Y_1 \le \cdots \le X_3$. In this case, the remaining angles must satisfy $Y_1 \le X_1 \le X_2 \le Y_2 \le Y_3 \le X_3$, by similar reasoning.

Let us analyze the possible Shape D configurations, based on the observation that at the apex the Y angle must exceed the X angle, while at the bases each X angle must exceed the neighboring Y angle.

In ordering (2), X_1 must be at the apex. If $Y_1 > X_2$, then shape D is impossible. Otherwise, we can construct a shape D circumscription. Among however many such configurations exist, one that is optimal has $Y_1 = \angle A'B'C'$, $Y_2 = \angle B'C'A'$, $Y_3 = \angle C'A'B'$, and $X_1 = \angle FDE$. For other orderings, again if shape D is possible, an optimal circumscription exists in which $Y_3 = \angle B'A'C'$. This somewhat narrows the list of cases that must be constructed.

Some trimming of cases can be conducted for the orderings (3), (4), and (5). These are left to the reader. In every case the largest triangle (if any) has X_1 and Y_3 at the apex.

Given the ordering (of angles X_j and Y_j), there is a shape C construction. For instance, in orderings of type (2), we may use the scheme illustrated on the left in Figure 3, while orderings of type (3) allow for the scheme illustrated on the right side of Figure 3.

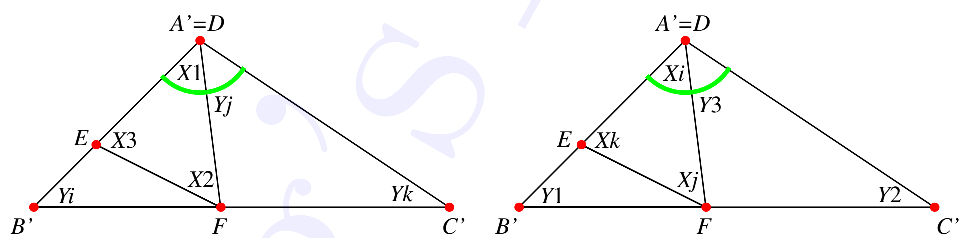


Figure 3. Orderings 2 and 3

Orderings (4) and (5) are slightly messier. Details are left to the reader.

From an initial marginal circumscription, we can twist $A'B'C'$ in one direction only, through a continuum of allowable strict circumscriptions, until another marginal circumscription is reached. During this process the size of the resulting triangle does one of the following: (i) decreases continuously, (ii) increases continuously, or (iii) increases to a maximum and then decreases.

In scenario (iii) we will show that the maximum occurs when the sides of $A'B'C'$ are respectively parallel to the line segments joining the three circumcenters O_A, O_B, and O_C. In (i) and (ii) it is impossible to reach a position where parallels occur. In (i) we move away from such a position; in (ii) we move toward such a position but are blocked from reaching it. Figure 1 illustrates (iii); note that the middle triangle is the desired maximal case, since its sides are parallel to the respective sides of the triangle of circumcenters. (Note that $O_AO_BO_C$ is similar to ABC.)

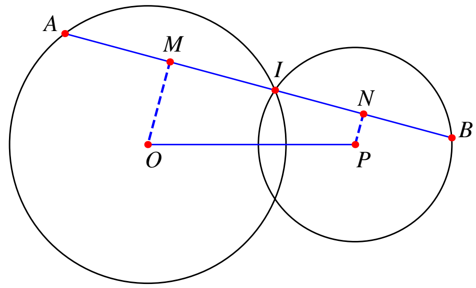


Figure 4. An inequality

When ABC is a 20-60-100 triangle and DEF is a 15-150-15 triangle, (iii) does not occur in any pairing because the parallel condition cannot be achieved.

In Figure 4 we show two circles, centered at O and P. In this figure, A is a point on circle (O), B is a point on circle (P), I is one of the intersections of the circles, and $\angle AIB$ is straight. Segments OM and PN are perpendicular to AIB. Since AI is a chord of circle (O), OM lies along its perpendicular bisector, so $AM = IM$. Likewise $BN = IN$. Thus MN is half as long as AB. Since MN is a projection of OP onto AIB, it is no longer than OP, and they have equal length only when AIB is parallel to OP. Thus among all choices of line AIB, the longest has length $2 \cdot OP$ and occurs when AIB is parallel to OP.

A "Program," then, for determining the largest circumscribing $A'B'C'$ is as follows. *For each of the six pairings of angles of ABC with those of DEF, create a case of marginal circumscription, and turn it through the continuum of strict circumscriptions to the marginal circumscription at the "other end." The largest of these circumscribing triangles occurs when the sides are parallel to the sides of the triangle of circumcenters, if that is possible. If this is not possible, then it occurs at the marginal case at the beginning or the end, whichever involves sides closer to the desired parallel directions. Select the largest of the six maximal cases. Then turn to the anomalous cases, if any, and determine the largest of these. The larger of the maximal cases for nonanomalous cases and the maximum of the anomalous cases is the largest circumscribing triangle.*

We have not developed an algorithm for conducting the above program in the general case.

Partially solved by M. Bataille (France), R. Boukharfane (Canada), O. Geupel (Germany), L. R. King, C. R. Pranesachar (India), and the proposer.

A Minimization with Sum and Product

11718 [2013, 570]. *Proposed by Arkady Alt, San Jose, CA.* Given positive real numbers $a_1, \ldots, a_n$ with $n \geq 2$, minimize $\sum_{i=1}^{n} x_i$ subject to the conditions that $x_1, \ldots, x_n$ are positive and that $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i x_i$.

Solution by Ronald E. Prather, Oakland, CA. Let

$$S = \left\{ (x_1, \ldots, x_n) : x_i > 0, \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i x_i \right\}.$$

First consider what happens when a point approaches the boundary of S. Writing the constraint as

$$1 = \sum_{i=1}^{n} \frac{a_i}{\prod_{j \neq i} x_j},$$

we see that each product $\prod_{j\neq i} x_i$ is bounded away from zero. If any x_k tends to zero, another one (in fact at least two) must tend to infinity, and then $\sum_{i=1}^{n} x_i$ tends to infinity. Therefore, $\sum_{i=1}^{n} x_i$ achieves a minimum value in the interior of S. We will find it using Lagrange multipliers.

Writing μ for the reciprocal of the usual Lagrange multiplier, we get $n+1$ equations

$$\prod_{j\neq i} x_j = a_i + \mu \quad \text{for } 1 \le i \le n, \quad \text{and} \prod_{i=1}^{n} x_n = \sum_{i=1}^{n} a_i x_i$$

in the $n+1$ unknowns $x_1, \dots, x_n$ and μ. Substituting the first n equations in the last yields an nth degree polynomial equation for μ, namely $f(\mu) = 0$, where

$$f(x) = \sum_{i=1}^{n} \frac{a_i}{a_i + x}.$$

Now f is continuous and monotonically decreasing, with $f(0) = n > 1$ and $f(\infty) = 0 < 1$, so there is a unique positive solution μ to the equation $f(\mu) = 1$. Multiplying the ith equation by x_i and summing over i, we get $n\prod_{j=1}^{n} x_j = \sum_{i=1}^{n} a_i x_i + \mu\sum_{i=1}^{n} x_i = \prod_{j=1}^{n} x_j + \mu\sum_{i=1}^{n} x_i$, so

$$\sum_{i=1}^{n} x_i = \frac{n-1}{\mu}\prod_{j=1}^{n} x_j.$$

Multiplying the first n equations yields $\prod_{i=1}^{n} x_i^{n-1} = \prod_{i=1}^{n}(a_i+\mu)$. Thus the minimum value of $\sum_{i=1}^{n} x_i$ is

$$\frac{n-1}{\mu}\left(\prod_{j=1}^{n}(a_j+\mu)\right)^{1/(n-1)}.$$

Editorial comment. The proposer notes that the problem is related to Problem 4 of the 2001 Vietnam Team Selection Test.

Also solved by R. Bagby, R. Boukharfane (Canada), P. Bracken, M. Dincă (Romania), N. Grivaux (France), O. Kouba (Syria), J. Martínez (Spain), N. C. Singer, R. Stong, T. Viteam (Chile), GCHQ Problem Solving Group (U. K.), and the proposer.

Euler and Bernoulli Make Integer Coefficients

11720 [2013, 660]. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* Let $E_n(t)$ be the Eulerian polynomial defined by

$$\sum_{k=0}^{\infty}(k+1)^n t^k = \frac{E_n(t)}{(1-t)^{n+1}},$$

and let B_n be the nth Bernoulli number. Show that $\big(E_{n+1}(t) - (1-t)^n\big)B_n$ is a polynomial with integer coefficients.

Solution by Josep Martínez, Spain. The assertion holds for $n = 0$, since $E_1(t) = 1$. We know that $B_n = 0$ when n is odd and greater than 1. By the von Staudt–Clausen theorem, when n is even and positive the denominator of B_n is the product of the prime numbers p such that $p-1$ divides n. This also holds for $n = 1$, since $B_1 = -1/2$.

Let p be a prime number such that $p-1$ divides n; we prove that p divides the coefficients of $E_{n+1}(t)-(1-t)^n$. We have

$$\frac{E_{n+1}(t)-(1-t)^n}{(1-t)^{n+2}} = \frac{E_{n+1}(t)}{(1-t)^{n+2}} - \frac{1}{(1-t)^2} = \sum_{k=1}^{\infty}(k^{n+1}-k)t^{k-1}.$$

If p does not divide k, then Fermat's little theorem gives $k^{p-1} \equiv 1 \pmod{k}$. Since $p-1$ divides n, we deduce that $k^{n+1}-k = k(k^n-1)$ is divisible by p. If p divides k, then p divides $k^{n+1}-k$. Thus every coefficient of

$$\frac{E_{n+1}(t)-(1-t)^n}{(1-t)^{n+2}}$$

is divisible by p, and therefore so is every coefficient of $E_{n+1}(t)-(1-t)^n$.

Also solved by R. Chapman (U. K.), G. C. Greubel, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Quaintance, R. Stong, R. Tauraso (Italy), T. P. Turiel, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

A Tangent Conic

11723 [2013, 661]. *Proposed by L. R. King, Davidson, NC.* Let A, B, and C be three points in the plane, and let D, E, and F be points lying on BC, CA, and AB, respectively. Show that there exists a conic tangent to BC, CA, and AB at D, E, and F, respectively, if and only if AD, BE, and CF are concurrent.

Solution by Raul A. Simon, Chile. Consider a conic meeting the sides in two points each, meeting BC at D_b and D_c, CA at E_c and E_a, and AB at F_a and F_b. Carnot's theorem for conics gives

$$\frac{AF_a \cdot AF_b \cdot BD_b \cdot BD_c \cdot CE_c \cdot CE_a}{AE_c \cdot AE_a \cdot BF_a \cdot BF_b \cdot CD_b \cdot CD_c} = 1,$$

where lengths are interpreted as signed. In the proposed problem, where each pair of points degenerates into a single point, we get

$$\left(\frac{AF \cdot BD \cdot CE}{AE \cdot BF \cdot CD}\right)^2 = 1.$$

Hence

$$\frac{AF \cdot BD \cdot CE}{AE \cdot BF \cdot CD} = \pm 1.$$

For the positive sign, Ceva's theorem shows that AD, BE, and CF are concurrent. For the negative sign, Menelaus' theorem shows that D, E, and F are collinear (and the conic has degenerated, so we exclude this case).

The converse follows from the same argument using the converses to Ceva's and Carnot's theorems.

Also solved by E. Bojaxhiu & E. Hysnelaj (Albania & Australia), O. Geupel (Germany), J.-P. Grivaux (France), A. Habil (Syria), K. Hanes, O. Kouba (Syria), O. P. Lossers (Netherlands), C. R. Pranesachar (India), R. Stong, M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

A Limit Computation

11724 [2013, 661]. *Proposed by Andrew Cusumano, Great Neck, NY.* Let $f(n) = \sum_{k=1}^{n} k^k$ and let $g(n) = \sum_{k=1}^{n} f(k)$. Find

$$\lim_{n\to\infty} \frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)}.$$

Solution by Hosam M. Mahmoud, The George Washington University, Washington, D.C. The limit is e. First we derive some asymptotics of the function $g(n)$. As $n \to \infty$,

$$\left(1+\frac{a}{n}\right)^n = e^a - \frac{a^2 e^a}{2n} + O(n^{-2}). \tag{1}$$

Now $f(n) = n^n + (n-1)^{n-1} + \sum_{k=1}^{n-2} k^k$, and the remaining series is bounded by a geometric series,

$$\sum_{k=1}^{n-2} k^k \le \sum_{k=1}^{n-2} n^k = \frac{n^{n-1}-n}{n-1} = O(n^{n-2}).$$

Using (1) we obtain

$$f(n) = n^n + n^{n-1}\left(1-\frac{1}{n}\right)^{n-1} + O(n^{n-2}) = n^n + \frac{n^{n-1}}{e} + O(n^{n-2}).$$

This leads to an asymptotic approximation for $g(n)$:

$$\begin{aligned} g(n) &= f(n) + f(n-1) + \sum_{k=1}^{n-2} O(k^k) \\ &= \left(n^n + \frac{n^{n-1}}{e} + O(n^{n-2})\right) + \left((n-1)^{n-1} + O(n^{n-2})\right) + O\left(\textstyle\sum_{k=1}^{n-2} k^k\right) \\ &= n^n + \frac{2}{e} n^{n-1} + O\left(n^{n-2}\right). \end{aligned}$$

Thus

$$\begin{aligned} \frac{g(n+1)}{g(n)} &= \frac{(n+1)^{n+1} + \frac{2}{e}(n+1)^n + O(n^{n-1})}{n^n + \frac{2}{e}n^{n-1} + O(n^{n-2})} \\ &= \frac{n(1+\frac{1}{n})^{n+1} + \frac{2}{e}(1+\frac{1}{n})^n + O(\frac{1}{n})}{1 + \frac{2}{en} + O(\frac{1}{n^2})} = e\,n + \frac{e}{2} + O(n^{-1}). \end{aligned}$$

Hence,

$$\frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)} = \left(e\,(n+1) + \frac{e}{2} + O(n^{-1})\right) - \left(e\,n + \frac{e}{2} + O(n^{-1})\right),$$

which simplifies to $e + O(n^{-1})$. Thus the required limit is indeed e.

Also solved by M. Bataille (France), P. Bracken, R. Chapman (U. K.), H. Chen, W. J. Cowieson, D. Fleischman, J.-P. Grivaux (France), A. Habil (Syria), E. A. Herman, B. Karaivanov, O. Kouba (Syria), C. W. Lienhard & M. Haner, J. H. Lindsey II, O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), P. Perfetti (Italy), C. R. Pranesachar (India), R. E. Prather, A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, J. Vinuesa (Spain), Z. Vörös (Hungary), J. Zacharias, and GCHQ Problem Solving Group (U. K.).

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before February 29, 2016. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11858. *Proposed by Arkady Alt, San Jose, CA.* Let D be a nonempty set and g be a function from D to D. Let n be an integer greater than 1. Consider the set X of all x in D such that $g^n(x) = x$, but $g^k(x) \neq x$ for $1 \leq k < n$. Prove that if X has exactly n elements, then there is no function f from D to D such that $f^n = g$. (Here, for $h\colon D \to D$, h^k denotes the k-fold composition of h with itself.)

11859. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Find all pairs (m, n) of positive integers for which there exists an $m \times n$ matrix A and an $n \times m$ matrix B, both with real entries, such that all diagonal entries of AB are positive and all off-diagonal entries are negative.

11860. *Proposed by Dimitris Vartziotis, NIKI MEPE Digital Engineering, Katsikas Ioannina, Greece.* Let ABC be a triangle. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Extend the ray DA beyond A to a point A', and similarly extend EB to B' and FC to C', in such a way that $\sqrt{3}|AA'| = |BC|$, $\sqrt{3}|BB'| = |CA|$, and $\sqrt{3}|CC'| = |AB|$. Prove that $A'B'C'$ is an equilateral triangle.

11861. *Proposed by Phu Cuong Le Van, College of Education, Hue, Vietnam.* Let n be a natural number and let f be a continuous function from $[0, 1]$ to $\mathbb{R}$ such that $\int_0^1 f(x)^{2n+1}\,dx = 0$. Prove that

$$\frac{(2n+1)^{2n+1}}{(2n)^{2n}}\left(\int_0^1 f(x)\,dx\right)^{2n} \leq \int_0^1 (f(x))^{4n}\,dx.$$

http://dx.doi.org/10.4169/amer.math.monthly.122.8.801

11862. *Proposed by David A. Cox and Uyen Thieu, Amherst College, Amherst, MA.* For positive integers n and k, evaluate

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{kn-in}{k+1}.$$

11863. *Proposed by Jeffrey C. Lagarias and Jeffrey Sun, University of Michigan, Ann Arbor, MI.* Consider integers a, b, c with $1 \le a < b < c$ that satisfy the following system of congruences:

$$\begin{aligned}(a+1)(b+1) &\equiv 1 \pmod{c}\\ (a+1)(c+1) &\equiv 1 \pmod{b}\\ (b+1)(c+1) &\equiv 1 \pmod{a}.\end{aligned}$$

(**a**) Show that there are infinitely many solutions (a, b, c) to this system.
(**b**) Show that under the additional condition that $\gcd(a, b) = 1$, there are only finitely many solutions (a, b, c) to the system, and find them all.

11864. *Proposed by Bakir Farhi, University of Béjaia, Béjaia, Algeria.* Let p be a prime number, and let $\langle u\rangle$ be the sequence given by $u_n = n$ for $0 \le n \le p-1$ and by $u_n = pu_{n+1-p} + u_{n-p}$ for $n \ge p$. Prove that for each positive integer n, the greatest power of p dividing u_n is the same as the greatest power of p dividing n.

11854. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata", Rome, Italy. (correction).* In the Euclidean plane, given distinct points $P_1, \dots, P_n$ and distinct lines $l_1, \dots, l_m$, prove that there is a half-line h such that for any point Q on h, any $k \in \{1, \dots, m\}$, and any $j \in \{1, \dots, n\}$, Q is nearer to l_k than to P_j.

SOLUTIONS

Taylor Approximation of the Logarithm

11725 [2013, 661]. *Proposed by Mher Safaryan, Yerevan State University, Yerevan, Armenia.* Let m be a positive integer. Show that, as $n \to \infty$,

$$\left|\log 2 - \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right| = \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots + \frac{C_m}{n^m} + o\left(\frac{1}{n^m}\right),$$

where

$$C_k = (-1)^k\sum_{i=1}^{k}\frac{1}{2^i}\sum_{j=1}^{i}(-1)^j\binom{i-1}{j-1}j^{k-1}$$

for $1 \le k \le m$.

Solution by Jean-Pierre Grivaux, Paris, France. Note that $\int_0^1 t^k\,dt = 1/(k+1)$. Hence,

$$\int_0^1 \frac{t^n}{1+t}\,dt = \sum_{k=n}^{\infty}(-1)^{k-n}\int_0^1 t^k\,dt = (-1)^n\left(\log 2 - \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right). \qquad (1)$$

Let I_n be the integral in (1). Integrating by parts k times yields $I_n = A_{n,k} + R_{n,k}$, where

$$A_{n,k} = \frac{1}{2(n+1)} + \frac{1}{4(n+1)(n+2)} + \cdots + \frac{(k-1)!}{2^k(n+1)(n+2)\cdots(n+k)}$$

and

$$R_{n,k} = \frac{k!}{(n+1)(n+2)\cdots(n+k)} \int_0^1 \frac{t^{n+k}}{(1+t)^{k+1}}\, dt.$$

Since

$$0 \le R_{n,k} \le \frac{k!}{n^k} \int_0^1 t^{n+k}\, dt = O\left(\frac{1}{n^{k+1}}\right),$$

it follows that

$$I_n = A_{n,k} + O\left(\frac{1}{n^{k+1}}\right).$$

What remains is to expand the terms in this asymptotic expansion for I_n using powers of $1/n$. First, we use partial fractions to expand $1/(x+1)(x+2)\cdots(x+r)$ and obtain

$$\frac{(r-1)!}{(n+1)(n+2)\cdots(n+r)} = \sum_{j=1}^{r} (-1)^{j-1} \binom{r-1}{j-1} \frac{1}{n+j}.$$

Hence,

$$I_n = \sum_{i=1}^{k} \frac{1}{2^i} \left(\sum_{j=1}^{i} (-1)^{j-1} \binom{i-1}{j-1} \frac{1}{n+j} \right) + O\left(\frac{1}{n^{k+1}}\right). \tag{2}$$

Now we expand the terms $1/(n+j)$. As n tends to infinity,

$$\frac{1}{n+j} = \frac{1}{n} \sum_{s=0}^{k-1} (-1)^s \frac{j^s}{n^s} + O\left(\frac{1}{n^{k+1}}\right).$$

Inserting this into (2) and grouping the powers of $1/n$ gives the result.

Editorial comment. Other solvers used a variety of sophisticated tools. Ulrich Abel generalized the problem to obtain, for $-1 \le x < 1$, the asymptotic expansion

$$\left| \log(1-x) + \sum_{k=1}^{n} \frac{x^k}{k} \right| \sim |x|^{n+1} \sum_{k=1}^{\infty} \frac{C_k(x)}{n^k},$$

with coefficients

$$C_k(x) = \sum_{i=1}^{k} (1-x)^{-i} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} j^{k-1}.$$

Also solved by U. Abel (Germany), M. Bataille (France), D. Beckwith, R. Chapman (U. K.), I. Gessel, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), P. Perfetti (Italy), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.

Jointly Discontinuous Only on a Cantor Set

11726 [2013, 754]. *Proposed by Stephen Scheinberg, Corona del Mar, CA.* Let K be Cantor's middle-third set. Let $K^* = K \times \{0\}$. Is there is a function F from $\mathbb{R}^2$ to $\mathbb{R}$ such that

1. for each $x \in \mathbb{R}$, the function $t \to F(x, t)$ is continuous on $\mathbb{R}$,
2. for each $y \in \mathbb{R}$, the function $s \to F(s, y)$ is continuous on $\mathbb{R}$, and
3. F is continuous on the complement of K^* and discontinuous on K^*?

Solution by Reiner Martin, Bad Soden-Neuenhain, Germany. We will construct such a function F. Define $d \colon \mathbb{R} \to \mathbb{R}$ so that $d(x)$ is the distance from x to K. Note that $d(x) = 0$ if and only if $x \in K$. Define $\phi \colon \mathbb{R} \to \mathbb{R}$ by $\phi(x) = \max\big\{\min\{x, 2 - x\}, 0\big\}$. Of course both d and ϕ are continuous on $\mathbb{R}$.

Now define F by $F(x, y) = 0$ for $x \in K$ and $F(x, y) = \phi(y/d(x))$ otherwise. Properties 1 and 2 and that F is continuous on the complement of K^* are straightforward. It remains to prove that F is discontinuous on K^*. Let $x \in K$ and $\varepsilon > 0$. The complement of K is dense in $\mathbb{R}$, so there is some $z \in \mathbb{R} \setminus K$ with $|z - x| < \varepsilon/2$. Now $d(z) < \varepsilon/2$, and $F(z, d(z)) = \phi(1) = 1$. Thus $F(x, 0) = 0$, but every neighborhood of $(x, 0)$ contains a point where F is 1. Hence, F is discontinuous at $(x, 0) \in K^*$.

Also solved by J.-P. Grivaux (France), K. P. Hart (Netherlands), B. Karaivanov, M. D. Meyerson, P. Perfetti (Italy), R. Stong, TCDmath Problem Group (Ireland), and the proposer.

An Apollonius Special Case

11727 [2013, 754]. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Let R be a circle with center O. Let R_1 and R_2 be circles with centers O_1 and O_2 inside R, such that R_1 and R_2 are externally tangent and both are internally tangent to R. Give a straight edge and compass construction of the circle R_3 that is internally tangent to R and externally tangent to R_1 and R_2.

Solution by Kit Hanes, Bellingham, WA. Let T be the point of tangency of R and R_1. Construct the line l incident with O, O_1, and T. Construct lines through T tangent to R_2 at P and Q. Construct the circle C centered at T incident with P and Q. Construct the line k through O_2 perpendicular to l. Construct lines m and n parallel to k and tangent to R_2. Since R_2 is orthogonal to C, the inverse of R_2 with respect to C is R_2 itself. The inverses of R_1 and R are lines perpendicular to l, and, since inversion preserves tangency, must be the lines m and n. Select one of the two circles tangent to m, n, and R_2, and construct U, V, and W, the respective points of tangency. Construct U', V', and W', the respective inverses of U, V, and W with respect to C. Construct the circle incident with U', V', and W'; it is one of the two possibilities for R_3.

Editorial comment. Charles Delorme noted that this construction works for any three pairwise tangent circles, and it also works if circle R is replaced by a line. J. Schaer noted that the problem is a special case of the Problem of Apollonius.

Also solved by R. Bagby, M. Bataille (France), L. Childers & C. Harden, P. P. Dályay (Hungary), C. Delorme (France), O. Geupel (Germany), J.-P. Grivaux (France), H. Guggenheimer, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Nita, J. Schaer (Canada), R. Stong, E. I. Verriest, Z. Vörös (Hungary), and the proposer.

Another Integral Sum

11728 [2013, 754]. *Proposed by Walter Blumberg, Coral Springs, FL.* Let p be a prime congruent to 7 modulo 8. Prove that

$$\sum_{k=1}^{p} \left\lfloor \frac{k^2+k}{p} \right\rfloor = \frac{2p^2+3p+7}{6}.$$

Solution by O. P. Lossers, Eindhoven, The Netherlands. Let $[x]$ denote the remainder of x upon division by p. It suffices to prove $\sum_{k=0}^{p-1}[k^2+k] = \frac{p(p-1)}{2}$, because then

$$\sum_{k=1}^{p} \left\lfloor \frac{k^2+k}{p} \right\rfloor = \sum_{k=1}^{p} \frac{k^2+k}{p} - \sum_{k=1}^{p} \frac{[k^2+k]}{p} = \sum_{k=1}^{p} \frac{k^2}{p} + \sum_{k=1}^{p} \frac{k}{p} - \sum_{k=0}^{p-1} \frac{[k^2+k]}{p}$$
$$= \frac{(p+1)(2p+1)}{6} + \frac{p+1}{2} - \frac{p-1}{2} = \frac{2p^2+3p+7}{6}.$$

Note that with $k \equiv m - \frac{p+1}{2} \pmod{p}$ we have

$$k^2 + k \equiv m^2 - m(p+1) + \frac{(p+1)^2}{4} + m - \frac{p+1}{2}$$
$$\equiv m^2 - \frac{p+1}{4}(-p-1+2) \equiv m^2 - \frac{p+1}{4} \pmod{p},$$

so we can compute $L = \sum_{m=0}^{p-1}[m^2 - \frac{p+1}{4}]$ instead.

Let $K = \sum_{k=0}^{p-1}[k^2]$. Note that $[m^2 - \frac{p+1}{4}] = [m^2] + p - \frac{p+1}{4}$ if $[m^2] < \frac{p+1}{4}$, so $L = K + (c_1+1)p - \frac{p+1}{4}p$, where c_1 is the number of m such that $0 < [m^2] < \frac{p}{4}$.

To study K, let us split the interval $(0, p)$ into four equal parts $(0, \frac{p}{4})$, $(\frac{p}{4}, \frac{p}{2})$, $(\frac{p}{2}, \frac{3p}{4})$, and $(\frac{3p}{4}, p)$. Denote respectively by K_1, K_2, K_3, and K_4 the subsums of K with terms in these intervals, and by c_1, c_2, c_3, and c_4 the number of terms in each of these subsums. Note that these terms come in pairs with equal values.

Because $p \equiv 7 \pmod{8}$, we know that 2 is a quadratic residue. It follows that $K = \sum_{k=0}^{p-1}[2k^2]$. Denote respectively by H_1 and H_2 the subsums of this sum with terms in $(0, \frac{p}{2})$ and in $(\frac{p}{2}, p)$. Note that H_1 has $c_1 + c_3$ terms and $H_1 = 2(K_1 + K_3) - c_3 p$, while H_2 has $c_2 + c_4$ terms and $H_2 = 2(K_2 + K_4) - c_4 p$. It follows readily that $K = (c_3 + c_4)p$.

Repeating this trick with $K = \sum_{k=0}^{p-1}[4k^2]$, we obtain $K = 2(H_1 + H_2) - (c_2 + c_4)p$, and hence $K = (c_2 + c_4)p$. It now follows that $c_2 = c_3$.

Because $p \equiv 3 \pmod{4}$, we know that -1 is a quadratic nonresidue. It follows that whenever a is a quadratic residue, $p - a$ is not. There are $\frac{p+1}{2}$ integers in the interval $(\frac{p}{4}, \frac{3p}{4})$. Every quadratic residue in $(\frac{p}{4}, \frac{p}{2})$ corresponds to a quadratic nonresidue in $(\frac{p}{2}, \frac{3p}{4})$ and vice versa. There are two terms in the sums with a particular quadratic residue as value, and this shows that $c_2 + c_3 = \frac{p+1}{2}$. Thus $c_2 = c_3 = \frac{p+1}{4}$.

From $L = K + (c_1+1)p - \frac{p(p+1)}{4}$ and $K = (c_3 + c_4)p$ it follows that $L = (c_1 + c_3 + c_4 + 1)p - \frac{p(p+1)}{4}$. Since $c_1 + c_2 + c_3 + c_4 = p - 1$ and $c_2 = \frac{p+1}{4}$, we obtain $L = \left(p - \frac{p+1}{4}\right)p - \frac{p(p+1)}{4} = p\left(p - \frac{p+1}{2}\right) = \frac{p(p-1)}{2}$, as required.

Also solved by R. Chapman (U. K.), E. J. Ionascu, Y. J. Ionin, B. Karaivanov, J. Martínez (Spain), M. A. Prasad (India), N. C. Singer, R. Tauraso (Italy), T. Viteam (South Africa), and the proposer.

Summing the Reciprocals of Normal Numbers Base b

11729 [2013 570]. *Proposed by Vassilis Papanicolaou, National Technical University of Athens, Athens, Greece.* An integer n is called *b-normal* if all digits $0, 1, \ldots, b-1$ appear the same number of times in the base-b expansion of n. Let $\mathcal{N}_b$ be the set of all b-normal integers. Determine those b for which

$$\sum_{n\in\mathcal{N}_b} \frac{1}{n} < \infty.$$

Solution by Nicole Grivaux, Paris, France. We denote by $\mathcal{N}_{b,k}$ the set of the elements of $\mathcal{N}_b$ written with kb digits. If n belongs to $\mathcal{N}_{b,k}$, then $b^{kb-1} \le n < b^{kb}$, so that $1/b^{kb} \le 1/n \le b(1/b^{kb})$. Since the first digit of an element of $\mathcal{N}_{b,k}$ cannot be 0, the number of elements of $\mathcal{N}_{b,k}$ is

$$|\mathcal{N}_{b,k}| = \binom{bk-1}{k} \times \binom{bk-k}{k} \times \cdots \times \binom{2k}{k} = \frac{b-1}{b} \times \frac{(bk)!}{(k!)^b}.$$

By Stirling's formula, $|\mathcal{N}_{b,k}| \sim \dfrac{Mb^{bk}}{k^{(b-1)/2}}$ as $k \to \infty$, where $M = \dfrac{(b-1)\sqrt{b}}{b(2\pi)^{(b-1)/2}}$. We compute

$$|\mathcal{N}_{b,k}|\frac{1}{b^{bk}} = \sum_{n\in\mathcal{N}_{b,k}} \frac{1}{b^{kb}} \le \sum_{n\in\mathcal{N}_{b,k}} \frac{1}{n} \le b \sum_{n\in\mathcal{N}_{b,k}} \frac{1}{b^{kb}} = b|\mathcal{N}_{b,k}|\frac{1}{b^{kb}},$$

which proves that $\displaystyle\sum_{k>0}\sum_{n\in\mathcal{N}_{b,k}} \frac{1}{n}$ and $\displaystyle\sum_{k>0} |\mathcal{N}_{b,k}|\frac{1}{b^{bk}}$ are both finite or both not. Since $|\mathcal{N}_{b,k}|\dfrac{1}{b^{bk}} \sim M\dfrac{1}{k^{(b-1)/2}}$, $\displaystyle\sum_{k>0} |\mathcal{N}_{b,k}|\frac{1}{b^{bk}}$ is finite if and only if $\dfrac{b-1}{b} > 1$, i.e., $b > 3$.

Also solved by R. Bagby, R. Boukharfane (Canada), R. Chapman (U. K.), W. Cowieson, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), Y. J. Ionin, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), G. Martin (Canada), R. Martin (Germany), M. A. Prasad (India), M. Safaryan (Armenia), N. C. Singer, R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), H. Wang & J. Wojdylo, Missouri State U. Problem Solving Group, NSA Problems Group, TCDmath Problems Group (Ireland), and the proposer.

A partition recurrence

11730 [2013, 755]. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let p be the *partition function* (counting the ways to write n as a sum of positive integers), extended so that $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Prove that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right) = 1.$$

Solution by Mark Wildon, Royal Holloway, University of London, Egham, U. K. Let

$$Q(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k x^{k(3k+1)/2+j}$$

and let

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n.$$

The coefficient of x^n in $Q(x)P(x)$ is

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k p\left(n-\frac{k(3k+1)}{2}-j\right).$$

Thus we need to prove that $Q(x)P(x) = 1/(1-x)$, or equivalently, $(1-x)Q(x) = 1/P(x)$. We have

$$\begin{aligned}(1-x)Q(x) &= \sum_{k=0}^{\infty}(-1)^k(1-x^{2k+1})x^{k(3k+1)/2}\\ &= \sum_{k=0}^{\infty}(-1)^k\left(x^{k(3k+1)/2}-x^{(k+1)(3(k+1)-1)/2}\right)\\ &= 1+\sum_{k=1}^{\infty}(-1)^k\left(x^{k(3k+1)/2}+x^{k(3k-1)/2}\right).\end{aligned}$$

By Euler's pentagonal number theorem, this is equal to $1/P(x)$.

Also solved by D. Beckwith, R. Boukharfane (Canada), R. Chapman (U. K.), D. Fleischman, O. Geupel (Germany), Y. J. Ionin, B. Karaivanov, O. P. Lossers (Netherlands), J. Martínez (Spain), M. A. Prasad, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), and the proposer.

The Integer Simplex

11731 [2013, 755]. *Proposed by Meijie Ma and Douglas B. West, Zhejiang Normal University, Jinhua, China.* The *integer simplex* with dimensions d and side-length m is the graph T_m^d whose vertices are the nonnegative integer $(d+1)$-tuples summing to m, with two vertices adjacent when they differ by 1 in two places and are equal in all other places. Determine the connectivity, the chromatic number, and the edge-chromatic number of T_m^d (the last when $m > d$).

Composite solution by Boris Karaivanov, Lexington, South Carolina, and the proposers. The connectivity is d, the chromatic number is $d+1$, and the edge-chromatic number is $(d+1)d$.

Connectivity: The $d+1$ *corner vertices* in T_m^d are those having 0 in all but one coordinate. Joining any two coordinate vertices u and v are d internally disjoint paths: one through convex combinations of u and v and $d-1$ through the other corner vertices. From any noncorner vertex, there are internally disjoint paths to the $d+1$ corner vertices; the (nonunique) path to the ith corner vertex increases coordinate i with each step. Hence deleting fewer than d vertices cannot separate any vertex from the set of corner vertices and cannot separate two corner vertices from each other, so T_m^d is d-connected. Equality holds, since each corner vertex has degree d.

Chromatic Number: Color each vertex $(x_0, \dots, x_d)$ with the congruence class modulo $d+1$ of $\sum_{k=0}^{d} kx_k$. Any adjacent vertices u and v differ in two coordinates s and t; their colors differ by $|s-t|$, so the coloring is proper. Also T_m^d contains T_1^d, a complete graph with chromatic number $d+1$, so $\chi(T_m^d) = d+1$.

Edge-Chromatic Number: Each edge involves two indices that change. The edges generated by one pair of indices form a disjoint union of paths. Devoting two colors to each such subgraph yields a proper edge coloring with $(d+1)d$ colors. This is optimal when $m > d$, since there then exist noncorner vertices with degree $(d+1)d$.

Also solved by R. Chapman (U. K.) and the SPARTA Problem Solving Group (Turkey). Partially solved by C. Delorme (France), O. P. Lossers (Netherlands), and R. Stong.

The Functional Equation $f(a^x) + f(b^x) = mx + n$

11732 [2013, 755]. *Proposed by Marcel Chirita, Bucharest, Romania.* Let a and b be real, with $1 < a < b$, and let m and n be real, with $m \neq 0$. Find all continuous functions f from $[0, \infty)$ to $\mathbb{R}$ such that for $x \geq 0$,

$$f(a^x) + f(b^x) = mx + n.$$

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. For $t \geq 0$ let

$$g(t) = \frac{f((ab)^t) - n/2}{m}. \tag{1}$$

For $x \geq 0$,

$$\begin{aligned} m\big(g(rx) + g((1-r)x)\big) + n &= mg(rx) + \frac{n}{2} + mg\big((1-r)x\big) + \frac{n}{2} \\ &= f(a^x) + f(b^x), \end{aligned}$$

where $r = \log_{ab} b > \log_{ab} a = 1 - r > 0$. Thus the problem becomes to show that for $x > 0$,

$$g(rx) + g((1-r)x) = x.$$

Let $g(t) = t + h(t)$, so $h(rx) = -h((1-r)x)$. Hence, for $t \geq 0$ and any positive integer n,

$$h(t) = -h(\alpha t) = h(\alpha^2 t) = \cdots = (-1)^n h(\alpha^n t),$$

where $\alpha = (1-r)/r \in (0, 1)$. Note that $h(0) = (-1)^n h(0)$ implies $h(0) = 0$. Since f is continuous at 1, also h is continuous at 0. For $t \geq 0$ it follows that

$$h(t) = \lim_{n\to\infty} (-1)^n h(\alpha^n t) = 0.$$

This holds since $\alpha^n t \to 0$ and $h(\alpha^n t) \to h(0) = 0$. The unique solution is given by $g(t) = t$ for $t \geq 0$. Substituting this result into (1) with $t = \log_{ab}(x)$ for $x \geq 1$, our unique solution is

$$f(x) = f((ab)^{\log_{ab}(x)}) = m \log_{ab}(x) + \frac{n}{2}.$$

Note that in this case the identity

$$f(a^x) + f(b^x) = mx + n$$

holds for all $x \in \mathbb{R}$.

Editorial comment. The proposer already published his solution in *Gazeta Matematica* No. 5 (2013) pp. 225–226.

Also solved by M. Aasila (France), I. Aburub (Jordan), M. Bataille (France), D. Beckwith, R. Boukharfane (Canada), P. Bracken, N. Caro (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), N. Grivaux (France), E. A. Herman, B. D. Hughes (Australia), E. J. Ionascu, Y. J. Ionin, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Martínez (Spain), T. L. McCoy, V. Mikayelyan (Armenia), V. Nita, M. Omarjee (France), S. K. Patel & H. D. Kamat (India), M. A. Prasad (India), M. Safaryan (Armenia), J. Schlosberg, R. Stong, N. Thornber, E. I. Verriest, T. Viteam (South Africa), H. Yousefi, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before March 31, 2016. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11865. *Proposed by Gary H. Chung, Clark Atlanta University, Atlanta, GA.* Let $\langle a_n \rangle$ be a monotone decreasing sequence of nonnegative real numbers. Prove that $\sum_{n=1}^{\infty} a_n/n$ is finite if and only if $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty}(a_n - a_{n+1}) \log n < \infty$.

11866. *Proposed by Arindam Sengupta, University of Calcutta, Kolkata, India.* Consider a finite set $\{\alpha_1, \ldots, \alpha_m\}$ of rational numbers in $(0, 1)$. For $0 < p < 1$ and $k \geq 1$, let Ω_k be the probability space for k independent flips of a coin that comes up heads with probability p. Show that there exists a positive integer k, a suitable p, and events $E_1, \ldots, E_m$ in Ω_k, such that for each j with $1 \leq j \leq m$, the probability of E_j is α_j.

11867. *Proposed by George Apostolopoulos, Messolonghi, Greece.* For real numbers a, b, c, let

$$f(a, b, c) = \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4}.$$

Prove that $f(a, b, c) + f(b, c, a) + f(c, a, b) \leq 3$.

11868. *Proposed by James Propp, University of Massachusetts Lowell, Lowell, MA.* For fixed positive integers a and b, let $m = ab - 1$ and let R be the set $\{1, \ldots, a\} \times \{1, \ldots, b\}$, indexed as p_0 through p_m in lexicographic order, so that $p_0 = (1, 1)$, $p_1 = (1, 2)$, and $p_m = (a, b)$. Define T from R to R as the map that sends p_0 to p_0 and p_m to p_m, and for $1 \leq i \leq m - 1$ sends p_i to p_j where $j \equiv ai \pmod{m}$. As a bijection, T partitions R into orbits. Show that the center of mass of each orbit lies on the line joining p_0 and p_m.

11869. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Prove that $|y \log y - x \log x| \leq |y - x|^{1-1/e}$ for $0 < x < y \leq 1$.

http://dx.doi.org/10.4169/amer.math.monthly.122.9.899

11870. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Suppose $0 \leq x \leq 1$, $y = 1 - x$, and a and b are unimodular complex numbers. Let $c_1 = 2(xa + yb)$ and $c_2 = 2(xa^2 + yb^2)$. Prove that $\left||c_1^2 + c_2| - 3|c_1|\right| \leq 3$, with equality if and only if $x = y = 1/2$ and $b\overline{a} = e^{2\pi i/3}$.

11871. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Ştefan Spătaru, Harvard University, Boston, MA.* Let ABC be a triangle in the Cartesian plane with vertices in $\mathbb{Z}^2$ (*lattice vertices*). Show that, if P is an interior lattice point of ABC, then at least one of the angles PAB, PBC, and PCA has a radian measure that is not a rational multiple of π.

11872. *Proposed by Phu Cuong Le Van, College of Education, Hue, Vietnam.* Let f be a continuous function from $[0, 1]$ into $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = 0$. Prove that for all positive integers n there exists $c \in (0, 1)$ such that $n \int_0^c x^n f(x)\,dx = c^{n+1} f(c)$.

SOLUTIONS

A Circumradial Inequality

11735 [2013, 854]. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ.* Let P be a point inside triangle ABC. Let d_A, d_B, and d_C be the distances from P to A, B, and C , respectively. Let r_A, r_B, and r_C be the radii of the circumcircles of PBC, PCA, and PAB, respectively. Prove that

$$\frac{1}{d_A} + \frac{1}{d_B} + \frac{1}{d_C} \geq \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

Solution by Traian Viteam, Cape Town, South Africa. Let O_A, O_B, and O_C be the circumcenters of PBC, PCA, and PAB, respectively. The line $O_A O_B$ is the perpendicular bisector of the common chord PC, and similarly for $O_A O_C$ and $O_B O_C$. Hence P is inside $O_A O_B O_C$. Also, r_A, r_B, and r_C are the distances from P to the vertices of triangle $O_A O_B O_C$, and $d_A/2$, $d_B/2$, and $d_C/2$ are the distances from P to the sides. Therefore the requested inequality is inequality (5) from A. Oppenheim, *The Erdős Inequality and Other Inequalities for a Triangle*, this MONTHLY **68** (1961) 226-230.

Also solved by M. Aassila (France), A. Alt, M. Dincă (Romania), O. Geupel (Germany), B. Karaivanov, J. Minkus, P. Nüesch (Switzerland), R. Stong, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Lucas numbers and roots of unity

11736 [2013, 855]. *Proposed by Mircea Mirca, University of Craiova, Craiova, Romania.* For $n \geq 1$, let f be the symmetric polynomial in variables $x_1, \ldots, x_n$ given by

$$f(x_1, \ldots, x_n) = \sum_{k=0}^{n-1} (-1)^{k+1} e_k(x_1 + x_1^2, x_2 + x_2^2, \ldots, x_n + x_n^2),$$

where e_k is the kth elementary polynomial in n variables. (For example, when $n = 6$, e_2 has 15 terms, each a product of two distinct variables.) Also, let ξ be a primitive nth root of unity. Prove that

$$f(1, \xi, \xi^2, \ldots, \xi^{n-1}) = L_n - L_0,$$

where L_k is the kth Lucas number ($L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 2$).

Solution by Borislav Karaivanov, Lexington, SC. Let

$$g(t_1, \ldots, t_n) = \sum_{k=0}^{n-1} (-1)^{k+1} e_k(t_1, \ldots, t_n) = (-1)^n \prod_{k=1}^{n} t_k - \prod_{k=1}^{n} (1 - t_k),$$

and let

$$p(x, y) = \prod_{k=0}^{n-1} (x - \xi^k y) = x^n - y^n.$$

Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, so that $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$, $\alpha\beta = -1$, and $L_n = \alpha^n + \beta^n$. We compute

$$\begin{aligned} f(1, \xi, \ldots, \xi^{n-1}) &= g(1 + 1, \xi + \xi^2, \ldots, \xi^{n-1} + \xi^{2(n-1)}) \\ &= (-1)^n \prod_{k=0}^{n-1} (\xi^k + \xi^{2k}) - \prod_{k=0}^{n-1} (1 - \xi^k - \xi^{2k}) \\ &= \prod_{k=0}^{n-1} (-\xi)^k \prod_{k=0}^{n-1} (1 + \xi^k) - \prod_{k=0}^{n-1} (1 - \alpha\xi^k) \prod_{k=0}^{n-1} (1 - \beta\xi^k) \\ &= p(0, 1) p(1, -1) - p(1, \alpha) p(1, \beta) \\ &= (-1)\bigl(1 - (-1)^n\bigr) - (1 - \alpha^n)(1 - \beta^n) \\ &= \bigl(-1 + (-1)^n\bigr) - \bigl(1 - \alpha^n - \beta^n + (\alpha\beta)^n\bigr) \\ &= \alpha^n + \beta^n - 2 = L_n - L_0. \end{aligned}$$

Also solved by D. Beckwith, R. Chapman (U. K.), D. Constales (Belgium), I. Gessel, Y. J. Ionin, O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), M. A. Prasad (India), R. Stong, R. Tauraso (Italy), T. Viteam (South Africa), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

An Infinite Matrix Product

11739 [2013, 855]. *Proposed by Fred Adams, Anthony Bloch, and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.* Let $B(x) = \left[\begin{smallmatrix} 1 & x \\ x & 1 \end{smallmatrix}\right]$. Consider the infinite matrix product

$$M(t) = B(2^{-t}) B(3^{-t}) B(5^{-t}) \cdots = \prod_p B(p^{-t}),$$

where the product runs over the primes, taken in increasing order. Evaluate $M(2)$.

Solution by Finbarr Holland, University College, Cork, Cork, Ireland. The value is $\frac{3}{2\pi^2} \left[\begin{smallmatrix} 7 & 3 \\ 3 & 7 \end{smallmatrix}\right]$. Note that $B(x) = QA(x)Q^{-1}$, where

$$A(x) = \begin{bmatrix} 1 + x & 0 \\ 0 & 1 - x \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

For each prime p and each complex number t with $\Re t > 1$, let $x_p = p^{-t}$. We compute

$$Q^{-1} M(t) Q \;=\; \prod_p A(x_p) \;=\; \prod_p \begin{bmatrix} 1 + x_p & 0 \\ 0 & 1 - x_p \end{bmatrix}$$

$$= \begin{bmatrix} \prod_p (1+x_p) & 0 \\ 0 & \prod_p (1-x_p) \end{bmatrix}$$

$$= \begin{bmatrix} \prod_p \left(\frac{1-x_p^2}{1-x_p}\right) & 0 \\ 0 & \prod_p (1-x_p) \end{bmatrix} = \begin{bmatrix} \frac{\zeta(t)}{\zeta(2t)} & 0 \\ 0 & \frac{1}{\zeta(t)} \end{bmatrix},$$

where, by Euler and Riemann,

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t} = \prod_p (1-x_p)^{-1}.$$

Since

$$\zeta(2) = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \frac{\pi^4}{90},$$

it follows that

$$M(2) = Q \begin{bmatrix} \frac{15}{\pi^2} & 0 \\ 0 & \frac{6}{\pi^2} \end{bmatrix} Q^{-1} = \frac{3}{2\pi^2} \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}.$$

Also solved by D. Beckwith, R. Boukharfane (Canada), M. A. Carlton, M. Chamberland, R. Chapman (U.K.), H. Chen, D. Constales (Belgium), C. Degenkolb, C. Delorme (France), E. S. Eyeson, D. Fleischman, O. Furdui (Romania), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), E. A. Herman, J. Iliams, M. Janas, B. Karaivanov, P. Khalili, J. C. Kieffer, J. H. Lindsey II, O. P. Lossers (Netherlands), G. Martin (Canada), R. Martin (Germany), J. Martínez (Spain), R. Molinari, R. Nandan, M. Omarjee (France), É. Pité (France), M. A. Prasad (France), A. J. Rosenthal, C. M. Russell, M. Safaryan (Armenia), E. Schmeichel, N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), N. Thornber, T. Trif (Romania), E. I. Verriest, C. Vignat & V. H. Moll (France & U.S.A.), J. Vinuesa (Spain), T. Viteam (South Africa), Z. Vörös (Hungary), T. Wiandt, L. Zhou, FAU Math Club, GCHQ Problem Solving Group (U.K.), GWstate Problem Solving Group, Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, NSA Problems Group, TCDmath Problem Group (Ireland), and the proposers.

An Infinite Set of Prime Ideals

11740 [2013, 941]. *Proposed by Cosmin Pohoata, Princeton University, Princeton NJ.* Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in a commutative Noetherian ring R with unity. Suppose that $\mathfrak{p} \subset \mathfrak{q}$. Let I be the set of all prime ideals $\mathfrak{j}$ in R such that $\mathfrak{p} \subset \mathfrak{j} \subset \mathfrak{q}$. Prove that I is either empty or infinite.

Solution by Borislav Karaivanov, Lexington, SC. Passing to the quotient ring $R/\mathfrak{p}$ and subsequent localization $(R/\mathfrak{p})_{\mathfrak{q}}$ at the prime ideal $\mathfrak{q}$ allows us to assume, without loss of generality, that R is a local (integral) domain, $\mathfrak{p} = (0)$, and $\mathfrak{q}$ is the only maximal ideal in R.

If I is nonempty, then $\mathfrak{q}$ is not a minimal prime ideal, and hence its height is at least 2. Let $a_1 \in \mathfrak{q}$ be nonzero, and let $\mathfrak{p}_1$ be a minimal prime ideal over the principal ideal (a_1). By Krull's Principal Ideal Theorem, the height of $\mathfrak{p}_1$ is 1. Therefore $\mathfrak{p}_1 \subsetneq \mathfrak{q}$.

We extend the construction by induction. Suppose we have constructed distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n-1}$, each strictly contained in $\mathfrak{q}$. By Krull's Prime Avoidance Lemma, $\bigcup_{l=1}^{n-1} \mathfrak{p}_k \subsetneq \mathfrak{q}$. Let $a_n \in \mathfrak{q} \ \bigcup_{l=1}^{n-1} \mathfrak{p}_k$, and let $\mathfrak{p}_{\mathfrak{n}}$ be a minimal prime ideal over (a_n). Clearly, $\mathfrak{p}_{\mathfrak{n}} \neq \mathfrak{p}_k$ for $1 \leq k \leq n-1$, since only $\mathfrak{p}_n$ contains a_n. By Krull's Principal Ideal Theorem again, the height of $\mathfrak{p}_n$ is 1, while that of $\mathfrak{q}$ is at least 2. Therefore, $\mathfrak{p}_n \subsetneq \mathfrak{q}$. This completes the induction argument, showing that I is infinite.

Editorial comment. Several solvers mentioned that Krull's results – and, in some cases, the stated problem – have appeared in textbooks on commutative algebra by Atiyah-Macdonald (*Introduction to Commutative Algebra*, 1994), Eisenbud *Commutative Algebra with an Eye Towards Algebraic Geometry*, 1995), Isaacs (*Algebra: A Graduate Course*, 1994, mentioned by the proposer himself), and Matsumura (*Commutative Ring Theory*, 1970), and in a survey article by Wiegand and Wiegand in *Ring and Module Theory* (2010).

Also solved by N. Caro (Brazil), R. Chapman (U. K.), O. Geupel (Germany), I. M. Isaacs, O. P. Lossers (Netherlands), J. Rosoff, R. Stong, M. Wildon (U. K.), TCDmath Problem Group (Ireland), and the proposer.

A Homomorphism to the Center?

11741 [2013, 941]. *Proposed by Chindea Filip-Andrei, University of Bucharest, Bucharest, Romania.* Given a ring A, let $Z(A)$ denote the *center* of A, which is the set of all $z \in A$ that commute with every element of A. Prove or disprove: For every ring A, there is a map $F : A \to Z(A)$ such that $f(1) = 1$ and $f(a + b) = f(a) + f(b)$ for all $a, b \in A$.

Solution by O. P. Lossers, Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands. The statement is false.

Let x and y be indeterminates, and consider the ring $B = \mathbb{Q}[x, y]$; its center is $\mathbb{Q}$. Let A be the subring of B generated over $\mathbb{Z}$ by x, y, and $\frac{2x-1}{3^i}$ for $i \in \mathbb{N}$. Note that $Z(A) \subseteq \mathbb{Q}$.

We claim that $Z(A) = \mathbb{Z}$. Assuming this to be true, suppose that $f\colon\ A \to Z(A)$ satisfies the stated conditions, namely $f(1) = 1$ and $f(a + b) = f(a) + f(b)$ for all $a, b \in A$. Observe that $f(2x - 1) = 3^i f\left(\frac{2x-1}{3^i}\right)$ for all i. Since $f(2x - 1)$ and $f((2x - 1)/3^i)$ are both integers, we deduce that $f((2x - 1)/3^i) \in \mathbb{Z}$ for all i, which is only possible if $f(2x - 1) = 0$. It follows that

$$2f(x) = f(2x) = f(2x - 1) + f(1) = 1.$$

This contradicts $f(x)$ being an integer. Hence $Z = \mathbb{Z}$ implies that there is no $f\colon\ A \to Z(A)$ satisfying $f(1) = 1$ and $f(a + b) = f(a) + f(b)$ for all $a, b \in A$.

It thus suffices to prove $Z(a) = \mathbb{Z}$. To show this, it suffices to show that if a polynomial with integral coefficients in x, y, and $\frac{2x-1}{3^i}$ is a rational number z, then it is an integer. If F is such a polynomial, then evaluating F at any $x, y \in \mathbb{Q}$ yields z. Since F is a polynomial, it depends on only finitely many of the values $\frac{2x-1}{3^i}$, say $F = F(x, y, \frac{2x-1}{3}, \frac{2x-1}{3^2}, \dots, \frac{2x-1}{3^n}) = z$. Evaluate F for $y = 0$ and $x = \frac{3^n+1}{2}$, an integer. We see that

$$z = F\left(x, y, \frac{2x-1}{3}, \frac{2x-1}{3^2}, \dots, \frac{2x-1}{3^n}\right) = F\left(\frac{3^n+1}{2}, 0, 3^{n-1}, 3^{n-2}, \dots, 1\right).$$

Since all coefficients of F are integral and all variables have integer values, we conclude that $z \in \mathbb{Z}$. Hence $Z = \mathbb{Z}$, as claimed.

Also solved by R. Stong, M. Towers & M. Wildon (U. K.), and the proposer.

Use the Jacobi Triple Product

11742 [2013, 941]. *Proposed by Alexandr Gromeko, Odessa, Ukraine.* For $0 \le p < q < 1$, find all zeros in $\mathbb{C}$ of the function f given by

$$f(z) = \sum_{n=-\infty}^{\infty} (aq^n, p/(aq^n); p)(-z)^n q^{n(n-1)/2},$$

where $(u, v; w) = \prod_{m=0}^{\infty}(1 - uw^m)(1 - vw^m)$.

Solution by Xinrong Ma, Soochow University, Suzhou, China. We prove that $f(z) = 0$ if and only if $z = q^k$ or $z = -ap^k q^{-k}$ for some $k \in \mathbb{Z}$. Let

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n), \qquad (a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty$$

for $0 < |q| < 1$. These infinite products converge absolutely, so they have value zero only if one of the factors is zero. The Jacobi theta function is defined by

$$\theta(z;q) = \sum_{n=-\infty}^{+\infty} \tau_q(n) z^n, \quad \text{where} \quad \tau_q(n) = (-1)^n q^{n(n-1)/2}.$$

We will use the Jacobi triple product identity:

$$\theta(z;q) = (q, z, q/z; q)_\infty \tag{I}$$

(for example (II.28) in G. Gasper, M. Rahman, *Basic Hypergeometric Series,* 2nd ed., Cambridge University Press, 2004) and the identities

$$\theta(z;q) = \theta(q/z;q) = \tau_q(n) z^n \theta(zq^n;q). \tag{II}$$

(Lemma 2.1.4 in B. C. Berndt, *Number Theory in the Spirit of Ramanujan,* American Mathematical Society, Providence, RI, 2006).

We now compute several expression for $F_p(z) = (p;p)_\infty f(z)$; steps citing 'def' are based on the definition of θ.

$$\begin{aligned}
F_p(z) &= \sum_{n\in\mathbb{Z}} \theta(aq^n;p) z^n \tau_q(n) = \sum_{n\in\mathbb{Z}} z^n \tau_q(n) \sum_{m\in\mathbb{Z}} \left(aq^n\right)^m \tau_p(m) && \text{by (I) and def} \\
&= \sum_{m\in\mathbb{Z}} a^m \tau_p(m) \sum_{n\in\mathbb{Z}} \left(zq^m\right)^n \tau_q(n) && \text{by absolute convergence} \\
&= \sum_{m\in\mathbb{Z}} a^m \tau_p(m) \theta(zq^m;q) = \theta(z;q) \sum_{m\in\mathbb{Z}} \frac{a^m \tau_p(m)}{z^m \tau_q(m)} && \text{by def and (II)} \\
&= \theta(z;q) \sum_{m\in\mathbb{Z}} \left(\frac{-a}{z}\right)^m \tau_{p/q}(m) = \theta(z;q)\theta(-a/z;p/q) && \text{def.}
\end{aligned}$$

Now $(p;p)_\infty \neq 0$, so $f(z) = 0$ if and only if $\theta(z;q)\theta(-a/z;p/q) = 0$. By the Jacobi triple product, this leads to $z = q^k, -ap^k q^{-k}, k \in \mathbb{Z}$, as claimed.

Also solved by R. Chapman (U. K.) and the proposer.

Inverse of a Tridiagonal Toeplitz Matrix

11743 [2013, 941]. *Proposed by François Capacès, Nancy, France.* Let n be a positive integer, let x be a real number, and let B be the n-by-n matrix with $2x$ in all diagonal entries, 1 in all sub- and super-diagonal entries, and 0 in all other entries. Compute the inverse of B, when it exists, as a function of x.

Solution by O. Geupel, Brühl, Germany. We will show that the (i,k)-entry of B^{-1} is

$$\frac{(-1)^{i+k}}{U_n(x)} \cdot U_{\min(i,k)-1} \cdot U_{n-\max(i,k)}$$

for $(i, k) \in [n] \times [n]$, where $U_n(x)$ is the n-th Chebyshev polynomial of the second kind, defined by the recursion

$$U_0(x) = 1, \ U_1(x) = 2x, \quad \text{and} \ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \ \text{for } n \geq 1$$

or by $U_n(\cos\theta) = \sin\big((n+1)\theta\big)/\sin\theta$.

We will write $B = B_n$ to emphasize the dependence on n. It is well-known (or follows easily from expanding along the first row), that $\det B_n = U_n(x)$. For $1 \leq i \leq k \leq n$, the (i, k)-minor of B_n has the block form

$$\det B_n^{(i,k)} = \begin{vmatrix} B_{i-1} & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B_{n-k} \end{vmatrix}$$

where D is a $(k - i) \times (k - i)$ upper triangular matrix with constant diagonal 1. Therefore,

$$\det B_n^{(i,k)} = U_{i-1}(x)U_{n-k}(x).$$

Similarly, for $1 \leq k < i \leq n$, we have $\det B_n^{(i,k)} = U_{k-1}(x)U_{n-i}(x)$. Thus the (i, k)-minor has general form

$$\det B_n^{(i,k)} = U_{\min(i,k)-1}(x)U_{n-\max(i,k)}(x)$$

and the result follows.

Editorial comment. Finding the inverse of a tridiagonal Toeplitz matrix is a standard result. The earliest appearance seems to be D. Moskovitz, "The numerical solution of Laplaces and Poissons equations," *Quart. Appl. Math.*, **2** (1944) 148–163. The explicit solution above is contained in D. Kershawl, "The Explicit Inverses of Two Commonly Occurring Matrices," *Math.Comp.* **23** (1969) no. 105, 189-191.

Also solved by M. Bataille (France), D. Beckwith, R. Chapman (U. K.), D. Constales (Belgium), S. Falcón & Á. Plaza (Spain), J.-P. Grivaux (France), E. A. Herman, S. Hitotumatu (Japan), R. A. Horn, B. Karaivanov, J. C. Kieffer, O. Kouba (Syria), J. H. Lindsey II, M. Omarjee (France), R. E. Prather, M. Safaryan (Armenia), N. C. Singer, R. Stong, J. L. Stuart, R. Tauraso (Italy), N. Thornber, E. I. Verriest, Z. Vörös (Hungary), T. Wiandt, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.

An Easy Inequality

11751 [2014, 83]. *Proposed by Carol Kempiak, Aliso Niguel High School, Aliso Viejo, CA, and Bogdan Suceavă, California State University, Fullerton, CA.* In a triangle with angles of radian measure A, B, and C, prove that

$$\frac{\csc A + \csc B + \csc C}{2} \geq \frac{1}{\sin B + \sin C} + \frac{1}{\sin C + \sin A} + \frac{1}{\sin A + \sin B},$$

with equality if and only if the triangle is equilateral.

Solution by Boris Karaivanov, Lexington, SC. Use the harmonic–arithmetic mean inequality. For any positive x, y, and z,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{\frac{1}{x} + \frac{1}{y}}{2} + \frac{\frac{1}{y} + \frac{1}{z}}{2} + \frac{\frac{1}{z} + \frac{1}{x}}{2} \geq \frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{x+z}$$

with equality if and only if $x = y = z$. Set $x = \sin A$, $y = \sin B$, $z = \sin C$.

Also solved by A. Alt, G. Apostolopoulos (Greece), H. I. Arshagi, R. Bagby, M. Bataille (France), D. M. Bătineţu-Giurgiu & N. Stanciu (Romania), D. Beckwith, E. Braŭne (Austria), R. Chapman (U. K.),

C. T. R. Conley, P. P. Dályáy (Hungary), A. Ercan (Turkey), E. S. Eyeson, D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, E. A. Herman, Y. J. Ionin, S. Kaczkowski, S. H. Kim (Korea), O. Kouba (Syria), P. T. Krasopoulos (Greece), W.-K. Lai & A. Khristyuk, K.-W. Lau (China), O. P. Lossers (Netherlands), R. Mabry, V. Mikayelyan (Armenia), D. J. Moore, Y. Oh (Korea), P. Perfetti (Italy), C. R. Pranesachar (India), M. Safaryan (Armenia), A. Salgarkar (India), E. Schmeichel, C. R. Selvaraj & S. Selvaraj, Y. Shim (Korea), Y. Song (Korea), R. Stong, T. P. Turiel, D. Vacaru (Romania), T. Viteam (South Africa), Z. Vörös (Hungary), M. Vowe (Switzerland), J. Wakem, T. Wiandt, J. Zacharias, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

The Product of n Distinct Derivatives

11753 [2014, 84]. *Proposed by Prapanpong Pongsriiam, Silpakorn University, Nakhon Pathom, Thailand.* Let f be a continuous map from $[0, 1]$ to $\mathbb{R}$ that is differentiable on $(0, 1)$, with $f(0) = 0$ and $f(1) = 1$. Show that for each positive integer n there exist distinct numbers $c_1, \ldots, c_n$ in $(0, 1)$ such that $\prod_{k=1}^{n} f'(c_k) = 1$.

Solution by John W. Hagood, Northern Arizona University, Flagstaff, AZ. The proof proceeds by induction on n. The case $n = 1$ follows from the Mean Value Theorem. Suppose $n > 1$ and the statement holds for $n - 1$. It then follows that there exist distinct $c_1, \ldots, c_{n-1}$ with $f'(c_1) \cdots f'(c_{n-1}) = 1$. If there is a number $c_n \notin \{c_1, c_2, \ldots, c_{n-1}\}$ such that $f'(c_n) = 1$, then the statement (for n) follows.

So, suppose $f'(x) \neq 1$ for all $x \notin \{c_1, c_2, \ldots, c_{n-1}\}$. By the Mean Value Theorem, $f'(c_i) = 1$ for some $i \in \{1, 2, \ldots, n-1\}$. We may assume $i = 1$ by reordering if need be, which leaves $\prod_{k=2}^{n-1} f'(c_i) = 1$. Now let $d_1 < d_2 < \cdots < d_m$ be the points in $(0, 1)$ such that $f'(d_i) = 1$ and put $d_0 = 0$ and $d_{m+1} = 1$. If it were the case that $f'(x) \leq 1$ on $(0, 1)$, then on each interval (d_{i-1}, d_i) we should have $f'(x) < 1$ and thus for each i, that $f(d_i) - f(d_{i-1}) < d_i - d_{i-1}$. This leads to a contradiction: $1 = f(1) - f(0) = \sum_{i=1}^{m+1} (f(d_i) - f(d_{i-1})) < \sum_{i=1}^{m+1} (d_i - d_{i-1}) = d_{m+1} - d_0 = 1$.

Thus f' assumes a value greater than 1 and f' takes on all values in some interval (a, b) with $a < 1 < b$. Choose c_n such that $f'(c_n) \in (\max\{a, 1/b\}, 1)$, but with $f'(c_n) \neq f'(c_i)$ and $f'(c_n) \neq 1/f'(c_i)$, for $2 \leq i \leq n-1$. Since $1/f'(c_n) \in (1, b)$, the intermediate value property assures the existence of a value c_{n+1} such that $f'(c_{n+1}) = 1/f'(c_n)$. Now we have that $\prod_{i=2}^{n+1} f'(c_i) = 1$ and $c_2, \ldots, c_{n+1}$ are distinct. This implies that the statement holds for case n, which completes the induction.

Also solved by M. Aassila (France), I. A. S. Aburub (Jordan), J. Boersema, M. W. Botsko, R. Boukharfane (Canada), P. Budney, N. Caro (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), A. Ercan (Turkey), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, S. J. Herschkorn, E. J. Ionascu, S. Kaczkowski, B. Karaivanov, E. Katsoulis, P. T. Krasopoulos (Greece), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Mabry, R. Martin (Germany), M. D. Meyerson, M. Omarjee (France), N. C. Overgaard (Sweden), V. Pambuccian, S. K. Patel (India), P. Perfetti (Italy), T. Persson & M. P. Sundqvist (Sweden), R. E. Prather, D. Ritter, M. Safaryan (Armenia), R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Viteam (South Africa), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before August 31, 2016. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11901. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* For $n \in \mathbb{Z}^+$, let $[n] = \{1, \ldots, n\}$. Define the functions $\uparrow$ and $\downarrow$ on $[n]$ by $\uparrow x = \min\{x+1, n\}$ and $\downarrow x = \max\{x-1, 1\}$. How many distinct mappings from $[n]$ to $[n]$ occur as compositions of $\uparrow$ and $\downarrow$?

11902. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania* Let $\{x\}$ denote $x - \lfloor x \rfloor$, the fractional part of x. Prove

$$\int_0^1 \int_0^1 \int_0^1 \left(\left\{ \frac{x}{y} \right\} \left\{ \frac{y}{z} \right\} \left\{ \frac{z}{x} \right\} \right)^2 dx\, dy\, dz$$
$$= 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{7\zeta(6)}{48} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{18} + \frac{\zeta(3)\zeta(4)}{12}.$$

11903. *Proposed by Paolo Perfetti, Universitá Degli Studi di Roma "Tor Vergata," Rome, Italy.* Find a homogeneous polynomial p of degree 2 in a, b, c, and d such that for $0 < -d < a < b < c$,

$$\int_0^a \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\, dx = \int_b^c \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\, dx$$

if and only if $\sqrt{-d(a+d)(b+d)(c+d)} = p(a, b, c, d)$.

11904. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let f be a function from $\mathbb{R}$ into $[0, \infty)$ such that $f^2(x+y) + f^2(x-y) = 2f^2(x) + 2f^2(y)$ for all x and y. Prove $f(x+y) \le f(x) + f(y)$ for all x and y.

http://dx.doi.org/10.4169/amer.math.monthly.123.4.399

11905. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* From a point P inside a triangle ABC, the perpendiculars PP_A, PP_B, and PP_C are drawn to its sides. Let R be the circumradius and r the inradius of the triangle. Prove

$$\frac{R}{2r} \le \frac{|PA|\,|PB|\,|PC|}{(|PP_B|+|PP_C|)(|PP_A|+|PP_C|)(|PP_A|+|PP_B|)}.$$

11906. *Proposed by Robert Bosch, Archimedean Academy, FL.* Let x, y, and z be positive numbers such that $xyz = 1$. Prove

$$\sqrt{\frac{x+1}{x^2-x+1}} + \sqrt{\frac{y+1}{y^2-y+1}} + \sqrt{\frac{z+1}{z^2-z+1}} \le 3\sqrt{2}.$$

11907. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.* Let A be an $n \times n$ positive-definite Hermitian matrix, with minimum and maximum eigvenvalues λ and ω, respectively. Prove

$$\left(\frac{1}{\omega}\frac{\mathrm{Tr}(A)}{n} + \frac{\omega n}{\mathrm{Tr}(A)}\right)^n \le \det\left(\frac{1}{\omega}A + \omega A^{-1}\right),$$

$$\left(\frac{1}{\lambda}\frac{n}{\mathrm{Tr}(A^{-1})} + \lambda\frac{\mathrm{Tr}(A^{-1})}{n}\right) \le \det\left(\frac{1}{\lambda}A + \lambda A^{-1}\right).$$

SOLUTIONS

When the *p*-norm is Strictly Decreasing Convex

11749 [2013, 83]. *Proposed by Branko Ćurgus, Western Washington University, Bellingham, WA.* For $\mathbf{x} \in \mathbb{C}^n$ and $p > 0$, let $\|\mathbf{x}\|_p$ denote the standard p-norm on $\mathbb{C}^n$. Prove that the function $p \mapsto \|\mathbf{x}\|_p$ is a strictly decreasing convex function on $(0, \infty)$ if and only if $\mathbf{x}$ is not of the form $c\mathbf{e}_k$, where $\mathbf{e}_k$ denotes the vector with 1 in the kth position and 0 elsewhere.

Solution by John W. Hagood, Northern Arizona University, Flagstaff, AZ. If $\mathbf{x} = c\mathbf{e}_k$, then $\|\mathbf{x}\|_p$ is constant with value $|c|$ and thus is not strictly decreasing. Otherwise, $\|\mathbf{x}\|_p$ has the form $\left(\sum_{k=1}^m a_k^p\right)^{1/p}$, where $2 \le m \le n$ and $a_k > 0$ for each k. Suppose further that we have indexed $a_1, \ldots, a_m$ so that $a_m = \max_{1 \le k \le m} a_k$.

Consider q with $q > p$. Since $a_k/a_m \le 1$ for all k, and $\sum_{k=1}^{m-1}(a_k/a_m)^p + 1 > 1$, we have

$$\|\mathbf{x}\|_p = a_m\left(\sum_{k=1}^{m-1}(a_k/a_m)^p + 1\right)^{1/p} \quad a_m\left(\sum_{k=1}^{m-1}(a_k/a_m)^p + 1\right)^{1/q}$$
$$\ge a_m\left(\sum_{k=1}^{m-1}(a_k/a_m)^q + 1\right)^{1/q} = \|\mathbf{x}\|_q,$$

which proves that the map Φ defined by $\Phi(p) = \|\mathbf{x}\|_p$ is strictly decreasing.

Since Φ is continuous, to show convexity it is enough to prove $\Phi\left(\frac{p+q}{2}\right) \le \frac{1}{2}(\Phi(p) + \Phi(q))$ for $q > p > 0$. By the Cauchy–Schwarz inequality,

$$\sum_{k=1}^m a_k^{(p+q)/2} = \sum_{k=1}^m a_k^{p/2} a_k^{q/2} \le \left(\sum_{k=1}^m a_k^p\right)^{1/2}\left(\sum_{k=1}^m a_k^q\right)^{1/2}.$$

Thus

$$\Phi\left(\frac{p+q}{2}\right) = \left(\sum_{k=1}^{m} a_k^{(p+q)/2}\right)^{2/(p+q)} \leq \left(\sum_{k=1}^{m} a_k^{p}\right)^{1/(p+q)} \left(\sum_{k=1}^{m} a_k^{q}\right)^{1/(p+q)}$$
$$= \|\mathbf{x}\|_p^{p/(p+q)} \|\mathbf{x}\|_q^{q/(p+q)} = \left(\|\mathbf{x}\|_q / \|x\|_p\right)^{q/(p+q)} \|\mathbf{x}\|_p$$
$$< \left(\|\mathbf{x}\|_q / \|\mathbf{x}\|_p\right)^{1/2} \|\mathbf{x}\|_p,$$

where we have used the inequalities $\|\mathbf{x}\|_q/\|\mathbf{x}\|_p < 1$ and $\frac{q}{p+q} > \frac{1}{2}$. Now the AM–GM inequality in the form $r^{1/2} \leq \frac{1}{2}(r+1)$ gives

$$\Phi\left(\frac{p+q}{2}\right) < \frac{1}{2}\left(\|\mathbf{x}\|_q/\|x\|_p + 1\right)\|\mathbf{x}\|_p = \frac{1}{2}(\Phi(p) + \Phi(q)).$$

Editorial comment. This result has appeared before, for example in E. F. Beckenbach, "An inequality of Jensen," THIS MONTHLY **53** (1946) 501–505, which proves more strongly that Φ is log-convex.

Also solved by D. Fleischman E. A. Herman, B. Karaivanov, J. C. Kieffer, O. Kouba (Syria), O. P. Lossers (Netherlands), T. Persson & M. P. Sundqvist (Sweden), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Diagonal coefficients

11757 [2014, 170]. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* Let $[x^a y^b]f(x, y)$ denote the coefficient of $x^a y^b$ in the Taylor series expansion of f. Show that

$$[x^n y^n]\frac{1}{(1-3x)(1-y-3x+3x^2)} = 9^n.$$

Solution I by Richard Stong, Center for Communications Research, San Diego, CA. Let a_n be the desired coefficient. Using the Taylor series expansion

$$\frac{1}{1-y-3x+3x^2} = \sum_{n=0}^{\infty} \frac{y^n}{(1-3x+3x^2)^{n+1}},$$

we obtain

$$a_n = [x^n]\frac{1}{(1-3x)(1-3x+3x^2)^{n+1}} = [x^{-1}]\frac{1}{(1-3x)(x-3x^2+3x^3)^{n+1}}.$$

By the Cauchy integral formula,

$$a_n = \frac{1}{2\pi i}\oint \frac{dz}{(1-3z)(z-3z^2+3z^3)^{n+1}}.$$

In the integral make the substitution $w = z - 3z^2 + 3z^3$. Since $1 - 9w = 1 - 9z + 27z^2 - 27z^3 = (1-3z)^3$, we have $\frac{dw}{1-9w} = \frac{dz}{1-3z}$, so

$$a_n = \frac{1}{2\pi i}\oint \frac{dw}{(1-9w)w^{n+1}} = [w^n]\frac{1}{1-9w} = 9^n.$$

Solution II by Robin Chapman, Mathematics Research Institute, University of Exeter, England, UK As in Solution I, a_n is the coefficient of x^{-1} in the Laurent series

$$h_n(x) = \frac{1}{(1-3x)(x-3x^2+3x^3)^{n+1}}.$$

To prove $a_n = 9^n$, note that $[x^{-1}]\phi'(x) = 0$ when $\phi(x)$ is a Laurent series. Letting $\phi_n(x) = \frac{1}{n(x-3x^2+3x^3)^n}$, we have

$$\begin{aligned} -\phi_n'(x) &= \frac{1-6x+9x^2}{(x-3x^2+3x^3)^{n+1}} \\ &= \frac{1-9x+27x^2-27x^3}{(1-3x)(x-3x^2+3x^3)^{n+1}} = h_n(x) - 9h_{n-1}(x). \end{aligned}$$

Since the coefficient of x^{-1} in $\phi_n'(x)$ is 0, we have

$$0 = [x^{-1}]h_n(x) - 9h_{n-1}(x) = a_n - 9a_{n-1}.$$

Thus $a_n = 9a_{n-1}$ for $n \geq 1$, and $a_0 = 1$, so $a_n = 9^n$.

Editorial comment. The proposer proved a more general result:

$$[x^n y^n]\frac{1}{(1-mx)^{m\alpha+1}\big(Q_m(x)-y\big)} = m^{2n}\binom{\alpha+n}{n}$$

when $Q_m(x) = \big(1-(1-mx)^m\big)/m^2x$. This problem is the case $(m,\alpha) = (3,0)$.

Also solved by D. Beckwith, A. Bostan & L. Dumont & P. Lairez (France), H. Chen, N. Grivaux (France), E. A. Herman, S. Kaczkowski, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), N. C. Singer, R. Tauraso (Italy), M. Vowe (Switzerland), and the proposer.

Twice a Prime Power is Enough

11761 [2014, 266]. *Proposed by Bob Tomper, University of North Dakota, Grand Forks, ND.* For each positive integer n, determine the least integer m such that

$$\operatorname{lcm}\{1,2,\dots,m\} = \operatorname{lcm}\{n,n+1,\dots,m\}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. If $n = 1$ or $n = 2$, then clearly $m = n$. For $n \geq 3$ we claim $m = 2q$, where q is the largest prime power smaller than n. Clearly the desired equality holds if and only if all of $1,\dots,n-1$ divide $\operatorname{lcm}\{n,\dots,m\}$. Since every number less than n factors into prime powers less than n, it suffices to find m such that every prime power less than n divides $\operatorname{lcm}\{n,\dots,m\}$, that is, divides one of $n,\dots,m$. Let q be the largest prime power smaller than n. Since we need a multiple of q in the interval $[n,m]$, we need $m \geq 2q$. By Bertrand's postulate, $2q > n$.

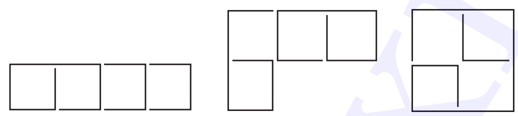
From Nagura's extension of Bertrand's postulate (for $n \geq 25$ there is a prime between n and $6n/5$) and checking some small cases, it follows for $n \notin \{3,7\}$ that $\frac{3n}{4} \leq q < n$, so $2q \geq \frac{3n}{2}$. For a prime power p smaller than n, let k be the largest integer such that $kp < n$. If $k = 1$, then $n \leq 2p \leq 2q$, so $2p$ is a multiple of p in $[n,2q]$. If $k \geq 2$, then $n \leq (k+1)p \leq \frac{3}{2}kp < \frac{3n}{2} \leq 2q$, so $(k+1)p$ is a multiple of p in $[n,2q]$. Thus $m = 2q$ suffices. The cases $n = 3$ and $n = 7$ can be checked explicitly.

Editorial comment. Many solvers failed to show that the interval from n to $2q$ is sufficiently large so that every prime power smaller than q has a multiple in that range. Nagura's extension can be found in J. Nagura, On the interval containing at least one prime number, *Proc. of the Japan Academy*, Series A 28 (1952), 177–181. It is available free at http://projecteuclid.org/euclid.pja/1195570997.

Also solved by N. Caro (Brazil), R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), Y. J. Ionin, B. Karaivanov, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), J. Schlosberg, N. C. Singer, R. Tauraso (Italy), T. Viteam (South Africa), Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Decomposing Partitions into Trails

11762 [2014, 266]. *Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA.* Let $f(n)$ be the least number of strokes needed to draw the Young diagrams of all the partitions of n. The figure below shows three of the five diagrams in an optimal set of drawings (using a total of 12 strokes) when $n = 4$ (the other two are reflections about the line $x + y = 0$ of the first two).



Let

$$F(x) = \sum_{n=1}^{\infty} f(n)x^n = x + 2x^2 + 5x^3 + 12x^4 + 21x^5 + 40x^6 + \cdots .$$

Find the coefficients $g(n)$ of the power series $G(x) = \sum g(n)x^n$ satisfying

$$F(x) = 1 + x + \frac{G(x)}{\prod_{i=1}^{\infty}(1 - x^i)}.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. The Young diagram of a partition can be viewed as a connected plane graph where every vertex has degree 2, 3, or 4. Drawing the diagram with k strokes means decomposing its graph into k trails. For $n = 1$, one trail suffices. For $n \geq 2$, there are always vertices of degree 3. From the characterization of Eulerian circuits in graphs, the minimum number of trails is half the number of vertices of odd degree.

The points of odd degree are the points on the boundaries of the diagram that are not corners. For a given partition λ, the boundary takes $2\lambda_1$ horizontal steps and $2k(\lambda)$ vertical steps, where λ_1 is the largest part and $k(\lambda)$ is the number of parts, so it has $2\lambda_1 + 2k(\lambda)$ vertices. If we follow the boundary clockwise, then the number of right turns is four more than the number of left turns, and the number of left turns is one less than the number of distinct parts of the partition. Thus the number of vertices of odd degree for λ is $2(\lambda_1 + k(\lambda) - d(\lambda) - 1)$, where $d(\lambda)$ is the number of distinct parts.

It follows that for $n \geq 2$,

$$f(n) = \sum_{|\lambda|=n} [\lambda_1 + k(\lambda) - d(\lambda) - 1].$$

Since $n = 0$ takes no strokes and $n = 1$ takes 1, when we introduce

$$\tilde{F}(x) = \sum_{n\geq 0} \left[\sum_{|\lambda|=n} \lambda_1 + k(\lambda) - d(\lambda) - 1 \right] x^n$$

we have $\tilde{F}(x) = F(x) - 1 - x$. Also, $\tilde{F} = G \cdot P$, where $P(x) = \sum_{n\geq 0} p(n)x^n$ and $p(n)$ is the number of partitions of n.

The terms in $f(n)$ yield $\tilde{F} = F_1 + F_2 - F_3 - F_4$. To determine F_1 and F_2, let $q(n, m)$ be the number of partitions of n with largest part m, and let $p(n, m)$ denote the number of partitions with m parts. By reflecting the Young diagram in the main diagonal (taking the "conjugate"), we have $q(n, m) = p(n, m)$, so $F_1 = F_2$. To form a generating function where the exponent on t records the number of parts, we introduce a factor for each part size and let

$$H(x, t) = \sum_{m,n} p(n, m)x^n t^m = \prod_{k=1}^{\infty} \frac{1}{1 - x^k t}.$$

Note that $H(x, 1) = P(x)$. Since each partition of n with m parts contributes m to the coefficient of x^n in $F_2(x)$, we have

$$F_1(x) = F_2(x) = \frac{\partial}{\partial t} H(x, t)|_{t=1} = \sum_{k=1}^{\infty} \frac{x^k}{1 - x^k} \cdot P(x).$$

Now let $F_3 = \sum m\, r(n, m)\, x^n$, where $r(n, m)$ is the number of partitions of n with m distinct parts. Using the same technique, we want the exponent on t to record the number of distinct parts. Let

$$J(x, t) = \sum_{m,n} r(n, m)x^n t^m = \prod_{k=1}^{\infty} \left(1 + \frac{tx^k}{1 - x^k}\right).$$

The factor for k is the contribution from parts equal to k, contributing 1 to the exponent on t when the number of these parts is positive. Note that $J(x, 1) = P(x)$. Again we extract m from the exponent on t:

$$F_3(x) = \frac{\partial}{\partial t} J(x, t)|_{t=1} = \sum_{k=1}^{\infty} x^k P(x) = \frac{x}{1 - x} \cdot P(x).$$

Finally, $F_4(x) = P(x)$, so $\tilde{F} = F_1 + F_2 - F_3 - F_4 = G \cdot P$ with

$$G(x) = \sum_{k=1}^{\infty} \left(\frac{2x^k}{1 - x^k} - \frac{x}{1 - x} - 1 \right).$$

From this we have $g(0) = -1$ and $g(n) = 2\tau(n) - 1$ for $n \geq 1$, where $\tau(n)$ is the number of divisors of n.

Also solved by R. Chapman (U. K.), B. Karaivanov, R. Stong, R. Tauraso (Italy), and the proposer.

A Differential Inequality

11763 [2014, 266]. *Proposed by Bessem Samet, Tunis College of Science and Techniques, Tunis, Tunisia.* Characterize the twice-differentiable, bounded functions f mapping $\mathbb{R}^+$ into itself and satisfying $xg''(x) + (1 + xg'(x))g'(x) \geq 0$ for all x, where $g = \log f$.

Solution by Patrick J. Fitzsimmons, University of California, San Diego, CA.

The only such functions are the constant functions with strictly positive value. Indeed, expressing g' and g'' in terms of f, f', and f'', the inequality imposed on

g is seen to be equivalent to the inequality $xf''(x) + f'(x) \geq 0$ for all $x > 0$. Since $xf''(x) + f'(x) = \frac{d}{dx}\left[xf'(x)\right]$, this means that $xf'(x)$ is a nondecreasing function of x.

Consequently, for fixed $b > 0$, for all $x \in (b, +\infty)$,

$$f(x) - f(b) = \int_b^x f'(t)\,dt = \int_b^x tf'(t) \cdot t^{-1} dt$$
$$\geq bf'(b) \int_b^x t^{-1} dt = bf'(b) \log \frac{x}{b}.$$

Because f is bounded (above), taking $x \to +\infty$ yields $f'(b) \leq 0$. Similarly, for $x \in (0, b)$, $f(x) \geq f(b) - bf'(b)\log(b/x)$, and again because f is bounded, taking $x \to 0$ yields $f'(b) \geq 0$. Thus $f'(b) = 0$ for all $b > 0$, and f is constant as claimed.

Editorial comment. A (subtler) alternate solution: If $\varphi(x, y) = f\left(\sqrt{x^2 + y^2}\right)$ for $(x, y) \neq (0, 0)$, then φ is subharmonic in the sense that $\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 \geq 0$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$. The only subharmonic functions bounded above in the punctured plane are the constant functions.

Also solved by R. Bagby, R. Boukharfane (France), E. A. Herman, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), N. C. Singer, R. Stong, E. I. Verriest, NSA Problems Group, and the proposer.

Reciprocal Catalan Sums

11765 [2014, 267]. *Proposed by David Beckwith, Sag Harbor, NY.*

Let C_n be the nth Catalan number, given by $C_n = \frac{1}{n+1}\binom{2n}{n}$. Show that:

(**a**) $\sum_{n=0}^{\infty} \frac{2^n}{C_n} = 5 + \frac{3}{2}\pi$;

(**b**) $\sum_{n=0}^{\infty} \frac{3^n}{C_n} = 22 + 8\sqrt{3}\pi$.

Solution by Ulrich Abel, Technische Hochschule Mittelhessen, University of Applied Sciences, Germany. Using the beta integral

$$\int_0^1 t^m (1-t)^n \, dt = \frac{m!\,n!}{(m+n+1)!},$$

for $|x| < 4$ we have

$$\sum_{n=0}^{\infty} \frac{x^n}{C_n} = 1 + \sum_{n=1}^{\infty} n(n+1)\frac{(n-1)!\,n!}{(2n)!}x^n = 1 + \sum_{n=1}^{\infty} n(n+1)x^n \int_0^1 t^{n-1}(1-t)^n\,dt$$
$$= 1 + \int_0^1 \sum_{n=1}^{\infty} n(n+1)x^n t^{n-1}(1-t)^n\,dt,$$

where the interchange of summation and integration is justified by uniform convergence. Using $\sum_{n=1}^{\infty} n(n+1)z^n = 2z/(1-z)^3$, we obtain

$$\sum_{n=0}^{\infty} \frac{x^n}{C_n} = 1 + \int_0^1 \frac{2x(1-t)}{(1-xt(1-t))^3}\,dt.$$

Direct calculation of the integral yields

$$\sum_{n=0}^{\infty} \frac{x^n}{C_n} = 1 - (x-10)x(4-x)^{-2} + 24\sqrt{x}(4-x)^{-5/2} \arctan\sqrt{\frac{x}{4-x}},$$

and setting $x = 1$ and $x = 2$ gives the desired formulas for (**a**) and (**b**).

Editorial comment. Several solvers noted that (**a**) and (**b**) follow easily from results in D. H. Lehmer, *Interesting series involving the central binomial coefficient*, THIS MONTHLY **92** (1985) 449–457:

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} = \frac{\pi}{2} + 1, \quad \sum_{n=1}^{\infty} \frac{n2^n}{\binom{2n}{n}} = \pi + 3$$

$$\sum_{n=1}^{\infty} \frac{3^n}{\binom{2n}{n}} = 2\nu + 3, \quad \sum_{n=1}^{\infty} \frac{n3^n}{\binom{2n}{n}} = 10\nu + 18,$$

where $\nu = 2\pi\sqrt{3}/3$.

As noted by several solvers, the formula

$$\sum_{n=0}^{\infty} \frac{x^n}{C_n} = 1 + \frac{x}{4-x} + \frac{6x}{(4-x)^2} + \frac{24\sqrt{x}}{(4-x)^{5/2}} \arcsin \frac{\sqrt{x}}{2}$$

is given by Thomas Koshy and Zhenguang Gao, *Convergence of a Catalan series*, College Math. J. **43** (2012) 141–146. Koshy and Gao also gave the formulas $\sum_{n=0}^{\infty} 1/C_n = 2 + 4\pi/9\sqrt{3}$ and

$$\sum_{n=0}^{\infty} (-1)^n/C_n = \frac{14}{25} - \frac{24\sqrt{5}}{125} \log\left(\frac{1+\sqrt{5}}{2}\right),$$

and Omran Kouba gave the additional formulas

$$\sum_{n=0}^{\infty} (-2)^n/C_n = \frac{1}{3} - \frac{1}{3\sqrt{3}} \log(2+\sqrt{3}),$$

$$\sum_{n=0}^{\infty} (-3)^n/C_n = \frac{10}{49} - \frac{36}{49\sqrt{21}} \log\left(\frac{5+\sqrt{21}}{2}\right).$$

An alternative formula for $\sum_{n=0}^{\infty} x^n/C_n$,

$$2\frac{\sqrt{4-x}(8+x) + 12\sqrt{x}\arctan\left(\frac{\sqrt{x}}{\sqrt{4-x}}\right)}{\sqrt{(4-x)^5}}$$

can be found on the `planetmath.org` web site.

Also solved by T. Amdeberhan & L. Jiu & V. H. Moll & C. Vignat, R. Bagby, M. Bataille (France), R. Boukharfane (Canada), K. N. Boyadzhiev, P. Bracken, B. Bradie, R. Chapman (U. K.), H. Chen, D. Constales (Belgium), B. E. Davis, C. Delorme (France), E. Deutsch, A. Ercan (Turkey), E. S. Eyeson, O. Furdui (Romania), M. L. Glasser, M. Goldenberg & M. Kaplan, N. Grivaux (France), E. A. Herman, B. Karaivanov, O. Kouba (Syria), H. Kwong, O. P. Lossers (Netherlands), L. M. J. Martinez (Spain), R. Molinari, M. Omarjee (France), P. Perfetti (Italy), A. Plaza (Spain), C. R. Pranesachar (India), R. Pratt, J. Schlosberg, A. Stenger, R. Stong, R. Tauraso (Italy), T. Trif (Romania), D. B. Tyler, M. Vowe (Switzerland), J. Zacharias, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, TCDmath Problem Group (Ireland), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before Sept 30, 2016. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11908. *Proposed by George. E. Andrews, The Pennsylvania State University, University Park, PA, and Emeric Deutsch, Polytechnic Institute of New York University, Brooklyn, NY.* Let n and k be nonnegative integers. Show that the number of partitions of n having k even parts is the same as the number of partitions of n in which the largest repeated part is k (defined to be 0 if the parts are all distinct). For example, 7 has three partitions with two even parts $(4+2+1 = 3+2+2 = 2+2+1+1+1)$ and also three partitions in which the largest repeated part is 2: $(3+2+2 = 2+2+2+1 = 2+2+1+1+1)$.

11909. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Prove that for every positive integer m there exists a polynomial P_m in two variables, with integer coefficients, such that for all integers n and r with $0 \le r \le n$,

$$\sum_{k=-r}^{r} \binom{n}{r+k}\binom{n}{r-k} k^{2m} = \frac{P_m(n,r)}{\prod_{j=1}^{m}(2n-2j+1)}\binom{2n}{2r}.$$

11910. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Let G_k be the reciprocal of the kth Fibonacci number; for example, $G_4 = 1/3$ and $G_5 = 1/5$. Find

$$\sum_{n=1}^{\infty} \left(\arctan G_{4n-3} + \arctan G_{4n-2} + \arctan G_{4n-1} - \arctan G_{4n}\right).$$

11911. *Proposed by Leonard Giugiuc, Drobotu Turnu Severin, Romania.* Let a, b, and c be positive numbers such that $1 + ab + bc + ca = a + b + c + 2abc$. Prove $a^3 + b^3 + c^3 + 5abc \ge 1$ and determine when equality holds.

http://dx.doi.org/10.4169/amer.math.monthly.123.5.504

11912. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let ω be the circumscribed circle of triangle ABC, and let R and r be the radii of its circumcircle and incircle, respectively. Let r_A, r_B, and r_C be the radii of the A-, B-, and C-mixtilinear incircles of ABC and ω, respectively. Prove that $4r \le r_A + r_B + r_C \le \frac{1}{4}(5R + 6r)$. (For the definition of a mixtilinear incircle see problem **11774**; that problem and its solution are found on the next page of this issue.)

11913. *Proposed by George Stoica, Saint John, New Brunswick, Canada.* Let ε be a positive constant, and let f map $(0, \infty)$ to $\mathbb{R}^+$. Given $\lim_{x\to\infty} x^{1/\varepsilon} f(x) = \infty$, prove

$$\liminf_{x\to\infty} \left| \frac{f'(x)}{f^{1+\varepsilon}(x)} \right| = 0.$$

11914. *Proposed by Robin Chapman, Mathematics Research Institute, University of Exeter, Exeter, (U. K.), and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Show that for all positive integers m and n,

$$\sum_{k=1}^{n} (-4)^{-k} \binom{n-k}{k-1} \sum_{j=1}^{3m} (-2)^{-j} \binom{n+1-2k}{j-1} \binom{m-k}{3m-j} = 0.$$

(Here $\binom{x}{k} = \frac{1}{k!} \prod_{i=0}^{k-1} (x - i)$ for $x \in \mathbb{R}$.)

SOLUTIONS

Compositions Having At Least One 1

11767 [2014, 267]. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Prove that

$$\sum \frac{(1 + t_1 + t_2 + \cdots + t_n)!}{(1 + t_1)!\, t_2! \cdots t_n!} = 2^n - F_n,$$

where the sum is over all nonnegative integer solutions to $t_1 + 2t_2 + \cdots + nt_n = n$ and F_k is the kth Fibonacci number.

Solution I by CMC 328, Carleton College, Northfield, MN. View the sum as over all partitions of $n + 1$ having at least one 1, treating $t_1 + 1$ as the number of copies of 1 and t_j as the number of copies of j for $2 \le j \le n$. The summand counts the ways to permute the parts, so the sum is the number of compositions of $n + 1$ having at least one 1.

The number of compositions of $n + 1$ is 2^n, so it suffices to prove that the number a_n of compositions of $n + 1$ with no 1 is F_n. This is clear for $n = 0$ and $n = 1$. When $n \ge 2$, these compositions have last part 2 or greater than 2. Deleting the last part shows that there are a_{n-2} of the first type, and subtracting 1 from the last part shows that there are a_{n-1} of the second type. By induction, $a_n = a_{n-1} + a_{n-2} = F_n$.

Solution II by Borislav Karaivanov, Lexington, SC. Rewrite the sum as

$$\sum \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1!\, t_2! \cdots t_n!},$$

summed over all integer solutions to $t_1 + 2t_2 + \cdots + nt_n = n + 1$ with $t_1 \ge 1$ and $t_i \ge 0$ for $i \ge 2$. This sum is the coefficient of x^{n+1} in the series

$$f(x) = \sum_{m=0}^{\infty} \left((x + x^2 + \cdots)^m - (x^2 + x^3 + \cdots)^m \right)$$

$$= \sum_{m=0}^{\infty} \left[\left(\frac{x}{1-x} \right)^m - \left(\frac{x^2}{1-x} \right)^m \right]$$

$$= \frac{1-x}{1-2x} - \frac{1-x}{1-x-x^2} = \frac{x(1-x)^2}{(1-2x)(1-x-x^2)}.$$

Hence we seek the coefficient of x^n in

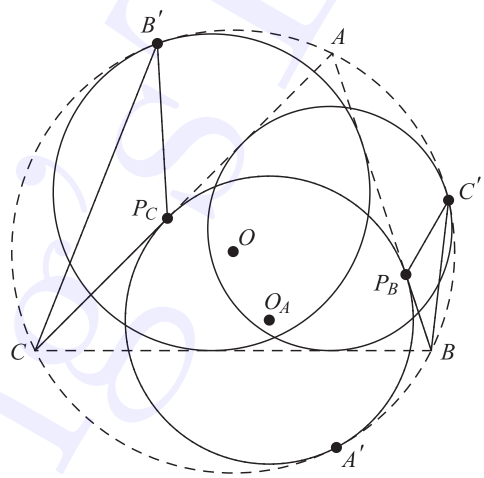
$$\frac{f(x)}{x} = \frac{(1-x)^2}{(1-2x)(1-x-x^2)} = \frac{1}{1-2x} - \frac{x}{1-x-x^2}.$$

The coefficient subtracted in the second term is the number of 1, 2-lists with sum $n-1$, well known to be F_n, so the answer is $2^n - F_n$.

Also solved by R. Bagby, D. Beckwith, R. Chapman (U. K.), M. Hoffman, Y. J. Ionin, O. P. Lossers (Netherlands), R. Martin (Germany), R. Molinari, M. Omarjee (France), N. C. Singer, J. H. Smith, R. Stong, R. Tauraso (Italy), T. Viteam (South Africa), T. Woodcock, GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

Mixtilinear Incircles

11774 [2015, 366]. *Proposed by Yunus Tunçbilek, Ataturk High School of Science, Istanbul, Turkey and Danny Lee, Herkimer Senior High School, New York, NY.* Let ω be the circumscribed circle of triangle ABC. The A-*mixtilinear incircle* of ABC and ω is the circle that is internally tangent to ω, AB, and AC, and similarly for B and C. Let A', P_B, and P_C be the points on ω, AB, and AC, respectively, at which the A-mixtilinear incircle touches. Define B' and C' in the same manner that A' was defined. (See figure.)



Prove that triangles $C'P_BB$ and CP_CB' are similar.

Solution by Radouan Boukharfane (student), Poitiers, France. Let a, b, and c be the sidelengths of ABC, and let s be its semiperimeter.

Lemma. *Let X be a point on the side AB of triangle ABC, and let Y be a point on the arc AB (not containing C) of the circumcircle ω of ABC. The rays CX and CY are isogonal in $\angle ACB$ if and only if $\frac{AX}{XB} \cdot \frac{AY}{YB} = \frac{AC}{BC}$.*

Proof. Suppose that CX and CY are isogonal, that is, $\angle XCA = \angle BCY$ and $\angle XCB = \angle ACY$. We also have $\angle CAX = \angle CAB = \angle CYB$ and $\angle CBX = \angle CBA = \angle CYA$

since they subtend the same arcs on ω. Thus we have similar triangles $CXA \sim CBY$ and $CXB \sim CAY$. Hence

$$\frac{AX}{YB} = \frac{AC}{YC} \text{ and } \frac{AY}{XB} = \frac{YC}{BC}.$$

The product of these is the claimed formula. For the converse, both the isogonality of CX and CY and the ratio $\frac{AX}{XB}$ uniquely determine a point X on side AB. ■

Let I map the extended plane by inverting through the circle with center C and radius $\sqrt{ab}$ and then reflecting across the angle bisector of BAC. Note that I swaps C with the point at infinity and swaps A with B. Hence it swaps line CA with CB and swaps line AB with the circumcircle ω of ABC. It also swaps the C-mixtilinear incircle with the C-excircle. Thus I swaps the tangency point C' of the C-mixtilinear incircle with ω and the tangency point, call it Q, of the C-excircle with AB. It follows that the rays CC' and CQ are isogonal; that is in $\angle ACB$. Thus by the lemma above

$$\frac{|BC|'}{|AC|'} = \frac{|AQ|}{|QB|} \cdot \frac{|BC|}{|AC|} = \frac{a(s-b)}{b(s-a)},$$

where we have used the well-known formulas $|AQ| = s - b$ and $|QB| = s - a$.

Furthermore, I swaps the tangency point, call it D, of the C-mixtilinear incircle with CA with the tangency point of the C-excircle with CB. This last tangency point is well known to be at distance s from C. It follows that $|CD| \cdot s = ab$. Hence $|CD| = \frac{ab}{s}$ and $|DA| = b - |CD| = \frac{b(s-a)}{s}$. Thus $\frac{|DA|}{|CD|} = \frac{s-a}{a}$.

Denote the points where the A- and B-mixtilinear incircles are tangent with AB by P_B and E, respectively. Analogs of the result of the previous paragraph yield

$$\frac{|BP_B|}{|P_BA|} \cdot \frac{|BE|}{|EA|} = \frac{s-b}{b} \cdot \frac{a}{s-a} = \frac{|BC'|}{|AC'|}.$$

Now consider the homothety with center B' that takes the B-mixtilinear incircle to ω. This map takes line AB, which is tangent to the B-mixtilinear incircle, to a parallel tangent to ω. Hence its image is the tangent to ω at the midpoint of arc AB. Since this tangency point is the image of E under the homothety, it follows that $B'E$ contains the midpoint of arc AB or, equivalently, that $B'E$ bisects $\angle AB'B$. The angle bisector theorem now yields $\frac{|BB'|}{|B'A|} = \frac{|BE|}{|EA|}$, and this gives

$$\frac{|BP_B|}{|P_BA|} \cdot \frac{|BB'|}{|B'A|} = \frac{|BC'|}{|AC'|}.$$

By the lemma above (applied to triangle ABC'), the rays $C'P_B$ and $C'B'$ are isogonal in $\angle AC'B$, and hence $\angle B'C'A = \angle BC'P_B$. Angles $\angle C'BP_B = \angle C'BA$ and $\angle C'B'A$ are also congruent since they subtend the same arc of ω. Hence, we see that triangles $C'P_BB$ and $C'AB'$ are similar. Analogously, triangles CP_CB' and $C'AB'$ are similar. Hence triangles $C'P_BB$ and CP_CB' are similar.

Also solved by C. Delorme (France), C. R. Pranesachar (India), R. Stong, H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposers.

A Partition Inequality

11775 [2014, 455]. *Proposed by Isaac Sofair, Fredericksburg, VA.* Let $A_1, \ldots, A_k$ be finite sets. For $J \subseteq \{1, \ldots, k\}$, let $N_J = \left|\bigcup_{j \in J} A_j\right|$, and let $S_m = \sum_{J:|J|=m} N_J$.

(**a**) Express in terms of $S_1, \ldots, S_k$ the number of elements that belong to exactly m of the sets $A_1, \ldots, A_k$.
(**b**) Same question as in (**a**), except that we now require the number of elements belonging to at least m of the sets $A_1, \ldots, A_k$.

Solution by Mark Meyerson, Naval Academy, Annapolis, MD.

(**a**) Let T_m be the desired value; we prove $T_m = \sum_{i=1}^{k}(-1)^{k+i+m+1}\binom{i}{k-m}S_i$. If an element x belongs to exactly i of $A_1, \ldots, A_k$, then x contributes $\binom{k}{m} - \binom{k-i}{m}$ to S_m. Therefore,

$$S_m = \sum_{i=1}^{k}\left(\binom{k}{m} - \binom{k-i}{m}\right)T_i.$$

It suffices to show that the inverse of the $k \times k$ matrix A with (m,i)-entry $\binom{k}{m} - \binom{k-i}{m}$ is the $k \times k$ matrix B with (s,m)-entry $(-1)^{k+m+s+1}\binom{m}{k-s}$ (interpreting $\binom{n}{j}$ as 0 when $j < 0$ or $j > n$). To see this, we compute the (s,i)-entry of BA:

$$\begin{aligned}
&\sum_{m=1}^{k}(-1)^{k+m+s+1}\binom{m}{k-s}\left(\binom{k}{m} - \binom{k-i}{m}\right)\\
&\quad = \sum_{m=0}^{k}(-1)^{k+m+s+1}\binom{m}{k-s}\binom{k}{m} - \sum_{m=0}^{k}(-1)^{k+m+s+1}\binom{m}{k-s}\binom{k-i}{m}\\
&\quad = \binom{k}{k-s}\sum_{m=0}^{k}(-1)^{k+m+s+1}\binom{s}{m-k+s} + \binom{k-i}{k-s}\sum_{m=0}^{k}(-1)^{k+m+s}\binom{s-i}{m-k+s}.
\end{aligned}$$

Since the alternating sum of a row of Pascal's triangle (other than the first) vanishes, the first sum in the last expression vanishes, as does the second except when $s = i$, in which case it is 1. Thus BA is the identity matrix.

(**b**) For the desired value U_m, we compute

$$\begin{aligned}
U_m &= \sum_{j=m}^{k} T_j = \sum_{j=m}^{k}\sum_{i=1}^{k}(-1)^{k+i+j+1}\binom{i}{k-j}S_i\\
&= \sum_{i=1}^{k}\sum_{j=m}^{k}(-1)^{k+i+j+1}\binom{i}{k-j}S_i = \sum_{i=1}^{k}(-1)^{k+i+m+1}\binom{i-1}{k-m}S_i,
\end{aligned}$$

where the last equality comes from $\sum_{j=m}^{k}(-1)^{j}\binom{i}{k-j} = (-1)^{m}\binom{i-1}{k-m}$, which is proved by induction on $k-m$.

Also solved by D. Beckwith, B. S. Burdick, R. Chapman (U. K.), Y. J. Ionin, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), Y. Shim (Korea), J. C. Smith, R. Stong, R. Tauraso (Italy), TCDmath Problem Group (Ireland), and the proposer.

A Line of Urns

11776 [2014, 455]. *Proposed by David Beckwith, Sag Harbor, NY.* Given urns $U_1, U_2, \ldots, U_n$ in a line, and plenty of identical blue and identical red balls, let a_n be the number of ways to put balls into the urns subject to the conditions that

(i) each urn contains at most one ball,

(ii) any urn containing a red ball is next to exactly one urn containing a blue ball, and

(iii) no two urns containing a blue ball are adjacent.

(**a**) Show that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1+t+2t^2}{1-t-t^2-3t^3}.$$

(**b**) Show that

$$a_n = \sum_{j\ge 0}\sum_{m\ge 0} 4^j \left[\binom{n-2m}{j}\binom{m}{j} + \binom{n-2m-1}{j}\binom{m}{j} + 2\binom{n-2m}{j}\binom{m-1}{j}\right].$$

Here $\binom{k}{l} = 0$ if $k < l$.

Solution by James Christopher Smith, Knoxville, TN.

(**a**) By explicit counting, $a_0 = 1$, $a_1 = 2$, $a_2 = 5$, and $a_3 = 10$. View solutions as strings of length n using E, B, R for empty, blue, and red, respectively, with e_n, b_n, r_n counting those beginning E, B, or R. Always $a_n = e_n + b_n + r_n$, and $e_n = a_{n-1}$ for $n \ge 1$. Also $r_n = b_{n-1}$ for $n \ge 2$. For $n \ge 3$, the solutions beginning B consist of e_{n-1} beginning BEE, BEB, or BER, plus e_{n-2} beginning BRE, plus r_{n-2} beginning BRR. Thus $b_n = e_{n-1} + e_{n-2} + r_{n-2}$, and

$$\begin{aligned} a_n &= e_n + b_n + r_n = a_{n-1} + e_{n-1} + e_{n-2} + r_{n-2} + b_{n-1} \\ &= a_{n-1} + a_{n-2} + a_{n-3} + b_{n-3} + e_{n-2} + e_{n-3} + r_{n-3} \\ &= a_{n-1} + a_{n-2} + a_{n-3} + b_{n-3} + a_{n-3} + e_{n-3} + r_{n-3} \\ &= a_{n-1} + a_{n-2} + 3a_{n-3}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(1-t-t^2-3t^3\right)\sum_{n=0}^{\infty} a_n t^n &= \sum_{n=0}^{\infty} a_n t^n - \sum_{n=1}^{\infty} a_{n-1} t^n - \sum_{n=2}^{\infty} a_{n-2} t^n - 3\sum_{n=3}^{\infty} a_{n-3} t^n \\ &= 1 + t + 2t^2 + \sum_{n=3}^{\infty} (a_n - a_{n-1} - a_{n-2} - 3a_{n-3}) t^n = 1 + t + 2t^2. \end{aligned}$$

(**b**) Use the identity $\sum_{m\ge 0}\binom{m}{k} t^m = t^k/(1-t)^{k+1}$ to obtain

$$\begin{aligned} \sum_{n\ge 0}\left(\sum_{m\ge 0}\binom{n-2m}{j}\binom{m}{j}\right) t^n &= \left(\sum_{m\ge 0}\binom{m}{j} t^{2m}\right)\left(\sum_{n\ge 0}\binom{n}{j} t^n\right) \\ &= \left(\frac{t^{2j}}{(1-t^2)^{j+1}}\right)\left(\frac{t^j}{(1-t)^{j+1}}\right) = \frac{1}{(1-t)(1-t^2)}\left(\frac{t^3}{(1-t)(1-t^2)}\right)^j. \end{aligned}$$

It follows that

$$\sum_{n\ge 0}\left(\sum_{j\ge 0}\sum_{m\ge 0} 4^j \binom{n-2m}{j}\binom{m}{j}\right) t^n = \sum_{j\ge 0} 4^j \sum_{n\ge 0}\left(\sum_{m\ge 0}\binom{n-2m}{j}\binom{m}{j}\right) t^n$$

$$= \frac{1}{(1-t)(1-t^2)} \sum_{j\geq 0} \left(\frac{4t^3}{(1-t)(1-t^2)} \right)^j$$

$$= \frac{1}{(1-t)(1-t^2)} \left(\frac{1}{1 - \frac{4t^3}{(1-t)(1-t^2)}} \right) = \frac{1}{1-t-t^2-3t^3}.$$

Hence, in the expansion of $1/(1-t-t^2-3t^3)$, the coefficients of t^{n-1} and t^{n-2} are, respectively,

$$\sum_{j\geq 0}\sum_{m\geq 0} 4^j \binom{n-1-2m}{j}\binom{m}{j} \quad \text{and} \quad \sum_{j\geq 0}\sum_{m\geq 0} 4^j \binom{n-2-2m}{j}\binom{m}{j}.$$

Shifting the index for m in the last expression and summing the various contributions now yields (**b**).

Also solved by R. Chapman (U. K.), M. Funkhouser, O. Geupel (Germany), O. Kouba (Syria), O. P. Lossers (Netherlands), Y. Shim (Korea), R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

The Beast

11777 [2014, 456]. *Proposed by Marian Dincă, Bucharest, Romania.* Let $x_1, \ldots, x_n$ be real numbers such that $\prod_{k=1}^n x_k = 1$. Prove that

$$\sum_{k=1}^n \frac{x_k^2}{x_k^2 - 2x_k\cos(2\pi/n) + 1} \geq 1.$$

Solution by Mazen Zarrouk, Montgomery College, Takoma Park, MD. When $n = 1$, the inequality becomes $\frac{1}{0} \geq 1$, which makes sense if we take $\frac{1}{0} = +\infty$. The inequality is not true for $n = 2$, as can be seen by taking $x_1 = x_2 = 1$. In the following it will be shown that the inequality is true for $n \geq 3$.

We will use the Shapiro inequality: If $y_i \geq 0$ for $1 \leq i \leq n$, with $y_{n+1} = y_1$ and $y_{n+2} = y_2$, then

$$\sum_{k=1}^n \frac{y_k}{y_{k+1}+y_{k+2}} \geq \begin{cases} n/2, & \text{for even } n \text{ at most 12 or odd } n \text{ at most 23,} \\ 0.49n, & \text{for all other } n. \end{cases}$$

(Reference: V. G. Drinfel'd, A cyclic inequality, *Math. Notes. Acad. Sci. USSR* **9** (1971) 68–71. H. S. Shapiro, **Monthly** Problem 4603, **61** (1954) 571. http://mathworld.wolfram.com/ShapiroCyclicSumConstant.html.)

Lemma. *Fix $n \in \mathbb{N}$ with $n \geq 4$. If $x_1, \ldots, x_n$ are positive real numbers with product 1, then*

$$\sum_{k=1}^n \left(\frac{x_k}{x_k+1} \right)^2 \geq 1.$$

Proof. Let $y_k = \prod_{j=k}^n x_j$ for $1 \leq k \leq n$, with $y_{n+1} = y_1$ and $y_{n+2} = y_2$. Note that $y_k > 0$ and $x_k = y_k/y_{k+1}$ for $1 \leq k \leq n$. Also,

$$\sum_{k=1}^n \frac{x_k}{x_k+1} = \sum_{k=1}^n \frac{y_k}{y_k+y_{k+1}} = \sum_{k=1}^n \frac{y_{k+1}}{y_{k+1}+y_{k+2}}$$

$$= \sum_{k=1}^{n} \frac{y_{k+1} - y_k}{y_{k+1} + y_{k+2}} + \sum_{k=1}^{n} \frac{y_k}{y_{k+1} + y_{k+2}} \geq \sum_{k=1}^{n} \frac{y_k}{y_{k+1} + y_{k+2}}. \qquad (1)$$

Note that since $y_k > 0$, setting $t = \max_{1 \leq k \leq n}\{y_{k+1} + y_{k+2}\} > 0$ yields

$$\sum_{k=1}^{n} \frac{y_{k+1} - y_k}{y_{k+1} + y_{k+2}} \geq \frac{1}{t} \sum_{k=1}^{n} (y_{k+1} - y_k) = \frac{1}{t}(y_{n+1} - y_1) = 0.$$

Thus, omitting this sum leads to the stated inequality in (1). Using the quadratic mean–arithmetic mean inequality, we obtain

$$\sum_{k=1}^{n} \left(\frac{x_k}{x_k + 1}\right)^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} \frac{x_k}{x_k + 1}\right)^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} \frac{y_k}{y_{k+1} + y_{k+2}}\right)^2.$$

The result now follows by applying the Shapiro inequality. ■

We now return to the original problem and prove the case $n = 3$, for which the lemma is not needed. Let x, y, z be real numbers such that $xyz = 1$. With $z = 1/xy$, the case $n = 3$ becomes

$$\frac{x^2}{x^2 + x + 1} + \frac{y^2}{y^2 + y + 1} + \frac{1}{x^2y^2 + xy + 1} \geq 1,$$

which is equivalent to

$$\frac{\frac{1}{4}(2x^2y^2 - x - y)^2 + \frac{3}{4}(x - y)^2}{(x^2 + x + 1)(y^2 + y + 1)(x^2y^2 + xy + 1)} \geq 0.$$

For each factor in the denominator, we have $t^2 + t + 1 = (t + \frac{1}{2})^2 + \frac{3}{4} > 0$. The desired inequality follows. This completes the case $n = 3$.

Now we consider the case $n \geq 4$. Let $x_1, x_2, \ldots, x_n$ be real numbers with product 1. For $1 \leq k \leq n$,

$$0 < 1 - \cos^2\left(\frac{2\pi}{n}\right) \leq x_k^2 - 2x_k \cos\left(\frac{2\pi}{n}\right) + 1 \leq x_k^2 + 2|x_k| + 1 = \big(|x_k| + 1\big)^2.$$

Applying Lemma 1 to $|x_1|, \ldots, |x_n|$, we obtain the required inequality

$$\sum_{k=1}^{n} \frac{x_k^2}{x_k^2 - 2x_k \cos(2\pi/n) + 1} \geq \sum_{k=1}^{n} \left(\frac{|x_k|}{|x_k| + 1}\right)^2 \geq 1.$$

Also solved by M. Aassila (France), P. P. Dályay (Hungary), D. Fleischman, Y. J. Ionin, O. P. Lossers (Netherlands), P. Perfetti (Italy), R. E. Prather, J. C. Smith, N. Stanciu (Romania), A. Stenger, R. Stong, R. Tauraso (Italy), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal, nor posted to the internet before the due date for solutions. Submitted solutions should arrive before Nov 30, 2016. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11915. *Proposed by Mark E. Kidwell and Mark D. Meyerson, U.S. Naval Academy, Annapolis, MD.* Given four points A, B, C, and D in order on a line in Euclidean space, under what conditions will there be a point P off the line such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure?

11916. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Universitá di Roma "Tor Vergata," Rome, Italy.* Show that if n, r, and s are positive integers, then

$$\binom{n+r}{n}\sum_{k=0}^{s-1}\binom{r+k}{r-1}\binom{n+k}{n} = \binom{n+s}{n}\sum_{k=0}^{r-1}\binom{s+k}{s-1}\binom{n+k}{n}.$$

11917. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let A be a 2×2 matrix with integer entries and both eigenvalues less than 1 in absolute value. Prove that $\log(I-A)$ has integer entries if and only if $A^2=0$. (Here $\log(I-X) = -X - X^2/2 - X^3/3 - \cdots$ when that sum converges.)

11918. *Proposed by Le Van Phu Cuong, College of Education, Hue University, Hue City, Vietnam.* Let f be n times continuously differentiable on $[0, 1]$, with $f(1/2)=0$ and $f^{(i)}(1/2)=0$ when i is even and at most n. Prove

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{(2n+1)2^{2n}(n!)^2}\int_0^1 f^{(n)}(x))^2\,dx.$$

11919. *Proposed by Arkady Alt, San Jose, CA.* For positive integers m and k with $k\ge 2$, prove

$$\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}(\min\{i_1,\dots,i_k\})^m = \sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}((n+1)^i - n^i)\sum_{j=1}^{n} j^{k+m-i}.$$

http://dx.doi.org/10.4169/amer.math.monthly.123.6.613

11920. *Proposed by Ángel Plaza and Sergio Falcón, University of Las Palmas de Gran Canaria, Spain.* For positive integer k, let $\langle F_k \rangle$ be the sequence defined by initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and the recurrence $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$. Find a closed form for $\sum_{i=0}^{n} \binom{2n+1}{i} F_{k,2n+1-2i}$.

11921. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Prove

$$\log^2(2) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)2^{k+1}} + \log(2) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2 2^k} + \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3 2^k}$$
$$= \frac{1}{4} \left((\zeta(4) + \log^4(2) \right).$$

(Here $H_k = \sum_{j=1}^{k} 1/j$ and ζ denotes the Riemann zeta function.)

SOLUTIONS

A Partition Inequality

11772 [2014, 366]. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let n be a positive integer. Prove that the number of integer partitions of $2n + 1$ that do not contain 1 as a part is less than or equal to the number of integer partitions of $2n$ that contain at least one odd part.

Solution by Jeffrey Olson, Norwich University, Northfield, VT. When p is a partition of $2n + 1$ with no parts equal to 1, let $\phi(p)$ be the partition of $2n$ obtained by replacing a smallest part k by $k - 1$ copies of 1. Note that ϕ is injective; p can be recovered from $\phi(p)$ by combining the copies of 1 into one copy of k if $\phi(p)$ has $k - 1$ copies of 1. This proves a stronger result: For each positive integer m, the number of partitions of $m + 1$ having no 1 is at most the number of partitions of m in which the number of copies of 1 is positive and less than the smallest other part.

Also solved by D. Beckwith, P. P. Dályay (Hungary), O. Geupel (Germany), B. Karaivanov, M. Krebs, K. Kusejko (Switzerland), O. P. Lossers (Netherlands), R. Martin (Germany), R. Molinari, R. E. Prather, E. Schmeichel, Y. Shim (Korea), A. V. Sills, N. C. Singer, R. Stong, J. Swenson, R. Tauraso (Italy), Z. Vörös (Hungary), E. A. Weinstein, H. Widmer (Switzerland), CMC 328, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

A Powerful Inequality

11780 [2014, 456]. *Proposed by Cézar Lupu, University of Pittsburgh, Pittsburgh, PA, and Tudorel Lupu, Decebal High School, Constanţa, Romania.* Let f be a positive-valued, concave function on [0, 1]. Prove that

$$\frac{3}{4} \left(\int_0^1 f(x)\, dx \right)^2 \leq \frac{1}{8} + \int_0^1 f^3(x)\, dx.$$

Solution by Roberto Tauraso, Università di Roma, Rome, Italy.
Note that for $t \geq 0$,

$$t^3 - \frac{3}{4}t^2 + \frac{1}{16} = \frac{1}{16}(4t + 1)(2t - 1)^2 \geq 0,$$

with a relative minimum at $t = 1/2$.

Hence, since $f(x)$ is nonnegative, it follows from this inequality that

$$\int_0^1 \left(f^3(x) - \frac{3}{4} f^2(x) + \frac{1}{16} \right) dx \ge 0.$$

Moving the middle term to the right and using the Cauchy–Schwarz inequality yields

$$\int_0^1 \left(f^3(x) + \frac{1}{16} \right) dx \ge \frac{3}{4} \int_0^1 f^2(x)\, dx \ge \frac{3}{4} \left(\int_0^1 f(x)\, dx \right)^2.$$

This inequality is stronger than the required inequality for all $f(x)$ for which the integrals exist. Thus, the condition that f be concave is not necessary, and equality holds if and only if $f \equiv 1/2$.

Also solved by R. A. Agnew, R. Bagby, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), A. Ercan (Turkey), P. J. Fitzsimmons, E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), P. Perfetti (Italy), J. C. Smith, A. Stenger, R. Stong, E. I. Verriest, H. Wang, GCHQ Problem Solving Group (U. K.), and the proposers.

An Application of Prime Density

11781 [2014, 456]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For $n \ge 2$, call a positive integer n*-smooth* if none of its prime factors is larger than n. Let S_n be the set of all n-smooth positive integers. Let C be a finite, nonempty set of nonnegative integers, and let a and d be positive integers. Let M be the set of all positive integers of the form $m = \sum_{k=1}^{d} c_k s_k$, where $c_k \in C$ and $s_k \in S_n$ for $k = 1, \dots, d$. Prove that there are infinitely many primes p such that $p^a \notin M$.

Solution by Reiner Martin, Bad Soden-Neuenhain, Germany. For a positive integer r and a prime p, the number of powers of p bounded by r is at most $1 + \ln r/\ln p$, which is in $O(\ln r)$. As the number of primes bounded by k is less than k, the number of n-smooth integers bounded by r is $O((\ln r)^n)$. Thus, the number of elements in M bounded by r is $O((\ln r)^{dn})$, where the constant depends only on n.

On the other hand, for fixed a the number of prime powers p^a bounded by r is the number of primes up to $r^{1/a}$, which is asymptotic to $ar^{1/a}/\ln r$ by the prime number theorem. Since for $c > 0$ we have

$$\lim_{r\to\infty} \left(\frac{ar^{1/a}}{\ln r} - c(\ln r)^{dn} \right) = \infty,$$

the claim follows.

Editorial comment. Christian Elsholtz and the proposer noted that the claim is true also when C may contain negative integers.

Also solved by R. Chapman (U. K.), C. Elsholtz (Austria), Y. J. Ionin, O. P. Lossers (Netherlands), R. Stong, NSA Problems Group, and the proposer.

Sparse binary representations

11782 [2014, 549]. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* A *signed binary representation* of an integer m is a finite list $a_0, a_1, \dots$ of elements of $\{-1, 0, 1\}$ such that $\sum a_l 2^l = m$. A signed binary representation is *sparse* if no two consecutive entries in the list are nonzero.
(**a**) Prove that every integer has a unique sparse representation.
(**b**) Prove that for all $m \in \mathbb{Z}$, every nonsparse signed binary representation of m has at least as many nonzero terms as the sparse representation.

Composite solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands, and Mark D. Meyerson, U. S. Naval Academy, Annapolis, MD.

We first show by induction on k that every integer having a sparse representation with k nonzero terms has a unique sparse representation (trailing 0s are appended as needed). When $k = 0$ we can represent only 0, and since the partial sum through $a_n 2^n$ has absolute value less than 2^{n+1} there is no other signed representation of 0.

For the induction step with $k > 0$, consider a sparse representation of m with k nonzero terms. Let a_r be the first nonzero term. Since $a_{r+1} = 0$, we have $m \equiv a_r 2^r \pmod{2^{r+2}}$. Thus every sparse representation of m has the same first nonzero term and is obtained by adding $a_r 2^r$ to a sparse representation of $m - a_r 2^r$, which by the induction hypothesis is unique.

Every integer has at least one signed representation (the usual binary representation or its negative). Hence to prove both (**a**) and (**b**) it suffices to obtain a sparse representation from any signed representation of m without increasing t, the number of nonzero terms. A nonsparse representation has a sublist of the form $1, 1, 0$ or $-1, 1$ or their negatives. Changing $1, 1, 0$ to $-1, 0, 1$ and $-1, 1$ to $1, 0$ neither changes the sum nor increases t, and similarly for their negatives. If such a change does not decrease t, then it moves a zero toward the front of the list. Thus finitely many steps yield a sparse representation with no more nonzero terms than the original representation.

Editorial comment. Donald Knuth and John C. Kieffer observed that this result was proved by George W. Reitwiesner [Binary arithmetic, *Advances in Computers*, Vol. 1, Academic Press, 1960, pp. 231–308], and a short proof is given in Knuth's *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, exercise 4.1-34. Kieffer also noted that U. Güntzer and M. Paul [Jump interpolation search trees and symmetric binary numbers, Inform. Process. Lett. 26 (1987), 193–204] showed that the sparse binary representation (also known as the *nonadjacent form* or *balanced binary representation*) of m arises by subtracting the usual binary representation of $m/2$ from the usual binary representation of $3m/2$. A short proof of this appears in Knuth's *The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 1*, exercise 7.1.3-35.

Also solved by A. Ali (India), R. Bagby, K. Banerjee, D. Beckwith, C. Blatter (Switzerland), G. Brown, N. Caro (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, J. Freeman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, D. Gove, E. A. Herman, E. J. Ionaşcu, Y. J. Ionin, B. Karaivanov, J. C. Kieffer, J. H. Lindsey II, R. Martin (Germany), J. Schlosberg, Y. Shim (Korea), J. C. Smith, J. H. Smith, R. Stong, H. Takeda (Japan), R. Tauraso (Italy), T. Viteam (India), J. Wakem, E. A. Weinstein, GCHQ Problem Solving Group (U. K.), NSA Problems Group, TCDmath Problem Group (Ireland), and the proposer.

Enumerating Subsets by Odd Runs

11785 [2014, 550]. *Proposed by Bhaskar Bagchi, India Statistics Institute, Bangalore, India.*

Let $[n] = \{1, \dots, n\}$. For a subset A of $[n]$, a *run* of A is a maximal subset of A consisting of consecutive integers. Let $O(A)$ denote the number of runs of A with an odd number of elements, and let $\mu(A) = \frac{1}{2}(|A| + O(A))$. (For instance, if $n = 9$ and $A = \{1, 3, 4, 5, 8, 9\}$, then A has three runs, $O(A) = 2$, and $\mu(A) = 4$.)

(**a**) Show that if $0 \le k \le n$ and $k/2 \le i \le k$, then the number $N_{i,k}$ of subsets A of $[n]$ such that $\mu(A) = i$ and $|A| = k$ is given by

$$N_{i.k} = \binom{n-i}{k-i}\binom{n-k+1}{2i-k}.$$

(**b**)* Prove or disprove that if m is a positive integer and $m+1 \le k \le 2m$, then the number of subsets A of $[3m+1]$ such that $|A| = k$ and $\mu(A) \le m$ is equal to the number of subsets B of $[3m+1]$ such that $|B| = 3m+1-k$ and $\mu(B) > m$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA.

(**a**) View $N_{i,k}$ as the number of k-subsets A in $[n+1]$ that omit $n+1$ and satisfy $\mu(A) = i$. Let $(a_1, \dots, a_{n+1}) \in \{0,1\}^{n+1}$ be the indicator vector for A. Encode A by a string using U, V, and W as follows: Begin with U if $(a_1, a_2) = (1, 0)$, with V if $(a_1, a_2) = (1, 1)$, and with W if $a_1 = 0$. Delete a_1 from the vector in the latter case; delete a_1 and a_2 in the first two cases. Repeat the process until the vector is exhausted.

Each run of length l becomes a string of length $\lceil l/2 \rceil$, all V except for ending in U if l is odd. Thus U appears $O(A)$ times and V occurs $(|A| - O(A))/2$ times. Together, they appear $\mu(A)$ times, each accounting for two positions in the vector, so W appears $n+1-2i$ times. Since $n+1 \notin A$, the string cannot end in the symbol V.

Conversely, any (U, V, W)-string with frequencies $(2i-k, k-i, n+1-2i)$ not ending in V encodes a desired subset. With $n-i+1$ entries in total, there are $\binom{n-i}{k-i}$ ways to place the V symbols and then $\binom{n-k+1}{2i-k}$ ways to place the U symbols, yielding the claimed formula for $N_{i,k}$.

(**b**) We prove that equality holds. Let $h(x, y, z) = \sum_{i,k,n} N_{i,k}(n) x^i y^k z^n$. For a particular string counted by $N_{i,k}$, each U, V, or W, respectively, contributes xyz^2, xy^2z^2, or z, since the exponents are their contributions to i, k, and n. The forced final W or U contributes options z or xyz^2, and we divide that by z so the string contributes to the coefficient of z^n rather than z^{n+1}. Before that the string has r terms in $\{U, V, W\}$, where $r \ge 0$. Thus

$$h(x, y, z) = (1 + xyz) \sum_r \left(z + xyz^2 + xy^2z^2\right)^r = \frac{1 + xyz}{1 - z - xyz^2 - xy^2z^2}.$$

We now find the generating functions for the two counts in (**b**). The number of subsets A of $[n]$ with $\mu(A) \le m$ and $|A| = k$ is the coefficient of $x^m y^k z^n$ in

$$f(x, y, z) = \frac{h(x, y, z)}{1 - x} = \frac{1 + xyz}{(1 - x)(1 - z - xyz^2 - xy^2z^2)}.$$

Restricting to $n = 3m + 1$ gives the coefficient of $x^m y^k$ in

$$F(x, y) = \frac{1}{2\pi i} \oint \frac{f(x/z^3, y, z)}{z^2}\, dz = \frac{1}{2\pi i} \oint \frac{xy + z^2}{(xy + xy^2 - z + z^2)(x - z^3)}\, dz,$$

where we interpret x and y as being small and the contour as being around $|z| = 1/2$. The only residue outside this contour is at $z = \left(1 + \sqrt{1 - 4xy - 4xy^2}\right)/2$, so, after some computation, we obtain

$$\begin{aligned} F(x, y) &= \left.\frac{xy + z^2}{(x - z^3)(2z - 1))}\right|_{z=\left(1+\sqrt{1-4xy-4xy^2}\right)/2} \\ &= \frac{1 - y + (1 + y)\sqrt{1 - 4xy - 4xy^2}}{2(1 - x(1 + y)^3)\sqrt{1 - 4xy - 4xy^2}}. \end{aligned}$$

Given $t(x) = \sum t_n x^n$, let $u_n = \sum_{k>n} t_k$ and $u(x) = \sum u_n x^n$. By clearing fractions and equating coefficients, it is easy to show that $u(x) = \frac{t(1) - t(x)}{1 - x}$. Applying this with $t(x) = h(x, y, z)$ in which y and z are treated as constants, the number of subsets B of

$[n]$ with $\mu(B) > m$ and $|B| = n - k$ is the coefficient of $x^m y^{n-k} z^n$ in $\frac{h(1,y,z)-h(x,y,z)}{1-x}$. By computation,

$$\frac{h(1, y, z) - h(x, y, z)}{1 - x} = \frac{yz}{(1 - z - yz)(1 - z - xyz^2 - xy^2z^2)}.$$

Hence we seek the coefficient of $x^m y^k z^n$ in $g(x, y, z)$, where

$$g(x, y, z) = \frac{z}{(1 - z - yz)(1 - yz - xyz^2 - xz^2)},$$

obtained by replacing y with $1/y$ and z with yz in the previous expression. Again specializing to $n = 3m + 1$ gives the coefficient of $x^m y^k$ in

$$F(x, y) = \frac{1}{2\pi i} \oint \frac{g(x/z^3, y, z)}{z^2}\, dz = \frac{1}{2\pi i} \oint \frac{dz}{(1 - z - yz)(z - x - xy - yz^2)}$$

along the same contour $|z| = 1/2$. The only residue inside this contour is at $z = (1 - \sqrt{1 - 4xy - 4xy^2})/(2y)$, so we get

$$G(x, y) = \left.\frac{1}{(1 - z - yz)(1 - 2yz))}\right|_{z=\left(1-\sqrt{1-4xy-4xy^2}\right)/(2y)}$$

$$= \frac{1 - y + (1 + y)\sqrt{1 - 4xy - 4xy^2}}{2(1 - x(1 + y)^3)\sqrt{1 - 4xy - 4xy^2}}.$$

The two generating functions agree, proving (**b**).

Part (a) also solved by D. Beckwith, R. Chapman (U. K.), O. Geupel (Germany), Y. J. Ionin, O. P. Lossers (Netherlands), and the proposer. No other solutions to (b) were received.

A Series Identity Involving Partitions and Divisors

11787 [2014, 550]. *Proposed by Mircea Merca, University of Craiova, Romania.* Prove that

$$\sum_{k=1}^{\infty} (-1)^{k-1} k p_k \left(n - \frac{1}{2}k(k+1)\right) = \sum_{k=-\infty}^{\infty} (-1)^k \tau \left(n - \frac{1}{2}k(3k-1)\right).$$

Here $p_k(n)$ denotes the number of partitions of n in which the greatest part is less than or equal to k (with $p_k(0) = 1$ and $p_k(n) = 0$ for $n < 0$) and $\tau(n)$ is the number of divisors of n (with $\tau(n) = 0$ for $n \le 0$).

Solution by O.P. Lossers, Eindhoven University of Technology, The Netherlands. Let $\tilde{p}_k(n)$ be the number of partitions of the integer n into *exactly* k parts, all distinct. Such partitions correspond bijectively to arbitrary partitions of $n - \frac{1}{2}k(k+1)$ with at most k parts, sending λ to λ' by subtracting i from the ith smallest part in λ, for $1 \le i \le k$. Also, partitions of $n - \frac{1}{2}k(k+1)$ with at most k parts correspond to partitions of $n - \frac{1}{2}k(k+1)$ with largest part at most k (by conjugation). Thus it suffices to show

$$\sum_{k=1}^{\infty} (-1)^{k-1} k \tilde{p}_k(n) = \sum_{k=-\infty}^{\infty} (-1)^k \tau \left(n - \frac{1}{2}k(3k-1)\right).$$

For fixed n, let a_n and b_n denote the left side and right side, respectively. We prove equality of the generating functions: $\sum_{n\ge0} a_n x^n = \sum_{n\ge0} b_n x^n$. With $P(t, x) = \sum_{k,n\ge0}, \tilde{p}_k(n) t^k x^n$, we have $\sum_{n\ge0} a_n x^n = \frac{\partial}{\partial t} P(t, x)\big|_{t=-1}$. To generate distinct parts and count 1 in the exponent on t for each part generated, we have $P(t, x) = \prod_{m=1}^{\infty} (1 + tx^m)$. Thus

$$\sum_{n \geq 0} a_n x^n = \frac{\partial}{\partial t} P(t, x)\Big|_{t=-1} = \left(\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} \right) \prod_{n=1}^{\infty} (1 - x^n).$$

We compute

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x^{mn} = \sum_{m,n \,:\, mn=k} x^k = \sum_{n=1}^{\infty} \tau(n) x^n,$$

and Euler's pentagonal theorem yields

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2} n(3n-1)}.$$

These computations combine to prove the desired result:

$$\sum_{n \geq 0} a_n x^n = \left(\sum_{n=1}^{\infty} \tau(n) x^n \right) \cdot \left(\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2} n(3n-1)} \right) = \sum_{n \geq 0} b_n x^n.$$

Also solved by R. Chapman (U. K.), K. Kusejko (Switzerland), R. Tauraso (Italy), and the proposer.

A Log Square-Root Inequality

11788 [2014, 550]. *Proposed by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece.* Let n be a positive integer, and suppose $0 < y_i \leq x_i < 1$ for $1 \leq i \leq n$. Prove that

$$\frac{\log x_1 + \cdots + \log x_n}{\log y_1 + \cdots + \log y_n} \leq \sqrt{\frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_n}{1 - y_n}}.$$

Solution by Borislav Karaivanov, Lexington, SC. First we prove the case $n = 1$, which is equivalent to $\log x_1 / \sqrt{1 - x_1} \geq \log y_1 / \sqrt{1 - y_1}$. We claim that the function f defined by $f(x) = \log x / \sqrt{1 - x}$ is strictly increasing on $(0, 1)$. The derivative is $f'(x) = \frac{x \log x + 2 - 2x}{2x(1-x)^{3/2}}$; the denominator $2x(1 - x)^{3/2}$ is positive on $(0, 1)$; and the numerator $g(x) = x \log x + 2 - 2x$ is positive since its derivative $g'(x) = \log x - 1$ is negative and $g(1) = 0$. Thus f is strictly increasing on $(0, 1)$, which proves the inequality. Equality holds only if $x_1 = y_1$.

Now consider $n \geq 2$. Rewrite and apply the case $n = 1$ to get

$$\frac{\log x_1 + \cdots + \log x_n}{\log y_1 + \cdots + \log y_n} \leq \sqrt{\frac{1 - x_1 \cdots x_n}{1 - y_1 \cdots y_n}} < \sqrt{\frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_n}{1 - y_n}}.$$

Here the last inequality is obtained by induction using the inequality

$$\frac{1 - x_1 x_2}{1 - y_1 y_2} < \frac{(1 - x_1) + (1 - x_2)}{1 - y_1 y_2} = \frac{1 - x_1}{1 - y_1 y_2} + \frac{1 - x_2}{1 - y_1 y_2} < \frac{1 - x_1}{1 - y_1} + \frac{1 - x_2}{1 - y_2}$$

for $x_1, x_2, y_1, y_2 \in (0, 1)$. Note that equality is never attained when $n \geq 2$.

Editorial comment.

Roberto Tauraso proved the inequality not only for the square-root (power $1/2$) but for any power t where $t \in [0, 1]$.

Also solved by M. Bataille (France), R. Chapman (U. K.), P. P. Dályay (Hungary), M. Dincă (Romania), J.-P. Grivaux (France), H. D. Gyu (Korea), E. A. Herman, Q. Hu (China), B. Karaivanov, O. Kouba (Syria), X. Lai (China), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), P. Perfetti (Italy), I. Pinelis, B. P. Pinto & W. Zhao, Á. Plaza & F. Perdomo (Spain), J. C. Smith, R. Stong, H. Takeda (Japan), R. Tauraso (Italy), Z. Vörös (Hungary), Z.-H. Yang (China), and the proposer.

A Series with Zetas

11793 [2014, 648]. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Prove that

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} = -\zeta'(2) + \sum_{n=3}^{\infty} (-1)^{n+1} \frac{\zeta(n)}{n-2},$$

where ζ denotes the Riemann zeta function and ζ' denotes its derivative.

Solution by FAU Problem Solving Group, Florida Atlantic University, Boca Raton, FL. The equality to be proved is the consequence of a change in the order of summation, using the standard formulas

$$\zeta'(s) = -\sum_{n=2}^{\infty} \frac{\log n}{n^s} \quad (s > 1), \quad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \quad (-1 < x \le 1).$$

Note that for $x = 1$, the second series converges only conditionally.

With $S = \sum_{n=3}^{\infty} (-1)^{n+1} \frac{\zeta(n)}{n-2}$, we now have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+2)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=1}^{\infty} \frac{1}{k^{n+2}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=2}^{\infty} \frac{1}{k^{n+2}} \\
&= \log 2 + \sum_{k=2}^{\infty} \frac{1}{k^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{nk^n} = \log 2 + \sum_{k=2}^{\infty} \frac{1}{k^2} \log\left(1 + \frac{1}{k}\right) \\
&= \log 2 + \sum_{k=2}^{\infty} \frac{1}{k^2} \log(k+1) - \sum_{k=2}^{\infty} \frac{1}{k^2} \log k = \sum_{k=1}^{\infty} \frac{1}{k^2} \log(k+1) + \zeta'(2).
\end{aligned}$$

Here the change of order of summation is justified by absolute convergence of the sum. The equality to be proved follows.

Editorial comment.

Submissions that omitted justifying the interchange of the order of summation were considered insufficient.

Also solved by R. Bagby, O. Furdui (Romania), C. Georghiou (Greece), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), E. A. Herman, O. Kouba (Syria), O. P. Lossers (Netherlands), J. C. Smith, A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems should be submitted online via http://www.americanmathematicalmonthly.submittable.com/submit. *Proposed solutions to the problems below should be submitted on or before January 31, 2017 at the same link. More detailed instructions are available online. Solutions to problems numbered 11921 or below should continue to be submitted via e-mail to monthlyproblems@math.tamu.edu. Proposed problems must not be under consideration concurrently to any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11922. *Proposed by Max Alekseyev, George Washington University, Washington, DC* Find every positive integer n such that both n and n^2 are palindromes when written in the binary numeral system (and with no leading zeros).

11923. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let f_p be the function on $(0, \pi/2)$ given by

$$f_p(x) = (1 + \sin x)^p - (1 - \sin x)^p - 2\sin(px).$$

Prove $f_p > 0$ for $0 < p < 1/2$ and $f_p < 0$ for $1/2 < p < 1$.

11924. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Calculate

$$\int_0^{\pi/2} \frac{\{\tan x\}}{\tan x}\,dx,$$

where $\{u\}$ denotes $u - \lfloor u \rfloor$.

11925. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let n be an integer with $n \geq 4$. Find the largest k such that for any list a of n real numbers that sum to 0,

$$\left(\sum_{j=1}^{n} a_j^2\right)^3 \geq k\left(\sum_{j=1}^{n} a_j^3\right)^2.$$

http://dx.doi.org/10.4169/amer.math.monthly.123.7.722

11926. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let k be an integer, $k \geq 2$. Find

$$\int_0^\infty \frac{\log|1-x|}{x^{(1+1/k)}}\,dx.$$

11927. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let O, G, I, and K be, respectively, the circumcenter, centroid, incenter, and symmedian point (also called Lemoine point or Grebe point) of triangle ABC. Prove $|OG| \leq |OI| \leq |OK|$, with equality if and only if ABC is equilateral.

11928. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For positive integers n and m and for a sequence $\langle a_i \rangle$, prove

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}a_{i+j} = \sum_{k=0}^{n+m}\binom{n+m}{k}a_k$$

and

$$\sum_{i<j}\binom{n}{i}\binom{n}{j}\binom{i+j}{n} = \sum_{i<j}\binom{n}{i}\binom{n}{j}^2.$$

SOLUTIONS

Special Multiples of an Integer

11789 [2014, 648]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.*

Let a and k be positive integers. Prove that for every positive integer d there exists a positive integer n such that d divides $ka^n + n$.

Solution by Mark Wildon, Royal Holloway, University of London, Egham, U. K.

We shall show by induction on d that there are infinitely many solutions. If $d = 1$, then any $n \in \mathbb{N}$ is a solution. Consider $d > 1$.

Suppose first that a is not a unit modulo d. Choose a prime p dividing $\gcd(a, d)$ and exponents α and δ such that $a = a'p^\alpha$ and $d = d'p^\delta$, where a' and d' are not divisible by p. Let $n = p^\delta m$. When m is sufficiently large, $ka^n + n \equiv n \equiv 0 \pmod{p^\delta}$. Therefore, $ka^n + n \equiv 0 \pmod{d}$ if and only if

$$ka^{p^\delta m} + p^\delta m \equiv 0 \pmod{d'},$$

or, equivalently, if and only if

$$\ell b^m + m \equiv 0 \pmod{d'},$$

where $b = a^{p^\delta}$ and $\ell \in \mathbb{N}$ is chosen so that $k \equiv \ell p^\delta \pmod{d'}$. By the induction hypothesis, we find infinitely many choices for m.

In the remaining case, $d > 1$ and a is a unit modulo d. Let $c = \gcd(\phi(d), d)$. By the induction hypothesis, there exists $m \in \mathbb{N}$ such that $ka^m + m \equiv 0 \pmod{c}$. Let $ka^m + m = rc$, where $r \in \mathbb{N}$, and choose $s, t \in \mathbb{N}$ so that $c = s\phi(d) + td$. Set

$n = m - rs\phi(d) + \lambda\phi(d)d$, where $\lambda \in \mathbb{N}$ is chosen so that $n \in \mathbb{N}$. Now $a^{\phi(d)} \equiv 1 \pmod{d}$, and hence,

$$\begin{aligned} ka^n + n &= ka^{m-rs\phi(d)+\lambda\phi(d)d} + m - rs\phi(d) + \lambda\phi(d)d \\ &\equiv ka^m + m - rs\phi(d)d \\ &= r(c - s\phi(d)) + \lambda\phi(d)d = rtd + +\lambda\phi(d)d, \end{aligned}$$

where the congruence is modulo d. Hence, d divides $ka^n + n$, as required. We now obtain infinitely many solutions by varying λ.

Also solved by J. H. Lindsey II, B. Maji (India), M. Omarjee (France), K. Razminia (Iran), N. Safei (Iran), J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), R. Viteam (India), University of Louisiana at Lafayette Math Club, and the proposers.

A Lacunary Recurrence for Bernoulli Numbers

11791 [2014, 648]. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.*

Show that for $r \geq 1$,

$$\sum_{s=1}^{r} \binom{6r+1}{6s-2} B_{6s-2} = -\frac{6r+1}{6},$$

where B_n denotes the nth Bernoulli number.

Solution by Allen Stenger, Boulder, CO. Let ρ be a primitive sixth root of unity so that $\rho^3 = -1$ and $\rho^2 - \rho + 1 = 0$. Recall that the nth Bernoulli polynomial $B_n(x)$ is defined by $B_n(x) = \sum_{s=0}^{n} \binom{n}{s} B_s x^{n-s}$. We first prove

$$\sum_{s=1}^{r} \binom{6r+1}{6s-2} B_{6s-2} = \frac{1}{6} \sum_{m=0}^{5} (-1)^m B_{6r+1}(\rho^m). \tag{1}$$

We know that $\sum_{m=0}^{5} \rho^{am}$ is 6 if a is divisible by 6 and is 0 otherwise. By the definition of Bernoulli polynomials, we have

$$\begin{aligned} \sum_{m=0}^{5} (-1)^m B_{6r+1}(\rho^m) &= \sum_{m=0}^{5} \rho^{-3m} \sum_{k=0}^{6r+1} \binom{6r+1}{k} B_k \cdot (\rho^m)^{6r+1-k} \\ &= \sum_{k=0}^{6r+1} \binom{6r+1}{k} B_k \sum_{m=0}^{5} \rho^{m(6r-2-k)} \\ &= 6 \sum_{\substack{0 \leq k \leq 6r+1 \\ 6 \mid k+2}} \binom{6r+1}{k} B_k = 6 \sum_{s=1}^{r} \binom{6r+1}{6s-2} B_{6s-2}, \end{aligned}$$

which proves (1).

From the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!},$$

it is easy to prove the well-known facts that $B_n(0) = B_n(1) = 0$ when n is odd and exceeds 1 and that $B_n(x+1) - B_n(x) = nx^{n-1}$ for all n. Thus, $B_n(-1) = -n$ when n is odd and exceeds 1. Now

$$\sum_{m=0}^{5}(-1)^m B_{6r+1}(\rho^m)$$
$$= 0 - (B_{6r+1}(\rho) - B_{6r+1}(\rho^2)) + (6r+1) + (B_{6r+1}(\rho^4) - B_{6r+1}(\rho^5)).$$

Using $\rho^2 = \rho - 1$, we evaluate the two grouped terms on the right as

$$\begin{aligned} B_{6r+1}(\rho) - B_{6r+1}(\rho^2) &= B_{6r+1}(\rho^2+1) = B_{6r+1}(\rho^2) \\ &= (6r+1)(\rho^2)^{6r} = 6r+1 \end{aligned}$$

and

$$\begin{aligned} B_{6r+1}(\rho^4) - B_{6r+1}(\rho^5) &= B_{6r+1}(-\rho) - B_{6r+1}(-\rho^2) \\ &= B_{6r+1}(-\rho) - B_{6r+1}(1-\rho) \\ &= -(6r+1)(-\rho)^{6r} = -(6r+1). \end{aligned}$$

Combining these facts yields

$$\sum_{m=0}^{5}(-1)^m B_{6r+1}(\rho^m) = -(6r+1).$$

Editorial comment. As noted by several solvers, this formula was proved by S. Ramanujan, *Some properties of Bernoulli's numbers*, J. Indian Math. Soc. 3 (1911), 219–234, and further work on related identities can be found in D. H. Lehmer, *Lacunary recurrence formulas for the numbers of Bernoulli and Euler*, Ann. of Math. (2) 36 (1935), 637–649 and F. T. Howard, *A general lacunary recurrence formula*, in Applications of Fibonacci Numbers, Vol. 9, Kluwer Acad. Publ., Dordrecht, 2004, pp. 121–135.

Also solved by U. Abel (Germany), D. Beckwith, R. Chapman (U.K.), C. Georghiou (Greece), F. T. Howard, B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), R. Tauraso (Italy), C. Vignat & V. H. Moll, M. Vowe (Switzerland), and GCHQ Problem Solving Group (U.K.).

A Functional Equation

11794 [2014, 648]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Find every twice differentiable function f on $\mathbb{R}$ such that for all nonzero x and y, $xf(f(y)/x) = yf(f(x)/y)$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We assume only that f is once differentiable. We claim that the solutions are $f(x) = a(x+a)$, where $a \in \mathbb{R}$.

The functional equation

$$xf\left(\frac{f(y)}{x}\right) = yf\left(\frac{f(x)}{y}\right) \tag{1}$$

can be written in the form

$$\frac{f\left(\frac{f(y)}{x}\right)}{\frac{f(y)}{x}} \cdot \frac{f(y)}{y} = f\left(\frac{f(x)}{y}\right) \tag{2}$$

when $f(y) \neq 0$.

Taking $x > 0$, $y < 0$ in (1), we conclude either that f has a zero or that f has both positive and negative values, so again f has a zero.

Case 1: $f(0) \neq 0$. If $f(x) = f(y) = 0$, then from (1) we get $x = y$, so the zero is unique in this case. If f satisfies (1), then $F(x) = a^2 f(x/a)$ also does, so we may assume $F(1) = 0$. Setting $x = 1$ and then $y = 0$ gives $F(F(0)) = 0$, so by the uniqueness, $F(0) = 1$. Setting $x = 1$ then yields $F(F(y)) = y$ for all y, so F is surjective. If we substitute z for $F(y)$ in (1) and differentiate with respect to x, we obtain

$$F\left(\frac{z}{x}\right) - \frac{z}{x} F'\left(\frac{z}{x}\right) = F'\left(\frac{F(x)}{F(z)}\right) F'(x).$$

Setting $z = x$ yields $F'(1)\big(1 + F'(x)\big) = 0$ for all x. It follows that $F(x) = 1 - x$. We conclude that the general solution in this case is $f(x) = a(a + x)$ with $a \neq 0$.

Case 2: $f(0) = 0$. We claim $f'(0) = 0$. If not, then by scaling $F(x) = a^2 f(x/a)$ as before, we produce an F so that $F'(0) = -1$. Set $x = -y$ in (2) and let y go to zero to obtain $F(1) \cdot (-1) = F(1)$, so $F(1) = 0$. Now set $x = 1$ in (2) and let y go to zero. We get $(F'(0))^2 = 0$, a contradiction; hence, $f'(0) = 0$.

Now we claim that f is identically zero. If not, then there exists a such that $f(a) \neq 0$. Set $y = f(x)/a$. We want to let $x \to 0$. Since f is continuous at 0, we have $f(x) \to 0$ as $x \to 0$. Thus, $y = f(x)/a \to 0$ as $x \to 0$. Now $f(y) \to 0$ as $x \to 0$. Since $f'(0) = 0$, also $f(x)/x \to 0$, so $y/x \to 0$ as $x \to 0$. Furthermore, $f(y)/y \to 0$ and $f(y)/x \to 0$ as $x \to 0$. Thus, by (2),

$$f(a) = f\left(\frac{f(x)}{y}\right) = \frac{f\left(\frac{f(y)}{x}\right)}{\frac{f(y)}{x}} \cdot \frac{f(y)}{y} \to f'(0) \cdot f'(0) = 0$$

as $x \to 0$, a contradiction. Hence, $f(x)$ is identically zero, and we get $a(x + a)$ in the last case $a = 0$.

Editorial comment. If the functional equation is required only for positive x, y, then there are many other solutions, such as $f(x) = (x^2 + 1)^{1/2}$.

Also solved by E. A. Herman, Y. J. Ionin, and R. Stong

A Trig Integral with Gamma

11796 [2015, 738]. *Proposed by Gleb Glebov, Simon Fraser University, Burnaby, Canada.* Find

$$\int_0^\infty \frac{\sin((2n+1)x)}{\sin x} e^{-\alpha x} x^{m-1}\, dx$$

in terms of α, m, and n, when $\alpha > 0$, $m \geq 1$, and n is a nonnegative integer.

Solution by Michel Bataille, Rouen, France. Define I to be the proposed integral. Then we can write it as,

$$I = \int_0^\infty \left(1 + 2\sum_{k=1}^\infty \cos(2kx)\right) e^{-\alpha x} x^{m-1}\, dx = I_0 + 2\sum_{k=1}^\infty I_k.$$

From the definition of the gamma function, we have

$$I_0 = \int_0^\infty e^{-\alpha x} x^{m-1}\, dx = \frac{\Gamma(m)}{\alpha^m},$$

and for $k \geq 1$,

$$I_k = \text{Re}\left(\int_0^\infty e^{-(\alpha-2ki)x} x^{m-1}\,dx\right) = \text{Re}\,\frac{\Gamma(m)}{(\alpha-2ki)^m} = \text{Re}\,\frac{\Gamma(m)\cdot(\alpha+2ki)^m}{(\alpha^2+4k^2)^m}.$$

To put this in real form, define $\theta_k = \tan^{-1}(2k/\alpha)$ so we can write $(\alpha+2ki)^m = (\sqrt{\alpha^2+4k^2})^m[\cos(m\theta_k) + i\sin(m\theta_k)]$. The real part I_k of the integral can be calculated from this, and

$$I = \Gamma(m)\left(\frac{1}{\alpha^m} + 2\sum_{k=1}^{m}\frac{\cos\left(m\tan^{-1}\frac{2k}{\alpha}\right)}{(\alpha^2+4k^2)^{m/2}}\right).$$

Also solved by U. Abel (Germany), K. F. Andersen (Canada), R. Bagby, D. Beckwith, R. Boukharfane (France), K. N. Boyadzhiev, P.Bracken, B. Bradie, M. A. Carlton, R. Chapman (U. K.), H. Chen, D. F. Connon (U. K.), B. E. Davis, J. L. Ekstrom, C. Georghiou (Greece), M. L. Glasser, E. A. Herman, M. Hoffman, B. Karaivanov & T. S. Vassilev (U.S.A & Canada), O. Kouba (Syria), K. D. Lathrop, M. Omarjee (France), P. Perfetti (Italy), C. M. Russell, R. Sargsyan (Armenia), M. A. Shayib & M. Misaghian, A. Stenger, R. Stong, R. Tauraso (Italy), N. Thornber, E. I. Verriest, Z. Vörös (Hungary), M. Vowe (Switzerland), H. Widmer (Switzerland), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Growth Rate for Solution

11799 [2014, 739]. *Proposed by Vicenţiu Rădulescu, King Abdulaziz University, Jeddah, Saudi Arabia.* Let a, b, and c be positive.
(**a**) Prove that there is a unique continuously differentiable function f from $[0,\infty)$ into $\mathbb{R}$ such that $f(0) = 0$ and, for all $x \geq 0$,

$$f'(x)\left(1 + a|f(x)|^b\right)^c = 1.$$

(**b**) Find, in terms of a, b, and c, the largest θ such that $f(x) = O(x^\theta)$ as $x \to \infty$.

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. (**a**) If f is a solution, then

$$f'(x)\left(1 + a|f(x)|^b\right)^c = 1 \tag{1}$$

implies $f'(x) > 0$ for all $x \geq 0$. Since $f(0) = 0$, we conclude that f is nonnegative and strictly increasing on $[0,\infty)$. We claim that f is unbounded. Indeed, if $f(x) \leq M$, then (1) shows $f'(x) \geq (1 + aM^b)^{-c} > 0$ for $x \geq 0$ so that

$$f(x) = \int_0^x f'(t)\,dt \geq \frac{x}{(1+aM^b)^c},$$

and thus, $f(x) > M$ for sufficiently large x, a contradiction. Thus, f is a bijection of $[0,\infty)$ onto itself, with a continuously differentiable inverse f^{-1} satisfying $f^{-1}(0) = 0$ and

$$\left(f^{-1}\right)'\left(f(x)\right)\, f'(x) = 1$$

for all $x \geq 0$. Combining this with (1) yields

$$\left(f^{-1}\right)'(y) = \left(1 + ay^b\right)^c$$

for all $y \in [0,\infty)$. Since $(1 + ay^b)^c$ is a continuous function of y, by the fundamental theorem of calculus,

$$f^{-1}(y) = \int_0^y \left(1 + at^b\right)^c dt.$$

Since inverses are unique, this uniquely determines f on $[0,\infty)$.

(**b**) Clearly, there is no maximal value of θ satisfying the stated requirement. If θ satisfies the requirement, then so does $\theta + 1$. We will show that the minimal value of θ satisfying the stated requirement is $(1+bc)^{-1}$. Note that

$$\lim_{y\to\infty} y^{-1-bc}\int_0^y \left(1+at^b\right)^c dt = \lim_{y\to\infty}\int_0^1 \left(y^{-b}+as^b\right)^c ds = \int_0^1 a^c s^{bc} ds = \frac{a^c}{bc+1}.$$

Thus,

$$\lim_{x\to\infty} x^{-\theta} f(x) = \lim_{y\to\infty}\left(\int_0^y \left(1+at^b\right)^c dt\right)^{-\theta} y$$

is finite if and only if $\theta \geq (1+bc)^{-1}$.

Editorial comment. In "simplifying" the statement, the editors mistakenly wrote "largest" instead of "smallest."

Also solved by R. Bagby, R. Chapman (U. K.), J.-P. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, I. Pinelis, A. Stenger, R. Stong, M. L. Treuden, E. I. Verriest, GCHQ Problem Solving Group (U. K.), and the proposer.

Arithmetic Minus Geometric Means

11800 [2014, 739]. *Proposed by Oleksiy Klurman, University of Montreal, Montreal, Canada.* Let f be a continuous function from [0, 1] into $\mathbb{R}^+$. Prove that

$$\int_0^1 f(x)\,dx - \exp\left[\int_0^1 \log f(x)\,dx\right] \leq \max_{0\leq x,y\leq 1}\left(\sqrt{f(x)}-\sqrt{f(y)}\right)^2.$$

Composite Solution by Rafik Sargsyan, Yerevan State University, Yerevan, Armenia, and Kenneth Schilling, Mathematics Department, University of Michigan–Flint, Flint, MI. We will prove a stronger result. Recall that the arithmetic, geometric, and harmonic means A, G, and H of f on [0, 1] satisfy

$$A = \int_0^1 f(x)\,dx \geq G = \exp\left[\int_0^1 \log f(x)\,dx\right] \geq H = \left[\int_0^1 \frac{dx}{f(x)}\right]^{-1}.$$

Let M and m be the maximum and minimum values of f, respectively. We show that $A - H \leq \left(\sqrt{M}-\sqrt{m}\right)^2$, which is stronger than the requested inequality $A - G \leq \left(\sqrt{M}-\sqrt{m}\right)^2$.

For $x \in [0, 1]$, define $s(x)$ by the relation

$$f(x) = s(x)\cdot m + (1-s(x))\cdot M$$

and let $t = \int_0^1 s(x)\,dx$. We have

$$\int_0^1 f(x)\,dx = tm + (1-t)M.$$

By convexity, $\frac{1}{f(x)} \leq \frac{s(x)}{m} + \frac{1-s(x)}{M}$, so

$$\int_0^1 \frac{dx}{f(x)} \leq \frac{t}{m} + \frac{1-t}{M} = \frac{tM+(1-t)m}{mM}.$$

Also note that

$$0 \le \left(t\sqrt{M} - (1-t)\sqrt{m}\right)^2 = tM + (1-t)m - t(1-t)\left(\sqrt{M}+\sqrt{m}\right)^2,$$

so $tM + (1-t)m \ge t(1-t)\left(\sqrt{M}+\sqrt{m}\right)^2$. Now

$$\begin{aligned} A - H &\le tm + (1-t)M - \frac{mM}{tM+(1-t)m} = \frac{t(1-t)(M-m)^2}{tM+(1-t)m} \\ &\le \frac{t(1-t)(M-m)^2}{t(1-t)\left(\sqrt{M}+\sqrt{m}\right)^2} = \left(\sqrt{M}-\sqrt{m}\right)^2, \end{aligned}$$

as claimed.

Note: The argument above applies even if f need only be a bounded, positive measurable function. In that version, equality holds in the strengthened inequality when f takes the value m on a set of measure $\frac{\sqrt{m}}{\sqrt{M}+\sqrt{m}}$ and the value M on a set of complementary measure $\frac{\sqrt{M}}{\sqrt{M}+\sqrt{m}}$.

Editorial comment. The discrete case of the strengthened inequality above appeared as Problem 11469 in this **Monthly** (problem in December 2009 and solution in May 2011) and in B. Meyer, "Some inequalities for elementary mean values," *Math. Comp.* **42** (1984) 193–194. The best possible upper bound for $A - G$ in terms of m and M can be proved by similar arguments to the ones above. This upper bound is the maximum of $tm + (1-t)M - m^t M^{1-t}$, which is attained at $t = \frac{\log[M\log(M/m)/(M-m)]}{\log(M/m)}$. This result is also proved in S. H. Tung, "On lower and upper bounds of the difference between the arithmetic and geometric mean," *Math. Comp.* **29** (1975) 834–836.

Also solved by R. Bagby, R. Boukharfane (France), R. Chapman (U. K.), E. A. Herman, B. Karaivanov & T. S. Vassilev (U.S.A. & Canada), J. H. Lindsey II, P. W. Lindstrom, M. Omarjee (France), P. Perfetti (Italy), I. Pinelis, R. Sargsyan (Armenia), K. Schilling, A. Stenger, R. Stong, S. Yi (Korea), Z. Zhang (China), FAU Problem Solving Group, Northwestern University Math Problem Solving Group, NSA Problems Group, University of Louisiana at Lafayette Math Club, and the proposer.

Rational Polynomials with no Nonnegative Zeros

11801 [2014, October]. *Proposed by David Carter, Nahant, MA.* Let f be a polynomial in one variable with rational coefficients that has no nonnegative real root. Show that there is a nonzero polynomial g with rational coefficients such that the coefficients of fg are all positive.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove the seemingly weaker statement that if a polynomial f has real coefficients and no nonnegative real root, then there is a polynomial h with real coefficients such that the coefficients of fh are nonnegative. To see that this suffices, note that if $f(x)h(x)$ has degree d and nonnegative coefficients, then $(x^d + x^{d-1} + \cdots + x + 1)f(x)h(x)$ has degree $2d$ and positive coefficients. Invoking continuity and the density of the rationals, we can then take g to be a polynomial with rational coefficients close enough to $(x^d + x^{d-1} + \cdots + x + 1)h(x)$ to solve the problem as stated (with a slightly weaker hypothesis on f).

The weaker fact has a multiplicative property: If polynomials h_1 and h_2 exist such that f_1h_1 and f_2h_2 have nonnegative coefficients, then $(f_1f_2)(h_1h_2)$ also has nonnegative coefficients. Thus, by factoring over the reals, it suffices to prove the result in three cases: for f a nonzero constant polynomial, for $f(x) = x + a$ with $a > 0$, and for $f(x) = (x - b)^2 + a$ with $a > 0$.

If $f(x) = c$ with $c \neq 0$, then set $h(x) = c$, yielding $f(x)h(x) = c^2 > 0$. If $f(x) = x + a$ with $a > 0$, then take $h(x) = 1$. For $f(x) = (x - b)^2 + a$ with $a > 0$, let α denote the root $b + i\sqrt{a}$ of f. We may also write $\alpha = re^{i\theta}$. Since α lies in the upper half-plane, $0 < \theta < \pi$, and hence, the origin lies in the convex hull of $\{e^{ik\theta} : 0 \leq k \leq \lfloor 2\pi/\theta \rfloor\}$.

Letting $d = \lfloor 2\pi/\theta \rfloor$, we can write $0 = \sum_{k=0}^{d} c_k e^{ik\theta}$ for nonnegative real constants $c_0, \ldots, c_d$ with sum 1. Rewriting this as $0 = \sum_{k=0}^{d} c_k r^{-k} x^k$ expresses α as a root of the polynomial p with nonnegative real coefficients defined by $p(x) = \sum_{k=0}^{d} c_k r^{-k} x^k$. Hence, f divides p, and p/f is the desired polynomial h.

Also solved by A. J. Bevelacqua, R. Chapman (U. K.), N. Grivaux (France), E. A. Herman, Y. J. Ionin, O. Kouba (Syria), R. E. Prather, N. C. Singer, A. Stenger, R. Tauraso (Italy), T. Viteam (India), Z. Wu (China), NSA Problems Group, and the proposer.

A Deranged Sum

11802 [2014, 739]. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let $H_{n,2} = \sum_{k=1}^{n} k^{-2}$, and let $D_n = n! \sum_{k=0}^{n} (-1)^k/k!$. (This is the *derangement number* of n, that is, the number of permutations of $\{1, \ldots, n\}$ that fix no element.) Prove that

$$\sum_{n=1}^{\infty} H_{n,2} \frac{(-1)^n}{n!} = \frac{\pi^2}{6e} - \sum_{n=0}^{\infty} \frac{D_n}{n!(n+1)^2}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA. Because the sum on the left side is absolutely convergent, the order of summation can be interchanged. Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} H_{n,2} \frac{(-1)^n}{n!} &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2} \right) \frac{(-1)^n}{n!} = \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sum_{n=k}^{\infty} \frac{(-1)^n}{n!} \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - \sum_{n=0}^{k-1} \frac{(-1)^n}{n!} \right) \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \right) - \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sum_{n=0}^{k-1} \frac{(-1)^n}{n!} \right) \\
&= \frac{\pi^2}{6} \frac{1}{e} - \sum_{k=1}^{\infty} \frac{D_{k-1}}{(k-1)!\, k^2} = \frac{\pi^2}{6e} - \sum_{k=0}^{\infty} \frac{D_k}{k!\,(k+1)^2}.
\end{aligned}$$

Also solved by U. Abel (Germany), A. Ali (India), K. F. Andersen (Canada), M. Andreoli, R. Bagby, S. Banerjee & B. Maji (India), M. Bataille (France), R. Boukharfane (France), K. N. Boyadzhiev, P. Bracken, M. A. Cariton, R. Chapman (U. K.), H. Chen, D. Fleischman, N. Fontes-Merz, O. Geupel (Germany), M. L. Glasser, J.-P. Grivaux (France), E. A. Herman, M. Hoffman, B. Karaivanov & T. Vassilev (Canada), P. M. Kayll, O. Kouba (Syria), J. Minkus, M. Omarjee (France), R. Sargsyan (Armenia), A. Stenger, R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), M. Wildon (U. K.), J. Zacharias, Armstrong Problem Solvers, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems should be submitted online via www.americanmathematicalmonthly.submittable.com. Proposed solutions to the problems below should be submitted on or before February 28, 2017 at the same link. More detailed instructions are available online. Solutions to problems numbered 11921 or below should continue to be submitted via email to monthlyproblems@math.tamu.edu. Proposed problems must not be under consideration concurrently to any other journal not be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part or a problem that indicates that no solution is currently available.

PROBLEMS

11929. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let a_n be the number of ways in which a rectangular box that contains $6n$ square tiles in three rows of length $2n$ can be split into two connected pieces of size $3n$ without cutting any tiles. Thus, $a_1 = 3$, $a_2 = 19$, and one of the 85 ways for $n = 3$ is shown.

Taking $a_0 = 1$, find a closed form for the generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$. What is the asymptotic nature of a_n as $n \to \infty$?

11930. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Find

$$\sum_{n=1}^{\infty} \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right).$$

11931. *Proposed by Igor Protasov, Kiev, Ukraine.* Given natural numbers m and r, prove that there is a finite connected graph G such that, for every r-coloring of its edge set $E(G)$, there is a monochromatic geodesic path of length m. (A path is *geodesic* if there is no shorter path with the same endpoints.)

11932. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let r be an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r.$$

http://dx.doi.org/10.4169/amer.math.monthly.123.08.831

11933. *Proposed by José M. Pacheco and Angel Plaza, University of Las Palmas de Gran Canaria, Spain.* For positive integer n, let $H_n = \sum_{k=1}^{n} 1/k$. Prove

$$\int_0^1 \frac{1}{x+1}\,dx \cdot \int_0^1 \frac{x+1}{x^2+x+1}\,dx \cdots \int_0^1 \frac{x^{n-2}+\cdots+x+1}{x^{n-1}+\cdots+x+1}\,dx \geq \frac{1}{H_n}.$$

11934. *Proposed by Leonard Giugiuc, Drobotu Turnu Severin, Romania.* Let ABC be an isosceles triangle, with $|AB| = |AC|$. Let D and E be two points on side BC such that $D \in BE$, $E \in DC$, and $m(\angle DAE) = \frac{1}{2}m(\angle A)$. Describe how to construct a triangle XYZ such that $|XY| = |BD|$, $|YZ| = |DE|$, and $|ZX| = |EC|$. Also, compute $m(\angle BAC) + m(\angle XYZ)$ (in radians).

11935. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Anastasios Kotronis, Athens, Greece, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let f be a function from $\mathbb{Z}^+$ to $\mathbb{R}^+$ such that $\lim_{n\to\infty} f(n)/n = a$, where $a > 0$. Find

$$\lim_{n\to\infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^{n} f(k)} \right).$$

SOLUTIONS

A Powerful Inequality

11804 [2015, 946]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Prove that $10|x^3+y^3+z^3-1| \leq 9|x^5+y^5+z^5-1|$ for real numbers x, y, and z with $x+y+z=1$. When does equality hold?

Solution I by Dain Kim, Yonsei University, Seoul, Korea. Since $x+y+z=1$, we have the following two identities:

$$x^3+y^3+z^3-1 = x^3+y^3+z^3-(x+y+z)^3 = -3(x+y)(y+z)(x+z),$$

$$x^5+y^5+z^5-1 = x^5+y^5+z^5-(x+y+z)^5$$

$$= -\frac{5}{2}(x+y)(y+z)(z+x)\left[(x+y)^2+(y+z)^2+(z+x)^2\right].$$

Thus, if the denominator is nonzero we obtain

$$\frac{|x^5+y^5+z^5-1|}{|x^3+y^3+z^3-1|} = \frac{5}{6}\cdot\left[(x+y)^2+(y+z)^2+(z+x)^2\right]$$

$$\geq \frac{5}{6}\frac{[2(x+y+z)]^2}{3} = \frac{10}{9}.$$

Equality holds if and only if $x=y=z=\frac{1}{3}$ or (x,y,z) is a permutation of $(t,-t,1)$ for some $t\in\mathbb{R}$.

Solution II by Oliver Geupel, NRW, Germany. Due to the constraint $x+y+z=1$, the required inequality is a consequence of the relation

$$10^2(x^3+y^3+z^3-1)^2 \leq 10^2(x^3+y^3+z^3-(x+y+z)^3)^2(x+y+z)^4$$

$$+\frac{225}{4}(x+y)^2(y-z)^2(z+x)^2((x-y)^2+(y-z)^2+(z-x)^2)\cdot$$
$$\cdot(3\cdot(x^2+y^2+z^2)+7\cdot(x+y+z)^2)$$
$$=9(x^5+y^5+z^5-(x+y+z)^5)^2.$$

Equality holds for the cases listed in the first solution.

Also solved by A. Ali (India), M. A. Carlton, S. Chakravarty, R. Chapman (U. K.), J. Duemmel, D. Fleischman, L. Giugiuc (Romania), E. A. Herman, E. J. Ionaşcu, Y. J. Ionin, B. Khadka, K.-W. Lau (China), S. Lee (Korea), O. P. Lossers (Netherlands), J. Loverde, J. R. Pentland, P. Perfetti (Italy), Á. Plaza (Spain), R. E.Prather, J. M. Sanders, R. Stong, M. L. Treuden, J. Van Hamme (Belgium), T. Viteam (India), Z. Vörös (Hungary), H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

Sums and Integral Related to Zeta

11805 [2014, 946]. *Proposed by Gleb Glebov, Simon Fraser University, Burnaby, Canada.*
(**a**) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}+\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{5\pi^3\sqrt{3}}{243}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}-\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{13}{18}\zeta(3).$$

(**b**) Prove that

$$\zeta(3)=\frac{9}{13}\int_0^1\frac{(\log x)^2}{x^3+1}\,dx-\frac{18}{13}\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}.$$

Here, ζ denotes the Riemann zeta function.

Solution I by the GCHQ Problem Solving Group, Cheltenham, U.K. (**a**) We denote the two required sums by

$$S_1=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}+\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{1}{1^3}+\frac{1}{2^3}-\frac{1}{4^3}-\frac{1}{5^3}+\cdots$$
$$S_2=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}-\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{1}{1^3}-\frac{1}{2^3}-\frac{1}{4^3}+\frac{1}{5^3}+\cdots.$$

The Fourier series of the odd function $x^3-\pi^2x$ on $[-\pi,\pi]$ yields

$$x^3-\pi^2x=12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nx)$$

for $-\pi\le x\le\pi$. In particular, setting $x=2\pi/3$ yields $-10\pi^3/27=6\sqrt{3}(-S_1)$, which is equivalent to the claimed value for S_1. For S_2, note that

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{5^3}+\cdots=\zeta(3)\left(1-\frac{1}{3^3}\right)=\frac{26}{27}\zeta(3).$$

Therefore,

$$\frac{26}{27}\zeta(3) - S_3 = 2\left(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{8^3} + \frac{1}{10^3} + \cdots\right) = \frac{13}{54}\zeta(3),$$

which yields the claimed value of S_2.

(**b**) Integration by parts yields $\int_0^1 x^n (\log x)^2 dx = 2/(m+1)^3$. So we have

$$\int_0^1 \frac{(\log x)^2}{x^3+1}\,dx = \int_0^1 (\log x)^3(1 - x^3 + x^6 - \cdots)dx$$
$$= 2\left(\frac{1}{1^3} - \frac{1}{4^3} + \frac{1}{7^3} - \cdots\right) = S_1 + S_2.$$

Finally,

$$\frac{9}{13}\left[\int_0^1 \frac{(\log x)^2}{x^3+1}\,dx - (S_1 - S_2)\right] = \frac{18}{13}S_2 = \zeta(3).$$

Solution II of (**a**) *by Ben Keigwin and John Zacharias, Alexandria VA.* Euler showed

$$\frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2} = \pi \cot \pi x$$

for $x \in (0, 1)$. Differentiate twice:

$$\frac{1}{x^3} + \sum_{k=1}^{\infty} \frac{1}{(x-k)^3} + \sum_{k=1}^{\infty} \frac{1}{(x+k)^3} = \frac{\pi^3 \cos \pi x}{\sin^3 \pi x}.$$

Next, set $x = 1/3$:

$$\sum_{k=0}^{\infty} \frac{1}{(3k+1)^3} - \sum_{k=0}^{\infty} \frac{1}{(3k+2)^3} = \frac{4\pi^3\sqrt{3}}{243}. \tag{1}$$

Let S_1, S_2 be as in Solution I. Thus,

$$S_1 - S_2 = 2\left(\sum_{k=0}^{\infty} \frac{1}{(2(3k+1))^3} - \sum_{k=0}^{\infty} \frac{1}{(2(3k+2))^3}\right) = 2 \cdot \frac{1}{2^3} S_2$$

or $S_1 = (5/4)S_2$. Combining this with (1), we get the first relation in (**a**).

Let

$$S_3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)^3}, \quad S_4 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)^3}, \quad S_5 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(3k)^3}.$$

Now $S_3 - S_4 + S_5 = 27S_5$. This in turn gives $\zeta(3) - 27S_5 = \zeta(3) - S_3 + S_4 - S_5 = 2\sum_{k=1}^{\infty}(2k)^{-3} = \frac{1}{4}\zeta(3)$. Finally, $S_5 = \frac{1}{36}\zeta(3)$ and $S_3 - S_4 = 26S_5 = \frac{13}{18}\zeta(3)$, the second relation in (**a**).

Editorial comment. A surprisingly wide range of methods was used for this problem: residue calculations, polylogarithm, polygamma, Hurwitz ζ, Dirichlet η.

Also solved by A. Ali (India), K. F. Andersen (Canada), R. Bagby, R. Bauer, D. Beckwith, D. Borwein & J. M. Borwein (Canada & Australia), P. Bracken, R. Brase, B. S. Burdick, M. A. Carlton, R. Chapman (U. K.), H. Chen, B. E. Davis, R. L. Doucette, J. Gatheral, C. Georghiou (Greece), O. Geupel (Germany), M. L. Glasser,

M. Goldenberg & M. Kaplan, E. A. Herman, M. Hoffman, F. Holland (Ireland), E. I. Ionaşcu, B. Keigwin & J. Zacharias, P. Khalili, O. Kouba (Syria), K. D. Lathrop, O. P. Lossers (Netherlands), M. Omarjee (France), J. Pentland & J. A. Green, P. Perfetti (Italy), Á. Plaza & F. Perdomo (Spain), P. G. Poonacha (India), N. C. Singer, M. Štofka (Slovakia), R. Stong, R. Tauraso (Italy), T. P. Turiel, J. Van Hamme (Belgium), Z. Vörös (Hungary), M. Vowe (Switzerland), S. Wagon, C. Y. Yıldırım (Turkey), FAU Problem Solving Group, NSA Problems Group, and the proposer.

Three of Four Sides Equal

11807 [2014, 947]. *Proposed by Robin Oakapple, Albany, OR.* Given a quadrilateral $ABCD$ inscribed in a circle K, and a point Z inside K, the rays AZ, BZ, CZ, and DZ meet K again at points E, F, G, and H, respectively, to yield another quadrilateral also inscribed in K. Develop a construction that takes as input A, B, C, and D and returns a point Z such that this second quadrilateral has (at least) three of its sides of equal length.

Solution by John Cade, University of Pikeville, Pikeville, KY. We assume the vertices of quadrilateral $ABCD$ are named in cyclical order. Consider Γ_A, the *circle of Apollonius*, at point A of triangle BAD. This circle has the following properties.

1) It is the locus of all points P for which $|PB|/|PD| = |AB|/|AD|$. This locus is a true circle except in the case that A is the apex of isosceles triangle BAD, in which case the locus is the principal axis of the triangle, i.e., the perpendicular bisector of BD.

2) The circle may be characterized as follows: Determine the points (say M and N) where the internal and external bisectors of angle A meet line BD. Then MN is a diameter of this circle.

3) The circle may also be characterized alternatively: Let the tangent line at A to the circumcircle K intersect line BD at T. The tangent line is perpendicular to the radius of the circumcircle from A to the center of K. And T is the center of the circle of Apollonius, which passes through A and is perpendicular there to K.

4) Because of the facts in 2) and 3), the circle of Apollonius may be constructed using Euclidean tools to find angle bisectors, perpendicular bisectors, etc., by well-known methods.

Let the circles of Apollonius Γ_B, Γ_C, Γ_D be constructed similarly. Point Z may be taken as any intersection inside K of two consecutive circles of Apollonius. If two such circles intersect at all, they intersect at two points, one inside K and the other outside; in fact, these two points are images of each other under inversion in K.

We must now prove two claims.

I) Some two consecutive circles of Apollonius intersect inside K.

II) Any such point of intersection is a suitable Z.

Proof of Claim I: The number of consecutive pairs of circles of Apollonius that intersect inside K can vary from one to four. We must show that it is not zero. Let VX be an edge of $ABCD$ that is no longer than either of its neighbors (for example, the shortest edge of $ABCD$ will do). Suppose that U, V, X, and Y denote the vertices of $ABCD$ in the same order and that U', V', X', and Y' denote E, F, G, and H in the corresponding order.

We have chosen VX so that it is no longer than VU and also no longer than XY. If equality holds in either case, the situation is simple because as noted earlier one (or both) corresponding "circle(s)" of Apollonius is(are) actually a straight line (passing through the center of K). So let us assume that VX is shorter than both VU and XY. We will argue that Γ_X and Γ_Y intersect inside K.

As noted above in property 3), Γ_X passes through X and is perpendicular there to K. The center, say T_X, of Γ_X is the intersection of the tangent to K at X with the line

VY. This intersection lies outside K and is closer to V than to Y because VX is shorter than XY. Thus, V lies inside Γ_X. By similar reasoning, Γ_V passes through V and is perpendicular there to K. Its center T_V lies on the line XU and is closer to X than to U because VX is shorter than VU. So X lies inside Γ_V. Thus, Γ_X and Γ_V must cross inside K.

Proof of Claim II: Let Z lie on both Γ_X and Γ_V. By the vantage point theorem (also known as the inscribed angle theorem) applied in circle K, $\angle VUZ = \angle ZVU'$; also $\angle VZU$ and $\angle U'ZV'$ are right angles. Thus, $\triangle VUZ$ is similar to $\triangle U'V'Z$, so (1) $\frac{|ZU|}{|VU|} = \frac{|ZV'|}{|U'V'|}$. In like manner, (2) $\frac{|ZX|}{|VX|} = \frac{|ZV'|}{|X'V'|}$.

Because Z lies on Γ_V, $\frac{|ZU|}{|ZX|} = \frac{|VU|}{|VX|}$ or $\frac{|ZU|}{|VU|} = \frac{|ZX|}{|VX|}$. Thus, since the left sides of (1) and (2) are equal, so are the right sides, i.e., $\frac{|ZV'|}{|U'V'|} = \frac{|ZV'|}{|X'V'|}$. Finally, $|U'V'| = |X'V'|$.

Because Z also lies on Γ_X, by similar reasoning, $|X'V'| = |X'Y'|$, i.e., the quadrilateral $U'V'X'Y'$ has three equal sides.

Editorial comment. Yuri Ionin noted the following: Let the disk K be interpreted as Klein's model of hyperbolic geometry, with A, B, C, and D as "ideal" points. Let Q denote the image of O, the center of K, under a (hyperbolic) half-turn about Z. In this context, the problem shows that Z can be chosen so that some three of the four angles AQB, BQC, CQD, and DQA are equal.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), Y. J. Ionin, R. Stong, and the proposer.

An Euler Sum

11810 [2015, 75]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $H_n = \sum_{k=1}^n 1/k$, and let ζ be the Riemann zeta function. Find

$$\sum_{n=1}^{\infty} H_n \left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3} \right).$$

Solution by C. Georghiou, University of Patras, Patras, Greece. The answer S is $2\zeta(3) - \zeta(2)$. To show this, we need the following well-known results:

$$\sum_{k=1}^{n-1} H_k = \sum_{k=1}^{n-1} \sum_{m=1}^{k} \frac{1}{m} = \sum_{m=1}^{n-1} \sum_{k=m}^{n-1} \frac{1}{m} = \sum_{m=1}^{n} \frac{n-m}{m} = nH_n - n$$

and

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3). \tag{E}$$

The latter identity is due to Euler (1775). To use these, we sum by parts with $b_n = \zeta(3) - \sum_{k=1}^n \frac{1}{k^3}$ and $A_n = \sum_{k=1}^n H_k$ to get

$$\sum_{k=1}^{n} H_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k) = A_n b_n + \sum_{k=1}^{n-1} \frac{H_{k+1} - 1}{(k+1)^2}.$$

Since $A_n = O(n \log n)$ and $b_n = O(n^{-2})$, this gives

$$S = \sum_{k=1}^{\infty} \frac{H_k - 1}{k^2} = 2\zeta(3) - \zeta(2).$$

Editorial comment. The sum (E) at the heart of this problem, and its generalizations have appeared multiple times in this **Monthly** (for example problems 4431, 10635, and 10754). Many solvers used these generalizations, particularly Euler's result for the Hurwitz zeta function $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$:

$$\zeta(b,1) = \sum_{n=1}^{\infty}\frac{H_{n-1}}{n^b} = \frac{b}{2}\zeta(b+1) - \frac{1}{2}\sum_{k=2}^{b-1}\zeta(b+1-k)\zeta(k),$$

to extend the stated result. For much more on Euler sums, see D. Borwein, J. M. Borwein, and R. Girgensohn, "Explicit evaluation of Euler sums," *Proc. Edinb. Math. Soc.*, **38** (1995) 277–294 or http://mathworld.wolfram.com/EulerSum.html.

Also solved by R. A. Agnew, R. Bagby, D. H. Bailey & J. M. Borwein (U.S. & Australia), D. Beckwith, R. Boukharfane (France), P. Bracken, B. Bradie, B. S. Burdick, M. A. Carlton, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), D. Fritze (Germany), J. Gatheral, O. Geupel (Germany), M. L. Glasser (Spain), G. C. Greubel, J.-P. Grivaux (France), E. A. Herman, M. Hoffman, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), I. Mezö (Hungary), M. Omarjee (France), P. Perfetti (Italy), I. Pinelis, M. A. Prasad (India), E. Schmeichel, B. Schmuland (Canada), N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), M. Wildon (U. K.), C. Y. Yıldırım (Turkey), GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, and the proposer.

What Are the Limits?

11811 [2015, 75]. *Proposed by Vazgen Mikayelyan, Yerevan State University, Yerevan, Armenia.* Let $\langle a\rangle$ and $\langle b\rangle$ be infinite sequences of positive numbers. Let $\langle x\rangle$ be the infinite sequence given for $n \ge 1$ by

$$x_n = \frac{a_1^{b_1}\cdots a_n^{b_n}}{\left(\frac{a_1b_1+\cdots+a_nb_n}{b_1+\cdots+b_n}\right)^{b_1+\cdots+b_n}}.$$

(**a**) Prove that $\lim_{n\to\infty} x_n$ exists.
(**b**) Find the set of all c that can occur as that limit, for suitably chosen $\langle a\rangle$ and $\langle b\rangle$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. (**a**) We first claim: x_n *is monotonic nonincreasing*. With notation

$$\Lambda_n = b_1 + \cdots + b_n, \quad \lambda_n = \frac{b_n}{\Lambda_n}, \quad G_n = \left(\prod_{k=1}^{n} a_k^{b_k}\right)^{1/\Lambda_n}, \quad A_n = \frac{1}{\Lambda_n}\sum_{k=1}^{n} b_k a_k,$$

we have $x_n = (G_n/A_n)^{\Lambda_n}$. Thus,

$$G_{n+1} = G_n^{1-\lambda_{n+1}} a_{n+1}^{\lambda_{n+1}} \text{ and } A_{n+1} = (1-\lambda_{n+1})A_n + \lambda_{n+1}a_{n+1},$$

so

$$\frac{G_{n+1}}{A_{n+1}} = \frac{G_n^{1-\lambda_{n+1}} a_{n+1}^{\lambda_{n+1}}}{(1-\lambda_{n+1})A_n + \lambda_{n+1}a_{n+1}} = \left(\frac{G_n}{A_n}\right)^{1-\lambda_{n+1}} \frac{A_n^{1-\lambda_{n+1}} a_{n+1}^{\lambda_{n+1}}}{(1-\lambda_{n+1})A_n + \lambda_{n+1}a_{n+1}},$$

or equivalently

$$\frac{x_{n+1}}{x_n} = \left(\frac{A_n^{1-\lambda_{n+1}} a_{n+1}^{\lambda_{n+1}}}{(1-\lambda_{n+1})A_n + \lambda_{n+1}a_{n+1}}\right)^{\Lambda_{n+1}}.$$

By the arithmetic mean–geometric mean inequality, if $a > 0$, $A > 0$, and $0 < \lambda < 1$, then $A^{1-\lambda}a^{\lambda} \le (1-\lambda)A + \lambda a$. We conclude that $x_{n+1} \le x_n$. Moreover, note that $x_1 = 1$ and $x_n > 0$ for all $n \ge 1$. Therefore, x_n converges, and the limit belongs to $[0, 1]$.

(**b**) It suffices to prove the following: *If $c \in [0, 1]$, then there exist sequences $\langle a\rangle$ and $\langle b\rangle$ such that x_n converges to c.* If $c = 0$, then let $b_n = 1$ and $a_n = \frac{1}{n(n+1)}$ for all n, so

$$x_n = \frac{(n+1)^n}{n! \cdot (n+1)!} \to 0.$$

If $c = 1$, then let $b_n = 1$ and $a_n = 1$ for all n, so $x_n = 1$ for all n and $x_n \to 1$. If $0 < c < 1$, then let $b_n = 1$ for all n, and let

$$a_1 = \frac{1+\sqrt{1-c}}{\sqrt{c}}, \quad a_2 = \frac{1-\sqrt{1-c}}{\sqrt{c}}, \quad a_n = \frac{1}{\sqrt{c}} \text{ for } n \ge 3.$$

Now $x_n = c$ for $n \ge 2$ and $x_n \to c$.

Also solved by R. A. Agnew, R. Bagby, D. Beckwith, R. Boukharfane (France), B. S. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), N. Grivaux (France), J. H. Lindsey II, P. Lindstrom, U. Milutinović (Slovenia), M. Omarjee (France), M. Omarjee & R. Tauraso (France & Italy), P. Perfetti (Italy), M. A. Prasad (India), B. Schmuland (Canada), R. Stong, and the proposer.

Bound an Integral

11812 [2015, 75]. *Proposed by Cristian Chiser, Craiova, Romania.* Let f be a twice continuously differentiable function from $[0, 1]$ into $\mathbb{R}$. Let p be an integer greater than 1. Given that $\sum_{k=1}^{p-1} f(k/p) = -\frac{1}{2}(f(0) + f(1))$, prove that

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{5!\,p^4}\int_0^1 (f''(x))^2\,dx.$$

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI. Subdivide the interval $[0, 1]$, and on each part, integrate $f(x)$ by parts:

$$\begin{aligned}\int_0^1 f(x)\,dx &= \sum_{k=0}^{p-1}\int_{k/p}^{(k+1)/p} f(x)\,dx \\ &= \sum_{k=0}^{p-1}\left[\left(x - \frac{2k+1}{2p}\right) f(x)\,\bigg|_{k/p}^{(k+1)/p} - \int_{k/p}^{(k+1)/p}\left(x - \frac{2k+1}{2p}\right) f'(x)\,dx\right] \\ &= \sum_{k=0}^{p-1}\left[\frac{1}{2p}\left(f\left(\frac{k}{p}\right) + f\left(\frac{k+1}{p}\right)\right) - \int_{k/p}^{(k+1)/p}\left(x - \frac{2k+1}{2p}\right) f'(x)\,dx\right].\end{aligned}$$

The sum of the first term is $\frac{1}{p}(\frac{1}{2}f(0) + \sum_{k=1}^{p-1} f(\frac{k}{p}) + \frac{1}{2}f(1))$, which is zero. Integrating the second term by parts yields

$$\begin{aligned}\int_0^1 f(x)\,dx &= \sum_{k=0}^{p-1}\left[-\int_{k/p}^{(k+1)/p}\left(x - \frac{2k+1}{2p}\right) f'(x)\,dx\right] \\ &= \sum_{k=0}^{p-1}\left[-\frac{1}{2}\left(x^2 - \frac{2k+1}{p}x + \frac{k^2+k}{p^2}\right) f'(x)\,\bigg|_{k/p}^{(k+1)/p}\right.\end{aligned}$$

$$+\int_{k/p}^{(k+1)/p} \frac{1}{2}\left(x^2 - \frac{2k+1}{p}x + \frac{k^2+k}{p^2}\right) f''(x)\,dx\Bigg].$$

However,

$$x^2 - \frac{2k+1}{p}x + \frac{k(k+1)}{p^2} = \left(x - \frac{k}{p}\right)\left(x - \frac{k+1}{p}\right),$$

evaluates to zero at the endpoints of the interval. Thus, the first term is again zero. The integral becomes

$$\begin{aligned}\int_0^1 f(x)\,dx &= \sum_{k=0}^{p-1} \int_{k/p}^{(k+1)/p} \frac{1}{2}\left(x - \frac{k}{p}\right)\left(x - \frac{k+1}{p}\right) f''(x)\,dx \\ &= \sum_{k=0}^{p-1} \int_{k/p}^{(k+1)/p} \frac{1}{2}\left[\left(x - \frac{k}{p}\right)^2 - \frac{1}{p}\left(x - \frac{k}{p}\right)\right] f''(x)\,dx. \qquad (1)\end{aligned}$$

Let $g(x)$ be the continuous periodic function with period $\frac{1}{p}$ and $g(x) = \frac{1}{2}(x^2 - \frac{1}{p}x)$ on $[0, \frac{1}{p}]$. Equation (1) becomes

$$\int_0^1 f(x)\,dx = \int_0^1 g(x) f''(x)\,dx.$$

By the Cauchy–Schwarz inequality,

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \int_0^1 g(x)^2\,dx \int_0^1 f''(x)^2\,dx. \qquad (2)$$

The integral of $g(x)^2$ can be evaluated exactly since $g(x)$ consists of p pieces, translates of $\frac{1}{2}(x^2 - \frac{1}{p}x)$ on $[0, \frac{1}{p}]$:

$$\int_0^1 g(x)^2\,dx = p\int_0^{1/p} \frac{1}{4}\left(x^2 - \frac{1}{p}x\right)^2 dx = \frac{1}{120\,p^4}.$$

Substituting this result into (2) yields the desired result.

Also solved by U. Abel (Germany), K. F. Andersen (Canada), R. Bagby, R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), J. Freeman, E. A. Herman, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), Á. Plaza & F. Perdomo (Spain), M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), FAU Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems should be submitted online via www.americanmathematicalmonthly.submittable.com. Proposed solutions to the problems below should be submitted on or before March 31, 2017 at the same link. More detailed instructions are available online. Solutions to problems numbered 11921 or below should continue to be submitted via email to monthlyproblems@math.tamu.edu. Proposed problems must not be under consideration concurrently to any other journal not be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part or a problem that indicates that no solution is currently available.

PROBLEMS

11936. *Proposed by William Weakley, Indiana University–Purdue University at Fort Wayne, Fort Wayne, Indiana.* Let S be the set of integers n such that there exist integers i, j, k, m, p with $i, j \geq 0$, $m, k \geq 2$, and p prime, such that $n = m^k = p^i + p^j$.
(**a**) Characterize S.
(**b**) For which elements of S are there two choices of (p, i, j)?

11937. *Proposed by Juan Carlos Sampedro, UPV/EHU-University of Basque Country, Leioa, Spain.* Let s be a complex number not a zero of the gamma function $\Gamma(s)$. Prove

$$\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y}{(1 - xy)\log(xy)}\, dx\, dy = \frac{\Gamma'(s)}{\Gamma(s)}.$$

11938. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, Macedonia.* Let a, b, c be the lengths of the sides of a triangle, and let A be its area. Let R and r be the circumradius and inradius of the triangle, respectively. Prove

$$a^2 + b^2 + c^2 \geq (a-b)^2 + (b-c)^2 + (c-a)^2 + 4A\sqrt{3 + \frac{R - 2r}{R}}.$$

11939. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Find

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log(k) - \gamma - \frac{1}{2k} + \frac{1}{12k^2}\right).$$

Here γ is Euler's constant.

http://dx.doi.org/10.4169/amer.math.monthly.123.9.941

11940. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $T_n = n(n+1)/2$ and $C(n,k) = (n-2k)\binom{n}{k}$. For $n \geq 1$, prove

$$\sum_{k=0}^{n-1} C(T_n, k)C(T_{n+1}, k) = \frac{n^3 - 2n^2 + 4n}{n+2}\binom{T_n}{n}\binom{T_{n+1}}{n}.$$

11941. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let

$$L = \lim_{n\to\infty} \int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx.$$

(**a**) Find L.
(**b**) Find

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx - L \right).$$

11942. *Proposed by Florin Parvanescu, Slat, Romania.* In acute triangle ABC, let D be the foot of the altitude from A, let E be the foot of the perpendicular from D to AC, and let F be a point on segment DE. Prove that AF is perpendicular to BE if and only if $|FE|/|FD| = |BD|/|CD|$.

SOLUTIONS

A Recurrence from Euler's Pentagonal Number Theorem

11795 [2014, 649]. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let p be the partition counting function on the set $\mathbb{Z}^+$ of positive integers, and let g be the function on $\mathbb{Z}^+$ given by $g(n) = \frac{1}{2}\lceil n/2 \rceil \lceil (3n+1)/2 \rceil$. Let $A(n)$ be the set of positive integer triples (i, j, k) such that $g(i) + j + k = n$. Prove for $n \geq 1$ that

$$p(n) = \frac{1}{n} \sum_{(i,j,k)\in A(n)} (-1)^{\lceil i/2 \rceil - 1} g(i) p(j) p(k).$$

Solution by Mark Wildon, Royal Holloway, Egham, U. K. In the statement, each instance of "positive integer" should be changed to "nonnegative integer."

Let $P(x) = \sum_{n=0}^{\infty} p(n)x^n$. Since

$$g(n) = \begin{cases} \dfrac{1}{2}m(3m+1) & \text{if } n = 2m \\ \dfrac{1}{2}m(3m-1) & \text{if } n = 2m-1, \end{cases}$$

we have $g(n) = \frac{1}{2}(-m)[3(-m)+1]$ when $n = 2m-1$, and hence

$$\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil - 1} x^{g(n)} = \sum_{m=-\infty}^{\infty} (-1)^{m-1} x^{\frac{1}{2}m(3m+1)} = \frac{-1}{P(x)},$$

where the final equality follows from Euler's pentagonal number theorem (see Corollary 1.7 of G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976). Differentiating yields

$$\sum_{n=1}^{\infty}(-1)^{\lceil n/2\rceil-1}g(n)x^{g(n)} = -x\frac{d}{dx}\frac{1}{P(x)} = \frac{xP'(x)}{P(x)^2}.$$

It follows that

$$\left(\sum_{n=1}^{\infty}(-1)^{\lceil n/2\rceil-1}g(n)x^{g(n)}\right)P(x)^2 = \sum_{n=1}^{\infty}np(n)x^n.$$

Comparing coefficients of x^n yields

$$\sum_{(i,j,k)\in A(n)}(-1)^{\lceil i/2\rceil-1}g(i)p(j)p(k) = np(n),$$

as desired.

Editorial comment. Substantial additional material on Euler's pentagonal number theorem can be found in J. Bell, A summary of Euler's work on the pentagonal number theorem, *Arch. Hist. Exact Sci.* **64** no. 3 (2010) 301–373.

Also solved by R. Chapman (U. K.), K. T. Gyun (Korea), Y. J. Ionin, K. Kusejko (Switzerland), O. P. Lossers (Netherlands), J. C. Smith, R. Stong, R. Tauraso (Italy), and the proposer.

Partitioning $\mathbb{N}$ into Four Sets

11803 [2014, 946]. *Proposed by Sam Speed, Germantown, PA.* Let $a_1(k,n) = (9^k(24n+5)-5)/8$, $a_2(k,n) = (9^k(24n+13)-5)/8$, $a_3(k,n) = (3\cdot 9^k(24n+7)-5)/8$, and $a_4(k,n) = (3\cdot 9^k(24n+23)-5)/8$. Show that for each nonnegative integer m there is a unique integer triple (j,k,n) with $j\in\{1,2,3,4\}$ and $k,n\ge 0$ such that $m = a_j(k,n)$.

Solution by Borislav Karaivanov, Lexington, SC, USA, and Tzvetalin S. Vassilev, Nipissing University, Ontario, Canada). After multiplying by 8 and adding 5, the problem becomes equivalent to showing that any integer of the form $8m+5$ belongs to exactly one of the four 2-parameter families $9^k(24n+5)$, $9^k(24n+13)$, $3\cdot 9^k(24n+7)$, and $3\cdot 9^k(24n+23)$. Since these families are clearly disjoint, it suffices to show for each m that there is a suitable (k,n). Since $8m+5$ is odd, by the Fundamental Theorem of Arithmetic we may write it as $3^s(24n+r)$, with r odd, at most 23, and relatively prime to 3. Let $R = \{1,5,7,11,13,17,19,23\}$.

We now reduce modulo 8. If s is even, then $8m+5 = 3^s(24n+r)$ reduces to $5\equiv r \bmod 8$; with $r\in R$, we must have $r\in\{5,13\}$. If s is odd, then $8m+5 = 3^s(24n+r)$ reduces to $5\equiv 3r \bmod 8$. Since $9\equiv 1 \bmod 8$, multiplying by 3 yields $15\equiv r \bmod 8$, which for $r\in R$ yields $r\in\{7,23\}$. Hence in all cases we find that m lies in one of the four specified classes.

Also solved by A. Ali (India), R. Brase, N. Caro (Brazil), R. Chapman (U. K.), C. Danivas (India), D. Fleischman, O. Geupel (Germany), J. A. Green, C. Heckman, E. A. Herman, E. J. Ionaşcu, Y. Ionin, O. Kouba (Syria), D. Lee (South Korea), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), Á. Plaza & F. Perdomo (Spain), R. E. Prather, R. Strong, N. Taylor, Z. Vörös (Hungary), H. Widmer (Switzerland), Florida Atlantic University Problem Solving, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Parseval and Kummer

11806 [2015, 947]. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Prove that

$$\int_0^{2\pi}\log\Gamma\left(\frac{x}{2\pi}\right)e^{\cos x}\sin(x+\sin x)\,dx = (e-1)(\log(2\pi)+\gamma)+\sum_{n=2}^{\infty}\frac{\log n}{n!}.$$

Here Γ denotes the gamma function and γ denotes the Euler–Mascheroni constant.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We will use Parseval's formula. The Fourier series for $\log\Gamma$,

$$\log\Gamma\left(\frac{x}{2\pi}\right)=\frac{\log(2\pi)}{2}+\sum_{n=1}^{\infty}\frac{1}{2n}\cos(nx)+\sum_{n=1}^{\infty}\frac{\gamma+\log(2\pi n)}{\pi n}\sin(nx),$$

for $0<x<2\pi$, is due to Kummer. (Beitrag zur theorie der function $\Gamma(x)$. *J. Reine Angew. Math.* **35** (1847) 1–4.) The other series is an exponential:

$$\sum_{n=1}^{\infty}\frac{\sin nx}{(n-1)!}=\operatorname{Im}\left(\sum_{n=1}^{\infty}\frac{e^{inx}}{(n-1)!}\right)=\operatorname{Im}\left(e^{ix}\sum_{n=0}^{\infty}\frac{e^{inx}}{n!}\right)$$
$$=\operatorname{Im}\left(e^{ix}e^{e^{ix}}\right)=\operatorname{Im}e^{cos x+i(x+\sin x)}=e^{\cos x}\sin(x+\sin x).$$

The two factors are both square-integrable, so Parseval's formula yields

$$\int_0^{2\pi}\log\Gamma\left(\frac{x}{2\pi}\right)e^{cos x}\sin(x+\sin x)=\sum_{n=1}^{\infty}\frac{\gamma+\log(2\pi n))}{\pi n(n-1)!}\int_0^{2\pi}\sin^2(nx)\,dx$$
$$=\sum_{n=1}^{\infty}\frac{\gamma+\log(2\pi)+\log n}{n!}=\big(\gamma+\log(2\pi)\big)\sum_{n=1}^{\infty}\frac{1}{n!}+\sum_{n=1}^{\infty}\frac{\log n}{n!}$$
$$=(e-1)\big(\gamma+\log(2\pi)\big)+\sum_{n=2}^{\infty}\frac{\log n}{n!}.$$

Also solved by D. Beckwith, R. Boukharfane (France), R. Chapman (U. K.), H. Chen, B. E. Davis, M. L. Glasser, R. Stong, FAU Problem Solving Group, and the proposer.

A Gamma Integral Limit

11808 [2015, 947]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let Γ be the gamma function. Compute

$$\lim_{n\to\infty}n^2\int_{((n+1)!)^{-1/(n+1)}}^{(n!)^{-1/n}}\Gamma(nx)\,dx.$$

Solution by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata," Rome, Italy. We will show that if f is a continuous function on a neighborhood of e, then

$$\lim_{n\to\infty}n^2\int_{((n+1)!)^{-1/(n+1)}}^{(n!)^{-1/n}}f(nx)\,dx=ef(e).$$

The given problem is the case $f=\Gamma$, and the limit in that case is $e\Gamma(e)$, which is equal to $\Gamma(e+1)$. Let $b_n=n(n!)^{-1/n}$ and $a_n=n((n+1)!)^{-1/(n+1)}$. By the mean value theorem for integrals, there exists $t_n\in(a_n,b_n)$ such that

$$n^2\int_{((n+1)!)^{-1/(n+1)}}^{(n!)^{-1/n}}f(nx)\,dx=n\int_{a_n}^{b_n}f(x)\,dx=n(b_n-a_n)f(t_n).$$

Now, by Stirling's approximation formula,

$$\ln n! = n\ln n - n + \frac{1}{2}\ln n + \ln\left(\sqrt{2\pi}\right) + O(1/n).$$

Hence

$$b_n = n\exp(-\ln(n!)/n) = e - \frac{e\ln n}{2n} - \frac{e\ln\left(\sqrt{2\pi}\right)}{n} + O\left(\ln^2 n/n^2\right)$$

and

$$b_n - a_n = b_n - \frac{nb_{n+1}}{n+1} = \frac{e}{n} + O(\ln^2 n/n^2).$$

These imply

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} n(b_n - a_n) = e.$$

Therefore the continuity of f at e gives

$$\lim_{n\to\infty} n(b_n - a_n) f(t_n) = ef(e).$$

Also solved by K. F. Andersen (Canada), G. E. Bilodeau, P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), O. Kouba (Syria), O. P. Lossers (Netherlands), I. Mezö (Hungary), M. Omarjee (France), P. Perfetti (Italy), R. Stong, M. Vowe (Switzerland), H. Widmer (Switzerland), and the proposers.

A Variant Alternating Series Test

11809 [2015, 947]. *Proposed by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria.* Let $\langle a_n\rangle$ be a sequence of real numbers.
(**a**) Suppose that $\langle a_n\rangle$ consists of nonnegative numbers and is nonincreasing, and $\sum_{n=1}^{\infty} a_n/\sqrt{n}$ converges. Prove that $\sum_{n=1}^{\infty}(-1)^{\lfloor\sqrt{n}\rfloor}a_n$ converges.
(**b**) Find a nonincreasing sequence $\langle a_n\rangle$ of positive numbers such that $\lim_{n\to\infty}\sqrt{n}a_n = 0$ and $\sum_{n=1}^{\infty}(-1)^{\lfloor\sqrt{n}\rfloor}a_n$ diverges.

Solution by John H. Lindsey II, Cambridge, MA.
(**a**) We claim that the sequence $\langle S_n\rangle$ defined by

$$S_n = \sum_{k=1}^{n}(-1)^{\lfloor\sqrt{k}\rfloor}a_k$$

is a Cauchy sequence. To bound $\sum_{k=m}^{n}(-1)^{\lfloor\sqrt{k}\rfloor}a_k$ above, if $\lfloor\sqrt{m}\rfloor$ is odd, delete all the terms involving $\lfloor\sqrt{m}\rfloor$; if $\lfloor\sqrt{m}\rfloor$ is even, insert all missing terms involving $\lfloor\sqrt{m}\rfloor$. Similarly delete or insert terms involving $\lfloor\sqrt{n}\rfloor$. (If $\lfloor\sqrt{n}\rfloor = \lfloor\sqrt{m}\rfloor$ and this is odd, then this tells us to delete all the terms to get 0 as an upper bound; otherwise, assume some terms remain.) Thus, given $m < n$, there exist i and j with $i \le j$ such that

$$S_n - S_{m-1} = \sum_{k=m}^{n}(-1)^{\lfloor\sqrt{k}\rfloor}a_k \le \sum_{k=(2i)^2}^{(2j+1)^2-1}(-1)^{\lfloor\sqrt{k}\rfloor}a_k$$

$$\le \sum_{k=(2i)^2}^{(2i+1)^2-1}a_k + \sum_{k=i+1}^{j}\left[2a_{(2k+1)^2-2} + \sum_{l=0}^{(2k)^2-(2k-1)^2-1}\left(a_{(2k)^2+l} - a_{(2k-1)^2+l}\right)\right]$$

$$\leq (4i+1)a_{(2i)^2} + \sum_{k=i+1}^{j} 2a_{(2k+1)^2-2} \leq (4i+1)a_{(2i)^2} + 2\sum_{k=i+1}^{j} \frac{1}{8k} \sum_{l=(2k-1)^2-1}^{(2k+1)^2-2} a_l$$

$$\leq (4i+1)a_{(2i)^2} + \sum_{k=i+1}^{j} \sum_{l=(2k-1)^2-1}^{(2k+1)^2-2} \frac{a_l}{\sqrt{l}} \leq (4i+1)a_{(2i)^2} + \sum_{l=(2i+1)^2-1}^{(2j+1)^2-2} \frac{a_l}{\sqrt{l}}.$$

Note that $i \to \infty$ as $m \to \infty$. We claim this last expression tends to 0 as $i \to \infty$. The second term tends to 0 by the assumption that $\sum a_l/\sqrt{l}$ converges. So we must show that the first term also tends to 0. Assume not. For some $\varepsilon > 0$, assume that

$$(4i+1)^2 a_{(2i)^2} > \varepsilon$$

infinitely often. Recursively take $i_k > 2i_{k-1}$ for $k \geq 1$, with

$$(4i_k+1)^2 a_{(2i_k)^2} > \varepsilon.$$

Now

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \geq \sum_{k=1}^{\infty} \sum_{l=i_k^2}^{(2i_k)^2} \frac{a_l}{\sqrt{l}} \geq \sum_{k=1}^{\infty} 3i_k^2 \frac{a_{(2i_k)^2}}{2i_k} \geq \frac{3}{2} \sum_{k=1}^{\infty} i_k a_{(2i_k)^2}$$

$$\geq \frac{3}{10} \sum_{k=1}^{\infty} (4i_k+1) a_{(2i_k)^2} \geq \frac{3}{10} \sum_{l=1}^{\infty} \varepsilon = +\infty,$$

a contradiction. This proves $\limsup_{m<n}(S_n - S_{m-1}) \leq 0$.

Similarly, bound below

$$S_n - S_{m-1} = \sum_{k=m}^{n} (-1)^{\lfloor\sqrt{k}\rfloor} a_k \geq \sum_{k=(2i-1)^2}^{(2j)^2-1} (-1)^{\lfloor\sqrt{k}\rfloor} a_k$$

and proceed as before to conclude $\liminf_{m<n} S_n - S_{m-1} \geq 0$. Thus $\langle S_n \rangle$ is a Cauchy sequence.

(**b**) For $k \geq 2$ and $(2k-1)^2 \leq n < (2k+1)^2$, let $a_n = \frac{1}{k \log k}$. Now

$$\sqrt{n} a_n = \frac{\sqrt{n}}{k \log k} \leq \frac{\sqrt{(2k+1)^2}}{k \log k} = \frac{2k+1}{k \log k} \to 0,$$

$$\sum_{n=9}^{(2i+1)^2-1} (-1)^{\lfloor\sqrt{n}\rfloor} a_n = \sum_{k=2}^{i} \sum_{n=(2k-1)^2}^{(2k+1)^2-1} (-1)^{\lfloor\sqrt{n}\rfloor} a_n = \sum_{k=2}^{i} \frac{2}{k \log k} \to +\infty.$$

Editorial comment. John Zacharias notes that Problem 11809 may be used to provide a solution for Problem 11384 (October, 2008): Let p_n be the nth prime. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{p_n}$$

converges. In (2010, 745) a defective solution was published, making an unjustified use of the alternating series test, as was noted in the End Notes (2011, 945). Zacharias notes that the elementary bound $p_n > (n \log n)/4$ can be combined with Problem 11809 to solve Problem 11384.

Also solved by R. Brase, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), D. Fleischman, O. P. Lossers (Netherlands), M. Omarjee (France) & R. Tauraso (Italy), K. Schilling, N. C. Singer, A. Stenger, R. Stong, J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer. Part (a) only solved by E. A. Herman. Part (b) only solved by P. Perfetti (Italy).

A Binary Operation Whose Closed Sets Form a Chain

11813 [2015, 76]. *Proposed by Greg Oman, University of Colorado–Colorado Springs, Colorado Springs, CO.* Let X be a set, and let $*$ be a binary operation on X (that is, a function from $X \times X$ to X). Prove or disprove: there exists an uncountable set X and a binary operation $*$ on X such that for any subsets Y and Z of X that are closed under $*$, either $Y \subseteq Z$ or $Z \subseteq Y$.

Solution by Gary Gruenhage, Auburn University, Auburn, AL. Such sets X and binary operations $*$ do exist, and $*$ can even be commutative. Let X be the set of countable ordinals. Define $0 * 0 = 0$, and for a finite ordinal n, define $n * n = 0$ and $n * i = i * n = i + 1$ for $i < n$. For infinite $\alpha \in X$, let $\beta_0, \beta_1, \ldots$ be a one-to-one indexing of the ordinals less than α, and define $\alpha * \alpha = \beta_0$ and $\alpha * \beta_i = \beta_i * \alpha = \beta_{i+1}$ for all $i < \omega$. This defines $\alpha * \beta$ for all $\alpha, \beta \in X$, and we note that $\alpha * \beta \le \max\{\alpha, \beta\}$.

The closure $\sigma(Y)$ under $*$ of a subset Y of X is the smallest subset of X containing Y that is closed under $*$, and Y is closed under $*$ if and only if $Y = \sigma(Y)$. One sees that the closure under $*$ of $\{\alpha\}$ is $\{\beta \in X : \beta \le \alpha\}$. It follows for any set Y that $\sigma(Y) = \{\alpha \in X : \alpha < \delta(Y)\}$, where $\delta(Y)$ is the least ordinal strictly greater than every element of Y (note that $\delta(Y) = \omega \notin X$ if Y is uncountable). Hence, if Y and Z are subsets of X that are closed under $*$, then $Y \subseteq Z$ if and only if $\delta(Y) \le \delta(Z)$.

Remark 1. There is no such $*$ on a set $X > \aleph_1$, and so no such $*$ exists on the set of real numbers if and only if the continuum hypothesis holds. To see this, for each $x \in X$ let $C(x)$ be the closure under $*$ of $\{x\}$. It is evident that $C(x)$ is countable. Now let $Y \subset X$ have cardinality $\aleph_1$ and choose $x \in X - \bigcup_{y \in Y} C(y)$. As Y is uncountable, there exists some $y \in Y - C(x)$. Thus $C(X)$ and $C(Y)$ are not comparable.

Remark 2. There is also no such $*$ that is associative. For, let X be uncountable with associative operation $*$. By associativity, x^n is uniquely defined for each positive integer n. Let $C(x)$ be the set of all positive integral powers of x; it follows that $C(x)$ is the closure of $\{x\}$.

If $C(x)$ is finite for every $x \in X$, then there is a positive integer k such that $\{x\colon |C(x)| = k\}$ is infinite. Pick any x with $|C(x)| = k$, and choose $y \notin C(x)$ such that $|C(y)| = k$. Now $C(x)$ and $C(y)$ are not comparable.

On the other hand, if $C(x)$ is infinite for some $x \in X$, then we claim $x^n \ne x^m$ for $n \ne m$. If not, then there exist p and q such that $p < q$ and $x^p = x^q$. Let $d = q - p$, and choose n with $n \ge q$. We have $x^n = x^q * x^{n-q} = x^p * x^{n-q} = x^{n-d}$. Thus $C(x) = \{x^m\colon m < q\}$, a contradiction. Finally, the sets $\{x^n\colon n \text{ even}\}$ and $\{x^n\colon n \equiv 0 \bmod 3\}$ are incomparable closed sets.

Editorial comment. Solvers Hart, Konieczny, and Schilling also proved that the statement fails when the cardinality of X exceeds $\aleph_1$; Burdick and Pagano/Tauraso also proved that the statement fails if $*$ is associative.

Also solved by B. Burdick, R. Chapman (U. K.), S. J. Garland, K. P. Hart (Netherlands), S. J. Herschkorn, J. Konieczny, C. Pagano (Netherlands) & R. Tauraso (Italy), P. G. Poonacha (India), K. Schilling, D. Ware, M. Wildon (U. K.), and the proposer.

A Symmetric Inequality

11815 [2015, 76]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let x, y, and z be positive numbers such that $x + y + z = 3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}\ge 3xyz.$$

Solution I by Ben Keigwin (student), West Potomac High School, Alexandria, VA. Note that $x^4+x^2+1=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$. Also $\frac{x^4+x^2+1}{x^2+x+1}=x^2-x+1=(x-1)^2+x\ge x$, so

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}\ge x+y+z=3.$$

By the AM–GM inequality, $1=\left(\frac{x+y+z}{3}\right)^3\ge xyz$, and the result follows upon multiplication.

Solution II by Ali Adnan, A.E.C.S.-4, Mumbai, India. Since $\frac{x^4+x^2+1}{x^2+x+1}=x^2-x+1$, the left hand side reduces to

$$x^2+y^2+z^2=3\left(\frac{x^2+y^2+z^2}{3}\right)\ge 3\left(\frac{x+y+z}{3}\right)^2=3\left(\frac{x+y+z}{3}\right)^3\ge 3xyz$$

by the QM–AM–GM inequality.

Also solved by A. Alt, T. Amdeberhan & V. H. Moll, M. Atasever (Turkey), R. Bagby, D. Bailey, E. Campbell, & C. Diminnie, M. Bataille (France), P. Bracken, B. Bradie, R. Chapman (U. K.), H. Chen, (S. Choi, D. Kim, S. Y. Kim, J. Lee, L. W. Lee, & S. Lee) (Korea), J. Christopher, P. P. Dályay (Hungary), (M. Dan-Ştefan, C. Cătălin-Emil, & O. Alexandru) (Romania), P. De (India), M. Dinca (Romania), H. Y. Far, O. Faynshteyn, J. N. Fitch, D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, N. Grivaux (France), E. A. Herman, S. J. Herschkorn, E. J. Ionaşcu, Y. J. Ionin, B. Karaivanov (U.S.) & T. S. Vassilev (Canada), D. Kasti, P. Khalili, O. Kouba (Syria), K. Kusejko (Switzerland), (W.-K. Lai, A. Kristyuk, C. Kalacanic, & J. Risher), K.-W. Lau (China), J. H. Lindsey II, G. Lord, O. P. Lossers (Netherlands), J. F. Loverde, U. Milutinović (Slovenia), S. G. Moreno (Spain), M. Omarjee (France), J. Pentland, P. Perfetti (Italy), H. H. Pham, I. Pinelis, Á. Plaza (Spain), M. A. Prasad (India), A. Ranallo (Italy), E. Schmeichel, B. Schmuland (Canada), M. A. Shayib, A. V. Singh (India), A. Stadler (Switzerland), N. Stanciu & T. Zvonaru (Romania), R. Stong, R. Tauraso (Italy), N. Thornber, D. B. Tyler, D. Văcaru (Romania), J. Van Hamme (Belgium), E. I. Verriest, T. Viteam (India), Z. Vörös (Hungary), G. White, T. Wiandt, H. Widmer (Switzerland), M. Wildon (U. K.), T. Woodcock, J. Zacharias, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), New York Math Circle, NSA Problems Group, Texas State University Problem Solvers Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Steven J. Miller, Mohamed Omar, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems should be submitted online via www.americanmathematicalmonthly.submittable.com. Proposed solutions to the problems below should be submitted on or before April 30, 2017 *at the same link. More detailed instructions are available online. Solutions to problems numbered* 11921 *or lower should continue to be submitted via email to monthlyproblems@math.tamu.edu. Proposed problems must not be under consideration concurrently to any other journal and must not be posted to the internet before the deadline date for solutions. An asterisk* (*) *after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11943. *Proposed by Keith Kearnes, University of Colorado, Boulder, CO, and Greg Oman, University of Colorado, Colorado Springs, CO.* Let X be a set, and let $\mathcal{F}$ be a collection of functions f from X into X. A subset Y of X is *closed* under $\mathcal{F}$ if $f(y) \in Y$ for all $y \in Y$ and f in $\mathcal{F}$. With the axiom of choice given, prove or disprove: There exists an uncountable collection $\mathcal{F}$ of functions mapping $\mathbb{Z}^+$ into $\mathbb{Z}^+$ such that
(a) every proper subset of $\mathbb{Z}^+$ that is closed under $\mathcal{F}$ is finite, and
(b) for every $f \in \mathcal{F}$, there is a proper infinite subset Y of $\mathbb{Z}^+$ that is closed under $\mathcal{F}\backslash\{f\}$.

11944. *Proposed by Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Let n be a positive integer, and let $[n] = \{1, \ldots, n\}$. For $i \in [n]$, let A_i, B_i, C_i be disjoint sets such that $A_i \cup B_i \cup C_i = [n] - \{i\}$ and $|A_i| = |B_i|$. Suppose also that

$$|A_i \cap B_j| + |B_i \cap C_j| + |C_i \cap A_j| = |B_i \cap A_j| + |C_i \cap B_j| + |A_i \cap C_j|$$

for $i, j \in [n]$. Prove that $i \in A_j$ if and only if $j \in A_i$ and, likewise, for the Bs and Cs.

11945. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, Macedonia.* Let a, b, and c be the lengths of the sides of triangle ABC opposite A, B, and C, respectively, and let w_a, w_b, w_c be the lengths of the corresponding angle bisectors. Prove

$$\frac{a}{w_a} + \frac{b}{w_b} + \frac{c}{w_c} \geq 2\sqrt{3}.$$

11946. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let f be a twice differentiable function from $[0, 1]$ to $\mathbb{R}$ with f'' continuous on $[0, 1]$ and $\int_{1/3}^{2/3} f(x)\,dx = 0$. Prove

http://dx.doi.org/10.4169/amer.math.monthly.123.10.1050

$$4860\left(\int_0^1 f(x)\,dx\right)^2 \le 11\int_0^1 f''(x)\,dx.$$

11947. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let n be a positive integer, and let $z_1, \dots, z_n$ be the zeros in $\mathbb{C}$ of $z^n + 1$. For $a > 0$, prove

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k - a|^2} = \frac{1 + a^2 + \cdots + a^{2(n-1)}}{(1 + a^n)^2}.$$

11948. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Find all surjective functions $f\colon \mathbb{R} \to \mathbb{R}^+$ such that (1) $f(x) \le x + 1$ for $f(x) \ge 1$, (2) $f(x) \ne 1$ for $x \ne 0$, and (3) for $x, y \in \mathbb{R}$,

$$f(xf(y) + yf(x) - xy) = f(x)f(y).$$

11949. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* Show that there exists a unique function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is differentiable, $2\cos(x + f(x)) - \cos x = 1$ for all real x, and $f(\pi/2) = -\pi/6$.

SOLUTIONS

Flett's Mean Value Theorem

11814 [2015, 76]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let ϕ be a continuously differentiable function from $[0, 1]$ into $\mathbb{R}$, with $\phi(0) = 0$ and $\phi(1) = 1$, and suppose that $\phi'(x) \ne 0$ for $0 \le x \le 1$. Let f be a continuous function from $[0, 1]$ into $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = \int_0^1 \phi(x)f(x)\,dx$. Show that there exists t with $0 < t < 1$ such that $\int_0^t \phi(x)f(x)\,dx = 0$.

Solution by New York Math Circle, NY. Define

$$h(s) = \int_0^s \phi(x)f(x)\,dx - \phi(s)\int_0^s f(x)\,dx.$$

Note that $h(0) = 0 = h(1)$. From Rolle's theorem, we obtain $h'(c) = 0$ for some $c \in (0, 1)$. Also, we compute

$$h'(s) = -\phi'(s)\int_0^s f(x)\,dx$$

and, in particular, $h'(0) = 0$. Since $\phi'(s) \ne 0$ for all $s \in (0, 1)$, the inverse function $\phi^{-1}(s)$ exists and is differentiable on $(0, 1)$. Letting $H(s) = h(\phi^{-1}(s))$, we see that

$$H'(s) = \frac{h'(\phi^{-1}(s))}{\phi'(\phi^{-1}(s))}.$$

Applying Flett's mean value theorem [*Math. Gaz.* **42** (1958) 38–39] to the function H on the interval $[0, \phi(c)]$, we have

$$\frac{H(T) - H(0)}{T - 0} = H'(T)$$

for some $T \in (0, \phi(c)) \subset (0, 1)$. Setting $t = \phi^{-1}(T) \in (0, 1)$, this becomes

$$\frac{1}{\phi(t)} \int_0^t \phi(x) f(x)\, dx - \int_0^t f(x)\, dx = \frac{h(t)}{\phi(t)} = \frac{h'(t)}{\phi'(t)} = -\int_0^t f(x)\, dx.$$

Thus $\int_0^t \phi(x) f(x)\, dx = 0$, as desired.

Also solved by K. F. Andersen (Canada), R. Bagby, M. W. Botsko, B. S. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), M. Dan-Ştefan & F. Cătălin-Emil & O. Alexandru (Romania), L. Giugiuc (Romania), D. Hancock, E. A. Herman, O. Kouba (Syria), J. K. Lindsey II, O. P. Lossers (Netherlands), A. Mingarelli & J. M. Pacheco & Á. Plaza (Spain), P. Perfetti (Italy), I. Pinelis, D. Ritter, B. Schmuland (Canada), R. Stong, R. Tauraso (Italy), T. P. Turiel, T. Wiandt, M. Wildon (U. K.), and the proposer.

Pascal's Theorem

11816 [2015, 76]. *Proposed by Sabin Tabirca, University College Cork, Cork, Ireland.* Let ABC be an acute triangle, and let B_1 and C_1 be the points where the altitudes from B and C intersect the circumcircle. Let X be a point on arc BC, and let B_2 and C_2 denote the intersections of XB_1 with AC and XC_1 with AB, respectively. Prove that the line B_2C_2 contains the orthocenter of ABC.

Solution by Adnan Ali, A.E.C.S.-4, Mumbai, India. The claim holds not only for the circumcircle of $\triangle ABC$ but also for any circumconic of the triangle—i.e. a conic circumscribing the triangle—as this problem is a special case of Pascal's theorem, according to which, if $ABCDEF$ is a hexagon with vertices on a conic, then the intersections of lines AB with ED, AF with CD, and EF with CB are collinear. (A point at infinity is allowed.)

Also solved by M. Atasever (Turkey), M. Bataille (France), B. S. Burdick, J. Cade, R. B. Campos (Spain), R. Chapman (U. K.), P. P. Dályay (Hungary), M. Dan-Ştefan & O. Alexandru & F. Cătălin-Emil (Romania), P. De (India), O. Faynshteyn, D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), J. G. Heuver (Canada), S. Hong (Korea), E. J. Ionaşcu, Y. J. Ionin, I. M. Isaacs, O. Kouba (Syria), G. Lord, O. P. Lossers (Netherlands), J. Minkus, M. A. Shayib, N. Stanciu & T. Zvonaru (Romania), R. Stong, T. Viteam (India), Z. Vörös (Hungary), T. Wiandt, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

Cycle Covers for Infinite Complete Graphs

11817 [2015, 175]. *Proposed by Mohammed Jahaveri, Siena College, Loudonville, NY.* A *cycle double cover* of a graph is a collection of cycles that, counting multiplicity, includes every edge exactly twice. Let X be an infinite set and let K_X be the complete graph on X. Construct a cycle double cover for X.

Solution I by I. M. Isaacs, Berkeley, CA. We construct a set of triangles covering each edge exactly once. Taking each triangle twice produces a cycle double cover. We may replace X by another set of the same cardinality, so we consider the set S of all nonempty finite subsets of X. For each edge AB in K_S, let $C = A \triangle B$ (the symmetric difference), and use the triangle with vertices A, B, C. Since $B \triangle C = A$ and $C \triangle A = B$, each edge lies in exactly one such triangle.

Solution II by Jerrold Grossman and László Lipták, Oakland University, Rochester, MI. We partition the edges into triangles. Use the axiom of choice to well order the edges of K_X using an ordinal that is also a cardinal. Transfinitely perform the following operation as long as there remains an edge not yet covered: For the least such uncovered edge uv, choose a vertex w not yet in any triangle, and add uvw to the set of triangles. Such a vertex w exists because the cardinality of the set of vertices used so far in the process is less than the cardinality of X.

Editorial comment. The method of Grossman and Lipták can be used to partition edges of K_X into copies of G for any finite graph G.

Also solved by E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), B. Burdick, R. Chapman (U. K.), B. Karaivanov & T. Vassilev (Canada), R. Stong, FAU Problem Solving Group, and the proposer.

Some Trig for the Nagel Cevians

11818 [2015, 175]. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* Let ABC be a triangle and let A_1, B_1, and C_1 be the points on sides opposite A, B, and C respectively at which the ecircles of the triangle are tangent to those sides. Let R and r be the circumradius and inradius of the triangle. Let the name of a vertex of ABC or of $A_1B_1C_1$ also stand for the radian measure of the corresponding angle. Prove that, wherever the expression is defined,

$$\frac{\cot A_1 + \cot(A/2)}{\cot A} + \frac{\cot B_1 + \cot(B/2)}{\cot B} + \frac{\cot C_1 + \cot(C/2)}{\cot C} = \frac{6R}{r}.$$

Solution by P. Nüesch, Switzerland. In fact, more is true: Each term on the left side of the identity equals $2R/r$. Write a, b, c for the side lengths of $\triangle ABC$, s for the semiperimeter, and F for the area. Write u, v, w for the side lengths of $\triangle A_1B_1C_1$ and F_1 for the area. Now

$$\frac{F}{F_1} = \frac{2R}{r}. \tag{1}$$

By the law of cosines,

$$\frac{\cot(A/2)}{\cot A} = \frac{1}{\cos A} + 1 = \frac{4s(s-a)}{b^2+c^2-a^2}. \tag{2}$$

The modified cosine laws

$$\cot A = \frac{b^2+c^2-a^2}{4F}, \qquad \text{and} \qquad \cot A_1 = \frac{v^2+w^2-u^2}{4F_1}$$

together with (1) give us

$$\frac{\cot A_1}{\cot A} = \frac{2R}{r}\,\frac{v^2+w^2-u^2}{b^2+c^2-a^2}. \tag{3}$$

Now we have to prove (3) + (2) $= 2R/r$, or equivalently,

$$\frac{2R}{r}\left[(b^2-v^2)+(c^2-w^2)-(a^2-u^2)\right] = 4s(s-a).$$

Observe that $b^2 - v^2 = 2(s-c)(s-a)(1+\cos B) = 4(s-c)(s-a)\cos^2(B/2) = bF/R$ and similarly for the other two sides. Therefore, as required,

$$\frac{2R}{r}\left[\frac{bF}{R} + \frac{cF}{R} - \frac{aF}{R}\right] = \frac{2R}{r}[b+c-a] = 2s \cdot 2(s-a) = 4s(s-a).$$

Editorial comment. Lines AA_1, BB_1, and CC_1 are the *Nagel cevians* of the triangle.

Also solved by A. Alt, R. Bagby, R. Chapman (U. K.), H. Y. Far, M. E. Kuczma (Poland), J. C. Smith, R. Stong, H. Widmer, and the proposer.

Twin Hölders

11819 [2015, 175]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let f be a continuous, nonnegative function on [0, 1]. Show that

$$\int_0^1 f^3(x)\,dx \geq 4\left(\int_0^1 x^2 f(x)\,dx\right)\left(\int_0^1 x f^2(x)\,dx\right).$$

Solution by Radouan Boukharfane, Université du Poitiers, Chasseneuil, France. We apply Hölder's inequality twice

$$\int_0^1 x^2 f(x)\,dx \leq \left(\int_0^1 x^3\,dx\right)^{2/3}\left(\int_0^1 f^3(x)\,dx\right)^{1/3}$$

$$\int_0^1 x f^2(x)\,dx \leq \left(\int_0^1 x^3\,dx\right)^{1/3}\left(\int_0^1 f^3(x)\,dx\right)^{2/3}.$$

Now multiply the inequalities

$$\int_0^1 x^2 f(x)\,dx \int_0^1 x f^2(x)\,dx \leq \left(\int_0^1 x^3\,dx\right)\left(\int_0^1 f^3(x)\,dx\right) = \frac{1}{4}\left(\int_0^1 f^3(x)\,dx\right).$$

Editorial comment. Several solvers proved generalizations. For example, the argument above, using the conjugate exponents $(a+b)/a$ and $(a+b)/b$, yields $\int f^a(x) g^b(x)\,dx \int f^b(x) g^a(x)\,dx \leq \int f^{a+b}(x)\,dx \int g^{a+b}(x)\,dx$.

Also solved by R. A. Agnew, A. Alt, T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), R. Bagby, M. Bataille (France), P. Bracken, M. A. Carlton, R. Chapman (U. K.), H. Chen, L. V. P. Cuong (Vietnam), P. J. Fitzsimmons, W. R. Green, N. Grivaux (France), E. A. Herman, B. Karaivanov (USA) & T. S. Vazzilev (Canada), O. Kouba (Syria), M. E. Kuczma (Poland), K.-W. Lau (China), J. H. Lindsey II, P. W. Lindstrom, M. Omarjee (France), X. Oudot (France), P. Perfetti (Italy), Á. Plaza & F. Perdomo (Spain), K. Schilling, J. G. Simmonds, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), H. Wang & J. Wojdylo, G. White, Q. Zhang (China), Z. Zhang (China), NSA Problems Group, and the proposer.

Noetherian Subrings

11820 [2015, 175]. *Proposed by Alborz Azarang, Shahid Chamran University of Ahvaz, Ahvaz, Iran.* Let K be a field and let R be a subring of $K[X]$ that contains K. Prove that R is noetherian, that is, that every ascending chain of ideals in R terminates.

Solution by the National Security Agency Problems Group, Fort Meade, MD. Since a finitely generated K-algebra is a quotient of $K[x_1, \ldots, x_n]$ for some n and, hence, is noetherian, it suffices to show that R is finitely generated as a K-algebra. We use a lemma of independent interest.

Lemma. Any set S of nonnegative integers that is closed under addition is finitely generated: That is, there are elements $d_1, \ldots, d_n \in S$ such that every $s \in S$ can be written as $s = \sum_{k=1}^n e_k d_k$ for some nonnegative integers $e_1, \ldots, e_n$.

Proof. This is clear if S is empty or equals $\{0\}$. Otherwise, let n be the least positive integer in S. For $1 \leq i < n$, let d_i be the least element of S congruent to i (modulo n), or $d_i = 0$ if S has no such element. We claim that $\{d_1, \ldots, d_{n-1}, n\}$ generates S. If $s \in S$, then $s \equiv d_i \bmod n$ for some i. Also, $s \geq d_i$. Hence, $s = d_i + kn$ for some nonnegative integer k.

Now let R be a subring of $K[X]$ that contains K. Let S be the degrees of the elements of R; note that S is closed under addition. By the lemma, there are integers $d_1, \ldots, d_n$ that generate S. For $a \leq i \leq n$, let f_i be a monic polynomial in

R of degree d_i. Using induction on the degree m of a polynomial in R, we prove $R = K[f_1, \ldots, f_n]$. For $m = 0$, note that the constant polynomials are in R. For $m > 1$, let a be the leading coefficient of a polynomial f in R. There are integers $e_i \geq 0$ such that $m = \sum_{k=1}^{n} e_k d_k$. Let $g = f - af_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}$. Note that g is in R and has degree less than m. By the induction hypothesis, $g \in K[f_1, \ldots, f_n]$. Hence, also, $f \in K[f_1, \ldots, f_n]$, as desired.

Editorial comment. Various solvers used theorems from commutative algebra such as the Eakin–Nagata theorem, the Artin–Tate theorem, and the Hilbert basis theorem, as well as the chicken McNugget theorem, which is also known as the Frobenius coin problem from number theory.

Also solved by A. J. Bevelacqua, T. Borislav (Canada) & V. Karaivanov, N. Caro (Brazil), R. Chapman (U. K.), I. M. Isaacs, J. H. Lindsey II, F. Perdomo & A. Francisco (Spain), J. C. Smith, R. Stong, D. Ware, and the proposer.

When a Composition of Polynomials Is Real

11822 [2015, 176]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Call a polynomial *real* if all its coefficients are real. Let P and Q be polynomials with complex coefficients such that the composition $P \circ Q$ is real. Show that if the leading coefficient of Q and its constant term are both real then P and Q are real.

Solution by Borislav Karaivanov, Sigma Space, Lanham, MD & Tzvetalin Vassilev, Nipissing University, North Bay, ON, Canada. Let P and Q be defined as $P(x) = \sum_{i=0}^{p} a_i x^i$ and $Q(x) = \sum_{i=0}^{q} b_i x^i$ with b_q and b_0 real. Since both b_q and the coefficient $a_p b_q^p$ of x^{pq} in $P(Q(x))$ are real, it can be concluded that a_p is real.

We first claim that Q is real. If not, then let k be the largest index for which b_k is not real. The coefficient of $x^{q(p-1)+k}$ in $P(Q(x))$ is real and has the form $a_p b_q^{p-1} b_k + M$, where M is a polynomial expression in a_p and b_i, for $k < i \leq q$, with integer coefficients, and thus real. Hence, b_k must be real. This contradiction shows that Q is real.

Next, we claim that P is real. If not, then let k be the largest index for which a_k is not real. Consider the coefficient of x^{qk} in $P(Q(x))$. By the premise of the problem, it is real. On the other hand, it has the form $a_k b_q^k + N$, where N is a polynomial expression in the real variables $a_{k+1}, \ldots, a_p$ and $b_0, \ldots, b_q$ with integer coefficients . Therefore, there is no such a_k, and P is real.

Also solved by B. Bekker (Russia) & Y. J. Ionin (USA), A. J. Bevelacqua, N. Caro (Brazil), R. Chapman (U.K.), P. De, B. Sury (India) & N. V. Tejaswi (Netherlands), D. Fleischman, J.-P. Grivaux (France), E. A. Herman, P. W. Lindstrom, R. Stong, R. Tauraso (Italy), N. Thornber, J. Van Hamme (Belgium), T. Viteam (Japan), and the proposer.

Inversion in a Circle?

11823 [2015, 176]. *Proposed by Sabin Tabirca, University College Cork, Cork, Ireland.* Let P be a point inside a circle C.

(**a**) Prove that there exists a point P' outside C such that, for all chords XY of C through P, $(|XP'| + |YP'|)/|XY|$ is the same. (Here, $|UV|$ denotes the distance from U to V.)

(**b**) Is P' unique?

Solution by Ahmad Habil, Damascus University, Damascus, Syria. Let O denote the center of C, r the radius of C, and p the distance of P from O. We must exclude the case $P = O$ (i.e., $p = 0$) because in that case there is no suitable P' at all. We (1) show this then (2) show that, for any other P inside C, the image of P under inversion in C can serve as P' and then finally (3) show that no other point can play the role of P' (so in answer to (**b**), P' is unique).

(1) Suppose $P = O$, and let Q be any "trial" point outside C. We will demonstrate that Q cannot meet the requirement for P' by finding two chords X_1Y_2 and X_2Y_2 through P for which the ratios $(|X_jQ| + |Y_jQ|)/|X_jY_j|$ are unequal. Let $q = |OQ|$. Let X_1, Y_1 be the diameter lying along OQ and let X_2, Y_2 be the perpendicular diameter. Now $(|X_1Q| + |Y_1Q|)/|X_1Y_1| = \frac{q}{r}$, but $(|X_2Q| + |Y_2Q|)/|X_2Y_2| = \sqrt{r^2 + q^2}/r > \frac{q}{r}$.

(2) Consider $P \neq O$, so $p \neq 0$. Let P' be the image of P under inversion in C. This is the point on ray OP such that $pp' = r^2$, where $p' = |OP'|$. When T is any point on C (but not on line OP), triangles TOP and $P'OT$ are similar, since they have a common angle at O and $|TP|/|OP| = |OP'|/|OT|$ or $\frac{r}{p} = \frac{p'}{r}$ from the definition of p'. Thus, $|TP'|/|OT| = |TP|/|OP|$, or $|TP'| = \frac{r}{p}|TP|$.

Now let XY denote any chord through P. We have both $|XP'| + \frac{r}{p}|XP|$ and $|YP'| = \frac{r}{p}|YP|$. Adding, $|XP'| + |YP'| = \frac{r}{p}(|XP| + |YP|)$. Since XPY is a straight line, $|XP| + |YP| = |XY|$. We conclude $(|XP'| + |YP'|)/|XY| = \frac{r}{p}$.

Note that the constant value of $(|XP'| + |YP'|)/|XY|$ must be $\frac{p'}{r}$, which equals $\frac{r}{p}$ and is the cosecant of half the angle intercepted by C at P'.

(3) Now let Q' be a point outside C such that $(|XQ'| + |YQ'|)/|XY|$ is constant for all chords XY containing P. We must show that Q' is the inversion image of P in C, that is, the point P' from part (2). Let Q be the image of Q' under inversion in C. Thus, $qq' = r^2$, where $q = |QO|$ and $q' = |Q'O|$. Inversion is self-dual, so our claim is that $Q = P$.

Applying part (2) starting with Q, we obtain that $(|XQ'| + |YQ'|)/|XY|$ is constant for all chords through Q. Consider the diameter X_1Y_1 containing Q and the chord X_2Y_2 lying along line PQ'. Label them so that X_i is closer than Y_i to Q' in each case. (These chords may be the same; in fact, we prove that they are.)

Now consider a chord $X'Y'$ through both P and Q. (If $P = Q$, as we will show, then there are infinitely many such chords, but in any case, there is at least one.)

Because both $X'Y'$ and X_1Y_1 include Q, we have $(|X'Q'| + |Y'Q'|)/|X'Y'| = (|X_1Q'| + |Y_1Q'|)/|X_1Y_1|$. Next, since both $X'Y'$ and X_2Y_2 include P, we have $(|X'Q'| + |Y'Q'|)/|X'Y'| = (|X_2Q'| + |Y_2Q'|)/|X_2Y_2|$. Therefore,

$$\frac{|X_1Q'| + |Y_1Q'|}{|X_1Y_1|} = \frac{|X_2Q'| + |Y_2Q'|}{|X_2Y_2|}.$$

Using $|Y_1Q'| + |X_1Q'| + |X_1Y_1|$ and $|Y_2Q'| = |X_2Q'| + |X_2Y_2|$, we get $(2|X_1Q'| + |X_1Y_1|)/|X_1Y_1| = (2|X_2Q'| + |X_2Y_2|)/|X_2Y_2|$. Subtracting 1 and dividing by 2 yields $|X_1Q'|/|X_1Y_1| = |X_2Q'|/|X_2Y_2|$. Inverting these ratios and adding 1 gives $(|X_1Q'| + |X_1Y_1|)/|X_1Q'| = (|X_2Q'| + |X_2Y_2|)/|X_2Q'|$. Since $Q'X_1Y_1$ and $Q'X_2Y_2$ are straight lines, $|Y_1Q'|/|X_1Q'| = |Y_2Q'|/|X_2Q'|$, or equivalently, $|Y_1Q'|/|Y_2Q'| = |X_1Q'|/|X_2Q'|$.

By the concurrent chords theorem, $|Y_2Q'|/|Y_1Q'| = |X_1Q'|/X_2Q'|$. Hence, $|Y_1Q'|/|Y_2Q'| = |Y_2Q'|/|Y_1Q'|$, so $|Y_2Q'| = |Y_1Q'|$, and in turn $|X_2Q'| = |X_1Q'|$. This implies that X_1Y_1 and X_2Y_2 are the same chord and, therefore, that P and Q are the same point. Finally, $Q' = P'$, and so the image of P under inversion is the unique point with the desired constant ratio property.

Editorial comment. Several solvers noted that circle C is the Circle of Apollonius determined by P and its image under inversion in C, using the ratio $(r-p)/(r+p)$ to get started. This observation provides another way to prove (**b**), that the image of P under inversion in C is a suitable P'.

Also solved by R. Bagby, M. Bataille (France), E. Bojaxhiu (Albania)& E. Hysnelaj (Australia), J. Cade, R. Chapman (U. K.), W. J. Cowieson, E. A. Herman, L. R. King, M. E. Kuczma (Poland), G. Lord, J. Schlosberg, J. C. Smith, N. Stanciu & T. Zvonaru (Romania), R. Stong, E. A. Weinstein, and the proposer.

A Binomial Coefficient Inequality

11826 [2015, 284]. *Proposed by Michel Bataille, Rouen, France.* Let m and n be positive integers with $m \le n$. Prove that

$$\sum_{k=m}^{n} 4^{n+1-k}\binom{m+k-1}{m-1}^2 \ge \sum_{k=m}^{n}\binom{m+n}{k}^2.$$

Solution by Timothy Woodcock, Stonehill College, Easton, MA. Equality holds when $n=m$ since $\binom{2m}{m} = 2\binom{2m-1}{m-1}$. Now suppose $n>m$, and inductively assume $\sum_{k=m}^{n-1} 4^{n-k}\binom{m+k-1}{m-1}^2 \ge \sum_{k=m}^{n-1}\binom{m+n-1}{k}^2$. We have

$$\begin{aligned}
\sum_{k=m}^{n} 4^{n+1-k}\binom{m+k-1}{m-1}^2 &= 4\binom{m+n-1}{m-1}^2 + 4\sum_{k=m}^{n-1} 4^{n-k}\binom{m+k-1}{m-1}^2 \\
&\ge 4\binom{m+n-1}{m-1}^2 + 4\sum_{k=m}^{n-1}\binom{m+n-1}{k}^2 = 4\sum_{k=m-1}^{n-1}\binom{m+n-1}{k}^2.
\end{aligned}$$

It now suffices to prove $4\sum_{k=m-1}^{n-1}\binom{m+n-1}{k}^2 \ge \sum_{k=m}^{n}\binom{m+n}{k}^2$. Since $(x+y)^2 \le 2(x^2+y^2)$ for $x, y \in \mathbb{R}$,

$$\begin{aligned}
\sum_{k=m}^{n}\binom{m+n}{k}^2 &= \sum_{k=m}^{n}\left(\binom{m+n-1}{k-1}+\binom{m+n-1}{k}\right)^2 \\
&\le \sum_{k=m}^{n} 2\left(\binom{m+n-1}{k-1}^2+\binom{m+n-1}{k}^2\right) \\
&= 4\binom{m+n-1}{n}^2 + 4\sum_{k=m}^{n-1}\binom{m+n-1}{k}^2 = 4\sum_{k=m}^{n}\binom{m+n-1}{k}^2.
\end{aligned}$$

Editorial comment. Allen Stenger notes that $(x+y)^p \le 2^{p-1}(x^p+y^p)$, valid for $x, y > 0$ and $p > 1$, may be used in place of $(x+y)^2 \le 2(x^2+y^2)$ to yield the generalization

$$\sum_{k=m}^{n} 2^{p(n+1-k)}\binom{m+k-1}{m-1}^p \ge \sum_{k=m}^{n}\binom{m+n}{k}^p.$$

Also solved by R. Chapman (U. K.), J. H. Lindsey II, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Daniel Cranston, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, and Fuzhen Zhang.

Proposed problems should be submitted online at
`http://www.americanmathematicalmonthly.submittable.com/submit.`
Proposed solutions to the problems below should be submitted on or before May 31, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11950. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Prove that for all positive integers a and b, there are infinitely many positive integers n such that n, $n + a$, and $n + b$ can all be expressed as a sum of two squares.

11951. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let ABC be a triangle that is not obtuse. Denote by a, b, and c the lengths of the sides opposite A, B, and C, respectively, and denote by h_a, h_b, and h_c the lengths of the altitudes dropped from A, B, and C, respectively. Prove that

$$\frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_c^2 + h_a^2} + \frac{c^2}{h_a^2 + h_b^2} < \frac{5}{2}.$$

Show also that $5/2$ is the smallest possible constant in this inequality.

11952. *Proposed by Z. K. Silagadze, Novosibirsk State University, Novosibirsk, Russia.* Prove that

$$\sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n+1} \left(\frac{(n-1)!}{(2n-1)!!} \right)^2 = \pi - 2,$$

where $(2n - 1)!!$ is defined as usual to be $\prod_{k=1}^{n}(2k - 1)$.

11953. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania.* Calculate

$$\int_0^\infty \int_0^\infty \frac{\sin x \, \sin y \, \sin(x + y)}{xy(x + y)} \, dx \, dy.$$

11954. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Determine the largest constant c and the smallest constant d such that, for all positive integers n,

$$\frac{1}{n - c} \le \sum_{k=n}^{\infty} \frac{1}{k^2} \le \frac{1}{n - d}.$$

http://dx.doi.org/10.4169/amer.math.monthly.124.1.83

11955. *Proposed by David Stoner, Aiken, SC.* Some boys and girls stand on some of the squares of an n-by-n grid. (Each square may contain several or no children.) Each child computes the fraction of children in his or her row whose gender matches his or her own and the fraction of children in his or her column whose gender matches his or her own. Then each child writes down the sum of the two numbers he or she obtains. Prove that the product of all numbers written down in such a manner is at least 1.

11956. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Show that

$$\sum_{n=1}^{\infty} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

converges, and find the sum.

SOLUTIONS

Summing to the Double Factorial

11821 [2015, 176]. *Proposed by Finbarr Holland and Claus Koester, University College Cork, Cork, Ireland.* Let p be a positive integer. Prove that

$$\lim_{n\to\infty} \frac{1}{2^n n^p} \sum_{k=0}^{n} (n-2k)^{2p} \binom{n}{k} = \prod_{j=1}^{p} (2j-1).$$

Solution I by Hongwei Chen, Christopher Newport University, Newport News, VA. Let $S_p(n) = \sum_{k=0}^{n} (n-2k)^{2p} \binom{n}{k}$ for $n, p \geq 0$. We compute

$$\begin{aligned} S_{p+1}(n) &= \sum_{k=0}^{n} (n-2k)^{2p} (n^2 - 4kn + 4k^2) \binom{n}{k} \\ &= n^2 S_p(n) - 4 \sum_{k=1}^{n-1} (n-2k)^{2p} k(n-k) \binom{n}{k} = n^2 S_p(n) - 4n(n-1) S_p(n-2). \end{aligned}$$

Let $M_p(n) = 2^{-n} S_p(n)$, so $M_0(n) = S_0(n) = 1$. We show by induction on p that $M_p(n)$ is a polynomial of degree p in n with leading coefficient $\prod_{j=1}^{p}(2j-1)$. From this the desired result follows immediately.

We use the common "double factorial" notation $(2p-1)!! = \prod_{j=1}^{p}(2j-1)$ and use $O(n^k)$ to indicate a polynomial of degree k in n. Letting c_p denote the coefficient of n^{p-1} in $M_p(n)$, the inductive computation for $p \geq 0$ is

$$\begin{aligned} M_{p+1}(n) &= n^2 M_p(n) - n(n-1) M_p(n-2) \\ &= (2p-1)!! n^{p+2} + c_p n^{p+1} - (n^2 - n)\big((2p-1)!!(n-2)^p + c_p (n-2)^{p-1}\big) + O(n^p) \\ &= c_p n^{p+1} - c_p n^{p+1} + n(2p-1)!!(n-2)^p + n^2 (2p)(2p-1)!! n^{p-1} + O(n^p) \\ &= (2p+1)(2p-1)!! n^{p+1} + O(n^p). \end{aligned}$$

Solution II by National Security Agency Problems Group, Fort Meade, MD. Let $X_1, \ldots, X_n$ be independent random variables, each taking the value $+1$ or -1 with probability $1/2$ each. Note that $\mathbb{E}[X_i] = 0$. Set $X = \sum X_i$. Note that $X = n - 2k$, where

$k = |\{i\colon X_i = -1\}|$. Hence $\mathbb{P}[X = n-2k] = \binom{n}{k}2^{-n}$. Therefore $\mathbb{E}[X^{2p}] = \frac{1}{2^n}\sum_{k=0}^{n}(n-2k)^{2p}\binom{n}{k}$, and we seek $\lim_{n\to\infty} n^{-p}\mathbb{E}[X^{2p}]$.

We also compute $\mathbb{E}[X^{2p}]$ another way:

$$\mathbb{E}[X^{2p}] = \mathbb{E}\left[\left(\sum X_i\right)^{2p}\right] = \sum \binom{2p}{i_1,\ldots,i_n}\mathbb{E}\left[\prod_{j=1}^{n} X_j^{i_j}\right] = \sum \binom{2p}{i_1,\ldots,i_n}\prod_{j=1}^{n}\mathbb{E}\left[X_j^{i_j}\right],$$

where the last step uses independence. Since any odd power of X_i equals X_i, it has expectation 0. Thus, the only nonzero terms in the last sum are those with all i_j even, where the expectation is 1. Hence $\mathbb{E}[X^{2p}]$ is the sum of the multinomial coefficients with all i_j even.

For $n \ge p$, there are $\binom{n}{p}$ terms in which each i_j is 0 or 2. The sum of these coefficients is $\binom{n}{p}\frac{(2p)!}{2^p}$, which equals $\frac{n!}{(n-p)!}(2p-1)!!$.

We claim that the contribution from other terms has lower order. The terms with some i_j greater than 2 have at most $p-1$ nonzero exponents. Let t_p be the number of partitions of p with at most $p-1$ parts. Each such partition can be arranged as $i_1,\ldots,i_n$ in fewer than $(p-1)!\binom{n}{p-1}$ ways, and the corresponding multinomial coefficient is less than $\binom{2p}{2,2,\ldots,2}$. Hence the total contribution of these terms to $\mathbb{E}[X^{2p}]$ is at most $t_p(p-1)!\binom{n}{p-1}(2p)!/2^p$. With p constant and n large, this is bounded by $O(n^{p-1})$.

It follows that for large n,

$$n^{-p}\mathbb{E}(X^{2p}) = (2p-1)!!\prod_{j=1}^{p}\left(1-\frac{j}{n}\right) + O\left(\frac{1}{n}\right),$$

and so $\lim_{n\to\infty} n^{-p}\mathbb{E}(X^{2p}) = (2p-1)!!$.

Editorial comment. Solvers also used various other approaches. Marcin E. Kuczma obtained a recurrence for the polynomial $M_p(n)$ by recognizing $M_p(n)$ as the coefficient of x^{2p} in the power series for $(\cosh x)^n$ and differentiating $(\cosh x)^n \cdot \cosh x$ using the product rule. John H. Lindsey expressed the sum essentially as a Riemann sum to approximate it with an integral involving an exponential function. Some others found explicit formulas for the sum (with or without the denominator), often involving the Stirling numbers, and then found the limit directly. Oliver Geupel gave a somewhat combinatorial proof, interpreting the quantities involved in terms of weighted Dyck paths.

Also solved by T. Amdeberhan & V. H. Moll, M. Bataille (France), R. Chapman (U. K.), R. Dutta (India), P. J. Fitzsimmons, D. Fritze (Germany), N. Grivaux (France), E. A. Herman, J. C. Kieffer, O. Kouba (Syria), M. E. Kuczma (Poland), J. H. Lindsey II, M. Omarjee (France), E. Omey (Belgium), N. C. Singer, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group, and the proposers.

Circular General Position

11824 [2015, 284]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI and Yusheng Luo, Harvard University, Cambridge, MA.* A set X of points in the plane is said to be in *circular general position* if it has the property that every circle or straight line in the plane misses at least two points of X. (Such sets must have at least five elements, and most five-element sets qualify.)

(**a**) Show that if X is a set in circular general position and contains at least seven points, then it has a five-element subset that is in circular general position.

(**b**) Show that there exist sets X in circular general position containing exactly six points for which there is no five-element subset in circular general position.

Solution by Edward Schmeichel, San Jose State University, San Jose, CA. We write "circles" to refer to both circles and straight lines, and we write "cgp" as an abbreviation for "circular general position."

(**a**) Let $n = |X|$. We proceed by induction on n, postponing the base step $n = 7$. For $n \geq 8$, consider a set X with $|X| = n$ in cgp. If every circle contains at most $n - 3$ points of X, then every subset of size $(n - 1)$ is in cgp, and by the induction hypothesis it contains a subset of size 5 in cgp. Otherwise, some circle Γ contains points $P_1, P_2, \ldots, P_{n-2}$ of X and misses two points A_1, A_2 of X. Since any circle through A_1 and A_2 meets Γ in at most two points and $(n - 2)/2 \geq 3$, at least three distinct circles $\Gamma_1, \Gamma_2, \Gamma_3$ through A_1, A_2 are required to cover the $n - 2$ points in $X \cap \Gamma$. Renumber so that $P_1 \in \Gamma_1 \setminus (\Gamma_2 \cup \Gamma_3)$, $P_2 \in \Gamma_2 \setminus (\Gamma_1 \cup \Gamma_3)$, $P_3 \in \Gamma_3 \setminus (\Gamma_1 \cup \Gamma_2)$. The 5-element subset $\{P_1, P_2, P_3, A_1, A_2\}$ will be in cgp. This completes the inductive step.

We now consider the base step. Let X be a 7-point set in cgp. If every circle contains at most three points of X, then any 5-element subset of X will be in cgp. If, on the other hand, some circle contains five points of X and misses two points of X, then the argument at the end of the inductive step above provides a 5-element subset of X in cgp.

Hence we may assume that some circle Γ contains four points of X, say P_1, P_2, P_3, P_4 in order around Γ, and misses three points A_1, A_2, A_3 of X. If any of the pairs (A_1, A_2), (A_1, A_3), or (A_2, A_3) require at least three circles through the pair to cover the four points in $X \cap \Gamma$, then the argument at the end of the inductive step will again provide a 5-element subset of X in cgp. Hence assume for each pair (A_i, A_j) with $1 \leq i < j \leq 3$, there are two circles $\Gamma^1_{i,j}$ and $\Gamma^2_{i,j}$ through A_i and A_j each containing two of the points of $X \cap \Gamma$. Without loss of generality, suppose that $\Gamma^1_{1,2}$ and $\Gamma^2_{1,2}$ contain, respectively, the nonconsecutive pairs (P_1, P_3) and (P_2, P_4) around Γ, while $\Gamma^1_{1,3}$, $\Gamma^2_{1,3}$, $\Gamma^1_{2,3}$, and $\Gamma^2_{2,3}$ contain, respectively, the consecutive pairs (P_1, P_2), (P_3, P_4), (P_1, P_4), and (P_2, P_3) around Γ. The points A_1, A_2 where $\Gamma^1_{1,2}$ and $\Gamma^2_{1,2}$ intersect occur in different connected components of $\mathbb{R}^2 \setminus \Gamma$. On the other hand, for $k \in \{1, 2\}$ the points A_k and A_3 where $\Gamma^1_{k,3}$ and $\Gamma^2_{k,3}$ intersect occur in the same component of $\mathbb{R}^2 \setminus \Gamma$. This is impossible, which completes the base step.

(**b**) Let $X = \{0, 1, 2\} \times \{0, 1\}$. This 6-point set has no 5-point subset in cgp, since every 5-element subset of X contains the four vertices of a rectangle.

Also solved by R. Chapman (U. K.), Y. J. Ionin, O. P. Lossers (Netherlands), M. Monea (Romania), M. A. Prasad (India), J. C. Smith, R. Stong, E. A. Weinstein, and the proposers.

A Rational Function Identity

11828 [2015, 285]. *Proposed by Roberto Tauraso, Universita di Roma "Tor Vergata," Rome, Italy.* Let n be a positive integer, and let z be a complex number that is not a kth root of unity for any k with $1 \leq k \leq n$. Let S be the set of all lists $(a_1, \ldots, a_n)$ of n nonnegative integers such that $\sum_{k=1}^{n} k a_k = n$. Prove that

$$\sum_{a \in S} \prod_{k=1}^{n} \frac{1}{a_k!\, k^{a_k} (1 - z^k)^{a_k}} = \prod_{k=1}^{n} \frac{1}{1 - z^k}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. First we suppose that z is a real number with $|z| < 1$. For any real number r with $|r| < 1$ we have

$$\sum_{k=1}^{\infty} \frac{r^k}{k(1 - z^k)} = \sum_{k=1}^{\infty} \frac{r^k}{k} \left(\sum_{n=0}^{\infty} z^{kn} \right) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(r z^n)^k}{k} \right)$$

$$= -\sum_{n=0}^{\infty} \log(1 - rz^n) = \log\left(\prod_{n=0}^{\infty} \frac{1}{1 - rz^n}\right).$$

It follows that

$$\prod_{k=1}^{\infty} \exp\left(\frac{r^k}{k(1 - z^k)}\right) = \prod_{n=0}^{\infty} \frac{1}{1 - rz^n}. \tag{1}$$

On the other hand, for a real number z with $|z| < 1$, consider the function f_z defined on the open unit disk $D(0, 1)$ in the complex plane by

$$f_z(w) = \prod_{n=0}^{\infty} \frac{1}{1 - wz^n}.$$

This product converges uniformly on every compact subset of $D(0, 1)$, so f_z is analytic in $D(0, 1)$. There is a Taylor series expansion $f_z(w) = \sum_{n=0}^{\infty} A_n(z)w^n$ for $w \in D(0, 1)$. Since $(1 - w)f_z(w) = f_z(zw)$,

$$(1 - w)\left(\sum_{n=0}^{\infty} A_n(z)w^n\right) = \sum_{n=0}^{\infty} A_n(z)z^n w^n.$$

It follows that $A_n(z) - A_{n-1}(z) = z^n A_n(z)$ for $n \geq 1$. Also $A_0(z) = 1$, so

$$A_n(z) = \prod_{k=1}^{n} \frac{1}{1 - z^k}, \quad \text{for } n \geq 1.$$

Thus, for $|r| < 1$, we have

$$\prod_{n=0}^{\infty} \frac{1}{1 - rz^n} = 1 + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{1}{1 - z^k}\right) r^n. \tag{2}$$

From (1) and (2),

$$\prod_{k=1}^{\infty} \left(\sum_{a_k=0}^{\infty} \frac{r^{ka_k}}{a_k!\, k^{a_k}(1 - z^k)^{a_k}}\right) = 1 + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{1}{1 - z^k}\right) r^n.$$

Equating the coefficient of r^n on both sides, we obtain

$$\sum_{a\in S} \prod_{k=1}^{n} \frac{1}{a_k!\, k^{a_k}(1 - z^k)^{a_k}} = \prod_{k=1}^{n} \frac{1}{1 - z^k}.$$

We have proved the required equality for real numbers z with $|z| < 1$. Each side is a rational function of z. The validity of the formula for every complex number z such that z is not a kth root of unity for any k with $1 \leq k \leq n$ follows by analytic continuation.

Also solved by T. Amdeberhan & R. P. Stanley, N. Caro (Brazil), R. Chapman (U. K.), S. M. Gagola Jr., O. P. Lossers (Netherlands), M. A. Prasad (India), J. C. Smith, R. Stong, M. Wildon (U. K.), and the proposer.

A Convergence Test

11829 (Corrected) [2015, 285; 2015, 605]. *Proposed by Paul Bracken, University of Texas-Pan American, Edinburg, TX.* Let $\langle a \rangle$ be a monotone decreasing sequence of real numbers that converges to 0. Prove that $\sum_{n=1}^{\infty} a_n/n < \infty$ if and only if $a_n = O(1/\log n)$ and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log n < \infty$.

Solution I by Moubinool Omarjee, Lycée Henri IV, Paris, France and Roberto Tauraso, Dipartimento de Matematica, Università di Roma, "Tor Vergata," Rome, Italy. Define two sequences $\{S_N\}_{N\geq 1}$ and $\{T_N\}_{N\geq 1}$ by

$$S_N = \sum_{n=1}^{N} \frac{a_n}{n} \quad \text{and} \quad T_N = \sum_{n=1}^{N} (a_n - a_{n+1}) \log n.$$

If $\{a_n\}_{n\geq 1}$ is positive and decreasing to 0, then both $\{S_N\}_{N\geq 1}$ and $\{T_N\}_{N\geq 1}$ are increasing. Let S and T be their respective limits (finite or $+\infty$). Notice that if $n \geq 2$, then $\log n - \log(n-1) = \int_{n-1}^{n} dx/x \in (1/n, 1/n-1)$.

(Necessity) If $S < \infty$, then for $N \geq 2$,

$$a_N \log N \leq a_N \sum_{n=2}^{N} \big(\log n - \log(n-1)\big) \leq a_N \sum_{n=2}^{N} \frac{1}{n-1} \leq \sum_{n=2}^{N} \frac{a_{n-1}}{n-1} = S_{N-1} \leq S.$$

This implies that $a_n = O(1/\log(n))$. Moreover,

$$\begin{aligned} T_N &= \sum_{n=1}^{N} (a_n - a_{n+1}) \log n = \sum_{n=2}^{N} a_n \log n - \sum_{n=2}^{N+1} a_n \log(n-1) \\ &= \sum_{n=2}^{N} a_n(\log n - \log(n-1)) - a_{N+1} \log N \leq \sum_{n=2}^{N} \frac{a_n}{n-1} \\ &\leq \sum_{n=2}^{N} \frac{a_{n-1}}{n-1} = S_{N-1} \leq S, \end{aligned}$$

hence $T < \infty$.

(Sufficiency) If $T < \infty$ and $a_N \log N \leq M$ for $N \geq 2$, then

$$\begin{aligned} S_N - a_1 &= \sum_{n=2}^{N} \frac{a_n}{n} \leq \sum_{n=2}^{N} a_n(\log n - \log(n-1)) = \sum_{n=2}^{N} a_n \log n - \sum_{n=2}^{N-1} a_{n+1} \log n \\ &= \sum_{n=2}^{N-1} (a_n - a_{n+1}) \log n + a_N \log N = T_{N-1} + a_N \log N \leq T + M, \end{aligned}$$

hence $S < \infty$.

Solution II by Traian Viteam, Osaka, Japan. Let $H_n = \sum_{i=1}^{n} 1/i$. We use the fact that $H_n \sim \log(n)$; that is, their ratio converges to 1. From the comparison test for series with nonnegative terms, we have $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log(n) < \infty$ if and only if $\sum_{n=1}^{\infty} (a_n - a_{n+1}) H_n < \infty$.

Suppose first that $\sum_{n=1}^{\infty} a_n/n < \infty$. Since $(a_n)_{n\geq 1}$ is decreasing, we have

$$a_N \sum_{n=1}^{N} \frac{1}{n} \leq \sum_{n=1}^{N} \frac{a_n}{n}$$

for all positive integers N. Since the right-hand side is bounded as $N \to \infty$, it follows that

$$a_N = O\left(\frac{1}{H_N}\right) = O\left(\frac{1}{\log N}\right).$$

Next note that

$$\sum_{n=1}^{N}(a_n - a_{n+1})H_n = \sum_{n=1}^{N} \frac{a_n}{n} - H_N a_{N+1}.$$

The latter expression is bounded, because the first sum is bounded and the second term satisfies $0 \le (1 + 1/2 + \cdots + 1/N)a_{N+1} \le (1 + 1/2 + \cdots + 1/N)a_N = O(1)$. Thus the partial sums of the series $\sum_{n=1}^{\infty}(a_n - a_{n+1})H_n$ are bounded. Since the terms are nonnegative, the series therefore converges, which implies the convergence of $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log n$.

Conversely, assume that $a_n = O(1/\log n)$ and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log n < \infty$. The first relation implies that $H_N a_{N+1}$ is bounded as $N \to \infty$, while the second implies $\sum_{n=1}^{\infty} (a_n - a_{n+1})H_n < \infty$. Thus the partial sums of this series are bounded. Now the conclusion that the partial sums of the series $\sum_{n=1}^{\infty} a_n/n$ are also bounded follows, since

$$\sum_{n=1}^{N} \frac{a_n}{n} = \sum_{n=1}^{N} (a_n - a_{n+1})H_n + H_N a_{N+1}$$

for all N. Since $\sum_{n=1}^{\infty} a_n/n$ is a series of nonnegative terms, the proof is complete.

Also solved by U. Abel (Germany), K. F. Andersen (Canada), R. Bagby, E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), R. Chapman (U. K.), H. Chen, G. H. Chung, M. Goldenberg & M. Kaplan, N. Grivaux (France), E. A. Herman, E. J. Ionaşcu, O. Kouba (Syria), K.-W. Lau, J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), E. Schmeichel, N. C. Singer, J. C. Smith, A. Stenger, R. Stong, J. Van Casteren (Belgium), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

A Circumcentric Triangle

11830 [2015, 285]. *Proposed by Leo Giugiuc, Drobeta-Turnu Severin, Romania, and Oai Thanh Dao, Quang Trung village, Kien Xuong district, Thai Binh Province, Vietnam.* Let A, B, C be the vertices of a triangle. Let P be a parabola tangent to the line BC at A_1, to CA at B_1, and to AB at C_1. Let A_2, B_2, and C_2 be the circumcenters of triangles AB_1C_1, BC_1A_1, and CA_1B_1, respectively.
(**a**) Show that there is a circle through A_2, B_2, C_2, and the focus of P.
(**b**) Show that the triangles ABC and $A_2B_2C_2$ are similar.

Solution by O. P. Lossers, Einhoven University of Technology, Eindhoven, The Netherlands. (**b**) We choose a Euclidean coordinate system such that the equation of the parabola is $x = y^2$. Let the tangents through B touch the parabola in $A_1 = (a^2, a)$ and $C_1 = (c^2, c)$, and let $B_1 = (b^2, b)$. Then B, the pole of the line A_1C_1, has the coordinates $(ac, (a + c)/2)$. We then compute

$$B_2 = \left(\frac{1}{4} + \frac{(a + c)^2}{2}, (a + c)\left(\frac{1}{4} - ac\right)\right).$$

The other points of interest for this problem may be obtained by cyclic permutations of a, b, c. Thus $C - B = (b - c)(a, 1/2)$ and

$$C_2 - B_2 = (b - c)\left(\frac{s}{2} + \frac{a}{2}, \frac{1}{4} - as\right), \quad \text{where } s = a + b + c.$$

For the length, $\overline{B_2C_2}^2 = (c - b)^2\left(s^2 + \frac{1}{4}\right)\left(a^2 + \frac{1}{4}\right) = \left(s^2 + \frac{1}{4}\right)\overline{BC}^2$. The same relation holds for the other sides, so the triangles ABC and $A_2B_2C_2$ are similar.

(**a**) The focus F is $(1/4, 0)$. To prove that F is on the circumcircle of $A_2B_2C_2$, we compare the cosines of $\angle A_2FC_2$ and $\angle A_2B_2C_2$. By (**b**), triangles ABC and $A_2B_2C_2$ have the same angles, and so

$$\cos\left(\angle A_2B_2C_2\right) = \cos\left(\angle ABC\right) = \pm\frac{\frac{1}{4}+ac}{\sqrt{\left(\frac{1}{4}+a^2\right)\left(\frac{1}{4}+c^2\right)}}.$$

To compute the angle A_2FC_2 we first compute the inner product of A_2F and C_2F:

$$A_2F = (b+c)\left(\frac{b+c}{2}, \frac{1}{4}-bc\right), \quad C_2F = (a+b)\left(\frac{a+b}{2}, \frac{1}{4}-ab\right),$$

$$\text{and } (A_2F)\cdot(C_2F) = (c+b)(b+a)\left(\frac{1}{4}+b^2\right)\left(\frac{1}{4}+ac\right).$$

The length of A_2F is $|c+b|\sqrt{\frac{1}{4}+c^2}\sqrt{\frac{1}{4}+b^2}$, so

$$\cos\left(\angle A_2FC_2\right) = \frac{\pm\left(\frac{1}{4}+ac\right)}{\sqrt{\left(\frac{1}{4}+a^2\right)\left(\frac{1}{4}+c^2\right)}}.$$

Thus $\cos\left(\angle A_2FC_2\right)$ and $\cos\left(\angle A_2B_2C_2\right)$ are equal in absolute value. Therefore F is either on the circumcircle of $A_2B_2C_2$ or on its mirror image under reflection in side A_2C_2. The same is true for the other sides, so F, A_2, B_2, C_2 are indeed on the same circle.

Editorial comment. Part (**b**) appears as a corollary on page 134 of R. A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

Also solved by R. Chapman (U. K.), J.-P. Grivaux (France), O. Kouba (Syria), J. McHugh, J. C. Smith, and R. Stong.

Uncountably Many Discontinuities, Again

11833 [2015, 390]. *Proposed by Mher Safaryan, Yerevan State University, Yerevan, Armenia, and Vahagn Aslanyan, University of Oxford, Oxford, U. K.* Let f be a real-valued function on an open interval (a, b) such that the one-sided limits $\lim_{t\to x^-} f(t)$ and $\lim_{t\to x^+} f(t)$ exist and are finite for all x in (a, b). Can the set of discontinuities of f be uncountable?

Solution by Klaas Pieter Hart, Delft University of Technology, Delft, Netherlands. No, the set of discontinuities is countable. Let D be the set of points at which f is discontinuous. Write $f(x^+) = \lim_{t\to x^+} f(t)$ and $f(x^-) = \lim_{t\to x^-} f(t)$. Because of the assumptions, a point x belongs to D for one of two reasons: $f(x^-) \neq f(x^+)$ or $f(x^-) = f(x^+) \neq f(x)$. Thus $D \subseteq A \cup B$, where $A = \{x : f(x) \neq f(x^-)\}$ and $B = \{x : f(x) \neq f(x^+)\}$. It suffices to show that A and B are countable. The arguments for A and B are mirror images of each other, so we concentrate on A.

For a natural number n, let $A_n = \{x \in A : |f(x^-) - f(x)| \geq 2^{-n}\}$. Since $A = \bigcup_n A_n$, it suffices to show each A_n is countable. We claim that if $p \in A_n$, then there is positive real number δ_p such that $(p - \delta_p, p) \cap A_n = \varnothing$. This claim suffices, because $\{(p - \delta_p, p) : p \in A_n\}$ is a pairwise disjoint family of intervals in the real line and hence countable, which implies that A_n itself is countable.

To prove the claim, choose $\delta_p > 0$ small enough that $\left|f(x) - f(p^-)\right| < (1/3)2^{-n}$ whenever $x \in (p - \delta_p, p)$. By the triangle inequality, we have $|f(y) - f(x)| < \frac{2}{3}\cdot 2^{-n}$ whenever $x, y \in (p - \delta_p, p)$. From this we get $|f(x^-) - f(x)| \leq \frac{2}{3}\cdot 2^{-n}$ whenever $x \in (p - \delta_p, p)$. So $(p - \delta_p, p) \cap A_n = \varnothing$, as claimed.

Editorial comment. A slightly stronger version of this result appeared as MONTHLY Problem 10979 [109 (2002), 921; solution 111 (2004), 630]. The solution given there (also by Hart) includes many references to this result in the literature.

Also solved by R. Acosta & J. Losada (Spain), M. Bataille (France), M. W. Botsko, R. Boukharfane (France), P. Budney, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), O. Geupel (Germany), N. Grivaux (France), J. W. Hagood, D. L. Hancock, Y. J. Ionin, B. Karaivano (U.S.A.) & T. S. Vassilev (Canada), J. H. Lindsey II, M. D. Meyerson, P. Perfetti (Italy), Á. Plaza & F. Perdomo (Spain), D. Ritter, K. S. Sarkaria, S. Scheinberg, K. Schilling, J. C. Smith, R. Stong, R. Tauraso (Italy), and the proposer.

A Putnam Addendum

11837 [2015, 391]. *Proposed by Iosif Pinelis, Michigan Technological University, Houghton, MI.* Let $a_0 = 1$, and, for $n \geq 0$, let $a_{n+1} = a_n + e^{-a_n}$. Let $b_n = a_n - \log n$. For $n \geq 0$, show that $0 < b_{n+1} < b_n$; also show that $\lim_{n\to\infty} b_n = 0$. (The proposer notes that the content of Problem B4 of the 73rd William Lowell Putnam Mathematical Competition—see, e.g., this MONTHLY, Volume 120, No. 8, pages 682 and 686—was the question of whether b_n has a finite limit as $n \to \infty$.)

Composite solution by Nicole Grivaux, Paris, France, and Oliver Geupel, Brühl, NRW, Germany. The sequence $(a_n)_{n\geq 0}$ satisfies $a_{n+1} = f(a_n)$, where $f(t) = t + e^{-t}$. The function f is increasing and positive on $[0, \infty)$. For any positive integer n, we have $\log(1 + 1/n) < 1/n$ and $\log(1 - 1/(n + 1)) < -1/(n + 1)$, and so

$$\log(n+1) < \log n + \frac{1}{n} \tag{1}$$

and

$$\frac{1}{n+1} < \log\left(1 + \frac{1}{n}\right) < \frac{1}{n}. \tag{2}$$

We prove by induction that $a_n > \log(n + 1)$ for all positive integers n. First, $a_0 = 1 > \log 2$. Next, if $a_n > \log(n + 1)$, then $a_{n+1} > f(\log(n + 1)) = \log(n + 1) + 1/(n + 1)$, and so (1) implies that $a_{n+1} > \log(n + 2)$.

Because $a_n > \log(n + 1) > \log n$, we have $b_n > 0$ for any positive integer n.

Because $a_{n+1} - a_n = e^{-a_n}$, we have $a_{n+1} - a_n < e^{-\log(n+1)} = 1/(n + 1)$ for any positive integer n. Using (2), we obtain $a_{n+1} - a_n < \log(n + 1) - \log(n)$, and so $b_{n+1} < b_n$.

Now a_n increases to infinity, so

$$\lim_{n\to\infty} \frac{e^{a_{n+1}} - e^{a_n}}{(n+1) - n} = \lim_{n\to\infty} \frac{e^{e^{-a_n}} - 1}{e^{-a_n}} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

By the Stolz–Cesàro lemma, it follows that $\lim_{n\to\infty} e^{a_n}/n = 1$. Consequently, $\lim_{n\to\infty} e^{b_n} = 1$ and $\lim_{n\to\infty} b_n = 0$.

Also solved by T. Amdeberhan & V. H. Moll, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), E. A. Herman, K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), G. Marks, M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), S. Roy & J. Bose (India), M. Sawhney, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Viteam (Japan), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Daniel Cranston, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, and Fuzhen Zhang.

Proposed problems should be submitted online at http://www.americanmathematicalmonthly.submittable.com/submit.

Proposed solutions to the problems below should be submitted on or before June 30, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11957. *Proposed by Éric Pité, Paris, France.* Let m and n be two integers with $n \geq m \geq 2$. Let $S(n, m)$ be the Stirling number of the second kind, i.e., the number of ways to partition a set of n objects into m nonempty subsets. Show that

$$n^m S(n, m) \geq m^n \binom{n}{m}.$$

11958. *Proposed by Kent Holing, Trondheim, Norway.*
(a) Find a condition on the side lengths a, b, and c of a triangle that holds if and only if the nine-point center lies on the circumcircle.
(b) Characterize the triangles whose nine-point center lies on the circumcircle and whose incenter lies on the Euler line.

11959. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that, for any n-by-n matrix A with (i, j)-entry $a_{i,j}$ and any $t_1, \ldots, t_n$, the permanent of A is

$$\frac{1}{2^n} \sum \prod_{i=1}^{n} \sigma_i \left(t_i + \sum_{j=1}^{n} \sigma_j a_{i,j} \right),$$

where the outer sum is over all 2^n choices of $(\sigma_1, \ldots, \sigma_n) \in \{1, -1\}^n$.

11960. *Proposed by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.* Let m and n be natural numbers, and, for $i \in \{1, \ldots m\}$, let a_i be a real number with $0 \leq a_i \leq 1$. Define

$$f(x) = \frac{1}{x^2} \left(\sum_{i=1}^{m} (1 + a_i x)^{mn} - m \prod_{i=1}^{m} (1 + a_i x)^n \right).$$

Let k be a nonnegative integer, and write $f^{(k)}$ for the kth derivative of f. Show that $f^{(k)}(-1) \geq 0$.

http://dx.doi.org/10.4169/amer.math.monthly.124.2.179

11961. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Evaluate

$$\int_0^{\pi/2} \frac{\sin x}{1+\sqrt{\sin(2x)}}\,dx.$$

11962. *Proposed by Elton Hsu, Northwestern University, Evanston, IL.* Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each taking the values ± 1 with probability 1/2. Find the distribution of the random variable

$$\sqrt{\frac{1}{2}+\frac{X_1}{2}\sqrt{\frac{1}{2}+\frac{X_2}{2}\sqrt{\frac{1}{2}+\cdots}}}\,.$$

11963. *Proposed by Gheorghe Alexe and George-Florin Serban, Braila, Romania.* Let $a_1,\ldots,a_n$ be positive real numbers with $\prod_{k=1}^n a_k = 1$. Show that

$$\sum_{i=1}^n \frac{(a_i+a_{i+1})^4}{a_i^2 - a_i a_{i+1} + a_{i+1}^2} \geq 12n,$$

where $a_{n+1} = a_1$.

SOLUTIONS

An Inequality with Squared Tangents

11778 [2014, 456]. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.* Let x, y, z be positive real numbers such that $x+y+z=\pi/2$. Let $f(x,y,z) = 1/(\tan^2 x + 4\tan^2 y + 9\tan^2 z)$. Prove that

$$f(x,y,z)+f(y,z,x)+f(z,x,y) \leq \frac{9}{14}\left(\tan^2 x + \tan^2 y + \tan^2 z\right).$$

Solution by Vazgen Mikayelyan, Department of Mathematics and Mechanics, Yerevan State University, Yerevan, Armenia. Letting $a = \tan x$, $b = \tan y$, and $c = \tan z$, we have $a, b, c > 0$, since $0 < x, y, z < \pi/2$, and $ab+bc+ca = 1$ since

$$a = \tan x = \cot(y+z) = \frac{1-\tan y \tan z}{\tan y + \tan z} = \frac{1-bc}{b+c}.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
3(a^2+4b^2+9c^2) &= \frac{3a^2b^2}{b^2}+\frac{11a^2b^2}{a^2}+\frac{b^2c^2}{c^2}+\frac{13b^2c^2}{b^2}+\frac{14c^2a^2}{a^2}\\
&= \frac{(3ab)^2}{3b^2}+\frac{(11ab)^2}{11a^2}+\frac{(bc)^2}{c^2}+\frac{(13bc)^2}{13b^2}+\frac{(14ca)^2}{14a^2}\\
&\geq \frac{(14ab+14bc+14ca)^2}{3b^2+11a^2+c^2+13b^2+14a^2} = \frac{14^2}{25a^2+16b^2+c^2}.
\end{aligned}$$

Hence,

$$f(x,y,z) = \frac{1}{a^2+4b^2+9c^2} \leq \frac{3(25a^2+16b^2+c^2)}{14^2}.$$

Adding this and the analogous inequalities obtained by cycling the variables, we obtain

$$f(x, y, z) + f(y, z, x) + f(z, x, y) \leq \frac{3(42a^2 + 42b^2 + 42c^2)}{14^2} = \frac{9(a^2 + b^2 + c^2)}{14},$$

which is the desired inequality.

Editorial comment. Paolo Perfetti proved the stronger result

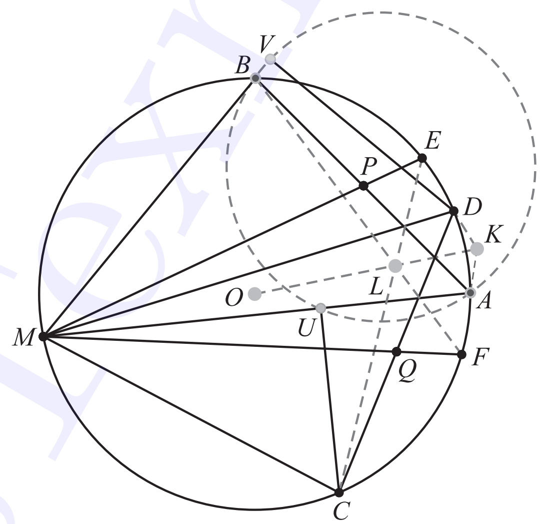
$$f(x, y, z) + f(y, z, x) + f(z, x, y) \leq \frac{9}{14} \leq \frac{9}{14}(\tan^2 x + \tan^2 y + \tan^2 z).$$

Also solved by A. Ali (India), S. Baek (Korea), R. Bagby, P. P. Dályay (Hungary), O. Geupel (Germany), P. Perfetti (Italy), R. Stong, R. Tauraso (Italy), and the proposer.

Concyclic or Collinear

11779 [2014, 456]. *Proposed by Michel Bataille, Rouen, France.*

Let M, A, B, C, and D be distinct points (in any order) on a circle Γ with center O. Let the medians through M of triangles MAB and MCD cross lines AB and CD at P and Q, respectively, and meet Γ again at E and F, respectively. Let K be the intersection of AF with DE, and let L be the intersection of BF with CE. Let U and V be the orthogonal projections of C onto MA and D onto MB, respectively, and assume $U \neq A$ and $V \neq B$. Prove that A, B, U, and V are concyclic if and only if O, K, and L are collinear.



Solution by Richard Stong, Center for Communications Research, San Diego, CA. The problem is not quite correct. We must also assume that E and F do not coincide, hence K and L do not coincide. (If K and L coincide, then O, K, L are clearly collinear, but A, U, B, V need not be concyclic.)

Let R be the radius of Γ, let N the point where lines AC and BD intersect, and let X and Y be the reflections of O across lines AC and BD, respectively. The claim is the equivalence of (1) A, B, U, V are concyclic, and (2) O, K, L are collinear. We show that each of these is equivalent to (3) M is equidistant from X and Y.

(1) $\iff$ (3). Note that A, B, U, V are concyclic if and only if the powers from M are equal: $|MA| \cdot |MU| = |MB| \cdot |MV|$. From trigonometry and the extended law of sines,

$$|MB| \cdot |MV| = 4R^2 \sin\left(\tfrac{1}{2}\angle MOB\right) \sin\left(\tfrac{1}{2}\angle MOD\right) \cos\left(\tfrac{1}{2}\angle BOD\right).$$

From the law of cosines applied to $\triangle MOY$, we find that $|MY|^2$ equals

$$\begin{aligned} &R^2 + 4R^2\cos^2\left(\tfrac{1}{2}\angle BOD\right) - 4R^2\cos\left(\tfrac{1}{2}\angle BOD\right)\cos\left(\tfrac{1}{2}\left(\angle MOB + \angle MOD\right)\right) \\ &\quad = R^2 + 8R^2\sin\left(\tfrac{1}{2}\angle MOB\right)\sin\left(\tfrac{1}{2}\angle MOD\right)\cos\left(\tfrac{1}{2}\angle BOD\right) \\ &\quad = R^2 + 2|MB| \cdot |MV|, \end{aligned}$$

and analogously, $|MX|^2 = R^2 + 2|MA| \cdot |MU|$. Thus, $|MX| = |MY|$ if and only if $|MA| \cdot |MU| = |MB| \cdot |MV|$.

(2) $\Longleftrightarrow$ (3). Lay down complex coordinates with Γ equal to the unit circle. If a point Z with coordinate z is on the unit circle and point W with coordinate w is any other point in the complex plane, then the second intersection of line WZ with the unit circle has coordinate $(z - w)/(\overline{w}z - 1)$. Hence, letting lower case letters denote coordinates of the points with the corresponding upper case letter, we compute

$$e = \frac{ab(a + b - 2m)}{2ab - am - bm} \qquad \text{and} \qquad f = \frac{cd(c + d - 2m)}{2cd - cm - dm}.$$

By Pascal's theorem applied to the hexagon $CEDBFA$, we see that K, L, and N all lie on the line

$$\begin{aligned}(ce + db + fa - ed - bf - ac)z + (abdf + acef + bcde - abcf - acde - bdef)\overline{z} \\ = ce(b + f) + db(a + c) + fa(e + d) - ed(f + a) - bf(c + e) - ac(b + d).\end{aligned}$$

If $K \neq L$, then this is the unique line through K and L. Hence, O, K, and L are collinear if and only if

$$ce(b + f) + db(a + c) + fa(e + d) - ed(f + a) - bf(c + e) - ac(b + d) = 0.$$

Plugging in the formulas for e and f above and factoring out $(a - b)(c - d)$, this becomes

$$m\left(\frac{1}{a} + \frac{1}{c} - \frac{1}{b} - \frac{1}{d}\right) + (a + c - b - d)\overline{m} = \frac{(ab - cd)(ad - bc)}{abcd}.$$

Since $x = a + c$ and $y = b + d$, this is the equation of the line perpendicular to XY through the midpoint $(a + b + c + d)/2$ of XY. Hence, O, K, and L are collinear if and only if $|MX| = |MY|$.

Also solved by R. Chapman (U. K.), J.-P. Grivaux (France), C. R. Pranesachar (India), and the proposer.

Altitudes of a Tetrahedron

11783 [2014, 549] and **11797** [2014, 738]. *Proposed by Zhang Yun, Xi'an City, Shaanxi, China.* Given a tetrahedron, let r denote the radius of its inscribed sphere. For $1 \le k \le 4$, let h_k denote the distance from the kth vertex to the plane of the opposite face. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \ge \frac{12}{5}.$$

Solution by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata," Rome, Italy. The volume of the tetrahedron is given by

$$\frac{h_k A_k}{3} = \frac{rS}{3},$$

where A_k is the area of the face opposite the kth vertex and $S = \sum_{k=1}^4 A_k$ is the surface area of the tetrahedron. Hence, $h_k = r/t_k$, where $t_k = A_k/S$. Since $0 < t_k < 1$ and the function $f(t) = (1 - t)/(1 + t)$ is convex on $[0, +\infty)$, we have

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} = \sum_{k=1}^{4} f(t_k) \ge 4\, f\left(\frac{1}{4}\sum_{k=1}^{4} t_k\right) = 4f\left(\frac{1}{4}\right) = \frac{12}{5}.$$

Editorial comment. The problem was inadvertently repeated as Problems 11783 and 11797. Several solvers noted that equality holds if and only if the face areas are equal. This does not require, however, that the tetrahedron be regular. Some solvers noted that the n-dimensional analogue of the inequality holds with lower bound $n(n+1)/(n+2)$.

Also solved by A. Ali (India), S. Baek (Korea), R. Bagby, M. Bataille (France), D. M. Bătinetu-Giurgiu & T. Zvonaru (Romania), I. Borosh, R. Boukharfane (Morocco), R. Chapman (U. K.), N. Curwen (U. K.), P. P. Dályay (Hungary), M. Dincă (Romania), D. Fleischman, H. S. Geun (Korea), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. G. Heuver (Canada), S. Hitotumatu (Japan), E. J. Ionaşcu, Y. J. Ionin, B. Karaivanov (U.S.A.) & T. S. Vassilev (Canada), O. Kouba (Syria), D. Lee (Korea), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, Y. Oh (Korea), P. Perfetti (Italy), I. Pinelis, C. R. Pranesachar (India), Y. Shim (Korea), J. C. Smith, R. Stong, T. Viteam (India), M. Vowe (Switzerland), T. Zvonaru & N. Stanciu (Romania), GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, University of Louisiana at Lafayette Math Club, and the proposer.

Circles around an Equilateral Triangle

11784 [2014, 549]. *Proposed by Abdurrahim Yilmuz, Middle East Technical University, Ankara, Turkey.* Let ABC be an equilateral triangle with center O and circumradius r. Given $R > r$, let ρ be a circle about O of radius R. All points named "P" are on ρ.
(**a**) Prove that $|PA|^2 + |PB|^2 + |PC|^2 = 3(R^2 + r^2)$.
(**b**) Prove that $\min_{P\in\rho} |PA|\,|PB|\,|PC| = R^3 - r^3$ and $\max_{P\in\rho} |PA|\,|PB|\,|PC| = R^3 + r^3$.
(**c**) Prove that the area of a triangle with sides of length $|PA|$, $|PB|$, and $|PC|$ is $\frac{\sqrt{3}}{4}(R^2 - r^2)$.
(**d**) Prove that if H, K, and L are the respective projections of P onto AB, AC, and BC, then the area of triangle HKL is $\frac{3\sqrt{3}}{16}(R^2 - r^2)$.
(**e**) With the same notation, prove that $|HK|^2 + |KL|^2 + |HL|^2 = \frac{9}{4}(R^2 + r^2)$.

Solution by TCDmath Problem Group, Trinity College, Dublin, Ireland.
(**a**) We represent the points by complex numbers: $A = r$, $B = r\omega$, $C = r\omega^2$, $O = 0$, $P = z$, where $r > 0$ and $\omega = e^{2\pi i/3}$. We compute

$$|PA|^2 = (z - r)(\bar{z} - r) = |z|^2 - r(z + \bar{z}) + r^2,$$
$$|PB|^2 = (z - r\omega)(\bar{z} - r\omega^2) = |z|^2 - r(z\omega + \bar{z}\omega^2) + r^2, \qquad \text{and}$$
$$|PC|^2 = (z - r\omega^2)(\bar{z} - r\omega) = |z|^2 - r(z\omega^2 + \bar{z}\omega) + r^2.$$

Summing these equations and using $1 + \omega + \omega^2 = 0$ yields

$$|PA|^2 + |PB|^2 + |PC|^2 = 3(R^2 + r^2).$$

(**b**) We have

$$|PA|\,|PB|\,|PC| = \left|(z - r)(z - r\omega)(z - r\omega^2)\right| = \left|z^3 - r^3\right|.$$

This formula takes its maximum value $R^3 + r^3$ when $z^3 = -R^3$, that is, when $z = Re^{i\theta}$ with $3\theta \equiv \pi \pmod{2\pi}$ or when P lies on one of the altitudes of the triangle on the opposite side to the vertex. It takes its minimum value $R^3 - r^3$ when $z^3 = R^3$, that is, when P lies on one of the three altitudes of the triangle on the same side as the vertex.
(**c**) Heron's formula for the area Δ of a triangle with sides a, b, c is

$$16\Delta^2 = 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) - \left(a^4 + b^4 + c^4\right)$$
$$= \left(a^2 + b^2 + c^2\right)^2 - 2\left(a^4 + b^4 + c^4\right).$$

In our case, we have $(|PA|^2 + |PB|^2 + |PC|^2)^2 = 9(R^2 + r^2)^2$ and

$$
\begin{aligned}
|PA|^4 + |PB|^4 + |PC|^4 &= \left(R^2 + r^2 - r(z+\overline{z})\right)^2 \\
&\quad + \left(R^2 + r^2 - r(z\omega + \overline{z}\omega^2)\right)^2 + \left(R^2 + r^2 - r(z\omega^2 + \overline{z}\omega)\right)^2 \\
&= 3\left(R^2+r^2\right)^2 - 2\left(R^2+r^2\right) r \left(z + \overline{z} + z\omega + \overline{z}\omega^2 + z\omega^2 + \overline{z}\omega\right) \\
&\quad + r^2\left((z+\overline{z})^2 + (z\omega + \overline{z}\omega^2)^2 + (z\omega^2 + \overline{z}\omega)^2\right) \\
&= 3\left(R^2+r^2\right)^2 + 6r^2 z\overline{z} = 3\left(R^2+r^2\right)^2 + 6R^2r^2.
\end{aligned}
$$

Thus, $16\Delta^2 = 3(R^2+r^2)^2 - 12(R^2r^2) = 3(R^2-r^2)^2$. Hence, $\Delta = \frac{\sqrt{3}}{4}(R^2-r^2)$.

(**d**) In coordinates (x, y), line BC has equation $x = -r/2$, and the projection PL has equation $y = b$, where b is the imaginary part of z. Thus,

$$L = \frac{1}{2}(-r + z - \overline{z}).$$

To determine H, we rotate the plane with the transformation $z \mapsto \omega z$ and then rotate back with $z \mapsto \omega^2 z$. This yields

$$H = \frac{\omega^2}{2}(-r + \omega z - \omega^2\overline{z}) = \frac{1}{2}(-\omega^2 r + z - \omega\overline{z}).$$

Similarly,

$$K = \frac{1}{2}(-\omega r + z - \omega^2\overline{z}).$$

Thus, using $|1-\omega| = |1-\omega^2| = |\omega - \omega^2| = \sqrt{3}$, we have

$$|HK| = \frac{\sqrt{3}}{2}|\overline{z} - r|, \quad |KL| = \frac{\sqrt{3}}{2}|\overline{z} - \omega r|, \quad \text{and} \quad |LH| = \frac{\sqrt{3}}{2}|\overline{z} - \omega^2 r|.$$

Replacing $\overline{z}$ with z in these expressions leaves the triple $\{|HK|, |KL|, |LH|\}$ unchanged. The triangle HKL is therefore similar to the triangle in part (**c**) with coefficient of similarity equal to $\sqrt{3}/2$. It follows that the area of $\triangle HKL$ is $3/4$ times the area of the earlier triangle, that is, $\frac{3\sqrt{3}}{16}(R^2-r^2)$.

(**e**) Similarly, comparing with the earlier triangle, $|HK|^2 + |KL|^2 + |HL|^2 = \frac{9}{4}(R^2+r^2)$.

Editorial comment. The Editors regret that the "16" in the denominator of part (**d**) was misprinted as "116."

All parts also solved by R. Bagby, M. Bataille (France), R. Boukharfane (Morocco), R. Chapman (U. K.), N. Curwen (U. K.), P. P. Dályay (Hungary), A. Ercan (Turkey), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), A. Habil (Syria), E. J. Ionaşcu, Y. J. Ionin, B. Karaivanov, O. Kouba (Syria), M. D. Meyerson, J. Minkus, C. R. Pranesachar (India), J. C. Smith, R. Stong, T. Zvonaru & N. Stanciu (Romania), GCHQ Problem Solving Group (U. K.), and the proposer. Some but not all parts solved by A. Ali (India), R. B. Campos (Spain), D. Fleischman, O. Geupel (Germany), P. Nüesch (Switzerland), J. Schlosberg, C. R. Selvaraj & S. Selvaraj, T. Viteam (India), and Z. Vörös (Hungary).

A Limit of a Ratio of Logarithms

11786 [2014, 550]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let $x_1, x_2, \ldots$ be a sequence of positive numbers such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \frac{\log x_n}{x_1+\cdots+x_n}$ is a negative number. Prove that $\lim_{n\to\infty} \frac{\log x_n}{\log n} = -1$.

Solution by Lixing Han, University of Michigan, Flint, MI. Suppose that

$$\lim_{n\to\infty} \log x_n/(x_1 + \cdots + x_n) = \beta < 0.$$

Then by the Stolz–Cesàro theorem, we have

$$\lim_{n\to\infty} \frac{\log(x_{n+1}/x_n)}{x_{n+1}} = \lim_{n\to\infty} \frac{\log x_{n+1} - \log x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\log x_n}{x_1 + \cdots + x_n} = \beta. \tag{1}$$

This implies that $x_{n+1} < x_n$ for large n since $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$ by assumption. Since x_n goes to zero as $n \to \infty$, (1) implies $\lim_{n\to\infty} \beta x_{n+1} = \lim_{n\to\infty} \log(x_{n+1}/x_n) = 0$. It follows that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1. \tag{2}$$

By the mean value theorem, there exists ζ_n such that

$$\log x_{n+1} - \log x_n = \frac{1}{\zeta_n}(x_{n+1} - x_n),$$

where ζ_n is between x_n and x_{n+1}. Thus, for large n, $x_{n+1}/x_n < \zeta_n/x_n < 1$. From (2), we conclude $\lim_{n\to\infty} \zeta_n/x_n = 1$. Using this result and (1) again,

$$\lim_{n\to\infty} \frac{\log x_{n+1} - \log x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{1}{\zeta_n} \cdot \frac{x_{n+1} - x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{x_n}{\zeta_n} \cdot \frac{x_{n+1} - x_n}{x_n x_{n+1}} = \beta.$$

This implies that for any $\epsilon \in (0, |\beta|)$, there exists a positive integer N such that

$$\beta - \epsilon < \frac{1}{x_n} - \frac{1}{x_{n+1}} < \beta + \epsilon$$

for all $n \geq N$. Summing these inequalities from $n = N$ to $N + m - 1$, we obtain

$$(\beta - \epsilon)m < \frac{1}{x_N} - \frac{1}{x_{N+m}} < (\beta + \epsilon)m.$$

Dividing by $N + m$ and taking the limit as $m \to \infty$,

$$\beta - \epsilon \leq \lim_{m\to\infty} \left(\frac{1}{(N+m)x_N} - \frac{1}{(N+m)x_{N+m}} \right) \leq \beta + \epsilon.$$

Since $\lim_{m\to\infty} \frac{1}{(N+m)x_N} = 0$,

$$\frac{1}{\beta + \epsilon} \leq -\lim_{m\to\infty} (N+m)x_{N+m} = -\lim_{n\to\infty} n x_n \leq \frac{1}{\beta - \epsilon}.$$

Let ϵ approach 0 to obtain $\lim_{n\to\infty} n x_n = -1/\beta > 0$. Thus,

$$\lim_{n\to\infty} \log(n x_n) = \lim_{n\to\infty} (\log n + \log x_n) = \log\left(-\frac{1}{\beta}\right).$$

However, if this holds, then, since $\log n \to \infty$, it must be the case that

$$\lim_{n\to\infty} \frac{\log n + \log x_n}{\log n} = \lim_{n\to\infty} \frac{\log(-\frac{1}{\beta})}{\log n} = 0.$$

Hence,

$$\lim_{n\to\infty} \left(1 + \frac{\log x_n}{\log n}\right) = 0,$$

so $\lim_{n\to\infty} \log x_n / \log n = -1$.

Also solved by P. Bracken, P. P. Dályay (Hungary), P. J. Fitzsimmons, E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), T. Persson & M. P. Sundqvist (Sweden), I. Pinelis, J. C. Smith, A. Stenger, R. Stong, and the proposer.

Sum of Medians of a Triangle

11790 [2014, 648]. *Proposed by Arkady Alt, San Jose, CA, and Konstantin Knop, St. Petersburg, Russia.* Given a triangle with semiperimeter s, inradius r, and medians of length m_a, m_b, and m_c, prove that $m_a + m_b + m_c \le 2s - 3(2\sqrt{3} - 3)r$.

Solution by James Christopher Smith, Knoxville, TN. Write R for the circumradius. We use two inequalities. The first is

$$(m_a + m_b + m_c)^2 \le 4s^2 - 16Rr + 5r^2,$$

due to Xiao-Guang Chu and Xue-Zhi Yang. (See J. Liu, "On an inequality for the medians of a triangle," *Journal of Science and Arts,* **19** (2012) 127–136.) The second is

$$s \le (3\sqrt{3} - 4)r + 2R,$$

known as Blundon's inequality. (See problem E1935, this MONTHLY, **73** (1966) 1122.)

Write $u = 2\sqrt{3} - 3$. From Blundon's inequality,

$$\begin{aligned}(2s - 3ur)^2 &= 4s^2 - 12sur + 9u^2r^2 \\ &\ge 4s^2 - 12ur\big((3\sqrt{3} - 4)r + 2R\big) + 9u^2r^2 \\ &= 4s^2 - 24uRr + \big(9u^2 - 12u(3\sqrt{3} - 4)\big)r^2 \\ &= 4s^2 - 16Rr + (16 - 24u)Rr + 3u(7 - 6\sqrt{3}\,)r^2.\end{aligned}$$

Next, we use Euler's inequality $R \ge 2r$ to get

$$\begin{aligned}\big(2s - 3ur\big)^2 &\ge 4s^2 - 16Rr + (16 - 24u)2r^2 + 3u(7 - 6\sqrt{3}\,)r^2 \\ &= 4s^2 - 16Rr + 5r^2,\end{aligned}$$

which is greater than or equal to $(m_a + m_b + m_c)^2$ by the Chu–Yang inequality.

Also solved by R. Boukharfane (Canada), O. Geupel (Germany), O. Kouba (Syria), R. Tauraso (Italy), M. Vowe (Switzerland), and T. Zvonaru & N. Stanciu (Romania).

A Middle Subspace

11792 [2014, 648]. *Proposed by Stephen Scheinberg, Corona del Mar, CA.* Show that every infinite-dimensional Banach space contains a closed subspace of infinite dimension and infinite codimension.

Solution by University of Louisiana at Lafayette Math Club, Lafayette, LA. Let V be an infinite-dimensional normed vector space (we do not require completeness). We construct a sequence of linearly independent vectors $v_0, v_1, \dots$ in V and a sequence of bounded linear functionals $\lambda_0, \lambda_1, \dots$ such that $\lambda_i(v_j) = \delta_{i,j}$ for all nonnegative integers i and j. Choose a nonzero $v_0 \in V$. By the Hahn–Banach theorem, there is a bounded linear functional λ_0 on V with $\lambda_0(v_0) = 1$. Suppose that nonzero vectors $v_0, \dots, v_k \in V$ and bounded linear functionals $\lambda_0, \dots, \lambda_k$ have been defined such that $\lambda_i(v_j) = \delta_{i,j}$ for $i, j \in \{1, \dots, k\}$. The vector subspace $\bigcap_{i=1}^k \ker \lambda_i$ has infinite dimension since it has finite codimension in V, which is infinite-dimensional. In particular, there exists nonzero $v_{k+1} \in \bigcap_{i=1}^k \ker \lambda_i$. The functional λ_{k+1} may be defined by $\lambda_{k+1}(v_j) = 0$ for $0 \le j \le k$ and $\lambda_{k+1}(v_{k+1}) = 1$ and then extended by the Hahn–Banach theorem to a bounded linear functional on V. The vectors $v_0, \dots, v_{k+1}$ are linearly independent since applying λ_j to $\sum c_i v_i = 0$ shows that $c_j = 0$. Continuing in this way, we construct the desired sequence $v_0, v_1, \dots$.

Let W be the closure of the linear span of $\{v_0, v_2, v_4, \dots\}$. The subspace W has infinite dimension, since the v_i are linearly independent. We claim also that W has infinite codimension, that is, that V/W is infinite-dimensional. We prove this by showing that the cosets $v_1 + W, v_3 + W, v_5 + W, \dots$ are linearly independent. Suppose otherwise, that there is some n and there are some scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with at least one of them nonzero such that $\sum_{i=1}^n \alpha_i v_{2i-1} \in W$. Say $\alpha_j \ne 0$. Since $\lambda_{2j-1}(v_i) = 0$ for even i, the linear functional λ_{2j-1} vanishes on their linear span and therefore on the closure W. This contradicts

$$\lambda_{2j-1}\left(\sum_{i=1}^n \alpha_i v_{2i-1}\right) = \alpha_j \ne 0.$$

Thus, W has infinite codimension.

Also solved by R. Chapman (U. K.), N. Eldredge, M. González & Á. Plaza (Spain), J. P. Grivaux (France), P. Perfetti (Italy), R. Tauraso (Italy), and the proposer.

Sums of Unit Vectors

11825 [2015, 284]. *Proposed by Marian Dinca, Vahalia University of Târgoviste, Bucharest, Romania, and Sorin Radulescu, Institute of Mathematical Statistics and Applied Mathematics, Bucharest, Romania.* Let E be a normed linear space. Given $x_1, \dots, x_n \in E$ (with $n \ge 2$) such that $\|x_k\| = 1$ for $1 \le k \le n$ and the origin of E is in the convex hull of $\{x_1, \dots, x_n\}$, prove that $\|x_1 + \cdots + x_n\| \le n - 2$.

Solution by Edward Schmeichel, San José State University, San José, CA. Since the origin is in the convex hull of $\{x_1, \dots, x_n\}$, there are nonnegative real numbers t_k for $1 \le k \le n$ with $\sum_{k=1}^n t_k = 1$ and $\sum_{k=1}^n t_k x_k = 0$. Since

$$t_k = \|t_k x_k\| = \left\| -\sum_{j \ne k} t_j x_j \right\| \le \sum_{j \ne k} t_j = 1 - t_k,$$

we see that $1 - 2t_k \ge 0$. Thus,

$$\|x_1 + \cdots + x_n\| = \left\| \sum_{k=1}^n (1 - 2t_k) x_k \right\| \le \sum_{k=1}^n (1 - 2t_k) = n - 2.$$

Editorial comment. This inequality seems to have first appeared in M. S. Klamkin and D. J. Newman, An inequality for the sums of unit vectors, *Univ. Beo. Publ. Elek. Fac., Ser. Mat. i. Fiz.* 338–352 (1971) 47–48. A more accessible reference is G. D. Chakerian and M. S. Klamkin, Inequalities for Sums of Distances, this MONTHLY **80** (1973) 1009–1017.

Also solved by M. Aassila (France), U. Abel (Germany), K. F. Andersen (Canada), R. Bagby, E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), F. Brulois, P. Budney, S. Byrd & R. Nichols, N. Caro (Brazil), R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), P. J. Fitzsimmons, N. Grivaux (France), E. A. Herman, Y. J. Ionin, E. G. Katsoulis, J. H. Lindsey II, O. P. Lossers (Netherlands), V. Muragan & A. Vinoth (India), M. Omarjee (France), M. A. Prasad (India), R. Stong, R. Tauraso (Italy), J. Van Hamme (Belgium), J. Zacharias, R. Zarnowki, New York Math Circle, and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Daniel Cranston, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, and Fuzhen Zhang.

Proposed problems should be submitted online at `http://www.americanmathematicalmonthly.submittable.com/submit`. *Proposed solutions to the problems below should be submitted on or before July 31, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11964. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Find all triples of integers (a, b, c) with $a \neq 0$ such that the function f defined by $f(x) = ax^2 + bx + c$ has the property that, for every positive integer n, there exists an integer m with $f(n)f(n+1) = f(m)$.

11965. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with circumradius R. Prove that there exists a point M on side BC such that $MA \cdot MB \cdot MC = 32R^3/27$ if and only if $2 \cot B \cot C = 1$.

11966. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania.* Prove that

$$\int_0^1 \frac{x \ln(1+x)}{1+x^2}\, dx = \frac{\pi^2}{96} + \frac{(\ln 2)^2}{8}.$$

11967. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fermat number $2^{2^n} + 1$. Find

$$\lim_{n\to\infty} \sqrt{6F_1 + \sqrt{6F_2 + \sqrt{6F_3 + \sqrt{\cdots + \sqrt{6F_n}}}}}.$$

11968. *Proposed by Christopher J. Hillar, Redwood Center for Theoretical Neuroscience, Berkeley, CA, Robert Krone, Queens University, Kingston, Ontario, Canada, and Anton Leykin, Georgia Tech University, Atlanta, GA.* Let F_n be the nth Fibonacci number, with $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. For $n \geq 1$, prove that $F_{5n}/(5F_n)$ is an integer congruent to 1 modulo 10.

11969. *Proposed by Askar Dzhumadil'daev, Kazakh-British Technical University, Almaty, Kazakhstan.* Let $x_1, \ldots, x_n$ be indeterminates, and let A be the n-by-n matrix with i, j-entry $\sec(x_i - x_j)$. Prove

$$\det A = (-1)^{\binom{n}{2}} \prod_{1\leq i<j\leq n} \tan^2(x_i - x_j).$$

http://dx.doi.org/10.4169/amer.math.monthly.124.3.274

11970. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let

$$\zeta_5(z) = 1 + 2^{-z} + 3^{-z} + 4^{-z} + 5^{-z},$$

where z is a complex number. Prove that $\zeta_5(z) \neq 0$ when the real part of z is greater than or equal to 0.9.

SOLUTIONS

Successors of Squares Without Large Prime Divisors

11831 [2015, 390]. *Proposed by Raitis Ozols, University of Latvia, Riga, Latvia.* Prove that for $\varepsilon > 0$ there exists an integer n such that the greatest prime divisor of $n^2 + 1$ is less than εn.

Solution by John P. Robertson, National Council on Compensation Insurance, Boca Raton, Florida. We show more generally that, for every integer k and every $\varepsilon > 0$, there exists an integer n such that the greatest prime divisor of $n^2 + k$ is less than εn.

When $k = 0$, it suffices to let n be a sufficiently large power of 2.

When $k \neq 0$, let $n = 4k^2m^3 + 3km$ for some positive integer m. We compute

$$n^2 + k = k(km^2 + 1)(4km^2 + 1)^2.$$

The largest prime p dividing $n^2 + k$ can be no larger than $|4km^2 + 1|$. In addition, $|4km^2 + 1|/n < \varepsilon$ when m is sufficiently large, so $p < \varepsilon n$.

Editorial comment. Some solutions involved the factorization of $x^{420} - 1$ into cyclotomic polynomials, and some used Pell's equation.

Also solved by D. Beckwith, B. Bekker (Russia) & Y. J. Ionin, R. Chapman (U. K.), V. De Angelis, J. Hosle, P. W. Lindstrom, O. P. Lossers (Netherlands), W. McDermott, M. Omarjee (France), L. Robitaille, C. P. Rupert, J. Schlosberg, N. C. Singer, R. Stong, R. Tauraso (Italy), E. Weinstein, M. Wildon (U. K.), and the proposer.

An Inequality When Two Triples Agree in Order

11834 [2015, 390]. *Proposed by Arkady Alt, San Jose, CA.* For nonnegative real numbers u, v, w, let $\Delta(u, v, w) = 2(uv + vw + wu) - (u^2 + v^2 + w^2)$. Say that two lists (a, b, c) and (x, y, z) *agree in order* if $(a - b)(x - y) \geq 0$, $(b - c)(y - z) \geq 0$, and $(c - a)(z - x) \geq 0$. Prove that if (x, y, z) and (a, b, c) agree in order, then $\Delta(a, b, c)\Delta(x, y, z) \geq 3\Delta(ax, by, cz)$.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA. The quantity Δ is invariant under permutation of its arguments. The relation "agree in order" and the inequality to be proved do not change when (a, b, c) and (x, y, z) undergo the same permutation. Therefore we may assume $a \geq b \geq c \geq 0$ and $x \geq y \geq z \geq 0$. One then sees that

$$\begin{aligned}\Delta(a, b, c)\Delta(x, y, z) - 3\Delta(ax, by, cz) &= (a - b)^2\big(4(x - y)^2 + 3(y - z)^2\big) \\ &+ 3(b - c)^2(x - y)^2 + 2(a - b)^2\big((x - y)(3y - z) + z(y - z)\big) \\ &+ 2(a - b)\big((x - y)^2(3b - c) + (x - y)(3by - cz) + cz(y - z)\big) \\ &+ 2c(b - c)\big((x - y)^2 + z(x - z) + z(y - z)\big)\end{aligned}$$

is nonnegative, since each term is nonnegative.

Also solved by R. Chapman (U. K.), H. Y. Far, J. F. Loverde, L. Matejíčka (Slovakia), J. C. Smith, R. Stong, S. Wagon, and the proposer.

A Functional Inequality

11835 [2015, 390]. *Proposed by George Stoica, University of New Brunswick, St. John, NB, Canada.* Find all functions f from $[0, \infty)$ to $[0, \infty)$ such that whenever $x, y \geq 0$,

$$\sqrt{3} f(2x) + 5f(2y) \leq 2f(\sqrt{3}x + 5y).$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We look at the slightly more general (a, b)-condition

$$af(x) + bf(y) \leq f(ax + by),$$

for some fixed pair (a, b) satisfying $0 < a < 1 < b$. The actual problem corresponds to the case $a = \sqrt{3}/2$ and $b = 5/2$, with $2x, 2y$ replaced by x, y. We claim that the only solutions are functions of the form $f(x) = qx$ with $q \geq 0$.

From $bf(0) \leq f(0)$ we have $f(0) = 0$, since $b > 1$. We claim that if $f(c) = 0$ for some $c > 0$, then $f(x) = 0$ for all x. Indeed, if x and y satisfy $ax + by = c$, then $af(x) + bf(y) = 0$, and so $f(x) = f(y) = 0$. It follows that $f(z) = 0$ for all z in the interval $[0, c/a]$ and then for all z in the interval $[0, c/a^2]$, and, continuing in this way, finally $f(z) = 0$ for all $z \geq 0$. In this case, $f(x) = 0 = qx$ for all x, where $q = 0$.

Now assume $f(x) > 0$ for all $x > 0$. Two special cases of our condition are $af(x) \leq f(ax)$ and $bf(x) \leq f(bx)$. For $x > 0$, these are equivalent to

$$\frac{f(x)}{x} \leq \frac{f(ax)}{ax} \quad \text{and} \quad \frac{f(x)}{x} \leq \frac{f(bx)}{bx}.$$

Let $q = \inf\{f(x)/x : 1 \leq x \leq b\}$. Using the second inequality, $q = \inf\{f(x)/x : 1 \leq x < \infty\}$. We then use the first inequality to obtain $q = \inf\{f(x)/x \ : \ 0 < x < \infty\}$.

Next let $g(x) = f(x) - qx$. Note that g is a function from $[0, \infty)$ to $[0, \infty)$ satisfying the (a, b)-condition, and $\inf\{g(x)/x : 1 \leq x \leq b\} = 0$. We claim $g(1) = 0$. Indeed, $g(a + by) \geq ag(1) + bg(y) \geq ag(1)$ for all y, so $g(z) \geq ag(1)$ for all $z \geq a$. Choose $\varepsilon > 0$ and $z \in [1, b]$ with $g(z) < \varepsilon$. Since $z \geq 1 > a$, we have $ag(1) < \varepsilon$. Since ε is arbitrary, we conclude $g(1) = 0$.

Thus $g(z) = 0$ when $z \geq 0$, which yields $f(x) = qx$ for all $x \geq 0$.

Also solved by R. Boukharfane (France), O. Bucicovschi, R. Chapman (U. K.), P. P. Dályay (Hungary), L. Matejíčka (Slovakia), M. Omarjee (France), M. Omarjee (France) & R. Tauraso (Italy), R. Stong, and the proposer.

From Nesbitt to Gerretsen

11836 [2015, 391]. *Proposed by Traian Viteam, Montevideo, Uruguay.* Let ABC be a triangle with sides of lengths a, b, and c, circumradius R, and inradius r. For $p, q, r > 0$, let $f(p, q, r) = pqr/(p + q)(r^2 - (p - q)^2)$. Prove that

$$\frac{R}{2r} \geq \frac{2}{3}\left(f(a, b, c) + f(b, c, a) + f(c, a, b)\right).$$

Solution by Mehtaab Sawhney (student), University of Pennsylvania, Philadelphia, PA. Let K and s denote the area and semiperimeter of the triangle, respectively. From the area formulas $K = rs = abc/(4R) = \sqrt{s(s-a)(s-b)(s-c)}$, we find

$$\frac{R}{r} = \frac{abc/4K}{K/s} = \frac{abcs}{4K^2} = \frac{2abc}{(a+b-c)(b+c-a)(c+a-b)}, \quad \text{or}$$

$$\frac{a+b-c}{a+b}\frac{R}{2r} = \frac{abc}{(a+b)(a-b+c)(b+c-a)} = f(a, b, c).$$

Thus

$$\sum_{\text{cyc}} f(a,b,c) = \frac{R}{2r}\sum_{\text{cyc}}\frac{a+b-c}{a+b} = \frac{R}{2r}\left(3-\sum_{\text{cyc}}\frac{a}{b+c}\right),$$

where the sums are taken over the cyclic permutations of (a, b, c). It follows that

$$\frac{3R}{4r} - \sum_{\text{cyc}} f(a,b,c) = \frac{R}{2r}\left(\left(\sum_{\text{cyc}}\frac{a}{b+c}\right)-\frac{3}{2}\right).$$

The problem is to show that this is nonnegative. By Engel's form of the Cauchy–Schwarz inequality,

$$\sum_{\text{cyc}}\frac{a}{b+c} = \sum_{\text{cyc}}\frac{a^2}{a(b+c)} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} = \frac{3}{2} + \sum_{\text{cyc}}\frac{(a-b)^2}{4(ab+bc+ca)} \geq \frac{3}{2}.$$

This proves the result, with equality if and only if $a = b = c$.

Editorial comment. The inequality $a/(b+c) + b/(c+a) + c/(a+b) \geq 3/2$ was first noted by A. M. Nesbitt (Problem 15144, *Educational Times*) in 1903. Its many proofs include: (i) averaging the two possible applications of the rearrangement inequality to the sequences $a \leq b \leq c$ and $1/(b+c) \leq 1/(c+a) \leq 1/(a+b)$, and (ii) normalizing to $a + b + c = 1$, then applying Jensen's inequality to $f(a) = a/(1-a)$ which is convex on $(0, 1)$. R. Boukharfane, P. P. Dálylay, and A. Gundamraj showed that the inequality in the problem is equivalent to $s^2 \geq 7r^2 + 10Rr$, thus relating Nesbitt's inequality to that of Gerretsen: $s^2 \geq 16Rr - 5r^2$.

Also solved by A. Ali (India), A. Alt, M. Bataille (France), R. Boukharfane (France), R. Chapman (U. K.), P. P. Dálylay (Hungary), H. Y. Far, D. Fleischman, O. Geupel (Germany), A. Gundamraj, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), B. Keigwin & J. Zacharias, D. Kim (Korea), J. F. Loverde, P. Perfetti (Italy), S. Roy & J. Bose (India), J. Schlosberg, J. C Smith, N. Stanciu & T. Zvonaru, R. Stong, T. Sun & S. Archer, R. Tauraso (Italy), L. Wimmer (Germany), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, and the proposer.

Forcing a Double Transversal

11838 [2015, 500]. *Proposed by Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA.* Let n be a positive integer. Find the least integer $f(n)$ with the following property: if M is an $n \times n$ matrix of nonnegative integers with every row and column sum equal to $f(n)$, then M contains n entries, all greater than 1, with no two of these n entries in the same row or column.

Solution by Mark Wildon, Royal Holloway, Egham, U. K. More generally, for $b \in \mathbb{N}$, we determine $f_b(n)$, defined to be the least t such that when the rows and columns sum to t there will always be a transversal whose entries exceed b, where a *transversal* is a set of n positions with one in each row and column. Let $g_b(n) = b\left\lfloor\frac{n+1}{2}\right\rfloor\left\lceil\frac{n+1}{2}\right\rceil$; we prove $f_b(n) = g_b(n) + 1$.

Given a nonnegative integer matrix M with rows and columns having sum t, let G be the corresponding bipartite graph with bipartition (X, Y) whose edge set is $\{x_i y_j : M_{i,j} > b\}$. It suffices to show that G must have a perfect matching when $t > g_b(n)$, and otherwise G may fail to have a perfect matching.

If G has no perfect matching, then by Hall's theorem there is a set $S \subseteq X$ such that $|N(S)| = |S| - 1$, where $N(S)$ is the set of vertices in Y having neighbors in S. Let $k = |N(S)|$. The submatrix M' with rows indexed by S and columns indexed by $N(S)$ has sum at most kt. Since entries in these rows outside these columns are at most b, the submatrix M' has sum at least $(k+1)(t-b(n-k))$.

Thus $(k+1)(t-b(n-k)) \le kt$, which simplifies to $t \le (k+1)b(n-k) \le g_b(n)$. Hence $t > g_b(n)$ implies that the desired transversal exists.

To show that $g_b(n)$ does not suffice, we consider odd and even n separately; note $g_b(2m-1) = bm^2$ and $g_b(2m) = bm(m+1)$. Let $J_{r,s}$ and $O_{r,s}$ be the r-by-s matrices that are all-1 and all-0, respectively. When $n = 2m-1$, let

$$M = b\begin{pmatrix} J_{m,m} & mJ_{m,m-1} \\ mJ_{m-1,m} & O_{m-1,m-1} \end{pmatrix}.$$

When $n = 2m$, let

$$M = b\begin{pmatrix} J_{m,m+1} & (m+1)J_{m,m-1} \\ mJ_{m,m+1} & O_{m,m-1} \end{pmatrix}.$$

Rows and columns sum to $g_b(n)$. The desired transversal does not exist, because the entries in the first m rows that exceed b occur in only $m-1$ columns.

An n-by-n matrix with constant row and column sums $g_b(n)-1$ may be obtained from either of the matrices above by taking a transversal of entries each at least 1, and reducing each of these entries by 1. This cannot create a new transversal with entries greater than b. Iterating this process shows that $f_b(n) > g_b(n)$.

Editorial comment. The generalization was also given by Robin Chapman, Pierre Lalonde, and John H. Smith.

Also solved by R. Chapman (U. K.), Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. E. Prather, J. H. Smith, R. Stong, T. Viteam (Japan), and the proposer.

Kooi Variant

11839 [2015, 500]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let R be the circumradius, r the inradius, and s the semiperimeter of a triangle. Prove that

$$16R^3 + 20R^2r + 15Rr^2 + 5r^3 \ge s^2(4R+r),$$

with equality if and only if the triangle is equilateral.

Solution by Peter Nüesch, École Polytechnique Fédérale, Lausanne, Switzerland. We deduce this as a consequence of Kooi's inequality

$$2(2R-r)s^2 \le R(4R+r)^2$$

(O. Kooi, Inequalities for the triangle, *Simon Stevin* **32** (1958) 97–101; O. Bottema, *Geometric Inequalities*, Groningen, 1969, 5.7). It suffices to show

$$\frac{R(4R+r)^2}{4R-2r} \le \frac{16R^3 + 20R^2r + 15Rr^2 + 5r^3}{4R+r}.$$

To prove this, note that

$$(16R^3 + 20R^2r + 15Rr^2 + 5r^3)(4R-2r) - R(4R+r)^3$$
$$= r^2(8R^2 - 11Rr - 10r^2) = r^2(R-2r)(8R+5r) \ge 0,$$

with equality only if $R = 2r$, i.e., the triangle is equilateral.

Also solved by A. Ali (India), A. Alt, S. Archer & T. Sun, E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Chapman (U. K.), A. Fanchini (Italy), O. Geupel (Germany), B. Karaivanov (U. S. A.) & T. S. Vassilev

(Canada), K.-W. Lau (China), J. H. Lindsey II, J. F. Loverde, P. Perfetti (Italy), M. Sawhney, J. Schlosberg, M. A. Shayib, J. C. Smith, R. Stong, M. Vowe (Switzerland), T. Zvonaru & N. Stanciu (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

A Complex Inequality

11840 [2015, 500]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let $z_1, \ldots, z_n$ be complex numbers. Prove that

$$\left(\sum_{k=1}^{n} |z_k|\right)^2 - \left|\sum_{k=1}^{n} z_k\right|^2 \geq \left(\sum_{k=1}^{n} |\operatorname{Re} z_k| - \left|\sum_{k=1}^{n} \operatorname{Re} z_k\right|\right)^2.$$

(Here $\operatorname{Re} z$ denotes the real part of z.)

Solution by Yongtao Li, Hunan Normal University, Changsha, China. Let $z_1, \ldots, z_n$ be complex numbers, and write $z_k = a_k + i b_k$ for $1 \leq k \leq n$, where a_k and b_k are real. The stated inequality may be written in the equivalent form

$$\left(\sum_{k=1}^{n} \sqrt{a_k^2 + b_k^2}\right)^2 - \left(\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n} b_k\right)^2\right) \geq \left(\sum_{k=1}^{n} |a_k| - \left|\sum_{k=1}^{n} a_k\right|\right)^2. \tag{1}$$

According to Minkowski's inequality and the fact that $|b_k| \geq b_k$, we have

$$\left(\sum_{k=1}^{n} \sqrt{a_k^2 + b_k^2}\right)^2 \geq \left(\sum_{k=1}^{n} |a_k|\right)^2 + \left(\sum_{k=1}^{n} |b_k|\right)^2 \geq \left(\sum_{k=1}^{n} |a_k|\right)^2 + \left(\sum_{k=1}^{n} b_k\right)^2.$$

This implies

$$\left(\sum_{k=1}^{n} \sqrt{a_k^2 + b_k^2}\right)^2 - \left(\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n} b_k\right)^2\right) \geq \left(\sum_{k=1}^{n} |a_k|\right)^2 - \left(\sum_{k=1}^{n} a_k\right)^2. \tag{2}$$

However,

$$\begin{aligned}\left(\sum_{k=1}^{n} |a_k|\right)^2 &= \left(\left(\sum_{k=1}^{n} |a_k| - \left|\sum_{k=1}^{n} a_k\right|\right) + \left|\sum_{k=1}^{n} a_k\right|\right)^2 \\ &\geq \left(\sum_{k=1}^{n} |a_k| - \left|\sum_{k=1}^{n} a_k\right|\right)^2 + \left(\sum_{k=1}^{n} a_k\right)^2.\end{aligned}$$

Consequently,

$$\left(\sum_{k=1}^{n} |a_k|\right)^2 - \left(\sum_{k=1}^{n} a_k\right)^2 \geq \left(\sum_{k=1}^{n} |a_k| - \left|\sum_{k=1}^{n} a_k\right|\right)^2. \tag{3}$$

Combining (2) and (3), we obtain the required inequality (1).

Also solved by K. F. Andersen (Canada), R. Chapman (U. K.), P. P. Dályay (Hungary), R. Dutta (India), D. Fleischman, E. A. Herman, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), M. Sawhney, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), E. I. Verriest, University of Louisiana at Lafayette Math Club, and the proposer.

A Quadrilateral's Perimeter Exceeds Its Diagonals

11841 [2015, 500]. *Proposed by Leonard Giugiuc, Drobeta Turnu, Romania.* Let $ABCD$ be a convex quadrilateral. Let E be the midpoint of AC, and let F be the midpoint of BD. Show that

$$|AB| + |BC| + |CD| + |DA| \geq |AC| + |BD| + 2|EF|.$$

(Here $|XY|$ denotes the distance from X to Y.)

Composite solution by many solvers. This is a consequence of Hlavka's inequality, which states (in one form) that for any three complex numbers x, y, z,

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |z + x|.$$

Let a, b, c, and d represent complex numbers associated with the points A, B, C, and D, respectively. Apply Hlavka's inequality with $x = b - a$, $y = c - b$, and $z = d - c$ to obtain the desired result.

Editorial comment. There is no need for the four points A, B, C, D to form a convex quadrilateral. Hlavka's inequality applies with any four distinct points in the plane irrespective of their position.

This inequality actually holds in all 2-dimensional real normed spaces: see A. Sudbery, The quadrilateral inequality in two dimensions, this MONTHLY **82** (1975) 629–632.

Solved by A. Ali (India), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), J. Cade, R. Chapman (U. K.), P. P. Dályay (Hungary), M. Dincă (Romania), R. Dutta (India), O. Geupel (Germany), E. J. Ionaşcu, Y. J. Ionin, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. P. Lossers (Netherlands), S. R. Mousavi (Iran), P. Nüesch (Switzerland), V. Pambuccian, R. Stong, R. Tauraso (Italy), T. Viteam (Japan), M. Vowe (Switzerland), University of Louisiana at Lafayette Math Club, and the proposer.

Creative Telescoping

11844 [2015, 501]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For nonnegative integers m and n, prove

$$\sum_{k=0}^{n} (m - 2k)\binom{m}{k}^3 = (m - n)\binom{m}{n} \sum_{j=0}^{m-1} \binom{j}{n}\binom{j}{m - n - 1}.$$

(Here $\binom{u}{v}$ is zero if $u < v$, and a sum is zero if its range of summation is empty.)

Solution I by Richard Stong, San Diego, CA. For $n = 0$ the identity reduces to $m = m$. Thus it suffices to show that both sides have the same differences for consecutive values of n, that is,

$$(m - 2n)\binom{m}{n}^3 = (m - n)\binom{m}{n} \sum_{j=0}^{m-1} \binom{j}{n}\binom{j}{m - n - 1}$$
$$- (m - n + 1)\binom{m}{n - 1} \sum_{j=0}^{m-1} \binom{j}{n - 1}\binom{j}{m - n}.$$

Observing that $(m - n + 1)\binom{m}{n-1} = n\binom{m}{n}$, we may cancel a factor of $\binom{m}{n}$ (equality clearly holds when $\binom{m}{n} = 0$) to reduce the identity to

$$(m - 2n)\binom{m}{n}^2 = \sum_{j=0}^{m-1}\left((m - n)\binom{j}{n}\binom{j}{m - n - 1} - n\binom{j}{n - 1}\binom{j}{m - n}\right). \qquad (*)$$

Multiplying the easily verifiable computation

$$\begin{aligned}(m-n)^2(j+1-n)-n^2(j+n+1-m)&=(m-2n)(mj+m+n^2-mn)\\&=(m-2n)\big((j+1)^2-(j+1-n)(j+n+1-m)\big)\end{aligned}$$

by $(j!)^2$ and dividing it by $n!(j+1-n)!(m-n)!(j+n+1-m)!$, we obtain

$$\begin{aligned}&(m-n)\binom{j}{n}\binom{j}{m-n-1}-n\binom{j}{n-1}\binom{j}{m-n}\\&\qquad=(m-2n)\left(\binom{j+1}{n}\binom{j+1}{m-n}-\binom{j}{n}\binom{j}{m-n}\right).\end{aligned}$$

Hence the sum on the right side of $(*)$ telescopes to

$$(m-2n)\left(\binom{m}{n}\binom{m}{m-n}-\binom{0}{n}\binom{0}{m-n}\right)=(m-2n)\binom{m}{n}^2,$$

as desired, where we used that the second term is nonzero only if $m=n=0$, in which case $m-2n=0$.

Solution II by Tewodros Amdeberhan, Tulane University, New Orleans, LA, and Shalosh B. Ekhad, U.S.A. Let $F_1(m,k)=(m-2k)\binom{m}{k}^3$, $F_2(m,j)=(m-n)\binom{m}{n}\binom{j}{n}\binom{j}{m-n-1}$, $f_1(m)=\sum_{k=0}^{n}F_1(m,k)$, and $f_2(m)=\sum_{j=0}^{m-1}F_2(m,j)$. We need to show $f_1(m)=f_2(m)$. Using Zeilberger's creative telescoping algorithm as described in the book "A=B" by M. Petkovšek, H. S. Wilf, and D. Zeilberger, we obtain the Wilf–Zeilberger mates $G_1(m,k)=(2m-k+2)\binom{m}{k-1}^3$ and $G_2(m,j)=(n+1)\binom{m}{n}\binom{j}{n+1}\binom{j}{m-n}$. These lead to the identities

$$\begin{aligned}&F_1(m+1,k)+F_1(m,k)=G_1(m,k+1)-G_1(m,k)\quad\text{and}\\&F_2(m+1,j)+F_2(m,j)=G_2(m,j+1)-G_2(m,j),\end{aligned}$$

which are easy (though tedious) to check. Summing respectively over k and j, using that G_1 and G_2 telescope and $G_1(m,0)=G_2(m,0)=0$, we find

$$\begin{aligned}&\sum_{k=0}^{n}\left(F_1(m+1,k)+F_1(m,k)\right)=G_1(m,n+1)=(2m-n+1)\binom{m}{n}^3\quad\text{and}\\&\sum_{j=0}^{m-1}\big(F_2(m+1,j)+F_2(m,j)\big)=G_2(m,m)=m\binom{m-1}{n}\binom{m}{n}^2.\end{aligned}$$

The first of these gives the recurrence $f_1(m+1)+f_1(m)=(2m-n+1)\binom{m}{n}^3$, while adding the term $F_2(m+1,m)$ to the second yields $f_2(m+1)+f_2(m)=m\binom{m-1}{n}\binom{m}{n}^2+(m+1-n)\binom{m+1}{n}\binom{m}{n}^2=(2m-n+1)\binom{m}{n}^3$. Thus f_1 and f_2 satisfy the same recurrence. Note: when $n\geq m$, we get $f_2(m)=0$ trivially, while $f_1(m)=\sum_{k=0}^{m}(m-k)\binom{m}{k}^3-\sum_{k=0}^{m}k\binom{m}{k}^3=\sum_{k=0}^{m}k\binom{m}{m-k}^3-\sum_{k=0}^{m}k\binom{m}{k}^3=0$. Since $f_1(0)=f_2(0)$, the identity $f_1(m)=f_2(m)$ follows.

Also solved by R. Bianconi & M. Elia, R. Chapman (U. K.), P. P. Dályay (Hungary), M. Omarjee (France), M. Wildon (U. K.), and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Daniel Cranston, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, and Fuzhen Zhang.

Proposed problems should be submitted online at
http://www.americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by August 31, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11971. *Proposed by Spiros P. Andriopoulos, Third High School of Amaliada, Eleia, Greece.* For $n \geq 2$, let $a_1, \ldots, a_n$ be positive real numbers. Prove

$$\left(\prod_{i=1}^{n}(1+a_i)\right)^{n-1} \geq \left(\prod_{i<j}\left(1+\frac{2a_ia_j}{a_i+a_j}\right)\right)^2.$$

11972. *Proposed by Yun Zhang, Xi'an Senior High School, Xi'an, China.* Let r be the radius of the sphere inscribed in a tetrahedron whose exscribed spheres have radii r_1, r_2, r_3, and r_4. Prove

$$r\left(\sqrt[3]{r_1}+\sqrt[3]{r_2}+\sqrt[3]{r_3}+\sqrt[3]{r_4}\right) \leq 2\sqrt[3]{r_1r_2r_3r_4}.$$

11973. *Proposed by Derek Orr, University of Pittsburgh, Pittsburgh, PA.* Catalan's constant G is defined to be $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. Prove

$$G = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n}\left(1-\frac{2}{4^n}\right),$$

where ζ is the Riemann zeta function, defined by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ for $s > 1$ and with $\zeta(0) = -1/2$ by analytic continuation.

11974. *Proposed by Haoran Chen, Gustavus Adolphus College, St. Peter, MN.* Any n points on a line divide that line into $n-1$ segments and two rays. If these $n-1$ segments all have the same length, then we say the line is *well-divided* by the set. Classify the arrangements consisting of a finite number of lines in the plane, no two parallel, such that each line is well-divided by its points of intersection with the other lines.

11975. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let x be a real number in $[0, 1)$, and let $L(x) = \int_0^1 \Gamma^x(t)\,dt$, where Γ is the gamma function defined by $\Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}\,dx$. Prove

$$\frac{(1-\gamma)^x}{1-x} \leq L(x) \leq \frac{1}{1-x},$$

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where γ is the Euler–Mascheroni constant $\lim_{n\to\infty}\left(-\ln n+\sum_{k=1}^{n} 1/k\right)$.

11976. *Proposed by Robert Bosch, Miami, FL.* Given a positive real number s, consider the sequence $\{u_n\}$ defined by $u_1=1$, $u_2=s$, and $u_{n+2}=u_n u_{n+1}/n$ for $n\ge 1$.

(a) Show that there is a constant C such that $\lim_{n\to\infty} u_n=\infty$ when $s>C$ and $\lim_{n\to\infty} u_n=0$ when $s<C$.

(b) Calculate $\lim_{n\to\infty} u_n$ when $s=C$.

11977. *Proposed by Joseph Foy, University of Chicago, Chicago, IL, Ali Hassani, Dearborn, MI, Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI, and Clark Zhang, University of Pennsylvania, Philadelphia, PA.*

(a) Suppose that a, b, c, and d are positive integers with $\gcd(a,b,c,d)=1$ and with $\sqrt{a}+\sqrt{b}=\sqrt{c}+\sqrt{d}$. Prove that if $\{a,b\}\neq\{c,d\}$, then each of a, b, c, and d is a perfect square.

(b)* More generally, suppose that k is an integer with $k\ge 3$, and suppose that a, b, c, and d are positive integers with $\gcd(a,b,c,d)=1$ and with $\sqrt[k]{a}+\sqrt[k]{b}=\sqrt[k]{c}+\sqrt[k]{d}$. Assuming $\{a,b\}\neq\{c,d\}$, must each of a, b, c, and d be a perfect kth power?

SOLUTIONS

Snake Oil Triumphs Again

11798 [2014, 738]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* For positive integers n, let f_n be the polynomial given by

$$f_n(x)=\sum_{r=0}^{n}\binom{n}{r}x^{\lfloor r/2\rfloor}.$$

(**a**) Prove that if $n+1$ is prime, then f_n is irreducible over $\mathbb{Q}$.

(**b**) Prove that for all n (whether $n+1$ is prime or not),

$$f_n(1+x)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}2^{n-2k}x^k.$$

Solution I by Mark Wildon, U. K. For nonnegative integer s, the coefficient of x^s in $f_n(x)$ is $\binom{n}{2s}+\binom{n}{2s+1}$, which equals $\binom{n+1}{2s+1}$.

(**a**) Note that $f_1(x)=2$. This is a unit, which usually is not considered irreducible even though it is not a product of two nonunits.

Let $n+1$ be an odd prime. If $m\neq 0$, then $m\binom{n+1}{m}$ equals $(n+1)\binom{n}{m-1}$, which is divisible by $n+1$. Hence $\binom{n+1}{m}$ is divisible by $n+1$ for $m\in\{1,\dots,n\}$. It follows that all coefficients of $f_n(x)$ are divisible by $n+1$ except the coefficient of $x^{n/2}$, which is 1. Also, $f_n(0)=\binom{n+1}{1}=n+1$, and $n+1$ is not divisible by $(n+1)^2$. By Eisenstein's criterion, $f_n(x)$ is irreducible over $\mathbb{Q}$.

(**b**) We have

$$f_n(1+x)=\sum_{s=0}^{\lfloor n/2\rfloor}\binom{n+1}{2s+1}(1+x)^s=\sum_{s=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{s}\binom{n+1}{2s+1}\binom{s}{k}x^k.$$

The required polynomial identity is therefore equivalent to

$$\sum_{s=k}^{\lfloor n/2\rfloor}\binom{n+1}{2s+1}\binom{s}{k}=\binom{n-k}{k}2^{n-2k} \tag{1}$$

for $n \geq 2k$. Fix k, and let a_n denote the left side of (1). Applying the "snake oil" method of H. S. Wilf, we form the generating function for $\{a_n\}_{n\geq 0}$, switch the order of summation, and use $\sum_{n=0}^{\infty}\binom{n+k}{k}y^n = (1-y)^{-(k+1)}$ and $\sum_{s=k}^{\infty}\binom{s}{k}z^s = z^k(1-z)^{-(k+1)}$ to compute

$$\begin{aligned}
\sum_{n=2k}^{\infty} a_n y^n &= \sum_{n=2k}^{\infty}\sum_{s=k}^{\lfloor n/2\rfloor}\binom{n+1}{2s+1}\binom{s}{k}y^n = \sum_{s=k}^{\infty}\sum_{n=2s}^{\infty}\binom{n+1}{2s+1}\binom{s}{k}y^n \\
&= \sum_{s=k}^{\infty}\binom{s}{k}\sum_{n=2s}^{\infty}\binom{n+1}{2s+1}y^n = \sum_{s=k}^{\infty}\binom{s}{k}\sum_{m=0}^{\infty}\binom{m+2s+1}{2s+1}y^{m+2s} \\
&= \sum_{s=k}^{\infty}\binom{s}{k}\frac{y^{2s}}{(1-y)^{2s+2}} = \frac{y^{2k}}{(1-y)^{2k+2}}\frac{1}{\left(1-\frac{y^2}{(1-y)^2}\right)^{k+1}} = \frac{y^{2k}}{(1-2y)^{k+1}} \\
&= \sum_{m=0}^{\infty}\binom{m+k}{k}2^m y^{m+2k} = \sum_{n=2k}^{\infty}\binom{n-k}{k}2^{n-2k}y^n.
\end{aligned}$$

The desired result follows by comparing the coefficients of y^n.

Solution II of part **(b)** *by Borislav Karaivanov, Lexington, SC, and Tzvetalin S. Vassilev, Nipissing University, North Bay Ontario, Canada.* We give a combinatorial proof of the needed identity (1) by showing that both sides count the ternary $(n+1)$-tuples having a 2 in exactly k positions such that the copies of 2 separate the string into $k+1$ portions each of which has an odd number of copies of 1.

On the left, start with $n+1$ positions, and choose an odd number (at least $2k+1$) to be nonzero. Choose k of the positions that are even-indexed relative to this sublist to receive 2. Between any two such positions, the number of copies of 1 is odd, and the number at the beginning or end is also odd.

On the right, begin with any ternary list of length $n-k$ having exactly k copies of 2. There are $\binom{n-k}{k}2^{n-2k}$ such lists; copies of 2 may be consecutive. Now insert one position immediately before each 2 and at the end. This position receives 1 or 0 as needed so that the number of copies of 1 in that portion between copies of 2 is odd. This choice is unique, so we obtain exactly one of the desired lists for each of the $\binom{n-k}{k}2^{n-2k}$ original lists of length $n-k$.

Editorial comment. Identity (1) needed for part (**b**) appears as identity (3.121) on p. 36 in H. W. Gould, *Combinatorial Identities*, Morgantown, WV, 1972. A closed form for $f_n(x)$ is $((1+\sqrt{x})^{n+1}-(1-\sqrt{x})^{n+1})/(2\sqrt{x})$.

Also solved by A. Ali (India), D. D'Addezio (Italy), C. Georghiou (Greece), O. Geupel (Germany), M. Goldenberg & M. Kaplan, Y. J. Ionin, O. Kouba (Syria), M. Omarjee (France), N. C. Singer, A. Stenger, R. Stong, B. Sury (India), R. Tauraso (Italy), M. Vowe (Switzerland), H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer. Part (a) also solved by D. Fleischman, E. A. Herman, Á. Plaza (Spain), and R. Sargsyan (Armenia). Part (b) also solved by D. Beckwith.

A Golden Series of Digammas

11842 [2015, 501]. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let ψ be the *digamma function*, that is, $\psi(x) = (\log\Gamma(x))'$. Let $\phi = (1+\sqrt{5})/2$. Prove that

$$\sum_{n=1}^{\infty}\frac{\psi(n+\phi)-\psi(n-1/\phi)}{n^2+n-1} = \frac{\pi^2}{2\sqrt{5}} + \frac{\pi^2\tan^2(\sqrt{5}\pi/2)}{\sqrt{5}} + \frac{4}{5}\pi\tan(\sqrt{5}\pi/2).$$

Solution by Rituraj Nandan, SunEdison, St. Peters, MO. The recurrence and reflection formulas for $\Gamma(x)$ lead to recurrence and reflection formulas for $\psi(x)$:

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{1}$$

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x). \tag{2}$$

Differentiate to get a reflection formula for the trigamma function $\psi^{(1)}(x) = \frac{d}{dx}\psi(x)$:

$$\psi^{(1)}(x) + \psi^{(1)}(1-x) = -\frac{\pi^2}{\sin^2(\pi x)}.$$

From another representation of the digamma function,

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right),$$

we get

$$\psi(\phi) - \psi(-1/\phi) = -\sum_{n=0}^{\infty}\left(\frac{1}{n+\phi} - \frac{1}{n-1/\phi}\right).$$

Differentiate to get

$$\psi^{(1)}(z) = \sum_{n=0}^{\infty}\frac{1}{(n+z)^2},$$

from which it follows that

$$\psi^{(1)}(\phi) + \psi^{(1)}(-1/\phi) = \sum_{n=0}^{\infty}\left(\frac{1}{(n+\phi)^2} + \frac{1}{(n-1/\phi)^2}\right).$$

Now we need to evaluate

$$S_1 = \sum_{n=1}^{\infty}\frac{1}{n^2+n-1},$$

$$S_2 = \sum_{n=1}^{\infty}\frac{1}{n^2+n-1}\sum_{k=0}^{n-1}\frac{1}{k^2+k-1}, \quad \text{and}$$

$$S_3 = \sum_{n=1}^{\infty}\frac{1}{(n^2+n-1)^2}.$$

First,

$$\begin{aligned}
S_1 &= \sum_{n=1}^{\infty}\frac{1}{n^2+n-1} = \frac{1}{1-2\phi}\sum_{n=0}^{\infty}\left(\frac{1}{n+\phi} - \frac{1}{n-1/\phi}\right) + 1 \\
&= -\frac{1}{\sqrt{5}}\big(\psi(\phi) - \psi(-1/\phi)\big) + 1 = -\frac{1}{\sqrt{5}}\big(\psi(\phi) - \psi(1-\phi)\big) + 1 \\
&= \frac{\pi}{\sqrt{5}}\cot(\pi\phi) + 1 = \frac{\pi}{\sqrt{5}}\tan\left(\frac{\pi}{2}\sqrt{5}\right) + 1.
\end{aligned}$$

Next,

$$S_3=\sum_{n=1}^{\infty}\frac{1}{(n^2+n-1)^2}=\frac{1}{(1-2\phi)^2}\sum_{n=0}^{\infty}\left(\frac{1}{n+\phi}-\frac{1}{n-1/\phi}\right)^2-1$$

$$=\frac{1}{5}\left(\psi^{(1)}(\phi)+\psi^{(1)}(-1/\phi)\right)-\frac{2}{5}\sum_{n=1}^{\infty}\frac{1}{n^2+n-1}-1$$

$$=\frac{1}{5}\left(-\frac{\pi^2}{\sin^2(\pi\phi)}\right)-\frac{2}{5}\left(\frac{\pi}{\sqrt{5}}\tan\left(\frac{\pi}{2}\sqrt{5}\right)+1\right)-1$$

$$=\frac{\pi^2}{5}+\frac{\pi^2}{5}\tan^2\left(\frac{\pi}{2}\sqrt{5}\right)-\frac{2}{5}\left(\frac{\pi}{\sqrt{5}}\tan\left(\frac{\pi}{2}\sqrt{5}\right)+1\right)-1.$$

Finally,

$$S_2=\frac{1}{2}\left(S_1^2-S_3\right)-S_1=\frac{\pi}{5\sqrt{5}}\tan\left(\frac{\pi}{2}\sqrt{5}\right)-\frac{\pi^2}{10}.$$

Now by (1) we have

$$\psi(n+\phi)=\psi(\phi)+\sum_{k=0}^{n-1}\frac{1}{k+\phi}$$

and similarly for $\psi(n-1/\phi)$. Thus

$$\psi(n+\phi)-\psi(n-1/\phi)=\psi(\phi)-\psi(-1/\phi)+\sum_{k=0}^{n-1}\left(\frac{1}{k+\phi}-\frac{1}{k-1/\phi}\right)$$

$$=\psi(\phi)-\psi(-1/\phi)+(1-2\phi)\sum_{k=0}^{n-1}\frac{1}{k^2+k-1}.$$

Applying (2) we get

$$\psi(n+\phi)-\psi(n-1/\phi)=-\pi\cot(\pi\phi)-\sqrt{5}\sum_{k=0}^{n-1}\frac{1}{k^2+k-1}.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\psi(n+\phi)-\psi(n-1/\phi)}{n^2+n-1}=\pi\tan\left(\frac{\pi}{2}\sqrt{5}\right)S_1-\sqrt{5}\,S_2$$

$$=\frac{\pi^2}{2\sqrt{5}}+\frac{\pi^2}{\sqrt{5}}\tan^2\left(\frac{\pi}{2}\sqrt{5}\right)+\frac{4\pi}{5}\tan\left(\frac{\pi}{2}\sqrt{5}\right).$$

Editorial comment. Some solvers used other methods for evaluation of the series S_1 and S_2, such as table lookup or residue calculus. The proposer did it by equating coefficients in the interesting infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2+n-1}\right)=\frac{1}{\cos((\pi/2)\sqrt{5})}\,\frac{\cos((\pi/2)\sqrt{5-4x^2})}{1-x^2}.$$

Also solved by R. Boukharfane (France), R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), M. L. Glasser, K. D. Lathrop, O. P. Lossers (Netherlands), M. Omarjee (France), R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), and the proposer.

Hidden Jensen

11843 [2015, 501]. *Proposed by Mihali Bencze, Bucharest, Romania.* Let n and k be positive integers, and let $x_j \geq 1$ for $1 \leq j \leq n$. Let $y = \prod_{j=1}^{n} x_j$. Show that

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \geq \sum_{j=1}^{n} \frac{1}{1+(x_j^{k-1}y)^{1/(n+k-1)}}.$$

Solution by James Christopher Smith, Knoxville, TN. The function $f(t) = 1/(1+e^t)$ is convex on $0 \leq t < \infty$, because $f''(t) = e^t(e^t-1)/(1+e^t)^3 > 0$ on $0 < t < \infty$. Therefore we may apply Jensen's inequality to see that for any numbers $z_1, \ldots, z_M$, all at least 1, we have

$$\sum_{i=1}^{M} \frac{1}{1+z_i} \geq \frac{M}{1+\sqrt[M]{z_1 z_2 \cdots z_M}}.$$

In particular, letting $M = n+k-1$ and taking $z_1, \ldots, z_M$ to consist of k copies of x_j and one copy of each of the other x_i, we get

$$\frac{k-1}{1+x_j} + \sum_{i=1}^{n} \frac{1}{1+x_i} \geq \frac{n+k-1}{1+(x_j^{k-1}y)^{1/(n+k-1)}}.$$

Summing this inequality over j, we get

$$(n+k-1)\sum_{i=1}^{n} \frac{1}{1+x_i} \geq (n+k-1)\sum_{j=1}^{n} \frac{1}{1+(x_j^{k-1}y)^{1/(n+k-1)}}.$$

Also solved by A. Ali (India), A. Alt, T. Amdeberhan, O. Bucicovschi, R. Chapman (U. K.), P. P. Dályay (Hungary), M. Dincă (Romania), O. Geupel (Germany), N. Grivaux (France), W.-K. Lai, O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), T. Viteam (Japan), and the proposer.

A Partition Suggested by a Regular Polygon

11845 [2015, 604]. *Proposed by Gregory Galperin, Eastern Illinois University, and Yury Ionin, Central Michigan University.*

(**a**) Let P be a convex polyhedron inside a sphere S, and let $e_1, \ldots, e_n$ be the edges of P. Let c_i be the chord of S containing edge e_i. Note that $c_i \backslash e_i$ is the union of two disjoint segments; we denote these by a_i and b_i. Prove that if all the edges of P have the same length, then the $2n$-element set consisting of the a_i and the b_i can be partitioned into two subsets such that the sum of the lengths of the elements in each part is the same.

(**b**) Let $A_0, A_1, \ldots, A_{n-1}$ be a regular n-gon inscribed in a circle γ. Let γ' be a circle containing γ, and let the tangent line to γ at A_i meet γ' at points X_i and Y_i. Prove that the $2n$-element set consisting of the segments A_iX_i and A_iY_i can be partitioned into two subsets such that the sum of the lengths of the elements in each part is the same.

Solution to part (**b**) *by Robin Chapman, University of Exeter, Exeter, UK.* We may assume that γ is the unit circle in the complex plane, and that $A_k = e^{2\pi ik/n}$.

Let the lengths of A_kX_k and A_kY_k be denoted by a_k and b_k; we distinguish between a_k and b_k by letting a_k be the length of the segment in the anticlockwise direction from A_k and b_k be the length of the segment in the clockwise direction from A_k. We claim that

$$\sum_{k=0}^{n-1} a_k = \sum_{k=0}^{n-1} b_k,$$

which displays the required partition. We need a formula for $a_k - b_k$.

Let the circle γ' have radius r and center $c + di$. The tangent to γ at A_0 is a vertical line that meets γ' at $1 + iy_+$ and $1 + iy_-$, where

$$y_\pm = d \pm \sqrt{r^2 - (1-c)^2}.$$

Now γ' surrounds γ, so $y_- < 0 < y_+$, and thus $a_0 - b_0 = y_+ + y_- = 2d$.

To find $a_k - b_k$ in general, let the configuration be rotated through $2\pi k/n$ radians clockwise about 0. Then circle γ remains the same, A_k is mapped to $z = 1$, and γ' is mapped to the circle with radius r and center $(c + di)e^{-2\pi ik/n}$. So $a_k - b_k = 2d_k$ with $d_k = \operatorname{Im}\big((c + di)e^{-2\pi ik/n}\big)$. Thus

$$\sum_{k=0}^{n-1}(a_k - b_k) = 2\,\operatorname{Im}\left((c + di)\sum_{k=0}^{n-1} e^{-2\pi ik/n}\right) = 2\,\operatorname{Im}(0) = 0.$$

Editorial comment. The claim in part (**a**) is false. The simplest and most commonly given counterexample is a small regular tetrahedron positioned near S so that 9 of the 12 disjoint segments are relatively short compared to the other 3, each of whose lengths is roughly 1/3 of the total sum of the lengths. No partition of the type desired can exist in this situation.

Part (**b**) also solved by E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), L. R. King, O. Kouba (Syria), O. P. Lossers (Netherlands), M. D. Meyerson, and the proposers.

Galois Groups of Polynomials with Missing Terms

11846 [2015, 604]. *Proposed by Kent Holing, Trondheim, Norway.* Let $f = \sum_{j=0}^{n} a_j x^j$ be an irreducible monic polynomial of odd degree with integer coefficients. Writing the terms $a_j x^j$ of f in order of increasing j, assume that either there is at least one term of f missing between two (nonmissing) terms of the same sign, or there is more than one term missing between two (nonmissing) terms of opposite sign. Prove that the Galois group G of f is not abelian. Also, prove that G is not a dihedral group if, in addition, $f(x) = 0$ has at least two real roots.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. By Descartes's rule of signs (applied to both $f(x)$ and $f(-x)$), the given condition implies that f has at most $n - 2$ real roots. Thus, f has at least one pair of complex roots, and hence complex conjugation is an element of order 2 in the Galois group G of f. Together with the lemma below, this will yield the desired conclusions.

Lemma. If an abelian group G acts transitively and faithfully on a set S, then each element of S has trivial stabilizer, and if S is finite then $|G| = |S|$.

Proof. If a nontrivial element $g \in G$ stabilizes s, then since the action is faithful and g is nontrivial there is an element $t \in S$ such that $g \cdot t \neq t$. Since the action is transitive, there is an element $h \in G$ such that $h \cdot s = t$. Now

$$hg \cdot s = h \cdot s = t, \text{ but } gh \cdot s = g \cdot t \neq t.$$

Hence $hg \neq gh$, contradicting the assumption that G is abelian. ■

We return now to the problem. If G is abelian, then by the lemma $|G|$ equals the number of roots of f. Since f has odd degree, we conclude that $|G|$ is odd, and hence G cannot contain an element of order 2.

Similarly, if G is dihedral with a cyclic subgroup H of index 2, then the lemma implies that no element of H can fix a root of f. Hence, $|H|$ is the number of roots of f, and the stabilizer of any root has order 2. In particular, $|H|$ is odd. Hence all elements of H are squares, and thus all elements of order 2 in G are conjugate. Moreover, each of these elements fixes the same number of roots. Since each root is fixed by only one element of order 2, it follows that each element of order 2 fixes exactly one root. In particular, complex conjugation fixes only one root, and we conclude that f has exactly one real root.

Also solved by A. J. Bevelacqua, R. Chapman (U. K.), and the proposer.

Bounds for a Sum

11847 [2015, 604]. *Proposed by Mihaly Bencze, Brasov, Romania.* Prove that for $n \geq 1$,

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\log^2(1+1/k)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}.$$

Solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, Logroño, Spain. From the inequality $\sqrt{t+1} < (t+2)/2$ for $t > 0$, we have

$$\frac{4}{(t+2)^2} < \frac{1}{1+t} < \frac{t+2}{2(t+1)^{3/2}}.$$

Integrating this from 0 to x yields

$$\frac{2x}{x+2} < \log(1+x) < \frac{x}{\sqrt{x+1}}.$$

It follows that

$$\frac{1+x}{x^2} < \frac{1}{\log^2(1+x)} < \frac{(x+2)^2}{4x^2}.$$

Taking $x = 1/k$ yields

$$k(k+1) < \frac{1}{\log^2(1+1/k)} < \frac{(2k+1)^2}{4} = \frac{1}{4} + k(k+1).$$

Using

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3},$$

we get the desired inequality.

Also solved by U. Abel (Germany), R. A. Agnew, A. Alt, T. Amdeberhan, K. F. Andersen (Canada), R. Bagby, E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), P. Bracken, M. A. Carlton, M. V. Channakeshava (India), R. Chapman (U. K.), H. Chen, N. Crwen (U. K.), P. P. Dályay (Hungary), B. E. Davis, R. Dutta (India), O. Geupel (Germany), M. Goldenberg & M. Kaplan, N. Grivaux (France), T. Horine, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), M. Lacruz (Spain), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), P. Perfetti (Italy), V. Rajasekaran & J. Grzesik, M. Sawhney, E. Schmeichel, N. C. Singer, A. Stenger, R. Stong, H. Takeda (Japan), R. Tauraso (Italy), D. B. Tyler, J. Van Casteren (Belgium), R. van der Veer (Netherlands), Z. Vörös (Hungary), M. Vowe (Switzerland), H. Wang & J. Wojdylo, L. Wimmer, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, and the proposer.

The Dilogarithm of an Exponential

11848 [2015, 605]. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Prove that

$$\frac{1}{2\pi}\mathrm{Li}_2\left(e^{-2\pi}\right) = \log(2\pi) - 1 - \frac{5\pi}{12} - \sum_{m=1}^{\infty} \frac{(-1)^m \zeta(2m)}{m(2m+1)}.$$

Here ζ is the Riemann zeta function, and $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$.

Solution by Ramya Dutta, Chennai Mathematical Institute, India. We write the series as

$$\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{k(2k+1)}=2\left(\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{2k}-\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{2k+1}\right)$$

and consider the two parts separately. For the first series, we compute

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{2k}&=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^k n^{-2k}}{2k}=-\frac{1}{2}\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^2}\right)\\&=-\frac{1}{2}\log\left(\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\right)=-\frac{1}{2}\log\left(\frac{\sinh\pi}{\pi}\right)\\&=\frac{1}{2}\log(2\pi)-\frac{\pi}{2}+\frac{1}{2}\log\left(\frac{1}{1-e^{-2\pi}}\right).\end{aligned}$$

Here we used the series $\log(1+x)=\sum_{k=1}^{\infty}(-1)^{k-1}x^k/k$ and the product expansion

$$\frac{\sinh(\pi x)}{\pi x}=\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right). \tag{$*$}$$

For the second series, we compute

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{2k+1}&=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^k n^{-2k}}{2k+1}=\int_0^1\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^k n^{-2k}x^{2k}\,dx\\&=\int_0^1\sum_{n=1}^{\infty}\left(-1+\frac{1}{1+\frac{x^2}{n^2}}\right)dx=\int_0^1\sum_{n=1}^{\infty}\frac{-x^2}{x^2+n^2}\,dx\\&=\frac{1}{2}\int_0^1(1-\pi x\coth(\pi x))\,dx.\end{aligned}$$

Here we used the partial fraction expansion $\pi\coth(\pi x)=1/x+2x\sum_{n=1}^{\infty}1/(n^2+x^2)$, obtained by differentiating the logarithm of $(*)$. Finally,

$$\begin{aligned}\int_0^1(1-\pi x\coth(\pi x))\,dx&=\int_0^1\left(1-\pi x\left(1+\frac{2e^{-2\pi x}}{1-e^{-2\pi x}}\right)\right)dx\\&=1-\frac{\pi}{2}-\int_0^1 2\pi x\sum_{n=1}^{\infty}e^{-2\pi n x}\,dx\\&=1-\frac{\pi}{2}-2\pi\sum_{n=1}^{\infty}\int_0^1 x e^{-2\pi n x}\,dx\\&=1-\frac{\pi}{2}-2\pi\sum_{n=1}^{\infty}\left(\frac{1}{4\pi^2 n^2}-\frac{e^{-2\pi n}}{2\pi n}-\frac{e^{-2\pi n}}{4\pi^2 n^2}\right)\\&=1-\frac{\pi}{2}-\frac{\pi}{12}+\log\left(\frac{1}{1-e^{-2\pi}}\right)+\frac{1}{2\pi}\mathrm{Li}_2(e^{-2\pi}).\end{aligned}$$

Combine the results to obtain the desired equation.

Also solved by R. Bagby, K. N. Boyadzhiev, P. Bracken, B. Bradie, R. Chapman (U. K.), H. Chen, B. E. Davis, O. Geupel (Germany), M. L. Glasser, M. Goldenberg & M. Kaplan, M. Hoffman, O. Kouba (Syria), M. Omarjee (France), S. Pathak (Canada), R. Stong, R. Tauraso (Italy), J. Van Casteren (Belgium), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Asymptotics for Coefficients

11849 [2015, 605]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Define numbers $a_0, a_1, \ldots$ by

$$\exp\left(\sum_{k=0}^{\infty} x^{2^k}\right) = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that $\liminf_{n\to\infty} \frac{\log a_n}{\log n} \le \frac{1}{\log 2} - 1 \le \limsup_{n\to\infty} \frac{\log a_n}{\log n}$.

Solution by Allen Stenger, Boulder, CO. Write $g(x)$ for the series $\sum_{n=0}^{\infty} a_n x^n$. We use the following result.

Proposition: Suppose $v_n \ge 0$ for all n and $\sum v_n = +\infty$. Suppose also that $\sum v_n x^n$ converges for $|x| < 1$. If $\lim_{n\to\infty} \sum_{k=0}^{n} u_k / \sum_{k=0}^{n} v_k = s$, then

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty} u_k x^k \Big/ \sum_{k=0}^{\infty} v_k x^k = s. \qquad \blacksquare$$

We also use the following application of the proposition: For $\alpha > 0$,

$$\lim_{x\to 1^-} (1-x)^{\alpha} \sum_{n=1}^{\infty} n^{\alpha-1} x^n = \Gamma(\alpha). \tag{1}$$

(See G. Pólya, G. Szegő, *Problems and Theorems in Analysis I*, Springer, Berlin, 1972, Problems I.88 and I.89.)

First we estimate $g(x)$ asymptotically as $x \to 1^-$. Let $u_n = 1$ if n is a power of 2 and $u_n = 0$ otherwise; and let $v_0 = 0$ and $v_n = 1/n$ for $n \ge 1$. Thus $\sum_{k=0}^{n} u_n = 1 + \lfloor \log_2 n \rfloor$ and $\sum_{k=0}^{n} v_n = \log n + O(1)$ as $n \to \infty$. Let $s = 1/\log 2$ in the proposition, which yields $\log g(x)/(-\log(1-x)) \to 1/\log 2$ or, equivalently,

$$\log g(x) \sim \frac{1}{\log 2} \log \frac{1}{1-x} \quad \text{as} \quad x \to 1^-. \tag{2}$$

Next we show $\liminf_{n\to\infty} \log a_n / \log n \le 1/\log 2 - 1$. Assume otherwise. There is $L > 1/\log 2 - 1$ such that $\log a_n / \log n > L$ for all sufficiently large n; that is, $a_n > n^L$. By (1) we have

$$\log g(x) > \log\left(\sum_{n=0}^{\infty} n^L x^n + O(1)\right) \sim \log \frac{\Gamma(L+1)}{(1-x)^{L+1}} \sim (L+1) \log \frac{1}{1-x}$$

as $x \to 1^-$. Since $L + 1 > 1/\log 2$, this contradicts (2).

The reasoning for the second inequality is similar. Assume, in order to obtain a contradiction, that $\limsup_{n\to\infty} \log a_n / \log n < 1/\log 2 - 1$. Hence there is a number U, less than $1/\log 2 - 1$, such that $\log a_n / \log n < U$ for all sufficiently large n; that is, $a_n < n^U$. By (1) we have

$$\log g(x) < \log\left(\sum_{n=0}^{\infty} n^U x^n + O(1)\right) \sim \log \frac{\Gamma(U+1)}{(1-x)^{U+1}} \sim (U+1) \log \frac{1}{1-x}$$

as $n \to 1^-$. Since $U + 1 < 1/\log 2$, this contradicts (2).

Also solved by R. Chapman (U. K.), M. Omarjee (France), R. Stong, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, and Fuzhen Zhang.

Proposed problems should be submitted online at http://www.americanmathematicalmonthly.submittable.com/submit. *Proposed solutions to the problems below should be submitted by September 30, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11978. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fibonacci number, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ when $n \geq 2$. Find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\cosh F_n \ \cosh F_{n+3}}.$$

11979. *Proposed by Zachary Franco, Houston, Texas.* Let O and I denote the circumcenter and incenter of a triangle ABC. Are there infinitely many nonsimilar scalene triangles ABC for which the lengths AB, BC, CA, and OI are all integers?

11980. *Proposed by George Stoica, Saint John, NB, Canada.* Let $a_1, \ldots, a_n$ be a nonincreasing list of positive real numbers, and fix an integer k with $1 \leq k \leq n$. Prove that there exists a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$\min_{1 \leq j \leq k} \sum_{i \in B_j} a_i \geq \frac{1}{2} \min_{1 \leq j \leq k} \frac{1}{k+1-j} \sum_{i=j}^{n} a_i.$$

11981. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Suppose that $f : [0, 1] \to \mathbb{R}$ is a differentiable function with continuous derivative and with

$$\int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx = 1.$$

Prove

$$\int_0^1 \left| f'(x) \right|^3 dx \geq \left(\frac{128}{3\pi} \right)^2.$$

11982. *Proposed by Ovidiu Furdui, Mircea Ivan, and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\lim_{x \to \infty} \left(\sum_{n=1}^{\infty} \left(\frac{x}{n} \right)^n \right)^{1/x}.$$

http://dx.doi.org/10.4169/amer.math.monthly.124.5.465

11983. *Proposed by Askar Dzhumadil'daev, Kazakh-British Technical University, Almaty, Kazakhstan.* Given a positive integer n, let $x_1, \dots, x_{n-1}$ and $y_1, \dots, y_n$ be indeterminates. Let A be the $2n$-by-$2n$ matrix that is antisymmetric with respect to both main diagonals and whose i, j-entry is $\sinh(x_i + y_j)$ when $i < j \le n$ and $\cosh(x_i + y_j)$ when $i < n < j \le 2n - i$. For example, when $n = 3$, the matrix A is

$$\begin{bmatrix} 0 & \mathrm{s}(x_1+y_2) & \mathrm{s}(x_1+y_3) & \mathrm{c}(x_1+y_3) & \mathrm{c}(x_1+y_2) & 0 \\ -\mathrm{s}(x_1+y_2) & 0 & \mathrm{s}(x_2+y_3) & \mathrm{c}(x_2+y_3) & 0 & -\mathrm{c}(x_1+y_2) \\ -\mathrm{s}(x_1+y_3) & -\mathrm{s}(x_2+y_3) & 0 & 0 & -\mathrm{c}(x_2+y_3) & -\mathrm{c}(x_1+y_3) \\ -\mathrm{c}(x_1+y_3) & -\mathrm{c}(x_2+y_3) & 0 & 0 & -\mathrm{s}(x_2+y_3) & -\mathrm{s}(x_1+y_3) \\ -\mathrm{c}(x_1+y_2) & 0 & \mathrm{c}(x_2+y_3) & \mathrm{s}(x_2+y_3) & 0 & -\mathrm{s}(x_1+y_2) \\ 0 & \mathrm{c}(x_1+y_2) & \mathrm{c}(x_1+y_3) & \mathrm{s}(x_1+y_3) & \mathrm{s}(x_1+y_2) & 0 \end{bmatrix},$$

where we have written $\mathrm{s}(z)$ for $\sinh(z)$ and $\mathrm{c}(z)$ for $\cosh(z)$. Prove $\det(A) = 0$ when n is odd and $\det(A) = 1$ when n is even.

11984. *Proposed by Daniel Sitaru, Drobeta-Turnu Severin, Romania.* Let a, b, and c be the lengths of the sides of a triangle with inradius r. Prove $a^6 + b^6 + c^6 \ge 5184r^6$.

SOLUTIONS

Orthogonal Functions

11850 [2015, 605]. *Proposed by Zafar Ahmed, Bhabha Atomic Research Centre, Mumbai, India.* Let

$$A_n(x) = \sqrt{\frac{2}{\pi}} \frac{1}{n!} (1+x^2)^{n/2} \frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Prove that $\int_{-\infty}^{\infty} A_m(x) A_n(x)\, dx = \delta(m,n)$ for nonnegative integers m and n. Here $\delta(m,n) = 1$ if $m = n$, and otherwise $\delta(m,n) = 0$.

Solution by Ramya Dutta, Chennai Mathematical Institute, Chennai, India. We have

$$\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right) = \frac{1}{2i}\frac{d^n}{dx^n}\left(\frac{1}{x-i} - \frac{1}{x+i}\right) = (-1)^n \frac{n!}{2i}\left(\frac{1}{(x-i)^{n+1}} - \frac{1}{(x+i)^{n+1}}\right).$$

Now let $\theta = \cot^{-1} x$, so $\cot\theta = x$ and $0 < \theta < \pi$. We have

$$x - i = \cot\theta - i = \frac{2ie^{-i\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{e^{-i\theta}}{\sin\theta}.$$

Similarly, $x + i = e^{i\theta}/\sin\theta$. Since $1 + x^2 = 1/\sin^2\theta$ and $\sin\theta > 0$,

$$\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right) = (-1)^n \frac{n!\,\sin^{n+1}\theta}{2i}\left(e^{i(n+1)\theta} - e^{-i(n+1)\theta}\right) = (-1)^n \frac{n!\,\sin((n+1)\theta)}{(1+x^2)^{(n+1)/2}}.$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} A_m(x)A_n(x)\,dx &= (-1)^{m+n}\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin((m+1)\theta)\,\sin((n+1)\theta)}{1+x^2}\,dx \\ &= (-1)^{m+n}\frac{2}{\pi}\int_0^{\pi}\sin((m+1)\theta)\,\sin((n+1)\theta)\,d\theta.\end{aligned}$$

The last integral is easily evaluated to yield $\delta(m,n)$.

Editorial comment. The problem as originally printed asserted that $A_n(x)$ is a polynomial. The editors are responsible for this error.

Also solved by T. Amdeberhan & H. Kilete-Seleste, T. Amdeberhan & S. B. Ekhad, R. Bagby, D. Beckwith, G. E. Bilodeau, R. Boukharfane (France), P. Bracken, H. Chen, P. P. Dályay (Hungary), P. J. Fitzsimmons, N. Grivaux (France), F. Holland (Ireland), O. Kouba (Syria), G. Kuldeep (India), O. P. Lossers (Netherlands), R. Stong, R. Tauraso (Italy), J. Van Hamme (Belgium), M. Vowe (Switzerland), H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

An Enumeration of the Positive Rationals

11852 [2015, 700]. *Proposed by Sam Northshield, SUNY Plattsburgh, Plattsburgh, NY.* For $n \in \mathbb{Z}^+$, let $\nu_n = k$ if 3^k divides n but 3^{k+1} does not. Let $X_1 = 2$, and for $n \geq 2$ let

$$X_n = 4\nu_n + 2 - \frac{2}{X_{n-1}},$$

so that $\langle X_n \rangle$ begins with $2, 1, 4, 3/2, 2/3, 3, \ldots$. Show that every positive rational number appears exactly once in the list $(X_1, X_2, \ldots)$.

Solution by László Lipták, Oakland University, Rochester, MI. Define linear fractional transformations S, P, Q, and R by

$$S(x) = x + 2, \quad P(x) = \frac{2x+2}{x+2}, \quad Q(x) = \frac{x}{x+1}, \quad \text{and} \quad R(x) = \frac{1}{x}.$$

The recurrence becomes

$$X_n = 4\nu_n + 2 - 2R(X_{n-1}),$$

and the following identities hold:

$$2 - 2R(S(x)) = P(x), \qquad 2 - 2R(P(x)) = Q(x), \qquad 2 - 2R(Q(x)) = -2R(x),$$

$$P^{-1}(w) = \frac{-2w+2}{w-2}, \qquad Q^{-1}(w) = \frac{-w}{w-1}, \qquad S^{-1}(w) = w - 2.$$

We first prove for $n \geq 1$ the three equalities

$$X_{3n} = S(X_n), \quad X_{3n+1} = P(X_n), \quad X_{3n+2} = Q(X_n). \tag{1}$$

This is clear for $n = 1$, and we proceed by induction. Consider $n > 1$. Since $\nu_{3n} = 1 + \nu_n$, we have

$$\begin{aligned} X_{3n} &= 4 + 4\nu_n + 2 - 2R(X_{3n-1}) \\ &= 4\nu_n + 2 + (2 - 2R(Q(X_{n-1})) + 2 \\ &= 4\nu_n + 2 - 2R(X_{n-1}) + 2 = X_n + 2 = S(X_n). \end{aligned}$$

Using this and $\nu_{3n+1} = 0$, we obtain

$$X_{3n+1} = 2 - 2R(X_{3n}) = 2 - 2R(S(X_n)) = P(X_n).$$

Using this and $\nu_{3n+2} = 0$, we obtain

$$X_{3n+2} = 2 - 2R(X_{3n+1}) = 2 - 2R(P(X_n)) = Q(X_n).$$

It is now immediate that every X_i is positive and that, for $n > 1$, we have

$$0 < X_{3n+2} < 1 < X_{3n+1} < 2 < X_{3n}. \tag{2}$$

For relatively prime positive integers a and b, define $\sigma(a/b) = a + b$. When a/b belongs to $(0, 1)$, $(1, 2)$, or $(2, \infty)$, let T denote Q, P, or S, respectively. We claim that always $\sigma(T^{-1}(a/b)) < \sigma(a/b)$, via the following computations.

- If $a/b \in (0, 1)$, then $a < b$ and $\sigma(Q^{-1}(a/b)) = \sigma(a/(b-a)) = b < \sigma(a/b)$.
- If $a/b \in (1, 2)$, then $b < a < 2b$ and $\sigma(P^{-1}(a/b)) = \sigma\left(\frac{2a-2b}{2b-a}\right) = a < \sigma(a/b)$.
- If $a/b \in (2, \infty)$, then $2b < a$ and $\sigma(S^{-1}(a/b)) = \sigma\left(\frac{a-2b}{b}\right) = a - b < \sigma(a/b)$.

Note that in each case $T^{-1}(a/b)$ is a ratio of relatively prime positive integers.

To prove the result, note first that by (2) the initial terms 2 and 1 cannot appear later in the sequence. Write any other positive rational in reduced form a/b. As T^{-1} is successively applied, the value of σ declines, which cannot continue forever. The process terminates only by reaching a value not in one of the intervals, namely $c \in \{1, 2\}$. That is, there is a list $T_1, \ldots, T_s$ with each $T_i \in \{Q, P, S\}$ such that

$$T_s^{-1} \cdots T_1^{-1}(a/b) = c \in \{1, 2\}. \tag{3}$$

Thus $a/b = T_1 \cdots T_s(c)$, and from (1) we have $a/b = X_m$ for some m.

It remains to prove uniqueness. With $a/b = X_m$, by (2) and (1) the transformation T_1 is S, P, or Q depending on whether the congruence class of m modulo 3 is 0, 1, or 2, respectively. Similarly, T_2 is determined by $\lfloor m/3 \rfloor \bmod 3$, and T_i is determined by $\lfloor m/3^{i-1} \rfloor \bmod 3$. Since the base 3 representation of a positive integer is unique, m is uniquely determined by a/b, and each positive rational occurs exactly once.

Editorial comment. Let $R(x) = \beta/x$, and let Q, S, and P denote linear fractional transformations such that the formulas given above for $2 - 2R(T(x))$ hold when $T \in \{Q, S, P\}$. A straightforward calculation shows that Q, S, and P are then given by

$$Q(x) = \frac{\beta x}{x + \beta}, \quad S(x) = \frac{\beta(2-\beta)x + 2\beta^2}{2(1-\beta)x + \beta(2-\beta)}, \quad P(x) = \frac{2\beta(x+\beta)}{(2-\beta)x + 2\beta}.$$

The determinants of S, P, Q, and R are β^4, $2\beta^3$, β^2, and $-\beta$, respectively. For $\beta = 1$ these reduce to the transformations used above. It might be of interest to determine for which β these transformations generate a free semigroup.

Lipták noted that the simpler sequence $\{1, 2, 1/2, 3/1, 2/3, 3/2, \ldots\}$ generated by $x_n = 2\nu_n + 1 - 1/x_{n-1}$ (where ν_n is "2-adic" rather than "3-adic") also has the full enumeration property, and that a proof can be based on the equalities $x_{2n} = x_n + 1$ and $x_{2n+1} = x_n/(x_n + 1)$.

Michael Josephy, O. P. Lossers, and Lipták indicated connections and similarities with the Calkin–Wilf tree (THIS MONTHLY **107** (2000) 360–363). A striking feature of the present problem is that the three equalities of (1) are compressed into the single recurrence of the problem. This compression theme seems to have been initiated by Moshe Newman (see the solution to Problem 10906, THIS MONTHLY **110** (2003) 642–643).

Enumerating the positive rationals in a nicely structured way has as its ultimate source the 19th-century paper of Abraham Stern, Über eine zahlentheoretische Funktion, *J. reine angew. Math.* **55** (1858) 193–220. Much of the current study of this topic can be viewed as an elaboration of this work. In fact, the simpler sequence given by Lipták does indeed yield the original Stern enumeration. The novelty in the present case is having a ternary rather than a binary tree.

It is also well known that the numerator and denominator sequences used by Stern (now known as Stern sequences) have many relations to the Fibonacci numbers. The paper T. Garrity, A multidimensional continued fraction generalization of Stern's diatomic sequence, *J. Integer Sequences* **16** (2013) 1–23 has some similarity in spirit with the present problem in that it introduces analogues of the Stern sequences related to the Tribonacci numbers.

Also solved by R. Chapman (U. K.), J. Gately, T. Horine, M. Josephy (Costa Rica), P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Tauraso (Italy), FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Hyperbolic Sine Series

11853 [2015, 700]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Find

$$\sum_{n=1}^{\infty} \frac{1}{\sinh 2^n}.$$

Solution I by Tewodros Amdeberhan, Tulane University, and Armin Straub, University of South Alabama. Dividing $\sinh x = \sinh(2x - x) = \sinh(2x)\cosh x - \cosh(2x)\sinh x$ through by $\sinh(2x)\sinh x$ yields $\frac{1}{\sinh 2x} = \coth x - \coth 2x$. A repeated application of this leads to a telescoping sum:

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n x)} = \sum_{n=1}^{\infty} \left[\coth(2^{n-1}x) - \coth(2^n x)\right] = \coth x - \lim_{n\to\infty} \coth(2^n x)$$
$$= \coth x - 1,$$

for all $x > 0$. The requested sum is the special case where $x = 1$.

Solution II by Rituraj Nandan, SunEdison, St. Peters, MO.

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n x)} = \sum_{n=1}^{\infty} \frac{2}{e^{2^n x} - e^{-2^n x}} = 2\sum_{n=1}^{\infty} \frac{e^{-2^n x}}{1 - e^{-2^{n+1}x}} = 2\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} e^{-2^n(2j+1)x}$$
$$= 2\sum_{k=1}^{\infty} e^{-2kx} = \frac{2}{e^{2x} - 1},$$

where in the penultimate step we have used the fact that every even, positive integer $2k$ can be written uniquely as $2k = 2^n(2j+1)$ for $n \geq 1$ and $j \geq 0$.

Editorial comment. This sum appears as 1.121.2 in Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products.*

Also solved by U. Abel (Germany), Z. Ahmed (India), A. Ali (India), K. Andersen (Canada), M. Arakelian (Armenia), H. I. Arshagi, M. Bataille (France), D. Beckwith, M. Bello & M. Benito & Ó. Ciaurri & E Fernández & L. Roncal (Spain), S. C. Bhoria (India), R. Boukharfane (France), P. Bracken, B. Bradie, N. Caro (Brazil), R. Chapman (U. K.), H. Chen, S. Choi (Korea), C. Curtis, N. Curwen (U. K.), P. P. Dályay (Hungary), B. E. Davis, R. Dutta (India), E. Errthum, D. Fleischman, J. Gaisser, O. Geupel (Germany), H. B. Ghaffari (Iran), M. L. Glasser, M. Goldenberg & M. Kaplan, N. Grivaux (France), J. A. Grzesik, M. Hoffman, F. Holland (Ireland), T. Horine, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), W. C. Lang, K.-W. Lau (China), L. Lipták, O. P. Lossers (Netherlands), J. Magliano, L. Matejíčka (Slovakia), V. Mikayelyan (Armenia), J. Mooney, M. Omarjee (France), S. Pathak (Canada), F. Perdomo & Á. Plaza (Spain), C. M. Russell, M. Sawhney, V. Schindler (Germany), N. C. Singer, J. Sorel (Romania), A. Stenger, R. Stong, H. Takeda (Japan), R. Tauraso (Italy), C. I. Vălean (Romania), G. Vidiani (France), J. Vinuesa (Spain), T. Viteam (Japan), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), M. Wildon (U. K.), J. Zacharias, L. Zhou, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, Northwestern University Math Problem Solving Group, PHP Solving Team, and the proposer.

Avoid the Parabolas

11854 [2015, 700] correction [2015, 802]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* In the Euclidean plane, given distinct points $P_1, \ldots, P_n$ and distinct lines $l_1, \ldots, l_m$, prove that there is a half-line h such that for any point Q on h, any $k \in \{1, \ldots, m\}$, and any $j \in \{1, \ldots, n\}$, Q is nearer to l_k than to P_j.

Solution by Irl C. Bivens and L. R. King, Davidson College, Davidson, NC. Let $S_{j,k}$ denote the closed interior of the parabola having focus P_j and directrix l_k. (If P_j is on line l_k, let $S_{j,k}$ denote the line through P_j perpendicular to l_k.) Any point not in $S_{j,k}$ is closer to l_k than to P_j. Thus it suffices to find a half-line h that avoids $S_{j,k}$ for all j and k.

Any line perpendicular to the directrix of a parabola intersects the parabola's closed interior in a ray; but any other line intersects the parabola's closed interior in a segment, a point, or not at all. Let g be any line not perpendicular to l_k for any k. The intersection of g with the union of all $S_{j,k}$ consists of finitely many points and finitely many segments, all of finite length. Thus when these points and segments are removed, there remain two half-lines of g, which have empty intersection with every $S_{j,k}$. Either may be chosen to be the required h.

Editorial comment. O. P. Lossers observed that the set of points P_j can be infinite, as long as it is bounded. Victor Pambuccian noted that the result here is false in hyperbolic geometry.

Also solved by R. Chapman (U. K.), R. Dutta (India), O. Geupel (Germany), T. Horine, Y. J. Ionin, J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, V. Pambuccian, J. Schlosberg, E. Schmeichel, R. Stong, L. Zhou, and the proposer.

A Momentous Inequality

11855 [2015, 700]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* For a continuous and nonnegative function f on [0, 1], let $\mu_n = \int_0^1 x^n f(x)\,dx$. Show that $\mu_{n+1}\mu_0 \geq \mu_n\mu_1$ for $n \in \mathbb{N}$.

Solution I by Ross Dempsey, student, Thomas Jefferson High School, Alexandria, VA. If $\mu_n = 0$ for some n, then f is identically zero. So we may assume $\mu_n > 0$ for all n.

For $n \geq 1$, consider the integral

$$\int_0^1 \left(x - \frac{\mu_n}{\mu_{n-1}}\right)^2 x^{n-1} f(x)\,dx.$$

The integrand is nonnegative, so

$$\begin{aligned} 0 &\leq \int_0^1 \left(x - \frac{\mu_n}{\mu_{n-1}}\right)^2 x^{n-1} f(x)\,dx \\ &= \int_0^1 x^{n+1} f(x)\,dx - 2\frac{\mu_n}{\mu_{n-1}}\int_0^1 x^n f(x)\,dx + \frac{\mu_n^2}{\mu_{n-1}^2}\int_0^1 x^{n-1} f(x)\,dx \\ &= \mu_{n+1} - \mu_n^2/\mu_{n-1}. \end{aligned}$$

It follows that $\mu_{n+1}/\mu_n \geq \mu_n/\mu_{n-1}$. By induction, $\mu_{n+1}/\mu_n \geq \mu_1/\mu_0$, which is equivalent to the required inequality.

Solution II by Oliver Geupel, Brühl, NRW, Germany. Let $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(x) = \int_0^x t^{n+1} f(t)\,dt \cdot \int_0^x f(t)\,dt - \int_0^x t^n f(t)\,dt \cdot \int_0^x t f(t)\,dt.$$

The function g is differentiable on [0, 1] with derivative

$$g'(x) = f(x)\int_0^x (x-t)(x^n - t^n) f(t)\,dt \geq 0.$$

Therefore, $g(x)$ is increasing on [0, 1], and since $g(0) = 0$, we have $g(x) \geq 0$. This implies that $g(1) \geq 0$, or $\mu_{n+1}\mu_0 - \mu_n\mu_1 \geq 0$.

Solution III by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany. We prove more generally that $\mu_m\mu_n \le \mu_{m+n}\mu_0$. The required inequality is the case $m = 1$. We have

$$\mu_m\mu_n = \frac{1}{2}(\mu_m\mu_n + \mu_n\mu_m) = \frac{1}{2}\int_0^1\int_0^1 (x^m y^n + x^n y^m) f(x) f(y)\, dx dy.$$

For $0 \le x, y \le 1$, we have $0 \le (x^m - y^m)(x^n - y^n) = (x^{m+n} + y^{m+n}) - (x^m y^n + x^n y^m)$, or $x^m y^n + x^n y^m \le x^{m+n} + y^{m+n}$, and this implies

$$\mu_m\mu_n \le \frac{1}{2}\int_0^1\int_0^1 (x^{m+n} + y^{m+n}) f(x) f(y)\, dx dy = \frac{1}{2}(\mu_{m+n}\mu_0 + \mu_0\mu_{m+n}).$$

Hence $\mu_m\mu_n \le \mu_{m+n}\mu_0$, as claimed.

Also solved by R. A. Agnew, A. Ali (India), T. Amdeberhan & A. Straub, K. F. Andersen (Canada), M. Andreoli, H. I. Arshagi, R. Bagby, M. Bataille (France), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), P. Bracken, M. A. Carlton, R. Chapman (U. K.), H. Chen, L. V. P. Cuong (Vietnam), C. Curtis, N. Curwen (U. K.), P. P. Dályay (Hungary), B. E. Davis, J. Duemmel, R. Dutta (India), D. L. Farnsworth, P. J. Fitzsimmons, D. Fleischman, L. Giugiuc (Romania), N. Grivaux (France), J. A. Grzesik, L. Han, E. A. Herman, F. Holland (Ireland), T. Horine, E. J. Ionaşcu, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), P. T. Krasopoulos (Greece), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), L. Matejíčka (Slovenia), V. Mikayelyan (Armenia), M. Omarjee (France), E. Omey (Belgium), D. Ritter, M. Sawhney, K. Schilling, N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), N. Thornber, R. van der Veer (Netherlands), E. I. Verriest, J. Vinuesa (Spain), J. Wakem, T. Wiandt, J. Zacharias, Z. Zhang (China), L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

The Number of Sylow Subgroups

11856 [2015, 700]. *Proposed by Keith Kearnes, University of Colorado, Boulder, CO.* Let G be a finite group. Show that the number of Sylow subgroups of G is at most $\frac{2}{3}|G|$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Let p be a prime, and let $s_p(G)$ denote the number of Sylow p-subgroups of G. It is well known that if P is a Sylow p-subgroup of G, then all Sylow p-subgroups of G are conjugates of P. It follows that $s_p(G)$ equals the index in G of the normalizer $N_G(P)$, which equals $|G|/|N_G(P)|$. In particular $P \subset N_G(P)$, so if p divides $|G|$ with multiplicity m, then $s_p(G) \le |G|/p^m$.

Now let A be the set of primes that divide $|G|$ with multiplicity 1. Note that A is the set of primes p such that every Sylow p-subgroup of G is cyclic of order p. Two such Sylow p-subgroups intersect only in the identity element, and thus G has $(p-1)s_p(G)$ elements of order p. Hence $\sum_{p\in A}(p-1)s_p(G) < |G|$.

If $p \in A$ and $s_p(G) = |G|/p$, then $N_G(P) = P$ for each Sylow p-subgroup P. Now $P \subset Z(N_G(P))$. By the Burnside transfer theorem, it follows that there exists a normal subgroup H of G with index p. Hence when q is a prime different from p, every Sylow q-subgroup is a Sylow subgroup of H. If $p \ge 3$, then by induction on $|G|$ for the number s of Sylow subgroups of G we compute

$$s \le \frac{|G|}{p} + \frac{2}{3}|H| = \frac{5}{3p}|G| < \frac{2}{3}|G|.$$

If $p = 2$, then there are $|G|/2$ elements of order 2, so every element of $G - H$ has order 2. Given such an element x, let h be an element of H. We have $(xh)^2 = 1$, so $xhx^{-1} = h^{-1}$. Thus, inversion is a group homomorphism from H to itself, and hence H is abelian. In this case the number of Sylow subgroups of H equals the number of distinct primes dividing

$|H|$ (note that $|H| = |G|/2$). Since $|H|$ is odd, there are at most $|H|/3$ such primes—with equality if and only if $|H| = 3$. Hence $s \le |G|/2 + |G|/6$.

We may therefore assume $s_p(G) < |G|/p$ for $p \in A$, and hence $s_p(G) \le |G|/(2p)$. Since $\frac{p-2}{p(p-1)} \le \frac{1}{6}$ for integer p, this implies $s_p(G) - \frac{|G|}{p^2} \le \frac{(p-1)}{6} s_p(G)$.

Summing over p, we compute

$$\begin{aligned}\sum_p s_p(G) &\le \sum_{p\in A} s_p(G) + \sum_{p\notin A} \frac{|G|}{p^2} = \sum_{p\in A}\left(s_p(G) - \frac{|G|}{p^2}\right) + \sum_p \frac{|G|}{p^2}\\ &\le \sum_{p\in A} \frac{p-1}{6} s_p(G) + \sum_p \frac{|G|}{p^2} < |G|\left(\frac{1}{6} + \sum_p \frac{1}{p^2}\right) < \frac{2}{3}|G|,\end{aligned}$$

where we have used the fact that $\sum_p \frac{1}{p}^2 \approx 0.452224742 < 1/2$.

The argument shows that equality occurs only when G is the group S_3 of order six.

Also solved by the proposer.

The Square Root of a Triangle

11857 [2015, 700]. *Proposed by Mehmet Şahin, Ankara University, Ankara, Turkey.* Let ABC be a triangle with corresponding sides of lengths a, b, and c, inradius r, and corresponding exradii r_a, r_b, and r_c. Let $A'B'C'$ be another triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. Show that $A'B'C'$ has area given by

$$\frac{1}{2}\sqrt{r(r_a + r_b + r_c)}.$$

Solution by Borislav Karaivanov, Sigma Space, Lanham, MD, and Tzvetalin S. Vassilev, Nipissing University, North Bay, ON, Canada. Write s for the semiperimeter of ABC. Using the formulas

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad r_a = \sqrt{\frac{s(s-b)(s-c)}{(s-a)}}$$

and similar formulas for r_b and r_c, we derive

$$rr_a = (s-b)(s-c), \quad rr_b = (s-c)(s-a), \quad rr_c = (s-a)(s-b). \tag{1}$$

Let Δ' denote the area of $A'B'C'$. Using Heron's formula, we find

$$\begin{aligned}16(\Delta')^2 &= \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(-\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\\ &= \left(\left(\sqrt{b}+\sqrt{c}\right)^2 - a\right)\left(a - \left(\sqrt{b}-\sqrt{c}\right)^2\right)\\ &= \left(2\sqrt{bc} + (b+c-a)\right)\left(2\sqrt{bc} - (b+c-a)\right)\\ &= 2ab + 2bc + 2ca - a^2 - b^2 - c^2\\ &= a^2 - (b-c)^2 + b^2 - (c-a)^2 + c^2 - (a-b)^2\\ &= 4\big((s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b)\big)\\ &= 4r(r_a + r_b + r_c),\end{aligned}$$

where we applied (1) in the final step.

Editorial comment. Sin Hitotumatu showed that for all triangles ABC, the triangle $A'B'C'$ is acute.

Also solved by Z. Ahmed (India), A. Ali (India), A. Alt, M. Bataille (France), B. S. Burdick, M. V. Channakeshava (India), R. Chapman (U. K.), C. Curtis, N. Curwen (U. K.), P. P. Dályay (Hungary), P. De (India), A. Fanchini (Italy), D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), J. A. Grzesik, J. G. Heuver (Canada), S. Hitotumatu (Japan), O. Hughes, Y. J. Ionin, L. R. King, O. Kouba (Syria), W.-K. Lai & J. Risher & W. D. Ethridge, K.-W. Lau (China), J. M. Lewis, J. H. Lindsey II, G. Lord, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), J. Minkus, D. J. Moore, R. Nandan, P. Nüesch (Switzerland), C. G. Petalas (Greece), M. Sawhney, V. Schindler (Germany), M. A. Shayib, I. Sofair, N. Stanciu & T. Zvonaru (Romania), R. Stong, W. Szpunar-Lojasiewicz, H. Takeda (Japan), R. Tauraso (Italy), T. Viteam (Japan), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, L. Wimmer, J. Zacharias, L. Zhou, GCHW Problem Solving Group (U. K.), and the proposer.

A Condition for Nonexistence of Compositional Roots

11858 [2015, 801]. *Proposed by Arkady Alt, San Jose, CA.* Let D be a nonempty set and g be a function from D to D. Let n be an integer greater than 1. Consider the set X of all x in D such that $g^n(x) = x$, but $g^k(x) \neq x$ for $1 \leq k < n$. Prove that if X has exactly n elements, then there is no function f from D to D such that $f^n = g$. (Here, for $h : D \to D$, h^k denotes the k-fold composition of h with itself.)

Composite solution by Janusz Konieczny, University of Mary Washington, Fredericksburg, VA, and NSA Problems Group, Fort Meade, MD. For $h\colon\ D \to D$, let $\Gamma(h)$ denote the functional digraph of h, with an edge from a to b if and only if $h(a) = b$. From the definition of X, we see that X induces a single cycle of length n in $\Gamma(g)$. Fix x on this cycle, and suppose that f exists. Since $f^{n^2}(x) = g^n(x) = x$, vertex x lies on a cycle in $\Gamma(f)$. Let C be this cycle, and let m be its length. Both f and g permute the vertices on C; it is a single cycle under f, and g produces the nth power of this cycle.

Thus g acts on C as a product of d disjoint cycles of equal length m/d, where $d = \gcd(m, n)$. One of these cycles contains x. We have seen that the cycle in $\Gamma(g)$ containing x has length n and contains all of X. Hence g on C must produce a single cycle of length n. This requires $d = 1$ and $m = n$, which in turn requires $n = 1$.

Also solved by K. Banerjee, P. Budney, B. S. Burdick, N. Caro (Brazil), S. Chan-Aldebol, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), H. B. Ghaffari (Iran), E. A. Herman, T. Horine, Y. J. Ionin, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), K. E. Lewis (Gambia), J. H. Lindsey II, J. Olson, J. M. Pacheco & Á. Plaza (Spain), A. J. Rosenthal, A. H. Sadeghimanesh (Denmark), J. Schlosberg, J. H. Smith, R. Stong, T. Viteam (Japan), GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

Avoiding Voids

11862 [2015, 802]. *Proposed by David A. Cox and Uyen Thieu, Amherst College, Amherst, MA.* For positive integers n and k, evaluate

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i}\binom{kn-in}{k+1}.$$

Solution I by Borislav Karaivanov, Sigma Space, Lanham, MD, and Tzvetalin S. Vassilev, Nipissing University, North Bay, Ontario, Canada. The value is $kn^{k-1}\binom{n}{2}$.

Consider a deck of kn cards, with n distinct cards in each of k suits. Both the summation and the value count the ways to pick $k+1$ cards with at least one card from each suit. For the value, we pick one of the k suits to contribute two cards and pick one card from each of the other suits.

For the summation, we use inclusion-exclusion. No suit can be omitted. When i specified suits are omitted, we choose $k+1$ cards from the remaining $kn - in$ cards. Hence the summand here is exactly the summand in the standard inclusion-exclusion computation to count the selections of $k+1$ cards omitting no suits.

Solution II by BSI Problems Group, Bonn, Germany. We use generating functions. Let $[z^n]$ denote the coefficient operator extracting the coefficient of z^n in a formal power series. Using the Binomial theorem, we obtain

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{kn-in}{k+1} &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}[z^{k+1}](1+z)^{n(k-i)} \\ &= [z^{k+1}]((1+z)^n-1)^k = [z]\left(\frac{(1+z)^n-1}{z}\right)^k \\ &= [z]\left(n+\binom{n}{2}z+\cdots\right)^k = kn^{k-1}\binom{n}{2}.\end{aligned}$$

Editorial comment. Extending Solution I, the FAU Problem Solving Group noted that choosing $k+2$ cards yields a similar formula:

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{kn-in}{k+2} = \binom{k}{2}\binom{n}{2}^2 n^{k-2} + \binom{k}{1}\binom{n}{3}n^{k-1}.$$

One could of course continue farther.

Several solutions used Stirling numbers and various identities. Others used finite differences. If the factor $\binom{kn-in}{k+1}$ is replaced by any polynomial in i of degree at most $k-1$, then the sum evaluates to 0. Thus, we need only compute the contribution from the coefficients of i^{k+1} and i^k in $\binom{kn-in}{k+1}$.

Also solved by U. Abel (Germany), A. Ali (India), M. Bataille (France), D. Beckwith, M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), R. Chapman (U. K.), P. P. Dályay (Hungary), R. Dutta (India), O. Geupel (Germany), M. L. Glasser, N. Grivaux (France), M. Hoffman, Y. J. Ionin, O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), J. H. Lindsey II, R. Nandan, M. Omarjee (France), M. Sawhney, E. Schmeichel, N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), H. Widmer (Swizerland), M. Wildon (U. K.), Armstrong Problem Solvers, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
http://www.americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by October 31, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11985. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* For fixed $s, t \in \mathbb{N}$ with $s \le t$, let $a_n = \binom{n}{s} + \binom{n}{s+1} + \cdots + \binom{n}{t}$. Prove that this sequence is log-concave, namely that $a_n^2 \ge a_{n-1}a_{n+1}$ for all $n \ge 1$.

11986. *Proposed by Martin Lukarevski, Goce Delčev University, Štip, Macedonia.* Let x, y, and z be positive real numbers. Prove

$$4(xy + yz + zx) \le \left(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}\right)\sqrt{(x+y)(y+z)(z+x)}.$$

11987. *Proposed by Shen-Fu Tsai, Redmond, WA.* Let $n_1, \ldots, n_k$ be positive integers. Let $S = [n_1] \times \cdots \times [n_k]$, where we write $[n]$ for $\{1, \ldots, n\}$. Define a binary relation on S by putting $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ whenever $x_i < y_i$ for every $i \in [k]$. An *antichain* A is a subset of S such that, for all x and y in A, neither $x < y$ nor $y < x$. An antichain is *maximal* if it is not a proper subset of any other antichain. Show that all maximal antichains in S have the same size.

11988. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle. Find the extrema of

$$\frac{AC^2 + CE^2 + EB^2 + BD^2 + DA^2}{AB^2 + BC^2 + CD^2 + DE^2 + EA^2}$$

over all points D and E in the plane of ABC. At which points D and E are these extrema attained?

11989. *Proposed by Spiros P. Andriopoulos, Third High School of Amaliada, Eleia, Greece.* Let x be a number between 0 and 1. Prove

$$\prod_{n=1}^{\infty}\left(1 - x^n\right) \ge \exp\left(\frac{1}{2} - \frac{1}{2(1-x)^2}\right).$$

http://dx.doi.org/10.4169/amer.math.monthly.124.6.563

11990. *Proposed by Nicuşor Minculete, Transilvania University of Braşov, Romania.* Let a, b, and c be the lengths of the sides of a triangle of area S. Weitzenböck's inequality states that $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$. Prove the following stronger inequality:

$$a^2 + b^2 + c^2 \geq \sqrt{3}\left(4S + (c-a)^2\right).$$

11991. *Proposed by Yongge Tian, Central University of Finance and Economics, Beijing, China.* Given two complex n-by-n positive definite matrices A and B, let $C = (A+B)/2$ and $D = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$; the matrices C and D are the arithmetic mean and geometric mean of A and B. Prove range$(C - D) =$ range$(A - B)$ and

$$\text{range}\begin{bmatrix} C & D \\ D & C \end{bmatrix} = \text{range}\begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

SOLUTIONS

Rational triples

11827 [2015, 284]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Show that there are infinitely many rational triples (a, b, c) such that $a + b + c = abc = 6$.

Solution by Boris Bekker, St. Petersburg State University, Russia, and Yury J. Ionin, Central Michigan University, Mt. Pleasant, MI. Given rational (x, y) satisfying $x^3 - 9x + 9 = y^2$, define

$$a = \frac{6}{3-x}, \quad b = \frac{6-3x+y}{3-x}, \quad c = \frac{6-3x-y}{3-x}.$$

Note that $a + b + c = (18 - 6x)/(3 - x) = 6$ and $abc = 6\left((6-3x)^2 - y^2\right)/(3-x)^3 = 6$, since $(6-3x)^2 - y^2 = 36 - 36x + 9x^2 - x^3 + 9x - 9 = (3-x)^3$. Also, (a, b, c) is a rational triple when (x, y) is rational. Hence it suffices to find infinitely many such rational (x, y).

Let E be the set of rational points of the elliptic curve given by $y^2 = x^3 - 9x + 9$; note by computation that $(9/4, -3/8) \in E$. We produce points $(x_n, y_n) \in E$, beginning with $(x_1, y_1) = (9/4, -3/8)$. Given (x_n, y_n), let (x_{n+1}, y_{n+1}) be the point at which the tangent line at (x_n, y_n) intersects the curve.

Using implicit differentiation, the equation for the tangent line is

$$y = \frac{3x_n^2 - 9}{2y_n}(x - x_n) + y_n.$$

Since intersection points must satisfy $x^3 - 9x + 9 - y^2 = 0$, we have

$$x^3 - 9x + 9 - \left(\frac{3x_n^2 - 9}{2y_n}(x - x_n) + y_n\right)^2 = 0.$$

Note that x_n is a double root of this equation, since the elliptic curve and tangent line are tangent at x_n. Since the three roots must sum to the negative of the coefficient of x^2, we have $x_n + x_n + x_{n+1} = \left((3x_n^2 - 9)/(2y_n)\right)^2$. Thus

$$x_{n+1} = \left(\frac{3x_n^2 - 9}{2y_n}\right)^2 - 2x_n = \frac{x_n^4 + 18x_n^2 - 72x_n + 81}{4(x_n^3 - 9x_n + 9)}.$$

For nonzero rational r, let $\nu(r)$ be the unique integer such that $2^{-\nu(r)}r$ equals a fraction with odd numerator and odd denominator. For nonzero rational r and s, we have $\nu(rs) = \nu(r) + \nu(s)$ and $\nu(r/s) = \nu(r) - \nu(s)$. Also, $\nu(r+s) = \nu(r)$ when $\nu(r) < \nu(s)$. Note that $\nu(x_1) = -2$. Inductively, with $\nu(x_n) < 0$,

$$\begin{aligned}\nu(x_{n+1}) &= \nu(x_n^4 + 18x_n^2 - 72x_n + 81) - \nu(4(x_n^3 - 9x_n + 9))\\ &= \min\{\nu(x_n^4), \nu(18x_n^2), \nu(72x_n), \nu(81)\} - (2 + \min\{\nu(x_n^3), \nu(9x_n), \nu(9)\})\\ &= \min\{4\nu(x_n), 1 + 2\nu(x_n), 3 + \nu(x_n), 0\} - 2 - \min\{3\nu(x_n), \nu(x_n), 0\}\\ &= 4\nu(x_n) - 2 - 3\nu(x_n) < \nu(x_n).\end{aligned}$$

We conclude that $(x_n, y_n)_{n\geq 1}$ has no repeated terms, and E is infinite.

Editorial comment. Other solutions used the machinery of elliptic curves. Several solvers cited a paper by A. Schinzel, Triples of positive integers with the same sum and the same product, *Serdica Math. J.* **22** (1996) 587–588, or one by J. B. Kelly, Partitions with equal products (II), *Proc. Amer. Math. Soc.* **107** (1989) 887–893.

Also solved by A. J. Bevelacqua, R. Boukharfane (France), R. Chapman (U. K.), J. Christopher, O. Geupel (Germany), M. Huibregtse, P. Lalonde (Canada), J. F. Loverde, G. Malisani (Italy), M. Omarjee (France), M. A. Prasad (India), F. Perdomo & Á. Plaza (Spain), J. P. Robertson, D. Singer, J. C. Smith, R. Stong, R. Tauraso (Italy), L. G. Vidiani, E. Weinstein, GCHQ Problem Solving Group (U. K.), New York Math Circle, NSA Problems Group, and the proposer.

Variations on the Euler-Mascheroni Constant

11851 [2015, 391]. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* For real a and b and integer $n \geq 1$, let $\gamma_n(a,b) = -\log(n+a) + \sum_{k=1}^{n} \frac{1}{k+b}$.
(**a**) Prove that $\gamma(a,b) = \lim_{n\to\infty} \gamma_n(a,b)$ exists and is finite.
(**b**) Find

$$\lim_{n\to\infty} \left(\log\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) \right)^n .$$

Solution by Ramya Dutta, Chennai Mathematical Institute, Chennai, India. The statement is not quite correct. If b is a negative integer, then $\gamma_n(a,b)$ is defined only for finitely many values of n and the problem does not make sense. We assume below that such b are excluded.

Let $\psi(x)$ denote the digamma function. We have

$$\gamma_n(a,b) = -\psi(b+1) + \psi(n+1+b) - \log(n+a).$$

From the asymptotic formula

$$\psi(z) = \log z - \frac{1}{2z} + O(z^{-2})$$

as $|z| \to \infty$, we get

$$\psi(n+1+b) = \log n + \frac{2b+1}{2n} + O(n^{-2}),$$

and therefore

$$\gamma_n(a,b) = -\psi(b+1) + \frac{2b-2a+1}{2n} + O(n^{-2}).$$

Hence we see that $\gamma(a,b) = \lim_{n\to\infty} \gamma_n(a,b) = -\psi(b+1)$ exists and is finite when b is not a negative integer.

Furthermore,

$$\log\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) = 1 + \gamma_n(a,b) - \gamma(a,b) = 1 + \frac{2b-2a+1}{2n} + O(n^{-2}).$$

Hence

$$\lim_{n\to\infty}\left(\log\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma(a,b)\right)^n$$
$$= \lim_{n\to\infty}\left(1 + \frac{2b-2a+1}{2n} + O(n^{-2})\right)^n = e^{(2b-2a+1)/2}.$$

Also solved by K. F. Andersen (Canada), R. Boukharfane (France), P. Bracken, B. Bradie, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), P. J. Fitzsimmons, D. Fleischman, M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), E. Omey & S. Van Gulck (Belgium), P. Perfetti (Italy), E. Saha (India), K. Schilling, R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), and the proposer.

Two Matrices

11859 [2015, 801]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Find all pairs (m,n) of positive integers for which there exists an $m\times n$ matrix A and an $n\times m$ matrix B, both with real entries, such that all diagonal entries of AB are positive and all off-diagonal entries are negative.

Solution by Edward Schmeichel, San Jose State University, San Jose, CA. Given (m,n), let such matrices A and B be called a *suitable pair*. Suitable pairs (A,B) exist for all pairs (m,n) except $(m,1)$ with $m\ge 3$ and $(m,2)$ with $m\ge 4$. The proof is by cases, depending on the value n.

$n=1$: Letting $A^t = B = \begin{bmatrix}1\end{bmatrix}$ and $A^t = B = \begin{bmatrix}1 & -1\end{bmatrix}$ provides suitable pairs for $m=1$ and $m=2$. For $m\ge 3$, a suitable pair (A,B) with $A = [a\ \ b\ \ c\ \cdots]^t$ and $B = [d\ \ e\ \ f\ \ \cdots]$ would require $bd<0$ and $a^2f^2bd = (bf)(ad)(af) > 0$, which is a contradiction.

$n=2$: For $m\in\{1,2,3\}$, obtain suitable pairs by letting A and B^t both equal one of the matrices below.

$$\begin{bmatrix}1 & -1\end{bmatrix} \qquad \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad \begin{bmatrix}2 & -1\\ -1 & 2\\ -1 & -1\end{bmatrix}$$

For $m\ge 4$, suppose that (A,B) is a suitable pair. We may let the rows of A be unit vectors $v_1,\dots,v_m$ and the columns of B be unit vectors $w_1,\dots,w_m$ in $\mathbb{R}^2$. Thus $AB = [v_i\cdot w_j]$, where $\cdot$ denotes the dot product. Now $v_i\cdot w_j > 0$ (respectively, <0) if and only if w_j lies inside the open (respectively, outside the closed) half of the unit circle centered at v_i. Since $m\ge 4$, some three (say w_1, w_2, w_3) lie in some half of the unit circle; say w_2 is between w_1 and w_3. Now $v_2\cdot w_2 > 0$, so v_2 lies in the half-circle centered at w_2. Also, either w_1 or w_3 lies in the closed half-circle centered at v_2. This contradicts the assertion that (A,B) is a suitable pair.

$n=3$: On a latitude L of the unit sphere in $\mathbb{R}^3$ positioned slightly south of the equator, consider unit vectors $v_1, v_2, \dots, v_m$ equally spaced around L. Let H_i denote the closed hemisphere centered at v_i, for $1 \le i \le m$. For each index i, there is a unit vector w_i due north of v_i, and slightly less than 90° from v_i, such that w_i is in the interior of H_i but in the exterior of all H_j with $j \ne i$. Let A be the $m \times 3$ matrix with row vectors $v_1, \dots, v_m$, and let B be the $3 \times m$ matrix with column vectors $w_1, \dots, w_m$. Now (A, B) is a suitable pair for $(m, 3)$.

$n \ge 4$: Given any $m \ge 1$, let (A_3, B_3) be a suitable pair for $(m, 3)$ as above. Define the $m \times n$ matrix A and $n \times m$ matrix B by

$$A = \begin{bmatrix} A_3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_3 \\ 0 \end{bmatrix}.$$

Now (A, B) is a suitable pair for (m, n).

Also solved by R. Chapman (U. K.), M. Javaheri, R. Stong, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposers.

A Power of an Integral versus an Integral of a Power

11861 [2015, 801]. *Proposed by Phu Cuong Le Van, College of Education, Hue, Vietnam.* Let n be a natural number, and let f be a continuous function from $[0, 1]$ to $\mathbb{R}$ such that $\int_0^1 f(x)^{2n+1}\,dx = 0$. Prove that

$$\frac{(2n+1)^{2n+1}}{(2n)^{2n}} \left(\int_0^1 f(x)\,dx \right)^{2n} \le \int_0^1 (f(x))^{4n}\,dx.$$

Solution by Paolo Perfetti, Departimento di Matematica, Università degli Studi di Roma, Rome, Italy. If $\int f(x)\,dx = 0$, then the inequality is trivially true. If $\int f(x)\,dx \ne 0$, then the inequality need not hold. To see this, consider λf with $\lambda > 0$. The right side goes to zero with λ according to λ^{4n}, while the left side goes to zero with λ according to λ^{2n}. Hence, for sufficiently small λ, the right side is smaller.

The correct statement is

$$\frac{(2n+1)^{2n+1}}{(2n)^{2n}} \left(\int_0^1 f(x)\,dx \right)^{4n} \le \int_0^1 (f(x))^{4n}\,dx.$$

To prove this, consider $c \in \mathbb{R}$. By the Cauchy–Schwarz inequality,

$$\left(\int_0^1 \left(c + f(x)^{2n}\right) f(x)\,dx \right)^2 \le \int_0^1 \left(c + f(x)^{2n}\right)^2 dx \int_0^1 f(x)^2 dx.$$

Since $\int_0^1 f^{2n+1}\,dx = 0$, this becomes

$$\begin{aligned} \left(\int_0^1 c f(x)\,dx \right)^2 &\le \int_0^1 \left(c + f(x)^{2n}\right)^2 dx \int_0^1 f(x)^2 dx \\ &= \left(c^2 + 2c \int_0^1 f(x)^{2n} dx + \int_0^1 f(x)^{4n} dx \right) \int_0^1 f(x)^2 dx, \end{aligned}$$

or

$$\begin{aligned} c^2 \left(\left(\int_0^1 f(x)^2 dx \right)^2 - \int_0^1 f(x)^2 dx \right) - 2c \int_0^1 f(x)^2 dx \int_0^1 f(x)^{2n} dx \\ \le \int_0^1 f(x)^{4n} dx \int_0^1 f(x)^2 dx. \end{aligned} \tag{1}$$

Let

$$A = \left(\int_0^1 f(x)\,dx\right)^2 - \int_0^1 f(x)^2 dx.$$

By the Cauchy–Schwarz inequality, $A \le 0$. Also, $A = 0$ only if f is constant, and since $\int f(x)^{2n+1}dx = 0$, the function f would be the constant 0. In this case, the inequality holds trivially. So assume $A < 0$. Now choose

$$c = \frac{1}{A}\int_0^1 f(x)^2 dx \int_0^1 f(x)^{2n} dx$$

(chosen to maximize the top line of (1)). Now (1) becomes

$$\int_0^1 f(x)^{4n} dx \int_0^1 f(x)^2 dx \ge \frac{-1}{A}\left(\int_0^1 f(x)^2 dx\right)^2 \left(\int_0^1 f(x)^{2n}\right)^2,$$

or

$$\int_0^1 f(x)^{4n} dx \ge \frac{-1}{A}\int_0^1 f(x)^2 dx \left(\int_0^1 f(x)^{2n} dx\right)^2.$$

Using the Hölder inequality, we get

$$\int_0^1 f(x)^{4n} dx \ge \frac{-1}{A}\int_0^1 f(x)^2 dx \left(\int_0^1 f(x)^2 dx\right)^{2n}.$$

With $p = \int_0^1 f(x)^2 dx$ and $p_0 = (\int_0^1 f(x)\,dx)^2$, this becomes

$$\int_0^1 f(x)^{4n} dx \ge \frac{p^{2n+1}}{p - p_0}, \quad \text{with } p > p_0.$$

Combining this with the fact that the function $p \mapsto p^{2n+1}/(p - p_0)$ has minimum value $p_0^{2n}(2n+1)^{2n+1}/(2n)^{2n}$, we see that

$$\int_0^1 f(x)^{4n} dx \ge \frac{(2n+1)^{2n+1}}{(2n)^{2n}} \left(\int_0^1 f(x)\,dx\right)^{4n},$$

which is the inequality to be proved.

Editorial comment. The proposer's original version indeed had $4n$, not $2n$.

Also solved by U. Abel (Germany), K. Andersen (Canada), P. P. Dályay (Hungary), M. Omarjee (France), M. Sawhney, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), and the proposer.

Circular Congruences with a Unique Solution

11863 [2015, 802]. *Proposed by Jeffrey C. Lagarias and Jeffrey Sun, University of Michigan, Ann Arbor, MI.* Consider integers a, b, c with $1 \le a < b < c$ that satisfy the following system of congruences:

$$(a+1)(b+1) \equiv 1 \pmod{c}$$
$$(a+1)(c+1) \equiv 1 \pmod{b}$$
$$(b+1)(c+1) \equiv 1 \pmod{a}.$$

(**a**) Show that there are infinitely many solutions (a, b, c) to this system.

(**b**) Show that under the additional condition that $\gcd(a, b) = 1$, there are only finitely many solutions (a, b, c) to the system, and find them all.

Solution by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.
(**a**) For $m \in \mathbb{N}$, letting $a = 6m$, $b = 12m$, and $c = 18m$ yields a solution.

(**b**) If $\gcd(b, c) = d > 1$, then $1 \equiv (a + 1)(b + 1) \equiv a + 1 \pmod{d}$, so $\gcd(a, b) \geq d$, a contradiction. Thus $\gcd(b, c) = 1$, and similarly $\gcd(a, c) = 1$. If a is even, then the congruence $(b + 1)(c + 1) \equiv 1 \pmod{a}$ implies that both b and c are even, yielding a common factor. Thus a is odd, and similarly b and c are also odd.

Note that $(a + 1)(b + 1)(c + 1)$ is congruent to 1 modulo a, b, and c. It is also congruent to 1 modulo abc since a, b, and c have no common factors. Hence $(a + 1)(b + 1)(c + 1) = 1 + nabc$ for some positive integer n. Dividing by abc yields

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} = n - 1. \tag{1}$$

If $a \geq 3$, then $b \geq 5$, $c \geq 7$, $ab \geq 15$, $ac \geq 21$, and $bc \geq 35$. Therefore the left side of (1) is at most $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{15} + \frac{1}{21} + \frac{1}{35}$, which equals $\frac{86}{105}$, so the sum cannot be a positive integer. Hence $a = 1$, and (1) reduces to

$$\frac{2}{b} + \frac{2}{c} + \frac{1}{bc} = n - 2. \tag{2}$$

If $b \geq 5$, then $c \geq 7$, and $\frac{2}{b} + \frac{2}{c} + \frac{1}{bc} \leq \frac{2}{5} + \frac{2}{7} + \frac{1}{35} = \frac{5}{7}$. Again the sum cannot be a positive integer. Hence $b = 3$, and (2) reduces to $\frac{2c+7}{3c} = n - 2$. Thus c divides 7, implying $c = 7$, so $(1, 3, 7)$ is the only solution to (**b**).

Editorial comment. T. Horine showed that the solutions in part (**a**) are the triples (a, b, c) that can be written as $a = a'd$, $b = b'd$, $c = c'd$ such that a', b', c' are pairwise relatively prime, $1 \leq a' < b' < c'$, and $d > 0$ satisfies the system

$$a'b'd \equiv -(a' + b') \pmod{c'}$$
$$a'c'd \equiv -(a' + c') \pmod{b'}$$
$$b'c'd \equiv -(b' + c') \pmod{a'},$$

which by the Chinese remainder theorem has a unique solution modulo $a'b'c'$.

Also solved by A. Ali (India), N. Bouchareb (Morocco), B. S. Burdick, R. Chapman (U. K.), N. Curwen (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), S. Hitotumatu (Japan), T. Horine, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), P. Lalonde (Canada), V. Mikayelyan (Armenia), J. P. Robertson, S. Roy (India), M. Sawhney, K. Schilling, J. Schlosberg, N. C. Singer, R. Stong, R. Tauraso (Italy), Z. Vörös (Hungary), E. T. White, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Congruence Property of an Integer Sequence

11864 [2015, 802]. *Proposed by Bakir Farhi, University of Béjaia, Béjaia, Algeria.* Let p be a prime number, and let $\langle u \rangle$ be the sequence given by $u_n = n$ for $0 \leq n \leq p - 1$ and $u_n = pu_{n+1-p} + u_{n-p}$ for $n \geq p$. Prove that for each positive integer n, the greatest power of p dividing u_n is the same as the greatest power of p dividing n.

Solution by John H. Lindsey II, Cambridge, Massachussetts. Let $\nu_p(n)$ be the greatest integer e such that p^e divides n. For $n \geq 1$, we first prove by induction that $\nu_p(n!) \leq n - 1$, with strict inequality for $p > 2$ and $n > 1$. This is immediate for $n < p$. Let $n = j + ip$ with $0 \leq j < p$ and $i > 0$. Using the induction hypothesis, we compute

$$\nu_p((j + ip)!) = \nu_p(p(2p) \cdots (ip)) = i + \nu_p(i!) \leq i + i - 1 \leq ip - 1,$$

with strict inequality for $p > 2$.

We now show $u_{j+ip} = \sum_{k=0}^{i} \binom{i}{k} p^k u_{j+k}$ for $i \ge 0$ and $j \ge 0$. We use induction on i. The result is trivial for $i = 0$, and for $i \ge 1$ we compute

$$u_{j+ip} = u_{j+(i-1)p} + pu_{j+1+(i-1)p} = \sum_{k=0}^{i-1} \binom{i-1}{k} p^k u_{j+k} + \sum_{k=0}^{i-1} \binom{i-1}{k} p^{1+k} u_{j+1+k}$$

$$= \sum_{k=0}^{i-1} \binom{i-1}{k} p^k u_{j+k} + \sum_{k=1}^{i} \binom{i-1}{k-1} p^k u_{j+k} = \sum_{k=0}^{i} \binom{i}{k} p^k u_{j+k}.$$

We must prove $\nu_p(u_n) = \nu_p(n)$ for $n \ge 1$. For $1 \le j < p$ and $i \ge 0$, we have $u_{j+ip} = u_j + \sum_{k=1}^{i} \binom{i}{k} p^k u_{j+k}$. Since $u_j = j$, we have that u_{j+ip} is not divisible by p. Also $j + ip$ is not divisible by p. Thus $\nu_p(u_{j+ip}) = 0 = \nu_p(j + ip)$.

Since $u_0 = 0$ and $u_1 = 1$, we have $u_{0+ip} = \sum_{k=0}^{i} \binom{i}{k} p^k u_k = ip + \sum_{k=2}^{i} \binom{i}{k} p^k u_k$. Suppose $p > 2$. When $k > 1$,

$$\nu_p\left(\binom{i}{k} p^k u_k\right) \ge \nu_p(i) - \nu_p(k!) + k > \nu_p(i) - k + 1 + k = \nu_p(ip).$$

Since the terms in the sum are divisible by higher powers of p, the powers of p dividing u_{ip} are just the powers dividing ip, and we have $\nu_p(u_{ip}) = \nu_p(ip)$.

The final case is $p = 2$. For $k = 2$, we have $\nu_2\left(\binom{i}{2} 2^2 u_2\right) \ge \nu_2(i) - 1 + 2 + 1 = \nu_2(2i) + 1$. For $k > 2$, we use the factors $i(i-1)(i-2)$ in the numerator of $\binom{i}{k}$ (with $i - 1$ or $i - 2$ being even) to obtain

$$\nu_2\left(\binom{i}{k} 2^k u_k\right) \ge \nu_2(i) + \nu_2((i-1)(i-2)) - \nu_2(k!) + k$$
$$\ge \nu_2(i) + 1 - k + 1 + k = \nu_2(2i) + 1.$$

As in the preceding case, we thus have $\nu_2(u_{2i}) = \nu_2(2i)$.

Editorial comment. Other solution techniques included using generating functions or the characteristic polynomial of the linear recursion.

Also solved by N. Caro (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), T. Horine, Y. J. Ionin, P. Lalonde (Canada), J. P. Robertson, N. C. Singer, R. Stong, R. Tauraso (Italy), J. Van hamme (Belgium), T. Viteam (Japan), and the proposer.

Two Sums Compared

11865 [2015, 899]. *Proposed by Gary H. Chung, Clark Atlanta University, Atlanta, GA.* Let $\langle a_n \rangle$ be a monotone decreasing sequence of nonnegative real numbers. Prove that $\sum_{n=1}^{\infty} a_n/n$ is finite if and only if $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log n < \infty$.

Solution by Robin Chapman, University of Exeter, Exeter, United Kingdom. If $\sum_{n=1}^{\infty} a_n/n$ is finite, then $\lim_{n\to\infty} a_n = 0$. Assuming $\lim_{n\to\infty} a_n = 0$, we prove that $\sum_{n=1}^{\infty} a_n/n < \infty$ if and only if $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log n < \infty$. Let $b_n = a_n - a_{n+1} \ge 0$. Since $\lim a_n = 0$, we have $a_n = \sum_{k=n}^{\infty} b_k$, and thus

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{b_k}{n}.$$

A nonnegative double series converges if and only if the series obtained by reversing the order of summation also converges. That series is

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_k}{n} = \sum_{k=1}^{\infty} b_k H_k,$$

where the harmonic numbers H_k given by $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ are asymptotic to $\log k$ as $k \to \infty$. By the limit comparison test, $\sum a_n/n < \infty$ if and only if $\sum_{k=1}^{\infty} b_k \log k < \infty$. This completes the proof.

Also solved by K. F. Andersen (Canada), B. S. Burdick, H. Chen, P. P. Dályay (Hungary), C. J. Dowd, P. J. Fitzsimmons, E. J. Ionaşcu, M. Javaheri, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), L. Madejíčka (Slovakia), T. L. McCoy, V. Mikayelyan (Armenia), A. Minasyan (Russia), M. Omarjee (France), M. Omarjee (France) & R. Tauraso (Italy), P. Perfetti (Italy), Á. Plaza (Spain), D. Ritter, S. M. Sasane & A. Sasane, K. Schilling, N. C. Singer, A. Stenger, R. Stong, D. B. Tyler, J. Vinuesa (Spain), T. Wiandt, GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposer.

Random Events Simulated by a Biased Coin

11866 [2015, 899]. *Proposed by Arindam Sengupta, University of Calcutta, Kolkata, India.* Consider a finite set $\{\alpha_1, \dots, \alpha_m\}$ of rational numbers in $(0, 1)$. For $0 < p < 1$ and $k \geq 1$, let Ω_k be the probability space for k independent flips of a coin that comes up heads with probability p. Show that there exists a positive integer k, a suitable p, and events $E_1, \dots, E_m$ in Ω_k, such that for each j with $1 \leq j \leq m$, the probability of E_j is α_j.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Let M be the least common multiple of the denominators of $\alpha_1, \dots, \alpha_m$. We show that there exists a suitable k and p so that the space Ω_k can be partitioned into M disjoint pieces each of probability $1/M$. (That is, we simulate a fair M-sided die using a fixed number k of tosses of a biased coin whose probability of heads is a fixed number p.) The required set E_j can then be taken as the union of $M\alpha_j$ of these pieces.

To build this partition we take k to be an odd prime congruent to $-1 \pmod{M}$. We choose p so that Ω_k can be partitioned into $k + 1$ pieces, each having probability $1/(k + 1)$. Grouping these in blocks of size $(k + 1)/M$ gives us a partition into pieces of probability $1/M$.

Now we describe the choice of p. The continuous function f defined on $[0, 1]$ by $f(t) = t^k + (1 - t)^k$ has minimum $f(1/2) = 2^{1-k} \leq 1/(k + 1)$ and maximum $f(0) = f(1) = 1$, so we can choose p so that $f(p) = 1/(k + 1)$.

Finally, we describe the partition into sets having probability $1/(k + 1)$. The first block Q_0 of our partition consists of the event that all flips agree (either k heads or k tails). So Q_0 has probability $p^k + (1 - p)^k$, which equals $1/(k + 1)$. The rest $\Omega_k \backslash Q_0$ has probability $k/(k + 1)$; we partition it into k sets of equal probability. For $1 \leq r < k$, the set Z_r of flips consisting of r heads and $k - r$ tails is made up of $\binom{k}{r}$ events having the same probability $p^k(1 - p)^{k-r}$. Since $\binom{k}{r}$ is a multiple of k, the set Z_r may be partitioned into k sets $Q_{1,r}, \dots, Q_{k,r}$ having the same probability. Repeating this for all r and setting $Q_i = \cup_{r=1}^{k-1} Q_{i,r}$ when $1 \leq i \leq k$, we partition $\Omega_k \backslash Q_0$ into k sets $Q_1, \dots, Q_k$ having equal probability.

Editorial comment. The GCHQ Problem Solving Group (U. K.) noted that, in general, it is not possible to take the probability p to be rational.

Also solved by N. Grivaux (France), O. P. Lossers (Netherlands), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at http://www.americanmathematicalmonthly.submittable.com/submit. *Proposed solutions to the problems below should be submitted by December 31, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

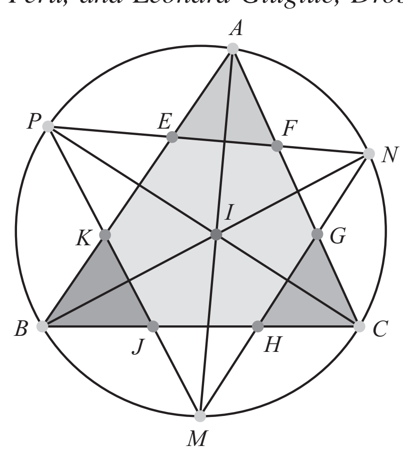
11992. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Prove that, for every positive integer n, there is a positive integer m such that $3^m + 5^m - 1$ is divisible by 7^n.

11993. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Prove

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}\,dx = -\frac{\pi^4}{240}.$$

11994. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with incenter I and circumcircle ω. Let M, N, and P be the second points of intersection of ω with lines AI, BI, and CI, respectively. Let E and F be the points of intersection of NP with AB and AC, respectively. Similarly, let G and H be the points of intersection of MN with AC and BC, respectively, and let J and K be the points of intersection of MP with BC and AB, respectively. Prove

$$EF + GH + JK \le KE + FG + HJ.$$

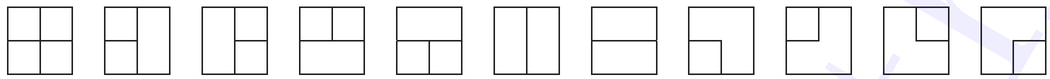


11995. *Proposed by Dan Ştefan Marinescu, National College "Iancu de Hunedoara," Hunedoara, Romania, and Mihai Monea, National College "Decebal," Deva, Romania.* Suppose $0 < x_0 < \pi$, and for $n \ge 1$ define $x_n = \frac{1}{n}\sum_{k=0}^{n-1} \sin x_k$. Find $\lim_{n\to\infty} x_n\sqrt{\ln n}$.

11996. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Consider all the tilings of a 2-by-n rectangle comprised of tiles that are either a unit square,

http://dx.doi.org/10.4169/amer.math.monthly.124.7.659

a domino, or a right tromino. Let f_n be the fraction of tiles among all such tilings that are unit squares. For example, $f_2 = 4/7$, because 16 out of the 28 tiles in the 11 tilings of a 2-by-2 rectangle are squares. What is $\lim_{n\to\infty} f_n$?



11997. *Proposed by Michael Drmota, Technical University of Vienna, Vienna, Austria; Christian Krattenthaler, University of Vienna, Vienna, Austria; and Gleb Pogudin, Johannes Kepler University, Linz, Austria.* Assume $|p| < 1$ and $pz \neq 0$. With $f(z) = \left(e^{(p-1)z} - e^{-z}\right)/(pz)$, define $f^*(z) = \prod_{k=0}^{\infty} f(p^k z)$, and then define $F_n(p)$ so that $f^*(z) = \sum_{n=0}^{\infty} F_n(p) z^n$. Prove the identity

$$\sum_{n=0}^{\infty} F_n(p)\, p^{\binom{n}{2}} = 0.$$

11998. *Proposed by Roger Cuculière, Lycée Pasteur, Neuilly-sur-Seine, France.* Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy $f(z) \le 1$ for some nonzero real number z and

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

for all real numbers x and y.

SOLUTIONS

Log-squared of the Catalan Generating Function

11832 [2015, 390]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let $C(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$ (thus $C(z)$ is the generating function of the Catalan numbers). Prove that

$$(\log C(z))^2 = \sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n}.$$

Here $H_k = \sum_{j=1}^{k} 1/j$; that is, H_k is the kth harmonic number.

Solution by James Christopher Smith, Knoxville, TN. From the well-known formula $C(z) = \frac{1-\sqrt{1-4z}}{2z}$, it follows that $\frac{C'(z)}{C(z)} = \frac{1}{2z}\left(\frac{1}{\sqrt{1-4z}} - 1\right)$. The generating function for the central binomial coefficients is also well known; it is $\sum_{k=0}^{\infty} \binom{2k}{k} z^k = \frac{1}{\sqrt{1-4z}}$. This yields

$$\frac{C'(z)}{C(z)} = \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} z^{k-1} \quad \text{and} \quad \log(C(z)) = \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{z^k}{k},$$

where we obtain the second formula by integration. After squaring and collecting terms of degree n, we obtain

$$(\log C(z))^2 = \sum_{n=1}^{\infty} \left(\frac{1}{4} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k(n-k)} \right) z^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{n} \left(\frac{1}{k} + \frac{1}{n-k} \right) \right) z^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} \right) \frac{z^n}{n}.$$

Thus it remains to show

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} = \binom{2n}{n} (H_{2n-1} - H_n).$$

Letting $H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k}$ and using $\binom{x}{k} = \left(\prod_{i=0}^{k-1} (x-i) \right) / k!$, for $m \le n$ we obtain

$$\frac{d}{dx} \binom{x+n}{m} = \binom{x+n}{m} (H_n(x) - H_{n-m}(x)).$$

In D. Merlini, R. Sprugnoli, M. C. Verri, Lagrange inversion: when and how, *Acta Applicandae Mathematica* **94** (2006) 233–249, Lagrange inversion is used to prove $\sum_{k=0}^{n} \frac{a}{a+qk} \binom{a+qk}{k} \binom{qn-qk}{n-k} = \binom{a+qn}{n}$. Setting $a = x$ and $q = 2$ yields

$$\sum_{k=0}^{n} \frac{x}{x+2k} \binom{x+2k}{k} \binom{2n-2k}{n-k} = \binom{x+2n}{n}.$$

Differentiation of both sides with respect to x yields

$$\sum_{k=1}^{n} \frac{d}{dx} \left(\frac{x}{x+2k} \binom{x+2k}{k} \right) \binom{2n-2k}{n-k} = \binom{x+2n}{n} (H_{2n}(x) - H_n(x)).$$

Expanding the differentiation of the product on the left side, evaluating at $x = 0$, and using $\frac{x}{x+2k} \frac{d}{dx} \binom{x+2k}{k} \Big|_{x=0} = 0$ and $\frac{d}{dx} \frac{x}{x+2k} \Big|_{x=0} = \frac{1}{2k}$, we obtain

$$\frac{1}{2} \sum_{k=1}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} = \binom{2n}{n} (H_{2n} - H_n).$$

Moving the term for $k = n$ to the other side yields the desired equation

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} = \binom{2n}{n} \left(H_{2n} - \frac{1}{2n} - H_n \right) = \binom{2n}{n} (H_{2n-1} - H_n).$$

Editorial comment. Solvers used a variety of techniques, including integrations and Zeilberger's algorithm with telescoping sums. The proposer commented more generally that if $f(z) = \sum_{n=0}^{\infty} \binom{tn}{n} \frac{z^n}{(t-1)n+1}$, then the coefficient of z^n in $(\ln f(z))^p$ is the coefficient of x^{p-1} in $\frac{p!}{x+tn} \binom{x+tn}{n}$. Several solvers noted that $(\log C(z))^2$ was ambiguously typeset as $\log(C(z))^2$ in the original publication of the problem.

Also solved by T. Amdeberhan & V. H. Moll, D. Beckwith, R. Chapman (U. K.), H. Chen, R. Dutta (India), M. L. Glasser, P. Lalonde (Canada), K. D. Lathrop, K.-W. Lau (China), L. Matejíčka (Slovakia), M. Omarjee (France), B. Salvy (France), A. Stenger, R. Stong, R. Tauraso (Italy), C. I. Vălean (Romania), M. Wildon, and the proposer.

Napoleon into the Fray

11860 [2015, 801]. *Proposed by Dimitris Vartziotis, NIKI MEPE Digital Engineerings, Katsikas Ioannina, Greece.* Let ABC be a triangle. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Extend the ray DA beyond A to a point A', and similarly extend EB to B' and FC to C', in such a way that $\sqrt{3}|AA'| = |BC|$, $\sqrt{3}|BB'| = |CA|$, and $\sqrt{3}|CC'| = |AB|$. Prove that $A'B'C'$ is an equilateral triangle.

Solution I by Irl C. Bivens and L. R. King, Davidson College, Davidson, NC. Let A^* be the reflection of A through the midpoint of the opposite side BC, and let B^* and C^* be defined similarly. The triangles A^*CB, CB^*A, and BAC^* are all congruent to the original triangle ABC. Therefore A, B, and C are the midpoints of the sides of triangle $A^*B^*C^*$, and so ABC is the medial triangle of $A^*B^*C^*$. The six triangles such as $A'AB^*$ are all $1:\sqrt{3}:2$ right triangles, so the points A', B', and C' defined in the problem are the centers of the equilateral triangles constructed outward on the sides of triangle $A^*B^*C^*$. Therefore, the Napoleon theorem guarantees that $A'B'C'$ is equilateral.

Solution II by TCDmath Problem Group, Trinity College, Dublin, Ireland. Set the construction in the complex plane, and use a corresponding lower-case letter to denote the complex number associated with the point whose name is an upper-case letter. For definiteness, let ABC be oriented positively. Let $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$, so that $\omega^3 = (\omega^2)^3 = 1$, $(\omega^2)^2 = \omega$, and $\omega^2 = -1/2 - i\sqrt{3}/2$.

Because of the positive orientation of ABC, the angle from BC to AA' is $\pi/2$. Since $\sqrt{3}|AA'| = |BC|$, we have $\sqrt{3}(a' - a) = i(c - b)$. Thus $\sqrt{3}a' = \sqrt{3}a - bi + ci$. Similarly, $\sqrt{3}b' = \sqrt{3}b - ci + ai$ and $\sqrt{3}c' = \sqrt{3}c - ai + bi$, so $(\sqrt{3}i/2)(b' - a') = \omega^2 a + \omega b + c$. Similarly, $(\sqrt{3}i/2)(c' - b') = \omega^2 b + \omega c + a$ and $(\sqrt{3}i/2)(a' - c') = \omega^2 c + \omega a + b$. Hence $b' - a' = \omega^2(c' - b') = \omega(a' - c')$, so $|A'B'| = |B'C'| = |C'A'|$, and thus $A'B'C'$ is equilateral.

Editorial comment. The problem did not specify that ABC is acute. When it is obtuse, the orthocenter lies outside of it. Some solutions did not allow for this possibility.

Also solved by A. Ali (India), H. Bailey, M. Bataille (France), B. S. Burdick, E. Chadraa, M. V. Channakeshava (India), R. Chapman (U. K.), K. Charatsaris (Greece), C. Curtis, N. Curwen (U. K.), P. P. Dályay (Hungary), S. N. Dinh (Germany), C. Effenberger (Germany), A. Fanchini (Italy), B. Fritzching (Germany), O. Geupel (Germany), M. Goldenberg & M. Kaplan, A. Gretsistas (Greece), A. Häcker (Germany), M. Hanselmann (Germany), I. Held (Germany), J. G. Heuver (Canada), S. Hitotumatu (Japan), S. Huggenberger (Germany), Y. J. Ionin, H. Jung (Korea), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), M. Kronenwelt (Germany), M. D. Meyerson, V. Mikayelyan (Armenia), J. Minkus, D. J. Moore, K. Nikolaou (Greece), P. Nüesch (Switzerland), C. R. Pranesachar (India), M. Sawhney, J. Schlosberg, N. Stanciu & T. Zvonaru (Romania), R. Stong, T. Toyonari (Japan), T. Viteam (Japan), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

A Symmetric Bound

11867 [2015, 899]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* For real numbers a, b, c, let

$$f(a, b, c) = \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4}.$$

Prove $f(a, b, c) + f(b, c, a) + f(c, a, b) \le 3$.

Solution by Vazgen Mikayelyan, Yerevan State University, Armenia. For real numbers x, y, and z, we have

$$f(x, y, z) = \left(\frac{x^2}{x^2 - xy + y^2}\right)^{1/4} \le \left(\frac{|x|^2}{|x|^2 - |x||y| + |y|^2}\right)^{1/4}.$$

Hence we may assume that a, b, c are all nonnegative.

If one of the numbers a, b or c is zero, for example $c = 0$, then

$$f(a, b, c) + f(b, c, a) + f(c, a, b) = \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4} + 1.$$

Thus, in this case, we need to prove

$$\frac{a^2}{a^2 - ab + b^2} \le 2^4.$$

This holds because

$$\frac{a^2}{a^2 - ab + b^2} = \frac{1}{(\frac{b}{a} - \frac{1}{2})^2 + \frac{3}{4}} \le \frac{4}{3} < 2^4.$$

Therefore, we may assume that a, b, c are all positive numbers. Note that for positive numbers x, y, and z,

$$\begin{aligned} f(x, y, z) &= \left(\frac{x^2}{x^2 - xy + y^2}\right)^{1/4} = \left(\frac{4x^2}{(x + y)^2 + 3(x - y)^2}\right)^{1/4} \\ &\le \left(\frac{4x^2}{(x + y)^2}\right)^{1/4} \le \sqrt{\frac{2x}{x + y}}. \end{aligned}$$

It follows that

$$f(a, b, c) + f(b, c, a) + f(c, a, b) \le \sqrt{\frac{2a}{a + b}} + \sqrt{\frac{2b}{b + c}} + \sqrt{\frac{2c}{a + c}}.$$

Using the Cauchy–Schwarz inequality and the inequality $x + y \ge 2\sqrt{xy}$, we get

$$\begin{aligned} &\sqrt{\frac{2a}{a + b}} + \sqrt{\frac{2b}{b + c}} + \sqrt{\frac{2c}{a + c}} \\ &= \sqrt{a + c}\sqrt{\frac{2a}{(a + b)(a + c)}} + \sqrt{a + b}\sqrt{\frac{2b}{(b + c)(a + b)}} + \sqrt{b + c}\sqrt{\frac{2c}{(a + c)(b + c)}} \\ &\le \sqrt{2(a + b + c)\left(\frac{2a}{(a + b)(a + c)} + \frac{2b}{(b + c)(a + b)} + \frac{2c}{(a + c)(b + c)}\right)} \\ &= 2\sqrt{\frac{2(a + b + c)(ab + bc + ac)}{(a + b)(b + c)(a + c)}} = 2\sqrt{\frac{2(a + b)(b + c)(a + c) + 2abc}{(a + b)(b + c)(a + c)}} \\ &= 2\sqrt{2 + \frac{2abc}{(a + b)(b + c)(a + c)}} \le 2\sqrt{2 + \frac{2abc}{2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac}}} = 3. \end{aligned}$$

Also solved by A. Ali (India), T. Amdeberhan, P. Bracken, B. Bradie, M. V. Channakeshava (India), R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), P. W. Gwanyama, T. Horine, O. Kouba (Syria), J. H. Lindsey II, J. Loverde, P. W. Lindstrom, O. P. Lossers (Netherlands), L. Matejíčka (Slovakia), T. L. McCoy, M. Omarjee (France) & R. Tauraso (Italy), P. Perfetti (Italy), J. Schlosberg, R. Stong, T. Wiandt, M. R. Yegan (Iran), GCHQ Problem Solving Group (U. K.), and the proposer.

Center of Mass of Multiplicative Orbits in a Grid

11868 [2015, 899]. *Proposed by James Propp, University of Massachusetts, Lowell, MA.* For fixed positive integers a and b, let $m = ab - 1$ and let R be the set $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$, indexed as p_0 through p_m in lexicographic order, so that $p_0 = (1, 1)$, $p_1 = (1, 2)$, and $p_m = (a, b)$. Define T from R to R as the map that sends p_0 to p_0 and p_m to p_m, and for $1 \le i \le m - 1$ sends p_i to p_j where $j \equiv ai \pmod{m}$. As a bijection, T partitions R into orbits. Show that the center of mass of each orbit lies on the line joining p_0 and p_m.

Solution by Jamie Simpson, Murdoch University, Perth, Australia. If $p_n = (x, y)$, then $n = b(x - 1) + y - 1$. Choose an orbit S under T and let $\overline{n}$, $\overline{x}$, and $\overline{y}$ be the average values of n, x, and y in S. The center of mass of S is then $(\overline{x}, \overline{y})$. Note that

$$\overline{n} = b(\overline{x} - 1) + (\overline{y} - 1). \tag{1}$$

Letting n' be the successor of n in the orbit, we have

$$n' \equiv ab(x - 1) + a(y - 1) \equiv x - 1 + a(y - 1) \pmod{m}.$$

Since the right side lies in the interval $[1, m]$, we have $n' = x - 1 + a(y - 1)$. The average value of n' over S is $\overline{n}$, so

$$\overline{n} = \overline{x} - 1 + a(\overline{y} - 1). \tag{2}$$

Together, (1) and (2) yield

$$\overline{y} = \frac{b - 1}{a - 1}(\overline{x} - 1) + 1,$$

and this is the equation of the line through $(1, 1)$ and (a, b).

Editorial comment. The solution shows that the average value of $(a - 1)y - (b - 1)x$ on each orbit is $a - b$. The phenomenon that prevails when the time-average of some quantity is the same for all orbits of a dynamical system is called *homomesy* and occurs in a broad range of contexts. For a catalog of examples, see Propp and Roby, Homomesy in products of two chains, *Electronic J. Combinatorics* **22** (2015) 3.4. The particular map T in this problem can be interpreted as a "re-reading map": for all k it maps the kth element of $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$ in lexicographic order to the kth element of $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$ in colexicographic order.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), A. Hammett & K. A. Roper, Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Stong, the GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

An Optimal Hölder Exponent

11869 [2015, 899]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Prove that $|y \log y - x \log x| \le |y - x|^{1-1/e}$ for $0 < x < y \le 1$.

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI. Using elementary calculus, we see that the function $f(t) = t^{1/t}$ for $t > 0$ has its unique maximum when $t = e$. Therefore, for $t > 0$, we have $t^{1/t} \le e^{1/e}$, so $t \le e^{t/e}$ with equality only when $t = e$. Putting $t = \log u$, we have $\log u \le u^{1/e}$ for $u > 1$, with equality only when $u = e^e$. With $u = 1/v$ we have $-\log v \le v^{-1/e}$ for $0 < v < 1$, with equality only when $v = e^{-e}$.

Let x and y be real numbers satisfying $0 < x < y \le 1$, and consider the difference quotient $S = (g(y) - g(x))/(y - x)$, where $g(t) = t \log t$. Because g is convex on $[0, 1]$, the lower bound of S for fixed y occurs as $x \to 0$. Also $S \le 1$ by the mean value theorem. So we have $\log y = g(y)/y < S \le 1$, which gives us

$$|S| \le \max\{-\log y, 1\} \le \max\{y^{-1/e}, 1\} < |y-x|^{-1/e}.$$

This yields the desired inequality.

Editorial comment. Omran Kouba noted that $1 - 1/e$ is the best possible exponent in the inequality. This is seen by choosing $y = e^{-e}$ and letting x tend to 0.

Also solved by K. F. Andersen (Canada), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), O. Kouba (Syria), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), L. Matejíčka (Slovakia), T. L. McCoy, V. Mikayelyan (Armenia), M. Omarjee (France) & R. Tauraso (Italy), E. Schmeichel, A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), and NSA Problems Group.

Bounds Related to Convex Univalent Functions

11870 [2015, 900]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Suppose $0 \le x \le 1$ and $y = 1 - x$, and let a and b be unimodular complex numbers. Let $c_1 = 2(xa + yb)$ and $c_2 = 2(xa^2 + yb^2)$. Prove $\big||c_1^2 + c_2| - 3|c_1|\big| \le 3$, with equality if and only if $x = y = 1/2$ and $b\overline{a} = e^{2\pi i/3}$.

Solution by the GCHQ Problem Solving Group, Cheltenham, UK. Correction: The last equation should read $b\overline{a} = e^{\pm 2\pi i/3}$. We can see that the stated equation is incorrect, since the inequality is invariant under swapping (a, x) and (b, y).

Lemma. $|c_1^2 + c_2| \ge 2\big||c_1|^2 - 1\big|$, *with equality if and only if* $(x, y) \in \big\{(0, 1), (1/2, 1/2), (1, 0)\big\}$ *or* $a = b$.

Proof. Use $|z|^2 = z\overline{z}$, $x + y = 1$, $\overline{a} = a^{-1}$, $\overline{b} = b^{-1}$ to obtain

$$\begin{aligned}
|c_1^2 + c_2|^2 - 4\big(|c_1|^2 - 1\big)^2 &= (c_1^2 + c_2)(\overline{c_1}^2 + \overline{c_2}) - 4\big(c_1\overline{c_1} - 1\big)^2 \\
&= \big(4(xa + yb)^2 + 2(xa^2 + yb^2)\big)\big(4(x\overline{a} + y\overline{b})^2 + 2(x\overline{a}^2 + y\overline{b}^2)\big) \\
&\quad - 4\big(4(xa + yb)(x\overline{a} + y\overline{b}) - 1\big)^2 \\
&= \big(4(xa + yb)^2 + 2(x + y)(xa^2 + yb^2)\big)\big(4(xa^{-1} + yb^{-1})^2 \\
&\quad + 2(x + y)(xa^{-2} + yb^{-2})\big) - 4\big(4(xa + yb)(xa^{-1} + yb^{-1}) - (x + y)^2\big)^2 \\
&= 12x^3ya^2b^{-2} - 24x^2y^2a^2b^{-2} + 12xy^3a^2b^{-2} - 48x^3yab^{-1} + 96x^2y^2ab^{-1} \\
&\quad - 48xy^3ab^{-1} + 72x^3y - 144x^2y^2 + 72xy^3 - 48x^3ya^{-1}b + 96x^2y^2a^{-1}b \\
&\quad - 48xy^3a^{-1}b + 12x^3ya^{-2}b^2 - 24x^2y^2a^{-2}b^2 + 12xy^3a^{-2}b^2 \\
&= 12xy(x - y)^2(a - b)^2(a^{-1} - b^{-1})^2 = 12xy(x - y)^2(a - b)^2(\overline{a} - \overline{b})^2 \\
&= 12xy(x - y)^2|a - b|^4 \ge 0,
\end{aligned}$$

with equality if and only if $x = 0$, $y = 0$, $x = y$, or $a = b$. ■

Using the triangle inequality, we have $|c_1| \le 2x|a| + 2y|b| = 2$, $|c_2| \le 2x|a^2| + 2y|b^2| = 2$, and

$$|c_1^2 + c_2| - 3|c_1| \le |c_1|^2 + |c_2| - 3|c_1| \le 2|c_1| + 2 - 3|c_1| = 2 - |c_1| < 3.$$

Thus the upper bound 3 is always strict. Now if $|c_1| < 1$, then

$$|c_1^2 + c_2| - 3|c_1| > 0 - 3 \cdot 1 = -3.$$

Thus the lower bound of -3 is strict in this case. For the case $|c_1| \ge 1$, we use the lemma to obtain

$$|c_1^2 + c_2| - 3|c_1| \ge 2\big||c_1|^2 - 1\big| - 3|c_1| = 2\big(|c_1|^2 - 1\big) - 3|c_1|$$
$$= \big(2|c_1| - 1\big)\big(|c_1| - 1\big) - 3 \ge -3.$$

According to the lemma, equality holds in the first inequality here if and only if $(x, y) \in \big\{(0, 1), (1/2, 1/2), (1, 0)\big\}$ or $a = b$. However, since $|c_1| \ge 1$, equality holds in the second inequality if and only if $|c_1| = 1$. This rules out $(x, y) = (0, 1)$, $(x, y) = (1, 0)$, and $a = b$, because all these imply $|c_1| = 2$. Hence in the $|c_1| \ge 1$ case, the lower bound of -3 is achieved if and only if $x = y = 1/2$ and $|c_1| = 1$. Putting these together, we obtain

$$\big||c_1^2 + c_2| - 3|c_1|\big| \le 3,$$

with equality if and only if $x = y = 1/2$ and $|c_1| = 1$. This equality condition is equivalent to $x = y = 1/2$ and $|a + b| = 1$. Equation $|a + b| = 1$ is equivalent to $a\overline{a} + a\overline{b} + \overline{a}b + b\overline{b} = -1$, i.e., $2\mathrm{Re}(b\overline{a}) = a\overline{b} + \overline{a}b = -1$. Since $b\overline{a}$ is unimodular, this is equivalent to $b\overline{a} = e^{\pm 2\pi i/3}$.

Editorial comment. The proposer notes that this is a toy version of an open problem: namely, the problem of finding sharp bounds for the functionals $\big||a_{n+1}| - |a_n|\big|$, $n = 0, 1, 2, \ldots$, where $a_0 = 0$, $a_1 = 1$, and $\sum_{n=0}^{\infty} a_n z^n$ is convex univalent on the open disk $\{z : |z| < 1\}$. The corresponding problem for starlike functions was resolved by Yuk Leung in 1977.

Also solved by P. Bracken, P. P. Dályay (Hungary), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Stong, and the proposer.

Not All Angles are Rational Multiples of Pi

11871 [2015, 900]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Ştefan Spătaru, Harvard University, Boston, MA.* Let ABC be a triangle in the Cartesian plane with vertices in $\mathbb{Z}^2$ (*lattice vertices*). Show that, if P is an interior lattice point of ABC, then at least one of the angles PAB, PBC, and PCA has a radian measure that is not a rational multiple of π.

Solution by L. R. King, Davidson, NC. The only rational values of $\tan(k\pi/n)$ when k/n is rational are 0 and ± 1. (See J. S. Calcut, Gaussian integers and arctangent identities for π, this MONTHLY **116** (2009) 515–530.) Since every angle of a triangle embedded in the integer lattice $\mathbb{Z}^2$ is either a right angle or has rational tangent, the possible angle measure for such an angle is reduced to $\pi/4$, $\pi/2$, and $3\pi/4$. As the sum of the radian measures of the named angles must be less than π, each must be $\pi/4$.

Assume $\angle PAB = \angle PBC = \pi/4$. We show $\angle PCA < \pi/4$. If $\angle CAP \ge \pi/4$ or $\angle ABP \ge \pi/4$, then we immediately get $\angle PCA < \angle BCA \le \pi/4$, since the sum of the radian measures of the angles in triangle ABC is π. Therefore, assume $\angle CAP < \pi/4$ and $\angle ABP < \pi/4$, and let I be the incenter of triangle ABC, where its angle bisectors meet. Since $\angle CAP < \pi/4 = \angle PAB$, ray AP is internal to angle CAI. Likewise, $\angle PBA < \pi/4 = \angle PBC$, so ray BP is internal to angle ABI. We conclude that line CP is internal to $\angle ACI$, and thus $\angle PCA < \angle ACI = \angle ACB/2 < (\pi/2)/2 = \pi/4$. Therefore, at least one of the angles PAB, PBC, and PCA has a radian measure that is not a rational multiple of π.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), M. Goldenberg & M. Kaplan, T. Horine, Y. J. Ionin, O. P. Lossers (Netherlands), V. Pambuccian, R. Stong, R. Tauraso (Italy), J. Zacharias & R. Dempsey, GCHQ Problem Solving Group (U. K.), and the proposers.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
http://www.americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by February 28, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11999. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)}.$$

12000. *Proposed by Mehtaab Sawhney, student, Massachusetts Institute of Technology, Cambridge, MA.* Let $H_k = \sum_{i=1}^{k} 1/i$. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n} H_k}$$

has no real zeroes.

12001. *Proposed by Marius Coman, Bucharest, Romania, and Florian Luca, Johannesburg, South Africa.* A *base-2 pseudoprime* is an odd composite number n that divides $2^n - 2$. Prove that if p is a prime number greater than 13, then there is a base-2 pseudoprime that divides $2^{p-1} - 1$.

12002. *Proposed by Florin Stanescu, Gaesti, Romania.* Let ABC be a triangle with area S, circumradius R, circumcenter O, and orthocenter H. Let D be the point of intersection of lines AO and BC. Similarly, let E be the point of intersection of lines BO and CA, and let F be the point of intersection of lines CO and AB. Let $T = \sqrt{(3R^2 - OH^2)^2 + 16S^2}/R^2$. Prove

$$T \le \frac{AH}{OD} + \frac{BH}{OE} + \frac{CH}{OF} \le 3 + \frac{T}{2}.$$

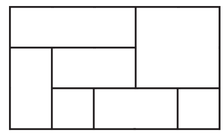
12003. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Given an odd positive integer n, compute

$$\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2(\pi k/n)}.$$

http://dx.doi.org/10.4169/amer.math.monthly.124.8.754

12004. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_1, a_2, \ldots$ be a strictly increasing sequence of real numbers satisfying $a_n \le n^2 \ln n$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} 1/(a_{n+1} - a_n)$ diverges.

12005. *Proposed by Donald E. Knuth, Stanford, CA.* A *tight m-by-n paving* is a decomposition of an m-by-n rectangle into $m + n - 1$ rectangular tiles with integer sides such that each of the $m - 1$ horizontal lines and $n - 1$ vertical lines within the rectangle is part of the boundary of at least one tile. For example, one of the 1,071 possible tight 3-by-5 pavings is pictured here:



Let $a_{m,n}$ denote the number of tight m-by-n pavings.
(a) Determine $a_{3,n}$ as a function of n.
(b) Show for $m \ge 3$ that $\lim_{n\to\infty} a_{m,n}/m^n$ exists, and compute its value.

SOLUTIONS

Another Mean Value Theorem

11872 [2015, 900]. *Proposed by Phu Cuong Le Van, Hue University, Hue, Vietnam.* Let f be a continuous function from [0, 1] into $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = 0$. Prove that for all positive integers n there exists $c \in (0, 1)$ such that $n\int_0^c x^n f(x)\,dx = c^{n+1} f(c)$.

Composite solution by Brian Bradie and Hongwei Chen, Christopher Newport University, Newport News, VA. In fact, n can be any positive real number.

If f is identically zero, then there is nothing to prove. Assume that $f(x)$ is not identically zero. Since $\int_0^1 f(x)\,dx = 0$, there exist distinct $a, b \in [0, 1]$ such that

$$f(a) = \max_{x\in[0,1]} f(x) > 0 \qquad \text{and} \qquad f(b) = \min_{x\in[0,1]} f(x) < 0.$$

Let

$$F(x) = \begin{cases} f(x) - \dfrac{n}{x^{n+1}} \displaystyle\int_0^x t^n f(t)\,dt, & x > 0; \\ f(0)\left(1 - \dfrac{n}{n+1}\right), & x = 0. \end{cases}$$

Since $\lim_{x\to 0^+} F(x) = F(0)$, the function F is continuous on [0, 1]. We claim $F(a) > 0$. This is clear when $a = 0$. For $a > 0$,

$$F(a) \ge f(a) - \frac{n}{a^{n+1}} \int_0^a t^n f(a)\,dt = \left(1 - \frac{n}{n+1}\right) f(a) > 0.$$

Similarly, $F(b) < 0$. The intermediate value theorem implies that there is a number c between a and b such that $F(c) = 0$. That is, $n\int_0^c x^n f(x)\,dx = c^{n+1} f(c)$.

Editorial comment. Chen noted that this problem is similar to Problem 11555 (February, 2011). The GCHQ Problem Solving Group showed that the hypothesis $\int f = 0$ can be replaced by the hypothesis that f has a zero in (0, 1). Moubinool Omarjee and Roberto Tauraso generalized the function x^n: If $u \in C^1[0, 1]$, $u(0) = 0$, and $u'(x) > 0$ for $x \in (0, 1)$, then there exists $c \in (0, 1)$ such that

$$\frac{u'(c)}{u(c)^2} \int_0^c u(x) f(x)\,dx = f(c).$$

Also solved by A. Ali (India), T. Amdeberhan, K. F. Andersen (Canada), M. W. Botsko, P. Bracken, K. Breeding & K. Bursac & C. Davis & T. McClanahan & R. Muller, B. S. Burdick, P. P. Dályay (Hungary), J. Dickerson & D. Harris & A. Young & J. Dodson, E. J. Ionaşcu, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), P. W. Lindstrom, O. P. Lossers (Netherlands), G. Macias & R. Smith, V. Mikayelyan (Armenia), M. Omarjee (France), M. Omarjee (France) & R. Tauraso (Italy), P. Perfetti (Italy), Á. Plaza (Spain), V. Rutherford-Rand, A. Stenger, R. Stong, T. Wiandt, GCHQ Problem Solving Group (U. K.), and the proposer.

Arithmetic Trigonometric Sums

11873 [2015, 1010]. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* Show that for $n \in \mathbb{N}$ with $n \geq 2$,

$$\sum_{j=1}^{n}\left(1-\frac{2j-1}{n}\right)\cot\frac{(2j-1)\pi}{2n} = \sum_{j=1}^{n}\csc j\pi n.$$

Correction: The right-hand side should have been $\sum_{j=1}^{n-1}\csc(j\pi/n)$.

Solution by Pierre Lalonde, Kingsey Falls, QC, Canada. Let z be an nth root of -1. Summing a geometric series gives

$$-\sum_{k=1}^{n-1} z^k = \frac{z-z^n}{z-1} = \frac{z+1}{z-1}.$$

If z is a $2n$th root of unity and $z \neq \pm 1$, then

$$\begin{aligned}(z^2-1)\sum_{j=1}^{n}(aj+b)z^{2j-1} &= \sum_{j=1}^{n}(aj+b)z^{2j+1} - \sum_{j=0}^{n-1}(aj+a+b)z^{2j+1}\\ &= (an+b)z^{2n+1} - bz - a\sum_{j=0}^{n-1}z^{2j+1} = anz - a\frac{z(z^{2n}-1)}{z^2-1} = anz.\end{aligned}$$

Therefore, setting $a=-2/n$ and $b=1+1/n$, we have $\sum_{j=1}^{n}\left(1-\frac{2j-1}{n}\right)z^{2j-1} = -\frac{2z}{z^2-1}$. Now let $\omega = e^{i\pi/2n}$. We compute

$$\begin{aligned}&\sum_{j=1}^{n}\left(1-\frac{2j-1}{n}\right)\cot\frac{(2j-1)\pi}{2n}\\ &\quad= i\sum_{j=1}^{n}\left(1-\frac{2j-1}{n}\right)\frac{\omega^{2(2j-1)}+1}{\omega^{2(2j-1)}-1} = -i\sum_{j=1}^{n}\left(1-\frac{2j-1}{n}\right)\sum_{k=1}^{n-1}\omega^{2(2j-1)k}\\ &\quad= -i\sum_{k=1}^{n-1}\sum_{j=1}^{n}\left(1-\frac{2j-1}{n}\right)\omega^{2(2j-1)k} = \sum_{k=1}^{n-1}\sum_{j=1}^{n}\frac{2i\omega^{2k}}{\omega^{4k}-1} = \sum_{k=1}^{n-1}\csc\frac{k\pi}{n}.\end{aligned}$$

Editorial comment. The GCHQ Problem Solving Group gave the generalization

$$\sum_{j=1}^{n}\frac{\sin((n-2j+1)y)}{\sin ny}\cot\frac{(2j-1)\pi}{2n} = \sum_{j=1}^{n-1}\csc\left(y+\frac{j\pi}{n}\right),$$

for any y that is not an integer multiple of π/n.

Also solved by T. Amdeberhan, R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), O. Kouba (Syria), M. Omarjee (France), S. Pathak (Canada), N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Minimal Zeta Knowledge Required

11874 [2015, 1010]. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Evaluate the limits below, where ζ denotes the Riemann zeta function and Γ denotes the gamma function:

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)}, \qquad \lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{\zeta(n-k)}{\Gamma(k)}.$$

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. The limits are e and $1/e$, respectively. More generally, let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be real numbers, and assume that $\lim_{n\to\infty} a_n$ exists and equals a, that some b_k is nonzero, and that $\sum_{k=1}^{\infty}|b_k|$ exists and equals B. When $n > N+1$,

$$\sum_{k=1}^{n-1} a_k b_{n-k} - a\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{N} a_k b_{n-k} + \sum_{k=N+1}^{n-1}(a_k - a)b_{n-k} - a\sum_{k=n-N}^{\infty} b_k. \qquad (*)$$

We show that the left side of $(*)$ tends to 0 as $n\to\infty$.

As $n\to\infty$, the first and third terms on the right side of $(*)$ tend to 0. Given $\epsilon > 0$, we can choose N so that $|a_k - a| < \epsilon/B$ for $k > N$. It then follows (applying the triangle inequality to the second term on the right) that

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{n-1} a_k b_{n-k} - a\sum_{k=1}^{\infty} b_k\right| < \epsilon.$$

Hence

$$\lim_{n\to\infty}\sum_{k=1}^{n-1} a_k b_{n-k} = a\sum_{k=1}^{\infty} b_k.$$

Letting $a_1 = 0$ and $a_k = \zeta(k)$ for $k \ge 2$, we have $a = 1$. For the first limit, set $b_k = 1/\Gamma(k) = 1/(k-1)!$ to obtain the value e. Since

$$\sum_{k=1}^{n-2}\frac{(-1)^{k-1}\zeta(n-k)}{\Gamma(k)} = \sum_{k=2}^{n-1}\frac{(-1)^{n-k-1}\zeta(k)}{\Gamma(n-k)},$$

setting $b_k = (-1)^{k-1}/\Gamma(k)$ shows that the second limit is $1/e$.

Editorial comment. One can avoid the epsilon argument by forcing the problem into the mold of the dominated convergence theorem or Tannery's theorem. Another approach is to work directly with inequalities. O. P. Lossers deduced the first limit directly from

$$\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)} - \sum_{k=2}^{n-1}\frac{1}{\Gamma(n-k)} < \frac{3}{2^{n/2}}e + \frac{3/2}{\Gamma(n/2)}$$

and the second limit from a similar inequality. As in Problem 11755 [2014, 170], there is a sophisticated approach using polylogarithms. The solution of Rituraj Nandan used the representation

$$\frac{1}{(n-1)!}\int_0^\infty \frac{x^{n-1}}{ze^x - 1}dx = \mathrm{Li}_n(1/2).$$

Also solved by T. Amdeberhan, P. Bracken, R. Chapman (U. K.), P. P. Dályay, D. Fleischman, M. L. Glasser, O. Kouba (Syria), L. Liptak, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), M. Omarjee (France) & R. Tauraso (Italy), C. Pathak (Canada), C. M. Russell, M. Sawhney, A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Stirling to the Rescue

11875 [2015, 1010]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $f_n = (1 + 1/n)^n\, ((2n-1)!!L_n)^{1/n}$. Find $\lim_{n\to\infty}(f_{n+1} - f_n)$. Here $n!! = \prod_{j=0}^{\lfloor (n-1)/2\rfloor}(n-2j)$, while L_n denotes the nth Lucas number, given by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \ge 2$.

Solution by László Lipták, Oakland University, Rochester, MI. The answer is $1 + \sqrt{5}$, which equals twice the golden ratio φ. We find the limit by giving an estimate of f_n with error $O(1/n)$. From Stirling's formula, we get

$$(2n-1)!! = \frac{(2n)!}{2^n n!} = \frac{\sqrt{2\pi}\,(2n)^{2n+\frac{1}{2}}\, e^{-2n}\, e^{O(1/n)}}{2^n\sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{O(1/n)}} = \sqrt{2}\left(\frac{2n}{e}\right)^n e^{O(1/n)}.$$

Hence $((2n-1)!!)^{1/n} = \sqrt[2n]{2}\,\frac{2n}{e}\, e^{O(1/n^2)}$. Since

$$\sqrt[2n]{2} = e^{\frac{\ln 2}{2n}} = 1 + \frac{\ln 2}{2n} + O\left(\frac{1}{n^2}\right),$$

this gives

$$((2n-1)!!)^{1/n} = \frac{2n}{e}\left(1 + \frac{\ln 2}{2n} + O\left(\frac{1}{n^2}\right)\right).$$

Similarly, using the estimate $\ln(1+x) = x - \frac{x^2}{2} + O(x^3)$ as $x \to 0$, we get

$$\left(1 + \frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)} = e^{1-\frac{1}{2n}+O\left(\frac{1}{n^2}\right)} = e\left(1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right).$$

Finally, $L_n = \varphi^n + (-\varphi)^{-n}$, so

$$L_n^{1/n} = \varphi\left(1 + (-\varphi^2)^{-n}\right)^{1/n} = \varphi\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

Hence as $n \to \infty$, we get

$$\begin{aligned} f_n &= e\left(1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right)\frac{2n}{e}\left(1 + \frac{\ln 2}{2n} + O\left(\frac{1}{n^2}\right)\right)\varphi\left(1 + O\left(\frac{1}{n^2}\right)\right) \\ &= 2n\varphi + \varphi(\ln 2 - 1) + O\left(\frac{1}{n}\right). \end{aligned}$$

This implies $f_{n+1} - f_n = 2\varphi + O\left(\frac{1}{n}\right)$; hence $\lim_{n\to\infty}(f_{n+1} - f_n) = 2\varphi$.

Also solved by T. Amdeberhan, R. Boukharfane (France), B. S. Burdick, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), D. Fleischman, O. Kouba (Syria), J. H. Lindsey II, V. Mikayelyan (Armenia), Á. Plaza (Spain), M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, J. Zacharias, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Limit Equals 1

11877 [2015, 1011]. *Proposed by George Stoica, University of New Brunswick, Saint John, NB, Canada.* Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}^+$ such that $\lim_{x\to\infty} x\frac{f'(x)}{f(x)} = 0$. Let g be a function on $\mathbb{R}$ such that $\lim_{x\to\infty} g(x) > -1$. Prove

$$\lim_{x\to\infty} \frac{f(x + xg(x))}{f(x)} = 1.$$

Solution by NSA Problems Group, Fort Meade, MD. Since $f(x) > 0$ for all x, we may define a function F by $F(x) = \log(f(x))$. Suppose $\lim_{x\to\infty} g(x) = -1 + \alpha$, where $\alpha > 0$. Observe that

$$\frac{f(x + xg(x))}{f(x)} = \exp[F(x + xg(x)) - F(x)].$$

Thus it suffices to prove $F(x + xg(x)) - F(x) \to 0$ as $x \to \infty$. To see this, note that for x much greater than zero, we have $\alpha/2 < 1 + g(x)$. Hence, for large x it follows that $x + xg(x) > \alpha x/2$. Since $\lim_{x\to\infty} xF'(x) = 0$, when $\epsilon > 0$ the bound $|xF'(x)| < \epsilon$ must hold for all sufficiently large x. Hence,

$$\begin{aligned}\left|F(x + xg(x)) - F(x)\right| &= \left|\int_x^{x+xg(x)} F'(t)\,dt\right| \le \left|\int_x^{x+xg(x)} |F'(t)|\,dt\right| \\ &\le \left|\int_x^{x+xg(x)} \frac{\epsilon}{t}\,dt\right| = \epsilon\,|\log(1 + g(x))|.\end{aligned}$$

However, $1 + g(x)$ approaches α as $x \to \infty$, so

$$|F(x + xg(x)) - F(x)| \le 2\epsilon\,|\log(\alpha)|$$

for sufficiently large x. Since $\epsilon > 0$ is arbitrary, this establishes the claim.

Also solved by K. F. Andersen (Canada), N. H. Bingham & A. Ostaszewski & A. Sasane, R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, N. Grivaux (France), E. A. Herman, B. Karaivanov (U. S. A) & T. S. Vassilev (Canada), O. Kouba (Syria), O. P. Lossers (Netherlands), T. L. McCoy, V. Mikayelyan (Armenia), E. Omey (Belgium), C. G. Petalas (Greece), A. Stenger, R. Stong, E. I. Verriest, GCHQ Problem Solving Group (U. K.), and the proposer.

A Medley of Chords

11878 [2015, 1011]. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* Consider a circle w and an exterior point D. Let B and F be the points where lines through D are tangent to w. Let E be another point exterior to w on line BF, and similarly let A and C be the points where lines through E are tangent to w.

(**a**) Prove that D, A, and C are collinear.

(**b**) Let KL be a chord passing through the intersection N of chords AC and BF. Prove that lines DK and EL intersect at a point R on w.

(**c**) Find choices of K and L on w that minimize, respectively maximize, the measure of angle KRL.

Solution to (**a**) *by S. Hitotumatu, Kyoto University, Kyoto, Japan.* Let w be the unit circle in the Cartesian plane. Let $D = (a, b)$ and $E = (c, d)$ lie outside w. The equation of the chord through the ends of the tangent segments from D is $ax + by = 1$. Likewise, the equation of the chord through the ends of the tangent segments from E is $cx + dy = 1$. The condition for E to lie on the first line is $ac + bd = 1$. This is also the condition for D to lie on the second line, so each implies the other.

Solution I to (**b**) *by Richard Stong, Center for Communications Research, San Diego, CA.* Let the tangents to w at K and L meet at P. Note that D, E, and P all lie on the polar to N. Applying Pascal's theorem (the theorem of the "mystic hexagon") to the (degenerate) hexagons $BBLFFK$ and $KKBLLF$, we see that the points $KB \cap LF$ and $LB \cap KF$ are collinear with both D and P. Hence, they lie on the polar of N as well. Let $R = DK \cap EL$. Opposite sides of the (degenerate) hexagon $RKBFFL$ meet at the collinear points $RK \cap EF = D$, $KB \cap LF$, and $BF \cap LR = E$. It follows from the Braikenridge–MacLaurin theorem (the converse of Pascal's theorem) that this hexagon is inscribed in a conic. Since five points determine a conic uniquely, this conic must be w. That is, R is on w as claimed.

Solution II to (**b**) *by the proposer.* Let us make a projective transformation of the plane to send line DE to infinity. The circle may become an ellipse, but then an affine transformation can convert it back to a circle. (We will continue to use the original notations for their images under these transformations.) Both BF and AC are diameters, and they are perpendicular. Also, N is the center of w so any chord KL passing through N is also a diameter. Since DK and EL are perpendicular (because they are respectively parallel to AC and BF), R lies on w by the converse of the theorem that a triangle inscribed in a semicircle is right.

Solution to (**c**) *by S. Hitotumatu.* The extreme values of angle KRL occur when KL is as short as possible, which means that KL must be perpendicular to a diameter through N. The angle will be maximized when R is on the minor arc between K and L, and it will be minimized when R is on the major arc. (The two cases result from interchanging K and L.) Note that the maximum and minimum angles are supplementary.

Editorial comment. Many readers pointed out that part (**a**) is essentially the content of La Hire's theorem, which says that a point X is on the polar of point Y if and only if Y is on the polar of X. Here, because DB and DF are tangent segments to w from D, we see that BF is the polar of D, and similarly AC is the polar of E. Thus if E lies on BF, then D lies on AC.

One or more parts also solved by A. Ali (India), B. S. Burdick, J. Cade, R. Chapman (U. K.), P. Chrysostom (Greece), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), P. Nüesch (Switzerland), M. Sawhney, L. Wimmer (Germany), and GCHQ Problem Solving Group (U.K.).

Law of Sines Conversely

11879 [2015, 1011]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* For positive a, b, and c, prove that there exist positive α, β, and γ with $\alpha + \beta + \gamma = \pi$ such that

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

if and only if $|b - c| < a < b + c$.

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. Assume that $|b - c| < a < b + c$, which after resolving the absolute value reduces to the three triangle inequalities $c < a + b$, $b < a + c$, and $a < b + c$. In this case, there is a triangle with side lengths a, b, and c. The opposite angles α, β, and γ satisfy the requirements by the law of sines.

Conversely, assume that angles with the given requirements exist. Take a line segment of length a, and draw half-lines through the endpoints subtending angles β and γ, respectively. Because $\alpha + \beta + \gamma = \pi$, we obtain a triangle with angles α, β, and γ and sides a, b', and c'. Using the law of sines for this triangle, we obtain $b = b'$ and $c = c'$. Hence

a, b, and c are the side-lengths of a triangle, and they satisfy the three triangle inequalities. Therefore $|b - c| < a < b + c$.

Also solved by A. Ali (India), K. F. Andersen (Canada), D. Bailey & E. Campbell & C. Diminnie, B. S. Burdick, R. Chapman (U. K.), H. J. Cho (Korea), J. Christopher, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), M. Hajja (Jordan), S. Hitotumatu (Japan), T. Horine, Y. J. Ionin, B. Karaivanov (U. S. A) & T. S. Vassilev (Canada), P. Kohn, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), T. L. McCoy, V. Mikayelyan (Armenia), A. Nakhash, M. G. Park & I. G. Yang (Korea), C. Petalas (Greece), M. Sawhney, R. Stong, R. Tauraso (Italy), D. B. Tyler, E. I. Verriest, Z. Vörös (Hungary), H. Widmer (Switzerland), M. R. Yegan (Iran), J. Zacharias, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), NSA Problems Group, Seton Hall Problem Solving Group, Skidmore College Problem Group, and the proposer.

A Parallelogram Circumscribing a Quadrilateral

11880 [2016, 97]. *Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let $ABCD$ be any plane quadrilateral (not necessarily convex or even simple). Let a parallelogram be created by constructing through the ends of each diagonal of $ABCD$ lines parallel to the other diagonal. Show that each diagonal of this parallelogram passes through the intersection point of a pair of opposite sides of $ABCD$.

Editorial comment. Several readers observed that if AC is parallel to BD, then the desired parallelogram cannot be constructed. Also, if two sides of the quadrilateral are parallel, then they must be considered to meet "at infinity," in which case the claim reduces to showing that the corresponding diagonal of the parallelogram also meets these sides "at infinity" (that is, is parallel to them).

In each of the three solutions below, we let $EFGH$ be the parallelogram with E the intersection of the sides through A and B, F the intersection of the sides through B and C, etc. It suffices to establish the claim for only one diagonal of the parallelogram; the claim regarding the other diagonal is established similarly.

Solution I by Victor Pambuccian, Arizona State University, Glendale, AZ. Let AB intersect FH at point P and CD intersect FH at point Q. Apply the Menelaus theorem to triangle EFH crossed by line ABP to obtain

$$\frac{EB}{BF} \cdot \frac{FP}{PH} \cdot \frac{HA}{AE} = -1.$$

Then apply the Menelaus theorem to triangle GFH crossed by line DCQ to obtain

$$\frac{GC}{CF} \cdot \frac{FQ}{QH} \cdot \frac{HD}{DG} = -1.$$

Because opposite sides of a parallelogram are equal in length, we have $AE = CF$, $HA = GC$, $BF = DG$, and $EB = HD$. Therefore, $FP/PH = FQ/QH$, so $P = Q$.

Solution II by Li Zhou, Polk State College, Winter Haven, FL. The dual in the projective plane of the theorem of Pappus states the following: Let a_1, a_2, a_3 be three concurrent lines, and let b_1, b_2, b_3 be three concurrent lines. Define c_1 as the line through $a_2 \cap b_3$ and $a_3 \cap b_2$, and define c_2 and c_3 similarly. The lines c_1, c_2, c_3 are concurrent.

Let EH, BD, and FG be the lines a_1, a_2, and a_3, respectively. They meet at infinity. Let EF, AC, and HG be the lines b_1, b_2, and b_3, respectively. They also meet at infinity. The lines c_1, c_2, and c_3 from the dual to Pappus's theorem are CD, FH, and AB, respectively. Hence they are concurrent, by the theorem. That is, the diagonal FH of the parallelogram passes through the intersection of sides AB and CD.

Solution III by the proposer. Consider the quadrilateral $ABDC$. Its diagonals are AD and BC. Let M be the midpoint of diagonal AD. Let N be the midpoint of diagonal BC. Let sides AB and DC meet at P, let sides BD and CA meet at Q, and let the midpoint of PQ be O. The points M, N, O are collinear: The line through them is known as the Newton–Gauss line of the quadrilateral $ABDC$. Now consider the dilation by factor 2 about Q. The image of M under this dilation is H because M is the midpoint of diagonal AD of paralellogram $AHDQ$. Hence M is also the midpoint of the other diagonal QH. Similarly, the image of N is F. The image of O is P because O is the midpoint of PQ. Hence H, P, and F are collinear. That is, the diagonal HF of the parallelogram $EFGH$ passes through P, which is the intersection of sides AB and DC. This is the claim to be proved.

Also solved by A. Ali (India), M. Bataille (France), B. S. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), J.-P. Grivaux (France), S. Hitotumatu (Japan), Y. J. Ionin, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, R. Pembroke, M. Sawhney, J. Schlosberg, J. C. Smith, R. Stong, B. D. Suceavă, R. Tauraso (Italy), T. Viteam (Denmark), GCHQ Problem Solving Group (U. K.), and Missouri State University Problem Solving Group.

Avoiding Left-Full Entries

11882. [2016, 97]. *Proposed by David Callan, University of Wisconsin, Madison, WI.* In a list of distinct positive integers, say that an entry a is *left-full* if the entries to the left of a include $1, \dots, a-1$. For example, the left-full entries in 241739 are 1 and 3. Show that the number of arrangements of n elements from $\{1, 2, \dots, 2n+1\}$ that contain 1 but no other left-full entry is equal to $(2n-1)!/n!$ times the sum of the entries of the $n \times n$ Hilbert matrix M with $M_{i,j} = 1/(i+j-1)$. (The seven arrangements for $n = 2$ are 13, 14, 15, 21, 31, 41, and 51.)

Solution by Adnan Ali, student, Atomic Energy Central School–4, Mumbai, India. The Hilbert matrix is constant along each antidiagonal, and the entries in each of the first n antidiagonals sum to 1. Thus, the entries of the matrix sum to $n + \sum_{j=1}^{n-1} \frac{n-j}{n+j}$.

Let S denote the set of all arrangements with the required properties, and let S_k denote the subset of S with 1 in position k (from the left). To construct an element of S_k, we first permute $k-1$ of the elements $2, 3, \dots, 2n+1$ and place them to the left of 1. Next we permute $n-k$ of the remaining elements and place them to the right of 1; to have a desired arrangement, it is necessary and sufficient that the latter $n-k$ elements exclude the smallest element not appearing to the left of 1 (since it would be left-full). This gives

$$|S_k| = \binom{2n}{k-1}(k-1)!\binom{2n-k}{n-k}(n-k)! = \frac{(2n-1)!}{n!}\frac{2n}{2n-k+1}.$$

Now

$$\begin{aligned} |S| = \sum_{k=1}^{n} |S_k| &= \sum_{k=1}^{n} \frac{(2n-1)!}{n!}\frac{2n}{2n-k+1} \\ &= \frac{(2n-1)!}{n!}\sum_{k=1}^{n}\left(1 + \frac{k-1}{2n-k+1}\right) \\ &= \frac{(2n-1)!}{n!}\left(n + \sum_{j=1}^{n-1}\frac{n-j}{n+j}\right). \end{aligned}$$

Also solved by D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), K. David, Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), J. C. Smith, R. Stong, R. Tauraso (Italy), L. Zhou, Armstrong Problem Solvers, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by May 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12013. *Proposed by David Stoner, student, Harvard University, Cambridge, MA.* Suppose that a, b, c, d, e, and f are nonnegative real numbers that satisfy $a+b+c=d+e+f$. Let t be a real number greater than 1. Prove that at least one of the inequalities

$$\begin{aligned} a^t+b^t+c^t &> d^t+e^t+f^t, \\ (ab)^t+(bc)^t+(ca)^t &> (de)^t+(ef)^t+(fd)^t, \text{ and} \\ (abc)^t &> (def)^t \end{aligned}$$

is false.

12014. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let a, b, c, and d be real numbers with $bc>0$. Calculate

$$\lim_{n\to\infty}\begin{bmatrix}\cos(a/n) & \sin(b/n)\\ \sin(c/n) & \cos(d/n)\end{bmatrix}^n.$$

12015. *Proposed by Dao Thanh Oai, Kien Xuong, Vietnam.* Let ABC be a triangle, let G be its centroid, and let D, E, and F be the midpoints of BC, CA, and AB, respectively. For any point P in the plane of ABC, prove

$$PA+PB+PC\le 2(PD+PE+PF)+3PG,$$

and determine when equality holds.

12016. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For nonnegative integers m, n, r, and s, prove

$$\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{r+k}{k}\binom{s}{k}=\sum_{k=0}^{r}\binom{m+s}{n-k}\binom{s+k}{k}\binom{r}{k}.$$

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12017. *Proposed by Mowaffaq Hajja, Philadelphia University, Amman, Jordan.* For $n \geq 2$, let R be the ring $F[t_1, \ldots, t_n]$ of polynomials in n variables over a field F. For j with $1 \leq j \leq n$, let $s_j = \sum \prod_{i=1}^{j} t_{m_i}$, where the sum is taken over all j-element subsets $\{m_1, \ldots, m_j\}$ of $\{1, \ldots, n\}$. This is the *elementary symmetric polynomial* of degree j in the variables $t_1, \ldots, t_n$. Let $f = \sum_{i=0}^{n} c_i s_i$ for some $c_0, \ldots, c_n$ in F with $c_1, \ldots, c_n$ not all 0. Show that f is reducible in R if and only if either $c_0 = \cdots = c_{n-1} = 0$ or $(c_0, \ldots, c_n)$ is a geometric progression, meaning that there is $r \in F$ such that $c_i = rc_{i-1}$ for all i with $1 \leq i \leq n$.

12018. *Proposed by Zachary Franco, Houston, TX.* For $n > 1$, let $k(n)$ be the largest integer k for which there exists a triangle with sides of length n^k, $(n+4)^k$, and $(n+5)^k$. Find $\lim_{n\to\infty} k(n)/n$.

12019. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Find all positive integers n such that $(2^n - 1)(5^n - 1)$ is a perfect square.

SOLUTIONS

Almost-Binary Expansions

11883 [2016, 97]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $|q| > 1$, prove that

$$\sum_{k=0}^{\infty} \frac{1}{(q^{2^0}+q)(q^{2^1}+q)\cdots(q^{2^k}+q)} = \frac{1}{q-1}\prod_{i=0}^{\infty} \frac{1}{q^{1-2^i}+1}.$$

Solution I by Adnan Ali (student), Atomic Energy Central School–4, Mumbai, India. Setting $q = 1/x$ converts the assertion to

$$\sum_{k=0}^{\infty}\prod_{j=0}^{k} \frac{x^{2^j}}{(1+x^{2^j-1})} = \frac{x}{1-x}\prod_{i=0}^{\infty}\frac{1}{1+x^{2^i-1}}$$

for $|x| < 1$. After rearranging and summing the exponents in the numerator, we seek

$$\frac{1}{x}\left(\prod_{i=0}^{\infty}(1+x^{2^i-1})\right)\sum_{k=0}^{\infty}\frac{x^{2^{k+1}-1}}{\prod_{j=0}^{k}(1+x^{2^j-1})} = \frac{1}{1-x}.$$

The left side of this equation simplifies to $\sum_{k=0}^{\infty} x^{2^{k+1}-2}\prod_{j=k+1}^{\infty}(1+x^{2^j-1})$.

Letting $F_n(x) = \sum_{k=0}^{n} x^{2^{k+1}-2}\prod_{j=k+1}^{n}(1+x^{2^j-1})$, where an empty product is 1, observe that $F_0(x) = 1$ and that $F_n(x) = (1+x^{2^n-1})F_{n-1}(x) + x^{2^{n+1}-2}$ for $n \geq 1$. Hence it follows by induction on n that $F_n(x) = \sum_{k=0}^{2^{n+1}-2} x^k$. Letting $n \to \infty$ yields both sides of the desired identity.

Solution II by GCHQ Problem Solving Group, Cheltenham, U. K. In Solution I, the identity is reduced to

$$\sum_{k=0}^{\infty} x^{2^{k+1}-1}\prod_{j=k+1}^{\infty}(1+x^{2^j-1}) = \frac{x}{1-x}.$$

As a formal power series, this is the statement that every positive integer has a unique expression as a sum of distinct numbers of the form $2^j - 1$ for $j \geq 1$, except that the smallest number used (expressed as $2^{k+1}-1$) can appear once or twice. We establish this by partitioning the positive integers into blocks of the form $[2^k - 1, 2^{k+1} - 2]$ for $k \geq 1$ and using induction on k.

For $k = 1$, the block is $[1, 2]$, and the partitions are 1 and $1 + 1$. For the block $[2^k - 1, 2^{k+1} - 2]$, one obtains such partitions by using $2^k - 1$ and $(2^{k+1} - 1) + (2^{k+1} - 1)$ for the two extreme elements and adding the part $2^k - 1$ to the partitions already found for 1 through $2^k - 2$. For uniqueness, note that the largest number that can be so partitioned without using a number at least $2^k - 1$ is in fact $2^k - 2$. Since $2^{k+1} - 1$ is too big for numbers in the block $[2^k - 1, 2^{k+1} - 2]$, partitions of numbers in this block must use one copy of $2^k - 1$, and then uniqueness follows inductively.

Given the identity as a formal power series, it then suffices to observe that the Taylor series for $x/(1 - x)$ converges when $|x| < 1$.

Also solved by T. Amdeberhan, P. Bracken, B. Bradie, R. Chapman (U. K.), P. P. Dályay (Hungary), R. S. Dubey, R. Dutta (India), O. Geupel (Germany), W. P. Johnson, O. Kouba (Syria), H. Kwong, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), M. Sawhney, J. C. Smith, R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

It's a Quartic Equation

11890 [2016, 197]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Find all x in $(1, \infty)$ such that

$$\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{1}{x^{2k-1}} + \left(\frac{x-1}{x+1}\right)^{2k-1}\right) = \frac{1}{2}\int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

Solution by Thomas Horine, Indiana University Southeast, New Albany, IN. The right side equals $\frac{1}{2}\sinh^{-1} x$, which equals $\frac{1}{2}\ln(x + \sqrt{x^2+1}\,)$. For $|x| < 1$, let $f(x) = \sum_{k=1}^{\infty} x^{2k-1}/(2k-1)$. This series converges absolutely, and $f'(x) = \sum_{k=0}^{\infty} x^{2k} = 1/(1 - x^2)$, so

$$f(x) = \int_0^x \frac{dt}{1-t^2} = \tanh^{-1} x = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|.$$

When $x > 1$, both $\frac{1}{x}$ and $\frac{x-1}{x+1}$ are in $(0, 1)$, so $f(\frac{1}{x}) = \frac{1}{2}\ln|\frac{x+1}{x-1}|$ and $f(\frac{x-1}{x+1}) = \frac{1}{2}\ln|x|$. Since the left side of the original equation is $f(\frac{1}{x}) + f(\frac{x-1}{x+1})$, we need to solve

$$\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\ln|x| = \frac{1}{2}\ln(x + \sqrt{x^2+1}\,),$$

which reduces to $x^4 - 2x^3 - 2x^2 - 2x + 1 = 0$. Dividing this equation by x^2 yields $x^2 + \frac{1}{x^2} - 2(x + \frac{1}{x}) - 2 = 0$. With $u = x + \frac{1}{x}$, we have $u^2 - 2u - 4 = 0$. Since $u > 1$, this implies $u = 1 + \sqrt{5}$. Finally, the only solution of $1 + \sqrt{5} = x + \frac{1}{x}$ with $x > 1$ is $x = \frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}$.

Also solved by A. Ali (India), R. Amdeberhan, K. F. Andersen (Canada), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), B. S. Burdick, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), B. E. Davis, P. De (India), O. Geupel (Germany), M. L. Glasser, M. Goldenberg & M. Kaplan, N. Grivaux (France), A. Hannan (India), E. A. Herman, R. Howard, B. Karaivanov (U. S. A.) & T. Vassilev (Canada), O. Kouba (Syria), D. López-Aguayo (Mexico), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), M. Panchatcharam (Ireland), R. Pratt, M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, M. Vowe (Switzerland), T. Wiandt, J. Zacharias, L. Zhou, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, San Francisco University High School Problem Solving Group, and the proposer.

A Mean-Value Point

11892 [2016, 198]. *Proposed by Francisco Perdomo and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.* Let f be a real-valued continuously differentiable function on $[a, b]$ with positive derivative on (a, b). Prove that for all pairs (x_1, x_2) with $a \leq x_1 < x_2 \leq b$ and $f(x_1)f(x_2) > 0$, there exists $\xi \in (x_1, x_2)$ such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \xi - \frac{f(\xi)}{f'(\xi)}.$$

Solution by Henry Ricardo, Tappan, NY. Suppose that (x_1, x_2) is a pair satisfying the given conditions. We note that $f'(x) > 0$ on (a, b) and $f(x_1)f(x_2) > 0$ imply $f(x) \neq 0$ for $x \in [x_1, x_2]$. Thus, if we define $F(x) = -x/f(x)$ and $G(x) = -1/f(x)$, then F and G are continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), with $G'(x) \neq 0$ for $x \in (x_1, x_2)$. Therefore, we may apply Cauchy's extended mean value theorem to conclude that there exists $\xi \in (x_1, x_2)$, such that

$$\frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)} = \frac{F'(\xi)}{G'(\xi)}.$$

This yields

$$\frac{-\dfrac{x_2}{f(x_2)} + \dfrac{x_1}{f(x_1)}}{-\dfrac{1}{f(x_2)} + \dfrac{1}{f(x_1)}} = \frac{\dfrac{-f(\xi) + \xi f'(\xi)}{(f(\xi))^2}}{\dfrac{f'(\xi)}{(f(\xi))^2}},$$

which simplifies to the desired equation.

Also solved by A. Ali (India), T. Amdeberhan, K. F. Andersen (Canada), G. Apostolopoulos (Greece), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Doncal (Spain), R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), J. Grivaux (France), J. W. Hagood, L. Han, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Markoe, V. Mikayelyan (Armenia), M. Omarjee (France), C. G. Petalas (Greece), J. C. Smith, R. Stong, R. Tauraso (Italy), E. I. Verriest, Z. Vőrős (Hungary), T. Wiandt, M. R. Yegan (Iran), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposers.

Constructing an Inscribed Quadrilateral

11893 [2016, 198]. *Proposed by Florin S. Pârvănescu, Slatina, Romania.* Let O be the center of a circle, let AB and CD be the perpendicular chords of this circle that do not contain O, let M be the intersection of these chords, and suppose that MA is longer than MB and MC is longer than MD. Give a compass and straightedge construction of a quadrilateral inscribed in the circle with sides of lengths $|MA| + |MB|$, $|MC| + |MD|$, $|MA| - |MB|$, and $|MC| - |MD|$.

Solution by James Christopher Smith, Knoxville, TN. Using a straightedge, draw ray AO and let E be its other intersection with the circle. Use a compass spiked at M to produce point F on AM such that $|MF| = |MB|$. Then use a compass spiked at A to produce point G on the circle such that $|AG| = |AF|$. Now $ABEG$ is a quadrilateral with the desired side lengths. (There are two choices for point G; one choice leads to a self-intersecting polygon, whereas the other does not. Either will meet the required conditions.)

Note that AB has length $|MA| + |MB|$ and that AG has the same length as AF, which is $|MA| - |MB|$. We show next that $|BE| = |MC| - |MD|$ and $|EG| = |MC| + |MD|$.

Let ray BO intersect the circle again at P, and let PE intersect CD at N. By symmetry, $|NC| = |MD|$, so $|MN| = |MC| - |MD|$. Since ABE is a right angle, BE and $DMNC$ are parallel (both being perpendicular to AMB). Also PNE is parallel to AMB; thus, $MNEB$ is a rectangle and $|BE| = |MN| = |MC| - |MD|$.

Since AOE is a diameter, both AGE and ABE are right triangles. Let the radius of the circle be r. Applying the Pythagorean theorem to AGE yields

$$4r^2 = |EG|^2 + |AG|^2 = |EG|^2 + (|MA| - |MB|)^2.$$

Applying the Pythagorean theorem to ABE yields

$$4r^2 = |AB|^2 + |BE|^2 = (|MA| + |MB|)^2 + (|MC| - |MD|)^2.$$

Thus,

$$\begin{aligned} |EG|^2 &= (|MC| - |MD|)^2 + (|MA| + |MB|)^2 - (|MA| - |MB|)^2 \\ &= |MC|^2 + |MD|^2 - 2|MC|\,|MD| + 4|MA|\,|MB| \\ &= |MC|^2 + |MD|^2 + 2|MC|\,|MD| \\ &= (|MC| + |MD|)^2, \end{aligned}$$

using for the penultimate equality the power-of-the-point theorem, which asserts that $|MA|\,|MB|$ is equal to $|MC|\,|MD|$. Thus, $|EG| = |MC| + |MD|$.

Editorial comment. O. P. Lossers pointed out that if chords AB and CD do not meet inside the circle, then they can be extended to cross at point M outside the circle, and the construction required is still possible. Oliver Geupel proved the following converse: For every inscribed convex quadrilateral $PQRS$ such that PR is a diameter, there are two perpendicular chords AB and CD (not containing O) with intersection M such that $|MA| - |MB| > 0$, $|MC| - |MD| > 0$, and the sides of $PQRS$ have lengths $|MA| + |MB|$, $|MC| + |MD|$, $|MA| - |MB|$, and $|MC| - |MD|$.

There was a misprint in the published statement of the problem. The third side length $|MA| - |MB|$ was inadvertently misprinted as $|MA| - |MD|$.

Also solved by R. Chapman (U. K.), O. Geupel (Germany), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, R. Stong, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Orthogonal Projection of Ellipsoids

11896 [2016, 296]. *Proposed by Ron Evans, University of California, San Diego, CA.* Let $n \geq 2$, and let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ellipsoid, by which we mean that E has n orthogonal semi-axis vectors. (For instance, E is an ellipse in $\mathbb{R}^3$ when $n = 2$.) Show that the projection of E onto an n-dimensional subspace of $\mathbb{R}^{n+1}$ is either an n-dimensional ellipsoid or a solid $(n-1)$-dimensional set bounded by an $(n-1)$-dimensional ellipsoid (when $n = 2$, the solid is a line segment.)

Solution by Richard Stong, Center for Communications Research, San Diego, CA. More generally, we show that any image of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ under any affine map is either an ellipsoid of dimension $n-1$ (if the map is nonsingular) or the convex hull of an ellipsoid of dimension $n-k-1$ (if the map is singular with a k-dimensional kernel). Since the given ellipsoid E is an affine image of S^{n-1} and the projection is affine with at most a 1-dimensional kernel, this implies the requested result.

Note that by composing with a translation, we may assume that the affine map is simply a linear map $L : \mathbb{R}^n \to \mathbb{R}^N$ defined by a matrix A. Also, note that by restricting to the image $L(\mathbb{R}^n)$, we may assume the linear map is onto.

First, suppose $n = N$, and hence L is nonsingular. In that case, $E = L(S^{n-1}) = \{Y : Y^T(A^{-1})^T A^{-1} Y = 1\}$. Since $\Sigma = (A^{-1})^T A^{-1}$ is a symmetric positive-definite matrix, it has an orthonormal basis of eigenvectors. It follows that E is an ellipsoid.

Next, suppose that $L\colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ has a k-dimensional kernel for some $k \geq 1$. Let K be the kernel of L, and let $\pi\colon \mathbb{R}^n \to K^\perp$ be the orthogonal projection of $\mathbb{R}^n$ onto the orthogonal complement $K^\perp$ of K. We can write $L = L' \circ \pi$ for some nonsingular linear map $L'\colon K^\perp \to \mathbb{R}^{n-k}$. The image of S^{n-1} under π is the closed unit ball $B^{n-k} \subset K^\perp$. By the previous case, L' sends the boundary S_{n-k-1} of this ball to an ellipsoid of dimension $n-k-1$, and hence it sends the convex hull B^{n-k} to the convex hull of the ellipsoid.

Also solved by J.-P. Grivaux (France), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Sawhney, J. C. Smith, GCHQ Problem Solving Group (U. K.), and the proposer.

A Product of Catalan Numbers

11897 [2016, 296]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Prove for $n \geq 0$ that

$$\sum_{k+l=n,\, k\geq 0,\, l\geq 0} \frac{\binom{2k}{k}\binom{2l+2}{l+1}}{k+1} = 2\binom{2n+2}{n}. \qquad (*)$$

Solution I by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI. Let $C_k = \frac{1}{k+1}\binom{2k}{k}$. It is well known that $\sum_{k+l=n} C_k C_l = C_{n+1}$ for $n \geq 0$. Note that

$$\sum_{k+l=n} lC_kC_l = \frac{1}{2}\sum_{k+l=n}(k+l)C_kC_l = \frac{n}{2}\sum_{k+l=n}C_kC_l = \frac{n}{2}C_{n+1}.$$

Now set $j = l+1$ (canceling the term $j = 0$ from the summation) to conclude

$$\sum_{k+l=n}\frac{\binom{2k}{k}\binom{2l+2}{l+1}}{k+1} = \sum_{k+l=n}(l+2)C_kC_{l+1} = \sum_{k+j=n+1}(j+1)C_kC_j - C_{n+1}$$
$$= \frac{n+1}{2}C_{n+2} + C_{n+2} - C_{n+1} = \frac{(n+3)(2n+4)!}{2(n+2)!(n+3)!} - \frac{(2n+2)!}{(n+1)!(n+2)!} = 2\binom{2n+2}{n}.$$

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Let

$$C(x) = \sum_{k=0}^{\infty} C_k x^{k+1} = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^{k+1}}{(k+1)}.$$

The recurrence $\sum_{k+l=n} C_kC_l = C_{n+1}$ for $n \geq 0$ with $C_0 = 1$ yields $C(x) - x = C(x)^2$. Hence, $C(x)C'(x) = \frac{1}{2}(C'(x)-1)$. The summand on the left side of $(*)$ is the product of the coefficients of x^{k+1} in $C(x)$ and x^{l+1} in $C'(x)$, summed over $(k+1)+(l+1) = n+2$, but lacking the term for $l = -1$. Since the constant term in $C'(x)$ is 1, the sum is the coefficient of x^{n+2} in $C(x)(C'(x)-1)$. We compute

$$C(x)(C'(x)-1) = \frac{1}{2}(C'(x)-1) - C(x) = \frac{1}{2}\sum_{k=1}^{\infty}\binom{2k}{k}x^k - \sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^{k+1}}{k+1}.$$

The coefficient of x^{n+2} is $\frac{1}{2}\binom{2n+4}{n+2} - \frac{1}{n+2}\binom{2n+2}{n+1}$, which equals $2\binom{2n+2}{n}$.

Solution III by John H. Smith, Needham, Massachusetts. Consider lattice paths in the xy-plane, consisting of unit steps in the positive x- or y-direction. It is well known that the number of such paths from $(0,0)$ to (n,n) not rising above the line $y=x$ is the Catalan number C_n, equal to $\frac{1}{n+1}\binom{2n}{n}$.

The total number of lattice paths from $(0,-1)$ to $(n+1,n+1)$ is $\binom{2n+3}{n+1}$. Among these, the number that first meet the line $y=x$ at the point (k,k) is $\frac{1}{k+1}\binom{2k}{k}\binom{2(n+1-k)}{n+1-k}$, since the initial portion does not rise above the line $y=x-1$. Therefore, the left side of (*) counts all paths from $(0,-1)$ to $(n+1,n+1)$ except those that first meet the line $y=x$ at $(n+1,n+1)$, since the term for $k=n+1$ is missing. Note also that $C_{n+1}=\frac{1}{n+2}\binom{2n+2}{n+1}=\binom{2n+2}{n+1}-\binom{2n+2}{n}$. Thus, to evaluate the sum we compute

$$\begin{aligned}\binom{2n+3}{n+1}-C_{n+1}&=\left(\binom{2n+2}{n+1}+\binom{2n+2}{n}\right)-\left(\binom{2n+2}{n+1}-\binom{2n+2}{n}\right)\\&=2\binom{2n+2}{n}.\end{aligned}$$

Also solved by U. Abel (Germany), A. Ali (India), T. Amdeberhan, M. Apagodu, M. Arakelian (Armenia), N. Balachandran & P. De (India), D. Beckwith, M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal, B. Bradie, R. Chapman (U. K.), H. Chen, J. Cigler (Austria), C. Georghiou (Greece), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), A. Hannan (India), M. Hoffman, O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), L. Mannion, R. Nandan, M. Omarjee (France), S. Pathak (Canada), Á. Plaza & S. Falcón (Spain), J. Schlosberg, E. Schmeichel, J. C. Smith, A. Stenger, M. Štofka (Slovakia), D. Stoner, R. Stong, M. Tang, R. Tauraso (Italy), Z. Vőrős (Hungary), M. Vowe (Switzerland), M. Wildon (U. K.), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

No Matter How You Slice It

11898 [2016, 297]. *Proposed by Richard Stanley, University of Miami, Coral Gables, FL.* Let n and k be integers, with $n \geq k \geq 2$. Let G be a graph with n vertices whose components are cycles of length greater than k. Let $f_k(G)$ be the number of k-element independent sets of vertices of G. Show that $f_k(G)$ depends only on k and n. (A set of vertices is independent if no two of them are adjacent.)

Solution by Edward Schmeichel, San Jose State University, San Jose, California. It suffices to show that the number of independent k-sets is the same when G consists of two cycles as when G is just one cycle. When G has more components, one can then repeatedly merge two cycles to reach a single cycle without changing the number of independent k-sets.

Let $V(C_s)=\{v_1,\dots,v_s\}$ and $V(C_t)=\{w_1,\dots,w_t\}$. Form C_{s+t} by replacing v_sv_1 and w_tw_1 with v_sw_1 and w_tv_1. The independent sets in C_{s+t} that do not contain $\{v_1,v_s\}$ or $\{w_1,w_t\}$ are the same as the independent sets in C_s+C_t that do not contain $\{v_s,w_1\}$ or $\{w_t,v_1\}$. It suffices to pair the remaining independent k-sets in C_{s+t} and C_s+C_t.

Let $S=\{v_1,v_s,w_1,w_t\}$. Let I be an independent k-set in C_{s+t}. In the remaining case, $I\cap S$ is $\{v_1,v_s\}$ or $\{w_1,w_t\}$. Let m be the least index such that $v_m,w_m\notin I$; note that m exists and is less than $\min\{s,t\}$, since otherwise $|I|\geq\min\{s,t\}>k$. Define I' by exchanging the incidence vector of I over $(v_1,\dots,v_m)$ with its incidence vector over $(w_1,\dots,w_m)$.

The result is an independent k-set in C_s+C_t, since $v_m,w_m\notin I'$ and $I'\cap S$ is $\{v_s,w_1\}$ or $\{w_t,v_1\}$. The map is also an involution. Hence, it produces a one-to-one correspondence between the two desired families of independent k-sets.

Editorial comment. The proposer and most solvers used generating functions. A substantial generalization has been proved inductively by Hannah Spinoza and Douglas West. They showed that the conclusion about independent k-sets also holds for every k-vertex subgraph, over all n-vertex graphs whose components are cycles with more than k vertices or paths with at least $k-1$ vertices, as long as the number of components that are paths is the same. They used this in determining when a graph with maximum degree 2 can be reconstructed from its multiset of k-vertex induced subgraphs.

Also solved by M. Arakelian (Armenia), D. Beckwith, R. Chapman (U. K.), Y. J. Ionin, P. Lalonde (Canada), J. H. Lindsey II, J. C. Smith, J. H. Smith, R. Stong, R. Tauraso (Italy), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

A Couple of Convolutions

11899 [2016, 297]. *Proposed by Julien Sorel, PNI, Piatra Neamt, Romania.* Show that for every positive integer n,

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{k-1}=\binom{4n+1}{2n}+\binom{2n}{n}^2.$$

Solution I by Li Zhou, Polk State College, Winter Haven, FL. We begin with the well-known Vandermonde convolution, comparing coefficients of x^{2n} in the expansions of $(1+x)^{4n+1}$ and $(1+x)^{2n}(1+x)^{2n+1}$:

$$\binom{4n+1}{2n}=\sum_{k=0}^{2n}\binom{2n}{2n-k}\binom{2n+1}{k}=\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n+1}{k}.$$

Manipulating the second term, we obtain

$$\begin{aligned}\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n+1}{k}&=\sum_{k=n+1}^{2n}\binom{2n}{k}\left(\binom{2n}{k-1}+\binom{2n}{k}\right)\\&=\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n}{k-1}+\sum_{j=n+2}^{2n+1}\binom{2n}{j-1}\binom{2n}{j-1}\\&=\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n}{k-1}+\sum_{k=n+1}^{2n+1}\binom{2n}{k-1}\binom{2n}{k-1}-\binom{2n}{n}^2\\&=\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{k-1}-\binom{2n}{n}^2,\end{aligned}$$

completing the proof.

Solution II by John H. Smith, Needham, MA. It suffices to show

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{2n+1-k}+\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{2n-k+1}-\binom{2n}{n}^2=\binom{4n+1}{2n+1}.$$

We count the ways of choosing $2n+1$ objects from $\{1,\ldots,4n+1\}$. The first sum counts the choices with at most n objects from the first $2n$. The second counts those having at least $n+1$ objects from the first $2n+1$. Each choice is counted in at least one of these sums. Those counted twice are the choices having exactly n from the first $2n$, plus n from the last $2n$, plus the element $2n+1$. There are thus $\binom{2n}{n}^2$ choices counted twice.

Also solved by A. Ali (India), T. Amdeberhan, B. Bradie, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), P. De (India), N. Fontes-Merz, N. Grivaux (France), T. Guan (China), M. Hoffman, Y. J. Ionin, B. Karivanov (U. S. A) & T. S. Vassilev (Canada), O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Madhyastha (India), R. Nandan, M. Nathanson, S. Pathak (Canada), Á. Plaza

(Spain), M. Sawhney, E. Schmeichel, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), Z. Vőrős (Hungary), M. Vowe (Switzerland), S. Y. Wang (Korea), G. Whieldon, M. Wildon (U. K.), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

Circles Next after Incircles

11900 [2016, 297]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let ABC be a triangle, and let I and r be the center and radius of its incircle. The circle with center and radius (I_A, r_A) is externally tangent to the incircle and internally tangent to sides AB and AC of ABC. Define (I_B, r_B) and (I_C, r_C) similarly. Prove for $n \geq 1$ that

$$\left(\frac{r+r_A}{r-r_A}\right)^n + \left(\frac{r+r_B}{r-r_B}\right)^n + \left(\frac{r+r_C}{r-r_C}\right)^n \geq 3 \cdot 2^n.$$

Solution by Oleh Faynshteyn, Leipzig, Germany. Let D and E be the feet of the perpendiculars dropped onto side AB from I and I_B, respectively. Let F be the foot of the perpendicular dropped onto segment ID from I_B. We have $IF = r - r_B$ and $II_B = r + r_B$. From the right triangle ΔIFI_B, we have

$$\frac{I\,I_B}{IF} = \frac{r+r_B}{r-r_B} = \csc(B/2).$$

Similarly,

$$\frac{r+r_A}{r-r_A} = \csc(A/2) \qquad \text{and} \qquad \frac{r+r_C}{r-r_C} = \csc(C/2).$$

Since $x^n + y^n + z^n \geq \frac{1}{3^{n-1}}(x+y+z)^n$,

$$\begin{aligned}\left(\frac{r+r_A}{r-r_A}\right)^n + \left(\frac{r+r_B}{r-r_B}\right)^n + \left(\frac{r+r_C}{r-r_C}\right)^n &= \csc^n(A/2) + \csc^n(B/2) + \csc^n(C/2) \\ &\geq \frac{1}{3^{n-1}}\big(\csc(A/2) + \csc(B/2) + \csc(C/2)\big)^n. \qquad (1)\end{aligned}$$

From the harmonic-geometric mean inequality, the identity

$$\sin(A/2)\sin(B/2)\sin(C/2) = \frac{r}{4R},$$

and Euler's inequality $R \geq 2r$, we obtain

$$\frac{3}{\csc(A/2) + \csc(B/2) + \csc(C/2)} \leq \sqrt[3]{\sin(A/2)\sin(B/2)\sin(C/2)} \leq \sqrt[3]{\frac{1}{8}} = \frac{1}{2}.$$

This implies

$$\csc(A/2) + \csc(B/2) + \csc(C/2) \geq 6. \qquad (2)$$

Substituting (2) into (1), we obtain

$$\csc^n(A/2) + \csc^n(B/2) + \csc^n(C/2) \geq \frac{1}{3^{n-1}} \cdot 6^n = 3 \cdot 2^n,$$

as required.

Also solved by J. J. Ahn (Korea), A. Ali (India), R. Bagby, R. Boukharfane (France), M. V. Channakeshava (India), R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, N. & J.-P. Grivaux (France), T. Guan (China), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), W.-K. Lai & J. Risher, K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, D. Pispinis (Saudi Arabia), M. Sawhney, V. Schindler (Germany), E. Schmeichel, J. C. Smith, N. Stanciu & T. Zvonaru (Romania), D. Stoner, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), J. Zacharias, L. Zhou, Armstrong Problem Solvers, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
http://www.americanmathematicalmonthly.submittable.com/submit
Proposed solutions to the problems below should be submitted by June 30, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12020. *Proposed by Erhard Braune, Linz, Austria*. Let α, β, and γ be the radian measures of the three angles of a triangle, and let ω be the radian measure of its Brocard angle. (The *Brocard angle* of triangle ABC is the angle TAB, where T is the unique point such that $\angle TAB$, $\angle TBC$, and $\angle TCA$ are congruent.) Yff's inequality asserts that $8\omega^3$ is a lower bound for $\alpha\beta\gamma$. Show that $\omega\pi^3/4$ is an upper bound for the same product.

12021. *Proposed by Omar Sonebi, Lycée Technique, Settat, Morroco*. Let ϕ be the Euler totient function. Given $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}^+$, show that there exists $n \in \mathbb{Z}^+$ such that $an + b$ is not in the range of ϕ.

12022. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania*. Let n be a positive integer, and let x be a real number not equal to -1 or 1. Prove

$$\sum_{k=0}^{n-1} \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} = n$$

and

$$\sum_{k=0}^{n-1}(-1)^k \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} x^{\binom{n-1-k}{2}} = nx^{\binom{n}{2}}.$$

12023. *Proposed by Vazgen Mikayelyan, Yerevan State University, Yerevan, Armenia*. Let α be a positive real number. Prove

$$\int_0^{\pi} x^{\alpha-2}\, \sin x\, dx \geq \pi^{\alpha} \frac{\alpha+6}{\alpha(\alpha+2)(\alpha+3)}.$$

12024. *Proposed by Marian Cucoaneş, Mărăşeşti, Romania, Marius Drăgan, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania*. Let x, y, and z be positive real numbers satisfying $xyz = 1$. Prove

$$(x^{10}+y^{10}+z^{10})^2 \geq 3(x^{13}+y^{13}+z^{13}).$$

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12025. *Proposed by Askar Dzhumadil'daev, S. Demirel University, Almaty, Kazakhstan*. The Chebyshev polynomials of the second kind are defined by the recurrence relation $U_0(x) = 1$, $U_1(x) = 2x$, and $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for $n \geq 2$. For an integer n with $n \geq 2$, prove

$$\det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ x & 0 & 1 & \cdots & 1 & 1 \\ x^2 & x & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & x^{n-4} & \cdots & 0 & 1 \\ x^{n-1} & x^{n-2} & x^{n-3} & \cdots & x & 0 \end{bmatrix} = (-1)^{n-1} x^{n/2} U_{n-2}(\sqrt{x}).$$

12026. *Proposed by Michel Bataille, Rouen, France*. For $n \in \mathbb{N}$, let $H_n = \sum_{k=1}^{n} 1/k$ and $S_n = \sum_{k=1}^{n} (-1)^{n-k}(H_1 + \cdots + H_k)/k$. Find $\lim_{n\to\infty} S_n/\ln n$ and $\lim_{n\to\infty}(S_{2n} - S_{2n-1})$.

SOLUTIONS

Expressing the Sum of Three Squares as the Sum of Two

11894 [2016, 296]. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA*. Let a, b, c, and d be integers such that $a^2 + b^2 + c^2 = d^2$ and $d \neq 0$. Let x, y, and z be three integers such that $ax + by + cz = 0$.
(**a**) Prove that $x^2 + y^2 + z^2$ can be written as the sum of two squares.
(**b**) Let $ABCD$ be a square in $\mathbb{R}^3$ with integer vertices A, B, C, and D. Show that the side lengths of $ABCD$ have the form $\sqrt{l}$, where l is the sum of two squares.

Solution by James Christopher Smith, Knoxville, TN.
(**a**) Since $d \neq 0$, at least one of a, b, and c is nonzero, so we may assume $c \neq 0$. Using $cz = -(ax + by)$ and algebraic manipulation, we obtain

$$\begin{aligned}(a^2 + c^2)c^2(x^2 + y^2 + z^2) &= (a^2 + c^2)\left(c^2x^2 + c^2y^2 + (ax + by)^2\right) \\ &= ((a^2 + c^2)x + aby)^2 + (a^2 + b^2 + c^2)c^2y^2 \\ &= ((a^2 + c^2)x + aby)^2 + (dcy)^2.\end{aligned}$$

It is well known that an integer can be written as the sum of two squares if and only if every prime congruent to 3 modulo 4 occurs in its prime factorization with even exponent. Also, every prime occurs with even exponent in the factorization of a square. Thus, each of the quantities c^2, $a^2 + c^2$, and $\left((a^2 + c^2)x + aby\right)^2 + (dcy)^2$ has the property that every prime congruent to 3 modulo 4 occurs with even exponent in its prime factorization. Also, none of these quantities equals 0. Hence after cancelation the same property holds for $x^2 + y^2 + z^2$, which proves that it can be written as the sum of two squares.
(**b**) Translate the square so that one of its vertices is at the origin; still all vertices are at integer points. Let (α, β, γ) and (x, y, z) be the coordinates of the two vertices of the square adjacent to the vertex at the origin, so $\alpha x + \beta y + \gamma z = 0$ and $l = \alpha^2 + \beta^2 + \gamma^2 = x^2 + y^2 + z^2$. Let (a, b, c) be the cross-product of (α, β, γ) and (x, y, z) (so $a = \beta z - \gamma y$, $b = \gamma x - \alpha z$, $c = \alpha y - \beta x$). We observe that (a, b, c) is orthogonal to (x, y, z) and has length l, so the claim follows from part (**a**).

Also solved by B. S. Burdick, R. Chapman (U. K.), R. Dempsey, Y. J. Ionin, O. P. Lossers (Netherlands), M. Omarjee (France), J. P. Robertson, M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), T. Viteam (Denmark), M. Wildon (U. K.), GCHQ Problem Solving Group (UK), and the proposer.

On the Definition of "Regularly Varying Function"

11895 [2016, 296]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada*. Let f be a *regularly varying* function from $(0, \infty)$ into $(0, \infty)$, with index $\rho > 0$, and let g be a function from $(0, \infty)$ into $(0, \infty)$ such that $\lim_{x\to\infty} g(x) = \infty$. (A function L on $\mathbb{R}^+$ is *regularly varying with index* ρ if $\lim_{x\to\infty} L(ax)/L(x) = a^\rho$.) Prove

$$\lim_{x\to\infty} \frac{f(x)}{f(g(x))} = L \text{ if and only if } \lim_{x\to\infty} \frac{x}{g(x)} = L^{1/\rho}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The problem is not correct without some additional hypothesis that $\lim_{x\to\infty} f(ax)/f(x) = a^\rho$ occurs "uniformly in a." We provide a counterexample.

A function $L : \mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$-*linear* if $L(ax + by) = aL(x) + bL(y)$ for all $x, y \in \mathbb{R}$ and $a, b \in \mathbb{Q}$. Let $L : \mathbb{R} \to \mathbb{R}$ be a discontinuous $\mathbb{Q}$-linear function. The graph of any such function is dense in the plane. Define $h : \mathbb{R} \to (-1, 1)$ by $h(x) = \tanh(L(x)/x)$ for $x \neq 0$, and define $h(0)$ arbitrarily. The graph of h is dense in $\mathbb{R} \times [-1, 1]$. Fix a real α, and let $\beta = L(\alpha)$. We have

$$\begin{aligned} h(x+\alpha) - h(x) &= \tanh \frac{L(x+\alpha)}{x+\alpha} - \tanh \frac{L(x)}{x} \\ &= \left(\tanh \frac{L(x+\alpha)}{x+\alpha} - \tanh \frac{L(x)}{x+\alpha}\right) - \left(\tanh \frac{L(x)}{x} - \tanh \frac{L(x)}{x+\alpha}\right) \\ &= \frac{\beta}{x+\alpha} \operatorname{sech}^2 \xi_1 - \frac{\alpha L(x)}{x(x+\alpha)} \operatorname{sech}^2 \xi_2 \end{aligned}$$

for some ξ_1 between $L(x)/(x+\alpha)$ and $(L(x)+\beta)/(x+\alpha)$ and some ξ_2 between $L(x)/(x+\alpha)$ and $L(x)/x$. The first term goes to 0 as $x \to \infty$ since the hyperbolic secant is bounded by 1. The second term also goes to 0 since $\xi_2 \operatorname{sech}^2 \xi_2$ is bounded and

$$\left|\frac{\alpha L(x)}{x(x+\alpha)\xi_2}\right| \leq \frac{|\alpha|}{\min\big(|x|, |x+\alpha|\big)} \to 0.$$

We conclude that $\lim_{x\to\infty} \big(h(x+\alpha) - h(x)\big) = 0$ for any α.

Now let

$$f(x) = x^\rho e^{h(\log x)}.$$

For any $a > 0$, we have

$$\frac{f(ax)}{f(x)} = a^\rho e^{h(\log x + \log a) - h(\log x)} \to a^\rho$$

as $x \to \infty$. Thus, f is "regularly varying" according to the definition given in the problem statement. Note that since β is unbounded, the rate of this convergence depends on a in a complicated way. Now from the density of the graph of h, there is a sequence $(x_n)_{n\geq 1}$ of positive reals that increases monotonically to ∞ such that $(-1)^n h(\log x_n) \to 1$. Again using the density, let y_n be chosen so that y_n is within 1 of x_n and $(-1)^n h(\log y_n) \to -1$. Define g by making its graph linearly interpolate the points $(0, 0)$, (x_1, y_1), (x_2, y_2), (x_3, y_3), ... We have $\lim_{x\to\infty} x/g(x) = 1$, but

$$\frac{f(x_n)}{f(g(x_n))} = \frac{f(x_n)}{f(y_n)} = \left(\frac{x_n}{y_n}\right)^\rho e^{h(\log x_n) - h(\log y_n)} \approx e^{2(-1)^n},$$

so $\lim_{x\to\infty} f(x)/f(g(x))$ fails to exist.

Editorial comment. Most solvers noted that the textbook definition of "regularly varying" usually includes an additional condition on f such as measurability. With that additional condition, the required assertion follows from a representation theorem of Karamata found in all textbooks on the subject. The GCHQ Problem Solving Group showed that the claimed conclusion holds for all functions g going to ∞ and all $L \in [0, \infty]$ if and only if, for all compact subsets $K \subseteq (0, \infty)$, $\lim_{x\to\infty} f(ax)/f(x)$ converges uniformly to a^ρ for $a \in K$.

Also solved by P. Bracken, P. J. Fitzsimmons, O. P. Lossers (Netherlands), M. Omarjee (France), E. Omey (Belgium), J. M. Sanders, J. C. Smith, GCHQ Problem Solving Group (U. K.), and the proposer.

A Finite Semigroup of Endofunctions

11901 [2016, 399]. *Proposed by Donald Knuth, Stanford, CA.* For $n \in \mathbb{Z}^+$, let $[n] = \{1, 2, \ldots, n\}$. Define the functions $\uparrow$ and $\downarrow$ on $[n]$ by $\uparrow x = \min\{x+1, n\}$ and $\downarrow x = \max\{x-1, 1\}$. How many distinct mappings from $[n]$ to $[n]$ occur as compositions of $\uparrow$ and $\downarrow$?

Solution by Rob Pratt, Washington, DC. We show that every such mapping f has the form

$$(f(1), \ldots, f(n)) = (\underbrace{i, i, \ldots, i}_{j}, \underbrace{i+1, i+2, \ldots i+d}_{d}, \underbrace{i+d, \ldots, i+d}_{n-d-j})$$

with $d \in \{0, 1, \ldots, n-1\}$, $i \in [n-d]$, and $j \in [n-d]$. Clearly composition with both $\uparrow$ and $\downarrow$ preserves this form. From the empty composition (itself the case $i = j = 1$ and $d = n-1$), application of $\uparrow^{i-1} \circ \downarrow^{n-d-1} \circ \uparrow^{n-d-j}$ achieves the specified function, given d, i, and j.

The allowable mappings are easily counted: for $d = 0$ there are n, and for $d \in [n-1]$ both i and j may take any value in $[n-d]$. Hence, the computation is

$$n + \sum_{d=1}^{n-1}(n-d)^2 = n + \sum_{m=1}^{n-1} m^2 = n + \frac{(n-1)n(2n-1)}{6} = \frac{n(2n^2-3n+7)}{6}.$$

Editorial comment. Many solvers excluded the empty composition from their count (for $n \neq 1$). The proposer's solution included it.

Also solved by J. Bartz, C. Blatter (Switzerland), N. Caro (Spain), P. P. Dályay (Hungary), K. David, F. Eckstrom (Sweden), S. Gagola, O. Geupel (Germany), J. Grossman & S. Kruk, Y. Ionin, O. Kouba (Syria), M. Kuczma (Poland), M. Lafond (France), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Meyerson, J. Olson, M. Patnott, J. Schlosberg, J. C. Smith, J. H. Smith, R. Stong, R. Tauraso (Italy), M. Wildon (U. K.), Armstrong Problem Solvers Group, Con Amore Problem Group, FAU Problem Solving Group, GCHQ Problem Solving Group, and the proposer.

A Row of Zetas

11902 [2016, 399]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania.* Let $\{x\}$ denote $x - \lfloor x \rfloor$, the fractional part of x. Prove

$$\int_0^1 \int_0^1 \int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 dx\,dy\,dz$$
$$= 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{7\zeta(6)}{48} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{18} + \frac{\zeta(3)\zeta(4)}{12}.$$

Solution by Rituraj Nandan, SunEdison, Maryland Heights, MO. For $(x, y, z) \in [0, 1]^3$, there are six possibilities for the ordering of x, y, and z. We write the requested integral as the sum of the integrals corresponding to these six orderings.

For $z \le y \le x$, we have $\{x/y\} = x/y - n$ whenever $x/(n+1) < y \le x/n$ and n is a positive integer, $\{y/z\} = y/z - m$ whenever $y/(m+1) < z \le y/m$ and m is a positive integer, and $\{z/x\} = z/x$. Therefore

$$\begin{aligned}
&\iiint_{0\le z\le y\le x\le 1} \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 \, dx\, dy\, dz \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^1 \int_{x/(n+1)}^{x/n}\int_{y/(m+1)}^{y/m} \left(\left(\frac{x}{y}-n\right)\cdot\left(\frac{y}{z}-m\right)\cdot\frac{z}{x}\right)^2 dz\, dy\, dx \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4n+1}{108m(m+1)^3 n^2(n+1)^4} \\
&\quad = \frac{1}{108}\sum_{n=1}^{\infty}\frac{4n+1}{n^2(n+1)^4}\sum_{m=1}^{\infty}\frac{1}{m(m+1)^3}. \qquad (1)
\end{aligned}$$

We evaluate the summations by partial fractions, obtaining

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{4n+1}{n^2(n+1)^4} &= \sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)-\frac{2}{(n+1)^3}-\frac{3}{(n+1)^4}\right) \\
&= 1 - 2(\zeta(3)-1) - 3(\zeta(4)-1) = 6 - 2\zeta(3) - 3\zeta(4)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=1}^{\infty}\frac{1}{m(m+1)^3} &= \sum_{m=1}^{\infty}\left(\left(\frac{1}{m}-\frac{1}{m+1}\right)-\frac{1}{(m+1)^2}-\frac{1}{(m+1)^3}\right) \\
&= 1 - (\zeta(2)-1) - (\zeta(3)-1) = 3 - \zeta(2) - \zeta(3).
\end{aligned}$$

Substituting into (1), we obtain

$$\begin{aligned}
&\iiint_{0\le z\le y\le x\le 1} \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 \, dx\, dy\, dz \\
&\quad = \frac{1}{108}\big(6 - 2\zeta(3) - 3\zeta(4)\big)\big(3 - \zeta(2) - \zeta(3)\big). \qquad (2)
\end{aligned}$$

For $x \le z \le y$ and $y \le x \le z$, the cyclic permutations of $z \le y \le x$, the integrals have the same form and therefore the same value.

For $y \le z \le x$, we have $\{x/y\} = x/y - n$ whenever $x/(n+1) < y \le x/n$ and n is a positive integer, $\{y/z\} = y/z$, and $\{z/x\} = z/x$. Therefore

$$\begin{aligned}
&\iiint_{0\le y\le z\le x\le 1} \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 \, dx\, dy\, dz \\
&\quad = \sum_{n=1}^{\infty}\int_0^1 \int_{x/(n+1)}^{x/n}\int_y^x \left(\left(\frac{x}{y}-n\right)\cdot\frac{y}{z}\cdot\frac{z}{x}\right)^2 dz\, dy\, dx \\
&\quad = \sum_{n=1}^{\infty}\frac{4n^2-1}{36n^2(n+1)^4}
\end{aligned}$$

$$= \frac{1}{36}\sum_{n=1}^{\infty}\left(4\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{n^2}-\frac{3}{(n+1)^2}-\frac{2}{(n+1)^3}+\frac{3}{(n+1)^4}\right)$$
$$= \frac{1}{36}\big(4-\zeta(2)-3(\zeta(2)-1)-2(\zeta(3)-1)+3(\zeta(4)-1)\big)$$
$$= \frac{1}{36}\big(6-4\zeta(2)-2\zeta(3)+3\zeta(4)\big). \tag{3}$$

As before, for $x \le y \le z$ and $z \le x \le y$, the cyclic permutations of $y \le z \le x$, the integrals have the same value.

Using (2) and (3), we obtain

$$\int_0^1\int_0^1\int_0^1\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 dx\,dy\,dz$$
$$= 3\cdot\frac{1}{108}\big(6-2\zeta(3)-3\zeta(4)\big)\big(3-\zeta(2)-\zeta(3)\big)+3\cdot\frac{1}{36}\big(6-4\zeta(2)-2\zeta(3)+3\zeta(4)\big)$$
$$= 1-\frac{\zeta(2)}{2}-\frac{\zeta(3)}{2}+\frac{\zeta(2)\zeta(4)}{12}+\frac{\zeta(2)\zeta(3)}{18}+\frac{\zeta(3)^2}{18}+\frac{\zeta(3)\zeta(4)}{12}$$
$$= 1-\frac{\zeta(2)}{2}-\frac{\zeta(3)}{2}+\frac{7\zeta(6)}{48}+\frac{\zeta(2)\zeta(3)}{18}+\frac{\zeta(3)^2}{18}+\frac{\zeta(3)\zeta(4)}{12}.$$

Editorial comment. Several solvers derived the more general formula

$$\int_0^1\int_0^1\int_0^1\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n dx\,dy\,dz$$
$$= 1-\frac{3\sum_{j=1}^{n}\zeta(j+1)}{2(n+1)}+\frac{\sum_{j=1}^{n}\zeta(j+1)}{(n+1)^2(n+2)}\left(\sum_{j=1}^{n}(j+1)\zeta(j+2)\right).$$

Also solved by T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), R. Boukharfane (France), R. Dempsey, R. Dutta (India), D. Fritze (Germany), M. L. Glasser, M. Hoffman, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), J. C. Smith, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (UK), and the proposer.

An Integral Identity

11903 [2016, 399]. *Proposed by Paolo Perfetti, Universitá Degli Studi di Roma "Tor Vergata," Rome, Italy*. Find a homogeneous polynomial p of degree 2 in a, b, c, and d such that for $0 < -d < a < b < c$,

$$\int_0^a \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx = \int_b^c \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx$$

if and only if $\sqrt{-d(a+d)(b+d)(c+d)} = p(a,b,c,d)$.

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. The integral written on the left does not exist due to its singularity at $x=-d$. We assume that instead of $0 < -d < a < b < c$, the condition on a, b, c, d is $0 < a < b < c$ and $d > 0$, and the condition we are aiming for is $\sqrt{d(a+d)(b+d)(c+d)} = p(a,b,c,d)$.

Let

$$p(a,b,c,d) = \frac{2ab+2bc+2ca+4ad+4bd+4cd-a^2-b^2-c^2+8d^2}{8}.$$

We show that, for $0 < a < b < c$ and $d > 0$, we have

$$\int_0^a \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx - \int_b^c \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx$$
$$= \pi \cdot p(a,b,c,d) - \pi\sqrt{d(a+d)(b+c)(c+d)},$$

which gives the result.

Let D be the complex plane, removing vertical half-lines starting at $0, a, b, c \in \mathbb{R}$ and going through the lower half-plane. As D is a simply-connected, open subset of the plane and $z \mapsto z(z-a)(z-b)(z-c)$ is a nonzero holomorphic function on D, it has a holomorphic square root, so there is a holomorphic s on D with $s(z)^2 = z(z-a)(z-b)(z-c)$ for $z \in D$. Restricting to the real axis, the argument of s decreases by $\pi/2 \pmod{2\pi}$ as s goes from 0^- to 0^+, from a^- to a^+, from b^- to b^+, and from c^- to c^+, with the argument locally constant at other points. Hence, if we take s to be positive real on $(-\infty, 0)$, then s is negative imaginary on $(0, a)$, negative real on (a, b), positive imaginary on (b, c), and positive real again on $(c, +\infty)$.

Now consider $f(z) = s(z)/(z+d)$. Note that f is holomorphic on $D \setminus \{-d\}$ with a simple pole at $-d$. Choose $R > \max\{c, d\}$ and $\varepsilon > 0$ small. Let C be the closed contour defined as follows. The contour goes from $-R$ to R on the real axis, taking semicircular detours $C_0, C_a, C_b, C_c, C_{-d}$ of radius ε around $0, a, b, c, -d$ into the upper half-plane, with semicircle C_R in the upper half-plane from R to $-R$. In calculating the counterclockwise contour integral of f around C, the contributions from C_0, C_a, C_b, C_c are $O(\varepsilon)$ and the contribution from C_{-d} is

$$-\pi i\sqrt{d(a+d)(b+d)(c+d)} + O(\varepsilon)$$

as $\varepsilon \to 0^+$. Hence, the integral along the part of the contour from $-R$ to R along the real axis with detours $C_{-d}, C_0, C_a, C_b, C_c$ is

$$\left(\int_{-R}^{-d-\varepsilon} + \int_{-d+\varepsilon}^{-\varepsilon} - i\int_{\varepsilon}^{a-\varepsilon} - \int_{a+\varepsilon}^{b-\varepsilon} + i\int_{b+\varepsilon}^{c-\varepsilon} + \int_{c+\varepsilon}^{R}\right) \frac{\sqrt{|x|\,|x-a|\,|x-b|\,|x-c|}}{x+d}\,dx$$
$$-\pi i\sqrt{d(a+d)(b+d)(c+d)} + O(\varepsilon).$$

Next, we determine the behavior of $f(z)$ as $z \to \infty$ in the upper half-plane. Since

$$s(z)^2 = z^4 + (-a-b-c)z^3 + (ab+bc+ca)z^2 - abcz,$$

we have

$$s(z) = z^2 + \frac{-a-b-c}{2}z + \frac{2ab+2bc+2ca-a^2-b^2-c^2}{8} + O(z^{-1}).$$

Since

$$\frac{1}{z+d} = z^{-1} - dz^{-2} + d^2z^{-3} + O(z^{-4}),$$

we obtain

$$f(z) = z + \frac{-a-b-c-2d}{2}$$
$$+ \frac{2ab+2bc+2ca+4ad+4bd+4cd-a^2-b^2-c^2+8d^2}{8}z^{-1} + O(z^{-2})$$
$$= z + \frac{-a-b-c-2d}{2} + p(a,b,c,d)z^{-1} + O(z^{-2}).$$

The integral of f along C_R is then $(a+b+c+2d)R + p(a,b,c,d)\pi i + O(R^{-1})$. By Cauchy's theorem, the contour integral of f around C is 0. Taking the imaginary part with $\varepsilon \to 0^+$ and $R \to \infty$, we get the claimed identity and hence the result.

Also solved by R. Stong, R. Tauraso (Italy), and the proposer.

The Triangle Inequality from the Parallelogram Law

11904 [2016, 399]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*. Let f be a function from $\mathbb{R}$ into $[0, \infty)$ such that $f^2(x+y) + f^2(x-y) = 2f^2(x) + 2f^2(y)$ for all x and y. Prove $f(x+y) \le f(x) + f(y)$ for all x and y.

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY. Setting $x = y = 0$ in the given functional equation yields $2f(0)^2 = 4f(0)^2$, implying $f(0) = 0$. Letting $x = 0$, we conclude that $f^2(y) + f^2(-y) = 2f^2(y)$, or (since f is nonnegative) $f(y) = f(-y)$ for all $y \in \mathbb{R}$.

If we define $g(x, y) = \left(f^2(x+y) - f^2(x-y)\right)/4$, then we see from the original equation that $g(x, x) = f^2(2x)/4 = f^2(x)$. Furthermore, the result of the previous paragraph gives $g(x, y) = g(y, x)$. Subtracting the functional equation for $(y-z, x)$ from the functional equation for $(x+y, z)$, then adding the functional equation for (y, z) and subtracting it for (x, y), we obtain

$$\begin{aligned} f^2(x+y+z) - f^2(x-y+z) &= 2f^2(x+y) - 2f^2(y-z) + 2f^2(z) - 2f^2(x) \\ &= f^2(x+y) - f^2(x-y) + f^2(y+z) - f^2(y-z). \end{aligned}$$

Thus, we conclude $g(x+z, y) = g(x, y) + g(z, y)$. Since g is symmetric, we conclude that it is also additive in its second argument. The additivity of g implies $g(nx, y) = ng(x, y)$ for all $n \in \mathbb{Z}$. It follows that $g(rx, y) = rg(x, y)$ for all $r \in \mathbb{Q}$. Thus for rational r we have

$$0 \le f^2(rx+y) = g(rx+y, rx+y) = r^2 g(x, x) + 2rg(x, y) + g(y, y).$$

Since the right-hand side is a polynomial in r, it follows that the inequality holds for all real r. Hence, the discriminant of this quadratic must be nonpositive, that is, $g^2(x, y) \le g(x, x)g(y, y)$ or equivalently $|g(x, y)| \le f(x)f(y)$. Hence

$$f^2(x+y) = g(x, x) + 2g(x, y) + g(y, y) \le (f(x) + f(y))^2,$$

which implies $f(x+y) \le f(x) + f(y)$.

Editorial comment. Allen Stenger noted that the problem appeared (in a slightly more general form) in this MONTHLY, Problem 5264 [1965, 193; 1966, 211], proposed by D. E. Knuth, with solutions by E. O. Buchman and W. G. Dotson, Jr.

Also solved by D. Bailey & E. Campbell & C. Diminnie, P. P. Dályay (Hungary), R. Ger (Poland), M. Goldenberg & M. Kaplan, J. W. Hagood, O. Kouba (Syria), O. P. Lossers (Netherlands), J. C. Smith, A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Strengthening the Mordell–Oppenheim Inequality

11905 [2016, 400]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA*. From a point P inside a triangle ABC, the perpendiculars PP_A, PP_B, and PP_C are drawn to its sides. Let R be the circumradius and r the inradius of the triangle. Prove

$$\frac{R}{2r} \le \frac{|PA|\,|PB|\,|PC|}{(|PP_B| + |PP_C|)(|PP_A| + |PP_C|)(|PP_A| + |PP_B|)}.$$

Solution by Mohammad Reza Yegan, Central Tehran Branch, Islamic Azad University, Tehran, Iran. We assume that P_A lies on segment BC, P_B lies on segment CA, and P_C lies on segment AB. Let PA divide angle A into A_1 and A_2 so that $|PP_B| = |PA| \sin A_1$ and $|PP_C| = |PA| \sin A_2$. It follows that $|PP_B| + |PP_C| = |PA|(\sin A_1 + \sin A_2)$. Similarly, $|PP_C| + |PP_A| = |PB|(\sin B_1 + \sin B_2)$ and $|PP_A| + |PP_B| = |PC|(\sin C_1 + \sin C_2)$. Hence

$$\begin{aligned}(|PP_A| &+ |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|) \\ &= |PA|\ |PB|\ |PC|(\sin A_1 + \sin A_2)(\sin B_1 + \sin B_2)(\sin C_1 + \sin C_2),\end{aligned}$$

or equivalently

$$\begin{aligned}(\sin A_1 &+ \sin A_2)(\sin B_1 + \sin B_2)(\sin C_1 + \sin C_2) \\ &= \frac{(|PP_A| + |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|)}{|PA|\ |PB|\ |PC|}.\end{aligned}$$

Since

$$\sin A_1 + \sin A_2 = 2(\sin(A_1 + A_2)/2)\,(\cos(A_1 - A_2)/2) \le 2 \sin A/2,$$

this implies

$$\frac{(|PP_A| + |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|)}{|PA|\ |PB|\ |PC|} \le 8 \sin A/2 \sin B/2 \sin C/2.$$

Taking reciprocals and using $\sin A/2 \sin B/2 \sin C/2 = r/4R$, we obtain the desired result.

Editorial comment. This inequality is (12.28) on p. 111 in *Geometric Inequalities* by O. Bottema et al. (1969). They reference L. J. Mordell, "On geometric problems of Erdös and Oppenheim," *Math. Gazette* **46** (1962) 213–215.

Using Euler's inequality $R \ge 2r$, we may conclude

$$|PA|\ |PB|\ |PC| \ge (|PP_A| + |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|),$$

which is the Mordell–Oppenheim inequality.

The problem statement is correct even if "side" is interpreted as "extended side," the angle at A, say, is obtuse, and the projection P_B of P onto line AC falls outside the segment AC. Define angles A_1 and A_2 as above. Note that $\angle PAP_B$ is equal not to $\angle PAC$ but rather to its supplement. However, the sine of an angle and the sine of its supplement are equal, so it remains true that $|PP_C| + |PP_B| = |PA|(\sin A_1 + \sin A_2)$, and therefore Yegan's proof continues to be valid.

Also solved by A. Ali (India), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), R. Bosch, M. V. Channakeshava (India), P. P. Dályay (Hungary), P. De (India), M. Dincă (Romania), D. Fleischman, S. Gayen (India), O. Geupel (Germany), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), W. Liu, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, P. Nüesch (Switzerland), M. Sawhney, J. C. Smith, R. Stong, R. Tauraso (Italy), Z. Vőrős (Hungary), M. Vowe (Switzerland), T. Wiandt, L. Wimmer (Germany), J. Zacharias, GCHQ Problem Solving Group (UK), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by July 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12027. *Proposed by Abdul Hannan, Chennai Mathematical Institute, Chennai, India.* Let ABC be a triangle with circumradius R and inradius r. Let D, E, and F be the points where the incircle of ABC touches BC, CA, and AB, respectively, and let X, Y, and Z be the second points of intersection between the incircle of ABC and AD, BE, and CF, respectively. Prove

$$\frac{|AX|}{|XD|}+\frac{|BY|}{|YE|}+\frac{|CZ|}{|ZF|}=\frac{R}{r}-\frac{1}{2}.$$

12028. *Proposed by Michael Elgersma, Minneapolis, MN, Ramin Naimi, Occidental College, Los Angeles, CA, and Stan Wagon, Macalester College, St. Paul, MN.* We have n coins, where $n=d+p+q$ for positive integers d, p, and q. Suppose that whenever any d of the coins are removed, the rest can be split into sets of size p and q that balance when placed on a balance with arm lengths q and p, respectively. That is, q times the weight of the p coins equals p times the weight of the q coins. Must all n coins have the same weight?

12029. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $a>0$, evaluate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(a+\frac{k}{n}\right).$$

12030. *Proposed by Jonathan Sondow, New York, NY.* Let S be the set of positive integers d such that, for some multiple m of d,

$$\binom{m+d}{d}\equiv 1 \pmod{m}.$$

(a) Does S contain a prime number?
(b) Does S contain a number with distinct prime factors?
(c)* Does S contain a nontrivial prime power?

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12031. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

(a) Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\} dx\, dy = 1 - \gamma,$$

where $\{a\}$ denotes the fractional part of a, and γ is Euler's constant.

(b) Let k be a nonnegative integer. Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\}^k dx\, dy = \int_0^1 \left\{ \frac{1}{x} \right\}^k dx.$$

12032. *Proposed by David Galante (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* For a positive integer n, compute

$$\sum_{p=0}^{n} \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{2p} \binom{n}{k} 2^{n-k}.$$

12033. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let $ABCD$ be a convex quadrilateral with area S. Prove

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \geq 8S + AB \cdot CD + BC \cdot AD - AC \cdot BD.$$

SOLUTIONS

A Radical Bound

11906 [2016, 400]. *Proposed by Robert Bosch, Archimedean Academy, FL.* Let x, y, and z be positive numbers such that $xyz = 1$. Prove

$$\sqrt{\frac{x+1}{x^2-x+1}} + \sqrt{\frac{y+1}{y^2-y+1}} + \sqrt{\frac{z+1}{z^2-z+1}} \leq 3\sqrt{2}.$$

Solution by Ramya Dutta, Chennai Mathematical Institute, Chennai, India. Since $xyz = 1$, we can choose $a, b, c \in \mathbb{R}^+$ such that $x = a/b$, $y = b/c$, and $z = c/a$. The inequality then becomes

$$\sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{a^2-ab+b^2}} \leq 3\sqrt{2},$$

where $\sum_{\text{cyc}} \tau(a,b,c)$ denotes the cyclic sum $\tau(a,b,c) + \tau(b,c,a) + \tau(c,a,b)$. Since

$$\sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{a^2-ab+b^2}} = \sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{\frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2}} \leq \sum_{\text{cyc}} 2\sqrt{\frac{b}{a+b}}$$

and since

$$\sum_{\text{cyc}} \sqrt{\frac{b}{a+b}} = \sum_{\text{cyc}} \sqrt{b+c} \cdot \sqrt{\frac{b}{(a+b)(b+c)}}$$

$$\leq \left(\sum_{\text{cyc}}(b+c)\right)^{1/2} \cdot \left(\sum_{\text{cyc}} \frac{b}{(a+b)(b+c)}\right)^{1/2} = \sqrt{2} \cdot \left[\sum_{\text{cyc}} \frac{b(a+b+c)}{(a+b)(b+c)}\right]^{1/2},$$

by the Cauchy–Schwarz inequality, it suffices to prove

$$\sum_{\text{cyc}} \frac{b(a+b+c)}{(a+b)(b+c)} \leq \frac{9}{4}.$$

This is equivalent to

$$\sum_{\text{cyc}} \left(1 - \frac{b(a+b+c)}{(a+b)(b+c)}\right) \geq \frac{3}{4}, \qquad \text{or} \qquad \sum_{\text{cyc}} \frac{ac}{(a+b)(b+c)} \geq \frac{3}{4},$$

which is in turn equivalent to

$$\sum_{\text{cyc}} ac(a+c) \geq \frac{3}{4}(a+b)(b+c)(a+c) = \frac{3}{4}\left(2abc + \sum_{\text{cyc}} ac(a+c)\right).$$

Solving this for the cyclic sum, we obtain

$$\sum_{\text{cyc}} ac(a+c) \geq 6abc.$$

This follows from the AM–GM inequality applied to the six terms a^2b, ab^2, a^2c, ac^2, b^2c, and bc^2. Therefore,

$$\sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{a^2-ab+b^2}} \leq 2\sum_{\text{cyc}} \sqrt{\frac{b}{a+b}} \leq 2\sqrt{2} \cdot \sqrt{\frac{9}{4}} = 3\sqrt{2}.$$

Also solved by R. A. Agnew, T. Amdeberhan & V. H. Moll, R. Boukharfane (France), P. Bracken, M. V. Channakeshava (India), H. Chen, P. P. Dályay (Hungary), M. Dincă (Romania), H. Evans, D. Fleischman, S. Gayen (India), J.-P. Grivaux (France), N. Grivaux (France), O. Kouba (Syria), M. E. Kuczma (Poland), K.-W. Lau (China), J. H. Lindsey II, S. Malikić (Canada), V. Mikayelyan (Armenia), M. Omarjee (France), Á. Plaza (Spain), J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, M. R. Yegan (Iran), Con Amore Problem Group (Denmark), FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, and the proposer.

An Inequality Applied To Eigenvalues

11907 [2016, 400]. *Proposed by Xiang-Qian Chang, Massachusetts College of Pharmacy and Health Sciences, Boston, MA.* Let A be an $n \times n$ positive-definite Hermitian matrix, with minimum and maximum eigvenvalues λ and ω, respectively. Prove

$$\left(\frac{1}{\omega}\frac{\text{Tr}(A)}{n} + \frac{\omega n}{\text{Tr}(A)}\right)^n \leq \det\left(\frac{1}{\omega}A + \omega A^{-1}\right),$$
$$\left(\frac{1}{\lambda}\frac{n}{\text{Tr}(A^{-1})} + \lambda\frac{\text{Tr}(A^{-1})}{n}\right)^n \leq \det\left(\frac{1}{\lambda}A + \lambda A^{-1}\right).$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The function $f(t) = \log(t + 1/t) = \log(t^2+1) - \log t$ has

$$f''(t) = \frac{1}{t^2} + \frac{2(1-t^2)}{(1+t^2)^2} \geq 0$$

for $0 < t \le 1$. Hence by Jensen's inequality, for any $x_1, \ldots, x_n \in (0, 1]$, we have

$$\left(\frac{x_1 + \cdots + x_n}{n} + \frac{n}{x_1 + \cdots + x_n}\right)^n \le \prod_{k=1}^{n} (x_k + x_k^{-1}).$$

Let the eigenvalues of A (with multiplicities) be $\lambda_1, \ldots, \lambda_n > 0$. Applying this inequality to $x_k = \lambda_k/\omega$, we obtain the first inequality requested. Applying it to $x_k = \lambda/\lambda_k \le 1$, we obtain the second inequality.

Editorial comment. The problem statement contained a typographical error: The exponent n was missing from the second inequality.

Also solved by H. Chen, P. P. Dályay (Hungary), R. Dutta (India), D. Fleischman, N. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France) & R. Tauraso (Italy), J. C. Smith, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Generalized Bijection for Partitions

11908 [2016, 504]. *Proposed by George E. Andrews, The Pennsylvania State University, University Park, PA, and Emeric Deutsch, Polytechnic Institute of New York University, Brooklyn, NY.* Let n and k be nonnegative integers. Show that the number of partitions of n having k even parts is the same as the number of partitions of n in which the largest repeated part is k (defined to be 0 if the parts are all distinct). For example, 7 has three partitions with two even parts ($4 + 2 + 1 = 3 + 2 + 2 = 2 + 2 + 1 + 1 + 1$) and also three partitions in which the largest repeated part is 2 ($3 + 2 + 2 = 2 + 2 + 2 + 1 = 2 + 2 + 1 + 1 + 1$).

Solution I by Meghana Madhyastha, International Institute of Information Technology, Bangalore, India. Fixing k, we find the generating functions of the two quantities, indexed by n. In a partition where k is the largest repeated part, each part smaller than k can appear any number of times, k appears at least twice, and parts larger than k appear at most once. Hence, the generating function is

$$\left(\prod_{i=1}^{k-1} \frac{1}{1 - x^i}\right) \frac{x^{2k}}{1 - x^k} \prod_{i=k+1}^{\infty} (1 + x^i).$$

For a partition with exactly k even parts, consider the even and odd parts separately. In the conjugate of the partition using the even parts, each part occurs an even number of times, and the largest part is k (occurring at least twice). There is no restriction on the use of odd parts. Hence, the generating function is

$$\left(\prod_{i=1}^{k-1} \frac{1}{1 - x^{2i}}\right) \frac{x^{2k}}{1 - x^{2k}} \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}}.$$

Straightforward manipulation shows that both generating functions equal

$$x^{2k} \prod_{i=1}^{\infty} \frac{1 - x^{2(k+i)}}{1 - x^i}.$$

Solution II by Nicolas Allen Smoot, Georgia Southern University, Statesboro, GA. We prove the following generalization: Given nonnegative integers n and k and a positive integer d, the number of partitions of n having exactly k parts divisible by d is the same as the number of partitions of n in which k is the largest part that occurs at least d times.

When $n = 0$, the claim is trivial, so assume $n > 0$. We construct a bijection. Let λ be a partition of n having exactly k parts divisible by d. Let A consist of all the parts in λ that are divisible by d, and let B consist of the other parts (A is empty when $k = 0$). Map A to its conjugate partition A^*, in which the largest part is k and every part occurs a multiple of d times.

We map B, which has no part divisible by d, to a partition B', in which no part occurs at least d times, bijectively. For this we use Glaisher's bijection (J. W. L. Glaisher, A theorem in partitions, *Messenger of Math.* **12** (1883) 158–170). This turns B into B' by iteratively combining d equal parts into one part until no instance of d identical parts remains. The proof that this is a bijection relies on the fact that every positive integer is expressible as a power of d times a number not divisible by d in a unique way.

Note that in the union of A^* and B', the largest part occurring at least d times is k. To invert the map, we separate a partition μ in which k is the largest part occurring at least d times into the contributions A^* and B', where A^* will have each part occurring a multiple of d times (k being the largest part) and B' will have no part occurring at least d times.

For each part i, occurring m_i times in μ, put $d\,\lfloor m_i/d \rfloor$ of the copies of i into A^*. Put the remaining copies into B'; no part occurs at least d times among these. This is the only way that μ can be separated into two partitions in the specified families. We can now invert the two maps separately and recombine the outcomes to retrieve the only partition λ that maps to μ under the given function.

Editorial comment. Mingjia Yang also proved the generalization in Solution II.

Also solved by D. Beckwith, K. David, Y. J. Ionin, P. Lalonde (Canada), P. W. Lindstrom, G. Lord, O. P. Lossers (Netherlands), R. Nandan, M. Sawhney, J. H. Smith, R. Stong, R. Tauraso (Italy), V. Walavalkar (India), E. T. White, M. Wildon (U. K.), M. Yang, GCHQ Problem Solving Group (U. K.), and the proposers.

Reciprocal Fibonacci Arctangents

11910 [2016, 504]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Let G_k be the reciprocal of the kth Fibonacci number, for example, $G_4 = 1/3$ and $G_5 = 1/5$. Find

$$\sum_{n=1}^{\infty} \left(\arctan G_{4n-3} + \arctan G_{4n-2} + \arctan G_{4n-1} - \arctan G_{4n}\right).$$

Solution by Ramya Dutta, Chennai Mathematical Institute, Chennai, India. The sum is $\pi/2 + \arctan((\sqrt{5}-1)/2)$. To see this, we write F_n for the nth Fibonacci number, and we make use of Catalan's identity $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$ and d'Ocagne's identity $F_nF_{n+1} + F_nF_{n-1} = F_{2n}$. Since

$$\arctan\frac{1}{F_{2n}} - \arctan\frac{1}{F_{2n+2}} = \arctan\frac{F_{2n+2}-F_{2n}}{F_{2n}F_{2n+2}+1} = \arctan\frac{F_{2n+1}}{F_{2n+1}^2} = \arctan\frac{1}{F_{2n+1}},$$

we have

$$\begin{aligned} \arctan\frac{1}{F_{4n-3}} + \arctan\frac{1}{F_{4n-1}} &= \arctan\frac{1}{F_{4n-4}} - \arctan\frac{1}{F_{4n-2}} \\ + \arctan\frac{1}{F_{4n-2}} - \arctan\frac{1}{F_{4n}} &= \arctan\frac{1}{F_{4n-4}} - \arctan\frac{1}{F_{4n}}. \end{aligned} \tag{1}$$

Equation (1) holds for all positive integers n, including $n = 1$, provided that we interpret $\arctan(1/0)$ to be $\pi/2$. We also have

$$\begin{aligned} \arctan\frac{F_{n-1}}{F_n} - \arctan\frac{F_n}{F_{n+1}} &= \arctan\frac{F_{n+1}F_{n-1} - F_n^2}{F_nF_{n-1} + F_nF_{n+1}} \\ &= \arctan\frac{(-1)^n}{F_{2n}} = (-1)^n \arctan\frac{1}{F_{2n}}. \end{aligned} \tag{2}$$

Thus,

$$\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-3}}+\arctan\frac{1}{F_{4n-2}}+\arctan\frac{1}{F_{4n-1}}-\arctan\frac{1}{F_{4n}}\right)$$
$$=\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-4}}-\arctan\frac{1}{F_{4n}}\right)+\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-2}}-\arctan\frac{1}{F_{4n}}\right)\quad\text{by (1)}$$
$$=\frac{\pi}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\arctan\frac{1}{F_{2n}}=\frac{\pi}{2}-\sum_{n=1}^{\infty}\left(\arctan\frac{F_{n-1}}{F_n}-\arctan\frac{F_n}{F_{n+1}}\right)\quad\text{by (2)}$$
$$=\frac{\pi}{2}+\lim_{n\to\infty}\arctan\frac{F_n}{F_{n+1}}=\frac{\pi}{2}+\arctan\frac{1}{\varphi},$$

where $\varphi=(1+\sqrt{5})/2$. This gives the claimed result.

Also solved by K. Adegoke (Nigeria) & Á. Plaza (Spain), B. Bradie, M. V. Channakeshava (India), P. P. Dályay (Hungary), D. Fleischman, D. Fritze (Germany), M. Goldenberg & M. Kaplan, S. Hitotumatu (Japan), O. Kouba (Syria), M. E. Kuczma (Poland), P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), A. Rajkumar & F. Mawyer, M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, M. Wildon (U. K.), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Symmetric Inequality for Real Triples

11911 [2016, 504]. *Proposed by Marian Cucoanes, Focşani, Romania, and Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let a, b, and c be positive real numbers such that $1+ab+bc+ca=a+b+c+2abc$. Prove $a^3+b^3+c^3+5abc\geq 1$, and determine when equality holds.

Solution by Marcin E. Kuczma, University of Warsaw, Poland. The claim holds trivially when $\max\{a,b,c\}\geq 1$. Hence we assume $a,b,c\in(0,1)$, which yields $abc<1$. In terms of the elementary symmetric polynomials $A=a+b+c$, $B=ab+bc+ca$, and $C=abc$, the constraint says $1+B=A+2C$. Let

$$X=a^3+b^3+c^3+5abc-1=A^3-3AB+8C-1=A^3+(4-3A)B+(3-4A).$$

We must show that X is nonnegative.

The AM–GM inequality implies $B/3\geq C^{2/3}$. Combining this with $C<1$ yields $B\geq 3C^{2/3}>3C>2C$, which implies $A=1+B-2C>1$. If $4-3A>0$, then $B=A+2C-1>A-1$ yields

$$X>A^3+(4-3A)(A-1)+(3-4A)=(A-1)^3>0.$$

If $4-3A\leq 0$, then the Cauchy–Schwarz inequality yields $A^2\leq 3(a^2+b^2+c^2)=3(A^2-2B)$, and thus $B\leq A^2/3$. Therefore,

$$X\geq A^3+(4-3A)(A^2/3)+(3-4A)=(2A-3)^2/3\geq 0.$$

Equality requires $2A-3=0$ as well as equality in the Cauchy–Schwarz application; the latter occurs when $a=b=c$. Thus equality holds if and only if $a=b=c=1/2$.

Also solved by A. Alt, P. P. Dályay (Hungary), M. Dincă (Romania), D. Fleischman, N. Grivaux (France), Y. Ionin, K.-W. Lau (China), J. H. Lindsey II, T. L. McCoy, R. Stong, T. Wiandt, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Bounds for the Sum of the Mixtilinear Radii

11912 [2016, 505]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let ω be the circumscribed circle of triangle ABC, and let R and r be the radii of its circumcircle and incircle, respectively. Let r_A, r_B, and r_C be the radii of the A-, B-, and C-mixtilinear incircles of ABC and ω, respectively. Prove $4r \le r_A + r_B + r_C \le (5R + 6r)/4$.

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY.

Lower bound: Let the angles at A, B, and C be α, β, and γ, respectively. We have $r_A = r\sec^2(\alpha/2)$, and similarly for r_B and r_C. (See, for example, L. Bankoff, A mixtilinear adventure, *Crux Mathematicorum* **9** (1983) 2–7.) Now we have

$$\sec^2\left(\frac{\alpha/2+\beta/2+\gamma/2}{3}\right) = \sec^2\frac{\pi}{6} = \frac{4}{3}.$$

For $x \in (0, \pi)$, let $f(x) = \sec^2(x/2)$. Since $f''(x) \ge 0$, we may apply Jensen's inequality to f and obtain

$$\begin{aligned}4r &= 3r\sec^2\left(\frac{\alpha/2+\beta/2+\gamma/2}{3}\right)\\ &\le 3r\frac{\sec^2(\alpha/2)+\sec^2(\beta/2)+\sec^2(\gamma/2)}{3} = r_A + r_B + r_C.\end{aligned}$$

Upper bound: Let a, b, c denote the side lengths opposite angles A, B, C, respectively. Let x, y, z be the distances from A, B, C, respectively, to the points of tangency of the incircle. Since $a = y + z$, $b = z + x$, and $c = z + y$, the semiperimeter s of the triangle is $x + y + z$. Now $\sec^2(\alpha/2) = \tan^2(\alpha/2) + 1 = r^2/x^2 + 1$, and similarly for β and γ. So $r_A = r(r^2/x^2 + 1)$, $r_B = r(r^2/y^2 + 1)$, and $r_C = r(r^2/z^2 + 1)$, and the desired inequality becomes

$$r^2\left(\frac{r^2}{x^2}+\frac{r^2}{y^2}+\frac{r^2}{z^2}\right) + 3r^2 \le \frac{3r^2}{2} + \frac{5Rr}{4}.$$

The area T of the triangle is given by any of the three formulae $T = rs$, $T = abc/4R$, or $T^2 = s(s-a)(s-b)(s-c) = sxyz$ (Heron's formula). From the first and the third, we obtain $r^2 = xyz/s$. From the first and second, we obtain $Rr = abc/4s$. Substituting these expressions into the desired inequality yields

$$\frac{xyz}{s}\left(\frac{xyz}{sx^2}+\frac{xyz}{sy^2}+\frac{xyz}{sz^2}\right) + \frac{3xyz}{2s} \le \frac{5abc}{16s}.$$

Expressing a, b, c in terms of x, y, z and rearranging, we produce the equivalent inequality

$$\begin{aligned}&3(2x^2y^2 + 2y^2z^2 + 2z^2x^2) + 2(2x^2yz + 2y^2zx + 2z^2xy)\\ &\le 5(x^3y + y^3x + y^3z + z^3y + z^3x + x^3z). \qquad (*)\end{aligned}$$

We now recall Muirhead's inequality, which asserts the following (in the case of three variables). Let (a, b, c) and (p, q, r) be two nonnegative triples satisfying the conditions $a \ge b \ge c$, $p \ge q \ge r$, $a \ge p$, $a + b \ge p + q$, and $a + b + c = p + q + r$. For all nonnegative real numbers x, y, and z, we have $\sum x^a y^b z^c \ge \sum x^p y^q z^r$, where the sums are taken over all $3! = 6$ permutations of the three variables x, y, z.

Applying Muirhead's inequality with $(a, b, c) = (3, 1, 0)$ and $(p, q, r) = (2, 2, 0)$, we get

$$x^3y + y^3x + y^3z + z^3y + z^3x + x^3z \ge 2x^2y^2 + 2y^2z^2 + 2z^2x^2.$$

Applying Muirhead's inequality with $(a, b, c) = (3, 1, 0)$ and $(p, q, r) = (2, 1, 1)$, we get

$$x^3y + y^3x + y^3z + z^3y + z^3x + x^3z \ge 2x^2yz + 2y^2zx + 2z^2xy.$$

Together these give the required inequality (*).

Also solved by O. Geupel (Germany), O. Kouba (Syria), M. E. Kuczma (Poland), R. Stong, J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

An Integral Inequality

11918 [2016, 613]. *Proposed by Phu Cuong Le Van, College of Education, Hue University, Hue City, Vietnam.* Let f be n times continuously differentiable on [0, 1], with $f(1/2) = 0$ and $f^{(i)}(1/2) = 0$ when i is even and at most n. Prove

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{(2n+1)2^{2n}(n!)^2}\int_0^1 (f^{(n)}(x))^2\,dx.$$

Solution by Patrick J. Fitzsimmons, University of California, San Diego, La Jolla, CA. Let F be an antiderivative of f. Using Taylor's theorem with remainder in integral form, we expand F in powers of $t - 1/2$ to obtain

$$F(t) = F(1/2) + \sum_{k=0}^{n-1} \frac{f^{(k)}(1/2)}{(k+1)!}\left(t - \frac{1}{2}\right)^{k+1} + \int_{1/2}^t \frac{f^{(n)}(x)}{n!}(t-x)^n dx$$

for any t in [0, 1]. In particular, with $t = 1$,

$$\int_{1/2}^1 f(x)\,dx = \sum_{k=0}^{n-1} \frac{f^{(k)}(1/2)}{(k+1)!}\left(\frac{1}{2}\right)^{k+1} + \int_{1/2}^1 \frac{f^{(n)}(x)}{n!}(1-x)^n dx,$$

and with $t = 0$,

$$\int_0^{1/2} f(x)\,dx = -\sum_{k=1}^{n-1} \frac{f^{(k)}(1/2)}{(k+1)!}\left(-\frac{1}{2}\right)^{k+1} + \int_0^{1/2} \frac{f^{(n)}(x)}{n!}(-x)^n dx.$$

When we add these, the terms for odd k cancel, while the terms for even k vanish by hypothesis. It follows that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x) f^{(n)}(x)\,dx,$$

where

$$g(x) = \begin{cases} (-x)^n/n! & \text{when } 0 \le x \le 1/2; \\ (1-x)^n/n! & \text{when } 1/2 \le x \le 1. \end{cases}$$

Now the desired inequality follows from the Cauchy–Schwarz inequality, because

$$\int_0^1 g(x)^2\,dx = \int_0^{1/2} \frac{x^{2n}}{(n!)^2}\,dx + \int_{1/2}^1 \frac{(1-x)^{2n}}{(n!)^2}\,dx = \frac{1}{(2n+1)2^{2n}(n!)^2}.$$

Also solved by U. Abel (Germany), K. F. Andersen (Germany), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), A. Harnist (France), E. A. Herman, K. Koo (China), O. Kouba (Syria), M. E. Kuczma (Poland), J. H. Lindsey II, O. P. Lossers (Netherlands), F. Marino (Italy), V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), Á. Plaza & F. Perdomo (Spain), M. A. Prasad (India), M. Sawhney, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit
Proposed solutions to the problems below should be submitted by August 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12034. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let N be any natural number that is not a multiple of 10. Prove that there is a multiple of N each of whose digits in base 10 is 1, 2, 3, 4, or 5.

12035. *Proposed by Dinh Thi Nguyen, Tuy Hòa, Vietnam.* Find the minimum value of

$$(a^2+b^2+c^2)\left(\frac{1}{(3a-b)^2}+\frac{1}{(3b-c)^2}+\frac{1}{(3c-a)^2}\right)$$

as a, b, and c vary over all real numbers with $3a \neq b$, $3b \neq c$, and $3c \neq a$.

12036. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.* Two metric spaces (X, d) and (X', d') are said to be *isometric* if there is a bijection $\phi : X \to Y$ such that $d(a, b) = d'(\phi(a), \phi(b))$ for all $a, b \in X$. Let X be an infinite set. Find all metrics d on X such that (X, d) and (X', d') are isometric for every subset X' of X of the same cardinality as X. (Here, d' is the metric induced on X' by d.)

12037. *Proposed by José Manuel Rodríguez Caballero, University of Quebec (UQAM), Montreal, QC, Canada.* For a positive integer n, let S_n be the set of pairs (a, k) of positive integers such that $\sum_{i=0}^{k-1}(a+i) = n$. Prove that the set

$$\left\{ n : \sum_{(a,k)\in S_n} (-1)^{a-k} \neq 0 \right\}$$

is closed under multiplication.

12038. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let ABC be an acute triangle with sides of length a, b, and c opposite angles A, B, and C, respectively, and with medians of length m_a, m_b, and m_c emanating from A, B, and C, respectively. Prove

$$\frac{m_a^2}{b^2+c^2}+\frac{m_b^2}{c^2+a^2}+\frac{m_c^2}{a^2+b^2} \geq 9\cos A\,\cos B\,\cos C.$$

doi.org/10.1080/00029890.2018.1438001

12039. *Proposed by Sandeep Silwal, Brookline, MA*. Let G be a graph with an even number of vertices. Show that there are two vertices in G with an even number of common neighbors.

12040. *Proposed by George Stoica, Saint John, NB, Canada*. Find all convergent series $\sum_{n=1}^{\infty} x_n$ of positive terms such that $\sum_{n=1}^{\infty} x_n x_{n+k}/x_k$ is independent of the positive integer k.

SOLUTIONS

A Weighted Vandermonde Convolution

11909 [2016, 504]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan*. Prove that for every positive integer m there exists a polynomial P_m in two variables, with integer coefficients, such that for all integers n and r with $0 \le r \le n$,

$$\sum_{k=-r}^{r} \binom{n}{r+k}\binom{n}{r-k} k^{2m} = \frac{P_m(n,r)}{\prod_{j=1}^{m}(2n-2j+1)}\binom{2n}{2r}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, California. A special case of the well-known Vandermonde convolution (proved by counting in two ways the choices of $2r$ balls from among n distinct black balls and n distinct white balls) is

$$\sum_{k=-r}^{r} \binom{n}{r+k}\binom{n}{r-k} = \binom{2n}{2r}.$$

Letting $x_{(j)} = \prod_{i=0}^{j-1}(x-i)$, we compute

$$\begin{aligned}
&\sum_{k=-r}^{r} \binom{n}{r+k}\binom{n}{r-k}(r+k)_{(m)}(r-k)_{(m)} \\
&\quad = (n_{(m)})^2 \sum_{k=m-r}^{r-m} \binom{n-m}{r+k-m}\binom{n-m}{r-k-m} \\
&\quad = (n_{(m)})^2 \binom{2(n-m)}{2(r-m)} = (n_{(m)})^2 \frac{\prod_{j=1}^{2m}(2r-j+1)}{\prod_{j=1}^{2m}(2n-j+1)}\binom{2n}{2r} \\
&\quad = \frac{n_{(m)} r_{(m)} \prod_{j=1}^{m}(2r-2j+1)}{\prod_{j=1}^{m}(2n-2j+1)}\binom{2n}{2r},
\end{aligned}$$

which has the desired form. Since, for fixed r, the polynomial $(-1)^m(r+k)_{(m)}(r-k)_{(m)}$ is monic of degree m in the variable k^2, we can write k^{2m} as a sum of integer multiples of these polynomials. Thus $\sum_{k=-r}^{r}\binom{n}{r+k}\binom{n}{r-k}k^{2m}$ also has the desired form.

Also solved by P. P. Dályay (Hungary), P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), and the proposer.

A Liminf Ratio

11913 [2016, 505]. *Proposed by George Stoica, Saint John, NB, Canada*. Let ϵ be a positive constant, and let f map $(0,\infty)$ to $\mathbb{R}^+$. Given $\lim_{x\to\infty} x^{1/\epsilon} f(x) = \infty$, prove

$$\liminf_{x\to\infty} \left| \frac{f'(x)}{f^{1+\epsilon}(x)} \right| = 0.$$

Solution by Nicole Grivaux, Paris, France. The statement of the problem should have included the assumption that f is differentiable. We proceed under this assumption.

For $x > 0$, let $g(x) = f(x)^{-\epsilon}$. The function g is positive, and

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to\infty} \frac{1}{(x^{1/\epsilon} f(x))^{\epsilon}} = 0.$$

By the triangle inequality,

$$\frac{|g(2x) - g(x)|}{x} \le 2\frac{g(2x)}{2x} + \frac{g(x)}{x}.$$

Thus $\lim_{x\to\infty} |g(2x) - g(x)|/x = 0$. By the mean value theorem,

$$\frac{|g(2x) - g(x)|}{x} \ge \inf_{t\in[x,2x]} |g'(t)|.$$

Thus $\lim_{x\to\infty} \inf_{t\ge x} |g'(t)| = 0$. Since $g'(t) = -\epsilon f'(t)/f^{1+\epsilon}(t)$, we obtain

$$\liminf_{x\to\infty} \left|f'(x)/f^{1+\epsilon}(x)\right| = 0,$$

as desired.

Editorial comment. The assumption that f is differentiable was included in the original problem proposal but was accidentally omitted in the editorial process.

Also solved by K. F. Andersen (Canada), P. P. Dályay (Hungary), O. Kouba (Syria), M. E. Kuczma (Poland), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), NSA Problems Group,, and the proposer.

Exchanging the Arguments

11916 [2016, 613]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Universitá di Roma "Tor Vergata," Rome, Italy*. Show that if n, r, and s are positive integers, then

$$\binom{n+r}{n}\sum_{k=0}^{s-1}\binom{r+k}{r-1}\binom{n+k}{n} = \binom{n+s}{n}\sum_{k=0}^{r-1}\binom{s+k}{s-1}\binom{n+k}{n}.$$

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. We use induction on s. For $s = 1$, we need $\binom{n+r}{n}r = (n+1)\sum_{k=0}^{r-1}\binom{n+k}{n}$, which holds by the standard identity $\sum_{k=0}^{r-1}\binom{n+k}{n} = \binom{n+r}{n+1}$.

For the induction step, note that the left side increases by $\binom{n+r}{n}\binom{r+s}{r-1}\binom{n+s}{n}$ when s changes to $s+1$. For the change in the right side, we compute

$$\begin{aligned}
&\binom{n+s+1}{n}\sum_{k=0}^{r-1}\binom{s+1+k}{s}\binom{n+k}{n} - \binom{n+s}{n}\sum_{k=0}^{r-1}\binom{s+k}{s-1}\binom{n+k}{n}\\
&= \left(\binom{n+s}{n} + \binom{n+s}{n-1}\right)\sum_{k=0}^{r-1}\binom{s+1+k}{s}\binom{n+k}{n} - \binom{n+s}{n}\sum_{k=0}^{r-1}\binom{s+k}{s-1}\binom{n+k}{n}\\
&= \binom{n+s}{n}\sum_{k=0}^{r-1}\binom{s+k}{s}\binom{n+k}{n} + \binom{n+s}{n-1}\sum_{k=0}^{r-1}\binom{s+1+k}{s}\binom{n+k}{n}\\
&= \binom{n+s}{n}\sum_{k=0}^{r-1}\left(\binom{s+k}{s}\binom{n+k}{n} + \binom{s+1+k}{s+1}\binom{n+k}{n-1}\right).
\end{aligned}$$

To complete the inductive step in s, it therefore suffices to show

$$\binom{n+r}{n}\binom{r+s}{r-1} = \sum_{k=0}^{r-1}\left(\binom{s+k}{s}\binom{n+k}{n} + \binom{s+1+k}{s+1}\binom{n+k}{n-1}\right). \tag{1}$$

We proceed by induction on r. When $r = 1$, this reduces to $n + 1 = 1 + n$, which is true. By considering the change in each side of (1) as r is incremented to $r + 1$, we see that we need to show

$$\binom{n+r+1}{n}\binom{r+s+1}{r} - \binom{n+r}{n}\binom{r+s}{r-1} = \binom{s+r}{s}\binom{n+r}{n} + \binom{s+1+r}{s+1}\binom{n+r}{n-1}.$$

Fortunately,

$$\begin{aligned}&\binom{n+r}{n}\binom{r+s}{r-1} + \binom{s+r}{s}\binom{n+r}{n} + \binom{s+1+r}{s+1}\binom{n+r}{n-1}\\ &\quad = \binom{n+r}{n}\binom{s+r+1}{s+1} + \binom{s+1+r}{s+1}\binom{n+r}{n-1} = \binom{n+r+1}{n}\binom{r+s+1}{r}.\end{aligned}$$

This completes the proof.

Editorial comment. Solvers used a variety of techniques. Moa Apagodu used the Wilf–Zeilberger Method. Robin Chapman and Omran Kouba generalized to polynomial identities. Christian Krattenthaler and Pierluigi Magli used a standard transformation formula from the theory of hypergeometric series.

Also solved by M. Apagodu, R. Chapman (U. K.), O. Kouba (Syria), C. Krattenthaler (Austria), P. Lalonde (Canada), J. H. Lindsey II, P. Magli (Italy), J. H. Smith, A. Stadler (Switzerland), R. Stong, and the proposers.

Rational Logarithms of Matrices

11917 [2016, 613]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let A be a 2×2 matrix with rational entries and both eigenvalues less than 1 in absolute value. Prove that $\log(I - A)$ has rational entries if and only if $A^2 = 0$. (Here $\log(I - X) = -X - X^2/2 - X^3/3 - \cdots$ when that sum converges.)

Solution by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND. Let λ and μ be the eigenvalues of A, with $|\lambda| < 1$ and $|\mu| < 1$. Since A is rational, the series for $\log(I - A)$ converges to a real 2×2 matrix with eigenvalues $\log(1 - \lambda)$ and $\log(1 - \mu)$. (See, for example, Jacobson's *Lectures in Abstract Algebra II: Linear Algebra*, Springer, 1953, p. 194.) The characteristic polynomial of a square rational matrix has rational coefficients. Hence, the eigenvalues of a square rational matrix are algebraic numbers.

If $\log(I - A)$ is a rational matrix, then both λ and $\log(1 - \lambda)$ are algebraic numbers. Since $1 - \lambda$ is an algebraic number, we have $\log(1 - \lambda) = 0$ by a theorem of Lindemann, which states that if α is a nonzero algebraic number, then e^α is transcendental. Hence $\lambda = 0$. By the same argument, $\mu = 0$. Since both eigenvalues of A are zero, $A^2 = 0$.

Conversely, if $A^2 = 0$ then $\log(I - A) = -A$, and $-A$ is a rational matrix.

Editorial comment. In the original problem statement, the word "rational" was errantly printed as "integer" (twice), rendering the problem nearly trivial.

Also solved by K. F. Andersen (Canada), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), P. P. Dályay (Hungary), S. de Luxán (Germany) & F. Perdomo (Spain) & Á. Plaza (Spain), D. Fleischman, C. Georghiou (Greece), N. Grivaux (France), J. Hartman, E. A. Herman, E. J. Ionaşcu, K. Koo (China), M. E. Kuczma (Poland), V. Kumar & R. Sarma (India), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), M. Sawhney, E. Schmeichel, A. Stadler (Switzerland), A. Stenger, R. Stong, J. Stuart,

R. Tauraso (Italy), N. S. Thornber, E. I. Verriest, Z. Vőrős (Hungary), T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), the Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

The Power of Minima

11919 [2016, 613]. *Proposed by Arkady Alt, San Jose, CA*. For positive integers m and k with $k \geq 2$, prove

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} ((n+1)^i - n^i) \sum_{j=1}^{n} j^{k+m-1}.$$

Solution by Pierre Lalonde, Kingsey Falls, QC, Canada. For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. The number of k-tuples $(i_1, \ldots, i_k) \in [n]^k$ such that $\min\{i_1, \ldots, i_k\} \geq r$ is $(n - r + 1)^k$, so the number of k-tuples such that $\min\{i_1, \ldots, i_k\} = r$ is $(n - r + 1)^k - (n - r)^k$. Thus

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{r=1}^{n} ((n - r + 1)^k - (n - r)^k) r^m.$$

Break this expression into two sums, setting $j = n - r + 1$ in the first sum and $j = n - r$ in the second. After recombining the summations, apply the binomial theorem to each term in the first factor and then interchange the order of summation. This gives

$$\begin{aligned}
\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m &= \sum_{r=1}^{n} (n - r + 1)^k r^m - \sum_{r=1}^{n} (n - r)^k r^m \\
&= \sum_{j=1}^{n} j^k (n - j + 1)^m - \sum_{j=1}^{n} j^k (n - j)^m = \sum_{j=1}^{n} ((n + 1 - j)^m - (n - j)^m) j^k \\
&= \sum_{j=1}^{n} \sum_{i=0}^{m} \binom{m}{i} \left((n+1)^i - n^i\right)(-j)^{m-i} j^k = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) \sum_{j=1}^{n} j^{k+m-i},
\end{aligned}$$

as desired.

Editorial comment. The identity also holds for $k = 1$. Ramya Dutta noted that the argument above implies

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \ldots, i_k\})^m = \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} (\min\{i_1, \ldots, i_m\})^k,$$

so these sums also equal $\sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} ((n+1)^i - n^i) \sum_{j=1}^{n} j^{k+m-i}$.

Also solved by U. Abel (Germany), T. Amdeberhan & V. H. Moll, D. Beckwith, P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), Y. J. Ionin, O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), M. Sawhney, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Generalization of a Fibonacci Identity

11920 [2016, 614]. *Proposed by Ángel Plaza and Sergio Falcón, University of Las Palmas de Gran Canaria, Spain*. For positive integer k, let $\langle F_k \rangle$ be the sequence defined by initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and the recurrence $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$. Find a closed form for $\sum_{i=0}^{n} \binom{2n+1}{i} F_{k,2n+1-2i}$.

Solution by Johann Cigler, University of Vienna, Vienna, Austria. The value of the sum is $(k^2+4)^n$.

We consider more generally the Fibonacci polynomial $f_n(x,s)$ defined by the recurrence $f_n(x,s) = xf_{n-1}(x,s) + sf_{n-2}(x,s)$ with $f_0(x,s) = 0$ and $f_1(x,s) = 1$. Let $\alpha = (x+\sqrt{x^2+4s})/2$ and $\beta = (x-\sqrt{x^2+4s})/2$, so that α and β are the roots of the characteristic equation $z^2 - xz - s = 0$ of the recurrence. Since $\alpha^2 = x\alpha + s$ and $\beta^2 = x\beta + s$, the well-known Binet formula $f_n(x,s) = (\alpha^n - \beta^n)/(\alpha - \beta)$ follows immediately by induction on n.

Since $s = -\alpha\beta$, we have $\alpha + \frac{s}{\alpha} = -(\beta + \frac{s}{\beta}) = \alpha - \beta = \sqrt{x^2+4s}$. Using the binomial theorem in the main step, we compute

$$\begin{aligned}
\sum_{i=0}^{n}\binom{2n+1}{i}s^i f_{2n+1-2i}(x,s) &= \frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{2n+1}{i}s^i(\alpha^{2n+1-2i} - \beta^{2n+1-2i}) \\
&= \frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{2n+1}{i}(-1)^i(\alpha^{2n+1-i}\beta^i - \beta^{2n+1-i}\alpha^i) \\
&= \frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{2n+1}{i}\alpha^{2n+1-i}(-\beta)^i + \frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{2n+1}{2n+1-i}\alpha^i(-\beta)^{2n+1-i} \\
&= \frac{(\alpha-\beta)^{2n+1}}{\alpha-\beta} = (x^2+4s)^n.
\end{aligned}$$

Editorial comment. The kth sequence $\{F_{k,n}\}_{n\geq 0}$ is known as the sequence of *k-Fibonacci numbers*. For $k = 1$, it is the usual Fibonnaci sequence, and Brian Bradie noted that this special case of the problem is part (b) of Problem O362 in *Mathematical Reflections*, Issue 1, 2016.

Also solved by T. Amdeberhan & V. H. Moll, D. Beckwith, B. Bradie, J. F. Buitrago Vélez, T. Cunningham, P. P. Dályay (Hungary), D. Fleischman, E. J. Ionaşcu, O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Nandan, M. A. Prasad (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. Terr, Z. Vőrős (Hungary), M. Vowe (Switzerland), L. Zhou, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposers.

A Harmonious Series

11921 [2016, 614]. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Prove

$$\log^2(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)2^{k+1}} + \log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k} + \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^k} = \frac{\zeta(4)+\log^4(2)}{4}.$$

(Here, $H_k = \sum_{j=1}^{k} 1/j$ and ζ denotes the Riemann zeta function.)

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. First note that by two applications of integration by parts, for $k \geq 1$ and $t > 0$ we have

$$\int_0^t x^k \log^2(x)\,dx = \frac{t^{k+1}\log^2(t)}{k+1} - \frac{2t^{k+1}\log(t)}{(k+1)^2} + \frac{2t^{k+1}}{(k+1)^3}. \tag{1}$$

Since

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x)$$

for $|x| < 1$, we have

$$\sum_{k=1}^{\infty} H_k x^k = \sum_{k=1}^{\infty} x^k \sum_{n=1}^{k} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} x^k = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{m=0}^{\infty} x^m = -\frac{\log(1-x)}{1-x}. \tag{2}$$

Let S denote the left side of the proposed equality. Using (1) with $t = 1/2$ and then (2), we find

$$\begin{aligned} S &= \sum_{k=1}^{\infty} H_k \left(\frac{\log^2(2)}{(k+1)2^{k+1}} + \frac{\log(2)}{(k+1)^2 2^k} + \frac{1}{(k+1)^3 2^k} \right) \\ &= \sum_{k=1}^{\infty} H_k \int_0^{1/2} x^k \log^2(x)\, dx = \int_0^{1/2} \sum_{k=1}^{\infty} H_k x^k \log^2(x)\, dx \qquad (3) \\ &= -\int_0^{1/2} \frac{\log(1-x)\log^2(x)}{1-x}\, dx, \end{aligned}$$

where the interchange of the integration and summation signs is justified by the positivity of the integrands.

Using integration by parts and then substituting x for $1-x$, we have

$$\begin{aligned} S &= -\int_0^{1/2} \frac{\log(1-x)\log^2(x)}{1-x}\, dx \\ &= \left[\frac{1}{2} \log^2(1-x)\log^2(x) \right]_0^{1/2} - \int_0^{1/2} \frac{\log^2(1-x)\log(x)}{x}\, dx \qquad (4) \\ &= \frac{1}{2}\log^4(2) - \int_{1/2}^{1} \frac{\log^2(x)\log(1-x)}{1-x}\, dx. \end{aligned}$$

Adding (3) and (4), we obtain

$$2S = \frac{1}{2}\log^4(2) + I, \qquad \text{where} \qquad I = -\int_0^1 \frac{\log^2(x)\log(1-x)}{1-x}\, dx. \tag{5}$$

To calculate I, we use (2) and then (1) with $t = 1$ to obtain

$$\begin{aligned} I &= \int_0^1 \log^2(x) \left(\sum_{k=1}^{\infty} H_k x^k \right) dx = \sum_{k=1}^{\infty} H_k \int_0^1 x^k \log^2(x)\, dx \\ &= \sum_{k=1}^{\infty} \frac{2H_k}{(k+1)^3} = 2\sum_{k=0}^{\infty} \frac{1}{(k+1)^3} \left(H_{k+1} - \frac{1}{k+1} \right) = 2\sum_{n=1}^{\infty} \frac{H_n}{n^3} - 2\zeta(4). \qquad (6) \end{aligned}$$

Since $H_n = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+n} \right) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)}$, we have

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \sum_{n,k \geq 1} \frac{1}{n^2 k(k+n)} = \sum_{n,k \geq 1} \frac{1}{nk^2(k+n)}.$$

Therefore,

$$2\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \sum_{n,k \geq 1} \left(\frac{1}{n^2 k(k+n)} + \frac{1}{nk^2(k+n)} \right) = \sum_{n,k \geq 1} \frac{1}{n^2 k^2} = \zeta(2)^2.$$

Substituting into (6) and (5) yields

$$I = \zeta(2)^2 - 2\zeta(4) = \frac{\pi^4}{36} - \frac{\pi^4}{45} = \frac{\pi^4}{180} = \frac{1}{2}\zeta(4)$$

and therefore $S = \frac{1}{4}\log^4(2) + \frac{1}{4}\zeta(4)$, which is the desired result.

Editorial comment. Several solvers found the values of the three summations in the original problem statement:

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)2^{k+1}} = \frac{1}{2}\log^2(2), \quad \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2 2^k} = \frac{1}{4}\zeta(3) - \frac{1}{3}\log^3(2),$$

$$\text{and} \quad \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3 2^k} = \frac{1}{4}\zeta(4) + \frac{1}{12}\log^4(2) - \frac{1}{4}\log(2)\zeta(3).$$

Also solved by P. Bracken, H. Chen, P. P. Dályay (Hungary), B. E. Davis, R. Dutta (India), M. L. Glasser, K.-W. Lau (China), R. Nandan, M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

Squares of Palindromes Are Not Palindromes

11922 [2016, 722]. *Proposed by Max Alekseyev, George Washington University, Washington, DC.* Find every positive integer n such that both n and n^2 are palindromes when written in binary (with no leading zeros).

Solution by Yury J. Ionin, Central Michigan University, Mt. Pleasant, MI. The only such integers n are 1 and 3. Since leading zeros are forbidden, palindromes must be odd. Since $n = 1$ and $n = 3$ obviously succeed, it suffices to assume $n \geq 5$. Let n be a palindrome with $n \geq 5$. We refer to the binary expansion of n as n itself.

Let $s = \lfloor \log_2 n \rfloor$, so n has $s+1$ bits and $2^s < n < 2^{s+1}$. We have $2^{2s} < n^2 < 2^{2s+2}$. If all bits of n are 1, then $n = 2^{s+1} - 1$ and $n^2 = 2^{2s+2} - 2^{s+2} + 1$. Since $s \geq 2$, we conclude that n^2 begins with 11 and ends with 01 and is not a palindrome.

Hence, we may let k and l be the positive integers such that n has k leading bits equal to 1 and the next l bits of n equal 0, followed by a 1. Using parenthesized superscripts for multiplicity, we thus have $n = 1^{(k)}0^{(l)} \cdots 10^{(l)}1^{(k)}$.

We use two facts. First, if a and b are odd and agree in their last m bits, so $a \equiv b \pmod{2^m}$, then a^2 and b^2 agree in their last $m+1$ bits (that is, $a^2 \equiv b^2 \pmod{2^{m+1}}$). If also $m \geq 2$ and they disagree in the previous bit, so $a \not\equiv b \pmod{2^{m+1}}$, then $a^2 \not\equiv b^2 \pmod{2^{m+2}}$ (using $a \equiv b + 2^m \pmod{2^{m+1}}$).

If $k \geq 3$, then $n > 2^s + 2^{s-1} + 2^{s-2}$ and $n \equiv 7 \pmod 8$. Therefore, $2^{2s+2} > n^2 > 2^{2s+1} + 2^{2s}$ and $n^2 \equiv 49 \pmod{16}$. Thus, n^2 begins with 11 and ends with 0001 and is not a palindrome.

If $k = 2$, then $2^{s+1} > n > 2^s + 2^{s-1} + 2^{s-l-2}$. At the low end, $n \equiv 3 \pmod{2^{l+2}}$ and $n \not\equiv 3 \pmod{2^{l+3}}$. Therefore, $n^2 \equiv 9 \pmod{2^{l+3}}$, but $n^2 \not\equiv 9 \pmod{2^{l+4}}$. Thus the last $l+4$ bits of n^2 are $10^{(l-1)}1001$. If n^2 is a palindrome, then it begins with $10010^{(l-1)}1$, and hence $n^2 < 2^{2s+1} + 2^{2s-2} + 2^{2s-l-1}$. This contradicts

$$n^2 > (2^s + 2^{s-1} + 2^{s-l-2})^2 > 2^{2s+1} + 2^{2s-2} + 2^{2s-l-1}.$$

Finally, if $k = 1$, then $n \equiv 1 \pmod{2^{l+1}}$ and $n \not\equiv 1 \pmod{2^{l+2}}$. Thus $n^2 \equiv 1 \pmod{2^{l+2}}$ and $n^2 \not\equiv 1 \pmod{2^{l+3}}$, so the last $l+3$ bits of n^2 are $10^{(l+1)}1$. If n^2 is a palindrome with $n^2 < 2^{2s+1}$, then $n^2 < 2^{2s} + 2^{2s-l}$. However, $n \geq 2^s + 2^{s-l-1}$, so $n^2 \geq 2^{2s} + 2^{2s-l} + 2^{2s-2l-2}$, a contradiction. Therefore, we may assume $n^2 > 2^{2s+1}$.

Since $(2^s + 2^{s-2})^2 = 25 \cdot 2^{2s-4} < 2^{2s+1}$, we have $n > 2^s + 2^{s-2}$, and hence $l = 1$. Now n^2 must begin with 1001 to be a palindrome, yielding $n^2 \geq 2^{2s+1} + 2^{2s-2} = 9 \cdot 2^{2s-2}$. This requires $n \geq 3 \cdot 2^{s-1} = 2^s + 2^{s-1}$, contradicting $k = 1$.

Also solved by R. Boukharfane (France), R. Chapman (U. K.), J. Christopher, E. Donelson, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada) & D. Dimitrov (Slovenia), K. T. L. Koo (China), J. H. Lindsey II, B. Randé (France), R. Stong, R. Tauraso (Italy), Z. Vőrős (Hungary), GCHQ Problem Solving Group (U. K.), and the proposer.

Sign of a Sine Expression

11923 [2016, 722]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria*. Let f_p be the function on $(0, \pi/2)$ given by

$$f_p(x) = (1 + \sin x)^p - (1 - \sin x)^p - 2\sin(px).$$

Prove $f_p > 0$ for $0 < p < 1/2$ and $f_p < 0$ for $1/2 < p < 1$.

Solution by M. A. Prasad, Mumbai, India. The binomial expansion of $(1 + z)^\alpha$ converges absolutely for any real α when $|z| < 1$. Since $0 < \tan \frac{x}{2} < 1$ on $(0, \frac{\pi}{2})$ and the expansion of $(1 + z)^\alpha - (1 - z)^\alpha$ has only odd powers of z, de Moivre's theorem yields

$$\begin{aligned} 2\sin px &= \frac{1}{i}\big((\cos x + i\sin x)^p - (\cos x - i\sin x)^p\big) \\ &= \frac{1}{i}\left(\left(\cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)\right)^{2p} - \left(\cos\left(\frac{x}{2}\right) - i\sin\left(\frac{x}{2}\right)\right)^{2p}\right) \qquad (1) \\ &= 2\cos^{2p}\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\binom{2p}{2k+1}(-1)^k\tan^{2k+1}\left(\frac{x}{2}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} (1 + \sin x)^p - (1 - \sin x)^p &= \left(\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right)^{2p} - \left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right)^{2p} \\ &= 2\cos^{2p}\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\binom{2p}{2k+1}\tan^{2k+1}\left(\frac{x}{2}\right). \qquad (2) \end{aligned}$$

When we subtract (2) from (1), all terms involving even values of k cancel, and thus

$$f_p(x) = 4\cos^{2p}\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\binom{2p}{4k+3}\tan^{4k+3}\left(\frac{x}{2}\right).$$

The result follows, since each term is positive if $0 < 2p < 1$ and negative if $1 < 2p < 2$.

Editorial comment. This solution brings to mind Hadamard's dictum: "The shortest path between two real truths in the real domain passes through the complex domain." Without going to the complex domain, several solvers observed that $f_p(x)$ is a solution to a second-order inhomogeneous differential equation with constant coefficients, and this leads to a representation of $f_p(x)$ as an integral transform. A detailed calculation yields

$$f_p(x) = \int_o^x K(p, t)\sin(p(x - t))\,dt,$$

where the kernel $K(p, t)$ has the desired positivity and negativity properties.

Also solved by E. Butler, R. Chapman (U. K.), V. Georges (France), N. Ghosh, K.-W. Lau (China), L. Matejička (Slovakia), M. Omarjee (France), A. Poplawski (Poland), B. Randé (France), A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by October 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12048. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Call an integer a *special Carmichael number* if it can be written as, $(6k+1)(12k+1)(18k+1)$ for some integer k, with each of $6k+1$, $12k+1$, and $18k+1$ being prime. Call an integer a *taxicab number* if it can be written as the sum of two positive integer cubes in two different ways. Show that 1729 is the only positive integer that is both a special Carmichael number and a taxicab number.

12049. *Proposed by Z. K. Silagadze, Novosibirsk State University, Novosibirsk, Russia.* For all nonnegative integers m and n with $m \le n$, prove

$$\sum_{k=m}^{n} \frac{(-1)^{k+m}}{2k+1}\binom{n+k}{n-k}\binom{2k}{k-m} = \frac{1}{2n+1}.$$

12050. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam.* Let ABC be a triangle, and let t be a real number with $1 < t < 2$. Let points D and G be chosen on side AB, points E and H on side BC, and points F and I chosen on side AC so that

$$\frac{AB}{BD} = \frac{AB}{AG} = \frac{BC}{CE} = \frac{BC}{BH} = \frac{CA}{AF} = \frac{CA}{CI} = t.$$

Let $A' = BF \cap CG$, $B' = CD \cap AH$, and $C' = AE \cap BI$.

(a) Prove that the Euler lines of ABC and $A'B'C'$ coincide.
(b) Let H and O denote the orthocenter and circumcenter, respectively, of ABC, and let H' and O' denote the orthocenter and circumcenter, respectively, of $A'B'C'$. Let $\phi = (1+\sqrt{5})/2$. Prove that $HO'/O'O = OH'/H'O'$ if and only if $t = \phi$, in which case $HO'/O'O = \phi$.
(c) Prove $A' = DH \cap EI$, $B' = FG \cap EI$, and $C' = DH \cap FG$ if and only if $t = \phi$.
(d) When $t = \phi$, compute the ratio of the area of ABC to the area of $A'B'C'$.

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12051. *Proposed by Pedro Ribeiro, student, University of Porto, Porto, Portugal.* Prove

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{4^n (2n+1)^3} = \frac{\pi^3}{48} + \frac{\pi}{4} \ln^2(2).$$

12052. *Proposed by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* An *m-dimensional parallelotope* is the image of an m-dimensional cube under a nondegenerate affine transformation. Let m and n be positive integers with $n \geq m$. Prove that the number of m-dimensional parallelotopes formed by the vertices of the n-dimensional cube is

$$\frac{2^{n-m}}{m!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m+1-i)^n .$$

12053. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let n be an integer greater than 1, and let $t, x_1, x_2, \ldots, x_{n+2}$ be positive real numbers with $x_{n+1} = x_1$ and $x_{n+2} = x_2$. Prove

$$\sum_{k=1}^{n} \frac{x_k}{x_{k+1}} \geq \sum_{k=1}^{n} \frac{x_k + t x_{k+1}}{x_{k+1} + t x_{k+2}}.$$

12054. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$\int_0^1 \frac{\arctan x}{x} \log\left(\frac{1+x^2}{(1-x)^2}\right) dx = \frac{\pi^3}{16}.$$

SOLUTIONS

Disappearing Power Series

11914 [2016, 505]. *Proposed by Robin Chapman, University of Exeter, Exeter, U. K., and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Show that for all positive integers m and n,

$$\sum_{k=1}^{n} (-4)^{-k} \binom{n-k}{k-1} \sum_{j=1}^{3m} (-2)^{-j} \binom{n+1-2k}{j-1} \binom{m-k}{3m-j} = 0.$$

(Here $\binom{x}{k} = \frac{1}{k!} \prod_{i=0}^{k-1} (x-i)$ for $x \in \mathbb{R}$.)

Solution by the proposers. We use the snake oil method popularized by Herbert Wilf in *generatingfunctionology*, A K Peters (2006). Let $S(m,n)$ be the specified sum, and let $[x^t]$ be the "coefficient operator" that extracts the coefficient of x^t from a formal power series in x. We introduce the formal variable x and express $S(m,n)$ using a double sum in which the terms contributing to the coefficient of x^{3m-1} are those, where $i = 3m - j$. We then separate the sums and apply the binomial formula to obtain

$$\begin{aligned} S(m,n) &= \sum_{k=1}^{n} (-4)^{-k} \binom{n-k}{k-1} [x^{3m-1}] \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} (-2)^{-j} \binom{n+1-2k}{j-1} x^{j-1} \binom{m-k}{i} x^i \\ &= \sum_{k=1}^{n} (-4)^{-k} \binom{n-k}{k-1} [x^{3m-1}] \left(\frac{-1}{2} \left(1 - \frac{x}{2}\right)^{n+1-2k} (1+x)^{m-k} \right). \end{aligned}$$

Continuing, we let $r = n - k$ and $s = k - 1$ and use $\binom{r}{s} = 0$ for $r < s$. The outer sum is then over pairs (r, s) such that $r + s = n - 1$ and $s \le r$:

$$S(m,n) = -\frac{1}{2} \sum_{\substack{r+s=n-1\\ s\le r}} (-4)^{-1-s} \binom{r}{s} [x^{3m-1}] \left(\left(1 - \frac{x}{2}\right)^{r-s} (1+x)^{m-s-1} \right)$$

$$= \frac{1}{8} \sum_{\substack{r+s=n-1\\ s\le r}} (-1)^s 2^{-r-s} \binom{r}{s} [x^{3m-1}] \left((2-x)^{r-s}(1+x)^{m-s-1}\right).$$

We now use snake oil again. For $m \ge 1$, let $F_m(y) = 8 \sum_{n=1}^{\infty} S(m,n)(2y)^{n-1}$. Again introducing a double sum, where the terms contributing to the coefficient of y^{n-1} are those with $r + s = n - 1$, we compute

$$F_m(y) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (-1)^s \binom{r}{s} [x^{3m-1}] \left((2-x)^{r-s}(1+x)^{m-s-1} y^{r+s}\right)$$

$$= [x^{3m-1}](1+x)^{m-1} \sum_{r=0}^{\infty} (y(2-x))^r \sum_{s=0}^{r} \binom{r}{s} \left(\frac{-y}{(1+x)(2-x)}\right)^s$$

$$= [x^{3m-1}](1+x)^{m-1} \sum_{r=0}^{\infty} (y(2-x))^r \left(1 - \frac{y}{(1+x)(2-x)}\right)^r$$

$$= [x^{3m-1}](1+x)^{m-1} \sum_{r=0}^{\infty} \left(y(2-x) - \frac{y^2}{1+x}\right)^r = [x^{3m-1}] \frac{(1+x)^{m-1}}{1 - y(2-x) + \frac{y^2}{1+x}}$$

$$= [x^{3m-1}] \frac{(1+x)^m}{1 + x - y(2-x)(1+x) + y^2} = [x^{3m-1}] \frac{(1+x)^m}{(1-y)^2 + (1-y)x + yx^2}$$

$$= \frac{1}{(1-y)^2} [x^{3m-1}] \frac{(1+x)^m}{1 + \frac{x}{1-y} + \frac{yx^2}{(1-y)^2}} = u^2 [x^{3m-1}] \frac{(1+x)^m}{1 + ux + (u^2-u)x^2},$$

where we let $u = \frac{1}{1-y}$ in the last step. Using partial fractions, we obtain

$$\frac{1}{1 + ux + (u^2-u)x^2} = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha - \beta} \left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right)$$

$$= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^{n+1} - \beta^{n+1}) x^n,$$

where α and β are the solutions for x (as formal power series) in the quadratic equation $0 = x^2 + ux + (u^2 - u) = x^2 + \frac{x}{1-y} + \frac{y}{(1-y)^2}$. These solutions are

$$\alpha = \frac{-u + \sqrt{4u - 3u^2}}{2} = \frac{-1 + \sqrt{1-4y}}{2(1-y)} \quad \text{and} \quad \beta = \frac{-u - \sqrt{4u - 3u^2}}{2} = \frac{-1 - \sqrt{1-4y}}{2(1-y)}.$$

Since α and β are solutions to $0 = x^2 + ux + (u^2 - u)$, we have $\alpha + \beta = -u$, and

$$\alpha^2 = -\alpha u + (u - u^2) = (u + \beta)u + (u - u^2) = u(1 + \beta).$$

Similarly, $\beta^2 = u(1+\alpha)$. Hence, $\alpha^2(1+\alpha) = u(1+\beta)(1+\alpha) = \beta^2(1+\beta)$, and

$$(\alpha-\beta)[x^{3m-1}]\frac{(1+x)^m}{1+ux+(u^2-u)x^2} = [x^{3m-1}](1+x)^m\sum_{n=0}^{\infty}(\alpha^{n+1}-\beta^{n+1})x^n$$

$$= [x^{3m-1}]\sum_{j=0}^{m}\binom{m}{j}x^{m-j}\sum_{n=0}^{\infty}(\alpha^{n+1}-\beta^{n+1})x^n$$

$$= \sum_{j=0}^{m}\binom{m}{j}(\alpha^{2m+j}-\beta^{2m+j}) = \alpha^{2m}\sum_{j=0}^{m}\binom{m}{j}\alpha^j - \beta^{2m}\sum_{j=0}^{m}\binom{m}{j}\beta^j$$

$$= \alpha^{2m}(1+\alpha)^m - \beta^{2m}(1+\beta)^m = \left(\alpha^2(1+\alpha)\right)^m - \left(\beta^2(1+\beta)\right)^m = 0.$$

Thus, every coefficient of $F_m(y)$ vanishes, and $S(m,n) = 0$ for all m and n.

Also solved by R. Stong and GCHQ Problem Solving Group (U. K.).

Strehl's Identity in Disguise

11928 [2016, 723]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For positive integers n and m and for a sequence $a_0, a_1, \ldots,$ prove

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{m}{j}a_{i+j} = \sum_{k=0}^{n+m}\binom{n+m}{k}a_k$$

and

$$\sum_{i<j}\binom{n}{i}\binom{n}{j}\binom{i+j}{n} = \sum_{i<j}\binom{n}{i}\binom{n}{j}^2.$$

Solution by Quentin Guignard, Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France, and Bernard Randé, Paris, France. Collecting terms involving a_k yields the first identity, using the well-known Vandermonde convolution

$$\sum_{i+j=k}\binom{n}{i}\binom{m}{j} = \binom{n+m}{k}. \tag{1}$$

The convolution can also be written as

$$\sum_{k\le n}\binom{i}{n-k}\binom{j}{k} = \binom{i+j}{n}.$$

Multiplying by $\binom{n}{i}\binom{n}{j}$ and summing over (i,j) such that $i<j\le n$ yields

$$\sum_{i<j\le n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n} = \sum_{k\le n}\sum_{i<j\le n}\binom{i}{n-k}\binom{j}{k}\binom{n}{i}\binom{n}{j}.$$

Let S denote the left-side of both this and the second desired identity. Since

$$\binom{i}{n-k}\binom{n}{i} = \binom{n}{k}\binom{k}{n-i} \quad\text{and}\quad \binom{j}{k}\binom{n}{j} = \binom{n}{k}\binom{n-k}{j-k},$$

we have

$$S = \sum_{k\le n}\sum_{i<j\le n}\binom{n}{k}^2\binom{k}{n-i}\binom{n-k}{j-k}.$$

In the inner sum, write $r = k + (i - j)$. Since $i < j$, we have $r < k$. Reindexing,

$$S = \sum_{r<k\le n} \binom{n}{k}^2 \sum_j \binom{k}{n+k-r-j}\binom{n-k}{j-k}.$$

Applying (1) once more, this time to the inner sum, gives

$$S = \sum_{r<k\le n} \binom{n}{k}^2 \binom{n}{n-r} = \sum_{r<k\le n} \binom{n}{r}\binom{n}{k}^2,$$

as desired.

Editorial comment. As noted by several readers, the second identity is equivalent to

$$\sum_k \binom{n}{k}^2 \binom{2k}{n} = \sum_k \binom{n}{k}^3,$$

which is equation (29) of V. Strehl, Binomial identities – combinatorial and algebraic aspects, *Discrete Mathematics* **136** (1994) 309–346. Two copies of the requested identity plus Strehl's identity yield

$$\sum_i \sum_j \binom{n}{i}\binom{n}{j}\binom{i+j}{n} = \sum_i \sum_j \binom{n}{i}\binom{n}{j}^2,$$

which can be proved by applying the Vandermonde convolution on both sides.

Strehl explored his identity in the context of hypergeometric techniques. It can also be proved by using the Vandermonde convolution twice along with various other identities, or by showing that both sides count the ways to start with n black cards and n white cards, designate an equal number of cards of each color as bad, and discard n bad cards.

Also solved by M. Apadogu, R. Chapman (U. K.), P. P. Dályay (Hungary), R. Dutta, N. Ghosh, Y. J. Ionin, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), P. Lalonde (Canada), J. Nieto (Venezuela), M. Prasad, J. H. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tiso, M. Wildon (U. K.), Y. Zhao, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Balanced Tilings of a Rectangle with Three Rows

11929 [2016, 831]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let a_n be the number of ways in which a rectangular box that contains $6n$ square tiles in three rows of length $2n$ can be split into two connected pieces of size $3n$ without cutting any tiles. Thus $a_1 = 3$, $a_2 = 19$, and one of the 85 ways for $n = 3$ is shown.

Taking $a_0 = 1$, find a closed form for the generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$. What is the asymptotic nature of a_n as $n \to \infty$?

Solution by the editors. The generating function is

$$A(z) = \frac{1+\sqrt{1-4z}}{(\sqrt{1-4z}+z)^2}\frac{1}{\sqrt{1-4z}} - \frac{1-z^2+2z^3}{(1-z)^3}.$$

The coefficients a_n are asymptotic to $4^{n+2}/\sqrt{\pi n}$.

In a splitting of a 3-by-m board into two connected pieces, call the piece containing more of the three cells in the first column *black* and the other piece *white*. Let $f_m(b, w)$ be the number of splittings having b black and w white cells and let $F(X, Y, Z) = \sum_{m=0}^{\infty} \sum_{b+w=3m} f_m(b, w) X^m Y^b Z^w$. We derive an explicit expression for $F(X, Y, Z)$ by relating f_m to paths in a directed multigraph G. The vertices of G represent cases for a column of the 3-by-m board using connectivity information from the cells to the left. We process columns of a tiling from left to right, using 11 states:

1. Start,
2. BBB, with no white cells anywhere to the left,
3. BBB, with some white cells to the left,
4. BBW or WBB,
5. BWB, with the two black cells connected via cells to the left,
6. BWB, with the two black cells not connected via cells to the left,
7. BWW or WWB,
8. WBW, with the two white cells connected via cells to the left,
9. WBW, with the two white cells not connected via cells to the left,
10. WWW,
11. End.

Due to the black-majority convention, *Start* leads next only to vertices 2, 4 (in two ways), 6, or 11. The possible transitions are encoded in the matrix M below. The entry in position (i, j) encodes a step from state i to state j as one column is added. When the transition is possible, it augments the power of X by 1 (for length) and the sum of the powers of Y and Z by 3 (for the three tiles). The coefficient is 2 when there are two ways to make the transition. The requirement that both pieces are connected is encoded by the impossibility of various transitions.

$$\begin{bmatrix}
0 & XY^3 & 0 & 2XY^2Z & 0 & XY^2Z & 0 & 0 & 0 & 0 & 1 \\
0 & XY^3 & 0 & 2XY^2Z & XY^2Z & 0 & 2XYZ^2 & 0 & XYZ^2 & XZ^3 & 1 \\
0 & 0 & XY^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & XY^3 & XY^2Z & 0 & 0 & XYZ^2 & 0 & XYZ^2 & XZ^3 & 1 \\
0 & 0 & XY^3 & 0 & XY^2Z & 0 & 2XYZ^2 & 0 & 0 & XZ^3 & 1 \\
0 & 0 & XY^3 & 0 & 0 & XY^2Z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & XY^3 & XY^2Z & 0 & XY^2Z & XYZ^2 & 0 & 0 & XZ^3 & 1 \\
0 & 0 & XY^3 & 2XY^2Z & 0 & 0 & 0 & XYZ^2 & 0 & XZ^3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & XYZ^2 & XZ^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & XZ^3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

For the example given, the state path is $\langle 1, 2, 5, 7, 4, 9, 10, 11 \rangle$. Splittings with m columns correspond to paths from state 1 to state 11 using $m+1$ transitions; we seek the coefficient of $X^{2n}Y^{3n}Z^{3n}$ in position $(1, 11)$ of M^{2n+1}. Thus, $F(X, Y, Z) = (I - M)^{-1}_{1,11}$.

The resulting expression for F is the fraction with numerator

$$\begin{aligned}
1 &- X^5Y^6Z^4(Y+Z)\left(Y^4 + 3Y^3Z + 2Y^2Z^2 - 2Z^4\right) \\
&+ X^4Y^4Z^2\left(Y^6 + 4Y^5Z + 7Y^4Z^2 + 7Y^3Z^3 + 6Y^2Z^4 - Z^6\right) \\
&- X^3Y^2Z\left(Y^6 + 3Y^5Z + 8Y^4Z^2 + 9Y^3Z^3 + 5Y^2Z^4 + YZ^5 + Z^6\right) \\
&+ X^2YZ\left(4Y^4 + 3Y^3Z + 3Y^2Z^2 + 3YZ^3 + 2Z^4\right) \\
&- X\left(Y^3 + 2YZ^2 + Z^3\right)
\end{aligned}$$

and denominator

$$\left(XY^3-1\right)^2\left(XZ^3-1\right)\left(XY^2Z-1\right)\left(XYZ^2-1\right)\left(XY^2Z+XYZ^2-1\right).$$

The coefficients of $X^{2n}Y^{3n}Z^{3n}$ for the first few values of n are 1, 3, 19, 85, 355, and 1435. The problem was first investigated in 2009, with these counts appearing in R. H. Hardin, number of ways to partition a $2n \times 3$ grid into 2 connected equal-area regions, oeis.org/A167242.

To extract the generating function $A(z)$, consider $H = F(X, Y, 1/Y)$. To have equal count in black and white, we seek the coefficient of Y^0 in H. Viewing H as a Laurent series in Y, we seek the constant term h_0 (an expression in X). The Cauchy coefficient formula applies to H (see P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, New York, 2009). We obtain $h_0 = \frac{1}{2\pi i}\oint_C \frac{1}{Y} H\,dY$, where C is a small counterclockwise circle around the origin. Now h_0 is the sum of the residues with respect to the Y-poles. Since the denominator of H is $(XY^3-1)^2(XY^{-3}-1)(XY-1)(XY^{-1}-1)(XY+XY^{-1}-1)$, the poles are at 0, X, the three cube roots of X, and $(1-\sqrt{1-4X^2})/(2X)$. There are eight additional poles, but they lie outside C when X is small. With the help of *Mathematica*, we find an exact expression for h_0, and then changing variables from X to $\sqrt{z}$ gives $A(z)$ as stated earlier.

The asymptotic behavior of a_n is governed by the singularity of $A(z)$ at $z = 1/4$. Write $A(z) = B(z) + 16/\theta - Q$ with $\theta = \sqrt{1-4z}$. We have $16/\theta = \sum_{n=1}^{\infty} 16\binom{2n}{n}z^n$, with coefficients asymptotic to $4^{n+2}/\sqrt{\pi n}$ by Stirling's formula. Setting Q equal to 3086/27 means that $B(z)/\theta$ is bounded in a disk of radius larger than 1/4. Hence, a "transfer theorem" applies: Use Theorem VI.4 of the book of Flajolet and Sedgewick cited above to deduce that $a_n = 4^{n+2}/\sqrt{\pi n} + O(4^n/n^{3/2})$. In that theorem, use $\zeta = 1/4$, $a = -1/2$, $\sigma(z) = 0$, and $\tau(z) = 1 - t$.

With more work, one can obtain a formula for a_n, with F_m denoting the mth Fibonacci number:

$$\begin{aligned} a_n = -3 + n - n^2 &- \frac{1}{5}\big((n-5)F_{3n+1} + (2n-1)F_{3n}\big) \\ &+ \frac{1}{5}\sum_{m=0}^{n}\binom{2(n-m)}{n-m}\big((3m+5)F_{3m+1} - (4m+3)F_{3m}\big). \end{aligned}$$

Also solved by J. Semonsen, R. Tauraso (Italy), GCHQ Problem Solving Group, and the proposer.

A Telescoping Series with Inverse Hyperbolic Sine

11930 [2016, 831]. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Find

$$\sum_{n=1}^{\infty} \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right).$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. Writing

$$\begin{aligned} \frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}} &= \frac{\sqrt{2^{n+2}+2}-\sqrt{2^{n+1}+2}}{2^{n+1}} \\ &= \sqrt{\frac{1}{2^n}}\cdot\sqrt{1+\frac{1}{2^{n+1}}} - \sqrt{\frac{1}{2^{n+1}}}\cdot\sqrt{1+\frac{1}{2^n}}, \end{aligned}$$

we see that

$$\sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right) = \sinh^{-1}\left(\sqrt{\frac{1}{2^n}}\right) - \sinh^{-1}\left(\sqrt{\frac{1}{2^{n+1}}}\right).$$

Since $\lim_{n\to\infty} \sinh^{-1}\sqrt{1/2^n} = 0$, the given sum S telescopes, and

$$S = \sum_{n=1}^{\infty} \left(\sinh^{-1}\left(\sqrt{\frac{1}{2^n}}\right) - \sinh^{-1}\left(\sqrt{\frac{1}{2^{n+1}}}\right)\right) = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

Also solved by M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), A. Berkane (Algeria), R. Boukharfane (Romania), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), B. Davis, H. Far, L. Giugiuc (Romania), J. Hartman, E. Herman, W. Johnson, B. Karaivanov (U. S. A.) and T. S. Vassilev (Canada), K. Kolczyńska-Przybycień (Poland), K. Koo (China), O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), O. P. Lossers (Netherlands), P. Magli (Italy), J. Magliano, S. Mandal (India), R. Molinari, R. Nandan, M. Omarjee (France), M. Prasad (India), A. N. Sharma (India), N. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), D. Tyler, J. Vinuesa (Spain), M. Vowe (Switzerland), H. Widmer (Switzerland), J. Zacharias, L. Zhou, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Geodesic Variation on Ramsey's Theorem for Paths

11931 [2016, 831]. *Proposed by Igor Protasov, Kiev, Ukraine.* Given natural numbers m and r, prove that there is a finite connected graph G such that, for every r-coloring of its edge set $E(G)$, there is a monochromatic geodesic path of length m. (A path is *geodesic* if there is no shorter path with the same endpoints.)

Solution I by Richard Stong, Center for Communications Research, San Diego, CA. Let Q_d be the d-dimensional hypercube: vertices are binary d-tuples, adjacent when they differ in exactly one coordinate. Given r, we will produce by induction on m a number n such that every r-coloring of $E(Q_n)$ has a monochromatic geodesic path of length m. The base case $m = 1$ is trivial; $n = 1$ suffices. For the induction step, suppose that every r-edge-coloring of Q_n contains a monochromatic geodesic of length m; it suffices to prove that, for $N = n(r2^{n-1} + 1)$, every r-coloring of Q_N contains a monochromatic geodesic of length $2m$.

To prove this, consider an r-edge-coloring of Q_N. Partition the N coordinates into $r2^{n-1} + 1$ sets of n consecutive coordinates. For each such n-tuple of coordinates, Q_N contains $2^{nr2^{n-1}}$ disjoint copies of Q_n, obtained by fixing the values in the other $nr2^{n-1}$ coordinates and allowing just these n to vary. Each such copy of Q_n contains a monochromatic geodesic of length m. Each such geodesic has two endpoints. Since

$$(r2^{n-1} + 1)2^{nr2^{n-1}}2 > r\,2^{n\,(1+r\,2^{n-1})} = r|V(Q_N)|,$$

by the pigeonhole principle some vertex of Q_N is an endpoint of at least $r + 1$ of these monochromatic geodesics. By the pigeonhole principle again, some two geodesics of the same color end at this vertex. Since they come from disjoint n-tuples of coordinates, their union is also a monochromatic geodesic, because a path in Q_d is a geodesic if and only if no coordinate changes twice along the path. We thus obtain a monochromatic geodesic of length $2m$.

Solution II by Stephen C. Locke, Florida Atlantic University, Boca Raton, FL. For any positive integers k and g, there is a finite k-regular graph with girth g (see P. Erdős and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 12 (1963) 251–258). Let G be a finite $2r$-regular graph with girth $2m + 1$. We may assume that G is connected; otherwise, we take a component of G containing a shortest cycle. By the pigeonhole principle, any partition of $E(G)$ into r spanning subgraphs has some subgraph with average degree at least 2. Such a subgraph H contains a cycle C, which has length at least $2m + 1$. A path of length m along C is a monochromatic geodesic of length m.

Editorial comment. The graphs of Solution I grow very rapidly in terms of r and m. Locke observed that Q_{rm} suffices. Because Q_{rm} is rm-regular, an r-edge-coloring yields some monochromatic subgraph H with average degree at least m. Now there is a geodesic in H with length at least m by Theorem 1.2 of I. Leader and E. Long, Long geodesics in subgraphs of the cube, *Discrete Mathematics* **326** (2014) 29–33.

Also solved by A. Poplawski (Poland), GCHQ Problem Solving Group (U. K.), and the proposer.

Geometric Mean Rates

11935 [2016, 832]. *Proposed by D. M. Bătinețu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania; Anastasios Kotronis, Athens, Greece; and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let f be a function from $\mathbb{Z}^+$ to $\mathbb{R}^+$ such that $\lim_{n\to\infty} f(n)/n = a$, where $a > 0$. Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^{n} f(k)}\right).$$

Solution by Marcin E. Kuczma, University of Warsaw, Warsaw, Poland. The limit is a/e. Let $x_n = f(n)/n$ so that x_n converges to a and hence $(x_1x_2\cdots x_n)^{1/n}$ also converges to a. We want to compute the limit of $g(n+1) - g(n)$, where

$$g(n) = \left(\prod_{k=1}^{n} f(k)\right)^{1/n} = (n!)^{1/n}(x_1x_2\cdots x_n)^{1/n} \sim \frac{an}{e}.$$

Let

$$c_n = \left(\frac{g(n+1)}{g(n)}\right)^{n+1} = \frac{(n+1)!(x_1x_2\cdots x_{n+1})}{(n!x_1x_2\cdots x_n)^{(n+1)/n}} = \frac{n+1}{(n!)^{1/n}}\cdot\frac{x_{n+1}}{(x_1x_2\cdots x_n)^{1/n}}.$$

Since $x_{n+1} \to a$, $(x_1x_2\cdots x_n)^{1/n} \to a$, and $(n!)^{1/n} \sim n/e$, we see that $c_n \to e$. Consequently,

$$\frac{g(n+1)}{g(n)} = c_n^{1/(n+1)} \to 1.$$

Finally,

$$\begin{aligned} g(n+1) - g(n) &= g(n)\left(\frac{g(n+1)}{g(n)} - 1\right) \sim g(n)\cdot\log\frac{g(n+1)}{g(n)} \\ &= g(n)\cdot\frac{\log c_n}{n+1} \sim \frac{an}{e}\cdot\frac{1}{n+1} \to \frac{a}{e}. \end{aligned}$$

Editorial comment. Several solvers pointed out that this result (and some generalizations) appear in Gh. Toader, Lalescu sequences, *Publikacije Elektrotehničkog fakulteta*, Serija Matematika **9** (1998) 25–28. This result is a consequence of Theorem 2 in that paper.

Also solved by A. Ali (India), K. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), L. Giugiuc (Romania), N. Grivaux (France), E. Herman, K. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Mortini, M. Omarjee (France) and R. Tauraso (Italy), Á. Plaza (Spain), M. A. Prasad (India), G.-F. Serban (Romania), A. Stenger, R. Stong, T. Wiandt, Y. Zhang, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, and Fuzhen Zhang.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by December 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12055. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let $a_1, a_2, \dots$ be a sequence of nonnegative integers with $a_1 \geq a_2 \geq \cdots$ and with finite sum. For a positive integer j, let b_j be the number of indices i such that $a_i \geq j$. (The sequence $b_1, b_2, \dots$ is the *conjugate* of $a_1, a_2, \dots$.) Prove that the multisets $\{a_1 + 1, a_2 + 2, \dots\}$ and $\{b_1 + 1, b_2 + 2, \dots\}$ are equal. For example, if $\langle a_i \rangle = \langle 5, 3, 2, 2, 0, 0, 0, \dots \rangle$, then $\langle b_j \rangle = \langle 4, 4, 2, 1, 1, 0, 0, \dots \rangle$, and the corresponding multisets are $\{6, 5, 5, 6, 5, 6, 7, 8, \dots\}$ and $\{5, 6, 5, 5, 6, 6, 7, 8, \dots\}$.

12056. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania, Kadir Altintas, Emirdağ, Turkey, and Florin Stanescu, Gaesti, Romania.* Let $ABCD$ be a rectangle inscribed in a circle S of radius R, and let P be a point inside S. The lines AP, BP, CP, and DP intersect S a second time at K, L, M, and N, respectively. Prove $AK^2 + BL^2 + CM^2 + DN^2 \geq 16R^4/(R^2 + OP^2)$.

12057. *Proposed by Peter Kórus, University of Szeged, Szeged, Hungary.*
(a) Calculate the limit of the sequence defined by $a_1 = 1$, $a_2 = 2$, and

$$a_{2k+1} = \frac{a_{2k-1} + a_{2k}}{2} \quad \text{and} \quad a_{2k+2} = \sqrt{a_{2k}\, a_{2k+1}}$$

for positive integers k.
(b) Calculate the limit of the sequence defined by $b_1 = 1$, $b_2 = 2$, and

$$b_{2k+1} = \frac{b_{2k-1} + b_{2k}}{2} \quad \text{and} \quad b_{2k+2} = \frac{2\, b_{2k}\, b_{2k+1}}{b_{2k} + b_{2k+1}}$$

for positive integers k.

12058. *Proposed by Max A. Alekseyev, George Washington University, Washington, DC.* Let b be an integer greater than 1. For a positive integer n, let $u_b(n)$ be the number of nonzero digits in the base b representation of n. Prove that for any positive integers n and k, there exists a positive integer m such that $u_b(mn) = u_b(n) + k$.

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12059. *Proposed by George Stoica, Saint John, NB, Canada.* Let n be an integer greater than 1, and let R be the ring of polynomials in the variables $x_1, \ldots, x_n$ with real coefficients. Let S be the ideal in R generated by the elementary symmetric polynomials $e_1, \ldots, e_n$, where

$$e_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

for $1 \le k \le n$. The *degree* of a monomial $x_1^{m_1} \cdots x_n^{m_n}$ is $m_1 + \cdots + m_n$. Prove that $n(n-1)/2$ is the largest degree among all monomials that do not belong to S.

12060. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $\zeta(3)$ be Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$, and let H_n be the nth harmonic number $1 + 1/2 + \cdots + 1/n$. Prove

$$\sum_{n=2}^{\infty} \frac{H_n H_{n+1}}{n^3 - n} = \frac{5}{2} - \frac{\pi^2}{24} - \zeta(3).$$

12061. *Proposed by Dao Thanh Oai, Thai Binh, Viet Nam, and Le Viet An, Hue, Viet Nam.* Two triangles ABC and $A'B'C'$ in the plane are *perspective from a point* if the lines AA', BB', and CC' are concurrent (the common point is the *perspector*) and are *perspective from a line* if the points of intersection of AB and $A'B'$, of AC and $A'C'$, and of BC and $B'C'$ are collinear (the common line is the *perspectrix*). Desargues's theorem states that two triangles are perspective from a point if and only if they are perspective from a line. Consider three triangles, each pair of which are perspective from a point, hence per Desargues's theorem perspective from a line. Show that the three perspectrices are identical if and only if the three perspectors are collinear.

SOLUTIONS

A Triangle out of Pieces

11934 [2016, 832]. *Proposed by Leonard Giugiuc, Drobotu Turnu Severin, Romania.* Let ABC be an isosceles triangle, with $|AB| = |AC|$. Let D and E be two points on side BC such that $D \in BE$, $E \in DC$, and $m(\angle DAE) = \frac{1}{2}m(\angle A)$. Describe how to construct a triangle XYZ such that $|XY| = |BD|$, $|YZ| = |DE|$, and $|ZX| = |EC|$. Also, compute $m(\angle BAC) + m(\angle YXZ)$ (in radians).

Solution by Pál Péter Dálymay, Szeged, Hungary. Write α, β, γ for the radian measures of the angles at A, B, C, respectively. Construct three circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with center A and radii r_1, r_2, r_3, respectively, with $r_1 = |AB|$, $r_2 = |AD|$, $r_3 = |AE|$. Let X be the intersection of the ray from A to the midpoint of BC with $\mathcal{C}_1$, let Y be the intersection of ray AE with $\mathcal{C}_2$, and let Z be the intersection of the ray AD with $\mathcal{C}_3$. We claim that $\triangle XYZ$ meets the required conditions.

Let $\triangle ABD$ be rotated around A by $\alpha/2$ to bring B to X and D to Y. Since $\triangle ABD$ is congruent to $\triangle AXY$, we have $|XY| = |BD|$ and $m(\angle AXY) = m(\angle ABD) = \beta$.

Similarly, let $\triangle ACE$ be rotated around A by $\alpha/2$ to bring C to X and E to Z. As before we conclude $|ZX| = |EC|$ and $m(\angle AXZ) = m(\angle ACE) = \gamma$.

Triangles ADE and AYZ are congruent, since they share an angle A, $|AY| = |AD|$, and $|AZ| = |AE|$. Thus $|YZ| = |DE|$, and triangle XYZ satisfies the required conditions.

Since $m(\angle YXZ) = m(\angle AXY) + m(\angle AXZ) = \beta + \gamma$, we have $m(\angle BAC) + m(\angle YXZ) = \alpha + (\beta + \gamma) = \pi$.

Editorial comment. The problem as originally published had $\angle XYZ$ for the last angle where $\angle YXZ$ was intended.

Also solved by R. B. Campos (Spain), R. Chapman (U. K.), I. Dimitric, J. Han (Korea), E. Ionascu, B. Karaivanov (U. S. A) & T. S. Vassilev (Canada), O. Kouba (Syria), O. P. Lossers (Netherlands), M. Meyerson, R. Stong, Armstrong State University Problem Solvers, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, and the proposer.

Hidden Mersenne

11936 [2016, 941]. *Proposed by William Weakley, Indiana University–Purdue University at Fort Wayne, Fort Wayne, IN.* Let S be the set of integers n such that there exist integers i, j, k, m, p with $i, j \geq 0$, $m, k \geq 2$, and p prime, such that $n = m^k = p^i + p^j$.
(a) Characterize S.
(b) For which elements of S are there two choices of (p, i, j)?

Solution by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND.
(**a**) The set S is the union of three sets: (1) $\{2^d : d \geq 2\}$, (2) $\{(2^t 3)^2 : t \geq 0\}$, and (3) $\{(2p^t)^k : t \geq 0 \text{ and } p = 2^k - 1 \text{ (a Mersenne prime)}\}$.

First, we prove that if $1 + p^d = v^k$, where p is a prime, $d \geq 1$, and $v, k \geq 2$, then either $p = 2$ and $d = 3$ (that is, $1 + 2^3 = 3^2$), or $p = 2^k - 1$ (a Mersenne prime) and $d = 1$ (so $1 + (2^k - 1) = 2^k$).

We prove this claim in two cases. Suppose first that $p = 2$ and $1 + 2^d = v^k$. In this case, $2^d = v^k - 1 = (v - 1)(v^{k-1} + \cdots + v + 1)$. Since all factors of 2^d are even, it follows that v is odd and $v^{k-1} + \cdots + v + 1$ is even, so k must be even, with $k = 2t$ for some $t \in \mathbb{N}$. This yields $2^d = v^k - 1 = (v^t - 1)(v^t + 1)$, so $v^t - 1$ and $v^t + 1$ are powers of 2 differing by 2. Thus they must be 2 and 4, so $v^t - 1 = 2$, and this implies $v = 3$, $t = 1$, and $d = 3$.

In the remaining case, p is an odd prime. Factor k as $p^t m$, where $t \geq 0$ and $p \nmid m$, and let $w = v^{p^t}$. We have $p^d = v^k - 1 = w^m - 1 = (w - 1)(w^{m-1} + \cdots + w + 1)$. If $w - 1 > 1$, then p divides both $w - 1$ and $w^{m-1} + \cdots + w + 1$, but then $w \equiv 1 \bmod p$ and $w^{m-1} + \cdots + w + 1 \equiv m \bmod p$. Hence p divides m, a contradiction. Therefore $w = 2$, and so $v = 2$ and $1 + p^d = 2^k$. Since we are given $k \geq 2$, it follows that $1 + p^d \equiv 0 \bmod 4$, so d is odd. If $d > 1$, then $d = qs$, where q is an odd prime and s is odd. We must have $s = 1$, since otherwise

$$2^k = p^d + 1 = (p^q + 1)\big((p^q)^{s-1} + (p^q)^{s-2} - \cdots - p^q + 1\big),$$

and the second factor is an odd number larger than 1. Thus d is an odd prime, and p has order $2d$ modulo 2^k, because $p^2 \not\equiv 1 \bmod 2^k$ (since $1 < p^2 < p^d < 2^k$) and $p^d \equiv -1 \bmod 2^k$. Thus $2d$ divides $\phi(2^k) = 2^{k-1}$, a contradiction. We conclude $d = 1$, and $p = 2^k - 1$ must be a Mersenne prime, finishing the proof of the claim.

Now consider the general case $n = m^k = p^i + p^j$. If $i = j$, then $m^k = 2p^i$, so $p = 2$ and $n = 2^{i+1}$. Thus n can be 2^d with $d \geq 2$ ($d = 1$ is excluded by $m, k \geq 2$). If $i < j$, then $m^k = p^i(1 + p^{j-i})$. Since p^i and $1 + p^{j-i}$ are relatively prime, we have $i = kt$ for some $t \geq 0$, and $1 + p^{j-i} = v^k$ for some $v \geq 2$. By our claim we have either $p = 2$ with $j - i = 3$ (so $v = 3$ and $k = 2$), or $p = 2^k - 1$ is a Mersenne prime with $j - i = 1$ (so $v = 2$). Thus $n = (2^t 3)^2 = 2^{2t} + 2^{2t+3}$ for some $t \geq 0$, or $n = (2p^t)^k = p^{kt} + p^{kt+1}$ for some $t \geq 0$, where $p = 2^k - 1$ is a Mersenne prime. Hence the set S is as claimed above.
(**b**) We further assume $i \leq j$ to exclude two such trivial representations obtained by switching i and j, so each member of (1), (2), or (3) has only one representation in that family.

Clearly, values of n in (1) and (2) cannot be the same. If n is in both (1) and (3), then $t = 0$ and $d = k$ (so $n = 2^k$, where $2^k - 1$ is a Mersenne prime), while if n has a representation in (2) and (3), then $p = 3$ (which is a Mersenne prime), $t = 1$ (in both representations), and $k = 2$ (so $n = 36$). Hence the only numbers in S with two different representations are 36 (represented as $2^2 + 2^5$ and $3^2 + 3^3$) and 2^k (represented as $2^{k-1} + 2^{k-1}$ and $(2^k - 1)^0 + (2^k - 1)^1$) whenever $2^k - 1$ is a Mersenne prime.

Editorial comment. To simplify the proof, several solvers referred to Catalan's conjecture (proved by Mihăilescu in 2004) that the only consecutive integers that are powers of integers with exponents at least 2 are 2^3 and 3^2.

Also solved by B. Karaivanov (U. S. A) & T. S. Vassilev (Canada), GCHQ Problem Solving Group (U. K.), and NSA Problems Group. Part (**a**) also solved by Y. J. Ionin, M. Josephy (Costa Rica), O. P. Lossers (Netherlands), R. Stong, and the proposer.

A Double Integral for the Digamma Function

11937 [2016, 941]. *Proposed by Juan Carlos Sampedro, University of the Basque Country, Leioa, Spain.* Let s be a complex number that is not a zero of the gamma function $\Gamma(s)$. Prove

$$\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y}{(1 - xy)\log(xy)}\, dx\, dy = \frac{\Gamma'(s)}{\Gamma(s)}.$$

Composite solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands, and Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. No finite complex number is a zero of $\Gamma(s)$, but we must assume $\operatorname{Re} s > 0$ for the integral to converge. Write the integral as

$$I(s) = -\int_0^1 \int_0^1 \frac{1 - (xy)^{s-1}}{1 - xy}\, \frac{dx\, dy}{\log(xy)} + \int_0^1 \int_0^1 \frac{1 - y}{1 - xy}\, \frac{dx\, dy}{\log(xy)}.$$

Notice that $(1 - (xy)^{s-1})/(1 - xy)$ has finite limit as $xy \to 1$, the functions x^{s-1} and y^{s-1} are integrable at 0, and $\int_0^1 \int_0^1 dx\, dy/|\log(xy)| < +\infty$. Therefore, the first integral converges absolutely. Since $0 \le (1 - y)/(1 - xy) \le 1$ whenever $0 < x, y < 1$, the second integral converges absolutely as well.

Now $I(s)$ is an analytic function of s in the right half-plane, so it suffices to prove the result for $0 < s < 1$. In this case, the integrand is real and has constant sign, so we may interchange the order of integration. Thus,

$$\begin{aligned}
I(s) &= \int_0^1 \left(\int_0^1 \frac{(xy)^{s-1} - y}{1 - xy}\, \frac{dx}{\log(xy)} \right) dy = \int_0^1 \left(\int_0^y \frac{t^{s-1} - y}{y(1 - t)\log t}\, dt \right) dy \\
&= \int_0^1 \frac{1}{(1 - t)\log t} \left(\int_t^1 \frac{t^{s-1} - y}{y}\, dy \right) dt \\
&= \int_0^1 \frac{-t^{s-1}\log t - (1 - t)}{(1 - t)\log t}\, dt = \int_0^1 \left(\frac{-t^{s-1}}{1 - t} - \frac{1}{\log t} \right) dt.
\end{aligned}$$

This is a well-known integral representation of the digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$ due to Gauss.

Also solved by M. Arnold, A. Berkane (Algeria), P. Bracken, R. Chapman (U. K.), H. Chen, B. Davis, C. Georghiou (Greece), G. Greubel, J.-P. Grivaux (France), J. A. Grzesik, E. Herman, R. Nandan, M. O'Brien, M. Omarjee (France), F. Perdomo & Á. Plaza (Spain), P. Perfetti (Italy), S. Sharma, A. Stadler (Switzerland),

R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), M. Wildon (U. K.), Y. Zhang, GCHQ Problem Solving Group (U. K.), and the proposer.

An Inequality for Triangles

11938 [2016, 941]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, Macedonia.* Let a, b, c be the lengths of the sides of a triangle, and let A be its area. Let R and r be the circumradius and inradius of the triangle, respectively. Prove

$$a^2 + b^2 + c^2 \geq (a-b)^2 + (b-c)^2 + (c-a)^2 + 4A\sqrt{3 + \frac{R-2r}{R}}.$$

Solution by John G. Heuver, Grande Prairie, AB, Canada. Let $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$. By the law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = (b-c)^2 + 2bc(1-\cos\alpha) = (b-c)^2 + 4A\tan\frac{\alpha}{2},$$

where we have used $2A = bc\sin\alpha$ and $(1-\cos\alpha)/\sin\alpha = \tan(\alpha/2)$. It follows that

$$a^2 + b^2 + c^2 = (a-b)^2 + (b-c)^2 + (c-a)^2 + 4A\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right).$$

We have

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \frac{4R+r}{s},$$

where s is the semiperimeter of the triangle. (This is equation 83 on page 59 of D. S. Mitrinovic (1989), *Recent Advances in Geometric Inequalities*, Dordrecht: Kluwer.) Kooi's inequality

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$$

(see, for example, item 5.7 in O. Bottema, et. al. (1969), *Geometric Inequalities,* Groningen: Wolters-Noordhoff) then gives

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq \sqrt{3 + \frac{R-2r}{R}}.$$

This completes the proof. Equality holds if and only if the triangle is equilateral.

Also solved by A. Ali (India), R. Boukharfane (France), P. P. Dályay (Hungary), L. Giugiuc (Romania), B. Karaivanov (U. S. A.) and T. S. Vassilev (Canada), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, D. Moore, R. Nandan, P. Nüesch (Switzerland), P. Perfetti (Italy), V. Schindler (Germany), M. Stănean (Romania), R. Stong, M. Vowe (Switzerland), T. Wiandt, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Summing Errors in Approximations to Euler's Constant

11939 [2016, 941]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Find

$$\sum_{k=1}^{\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log(k) - \gamma - \frac{1}{2k} + \frac{1}{12k^2}\right).$$

Here γ is Euler's constant.

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. Let $H_k = 1 + 1/2 + \cdots + 1/k$. We have

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} = \sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n} 1 = \sum_{j=1}^{n}\frac{n+1-j}{j} = (n+1)H_n - n.$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n}\left(H_k - \log(k) - \gamma - \frac{1}{2k}\right) &= (n+1)H_n - n - \log(n!) - n\gamma - \frac{H_n}{2}\\ &= \left(n+\frac{1}{2}\right)\left(\log(n) + \gamma + \frac{1}{2n} + O(1/n^2)\right) - n - n\gamma\\ &\quad - \left(n\log(n) - n + \frac{\log(2\pi)}{2} + \frac{\log(n)}{2} + O(1/n)\right)\\ &= \frac{1+\gamma-\log(2\pi)}{2} + O(1/n),\end{aligned}$$

where we have used the approximations $H_n = \log(n) + \gamma + \frac{1}{2n} + O(1/n^2)$ and $\log(n!) = n\log(n) - n + \frac{\log(2\pi)}{2} + \frac{\log(n)}{2} + O(1/n)$. Also,

$$\sum_{k=1}^{\infty}\frac{1}{12k^2} = \frac{1}{12}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{72}.$$

Combining these results, we obtain

$$\sum_{k=1}^{\infty}\left(H_k - \log(k) - \gamma - \frac{1}{2k} + \frac{1}{12k^2}\right) = \frac{1+\gamma-\log(2\pi)}{2} + \frac{\pi^2}{72}.$$

Editorial comment. Several solvers noted that the requested sum, without the final term $1/(12k^2)$, appears as Problem 3.42 on page 195 of O. Furdui (2013), *Limits, Series, and Fractional Part Integrals: Problems in Mathematical Analysis*, New York: Springer. The more general formula

$$\sum_{k=1}^{\infty}\left(H_{pk} - \log(pk) - \gamma - \frac{1}{2pk}\right) = \frac{\log(p)+\gamma-\log(2\pi)}{2} + \frac{1}{2p} + \frac{\pi}{2p^2}\sum_{k=1}^{p-1} k\cot\left(\frac{k\pi}{p}\right),$$

where p is a positive integer, appears in O. Kouba (2016), Inequalities for finite trigonometric sums. An interplay: with some series related to harmonic numbers, *J. Inequal. Appl.*, Paper No. 173, 15 pp.

Also solved by A. Balfaqih (Yemen), A. Berkane (Algeria), R Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, R. Guculiére (France), R. Dutta (India), O. Furdui (Romania), N. Ghosh, M. L. Glasser, J. A. Grzesik, L. Han, E. A. Herman, E. J. Ionaşcu, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), C. W. Lienhard, O. P. Lossers (Netherlands), G. N. Macris, P. Magli (Italy), C. R. McCarthy, R. Nandan, P. Perfetti (Italy), F. A. Rakhimjanovich (Uzbekistan), E. Schmeichel, A. Stadler (Switzerland), A. Stenger, R. Stong, M. Vowe (Switzerland), S. Wagon, H. Widmer (Switzerland), J. Zacharias, Y. Zhang, GCHQ Problem Solving Group (U. K.), and the proposer.

A Hypergeometric Identity

11940 [2016, 942]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $T_n = n(n+1)/2$ and $C(n,k) = (n-2k)\binom{n}{k}$. For $n \geq 1$, prove

$$\sum_{k=0}^{n-1} C(T_n, k)C(T_{n+1}, k) = \frac{n^3 - 2n^2 + 4n}{n+2} \binom{T_n}{n} \binom{T_{n+1}}{n}.$$

Solution I by Pierre Lalonde, Kingsey Falls, QC, Canada. Let m be a positive integer. We prove by induction on m the more general formula

$$\sum_{k=0}^{m-1} C(T_n, k)C(T_{n+1}, k) = \frac{m^2(n^2 + 2n - 4m + 4)}{n(n+2)} \binom{T_n}{m} \binom{T_{n+1}}{m}.$$

For $m = 1$ both sides give $T_n T_{n+1}$. Given the formula for m, we compute

$$\begin{aligned}
\sum_{k=0}^{m} C(T_n, k)C(T_{n+1}, k) &= \sum_{k=0}^{m-1} C(T_n, k)C(T_{n+1}, k) + C(T_n, m)C(T_{n+1}, m) \\
&= \left(\frac{m^2(n^2 + 2n - 4m + 4)}{n(n+2)} + (T_n - 2m)(T_{n+1} - 2m) \right) \binom{T_n}{m} \binom{T_{n+1}}{m} \\
&= \frac{(n^2 + 2n - 4m)}{n(n+2)} \frac{(n^2 + n - 2m)(n^2 + 3n - 2m + 2)}{4} \binom{T_n}{m} \binom{T_{n+1}}{m} \\
&= \frac{(m+1)^2(n^2 + 2n - 4m)}{n(n+2)} \frac{(T_n - m)(T_{n+1} - m)}{(m+1)^2} \binom{T_n}{m} \binom{T_{n+1}}{m} \\
&= \frac{(m+1)^2(n^2 + 2n - 4m)}{n(n+2)} \binom{T_n}{m+1} \binom{T_{n+1}}{m+1},
\end{aligned}$$

where the step from the second to the third line is easy (though tedious) to check. The special case $m = n$ gives the desired result.

Solution II by Akalu Tefera, Grand Valley State University, Allendale, MI. Dividing both sides of the desired equality by its right side yields $\sum_{k=0}^{n-1} F(n, k) = 1$, where

$$F(n, k) = \frac{n+2}{n^3 - 2n^2 + 4n} \frac{C(T_n, k)C(T_{n+1}, k)}{\binom{T_n}{n}\binom{T_{n+1}}{n}}.$$

Applying Gosper's algorithm to $F(n, k)$ produces a rational function

$$R(n, k) = \frac{4k^2(n^2 + 2n - 4k + 4)}{n(n+2)(n^2 + n - 4k)(n^2 + 3n - 4k + 2)}$$

such that setting $G(n, k) = F(n, k)R(n, k)$ yields $F(n, k) = G(n, k+1) - G(n, k)$, which can be confirmed easily. Summing both sides of this equality with respect to k then gives the telescoping sum

$$\sum_{k=0}^{n-1} F(n, k) = \sum_{k=0}^{n-1} \big(G(n, k+1) - G(n, k) \big) = G(n, n) - G(n, 0) = 1.$$

Also solved by R. Chapman (U. K.), R. Stong, R. Tauraso (Italy), and the proposer.

Rate of Convergence for an Integral

11941 [2016, 492]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let

$$L = \lim_{n\to\infty} \int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx.$$

(**a**) Find L.
(**b**) Find

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx - L \right).$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. (**a**) We prove $L = 3/4$. To see this, let $I_n = \int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx$. We have

$$\begin{aligned} I_n &= \int_0^{1/2} \sqrt[n]{x^n + (1-x)^n}\, dx + \int_{1/2}^1 \sqrt[n]{x^n + (1-x)^n}\, dx \\ &\geq \int_0^{1/2} (1-x)\, dx + \int_{1/2}^1 x\, dx = \frac{3}{4}. \end{aligned}$$

On the other hand, since $x \leq 1-x$ for $x \in [0, 1/2]$ and $1-x \leq x$ for $x \in [1/2, 1]$,

$$\begin{aligned} I_n &= \int_0^{1/2} \sqrt[n]{x^n + (1-x)^n}\, dx + \int_{1/2}^1 \sqrt[n]{x^n + (1-x)^n}\, dx \\ &\leq \int_0^{1/2} \sqrt[n]{2}\,(1-x)\, dx + \int_{1/2}^1 \sqrt[n]{2}\, x\, dx = \frac{3}{4}\sqrt[n]{2}. \end{aligned}$$

The squeeze theorem implies that $L = \lim_{n\to\infty} I_n = 3/4$.

(**b**) The limit is $\pi^2/48$. Notice that $\int_0^{1/2} (1-x) = \int_{1/2}^1 x\, dx = 3/8$. We claim

$$\lim_{n\to\infty} n^2 \left(\int_0^{1/2} \sqrt[n]{x^n + (1-x)^n}\, dx - \frac{3}{8} \right) = \frac{\pi^2}{96} \tag{1}$$

and

$$\lim_{n\to\infty} n^2 \left(\int_{1/2}^1 \sqrt[n]{x^n + (1-x)^n}\, dx - \frac{3}{8} \right) = \frac{\pi^2}{96}, \tag{2}$$

from which the required limit follows. To prove (1), we compute

$$\begin{aligned} &\lim_{n\to\infty} n^2 \left(\int_0^{1/2} \left(\sqrt[n]{x^n + (1-x)^n} - (1-x) \right) dx \right) \\ &= \lim_{n\to\infty} n^2 \left(\int_0^{1/2} (1-x) \left(\sqrt[n]{1 + \left(\frac{x}{1-x} \right)^n} - 1 \right) \right) dx \\ &= \lim_{n\to\infty} n^2 \left(\int_0^1 \frac{1}{(1+t)^3} \left(\sqrt[n]{1+t^n} - 1 \right) \right) dt \qquad (\text{letting } t = x/(1-x)) \\ &= \lim_{n\to\infty} n \left(\int_0^1 \frac{1}{(1+u^{1/n})^3} \left(\sqrt[n]{1+u} - 1 \right) u^{1/n-1} \right) du \qquad (\text{letting } u = t^n) \\ &= \int_0^1 \lim_{n\to\infty} \frac{1}{(1+u^{1/n})^3}\, n \left(\sqrt[n]{1+u} - 1 \right) u^{1/n-1}\, du \\ &= \frac{1}{8} \int_0^1 \frac{\ln(1+u)}{u}\, du = \frac{1}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{96}. \end{aligned}$$

Equation (2) follows from (1) upon substituting $1-x$ for x.

Editorial comment. Chen noted that the results can be generalized as follows. For part **(a)**: If f and g are nonnegative and integrable on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b \sqrt[n]{f(x)^n + g(x)^n}\, dx = \int_a^b \max\{f(x), g(x)\}\, dx.$$

For part **(b)**: If f is a positive continuous function on $[0, 1]$ with $f(0) = 1$ and $g(x)$ is continuous on $[0, 1]$, then

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{f(x^n)} g(x)\, dx - \int_0^1 g(x)\, dx \right) = g(1) \int_0^1 \frac{\ln f(x)}{x}\, dx.$$

Letting $f(x) = 1 + x$ and $g(x) = 1/(1 + x)^3$ yields the result in part **(b)**.

Also solved by R. Agnew, K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), B. E. Davis, R. Dutta (India), D. Fleischman, N. Ghosh, J.-P. Grivaux (France), L. Han, F. Holland (Ireland), E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), S. de Luxán (Germany) & Á. Plaza (Spain), M. Omarjee (France), N. Osipov (Russia), P. Perfetti (Italy), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

On Perpendicularity

11942 [2016, 492]. *Proposed by Florin Parvanescu, Slat, Romania.* In acute triangle ABC, let D be the foot of the altitude from A, let E be the foot of the perpendicular from D to AC, and let F be a point on segment DE. Prove that AF is perpendicular to BE if and only if $|FE|/|FD| = |BD|/|CD|$.

Solution by Wei-Kai Lai and John Risher (student), University of South Carolina Salkehatchie, Walterboro, SC. Note that since $\overrightarrow{AD} \cdot \overrightarrow{BD} = 0$,

$$\begin{aligned} \overrightarrow{AF} \cdot \overrightarrow{BE} &= (\overrightarrow{AD} + \overrightarrow{DF}) \cdot (\overrightarrow{BD} + \overrightarrow{DE}) = \overrightarrow{AD} \cdot \overrightarrow{DE} + \overrightarrow{DF} \cdot \overrightarrow{BD} + \overrightarrow{DF} \cdot \overrightarrow{DE} \\ &= (\overrightarrow{AE} - \overrightarrow{DE}) \cdot \overrightarrow{DE} + |DF|\, |BD| \cos(\angle EDC) + |DF|\, |DE| \\ &= -|DE|^2 + |DF|\, |BD| \frac{|DE|}{|DC|} + |DF|\, |DE|. \qquad (1) \end{aligned}$$

Consider first the necessity of the condition. When $AF \perp BE$, (1) yields $|DF|\, |BD| + |DF|\, |DC| = |DE|\, |DC|$. Since $|DE| = |DF| + |FE|$, we get

$$|DF|\, |BD| + |DF|\, |DC| = |DF|\, |DC| + |FE|\, |DC|,$$

which implies $|DF|\, |BD| = |FE|\, |DC|$ as required.

Now consider the sufficiency of the condition. Since $|DE| = |DF| + |FE|$, and $|FE|/|FD| = |BD|/|CD|$ is assumed, we can write (1) in the equivalent form

$$\begin{aligned} \overrightarrow{AF} \cdot \overrightarrow{BE} &= -(|DF| + |FE|)^2 + |DF|\, |DE| \cdot \frac{|FE|}{|FD|} + |DF|(|DF| + |FE|) \\ &= -|DF|^2 - 2|DF| \cdot |FE| - |FE|^2 + (|DF| + |FE|)|FE| + |DF|^2 + |DF| \cdot |FE|. \end{aligned}$$

This equals zero, and hence AF is perpendicular to BE, as desired.

Also solved by A. Ali (India), H. Bailey, R. Chapman (U. K.), P. P. Dályay (Hungary), P. De (India), I. Dimitrić, A. Fanchini, D. Fleischman, O. Geupel, L. Giugiuc (Romania), N. Grivaux (France), J. Han (South Korea), E. A. Herman, S. Hitotumatu (Japan), E. J. Ionaşcu, Y. Ionin, S.-H. Jeong (Korea), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, I. Mihăilă, J. Minkus, R. Nandan, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, T. Zvonaru & N. Stanciu (Romania), Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, and Fuzhen Zhang.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by February 28, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12062. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yuri J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let l be a line, and let P be a point off l. Call a triangle QPQ' with $\{Q, Q'\} \subset l$ *nice* if it is isosceles or if $\angle QPQ'$ is a right angle. Let n be an integer with $n \geq 2$. Of all the subsets X of l of size n, what is the largest possible number of nice triangles with two vertices in X?

12063. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let p and q be real numbers with $p > 0$ and $q > -p^2/4$. Let $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = pU_{n+1} + qU_n$ for $n \geq 0$. Calculate

$$\lim_{n\to\infty} \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}}.$$

12064. *Proposed by Cesar Adolfo Hernandez Melo, State University of Maringá, Maringá, Brazil.* Let f be a convex, continuously differentiable function from $[1, \infty)$ to $\mathbb{R}$ such that $f'(x) > 0$ for all $x \geq 1$. Prove that the improper integral $\int_1^\infty 1/f'(x)\,dx$ is convergent if and only if the series $\sum_{n=1}^{\infty}\big(f^{-1}(f(n)+\epsilon) - n\big)$ is convergent for all positive ϵ.

12065. *Proposed by Hojoo Lee, Seoul National University, Seoul, South Korea.* Let n be a positive integer, and let $x_1, \ldots, x_n$ be a list of n positive real numbers. For $k \in \{1, \ldots, n\}$, let $y_k = x_k(n+1)/(n+1-k)$ and let

$$z_k = \frac{(k!)^{1/k}}{k+1}\left(\prod_{j=1}^{k} y_j\right)^{1/k}.$$

Prove that the arithmetic mean of $x_1, \ldots, x_n$ is greater than or equal to the arithmetic mean of $z_1, \ldots, z_n$, and determine when equality holds.

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12066. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.* Let n and k be integers greater than 1, and let A be an n-by-n positive definite Hermitian matrix. Prove

$$(\det A)^{1/n} \le \left(\frac{\operatorname{trace}^k(A) - \operatorname{trace}(A^k)}{n^k - n} \right)^{1/k}.$$

12067. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* For a positive integer n, let $\gamma_n = \left(\sum_{k=1}^{n} 1/k\right) - \ln n$, so that $\lim_{n\to\infty} \gamma_n$ is Euler's constant γ. Let $\beta_n = 6n + 12n^2(\gamma - \gamma_n)$. Prove that $\beta_{n+1} > \beta_n$ for all n.

12068. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Consider a triangle with altitudes h_a, h_b, and h_c and corresponding exradii r_a, r_b, and r_c. Let s, r, and R denote the triangle's semiperimeter, inradius, and circumradius, respectively.

(a) Prove

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_a} r_c^2 \ge 2s^2.$$

(b) Prove

$$\frac{r_b + r_c}{r_a} h_a^2 + \frac{r_c + r_a}{r_b} h_b^2 + \frac{r_a + r_b}{r_a} h_c^2 \ge \frac{4s^2 r}{R}.$$

SOLUTIONS

Subsets Closed Under a Family of Functions

11943 [2016, 1050]. *Proposed by Keith Kearnes, University of Colorado, Boulder, CO, and Greg Oman, University of Colorado, Colorado Springs, CO.* Let X be a set, and let $\mathcal{F}$ be a collection of functions f from X into X. A subset Y of X is *closed* under $\mathcal{F}$ if $f(y) \in Y$ for all $y \in Y$ and f in $\mathcal{F}$. With the axiom of choice given, prove or disprove: There exists an uncountable collection $\mathcal{F}$ of functions mapping $\mathbb{Z}^+$ into $\mathbb{Z}^+$ such that
(a) every proper subset of $\mathbb{Z}^+$ that is closed under $\mathcal{F}$ is finite, and
(b) for every $f \in \mathcal{F}$, there is a proper infinite subset Y of $\mathbb{Z}^+$ that is closed under $\mathcal{F}\setminus\{f\}$.

Solution by Klaas Pieter Hart, Delft University of Technology, Delft, Netherlands. There is no family satisfying the specified conditions. Let $\mathcal{F}$ be an uncountable family of functions from $\mathbb{N}$ to $\mathbb{N}$. We claim that there exists $f \in \mathcal{F}$ such that, for all n, there exists $g \in \mathcal{F} \setminus \{f\}$ such that $g(i) = f(i)$ for all $i \le n$.

Once this claim is proved, it follows that **(b)** fails for this choice of f if $\mathcal{F}$ satisfies **(a)**. Indeed, if A is closed under $\mathcal{F} \setminus \{f\}$, then A is also closed under f. To see this, take $n \in A$ and take $g \in \mathcal{F} \setminus \{f\}$ such that $g(i) = f(i)$ for $i \le n$. In particular, $f(n) = g(n) \in A$. It follows that exactly the same sets are closed under $\mathcal{F}$ and under $\mathcal{F} \setminus \{f\}$ and this means that once **(a)** holds for $\mathcal{F}$, condition **(b)** fails for f.

To prove the claim, let $\mathcal{C}$ denote the set of $f \in \mathcal{F}$ that do not have the desired property. For each $f \in \mathcal{C}$, let n_f be the first natural number n for which there is no $g \in \mathcal{F} \setminus \{f\}$ such that $g(i) = f(i)$ for $i \le n$. Let s_f denote the finite list $\langle f(i) : i \le n_f \rangle$. The map $f \mapsto s_f$ is injective from $\mathcal{C}$ to the set of finite lists of natural numbers. The latter set is countable, hence so is $\mathcal{C}$. Thus $\mathcal{F} \setminus \mathcal{C}$ is uncountable and therefore nonempty, as desired.

Editorial comment. Hart pointed out that the claim is a special case of the general fact that an uncountable subset of a separable and metrizable space has nonisolated points. The argument of the middle paragraph shows that if f is a nonisolated point of $\mathcal{F}$, then a set is closed under $\mathcal{F}$ if and only if it is closed under $\mathcal{F} \setminus \{f\}$.

Also solved by W. Chen, A. Dow, F. Galvin, G. Gruenhage, GCHQ Problem Solving Group (U. K.), and the proposers.

A Special Instance of Rock–Paper–Scissors

11944 [2016, 1050]. *Proposed by Yury Ionin, Central Michigan University, Mount Pleasant, MI.* Let n be a positive integer, and let $[n] = \{1, \dots, n\}$. For $i \in [n]$, let A_i, B_i, and C_i be disjoint sets such that $A_i \cup B_i \cup C_i = [n] - \{i\}$ and $|A_i| = |B_i|$. Suppose also that

$$\left|A_i \cap B_j\right| + \left|B_i \cap C_j\right| + \left|C_i \cap A_j\right| = \left|B_i \cap A_j\right| + \left|C_i \cap B_j\right| + \left|A_i \cap C_j\right|$$

for $i, j \in [n]$. Prove that $i \in A_j$ if and only if $j \in A_i$ and, likewise, for the Bs and Cs.

Solution by Kyle Hansen (student), Westmont College, Santa Barbara, CA. Because A_i, B_i, and C_i are disjoint, we have $C_i = [n] - (\{i\} \cup A_i \cup B_i)$. Therefore the left side of the given sum becomes

$$\begin{aligned}
&\left|A_i \cap B_j\right| + \left|B_i \cap C_j\right| + \left|A_j \cap C_i\right| \\
&\qquad = \left|A_i \cap B_j\right| + \left|B_i - \left(\{j\} \cup A_j \cup B_j\right)\right| + \left|A_j - (\{i\} \cup A_i \cup B_i)\right| \\
&\qquad = \left|A_i \cap B_j\right| + |B_i| - |B_i \cap \{j\}| - \left|B_i \cap A_j\right| - \left|B_i \cap B_j\right| \\
&\qquad\qquad + \left|A_j\right| - \left|A_j \cap \{i\}\right| - \left|A_j \cap B_i\right| - \left|A_j \cap A_i\right|.
\end{aligned}$$

Similarly, the right side becomes

$$\begin{aligned}
&\left|A_j \cap B_i\right| + \left|B_j \cap C_i\right| + \left|A_i \cap C_j\right| \\
&\qquad = \left|A_j \cap B_i\right| + \left|B_j - (\{i\} \cup A_i \cup B_i)\right| + \left|A_i - \left(\{j\} \cup A_j \cup B_j\right)\right| \\
&\qquad = \left|A_j \cap B_i\right| + \left|B_j\right| - \left|B_j \cap \{i\}\right| - \left|B_j \cap A_i\right| - \left|B_j \cap B_i\right| \\
&\qquad\qquad + |A_i| - |A_i \cap \{j\}| - \left|A_i \cap B_j\right| - \left|A_i \cap A_j\right|.
\end{aligned}$$

Equating these expressions and rearranging using $|A_i| = |B_i|$ yields

$$3\left(\left|A_i \cap B_j\right| - \left|B_i \cap A_j\right|\right) = |B_i \cap \{j\}| - \left|B_j \cap \{i\}\right| + \left|A_j \cap \{i\}\right| - |A_i \cap \{j\}|.$$

The left side is an integer multiple of 3, and the right side has absolute value at most 2, so the right side must be 0. Since $A_i \cap B_i = \emptyset$, we must have $|A_i \cap \{j\}| = \left|A_j \cap \{i\}\right|$ and $|B_i \cap \{j\}| = \left|B_j \cap \{i\}\right|$. That is, $i \in A_j$ if and only if $j \in A_i$ and likewise for the Bs. For the final conclusion, if $i \in C_j$ and $j \notin C_i$, then $j \neq i$ and $j \in A_i \cup B_i$, which means $i \in A_j \cup B_j$, contradicting $i \in C_j$.

Editorial comment. The GCHQ Problem Solving Group interpreted the question using the children's game of rock–paper–scissors. In this game, each of two players picks one of these three options. When both choose the same option, neither wins. When the choices are different, the winner is as follows: rock beats (breaks) scissors, scissors beats (cuts) paper, and paper beats (covers) rock.

Now imagine that n players play n rounds of this game, except that player i does not play in round i, for each $i \in [n]$. On a given round, all choose one of the three options simultaneously, and the winner is determined for each pair. Assume that each player plays rock and paper equally often, and that for every two players X and Y, the number of times that player X beats player Y in the pairwise comparison is the same as the number of times that Y beats X (in the rounds where they both play).

Letting A_i, B_i, and C_i be the sets of rounds on which player i chooses rock, paper, and scissors, respectively, the question posed becomes equivalent to showing that for every two players, the options each chooses in the round skipped by the other are the same.

Also solved by P. P. Dályay (Hungary), T. Hakobyan, O. P. Lossers (Netherlands), S. Patel (India), J. C. Smith, R. Stong, Y. Zhao, GCHQ Problem Solving Group (U. K.), Texas State Problem Solving Group, and the proposer.

A Bisector Inequality

11945 [2016,1050]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, Macedonia.* Let a, b, and c be the lengths of the sides of triangle ABC opposite A, B, and C, respectively, and let w_a, w_b, and w_c be the lengths of the corresponding angle bisectors. Prove

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\geq 2\sqrt{3}.$$

Solution by Dmitry Fleischman, Santa Monica, CA. Writing twice the area of triangle ABC in two ways, we get $w_a(b+c)\sin(A/2)=bc\sin A$, from which follows the known formula

$$w_a=\frac{2bc}{b+c}\cos\left(\frac{A}{2}\right).$$

From the AM-GM inequality used multiple times, we obtain

$$\begin{aligned}\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}&\geq 3\left(\frac{abc}{w_aw_bw_c}\right)^{1/3}\\&=3\left(\frac{(a+b)(b+c)(c+a)}{8abc\cos(A/2)\cos(B/2)\cos(C/2)}\right)^{1/3}\\&\geq\frac{3}{(\cos(A/2)\cos(B/2)\cos(C/2))^{1/3}}.\end{aligned}$$

Now Jensen's inequality applied to $\log\cos x$ yields

$$\cos(A/2)\cos(B/2)\cos(C/2)\leq\cos^3(\pi/6)=\frac{3\sqrt{3}}{8},$$

so

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\geq 2\sqrt{3}.$$

Editorial comment. Radouan Boukharfane noted that the stronger inequality

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\geq 2\sqrt{3}+\frac{3}{2}\left(1-\frac{2r}{R}\right)$$

appears in S. H. Wu and Z. H. Zhang (2006), A class of inequalities related to angle bisectors and the sides of a triangle, *J. Inequal. Pure Appl. Math.* 7(3): 1–16. Several solvers noted that the requested inequality is an easy consequence of the inequality $aw_a+bw_b+cw_c\leq(a+b+c)^2/(2\sqrt{3})$, which appears as item 11.5 in D. S. Mitrinović et. al. (1989), *Recent Advances in Geometric Inequalities*, Dordrecht: Kluwer. Adnan Ali and Albert Stadler (independently) proved the stronger inequality $a/m_a+b/m_b+c/m_c\geq 2\sqrt{3}$, where m_a, m_b, and m_c denote the lengths of the corresponding medians.

Also solved by A. Ali (India), G. Apostolopoulos (Greece), R. Bagby, D. Bailey & E. Campbell & C. Diminnie, D. M. Bătineţu & D. Sitaru (Romania), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), S. Brown, D. Chakerian, M. V. Channakeshava (India), R. Chapman (U. K.), P. P. Dályay (Hungary), M. Drăgan & N. Stanciu & T. Zvonaru (Romania), H. Y. Far, S. Gayen (India), O. Geupel (Germany), L. Giugiuc (Romania), M. Goldenberg and M. Kaplan, J.-P. Grivaux (France), N. Grivaux (France), J. G. Heuver (Canada), F. Holland (Ireland), A. Kadaveru & J. Zacharias, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), D. Moore, G. Musinu (Italy), R. Nandan, P. Nüesch (Switzerland), P. Perfetti (Italy), M. Reid, D. Ritter, V. Schindler (Germany), C. R. Selvaraj & S. Selvaraj, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), Skidmore College Problem Group, and the proposer.

A Second-Derivative Integral Inequality

11946 [2016, 1050]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let f be a twice differentiable function from $[0, 1]$ to $\mathbb{R}$ with f'' continuous on $[0, 1]$ and $\int_{1/3}^{2/3} f(x)\,dx = 0$. Prove

$$4860\left(\int_0^1 f(x)\,dx\right)^2 \le 11\int_0^1 \left(f''(x)\right)^2\,dx.$$

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. Consider the function g defined by

$$g(x) = \begin{cases} x^2/2, & \text{when } 0 \le x \le 1/3; \\ -x^2 + x - 1/6, & \text{when } 1/3 \le x \le 2/3; \\ x^2/2 - x + 1/2, & \text{when } 2/3 \le x \le 1. \end{cases}$$

Observe that g is continuously differentiable with $g(0) = g(1) = g'(0) = g'(1) = 0$ and that the second derivative is piecewise continuous, taking values 1, -2, and 1 on $[0, 1/3)$, $(1/3, 2/3)$, and $(2/3, 1]$, respectively, with discontinuities at $1/3$ and $2/3$. A calculation gives $\int_0^1 g^2\,dx = 11/4860$. Using integration by parts twice, we have

$$\begin{aligned}\int_0^1 gf''\,dx &= gf'\Big|_0^1 - \int_0^1 g'f'\,dx \\ &= -\left(g'f\Big|_0^{1/3} - \int_0^{1/3} g''f\,dx + g'f\Big|_{1/3}^{2/3}\right. \\ &\qquad\qquad \left. - \int_{1/3}^{2/3} g''f\,dx + g'f\Big|_{2/3}^1 - \int_{2/3}^1 g''f\,dx\right) \\ &= \int_0^1 f\,dx - 3\int_{1/3}^{2/3} f\,dx = \int_0^1 f\,dx.\end{aligned}$$

The Cauchy–Schwarz inequality then yields

$$\left(\int_0^1 f\,dx\right)^2 = \left(\int_0^1 gf''\,dx\right)^2 \le \frac{11}{4860}\int_0^1 (f'')^2\,dx.$$

Editorial comment. There was an error in the original problem statement, omitting the square on f''.

Also solved by K. F. Andersen (Canada), R. Bagby, A. Berkane (Algeria), R Boukharfane (France), R. Chapman (U. K.), P. P. Dályay (Hungary), P. Fitzsimmons, K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II,

O. P. Lossers (Netherlands), R. Nandan, S. Patel, P. Perfetti (Italy), J. C. Smith, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (U. K.), and the proposer.

Reality After Logarithmic Differentiation

11947 [2016, 1051]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let n be a positive integer, and let $z_1, \ldots, z_n$ be the zeros in $\mathbb{C}$ of $z^n + 1$. For $a > 0$, prove

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{|z_k - a|^2} = \frac{1 + a^2 + \cdots + a^{2(n-1)}}{(1 + a^n)^2}.$$

Solution by Finbarr Holland, University College Cork, Cork, Ireland. We prove a more general result, assuming only that $a \notin \{z_1, \ldots, z_n\}$. After logarithmic differentiation of

$$z^n + 1 = \prod_{k=1}^{n}(z - z_k),$$

setting $z = a$ yields

$$\frac{2na^n}{a^n + 1} = \sum_{k=1}^{n} \frac{(a + z_k) + (a - z_k)}{a - z_k}$$

$$= n + \sum_{k=1}^{n} \frac{a + z_k}{a - z_k}.$$

Division by n gives

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a + z_k}{a - z_k} = \frac{a^n - 1}{a^n + 1}.$$

Take the real part of both sides using

$$\operatorname{Re}\left(\frac{u + v}{u - v}\right) = \frac{|u|^2 - |v|^2}{|u - v|^2}$$

to obtain

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|a|^2 - 1}{|a - z_k|^2} = \frac{|a^n|^2 - 1}{|a^n + 1|^2}.$$

Factoring the numerator of the fraction on the right yields

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{|z_k - a|^2} = \frac{1 + |a|^2 + \cdots + |a|^{2(n-1)}}{|1 + a^n|^2}.$$

The case $|a| = 1$ follows by continuity. The case $a > 0$ is the desired result.

Also solved by K. F. Andersen (Canada), D. Beckwith, A. Berkane (Algeria), A. J. Bevelacqua, P. Bracken, D. Chakerian, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), N. Ghosh, J.-P. Grivaux (France), J. Grzesik, E. Herman, Y. Ionin, M. Kaplan, K. T. L. Koo (China), O. Kouba (Syria), D. Kyle, P. Lalonde (Canada), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), P. Magli (Italy), P. Perfetti (Italy), V. Schindler (Germany), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), J. Van hamme (Belgium), M. Vowe (Switzerland), T. Wiandt, M. Wildon (U. K.), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Functional Equation

11948 [2016, 1051]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Find all surjective functions $f\colon \mathbb{R} \to \mathbb{R}^+$ such that (1) $f(x) \le x + 1$ for $f(x) \ge 1$, (2) $f(x) \ne 1$ for $x \ne 0$, and (3) for $x, y \in \mathbb{R}$,

$$f(xf(y) + yf(x) - xy) = f(x)f(y).$$

Composite solution by Robin Chapman, University of Exeter, Exeter, U. K., Richard Stong, Center for Communications Research, San Diego, CA, and GCHQ Problem Solving Group, U. K. For $a > 0$, define $F_a : \mathbb{R}^+ \to \mathbb{R}$ by the equation $F_a(x) = x - 1/x^a$. Note that $F_a'(x) > 0$, $F_a(x) \to -\infty$ as $x \to 0^+$, and $F_a(x) \to \infty$ as $x \to \infty$. Thus F_a is an increasing bijection. Therefore it has an inverse function $f_a : \mathbb{R} \to \mathbb{R}^+$, which is also an increasing bijection. We claim that the solutions are precisely the functions f_a.

We begin by verifying that f_a satisfies all of the given conditions. If $f_a(x) \ge 1$, then

$$x = F_a(f_a(x)) = f_a(x) - \frac{1}{(f_a(x))^a} \ge f_a(x) - 1,$$

so $f_a(x) \le x + 1$. This shows that (1) holds for f_a. We have $F_a(1) = 0$, so $f_a(0) = 1$, and since f_a is a bijection, this implies (2). Finally, consider any $x, y \in \mathbb{R}$, and let $u = f_a(x)$, $v = f_a(y)$. Since $x = F_a(u) = u - 1/u^a$, we have $1/u^a = u - x$, and similarly $1/v^a = v - y$. Therefore

$$\begin{aligned} F_a(uv) &= uv - \frac{1}{(uv)^a} = uv - (u - x)(v - y) \\ &= xv + yu - xy = xf_a(y) + yf_a(x) - xy, \end{aligned}$$

so

$$f_a(xf_a(y) + yf_a(x) - xy) = f_a(F_a(uv)) = uv = f_a(x)f_a(y),$$

which is (3).

Next we claim that the functions f_a are the only ones that satisfy the given conditions. Suppose that f satisfies the conditions. For any real number x, we have $f(x) > 0$, so since f is surjective, there is some $y \in \mathbb{R}$ such that $f(y) = 1/f(x)$. Therefore by (3), $f(xf(y) + yf(x) - xy) = f(x)f(y) = 1$. It follows by (2) that $xf(y) + yf(x) - xy = 0$, and thus $(f(x) - x)(f(y) - y) = 1$. Therefore $f(x) \ne x$; in other words, f has no fixed points. Also, if there is some x' such that $f(x') = f(x)$, then $(f(x) - x')(f(y) - y) = 1 = (f(x) - x)(f(y) - y)$ and hence $x' = x$. We conclude that f is a bijection, so we can let $F : \mathbb{R}^+ \to \mathbb{R}$ be the inverse of f.

For arbitrary positive real numbers u and v, let $x = F(u)$ and $y = F(v)$. Since $f(x) = u$ and $f(y) = v$, by (3), we have

$$f(F(u)v + F(v)u - F(u)F(v)) = f(xf(y) + yf(x) - xy) = f(x)f(y) = uv,$$

and therefore

$$uv - F(uv) = uv - F(u)v - F(v)u + F(u)F(v) = (u - F(u))(v - F(v)).$$

In other words, if we define $g : \mathbb{R}^+ \to \mathbb{R}$ by $g(x) = x - F(x)$, then

$$g(uv) = g(u)g(v)$$

for all $u, v \in \mathbb{R}^+$. In particular, $g(u) = (g(\sqrt{u}))^2 \ge 0$. Since f has no fixed points, F also has none, and therefore g never takes the value 0. Thus, g takes only positive values.

Define $L : \mathbb{R} \to \mathbb{R}$ by the equation $L(x) = \log(g(e^x))$, so that $g(x) = e^{L(\log(x))}$ for $x > 0$. For all $x, y \in \mathbb{R}$,

$$L(x+y) = \log(g(e^x e^y)) = \log(g(e^x)) + \log(g(e^y)) = L(x) + L(y),$$

and this implies that $L(rx) = rL(x)$ for every rational number r. In particular, if we let $c = L(1)$, then $L(r) = cr$ for every rational number r.

We claim next that L is a weakly decreasing function. To see why, suppose $x, y \in \mathbb{R}$ and $x \le y$. Let $t = y - x$, $u = e^t \ge 1$, and $z = F(u)$. Since $f(z) = u \ge 1$, by condition (1), we have $f(z) \le z + 1$, which means $u \le F(u) + 1$. Therefore $g(u) = u - F(u) \le 1$, so $L(t) = \log(g(u)) \le 0$. We conclude that $L(y) = L(x+t) = L(x) + L(t) \le L(x)$.

We can now determine all values of L. For any $x \in \mathbb{R}$, if r_1 and r_2 are rational numbers with $r_1 \le x \le r_2$ then

$$cr_1 = L(r_1) \ge L(x) \ge L(r_2) = cr_2.$$

The only possible value for $L(x)$ is therefore $L(x) = cx$. Thus $g(x) = e^{L(\log(x))} = e^{c\log(x)} = x^c$ and $F(x) = x - x^c$.

Since L is weakly decreasing, $c \le 0$. If $c = 0$, then $F(x) = x - 1$, which contradicts the fact that F is a bijection from $\mathbb{R}^+$ to $\mathbb{R}$. Therefore $c < 0$; say $c = -a$, where $a > 0$. Then $F(x) = x - 1/x^a$, so $F = F_a$, and therefore $f = f_a$.

Also solved by W. Chen, P. P. Dályay (Hungary), N. Ghosh, O. P. Lossers (Netherlands), NSA Problems Group, and the proposer.

A Functional Equation with Cosines

11949 [2016, 1051]. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* Show that there exists a unique function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is differentiable, $2\cos(x + f(x)) - \cos x = 1$ for all real x, and $f(\pi/2) = -\pi/6$.

Solution by GCHQ Problem Solving Group, U. K. The arccos function from $[-1, 1]$ to $[0, \pi]$ is differentiable with $(\arccos\, x)' = -1/\sqrt{1-x^2}$ when $x \ne \pm 1$. We first prove existence of f. For $x \in [0, 2\pi]$, define

$$f(x) = \arccos\left(\frac{1+\cos x}{2}\right) - x.$$

By definition, f satisfies the functional equation for all $x \in [0, 2\pi]$ and $f(\pi/2) = -\pi/6$. By the chain rule, f is differentiable on $(0, 2\pi)$, since $(1+\cos x)/2 \ne \pm 1$. Now $\arccos(1-y) \sim \sqrt{2y}$ as $y \to 0^+$. To see this, set $z = \arccos(1-y)$, which by Taylor's theorem yields $z \sim \sqrt{2(1-\cos z)} = \sqrt{2y}$ as $z \to 0^+$. Since $(1-\cos x)/2 \sim x^2/4$ as $x \to 0^+$,

$$f(x) + x = \arccos\left(1 - \frac{1-\cos x}{2}\right) \sim \sqrt{\frac{2x^2}{4}} = \frac{x}{\sqrt{2}} \qquad \text{as } x \to 0^+.$$

Let $\alpha = 1/\sqrt{2}$. Since $f(0) = 0$, we conclude that f is right differentiable at 0 with right derivative $\alpha - 1$. A similar argument shows f is left differentiable at 2π with left derivative $-\alpha - 1$. Now extend f to $[-2\pi, 2\pi]$ by taking f to be odd on this interval, which is possible since $f(0) = 0$. The equation $2\cos(x + f(x)) - \cos x = 1$ now holds for all $x \in [-2\pi, 2\pi]$. Further, since

$$\frac{f(-x) - f(0)}{-x - 0} = \frac{f(x)}{x} = \frac{f(x) - f(0)}{x},$$

we see that f is left differentiable at 0 with the same value as its right derivative, $\alpha - 1$.

Hence, f is differentiable at 0, and since f is differentiable on $(0, 2\pi)$ and is odd on $[-2\pi, 2\pi]$, we conclude that f is in fact differentiable on $(-2\pi, 2\pi)$. We also know that f is left differentiable at 2π with left derivative $-\alpha - 1$ and, since f is odd on $[-2\pi, 2\pi]$, f is right differentiable at -2π with right derivative also equal to $-\alpha - 1$. Also $f(2\pi) - f(-2\pi) = 2f(2\pi) = -4\pi$. Hence, we may extend f to a continuous function on $\mathbb{R}$ by means of the relation $f(x + 4\pi) = f(x) - 4\pi$. The equation $2\cos(x + f(x)) - \cos x = 1$ is easily seen to hold on $\mathbb{R}$.

Since f is differentiable on $(-2\pi, 2\pi)$, the function f is differentiable on all of $\mathbb{R}$, except possibly at $2\pi + 4n\pi$ for integer n. To show that f is differentiable everywhere, it suffices to show that f is differentiable at 2π. We know that f is differentiable at 2π with left derivative $-\alpha - 1$. Since $x + f(x)$ is 4π-periodic and f is right differentiable at -2π with right derivative $-\alpha - 1$, the same is true at 2π. Hence, f is differentiable at 2π. This shows existence.

Next, we prove uniqueness. Suppose that f is as described above and $g : \mathbb{R} \to \mathbb{R}$ also satisfies the given conditions. Let $F(x) = x + f(x)$ and $G(x) = x + g(x)$, so F and G are differentiable with $2\cos(F(x)) - \cos x = 1$ and $2\cos(G(x)) - \cos x = 1$ for all $x \in \mathbb{R}$, and $F(\pi/2) = G(\pi/2) = \pi/3$. We first show $F(x) = G(x)$ for all $x \geq \pi/2$. Assume otherwise. The set $\{x \geq \pi/2 \mid F(x) \neq G(x)\}$ is nonempty and has an infimum $x_0 \geq \pi/2$. If this inequality is strict, then $F(x) = G(x)$ for $x \in [\frac{\pi}{2}, x_0)$, and by continuity we have $F(x_0) = G(x_0)$. This also holds if $x_0 = \pi/2$, by assumption.

First suppose x_0 is not an integer multiple of 2π. Since $\cos x_0 \neq 1$, we have $\cos F(x_0) \neq \pm 1$, so $F(x_0)$ is not an integer multiple of π. Since $F(x_0) + G(x_0) = 2F(x_0)$, this quantity is not an integer multiple of 2π. We conclude that $F + G$ is bounded away from the set $2\pi\mathbb{Z}$ in some open interval I containing x_0. Since $\cos F(x) = \cos G(x)$ for all x, which implies $F(x) + G(x)$ or $F(x) - G(x)$ is an integer multiple of 2π, we must have the latter on I. Since $F(x) - G(x)$ is continuous, equals 0 at x_0, and is an integer multiple of 2π on I, we conclude $F(x) - G(x) = 0$ on I, contradicting the definition of x_0 an infimum.

On the other hand, suppose that x_0 is a multiple of 2π. By construction, $F(x_0) = 0$, and F is differentiable at x_0 with derivative α or $-\alpha$. Since G agrees with F up to x_0, G is left differentiable at x_0 with left derivative $F'(x_0)$. We know that $G(x)$ is differentiable everywhere, so $G'(x_0) = F'(x_0) \neq 0$. Since $F(x_0) = G(x_0) = 0$, and F and G are continuous, we may find an open interval J containing x_0 such that $F - G$ is not a nonzero integer multiple of 2π on J. By the definition of x_0, we can find a sequence $x_t \in J$ with $x_t \to x_0^+$ and $F(x_t) \neq G(x_t)$. Since $\cos F(x_t) = \cos G(x_t)$ for all t, we have $F(x_t) + G(x_t) = 2\pi n_t$ for some $n_t \in \mathbb{Z}$. As $t \to \infty$, the left side tends to zero, which means $n_t = 0$ for t sufficiently large. Therefore, using $F(x_0) = G(x_0) = 0$ again,

$$\frac{F(x_t) - F(x_0)}{x_t - x_0} = -\frac{G(x_t) - G(x_0)}{x_t - x_0}$$

for sufficiently large t. This implies $F'(x_0) = -G'(x_0)$, contradicting $F'(x_0) = G'(x_0) \neq 0$. This contradiction establishes that $F(x) = G(x)$ for all $x \geq \pi/2$.

A similar argument shows $F(x) = G(x)$ for all $x \leq \pi/2$. Hence, $F(x) = G(x)$ for all $x \in \mathbb{R}$, and so $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Also solved by K. F. Andersen (Canada), R. Boukharfane (France), P. Bracken, D. Chakerian, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), B. Davis, D. Fleischman, J.-P. Grivaux (France), T. Hakobyan, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), I. Mihăilă, S. Muthiah, M. Omarjee (France), P. Perfetti (Italy), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), H. Wang, T. Wiandt, Missouri State University Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, and Fuzhen Zhang.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by March 31, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12069. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Place n nonattacking rooks on an n-by-n chessboard in such a way as to maximize the sum of the Euclidean distances from the rooks to the center of the chessboard. (Regard a rook as a point positioned at the center of its square.) How many placements attain this maximum?

12070. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$\int_0^{\pi/4}\int_0^{\pi/4} \frac{\cos x \cos y\,(y\,\sin y\,\cos x - x\,\sin x\,\cos y)}{\cos(2x)-\cos(2y)}\,dx\,dy = \frac{7\zeta(3)+4\pi\ln 2}{64},$$

where ζ is the Riemann zeta function.

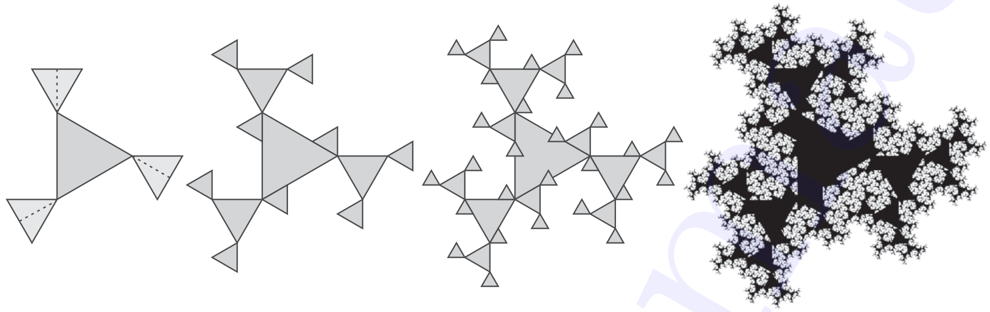
12071. *Proposed by Paul Hagelstein and Daniel Herden, Baylor University, Waco, TX.* For a positive integer n, let $Q(n)$ denote the greatest integer with the following property: Any family of n closed squares in the plane whose sides are parallel to the coordinate axes contains either a subfamily of $Q(n)$ squares with a nonempty intersection or a subfamily of $Q(n)$ squares that are pairwise disjoint.
(a) Prove $Q(n)/\sqrt{n} \geq 0.5$ for all n.
(b) Prove $\limsup_{n\to\infty} Q(n)/\sqrt{n} \leq \sqrt{0.8}$.

12072. *Proposed by Stephen Scheinberg, Corona del Mar, CA.*
(a) Let X be a connected Hausdorff topological space with the property that every point has a neighborhood whose cardinality c is that of the continuum. Assume the following: For every $x \in X$ and $Y \subset X$ with $x \in \text{closure}(Y)$, there exists a sequence $(y_n)_{n=1}^{\infty}$ in Y with $\lim_{n\to\infty} y_n = x$. Prove that the cardinality of X is c.
(b) Give an example of a connected, locally connected, locally compact Hausdorff topological space whose cardinality is greater than c but every one of whose points has a neighborhood of cardinality c.

12073. *Proposed by Hakan Karakus, Antalya, Turkey.* Given a scalene triangle ABC, let G denote its centroid and H denote its orthocenter. Let P_A be the second point of intersection of the two circles through A that are tangent to BC at B and at C. Similarly define P_B and P_C. Prove that G, H, P_A, P_B, and P_C are concyclic.

doi.org/10.1080/00029890.2018.1507359

12074. *Proposed by E. Paul Goldenberg, Education Development Center, Waltham, MA.* Start with an equilateral triangle of area 1. Attach externally three equilateral triangles to the vertices of the original triangle as in the first picture below, so that the altitude of each new triangle is an extension of one side of the original triangle and half its length. Always use the side that is counterclockwise from the vertex. Continue this process, producing each new generation by attaching three triangles to each triangle of the previous generation. Let τ_n be the union of all triangles (and their interiors) produced through generation n. What is the area of $\cup_{n=1}^{\infty} \tau_n$?



τ_2 τ_3 τ_4 τ_{10}

12075. *Proposed by George Stoica, Saint John, NB, Canada.* For $n \geq 1$, let $x_n = \sum_{k=1}^{\infty} k^n/e^k$. Prove that $\lim_{n\to\infty} x_n/n!$ equals 1 but the sequence $(x_n - n!)_{n\geq 1}$ is unbounded.

SOLUTIONS

Sums of Squares

11950 [2017, 84]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Prove that for all positive integers a and b, there are infinitely many positive integers n such that n, $n + a$, and $n + b$ can each be expressed as a sum of two squares.

Solution by John P. Robertson, National Council on Compensation Insurance, Boca Raton, FL. Fix $a, b \in \mathbb{N}$. By parametrizing the problem, we combine several cases into one. If $a \equiv 2 \pmod 4$ and $b \equiv 2 \pmod 4$, then let $c = b - a$ and $d = -a$. We now seek integers m such that each element of $\{m, m + c, m + d\}$ is a sum of two squares; the desired integer n is then $m - a$. In this case, $c \equiv 0 \pmod 4$.

If a and b are not both congruent to 2 modulo 4, then by symmetry we may assume $a \not\equiv 2 \pmod 4$. We again seek m such that each of $\{m, m + c, m + d\}$ is a sum of two squares, where now $c = a$ and $d = b$. The desired integer n will equal m. In this case, c is odd or divisible by 4, but not congruent to 2 modulo 4.

Hence in all cases $c \not\equiv 2 \pmod 4$. Let $i = c/4 - 1$ and $j = c/4 + 1$ when 4 divides c. Let $i = (c - 1)/2$ and $j = (c + 1)/2$ when c is odd. In each case, i and j are integers and $j^2 - i^2 = c$.

Let ℓ be an integer with opposite parity to $i^2 + d$. Set $r = (\ell^2 - i^2 - d + 1)/2$ and $k = r - 1$. Letting $m = r^2 + i^2$, we have $m + c = r^2 + j^2$ and $m + d = k^2 + \ell^2$. Since we have infinitely many choices for ℓ, we obtain infinitely many choices for m having the desired property.

Editorial comment. The problem is solved in Hooley, C. (1973), On the intervals between numbers that are the sum of two squares II, *J. Number Theory*, 5: 215–217.

Also solved by A. Bevelacqua, R Boukharfane (France), T. Horine, K. Koo (China), O. P. Lossers (Netherlands), B. Randé (France), M. Reid, J. C. Smith, A. Stadler (Switzerland), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

An Inequality for the Altitudes

11951 [2017, 84]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let ABC be a triangle that is not obtuse. Denote by a, b, and c the lengths of the sides opposite A, B, and C, respectively, and denote by h_a, h_b, and h_c the lengths of the altitudes dropped from A, B, and C, respectively. Prove

$$\frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_c^2+h_a^2}+\frac{c^2}{h_a^2+h_b^2}<\frac{5}{2}.$$

Show also that $5/2$ is the smallest possible constant in this inequality.

Solution by Haoran Chen, Gustavus Adolphus College, St. Peter, MN. Let A, B, and C denote the angles of the triangle. Since $h_a = b\sin C = c\sin B$ and similarly for h_b and h_c, the requested inequality becomes

$$\frac{1}{\sin^2 B+\sin^2 C}+\frac{1}{\sin^2 C+\sin^2 A}+\frac{1}{\sin^2 A+\sin^2 B}<\frac{5}{2}.$$

If $A=\pi/2$, $B\to\pi/2$, and $C\to 0$, then the left side tends to $5/2$, hence the constant $5/2$ cannot be lowered.

Now assume $\pi/2 \ge A \ge B \ge C$. The inequality becomes

$$\frac{1}{\sin^2 A+\sin^2 B}-\frac{1}{2}<1-\frac{1}{\sin^2 C+\sin^2 A}+1-\frac{1}{\sin^2 B+\sin^2 C},$$

or

$$\frac{\cos^2 A+\cos^2 B}{2(\sin^2 A+\sin^2 B)}<\frac{\sin^2 C-\cos^2 A}{\sin^2 C+\sin^2 A}+\frac{\sin^2 C-\cos^2 B}{\sin^2 B+\sin^2 C}.$$

Since $0\le A-C\le B$, we have $\cos(A-C)\ge\cos B$, and hence

$$\sin^2 C-\cos^2 A=-\cos(A+C)\cos(A-C)=\cos B\cos(A-C)\ge\cos^2 B.$$

Thus it suffices to show

$$\frac{\cos^2 A+\cos^2 B}{2(\sin^2 A+\sin^2 B)}<\frac{\cos^2 B}{\sin^2 C+\sin^2 A}+\frac{\cos^2 A}{\sin^2 B+\sin^2 C}.$$

This follows directly, since both denominators on the right are smaller than the denominator on the left.

Also solved by R. Chapman (U. K.), M. Drągan & N. Stanciu (Romania), Y. Ionin, K. Koo (China), J. H. Lindsey II, O. P. Lossers (Netherlands), G. Musinu (Italy), M. Reid, V. Schindler (Germany), A. Schwenk, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), and the proposer.

Telescoping Without Partial Fractions

11952 [2017, 83]. *Proposed by Z. K. Silagadze, Novosibirsk State University, Novosibirsk, Russia.* Prove

$$\sum_{n=1}^{\infty}\frac{2^{2n-1}}{2n+1}\left(\frac{(n-1)!}{(2n-1)!!}\right)^2=\pi-2,$$

where $(2n-1)!!$ is defined as usual to be $\prod_{k=1}^{n}(2k-1)$.

Solution by Chikkanna R. Selvaraj and Suguna Selvaraj, Pennsylvania State University–Shenango, Sharon, PA. Let a_n be the nth term of the series, and let

$$b_n = 2\left(\frac{2^2}{1\cdot 3}\cdot\frac{4^2}{3\cdot 5}\cdot\cdots\cdot\frac{(2n-2)^2}{(2n-3)(2n-1)}\right) \qquad \text{and} \qquad c_n = \frac{1}{(2n-1)(2n+1)}.$$

We then have $a_n = b_n c_n$. Instead of the partial fraction decomposition of c_n, write $c_n = \frac{(2n)^2}{(2n-1)(2n+1)} - 1$ to obtain $a_n = b_{n+1} - b_n$. The Nth partial sum then telescopes:

$$\sum_{n=1}^{N} a_n = b_{N+1} - b_1 = b_{N+1} - 2.$$

By Wallis's formula, $\lim_{N\to\infty} b_{N+1} = 2\,(\pi/2) = \pi$, and the result follows.

Editorial comment. Some solvers rewrote the formula

$$(\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{2^{n-1}}{n}\frac{(n-1)!}{(2n-1)!!}x^{2n} \quad \text{for} -1 \le x \le 1,$$

as

$$t^2 = \sum_{n=1}^{\infty} \frac{2^{n-1}}{n}\frac{(n-1)!}{(2n-1)!!}\,\sin^{2n} t \quad \text{for} \frac{-\pi}{2} \le t \le \frac{\pi}{2},$$

multiplied by $\sin t$, and then integrated from 0 to $\pi/2$. Multiplying by $(\sin t)^{2k+1}$ instead of $\sin t$, Omran Kouba obtained the generalization

$$\sum_{n=1}^{\infty} \frac{2^{2n+k-1}}{n}\frac{(n-1)!}{(2n-1)!!}\frac{(n+k)!}{(2n+2k+1)!!} = A_k\pi - B_k,$$

where

$$A_k = \frac{1}{2^{2k}}\sum_{j=0}^{k}\frac{\binom{2k+1}{k-j}}{(2j+1)^2} \quad \text{and} \quad B_k = \frac{1}{2^{2k-1}}\sum_{j=0}^{k}(-1)^j\frac{\binom{2k+1}{k-j}}{(2j+1)^3}.$$

When $k = 0$, the right side reduces to the desired value $\pi - 2$.

The problem was suggested by the paper of Friedmann, T. and Hagen, C. R. (2015), Quantum mechanical derivation of the Wallis formula for π, *J. Math. Phys.* 56: 112101.

Also solved by A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, D. Fritze (Germany), O. Kouba (Syria), O. P. Lossers (Netherlands), P. Magli (Italy), V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), P. Perfetti (Italy), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), C. I. Vąlean (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Using Plancherel to Evaluate a Double Integral

11953 [2017, 84]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania.* Calculate

$$\int_0^\infty\int_0^\infty \frac{\sin x\ \sin y\ \sin(x+y)}{xy(x+y)}\,dx\,dy.$$

Solution by Robin Chapman, University of Exeter, Exeter, UK. We claim

$$\int_0^\infty\int_0^\infty \frac{\sin x \sin y \sin(x+y)}{xy(x+y)}\,dx\,dy = \frac{\pi^2}{6}.$$

For convenience, set $S(x) = (\sin x)/x$. We first calculate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x)S(y)S(x+y)\,dx\,dy,$$

using some of the theory of the Fourier transform. Define $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt$ whenever f is an L^1 function. Note that $S = \hat{h}/2$, where h is the indicator function of the interval $[-1, 1]$.

The Plancherel theorem states that $\int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)}\,dx = 2\pi \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt$ whenever f and g are both L^1 and L^2 functions. For each real y,

$$S(x+y) = \frac{1}{2}\int_{-1}^{1} e^{-i(x+y)t}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(x+y)t}h(t)\,dt,$$

and so $x \mapsto S(x+y)$ is the Fourier transform of the function $t \mapsto (1/2)h(t)e^{-iyt}$. Applying the Plancherel theorem to this function and to $h/2$ gives

$$\int_{-\infty}^{\infty} S(x)S(x+y)\,dx = \frac{\pi}{2}\int_{-1}^{1} e^{-iyt}\,dt = \pi S(y).$$

Thus we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x)S(y)S(x+y)\,dx\,dy = \pi\int_{-\infty}^{\infty} S(y)^2\,dy = \pi^2.$$

We now show that the function $(x, y) \mapsto S(x)S(y)S(x+y)$ is an L^1 function. Observe that $S(x)$ is even, is bounded near 0, and is in $O(1/|x|)$ as $|x| \to \infty$. Hence there exists A such that $S(x) \le A/(1+|x|)$ as $|x| \to \infty$. Therefore, to show that $(x, y) \mapsto S(x)S(y)S(x+y)$ is L^1, it suffices to show that

$$(x, y) \mapsto \frac{1}{(1+|x|)(1+|y|)(1+|x+y|)}$$

is L^1. First of all, for $y > 0$,

$$\begin{aligned}\int_0^{\infty} \frac{dx}{(1+|x|)(1+|y|)(1+|x+y|)} &= \frac{1}{1+y}\int_0^{\infty}\frac{dx}{(1+x)(1+x+y)}\\ &= \frac{1}{y(1+y)}\int_0^{\infty}\left(\frac{1}{1+x} - \frac{1}{1+x+y}\right)dx = \frac{\log(1+y)}{y(1+y)}.\end{aligned}$$

As $y \to 0$, we have $\log(1+y)/(y(1+y)) \to 1$. Hence

$$\int_0^{\infty}\int_0^{\infty}\frac{dx\,dy}{(1+|x|)(1+|y|)(1+|x+y|)} = \int_0^{\infty}\frac{\log(1+y)}{y(1+y)}\,dy$$

is finite. Therefore, the integral $I = \int_0^{\infty}\int_0^{\infty} S(x)S(y)S(x+y)\,dx\,dy$ is absolutely convergent.

We can therefore freely apply changes of variable to I:

$$\begin{aligned} I &= \int_0^{\infty}\int_0^{\infty} S(x)S(y)S(x+y)\,dx\,dy \qquad (1)\\ &= \int_0^{\infty}\int_{-\infty}^{-y} S(-x'-y)S(y)S(-x')\,dx'\,dy\\ &= \int_0^{\infty}\int_{-\infty}^{-y} S(x'+y)S(y)S(x')\,dx'\,dy\end{aligned}$$

on setting $x' = -x - y$ and using the fact that S is even. Therefore,

$$I = \int_0^\infty \int_{-\infty}^{-y} S(x)S(y)S(x+y)\,dx\,dy. \tag{2}$$

Now mapping (x, y) to $(-y, -x)$ gives

$$I = \int_0^\infty \int_{-y}^{0} S(x)S(y)S(x+y)\,dx\,dy. \tag{3}$$

By summing (1)–(3),

$$3I = \int_0^\infty \int_{-\infty}^{\infty} S(x)S(y)S(x+y)\,dx\,dy.$$

By the evenness of the integrand,

$$6I = \int_{-\infty}^\infty \int_{-\infty}^{\infty} S(x)S(y)S(x+y)\,dx\,dy = \pi^2.$$

We conclude $I = \pi^2/6$.

Also solved by A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, B. Davis, J. Grzesik, M. Hoffman, F. Holland (Ireland), O. Kouba (Syria), V. Mikayelyan (Armenia), M. Omarjee (France), P. Perfetti (Italy), V. Schindler (Germany), A. N. Sharma (India), S. Sharma, J. C. Smith, A. Stadler (Switzerland), S. Stewart (Australia), R. Stong, R. Tauraso (Italy), E. I. Verriest, H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

A Tale of the Tails of $\zeta(2)$

11954 [2017, 83]. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Determine the largest constant c and the smallest constant d such that, for all positive integers n,

$$\frac{1}{n-c} \le \sum_{k=n}^{\infty} \frac{1}{k^2} \le \frac{1}{n-d}.$$

Composite solution by Douglas B. Tyler, Torrance, CA, and NSA Problems Group, Fort Meade, MD. The answer is $c = 1 - 6/\pi^2 \approx 0.39$ and $d = 1/2$. To prove this, set $T_n = \sum_{k \ge n} 1/k^2$. The required inequality is equivalent to $c \le n - (1/T_n) \le d$. Thus $c = \inf\{n - 1/T_n \colon\ n = 1, 2, \ldots\}$ and $d = \sup\{n - 1/T_n \colon\ n = 1, 2, \ldots\}$.

Note that the function $x \mapsto 1/x^2$ is convex on $(0, \infty)$. Since the trapezoid rule overestimates the integral of a convex function,

$$T_n = \sum_{k=n}^{\infty} \frac{1}{k^2} > \frac{1}{2n^2} + \int_n^\infty \frac{dx}{x^2} = \frac{2n+1}{2n^2} = \frac{1}{n - 1/2 + 1/(4n+2)}. \tag{1}$$

Similarly, the midpoint rule underestimates the integral of a convex function, so

$$T_n = \sum_{k=n}^{\infty} \frac{1}{k^2} < \int_{n-1/2}^\infty \frac{dx}{x^2} = \frac{1}{n-1/2}. \tag{2}$$

Combining (1) and (2), we get

$$\frac{1}{2} - \frac{1}{4n+2} < n - \frac{1}{T_n} < \frac{1}{2}.$$

These inequalities imply $d = 1/2$. Also, for $n \geq 2$,

$$n - \frac{1}{T_n} > \frac{1}{2} - \frac{1}{4n+2} \geq \frac{2}{5} > 1 - \frac{6}{\pi^2} = 1 - \frac{1}{T_1}.$$

It follows that $c = 1 - 1/T_1 = 1 - 6/\pi^2$.

Also solved by K. F. Andersen (Canada), R. Bagby, B. Burdick, R. Chapman (U. K.), N. Grivaux (France), J. Grzesik, E. Herman, O. Kouba (Syria), J. H. Lindsey II, J. Lockhart, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), B Randé (France), M. Reid, J. C. Smith, A. Stadler (Switzerland), A. Stenger, G. Stoica (Canada), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Boys and Girls in Disarray

11955 [2017, 84]. *Proposed by David Stoner, Aiken, SC.* Some boys and girls stand on some of the squares of an n-by-n grid. (Each square may contain several or no children.) Each child computes the fraction of children in his or her row whose gender matches his or her own and the fraction of children in his or her column whose gender matches his or her own. Each child writes down the sum of the two numbers he or she obtains. Prove that the product of all numbers written down in such a manner is at least 1.

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. We prove that the claim holds more generally for any m-by-n grid with $m, n \geq 1$. Discarding empty rows or columns, we may assume that each row and column contains at least one child. Let $x_{i,j}$ be the number of girls and $y_{i,j}$ be the number of boys on square i, j. Let $x_{i\bullet} = \sum_j x_{i,j}$ and $x_{\bullet j} = \sum_i x_{i,j}$, and similarly for $y_{i\bullet}$ and $y_{\bullet j}$. Our task is to prove

$$\prod_{i,j} \left(\frac{x_{i\bullet}}{x_{i\bullet} + y_{i\bullet}} + \frac{x_{\bullet j}}{x_{\bullet j} + y_{\bullet j}} \right)^{x_{ij}} \left(\frac{y_{i\bullet}}{x_{i\bullet} + y_{i\bullet}} + \frac{y_{\bullet j}}{x_{\bullet j} + y_{\bullet j}} \right)^{y_{ij}} \geq 1,$$

where by convention $0^0 = 1$. Since $a + b \geq 2\sqrt{ab}$ for nonnegative a and b, the product on the left is at least

$$\prod_{i,j} \left(2\sqrt{\frac{x_{i\bullet}}{x_{i\bullet} + y_{i\bullet}} \cdot \frac{x_{\bullet j}}{x_{\bullet j} + y_{\bullet j}}} \right)^{x_{ij}} \left(2\sqrt{\frac{y_{i\bullet}}{x_{i\bullet} + y_{i\bullet}} \cdot \frac{y_{\bullet j}}{x_{\bullet j} + y_{\bullet j}}} \right)^{y_{ij}},$$

which rearranges to

$$\sqrt{\prod_i \left(\frac{2x_{i\bullet}}{x_{i\bullet} + y_{i\bullet}} \right)^{x_{i\bullet}} \left(\frac{2y_{i\bullet}}{x_{i\bullet} + y_{i\bullet}} \right)^{y_{i\bullet}} \prod_j \left(\frac{2x_{\bullet j}}{x_{\bullet j} + y_{\bullet j}} \right)^{x_{\bullet j}} \left(\frac{2y_{\bullet j}}{x_{\bullet j} + y_{\bullet j}} \right)^{y_{\bullet j}}}.$$

In this last expression, each factor corresponding to i or j has the form

$$\left(2z^z(1-z)^{1-z}\right)^k,$$

where z is the fraction of girls and k is the total number of children in the row or column. Finally, we show that $2z^z(1-z)^{1-z}$ is at least 1 for every $z \in [0, 1]$. This is equivalent to showing that $z \log_2 z + (1-z)\log_2(1-z)$ is at least -1 for every $z \in [0, 1]$. Let $f(x) = x \log_2 x$ on $[0, 1]$ (extended to $x = 0$ by $f(0) = 0$). Since f is convex, $f(x) + f(1-x) \geq 2f(1/2) = -1$. Hence, the desired quantity is a product of numbers each of which is at least 1.

Also solved by P. P. Dályay (Hungary), N. Grivaux (France), M. A. Prasad (India), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

A Hyperbolic Arctangent Series

11956 [2017, 85]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Show that

$$\sum_{n=1}^{\infty} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

converges, and find the sum.

Solution by Pierluigi Magli, Liceo "Marzolla-Leo-Simone-Durano," Brindisi, Italy. We write $A(x)$ for $\arctan x$. From the identities $A(u) + A(1/u) = \pi/2$ and $A(u) - A(v) = A\left((u-v)/(1+uv)\right)$ for $u, v > 0$, we obtain

$$A(\sinh n) = A\left(\frac{e^n - e^{-n}}{2}\right) = A(e^n) - A(e^{-n}) = 2A(e^n) - \frac{\pi}{2}$$

and

$$A\left(\frac{\sinh 1}{\cosh n}\right) = A\left(\frac{e^{n+1} - e^{n-1}}{1 + e^{2n}}\right) = A(e^{n+1}) - A(e^{n-1}).$$

Thus the partial sum of the required series is given by

$$\begin{aligned} S_M &= 2\sum_{n=1}^{M}\Big(A(e^{n+1})A(e^n) - A(e^n)A(e^{n-1})\Big) - \frac{\pi}{2}\sum_{n=1}^{M}\Big(A(e^{n+1}) - A(e^{n-1})\Big) \\ &= 2\Big(A(e^{M+1})A(e^M) - \frac{\pi}{4}A(e)\Big) - \frac{\pi}{2}\Big(A(e^M) + A(e^{M+1}) - A(e) - \frac{\pi}{4}\Big) \\ &= 2A(e^{M+1})A(e^M) - \frac{\pi}{2}A(e^M) - \frac{\pi}{2}A(e^{M+1}) + \frac{\pi^2}{8}. \end{aligned}$$

It follows that $\lim_{M\to+\infty} S_M = \pi^2/8$.

Also solved by A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), N. Grivaux (France), E. Ionaşcu, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), K. Koo (China), O. Kouba (Syria), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), P. Perfetti (Italy), A. N. Sharma (India), N. Singer, J. C. Smith, A. Stadler (Switzerland), S. Stewart (Australia), R. Stong, R. Tauraso (Italy), C. I. Vălean (Romania), T. Wiandt, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), NSA Problems Group, and the proposer.

Are Surjections More Probable than Injections?

11957 [2017, 179]. *Proposed by Éric Pité, Paris, France.* Let m and n be two integers with $n \ge m \ge 2$. Let $S(n, m)$ be the Stirling number of the second kind, i.e., the number of ways to partition a set of n objects into m nonempty subsets. Show

$$n^m S(n, m) \ge m^n \binom{n}{m}.$$

Solution I by Thomas Horine, Indiana University Southeast, New Albany, IN. To partition the set $[n]$ into m nonempty unlabelled sets, first choose m elements and place one in each set, which can be done in $\binom{n}{m}$ ways. The remaining $n - m$ elements can be assigned to those m sets in m^{n-m} ways to complete a partition. However, a partition with part sizes $s_1, \ldots, s_m$ has been counted $\prod_{i=1}^{m} s_i$ times, since each part can be initiated by any of its s_i elements. In each case $\sum_{i=1}^{m} s_i = n$, so the arithmetic-geometric mean inequality yields $\prod_{i=1}^{m} s_i \le (n/m)^m$ for each partition. Thus

$$S(n, m) \ge \frac{\binom{n}{m} \cdot m^{n-m}}{\left(\frac{n}{m}\right)^m} = \binom{n}{m}\frac{m^n}{n^m}.$$

Solution II by Mark Wildon, Royal Holloway, University of London, Egham, U. K. We interpret the desired inequality as comparing the sizes of certain sets.

The left side, $n^m S(n,m)$, is the number of ways to form a partition $\mathcal{P}$ of $[n]$ into m parts and assign to each part a number in $[n]$ via a function f. The right side, $m^n\binom{n}{m}$, is the number of ways to choose a set Z of m elements in $[n]$ and assign to each member of $[n]$ one element of Z via a function g.

Let $\mathcal{L}_r$ be the set of such pairs $(\mathcal{P}, f)$ such that exactly r elements of $[n]$ are used in the image of f, and let $\mathcal{R}_r$ be the set of such pairs (Z, g) such that exactly r elements of $[m]$ are used in the image of g. It suffices to prove $|\mathcal{L}_r| \geq |\mathcal{R}_r|$ for all r. Since $m \leq n$, we may assume $r \leq m$.

For $(Z, g) \in \mathcal{R}_r$, the preimages of elements of $[n]$ under g form a partition $\mathcal{Q}$ of $[n]$ into r parts. Since the image of g is contained in Z, we can build the pair (Z, g) by first choosing an r-element subset W of $[n]$ for the image of g, then $m - r$ additional members of $[n]$ to complete Z, then the partition $\mathcal{Q}$, and finally the bijective assignment of elements of Z to parts of $\mathcal{Q}$. Thus

$$|\mathcal{R}_r| = \binom{n}{r}\binom{n-r}{m-r} S(n,r) r!.$$

To count $\mathcal{L}_r$, note that for any partition $\mathcal{P}$ of $[n]$, we can build the function f by first choosing an r-element subset X of $[n]$ to be the image of f, then a partition of the m parts of $\mathcal{P}$ into r nonempty blocks, and finally the bijective assignment of elements of X to these blocks. Thus,

$$|\mathcal{L}_r| = S(n,m)\binom{n}{r} S(m,r) r!.$$

To prove $|\mathcal{L}_r| \geq |\mathcal{R}_r|$, it thus suffices to prove

$$S(n,m)S(m,r) \geq \binom{n-r}{m-r} S(n,r).$$

Given a partition $\mathcal{Q}$ of $[n]$ with r parts, let $M(\mathcal{Q})$ be the r-subset of $[n]$ consisting of the largest element of each part. Choose also a set T of $m - r$ elements from $[n] - M(Q)$. The right side is the number of such pairs $(\mathcal{Q}, T)$.

Given such a pair $(\mathcal{Q}, T)$, we define a partition $\mathcal{P}$ of $[n]$ into m parts such that $\mathcal{P}$ refines $\mathcal{Q}$. Simply extract each element of T from its part in $\mathcal{Q}$ and make it a new singleton part.

Next consider all the pairs $(\mathcal{Q}, \mathcal{P})$ of partitions of $[n]$ such that $\mathcal{Q}$ has r parts, $\mathcal{P}$ has m parts, and $\mathcal{P}$ is a refinement of $\mathcal{Q}$. We can build such pairs by first choosing $\mathcal{P}$ and then grouping the parts of $\mathcal{P}$ into a partition with r parts. Hence, there are $S(n,m)S(m,r)$ such pairs. It therefore suffices to show that our map obtaining $(\mathcal{Q}, \mathcal{P})$ from $(\mathcal{Q}, T)$ is injective.

If $(\mathcal{Q}, \mathcal{P})$ arises from $(\mathcal{Q}, T)$ by this map, then $\mathcal{P}$ has at least $m - r$ singleton parts. The element x of a singleton part lies in T if and only if $x \notin M(Q)$. Thus, we can reconstruct $(\mathcal{Q}, T)$ from $(\mathcal{Q}, \mathcal{P})$, and the map is injective, as desired.

Editorial comment. This problem was also posted on MathOverflow by Filip Nikšić (mathoverflow.net/questions/268544), who noted that when the inequality is rewritten as

$$\frac{S(n,m)m!}{m^n} \geq \frac{n_{(m)}}{n^m},$$

where $n_{(m)} = n!/(n-m)!$, the left side is the probability that a uniformly chosen random function $[n] \to [m]$ is surjective, while the right side is the probability that a uniformly chosen random function $[m] \to [n]$ is injective.

Also solved by N. Grivaux (France), O. P. Lossers (Netherlands), F. Nikšić (Germany), M. Omarjee (France), E. Schmeichel, and R. Tauraso (Italy).

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by April 30, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12076. *Proposed by Tibor Beke, University of Massachusetts, Lowell, MA.* From each of the three feet of the altitudes of an arbitrary triangle, produce two points by projecting this foot onto the other two sides. Show that the six points produced in this way are concyclic.

12077. *Proposed by Max A. Alekseyev, George Washington University, Washington, DC.* Let $f(x)$ be a monic polynomial of degree n with distinct zeros $a_1, \ldots, a_n$. Prove

$$\sum_{i=1}^{n} \frac{a_i^{n-1}}{f'(a_i)} = 1.$$

12078. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $\binom{n}{m}_q$ be the *q-binomial coefficient* defined by

$$\binom{n}{m}_q = \prod_{i=0}^{m-1} \frac{1 - q^{n-i}}{1 - q^{i+1}}.$$

For a positive integer s and for $0 < q < 1$, prove

$$\sum_{n=1}^{\infty} \frac{q^{sn}}{\binom{s+n}{s+1}_q} = \frac{q^s(1 - q^{s+1})}{1 - q^s}.$$

12079. *Proposed by Moubinoul Omarjee, Lyceé Henri IV, Paris, France.* Choose x_1 in $(0, 1)$, and let $x_{n+1} = (1/n)\sum_{k=1}^{n} \ln(1 + x_k)$ for $n \geq 1$. Compute $\lim_{n\to\infty} x_n \ln n$.

12080. *Proposed by Daniel Sitaru, Drobeta Turnu Severin, Romania.* Let ABC be a scalene acute triangle with semiperimeter s. Let A_1, A_2, and A_3 be the points on BC such that AA_1 is an altitude, AA_3 is a median (i.e., A_3 is the midpoint of BC), and AA_2 is a symmedian (i.e., the ray AA_2 is the reflection of the ray AA_3 across the angle bisector at A). Define B_1, B_2, B_3 and C_1, C_2, C_3 similarly. Prove

$$\frac{A_2A_3}{A_1A_2} + \frac{B_2B_3}{B_1B_2} + \frac{C_2C_3}{C_1C_2} > \frac{4s^2}{a^2 + b^2 + c^2}.$$

doi.org/10.1080/00029890.2018.1524660

12081. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Let A and B be complex n-by-n matrices such that $AB - BA$ is invertible and such that $A^2 + B^2 = c(AB - BA)$ for some rational number c. Prove $c \in \{-1, 0, 1\}$, and show that n is a multiple of 4 when $c \neq 0$.

12082. *Proposed by Stan Wagon, Macalester College, St. Paul, MN, and Piotr Zielinski, Boston, MA.* Alice, Bob, and Charlie are prisoners in the care of a warden who lines them up in order, Charlie in front of Bob and Bob in front of Alice. The warden has k differently colored hats with $k \geq 3$ and places one hat on each prisoner's head, making the selection at random and discarding the $k - 3$ unused hats. The prisoners know what the k colors are but see only the hats of the prisoners in front of them (i.e., Alice sees two hats, Bob sees one, and Charlie sees none). The prisoners then guess the colors of their hats in turn, first Alice, then Bob, then Charlie. All prisoners hear the guesses. If the three guesses are correct, then the prisoners will all be freed.

The prisoners know the rules and can devise a strategy in advance. No communication other than the guesses is allowed once the hats are placed. What is the best possible strategy for the prisoners?

SOLUTIONS

When the Nine-Point Center Lies on the Circumcircle

11958 [2017, 179]. *Proposed by Kent Holing, Trondheim, Norway.* **(a)** Find a condition on the side lengths a, b, and c of a triangle that holds if and only if the nine-point center lies on the circumcircle.
(b) Characterize the triangles whose nine-point center lies on the circumcircle and whose incenter lies on the Euler line.

Solution by Koupa Tak Lun Koo, Beacon College, Hong Kong, China.
(a) Let O, H, and N be the circumcenter, orthocenter, and nine-point center, respectively. It is well known that N is the midpoint of OH and that $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$, where R is the circumradius. Note that N lies on the circumcircle if and only if $ON = R$, or $9R^2 - (a^2 + b^2 + c^2) = OH^2 = (2R)^2$. Substituting $R = abc/(4K)$, where K is the area of the triangle, this condition becomes

$$a^2 + b^2 + c^2 = 5R^2 = \frac{5a^2b^2c^2}{16K^2} = \frac{5a^2b^2c^2}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)},$$

which is equivalent to

$$a^6 + b^6 + c^6 + a^2b^2c^2 = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2). \qquad (*)$$

(b) It is well known that the incenter lies on the Euler line if and only if the triangle is isosceles. Letting $b = c$ in $(*)$, we have

$$a^6 + 2b^6 + a^2b^4 = (a^2 + b^2)(2b^2)(a^2 + b^2),$$

or $a^2(a^2 + b^2)(a^2 - 3b^2) = 0$. Hence the triangle satisfies the required conditions if and only if $a^2 = 3b^2 = 3c^2$ so that the angles of the triangle are $\pi/6$, $\pi/6$, and $2\pi/3$.

Editorial comment. For more on the triangles satisfying the condition in part (a), see O. Bottema (2008), *Topics in Elementary Geometry*, New York: Springer, page 76. There one finds the equivalent condition $t = -3/8$ where $t = \cos A \cos B \cos C$. Also, if the nine-point circle and the circumcircle intersect at angle θ, then $t = -\sin^2(\theta/2)$. For more on the location of the incenter relative to the Euler line, see A. P. Guinand (1984), Euler lines, tritangent centers, and their triangles, this MONTHLY, 91(5): 290–300.

Also solved by M. Bataille (France), B. S. Burdick, P. P. Dályay (Hungary), O. Geupel (Germany), S. Hitotumatu, (Japan), T. Horine, M. Goldenberg, M. Kaplan, O. Kouba (Syria), J. Minkus, C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, Z. Vőrős (Hungary), M. Vowe (Switzerland), T. Wiandt, L. Zhou, T. Zvonaru (Romania), N. Stanciu (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

A Permanent Solution

11959 [2017, 179]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that, for any n-by-n matrix A with (i, j)-entry $a_{i,j}$ and any $t_1, \dots, t_n$, the permanent of A is

$$\frac{1}{2^n}\sum\prod_{i=1}^{n}\sigma_i\left(t_i+\sum_{j=1}^{n}\sigma_j a_{i,j}\right),$$

where the outer sum is over all 2^n choices of $(\sigma_1, \dots, \sigma_n) \in \{1, -1\}^n$.

Solution by Fredrik Ekström, University of Oulu, Oulu, Finland. We use the signs in the choices for $(\sigma_1, \dots, \sigma_n)$ to cancel spurious contributions to the sum, leaving $2^n\,\mathrm{perm}(A)$. Let $[n] = \{1, \dots, n\}$.

With $\sigma_0 = 1$ and $a_{i,0} = t_i$, we have $\sigma_i(t_i + \sum_{j=1}^{n}\sigma_j a_{i,j}) = \sum_{j=0}^{n}\sigma_i\sigma_j a_{i,j}$. In expanding the product over i, we select an index $\varphi(i)$ for each i, where φ is a function from $[n]$ to $\{0, \dots, n\}$. Hence $\prod_{i=1}^{n}\sum_{j=0}^{n}\sigma_i\sigma_j a_{i,j} = \sum_{\varphi}\prod_{i=1}^{n}\sigma_i\sigma_{\varphi(i)}a_{i,\varphi(i)}$. where the sum is over all such functions φ. After interchanging the order of summation, the sum in the problem statement becomes $\sum_{\varphi}\sum_{\sigma}\prod_{i=1}^{n}\sigma_i\sigma_{\varphi(i)}a_{i,\varphi(i)}$, where σ runs through all elements of $\{1\} \times \{1, -1\}^n$.

Consider φ whose image omits some $k \in [n]$. In the sum over σ for such φ, the terms when $\sigma_k = 1$ and $\sigma_k = -1$ cancel. After grouping terms by the least such k, the sum over φ reduces to the sum over terms where $(\varphi(1), \dots, \varphi(n))$ is a permutation of $[n]$, eliminating the dependence on $t_1, \dots, t_n$. Summing over permutations π of $[n]$, we then claim

$$\sum_{\varphi}\sum_{\sigma}\prod_{i=1}^{n}\sigma_i\sigma_{\varphi(i)}a_{i,\varphi(i)} = \sum_{\pi}\sum_{\sigma}\prod_{i=1}^{n}\sigma_i\sigma_{\pi(i)}a_{i,\pi(i)} = 2^n\sum_{\pi}\prod_{i=1}^{n}a_{i,\pi(i)} = 2^n\mathrm{perm}(A).$$

For the central equality above, note that the product for a particular choice of σ accesses $2n$ factors in the image of σ, each one twice. Therefore, for each σ the coefficient on $\prod_{i=1}^{n}a_{i,\pi(i)}$ is 1, and there are 2^n choices for σ.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), T. Horine, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), S. Navasardyan, M. Omarjee (France), M. A. Prasad (India), J. C. Smith, R. Stong, R. Tauraso (Italy), and the proposer.

Hurwitz to the Rescue

11960 [2017, 179]. *Proposed by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.* Let m and n be natural numbers, and, for $i \in \{1, \dots, m\}$, let a_i be a real number with $0 \le a_i \le 1$. Define

$$f(x) = \frac{1}{x^2}\left(\sum_{i=1}^{m}(1+a_i x)^{mn} - m\prod_{i=1}^{m}(1+a_i x)^n\right).$$

Let k be a nonnegative integer, and write $f^{(k)}$ for the kth derivative of f. Show that $f^{(k)}(-1) \ge 0$.

Solution by the editors. Let S be the collection of functions f that are infinitely differentiable at -1 and such that $f^{(k)}(-1) \geq 0$ for all integers $k \geq 0$. Note that S is closed under sums, products, and nonnegative scalar multiples. For $0 \leq a_i \leq 1$, we claim that the function $g(x) = 1 + a_i x$ belongs to S. Indeed, $g(-1) = 1 - a_i \geq 0$, $g'(-1) = a_i \geq 0$, and $g^{(k)}(-1) = 0$ for $k \geq 2$.

For $m = 1$, we have $f(x) = 0$, so $f \in S$.

For $m = 2$, we have

$$\begin{aligned} f(x) &= x^{-2}\left((1+a_1x)^{2n} + (1+a_2x)^{2n} - 2(1+a_1x)^n(1+a_2x)^n\right) \\ &= (a_1-a_2)^2\left(\frac{(1+a_1x)^n - (1+a_2x)^n}{(1+a_1x)-(1+a_2x)}\right)^2 \\ &= (a_1-a_2)^2\left(\sum_{i=0}^{n-1}(1+a_1x)^i(1+a_2x)^{n-1-i}\right)^2. \end{aligned}$$

Thus, by the closure properties, $f \in S$.

Finally, let $m \geq 3$. We cite the 3-page note on the AM–GM inequality due to Hurwitz, A. (1891), Uber den Vergleich des arithmetischen und des geometrischen Mittels, *J. Reine Ange. Math.*, 108: 266–268, in which it is proved that the polynomial $X_1^m + X_2^m + \cdots + X_m^m - mX_1X_2\cdots X_m$ has a representation in the form

$$\sum_{1\leq i<j\leq m}(X_i - X_j)^2\, P_{i,j}(X_1,\ldots,X_m),$$

where $P_{i,j}$ are polynomials with all nonnegative coefficients. So

$$f(x) = \sum_{1\leq i<j\leq m}\frac{\left((1+a_ix)^n - (1+a_jx)^n\right)^2}{x^2}\, P_{i,j}\left((1+a_1x)^n,\ldots,(1+a_mx)^n\right).$$

Now $x^{-2}((1+a_ix)^n - (1+a_jx)^n)^2 \in S$ by the case $m = 2$, and

$$P_{i,j}((1+a_1x)^n,\ldots,(1+a_mx)^n) \in S$$

by the closure properties, so finally $f \in S$ again by the closure properties.

No solutions were received for $m \geq 3$.

A Radical Distribution

11962 [2017, 180]. *Proposed by Elton Hsu, Northwestern University, Evanston, IL.* Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each taking the values ± 1 with probability 1/2. Find the distribution of the random variable

$$\sqrt{\frac{1}{2} + \frac{X_1}{2}\sqrt{\frac{1}{2} + \frac{X_2}{2}\sqrt{\frac{1}{2} + \cdots}}}.$$

Solution by Li Zhou, Polk State College, Winter Haven, FL. The probability that the random variable is at most x is $(2/\pi)\arcsin x$, for $x \in [0,1]$.

Let $Y_0, Y_1, \ldots$ and $Z_0, Z_1, \ldots$ be random variables defined by

$$Y_0 = \sqrt{\frac{1}{2}},\quad Y_1 = \sqrt{\frac{1}{2} + \frac{X_1}{2}\sqrt{\frac{1}{2}}},\quad Y_2 = \sqrt{\frac{1}{2} + \frac{X_1}{2}\sqrt{\frac{1}{2} + \frac{X_2}{2}\sqrt{\frac{1}{2}}}},\quad \ldots$$

and

$$Z_0 = \sqrt{\frac{1}{2}}, \quad Z_1 = \sqrt{\frac{1}{2} + \frac{X_2}{2}\sqrt{\frac{1}{2}}}, \quad Z_2 = \sqrt{\frac{1}{2} + \frac{X_2}{2}\sqrt{\frac{1}{2} + \frac{X_3}{2}\sqrt{\frac{1}{2}}}}, \quad \ldots.$$

For $n \geq 0$, the variable Z_n is independent of X_1 and has the same distribution as Y_n. Furthermore, note that $Y_{n+1} = \sqrt{(1 + X_1 Z_n)/2}$.

We first prove inductively that Y_n and Z_n take the value $\cos(k\pi/2^{n+2})$ with probability $1/2^n$, for $k \in \{1, 3, 5, \ldots, 2^{n+1} - 1\}$. The claim is immediate for $n = 0$. If it holds for some value n, then for $k \in \{1, 3, 5, \ldots, 2^{n+1} - 1\}$ the random variable Y_{n+1} takes the value $\sqrt{\big(1 + \cos(k\pi/2^{n+2})\big)/2}$ or $\sqrt{\big(1 - \cos(k\pi/2^{n+2})\big)/2}$ with probability $1/2^{n+1}$ each. Since

$$\sqrt{\frac{1}{2} + \frac{1}{2}\cos\frac{k\pi}{2^{n+2}}} = \cos\frac{k\pi}{2^{n+3}} \quad \text{and} \quad \sqrt{\frac{1}{2} - \frac{1}{2}\cos\frac{k\pi}{2^{n+2}}} = \sin\frac{k\pi}{2^{n+3}} = \cos\frac{(2^{n+2} - k)\pi}{2^{n+3}},$$

this completes the induction.

Therefore, the probability that Y_n is at most x is $|A_n(x)|/2^n$, where

$$A_n(x) = \left\{k \in \{1, 3, 5, \ldots, 2^{n+1} - 1\} : \ 0 \leq \cos\frac{k\pi}{2^{n+2}} \leq x\right\}.$$

As $n \to \infty$,

$$|A_n(x)| \sim \frac{2^{n+1}}{\pi}(\arccos 0 - \arccos x) = \frac{2^{n+1}}{\pi}\arcsin x,$$

which yields the claimed limiting distribution.

Also solved by R. Agnew, E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Chapman (U. K.), P. J. Fleischman, N. Grivaux (France), J. A. Grzesik, J. C. Keiffer, O. Kouba (Syria), J. H. Lindsey II, M. A. Prasad (India), K. Schilling, N. C. Singer, J. C. Smith, R. Stong, D. B. Tyler, E. I. Verriest, T. Wiandt, and the proposer.

A Summation Inequality

11963 [2017, 180]. *Proposed by Gheorghe Alexe and George-Florin Serban, Braila, Romania.* Let $a_1, \ldots, a_n$ be positive real numbers with $\prod_{k=1}^n a_k = 1$. Show that

$$\sum_{i=1}^{n} \frac{(a_i + a_{i+1})^4}{a_i^2 - a_i a_{i+1} + a_{i+1}^2} \geq 12n,$$

where $a_{n+1} = a_1$.

Solution by Leonard Giugiuc, Drobeta Turnu Severin, Romania. We first observe that if a and b are positive real numbers, then

$$\frac{(a+b)^4}{a^2 - ab + b^2} \geq 12ab.$$

This follows from

$$\frac{(a+b)^4}{a^2 - ab + b^2} - 12ab = \frac{(a^2 - 4ab + b^2)^2}{a^2 - ab + b^2} \geq 0.$$

Applying this fact to each term of the summation and then using the AM–GM inequality, we obtain

$$\sum_{i=1}^{n} \frac{(a_i + a_{i+1})^4}{a_i^2 - a_i a_{i+1} + a_{i+1}^2} \geq 12\sum_{i=1}^{n} a_i a_{i+1} \geq 12n\sqrt[n]{\left(\prod_{i=1}^{n} a_i\right)^2} = 12n.$$

Editorial comment. Several solvers noted that the inequality in the problem is an equality when n is even and the terms of the sequence $a_1, \ldots, a_n$ alternate between $(\sqrt{6}+\sqrt{2})/2$ and $(\sqrt{6}-\sqrt{2})/2$. When n is odd, the inequality is always strict.

Also solved by A. Ali (India), A. Alt, R. Boukharfane (France), P. Bracken, E. Braune (Austria), R. Chapman (U. K.) P. P. Dályay (Hungary), D. Fleischman, N. Ghosh, T. Horine, K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, P. Perfetti (Italy), M. Reid, E. Schmeichel, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Condition for a Certain Point on a Triangle

11965 [2017, 274]. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with circumradius R. Prove that there exists a point M on side BC such that $MA \cdot MB \cdot MC = 32R^3/27$ if and only if $2\cot B \cot C = 1$.

Solution by Dain Kim, Seoul Science High School, Seoul, South Korea. We begin with the following result.

Lemma. *Let ABC be inscribed in the circle Ω. If AD is the diameter of Ω through A, and M is the point where BC and AD meet, then*

$$\cot B \cdot \cot C = \frac{MB \cdot MC}{MA^2}.$$

Proof. Because AD is a diameter, $\angle ABD = \angle ACD = \pi/2$, and by the inscribed angle theorem it follows that $\angle BAD = \angle BCD = \pi/2 - C$ and $\angle CAD = \angle CBD = \pi/2 - B$. The law of sines gives

$$\frac{\cos C}{\sin B} = \frac{\sin(\frac{\pi}{2} - C)}{\sin B} = \frac{MB}{MA} \quad \text{and} \quad \frac{\cos B}{\sin C} = \frac{\sin(\frac{\pi}{2} - B)}{\sin C} = \frac{MC}{MA}.$$

Multiplying these two equations together yields the desired result. ■

We now address separately the necessity and sufficiency of $2\cot B \cot C = 1$.

(Necessity) Assume there exists a point M on side BC such that $MA \cdot MB \cdot MC = 32R^3/27$. Let line MA intersect the circumcircle of triangle ABC again at D. Let $MA = x$ and $MD = y$. Since $2R$ is the diameter of the circumcircle, $x + y = AD \le 2R$. By the power-of-the-point theorem, $MB \cdot MC = MA \cdot MD$. Multiplying this on both sides by MA yields $MA \cdot MB \cdot MC = MA^2 \cdot MD = x^2 y$. The AM–GM inequality then implies

$$MA \cdot MB \cdot MC = x^2 y \le \frac{1}{2}\left(\frac{x + x + 2y}{3}\right)^3 \le \frac{(2(2R))^3}{54} = \frac{32R^3}{27}.$$

Equality holds only when $x = 2y$ and $x + y = 2R$, in which case $x = (4/3)R$ and $y = (2/3)R$. Thus AD is a diameter.

Applying the lemma,

$$\cot B \cdot \cot C = \frac{MB \cdot MC}{MA^2} = \frac{MA \cdot (MD)^2}{MA \cdot MB \cdot MC} = \frac{xy^2}{x^2 y} = \frac{y}{x} = \frac{1}{2}.$$

This is the required condition $2\cot B \cdot \cot C = 1$.

(Sufficiency) Assume $2\cot B \cot C = 1$. Let AD be the diameter of the circumcircle through A, and let M be the point where BC and AD meet. We must show that $MA \cdot MB \cdot MC = 32R^3/27$. As before, $MA \cdot MB \cdot MC = x^2 y$, with $x = MA$ and $y = MD$. From the lemma,

$$x^2 y = MA \cdot MB \cdot MC = \frac{MB \cdot MC}{MA^2} \cdot MA^3 = \cot B \cot C \cdot MA^3 = \frac{x^3}{2}.$$

Therefore $y = x/2$, so $x + x/2 = 2R$. It follows that $x = 4R/3$ and $x^3/2 = 32R^3/27$.

Also solved by M. Bataille (France) D. Fleischman, T. Horine, O. Kouba (Syria), O. P. Lossers (Netherlands), R. Nandan, V. Schindler (Germany), J. C. Smith, R. Stong, E. I. Verriest, L. Zhou, and the proposer.

Evaluate an Integral

11966 [2017, 274]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania.* Prove

$$\int_0^1 \frac{x\ln(1+x)}{1+x^2}\,dx = \frac{\pi^2}{96} + \frac{(\ln 2)^2}{8}.$$

Solution by Juan Manuel Sánchez (student), Universidad de Antioquia, Medellin, Colombia. Since $\int_0^1 \frac{x\,dt}{1+xt} = \ln(1+x)$,

$$\begin{aligned}
\int_0^1 \frac{x\ln(1+x)}{1+x^2}dx &= \int_0^1\int_0^1 \frac{x^2}{(1+x^2)(1+xt)}dt\,dx = \int_0^1\int_0^1 \frac{x^2}{(1+x^2)(1+xt)}dx\,dt \\
&= \int_0^1 \frac{1}{1+t^2}\int_0^1 \left(\frac{(tx-1)}{1+x^2} + \frac{1}{1+tx}\right)dx\,dt \\
&= \int_0^1 \frac{1}{1+t^2}\left(\frac{t\ln 2}{2} - \frac{\pi}{4} + \frac{\ln(1+t)}{t}\right)dt \\
&= \frac{\ln 2}{2}\int_0^1 \frac{t}{1+t^2}dt - \frac{\pi}{4}\int_0^1 \frac{1}{1+t^2}dt + \int_0^1 \frac{\ln(1+t)}{t(1+t^2)}dt \\
&= \frac{(\ln 2)^2}{4} - \frac{\pi^2}{16} + \int_0^1 \ln(1+t)\left(\frac{1}{t} - \frac{t}{1+t^2}\right)dt \\
&= \frac{(\ln 2)^2}{4} - \frac{\pi^2}{16} + \int_0^1 \frac{\ln(1+t)}{t}dt - \int_0^1 \frac{t\ln(1+t)}{1+t^2}dt.
\end{aligned}$$

Solving this for $\int_0^1 \frac{x\ln(1+x)\,dx}{1+x^2}$ yields

$$\int_0^1 \frac{x\ln(1+x)}{1+x^2}dx = \frac{(\ln 2)^2}{8} - \frac{\pi^2}{32} + \frac{1}{2}\int_0^1 \frac{\ln(1+t)}{t}dt.$$

Since

$$\begin{aligned}
\int_0^1 \frac{\ln(1+t)}{t}dt &= \int_0^1 \sum_{i=0}^{\infty} \frac{(-1)^i t^i}{i+1}dt = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^2} \\
&= \sum_{i=1}^{\infty} \frac{1}{i^2} - 2\sum_{i=1}^{\infty} \frac{1}{(2i)^2} = \frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{12},
\end{aligned}$$

we obtain

$$\int_0^1 \frac{x\ln(1+x)}{1+x^2}dx = \frac{(\ln 2)^2}{8} - \frac{\pi^2}{32} + \frac{1}{2}\cdot\frac{\pi^2}{12} = \frac{\pi^2}{96} + \frac{(\ln 2)^2}{8}.$$

Also solved by T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), A. Arenas & M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & E. Labarga & L. Roncal (Spain), M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (France), K. N. Boyadzhiev, P. Bracken, B. Bradie, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), B. E. Davis, D. Fritze (Germany), S. Gao, C. Georghiou (Greece), N. Ghosh, M. L. Glasser (Spain),

H. Grandmontagne (France), J. M. Groah, J. A. Grzesik, T. Hakobyan & S. Navasardyan (Armenia), A. Hannan (India), E. A. Herman, T. Horine, M. Ivan (Romania), S. Kaczkowski, K. T. L. Koo (China), O. Kouba (Syria), A. Kourdouklas (Greece), K.-W. Lau (China), H. Lee (South Korea), O. P. Lossers (Netherlands), V. Lucic (U. K.), L. Kempeneers & J. Van Casteren (Belgium), P. Magli (Italy), V. Mikayelyan (Armenia), R. Nandan, G. Negri (Italy), M. Omarjee (France), P. Perfetti (Italy), R. Poodiack, F. A. Rakhimjanovich (Uzbekistan), H. Ricardo, S. Seales, A. N. Sharma (India), S. Sharma (India), S. Silwal & N. Taylor, J. Singh (India), J. C. Smith, A. Stadler (Switzerland), J. Steier, R. Stong, R. Tauraso (Italy), E. I. Verriest, H. Wang & J. Wojdylo, T. Wiandt, M. Wildon (U. K.), M. R. Yegan (Iran), Y. Zhao, L. Zhou, GCHQ Problem Solving Group (U. K.), Get Stoked Student Problem Solving Group, and the proposer.

Zeros of a Truncated Riemann Zeta Function

11970 [2017, 275]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let

$$\zeta_5(z) = 1 + 2^{-z} + 3^{-z} + 4^{-z} + 5^{-z},$$

where z is a complex number. Prove that $\zeta_5(z) \neq 0$ when the real part of z is greater than or equal to 0.9.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $z = x + iy$ with $x \geq 0.9$. Since $|3^{-z} + 5^{-z}| \leq |3^{-z}| + |5^{-z}| = 3^{-x} + 5^{-x} \leq 3^{-0.9} + 5^{-0.9} < 0.61$, it suffices to show $\left|1 + 2^{-z} + 4^{-z}\right| > 0.61$. To this end, we compute

$$\begin{aligned}
\left|1 + 2^{-z} + 4^{-z}\right|^2 &= \left(1 + \frac{\cos(y \ln 2)}{2^x} + \frac{\cos(2y \ln 2)}{4^x}\right)^2 + \left(\frac{\sin(y \ln 2)}{2^x} + \frac{\sin(2y \ln 2)}{4^x}\right)^2 \\
&= 1 + \frac{1}{4^x} + \frac{1}{16^x} + 2\left(\frac{\cos(y \ln 2)}{2^x} + \frac{\cos(2y \ln 2)}{4^x} + \frac{\cos(y \ln 2)}{8^x}\right) \\
&= 4\left(\frac{\cos(y \ln 2)}{2^x} + \frac{1}{4}\left(1 + \frac{1}{4^x}\right)\right)^2 + \frac{3}{4}\left(1 - \frac{1}{4^x}\right)^2 \\
&\geq \frac{3}{4}\left(1 - \frac{1}{4^{0.9}}\right)^2 > 0.61^2,
\end{aligned}$$

completing the proof.

Editorial comment. A discussion, with plots, of the zeros of truncated zeta functions $\zeta_k(z)$ can be found in Borwein, P., Fee, G., Ferguson, R., van der Waall, A. (2007), Zeros of partial sums of the Riemann zeta function, *Experiment. Math.* 16(1): 21–39.

Also solved by P. Bracken, T. Horine, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Stong, E. I. Verriest, and the proposer.

A Mean Inequality

11971 [2017, 369]. *Proposed by Spiros P. Andriopoulos, Third High School of Amaliada, Eleia, Greece.* For $n \geq 2$, let $a_1, \ldots, a_n$ be positive real numbers. Prove

$$\left(\prod_{i=1}^{n}(1 + a_i)\right)^{n-1} \geq \left(\prod_{i<j}\left(1 + \frac{2a_i a_j}{a_i + a_j}\right)\right)^2.$$

Solution by Koopa Tak Lun Koo, Beacon College, Hong Kong, China. In the product $\prod_{i<j}(1 + a_i)(1 + a_j)$, there are precisely $n - 1$ occurrences of $(1 + a_k)$ for each k in

$\{1, \ldots, n\}$. Therefore, using both the AM–GM and GM–HM inequalities, we can write

$$\begin{aligned}\left(\prod_{i=1}^{n}(1+a_i)\right)^{n-1} &= \prod_{1\le i<j\le n}(1+a_i)(1+a_j) = \prod_{1\le i<j\le n}(1+a_i+a_j+a_ia_j)\\ &\ge \prod_{1\le i<j\le n}(1+2\sqrt{a_ia_j}+a_ia_j) = \prod_{1\le i<j\le n}(1+\sqrt{a_ia_j})^2\\ &\ge \prod_{1\le i<j\le n}\left(1+\frac{2}{\frac{1}{a_i}+\frac{1}{a_j}}\right)^2 = \left(\prod_{1\le i<j\le n}\left(1+\frac{2a_ia_j}{a_i+a_j}\right)\right)^2.\end{aligned}$$

Equality occurs if and only if all the real numbers a_i are equal.

Also solved by M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (France), R. Chapman (U. K.), P. P. Dályay (Hungary), S. Dubey, D. Fleischman, O. Geupel (Germany), N. Ghosh, L. Giugiuc (Romania), M. Goldenberg & M. Kaplan, T. Hakobyan & S. Navasardyan (Armenia), A. Hannan (India), E. A. Herman, T. Horine, S. Hwang (South Korea), J. Kim (South Korea), O. Kouba (Syria), H. Kwong, W. Lai & J. Risher, J. H. Lindsey ll, O. P. Lossers (Netherlands), D. Marinescu (Romania), R. Martin (Germany), V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), P. Perfetti (Italy), F. A. Rakhimjanovich (Uzbekistan), E. Schmeichel, J. C. Smith, A. Stadler (Switzerland), N. Stanciu & T. Zvonaru (Romania), R. Stong, R. Tauraso (Italy), D. B. Tyler, E. I. Verriest, H. Widmer (Switzerland), L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problem Solving Group, and the proposer.

Inradius and Exradii of a Tetrahedron

11972 [2017, 369]. *Proposed by Yun Zhang, Xi'an Senior High School, Xi'an China.* Let r be the radius of the sphere inscribed in a tetrahedron whose exscribed spheres have radii r_1, r_2, r_3, and r_4. Prove

$$r\left(\sqrt[3]{r_1}+\sqrt[3]{r_2}+\sqrt[3]{r_3}+\sqrt[3]{r_4}\right) \le 2\sqrt[3]{r_1r_2r_3r_4}.$$

Solution by Hansruedi Widmer, Baden, Switzerland. By the AM–GM inequality,

$$\frac{1}{3}\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}\right) \ge \sqrt[3]{\frac{1}{r_ir_jr_k}}$$

for $1 \le i, j, k \le 4$. Summing over the four triplets $\{i, j, k\}$ yields

$$\sum_{k=1}^{4}\frac{1}{r_k} \ge \sum_{k=1}^{4}\frac{\sqrt[3]{r_k}}{\sqrt[3]{r_1r_2r_3r_4}}.$$

The result follows from the identity $\sum_{k=1}^{4} 1/r_k = 2/r$. (See Toda, A. A. (2014). Radii of the inscribed and escribed spheres of a simplex. *Inter. J. Geom.* 3(2): 5–13.)

Editorial comment. Equality occurs if and only if $r_1 = \cdots = r_4$, e.g., for isosceles tetrahedra. The n-dimensional analogues are $\sum_{k=1}^{n+1} 1/r_k = (n-1)/r$ (Toda, 2014) and

$$r(\sqrt[n]{r_1}+\sqrt[n]{r_2}+\cdots+\sqrt[n]{r_{n+1}}) \le (n-1)\sqrt[n]{r_1r_2\cdots r_{n+1}}.$$

Also solved by M. Bataille (France), R. Boukharfane (France), R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), L. Giugiuc (Romania), T. Horine, K. T. L. Koo (China), O. Kouba (Syria), O. P. Lossers (Netherlands), D. Marinescu (Romania), R. Nandan, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, L. Zhou, and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by May 31, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12083. *Proposed by Alijadallah Belabess, Khemisset, Morocco.* Let x, y, and z be positive real numbers. Prove

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{3\sqrt{3}}{2\sqrt{x^2+y^2+z^2}}.$$

12084. *Proposed by George Stoica, Saint John, NB, Canada.* Let $a_1, a_2, \ldots$ be a sequence of nonnegative numbers. Prove that $(1/n)\sum_{k=1}^{n} a_k$ is unbounded if and only if there exists a decreasing sequence $b_1, b_2, \ldots$ such that $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n$ is finite, and $\sum_{n=1}^{\infty} a_n b_n$ is infinite. Is the word “decreasing” essential?

12085. *Proposed by Joseph DeVincentis, Salem, MA, Stan Wagon, Macalester College, St. Paul, MN, and Michael Elgersma, Plymouth, MN.* For which positive integers n can $\{1, \ldots, n\}$ be partitioned into two sets A and B of the same size so that

$$\sum_{k\in A} k=\sum_{k\in B} k,\quad \sum_{k\in A} k^2=\sum_{k\in B} k^2,\quad\text{and}\quad \sum_{k\in A} k^3=\sum_{k\in B} k^3?$$

12086. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with right angle at A, and let H be the foot of the altitude from A. Let M, N, and P be the incenters of triangles ABH, ABC, and ACH, respectively. Prove that the ratio of the area of triangle MNP to the area of triangle ABC is at most $(\sqrt{2}-1)^3/2$, and determine when equality holds.

12087. *Proposed by M. L. J. Hautus, Heeze, Netherlands.* Let K be a field, and let A be a linear map from K^n into itself. The equation $X^2 = AX$ has the trivial solutions $X = 0$ and $X = A$. Show that it has a nontrivial solution if and only if the characteristic polynomial $\det(\lambda I - A)$ is reducible, with the following sole exception: If K has two elements, $n = 2$, and A is nilpotent and nonzero, then the characteristic polynomial is reducible, yet $X^2 = AX$ has no nontrivial solutions.

doi.org/10.1080/00029890.2018.1537413

12088. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Let k be a positive integer with $k \geq 2$, and let $f : [0, 1] \to \mathbb{R}$ be a function with continuous kth derivative. Suppose $f^{(k)}(x) \geq 0$ for all $x \in [0, 1]$, and suppose $f^{(i)}(0) = 0$ for all $i \in \{0, 1, \dots, k-2\}$. Prove

$$\int_0^1 x^{k-1} f(1-x)\, dx \leq \frac{(k-1)!k!}{(2k-1)!} \int_0^1 f(x)\, dx.$$

12089. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO, and Adam Salminen, University of Evansville, Evansville, IN.* All rings in this problem are assumed to be commutative with a nonzero multiplicative identity. A *homomorphism* from a ring R to a ring S is an identity-preserving map $\phi : R \to S$ such that $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in R$. Consider the following two properties of a ring R:

(1) For every proper ideal I of R, there is an injective homomorphism $\phi : R/I \to R$.
(2) For every proper ideal I of R, there is an injective homomorphism $\phi : R \to R/I$.

(a) Must a ring that enjoys property (1) be a field?
(b) Must a ring that enjoys property (2) be a field?
(c) Must a ring that enjoys properties (1) and (2) be a field?

SOLUTIONS

A Trigonometric Integral

11961 [2017, 180]. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Evaluate

$$\int_0^{\pi/2} \frac{\sin x}{1 + \sqrt{\sin(2x)}}\, dx.$$

Solution by Koopa Tak Lun Koo, Beacon College, Hong Kong, China. The integral equals $(\pi/2) - 1$. To see this, denote the integral by I. The substitution $x \mapsto (\pi/2) - x$ yields

$$I = \int_0^{\pi/2} \frac{\sin x}{1 + \sqrt{\sin 2x}}\, dx = \int_0^{\pi/2} \frac{\cos x}{1 + \sqrt{\sin 2x}}\, dx.$$

Adding these two integrals, substituting $u = \cos x - \sin x$, and noting that $u^2 = \cos^2 x - 2\sin x \cos x + \sin^2 x = 1 - \sin 2x$ gives

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{1 + \sqrt{\sin 2x}}\, dx = \int_{-1}^1 \frac{1}{1 + \sqrt{1-u^2}}\, du = 2\int_0^1 \frac{1}{1 + \sqrt{1-u^2}}\, du.$$

To compute this integral, substitute $u = \sin\theta$ to obtain

$$\int_0^1 \frac{1}{1 + \sqrt{1-u^2}}\, du = \int_0^{\pi/2} \frac{\cos\theta}{1 + \cos\theta}\, d\theta = \left[\theta - \tan\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2} - 1.$$

Editorial comment. Several solvers noted a more general result, with essentially the same proof: If f is continuous on $[0, \pi/2]$, then

$$\int_0^{\pi/2} f(\sin 2x) \sin x\, dx = \int_0^{\pi/2} f(\cos^2\theta) \cos\theta\, d\theta.$$

This has appeared in Besge, M. (1853), Addition à la note sur une transformation d'intégrales défines, insérée dans le cahier de mars, *J. de Math. Pures et Appl.* 18: 168.

Also solved by P. Acosta, A. Ali (India), A. Alt, K. F. Andersen (Canada), G. Apostolopoulos (Greece), M. Bataille (France), A. Berkane (Algeria), R. Boukharfance (France), B. Bowers & T. Fauss & R. Melton, P. Bracken, D. Bronicki, K. Bryant & A. Cathers & B. Zaretzky, B. Burdick, M. V. Channakeshava (India), R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), B. E. Davis, P. De (India), H. Y. Far, N. Ghosh, L. Giugiuc (Romania), M. L. Glasser, M. Goldenberg & M. Kaplan, J. M. González (Chile), H. Grandmontagne (France), J. M. Groah, L. Han, P. Hauber (Germany), E. A. Herman, M. Hoffman, T. Horine, E. J. Ionaşcu, Y. J. Ionin, W. P. Johnson, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), B. Prescott, H. Ricardo, V. Schindler (Germany), S. Sharma (India), J. Showmaker, N. C. Singer, J. C. Smith, A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Australia), R. Stong, J. Swenson, R. Tauraso (Italy), D. B. Tyler, E. I. Verriest, L. Walker, H. Wang & J. Wojdylo, T. Wiandt, H. Widmer (Switzerland), M. R. Yegan (Iran), L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Almost Perfect Squares

11964 [2017, 274]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Find all triples of integers (a, b, c) with $a \neq 0$ such that the function f defined by $f(x) = ax^2 + bx + c$ has the property that, for every positive integer n, there exists an integer m with $f(n)f(n+1) = f(m)$.

Solution by Allen Stenger, Boulder, CO. Let $(*)$ denote $f(n)f(n+1) = f(m)$. The triples (a, b, c) such that $(*)$ is solvable for all n are of two types: (i) $a = 1$ with b and c arbitrary, and (ii) $a \neq 1$ with $(a, b, c) = (r^2, 2rs, s^2)$, where r and s are integers such that r divides $s(s-1)$ or $s(s+1)$.

Observe first that $4f(n)f(n+1)$ is "near" a perfect square: Expanding both sides and canceling like terms yields

$$4f(n)f(n+1) = (2an^2 + (2a+2b)n + (b+2c))^2 - (b^2 - 4ac). \tag{1}$$

Thus $(*)$ requires $4f(m) = y^2 - (b^2 - 4ac)$, where $y = 2an^2 + (2a+2b)n + (b+2c)$. Multiplying by a leads to

$$ay^2 - a(b^2 - 4ac) = 4af(m) = (2am + b)^2 - (b^2 - 4ac).$$

Writing $x = 2am + b$ and rearranging yields

$$x^2 - ay^2 = (1-a)(b^2 - 4ac), \tag{2}$$

which has the form of Pell's equation. Note that $a > 0$, because if $a < 0$, then $f(m)$ is bounded above while $f(n)f(n+1) \to +\infty$ as $n \to \infty$.

There are three cases: $(1-a)(b^2 - 4ac) \neq 0$, or $a = 1$, or $b^2 - 4ac = 0$. The first case is the usual Pell's equation. Solutions to Pell's equation are sparse, with successive values growing exponentially, so the number of solutions with $0 \leq x \leq X$ is $O(\ln X)$ — see pp. 205–206 in LeVeque, W. J. (1996), *Fundamentals of Number Theory*, New York: Dover. On the other hand, $(*)$ implies $m \sim \sqrt{a}n^2$, where $p \sim q$ means that p is asymptotic to q, so $x \sim 2a^{3/2}n^2$. If $(*)$ has an integral solution m for every positive integer n, then the number of x with $0 \leq x \leq X$ is asymptotic to $\sqrt{X/(2a^{3/2})}$. Since this expression grows much faster than the upper bound $O(\ln X)$, in this case $(*)$ cannot have an integer solution m for every n.

When $a = 1$, setting $m = n^2 + (b+1)n + c$ yields a solution to $(*)$, verified by multiplying out both sides of $(*)$.

When $b^2 - 4ac = 0$, we have $x^2 = ay^2$. Therefore a is a perfect square (set $a = r^2$), and $4ac = b^2$ implies that c is also a perfect square. Set $c = s^2$. Replacing $-r$ with r if necessary, this gives $b = 2rs$. With these substitutions, $f(x)$ equals $(rx + s)^2$ and is also a perfect square. Substituting these into (1), we see that satisfying $(*)$ requires

$$(rm + s)^2 = (r^2n^2 + (r^2 + 2rs)n + (rs + s^2))^2.$$

Taking square roots yields

$$|rm + s| = |r^2n^2 + (r^2 + 2rs)n + (rs + s)^2|. \tag{3}$$

All the summands are divisible by r except the s on the left and the s^2 on the right. Hence, depending on the choice of sign, we have $\pm s \equiv s^2 \bmod r$, so the existence of an integer solution m to $(*)$ requires r divides $s(s \mp 1)$.

Conversely, choose

$$m = rn^2 + (r + 2s)n + (s + s(s-1)/r) \quad \text{if } r \text{ divides } s(s-1),$$

and

$$m = -\big(rn^2 + (r + 2s)n + (s + s(s+1)/r)\big) \quad \text{if } r \text{ divides } s(s+1).$$

In either case, the divisibility assumptions make m an integer. In the first case,

$$rm + s = r(rn^2 + (r + 2s)n + (s + s(s-1)/r)) + s,$$

so

$$rm + s = r^2n^2 + (r^2 + 2rs)n + (rs + s(s-1)) + s = r^2n^2 + (r^2 + 2rs)n + (rs + s^2),$$

which satisfies (3), as required. Similarly, in the second case

$$rm + s = -\big(r^2n^2 + (r^2 + 2rs)n + (rs + s^2)\big),$$

which again satisfies (3).

Therefore, if r and s are chosen such that r divides $s(s-1)$ or $s(s+1)$, then $(a, b, c) = (r^2, 2rs, s^2)$ is a triple such that $(*)$ has an integer solution m for all n.

Editorial comment. Several solvers used the theorem that if some polynomial $P(x)$ evaluates to a square integer whenever x is a positive integer, then $P(x)$ is the square of a polynomial. A good treatment of this result appears in Murty, M. R. (2002), Prime numbers and irreducible polynomials, this MONTHLY, 109(5): 452–458.

Also solved by T. Hakobyan (Armenia), T. Horine, E. J. Ionaşcu, Y. J. Ionin, S. Jung (South Korea), O. Kouba (Syria), O. P. Lossers (Netherlands), C. R. Pranesachar (India), M. Reid, J. Robertson, C. Schacht, O. Senobi (Morroco), N. C. Singer, J. C. Smith, R. Stong, E. I. Verriest, L. Zhou, and the proposer.

A Divisibility Property of Fibonacci Numbers

11968 [2017, 274]. *Proposed by Christopher J. Hillar, Redwood Center for Theoretical Neuroscience, Berkeley, CA, Robert Krone, Queens University, Kingston, Ontario, Canada, and Anton Leykin, Georgia Tech University, Atlanta, GA.* Let F_n be the nth Fibonacci number, with $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. For $n \geq 1$, prove that $F_{5n}/(5F_n)$ is an integer congruent to 1 modulo 10.

Solution by Michael Tang (student), Massachusetts Institute of Technology, Cambridge, MA. Let r and s be the roots of $x^2 - x - 1$ with $r = (1 + \sqrt{5})/2$. Note that $rs = -1$. Binet's formula gives $\sqrt{5}F_k = r^k - s^k$. Hence

$$r^{2n} + s^{2n} = \big(r^n - s^n\big)^2 + 2r^ns^n = 5F_n^2 + 2(-1)^n$$

and

$$
\begin{aligned}
r^{4n}+s^{4n} &= \left(r^{2n}+s^{2n}\right)^2-2r^{2n}s^{2n} \\
&= \left(5F_n^2+2(-1)^n\right)^2-2=25F_n^4+20(-1)^nF_n^2+2.
\end{aligned}
$$

Using the preceding identities,

$$
\begin{aligned}
F_{5n} = F_n\frac{r^{5n}-s^{5n}}{r^n-s^n} &= F_n\left(r^{4n}+r^{3n}s^n+r^{2n}s^{2n}+r^ns^{3n}+s^{4n}\right) \\
&= F_n\left(r^{4n}+(-1)^nr^{2n}+1+(-1)^ns^{2n}+s^{4n}\right) \\
&= F_n\left(25F_n^4+(-1)^n25F_n^2+5\right)=5F_n\left(5F_n^2\left(F_n^2\pm 1\right)+1\right).
\end{aligned}
$$

Because $F_n^2\left(F_n^2\pm 1\right)$ is even, the preceding identity shows that $F_{5n}/(5F_n)$ is an integer congruent to 1 modulo 10.

Editorial comment. Radouan Boukharfane showed that if $a_n = F_{5n}/(5F_n)$, then

$$
\sum_{n=0}^{\infty} a_nx^n = \frac{1-4x-9x^2+6x^3+x^4}{1-5x-15x^2+15x^3+5x^4-x^5}.
$$

This implies

$$
\sum_{n=0}^{\infty}(a_n-1)\,x^n = \frac{10x^2(x+1)}{1-5x-15x^2+15x^3+5x^4-x^5}.
$$

It follows that a_n-1 is an integer divisible by 10.

Eugen Ionaşcu observed that when p is prime and $1\le n\le p-1$, the ratio $F_{pn}/(F_pF_n)$ seems to be an integer congruent to 0 or 1 (mod p). The cases where only the residue 1 occurs (with $n\le p-1$) appear to be the set of primes in oeis.org/A000057, plus $p=5$. The case $p=5$ is exceptional in that $F_5=5$ and the desired property holds for all n.

More information on $F_{5n}/(5F_n)$ can be found at oeis.org/A088545.

Also solved by D. Bailey & E. Campbell & C. Diminnie, M. Bataille (France), R. Boukharfane (France), B. Bradie, R. Chapman (U. K.), J. Christopher, P. P. Dályay (Hungary), D. Fleischman, C. Georghiou (Greece), O. Geupel (Germany), M. Goldenberg & M. Kaplan, T. Hakobyan (Armenia), A. Hannan (India), T. Horine, E. J. Ionascu, Y. J. Ionin, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), D. E. Knuth, K. T. L. Koo (China), O. Kouba (Syria), H. Kwong, W.-K. Lai, P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), R. Nandan, M. Omarjee (France), Á. Plaza (Spain), M. Reid, J. C. Smith, A. Stadler (Switzerland), R. Stong, J. Swenson, R. Tauraso (Italy), D. Terr, E. I. Verriest, T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Trigonometric Determinant

11969 [2017, 274]. *Proposed by Askar Dzhumadil'daev, Kazakh-British Technical University, Almaty, Kazakhstan.* Let $x_1,\dots,x_n$ be indeterminates, and let A be the n-by-n matrix with i,j-entry $\sec(x_i-x_j)$. Prove

$$
\det A = (-1)^{\binom{n}{2}}\prod_{1\le i<j\le n}\tan^2(x_i-x_j).
$$

Solution by Oliver Geupel, Brühl, Germany. We use induction on n for $n\ge 2$. Let A_n denote the n-by-n matrix A. The case $n=2$ follows from $\tan^2\theta = 1+\sec^2\theta$.

For $n > 2$, without changing the value of $\det A_n$, we subtract $\sec(x_i - x_n)$ times the last row from the ith row, for $1 \le i < n$. The last column now is all 0 except for 1 in the last row. For $i, j \in \{1, \dots, n-1\}$, the resulting i, j-entry is

$$\sec(x_i - x_j) - \sec(x_i - x_n)\sec(x_n - x_j).$$

We compute

$$\begin{aligned}
\sec(x_i - x_j) &- \sec(x_i - x_n)\sec(x_n - x_j)\\
&= \frac{\cos(x_i - x_n)\cos(x_j - x_n) - \cos(x_i - x_j)}{\cos(x_i - x_j)\cos(x_i - x_n)\cos(x_j - x_n)}\\
&= \frac{-\sin(x_i - x_n)\sin(x_j - x_n)}{\cos(x_i - x_j)\cos(x_i - x_n)\cos(x_j - x_n)}\\
&= -\tan(x_i - x_n)\tan(x_j - x_n)\sec(x_i - x_j).
\end{aligned}$$

All entries of row i have the factor $\tan(x_i - x_n)$, while all entries of column j have the factor $\tan(x_j - x_n)$. We extract these factors and expand the determinant along the last column to obtain, using the induction hypothesis,

$$\begin{aligned}
\det A_n &= (-1)^{n-1} \prod_{1 \le i < n} \tan^2(x_i - x_n) \cdot \det A_{n-1}\\
&= (-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} \tan^2(x_i - x_j).
\end{aligned}$$

Editorial comment. The problem can also solved using the Cauchy matrix. Robin Chapman proved a generalization where the matrix has i, j-entry $\sec(x_i + y_j)$.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, E. A. Herman, T. Horine, W. P. Johnson, K. T. L. Koo (China), O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), S. Navasardyan (Armenia), M. Omarjee (France), O. Sonebi (Morocco), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Walsh, L. Zhou, and the proposer.

An Application of Partial Fractions

11973 [2017, 369]. *Proposed by Derek Orr, University of Pittsburgh, Pittsburgh, PA.* Catalan's constant G is defined to be $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. Prove

$$G = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n}\left(1 - \frac{2}{4^n}\right),$$

where ζ is the Riemann zeta function, defined by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ for $s > 1$ and $\zeta(0) = -1/2$ by analytic continuation.

Composite solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria, and Mark Wildon, Royal Holloway, Egham, U. K. Expanding $\arctan x$ in a Maclaurin series yields

$$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \int_0^1 \frac{\arctan x}{x}\,dx.$$

After setting $x = \tan\theta$ and using $\sin(2\theta) = 2\sin\theta\cos\theta$, we obtain

$$G = \int_0^{\frac{\pi}{4}} \frac{\theta\, d\theta}{\sin\theta\cos\theta} = \frac{1}{2}\int_0^{\frac{\pi}{2}} s\csc s\, ds = \frac{\pi}{2}\int_0^{\frac{1}{2}} \pi z\csc(\pi z)\, dz.$$

By the partial fraction expansion of the cosecant and the formula for the sum of a geometric series,

$$\begin{aligned}
\pi z \csc(\pi z) &= \sum_{k=-\infty}^{\infty} (-1)^{k-1} \frac{z^2}{k^2 - z^2} = 1 + 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^2/k^2}{1 - z^2/k^2} \\
&= 1 + 2 \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} \right) z^{2n} \\
&= 1 + 2 \sum_{n=1}^{\infty} \left(\left(1 - \frac{2}{2^{2n}} \right) \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \right) z^{2n} \\
&= 1 + 2 \sum_{n=1}^{\infty} \left(1 - \frac{2}{2^{2n}} \right) \zeta(2n) z^{2n},
\end{aligned}$$

where the interchange of summation is easily justified. With $\zeta(0) = -1/2$,

$$\pi z \csc(\pi z) = 2 \sum_{n=0}^{\infty} \left(1 - \frac{2}{2^{2n}} \right) \zeta(2n) z^{2n}.$$

The result now follows, since the integral from 0 to $1/2$ of this final summation is

$$2 \sum_{n=0}^{\infty} \left(1 - \frac{2}{2^{2n}} \right) \frac{\zeta(2n)}{2^{2n+1}(2n+1)}.$$

Editorial comment. The Clausen function, defined by $Cl_2(\theta) = \sum_{k=1}^{\infty} \sin(k\theta)/k^2$, equals G when $\theta = \pi/4$. Several solvers employed known identities (or produced identities) for this function to obtain the solution. Others used Bernoulli numbers. Richard Stong employed the Barnes gamma function. For an enlightening look at the derivation and application of partial fraction expansions, see the chapter on analysis in Aigner M., Ziegler, G. M. (2018), *Proofs from the BOOK*, 6th ed., Berlin: Springer.

Also solved by A. Berkane (Algeria), R. Boukharfane (France), K. N. Boyadzhiev, P. Bracken, B. Bradie, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), C. Georghiou (Greece), M. L. Glasser, M. Kaplan, O. P. Lossers (Netherlands), S. Singh, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Integrating a Power of the Gamma Function

11975 [2017, 369]. *Proposed by István Mező, Nanjing University of Information Science and Technology, Nanjing, China.* Let x be a real number in $[0, 1)$, and let $L(x) = \int_0^1 \Gamma^x(t)\, dt$, where Γ is the gamma function defined by $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, dx$. Prove

$$\frac{(1-\gamma)^x}{1-x} \le L(x) \le \frac{1}{1-x},$$

where γ is the Euler–Mascheroni constant $\lim_{n\to\infty} \left(-\ln n + \sum_{k=1}^{n} 1/k \right)$.

Solution by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX. We claim

$$e^{(s-1)(1-\gamma)} \le \Gamma(1+s) \le 1 \tag{1}$$

when $s \in [0, 1]$. The gamma function is strictly convex on the positive real axis, so the upper bound in (1) follows from $\Gamma(1) = \Gamma(2) = 1$. For $s \in (0, 1)$, the mean value theorem asserts that there exists $\tau \in (s, 1)$ such that

$$\frac{\log(\Gamma(2)) - \log(\Gamma(1+s))}{1-s} = \frac{d}{ds}\log(\Gamma(1+s))\Big|_{s=\tau} = \psi(1+\tau),$$

where ψ is the digamma function. Since $\psi(x)$ is increasing for $x > 0$, this implies

$$\log(\Gamma(1+s)) > (s-1)\psi(2).$$

Using $\psi(2) = 1 - \gamma$, we obtain the lower bound in (1), with equality when $s = 1$.

Next we claim $e^{(s-1)(1-\gamma)} > 1 - \gamma$. Indeed, this is a calculation for $s = 0$, and the left side is an increasing function of s. Hence (1) implies $1 - \gamma \le \Gamma(1+s) \le 1$. Since $\Gamma(1+s) = s\Gamma(s)$, we have

$$\frac{1-\gamma}{s} \le \Gamma(s) \le \frac{1}{s}$$

when $s \in (0, 1]$. For $x \in [0, 1)$, it follows that

$$\frac{(1-\gamma)^x}{s^x} \le \Gamma^x(s) \le \frac{1}{s^x}.$$

Integration with respect to s from 0 to 1 produces the desired result.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfance (France), R. Chapman (U. K.), P. P. Dályay (Hungary), A. Hannan (India), O. Kouba (Syria), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France) & R. Tauraso (Italy), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, L. Zhou, and the proposer.

A Nonlinear Recurrence with Fibonacci Exponents

11976 [2017, 370]. *Proposed by Robert Bosch, Miami, FL.* Given a positive real number s, consider the sequence $\{u_n\}$ defined by $u_1 = 1$, $u_2 = s$, and $u_{n+2} = u_n u_{n+1}/n$ for $n \ge 1$.

(a) Show that there is a constant C such that $\lim_{n\to\infty} u_n = \infty$ when $s > C$ and $\lim_{n\to\infty} u_n = 0$ when $s < C$.

(b) Calculate $\lim_{n\to\infty} u_n$ when $s = C$.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.

(**a**) We may compute the next few terms:

$$u_3 = s, \qquad u_4 = \frac{s^2}{2}, \qquad u_5 = \frac{s^3}{2\cdot 3},$$

$$u_6 = \frac{s^5}{2^2\cdot 3\cdot 4}, \qquad u_7 = \frac{s^8}{2^3\cdot 3^2\cdot 4\cdot 5}, \qquad u_8 = \frac{s^{13}}{2^5\cdot 3^3\cdot 4^2\cdot 5\cdot 6}.$$

Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It follows by induction that $u_{n+2} = s^{F_{n+1}}/b_n$, where

$$b_n = \prod_{k=2}^{n} k^{F_{n+1-k}} = 2^{F_{n-1}} \cdot 3^{F_{n-2}} \cdot \cdots \cdot (n-1)^{F_2} \cdot n^{F_1}.$$

Let $\phi = (1+\sqrt{5})/2$, let $l = \sum_{n=1}^{\infty} \ln k/\phi^k \approx 1.16345$, and let $C = e^l \approx 3.20096$. We now show $b_n \sim C^{F_{n+1}}$ as $n \to \infty$. Indeed, since

$$\ln b_n = \sum_{k=1}^{n} F_{n+1-k} \ln k$$

and $F_n = (\phi^n - \tau^n)/\sqrt{5}$, where $\tau = -1/\phi$, we have

$$\ln b_n = \frac{1}{\sqrt{5}}\left(\sum_{k=1}^{n}(\phi^{n+1-k} - \tau^{n+1-k})\ln k\right)$$
$$= \frac{1}{\sqrt{5}}\phi^{n+1}\sum_{k=1}^{n}\phi^{-k}\ln k - \frac{1}{\sqrt{5}}\sum_{k=1}^{n}\tau^{n+1-k}\ln k.$$

Therefore

$$\frac{\sqrt{5}\ln b_n}{\phi^{n+1}} = \sum_{k=1}^{n}\frac{\ln k}{\phi^k} - \frac{1}{\phi^{n+1}}\sum_{k=1}^{n}\tau^{n+1-k}\ln k.$$

Since $-1 < \tau < 0$, we have

$$\left|\sum_{k=1}^{n}\tau^{n+1-k}\ln k\right| \leq \sum_{k=1}^{n}\ln k = \ln n! < n\ln n,$$

and so

$$\frac{1}{\phi^{n+1}}\left|\sum_{k=1}^{n}\tau^{n+1-k}\ln k\right| \to 0$$

as $n \to \infty$. Thus $(\sqrt{5}\ln b_n)/\phi^{n+1} \sim l$ and $(\ln b_n)/F_{n+1} \sim l$ as $n \to \infty$. When $\ln s \neq l$,

$$\ln u_{n+2} = F_{n+1}\ln s - \ln b \sim F_{n+1}(\ln s - l) = F_{n+1}\ln\frac{s}{C}.$$

Therefore

$$\lim_{n\to\infty} u_n = \begin{cases} 0 & \text{if } s < C; \\ \infty & \text{if } s > C. \end{cases}$$

(**b**) When $s = C$,

$$\ln u_{n+2} = F_{n+1}l - \ln b_n$$
$$= \frac{1}{\sqrt{5}}\phi^{n+1}\sum_{k=n+1}^{\infty}\frac{\ln k}{\phi^k} - \frac{1}{\sqrt{5}}\tau^{n+1}l + \frac{1}{\sqrt{5}}\sum_{k=1}^{n}\tau^{n+1-k}\ln k$$
$$\sim \frac{1}{\sqrt{5}}\phi^{n+1}\sum_{k=n+1}^{\infty}\frac{\ln k}{\phi^k} \geq \frac{\phi}{\sqrt{5}}\ln(n+1),$$

and so $\lim_{n\to\infty} u_n = \infty$.

Editorial comment. Douglas B. Tyler provided a generalization: If the initial condition is relaxed to $u_1 = t > 0$, then $u_n \to \infty$ when $t^{\phi-1}s \geq C$ and $u_n \to 0$ when $t^{\phi-1}s < C$.

Also solved by A. Berkane (Algeria), P. Bracken, N. Caro (Brazil), R. Chapman (U. K.), P. P. Dályay (Hungary), B. Golosio & G. Stegel (Italy), J. Grivaux (France), T. Hakobyan & S. Navasardyan (Armenia), T. Horine, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), R. K. Schwartz, J. C. Smith, A. Stadler (Switzerland) A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at americanmathematicalmonthly.submittable.com/submit. *Proposed solutions to the problems below should be submitted by June 30, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

12090. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* The Pell–Lucas numbers Q_n satisfy $Q_0 = 2$, $Q_1 = 2$, and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Prove

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{Q_n}\right) \arctan\left(\frac{2}{Q_{n+1}}\right) = \frac{\pi^2}{32}.$$

12091. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{i!\,j!\,k!}{ij(i+j+k)!}\left(H_{i+j+k} - H_k\right) = \zeta(3),$$

where H_k is the kth harmonic number and ζ is the Riemann zeta function.

12092. *Proposed by Michael Diao, student, University High School, Irvine, CA, and Andrew Wu, student, St. Albans School, Washington, DC.* Let ABC be a triangle, and let P be a point in the plane of the triangle satisfying $\angle BAP = \angle CAP$. Let Q and R be diametrically opposite P on the circumcircles of $\triangle ABP$ and $\triangle ACP$, respectively. Let X be the point of concurrency of line BR and line CQ. Prove that XP and BC are perpendicular.

12093. *Proposed by Melih Üçer, Yildirim Beyazit University, Ankara, Turkey.* Let S be a finite set of points in the plane no three of which are collinear and no four of which are concyclic. A coloring of the points of S with colors red and blue is *circle-separable* if there is a circle whose interior contains all the red points of S and whose exterior contains all the blue points of S. Determine the number of circle-separable colorings of S.

12094. *Proposed by Pablo Fernández Refolio, Madrid, Spain.* Let G be Catalan's constant, defined to be $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. Prove

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n(n+1)^3} = 16\log 2 - \frac{32G}{\pi} + \frac{48}{\pi} - 16.$$

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12095. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x+1) = f(x) + 1$ and $f(x^4 - x^2) = f(x)^4 - f(x)^2$ for all x.

12096. *Proposed by Dan Ştefan Marinescu, Hunedoara, Romania, and Mihai Monea, Deva, Romania.* Let a and b be real numbers with $a < b$. Given a function $f : (a, b) \to \mathbb{R}$, we let $g(x) = (x - a)f(x)$ and $h(x) = (x - b)f(x)$. Prove that if g and h are convex, then f is differentiable.

SOLUTIONS

A Continued Radical of Fermat Numbers

11967 [2017, 274]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fermat number $2^{2^n} + 1$. Find

$$\lim_{n\to\infty} \sqrt{6F_1 + \sqrt{6F_2 + \sqrt{6F_3 + \sqrt{\cdots + \sqrt{6F_n}}}}}.$$

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. The desired limit L exists and equals $13/2$. Let

$$a_n = \sqrt{6F_1 + \sqrt{6F_2 + \sqrt{6F_3 + \sqrt{\cdots + \sqrt{6F_n}}}}}.$$

We first prove $a_n \le 13/2$ for $n \in \mathbb{N}$. Since $a_{n+1} > a_n$ and every bounded increasing sequence has a limit, this will imply that L exists and is at most $13/2$.

To study a_n, let $b_j = \sqrt{6F_j + \sqrt{\cdots + \sqrt{6F_n}}}$ for $1 \le j \le n$. We use induction on $n - j$ to prove $b_j < 3 \cdot 2^{2^{j-1}} + (1/2)^{2^{j-1}}$. For $j = n$, since $6F_j = 6 \cdot 2^{2^j} + 6 < 9 \cdot 2^{2^j} + 6 + (1/2)^{2^j}$, we have $b_n = \sqrt{6F_n} < 3 \cdot 2^{2^{n-1}} + (1/2)^{2^{n-1}}$. For smaller j, we use the induction hypothesis to compute $6F_j + b_{j+1} < 9 \cdot 2^{2^j} + 6 + (1/2)^{2^j}$. Thus $b_j = \sqrt{6F_{n-1} + b_{j+1}} < 3 \cdot 2^{2^{j-1}} + (1/2)^{2^{j-1}}$. Hence $a_n = b_1 < 3 \cdot 2^{2^0} + (1/2)^{2^0} = 13/2$.

It remains to prove $L \ge 13/2$. Note that $L = \sqrt{t_1}$, where

$$t_n = 6F_n + \sqrt{6F_{n+1} + \sqrt{6F_{n+2} + \cdots}}$$

for $n \in \mathbb{N}$. Let $u_n = \sqrt{t_n}$ for all n. Starting from a fixed term u_n, we will use induction on k to prove $u_{n-k} > 3 \cdot 2^{2^{n-1-k}} + \frac{2^k - 1}{2^k} \frac{1}{2^{2^{n-1-k}}}$. With $k = n - 1$ in this particular induction, we obtain $u_1 > 3 \cdot 2^{2^0} + \frac{2^{n-1}-1}{2^{n-1}}(1/2)^{2^0}$. Since this inequality on u_1 holds for all n, we obtain $L = u_1 \ge 3 \cdot 2^{2^0} + (1/2)^{2^0} = 13/2$.

First note $F_n = 2^{2^n} + 1 > 2^{2^n}$, so

$$u_n > \sqrt{6 \cdot 2^{2^n} + \sqrt{6 \cdot 2^{2^{n+1}} + \sqrt{6 \cdot 2^{2^{n+2}} + \cdots}}} = \sqrt{2^{2^n}\left(6 + \sqrt{6 + \sqrt{6 + \cdots}}\right)}.$$

Let $s = 6 + \sqrt{6 + \sqrt{6 + \cdots}}$. With $s_0 = 6$ and $s_n = 6 + \sqrt{s_{n-1}}$ for $n > 0$, it is immediate inductively that $\{s_n\}_{n\ge0}$ is an increasing sequence bounded by 9, so the limit s exists. Also

s must satisfy $(s-6)^2 = s$, and $s > 6$, so $s = 9$. Hence $u_n > 3 \cdot 2^{2^{n-1}}$. This is the base step ($k = 0$) for the induction.

For $k > 0$, we use the induction hypothesis to compute

$$\begin{aligned}(u_{n-k})^2 = t_{n-k} = 6F_{n-k} + \sqrt{t_{n-k+1}} &= 6 \cdot (2^{2^{n-k}} + 1) + u_{n-k+1} \\ &> 6 \cdot 2^{2^{n-k}} + 6 + 3 \cdot 2^{2^{n-k}} + \frac{2^{k-1}-1}{2^{k-1}} \frac{1}{2^{2^{n-k}}} \\ &= 9 \cdot 2^{2^{n-k}} + 6 + \frac{1}{2^{2^{n-k}}} - \frac{1}{2^{k-1}} \frac{1}{2^{2^{n-k}}} = A^2 - \frac{1}{2^{k-1+2^{n-k}}},\end{aligned}$$

where $A = 3 \cdot 2^{2^{n-1-k}} + (1/2)^{2^{n-1-k}}$. A simple calculation shows

$$A^2 - \frac{1}{2^{k-1+2^{n-k}}} > \left(A - \frac{1}{2^{k+2^{n-1-k}}}\right)^2,$$

and it follows that $u_{n-k} > A - (1/2)^{k+2^{n-1-k}}$, as required.

Also solved by A. Berkane (Algeria), P. P. Dályay (Hungary), D. Fleischman, M. Goldenberg & M. Kaplan, J. A. Grzesik, T. Horine, O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), M. Omarjee (France), M. Reid, A. Stadler, A. Stenger, R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

Well-divided Arrangements

11974 [2017, 369]. *Proposed by Haoran Chen, Gustavus Adolphus College, St. Peter, MN.* Any n points on a line divide that line into $n-1$ segments and two rays. If these $n-1$ segments all have the same length, then we say the line is *well-divided* by the set. Classify the arrangements consisting of a finite number of lines in the plane, no two parallel, such that each line is well-divided by its points of intersection with the other lines.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We show that all such arrangements are either (1) n concurrent lines, or (2) $n-1$ concurrent lines that well-divide an nth line.

Such arrangements clearly satisfy the conditions. Suppose that another type of arrangement satisfies them. Since it is not type (1), some three lines form a triangle. Let XYZ be a triangle with smallest area formed by three of the lines. Any line cutting a triangle produces a piece that is again a triangle and has smaller area, so no other line cuts XYZ.

Consider barycentric coordinates with respect to triangle XYZ such that $X = (1, 0, 0)$, $Y = (0, 1, 0)$, and $Z = (0, 0, 1)$. Points on the lines YZ, ZX, and XY have coordinates of the form $(0, 1-k, k)$, $(\ell, 0, 1-\ell)$, and $(1-m, m, 0)$, respectively. Setting k, ℓ, or m to 0 or 1 yields a point in $\{X, Y, Z\}$. Since no line cuts into XYZ, the corners of the triangle are consecutive intersection points on their respective lines. Thus the intersection points on these lines correspond to values of k, ℓ, and m that are contiguous blocks of integers including $\{0, 1\}$.

Since the arrangement of lines is not type (2), one of these lines has another intersection point S with a fourth line passing through it. By symmetry, we may assume that S is on YZ and has coordinates $(0, -1, 2)$. Let the fourth line intersect lines XZ and XY at points R and T, with coordinates $(\ell, 0, 1-\ell)$ and $(1-m, m, 0)$, respectively. The theorem of Menelaus requires

$$\frac{XT}{TY} \cdot \frac{YS}{SZ} \cdot \frac{ZR}{RX} = -1,$$

where we use signed lengths of segments. Therefore,

$$\frac{-m}{m-1} \cdot \frac{-2}{1} \cdot \frac{\ell}{1-\ell} = -1,$$

which simplifies to $(\ell+1)(m+1) = 2$. Thus $\{\ell, m\} = \{0, 1\}$ or $\{\ell, m\} = \{-3, -2\}$.

The case $(\ell, m) = (0, 1)$ gives line YZ, which cannot be the fourth line.

The case $(\ell, m) = (1, 0)$ gives line XS as the fourth line. In this case triangle XZS also has minimal area, so we can use XZS instead of XYZ. In fact, every triangle formed by X and two consecutive intersection points on YZ has minimal area. If every line through every intersection point on YZ that neighbors a consecutive pair yields the case $\{\ell, m\} = \{0, 1\}$ and passes through X, then the arrangement is type (2), a contradiction. Hence we may consider a smallest triangle yielding the case $\{\ell, m\} = \{-3, -2\}$.

If $(\ell, m) = (-2, -3)$, then $R = (-2, 0, 3)$ and $T = (4, -3, 0)$ (see the left side of Figure 1). Here triangle ZRS has twice the area of triangle XYZ, and some fifth line in the arrangement must pass through the midpoint $(-1, 0, 2)$ of RZ, since the intersection points are equally spaced on line XZ. Letting $M = (-1, 0, 2)$, this fifth line cannot be MS, because that is parallel to XY. Hence it intersects the interior of ZS or SR, forming a triangle smaller than XYZ, a contradiction. Thus this case is not possible.

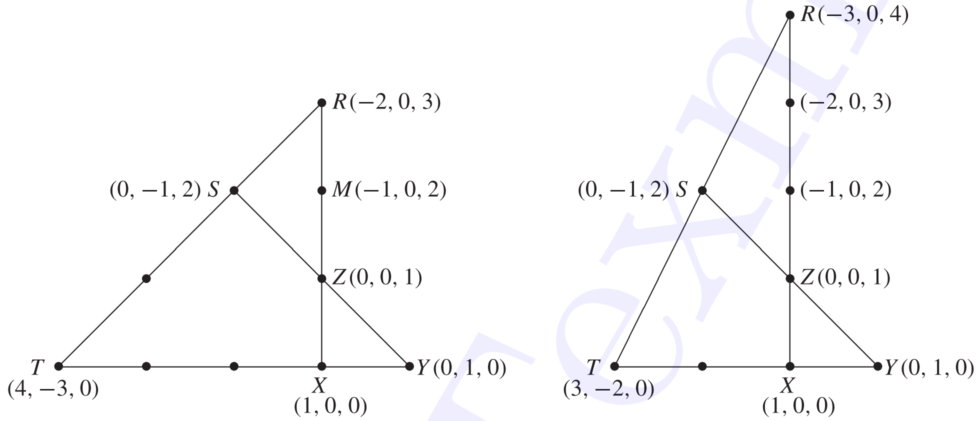

Figure 1. $(\ell, m) = (-2, -3)$ (left) and $(\ell, m) = (-3, -2)$ (right).

Finally, if $(\ell, m) = (-3, -2)$, then $R = (-3, 0, 4)$ and $T = (3, -2, 0)$ (see the right side of Figure 1). The area of triangle ZRS is three times that of XYZ, and two more lines must pass through $(-1, 0, 2)$ and $(-2, 0, 3)$, respectively. To avoid parallel lines and triangles smaller than XYZ, the line through $(-1, 0, 2)$ must intersect the interior of segment RS, forming a triangle with area less than twice that of XYZ. The line through $(-2, 0, 3)$ then cuts this triangle, forming a triangle smaller than XYZ, which is again a contradiction.

Also solved by Y. J. Ionin and the proposer.

Distinct Roots with Equal Sums Are Integers

11977 [2017, 370]. *Proposed by Joseph Foy, University of Chicago, Chicago, IL, Ali Hassani, Dearborn, MI, Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI, and Clark Zhang, University of Pennsylvania, Philadelphia, PA.*

(a) Suppose that a, b, c, and d are positive integers with $\gcd(a, b, c, d) = 1$ and with $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$. Prove that if $\{a, b\} \neq \{c, d\}$, then each of a, b, c, and d is a perfect square.

(b)* More generally, suppose that k is an integer with $k \geq 3$, and suppose that a, b, c, and d are positive integers with $\gcd(a, b, c, d) = 1$ and with $\sqrt[k]{a} + \sqrt[k]{b} = \sqrt[k]{c} + \sqrt[k]{d}$. Assuming $\{a, b\} \neq \{c, d\}$, must each of a, b, c, and d be a perfect kth power?

Solution to **(a)** *by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.* Squaring $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ leads to $\sqrt{ab} + x = \sqrt{cd}$, where x is some rational number. Squaring again gives $ab + x^2 + 2\sqrt{ab} = cd$, so $\sqrt{ab}$ and thus $\sqrt{cd}$ are rational. By applying this procedure to $\sqrt{a} - \sqrt{c} = \sqrt{d} - \sqrt{b}$ and $\sqrt{a} - \sqrt{d} = \sqrt{c} - \sqrt{b}$

we obtain that $\sqrt{ac}$, $\sqrt{bd}$, $\sqrt{ad}$, and $\sqrt{bc}$ are rational. Hence ab, ac, ad, bc, bd, and cd are all perfect squares. If a is not a perfect square, then for some prime p, the largest e such that p^e divides a is odd. This implies that all of b, c, d are also divisible by p, contradicting the gcd condition. Hence a is a perfect square. By symmetry, the result follows.

Solution to **(b)** *by Richard Stong, Center for Communications Research, San Diego, CA.* We prove a stronger fact.

Claim. *For $k \geq 2$, if $m_1, \ldots, m_n$ are distinct positive integers with no divisors that are kth powers, and $r_1, \ldots, r_n$ are rational numbers such that $r_1 \sqrt[k]{m_1} + \cdots + r_n \sqrt[k]{m_n} = 0$, then $r_1 = \cdots = r_n = 0$.*

Proof of Claim. Consider a counterexample such that k is minimal and such that n is minimal among counterexamples with this k. Since n is minimized, all of $r_1, \ldots, r_n$ are nonzero. Since $m_1, \ldots, m_n$ are distinct and kth-power-free, the ratio of any two terms in the sum is irrational.

Let $\omega = e^{2\pi i/k}$ and $K = \mathbb{Q}[\omega]$. Fix g in the Galois group of $K[\sqrt[k]{m_1}, \ldots, \sqrt[k]{m_n}]$ over K. We have $g(\sqrt[k]{m_j}) = \omega^s \sqrt[k]{m_j}$ for some s with $0 \leq s < k$. Let S_s denote the sum of all terms in the given sum with this power of ω. Applying g^t to the given identity yields

$$S_0 + \omega^t S_1 + \cdots + \omega^{st} S_s + \cdots + \omega^{(k-1)t} S_{k-1} = 0.$$

By a Fourier transformation or by noting that the coefficient matrix of the resulting system for $0 \leq t < k$ is an invertible Vandermonde matrix, we conclude $S_s = 0$ for all s. By the minimality of n, it follows that only one of the sums $S_0, \ldots, S_{k-1}$ is nontrivial. Thus every element g of the Galois group acts by the same power of ω on each of the kth roots and hence fixes $\sqrt[k]{m_j/m_n}$ for all j. Therefore, $\sqrt[k]{m_j/m_n}$ is in K for all j.

Now fix j, and write $a = m_j/m_n$. Suppose that $\sqrt[k]{a}$ has degree d over $\mathbb{Q}$. Let $P(X)$ be the degree d monic minimal polynomial of $\sqrt[k]{a}$. Note that $P(X)$ divides the polynomial $X^k - a$, whose roots are $\sqrt[k]{a}$ times a kth root of unity. Thus the magnitude of the constant term of P is $a^{d/k}$, which must therefore be rational. If $m = \gcd(d, k)$, then we can write $a^{m/k} = a^{(rd+sk)/k} = \left(a^{d/k}\right)^r a^s$ for some integers r and s. Hence $a^{m/k}$ is rational and $\sqrt[k]{a} = \sqrt[m]{a^{m/k}}$ is a root of a polynomial of degree m. It follows that $d = m$ and $d \mid k$. Since $[K : \mathbb{Q}] = \phi(k)$, where ϕ is the Euler ϕ-function, we have $d \mid l$, where $l = \gcd(k, \phi(k)) < k$. Furthermore, $\sqrt[k]{a}$, which equals $\sqrt[d]{a^{d/k}}$, is a dth root of a rational and hence an lth root of a rational as well.

We conclude that $\sqrt[k]{m_j/m_n}$ is an lth root of a rational for all j. Dividing the given identity by $\sqrt[k]{m_n}$ and rewriting, we obtain a similar sum with l instead of k. We cannot have $l = 1$, since the ratio of any two distinct terms is still irrational, so this contradicts the minimality of k. This proves the claim. ■

From the claim, the desired result is simple bookkeeping. Writing each of a, b, c, d as a kth power of an integer times a kth-power-free integer and collecting terms with the same kth-power-free part would give a sum as in the claim, so this sum must be trivial. Thus each collected coefficient must vanish; in particular, each kth-power-free part that occurs must occur at least twice. If two distinct kth-power-free parts each occur exactly twice, then the terms of $\sqrt[k]{a} + \sqrt[k]{b} = \sqrt[k]{c} + \sqrt[k]{d}$ cancel in pairs, which can only happen if $\{a, b\} = \{c, d\}$. Otherwise, all four of a, b, c, d have the same kth-power-free part, which by the gcd condition must be 1, and hence a, b, c, d are all kth powers.

Also solved in full by A. J. Bevelacqua. Part **(a)** also solved by R. Boukharfane (France), P. Budney, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, S. M. Gagola, Jr., N. Ghosh, Y. J. Ionin, K. T. L. Koo (China), J. C. Smith, J. H. Smith, M. Tetiva (Romania), H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Sum of Hyperbolic Cosines of Fibonacci Numbers

11978 [2017, 465]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fibonacci number, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ when $n \geq 2$. Find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\cosh F_n \, \cosh F_{n+3}}.$$

Solution by Kyle Gatesman, Thomas Jefferson High School for Science and Technology, Alexandria, VA. The value of the sum is $(2\cosh^2 1)^{-1}$.

Using the identity $\cosh(\alpha+\beta) + \cosh(\alpha-\beta) = 2\cosh\alpha\cosh\beta$ and the identities $F_{n+3} = F_{n+2} + F_{n+1}$ and $F_n = F_{n+2} - F_{n+1}$, we obtain

$$\cosh F_n + \cosh F_{n+3} = 2\cosh F_{n+1} \cosh F_{n+2}.$$

Noting that

$$\lim_{n\to\infty} \frac{1}{\cosh F_n \cosh F_{n+3}} = 0$$

and rewriting the sum as a telescoping series, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{\cosh F_n \cosh F_{n+3}} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cosh F_n + \cosh F_{n+3}} \left(\frac{1}{\cosh F_n} + \frac{1}{\cosh F_{n+3}} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2\cosh F_{n+1} \cosh F_{n+2}} \left(\frac{1}{\cosh F_n} + \frac{1}{\cosh F_{n+3}} \right) \\
&= \frac{1}{2\cosh F_0 \cosh F_1 \cosh F_2} + \sum_{n=0}^{\infty} \frac{(-1)^n + (-1)^{n+1}}{2\cosh F_{n+1} \cosh F_{n+2} \cosh F_{n+3}} \\
&= \frac{1}{2\cosh^2 1} = \frac{2e^2}{e^4 + 2e^2 + 1}.
\end{aligned}$$

Editorial comment. Several solvers noted that the Fibonacci numbers can be replaced more generally with any sequence G_n satisfying the recurrence $G_{n+2} = G_{n+1} + G_n$. The resulting sum is $(2\cosh G_0 \cosh G_1 \cosh G_2)^{-1}$. Examples include $G_n = aF_n$ (for $a > 0$), $G_n = L_n$ (the Lucas numbers), and $G_n = \phi^n$ (powers of the golden ratio).

Also solved by A. Berkane (Algeria), R. Boukharfane (France), B. S. Burdick, R. Chapman (U. K.), P. P. Dályay (Hungary), G. Fera (Italy), M. Goldenberg & M. Kaplan, O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Martin (Germany), R. Nandan, M. Omarjee (France), F. A. Rakhimjanovich (Uzbekistan), R. Stong, R. Tauraso (Italy), D. B. Tyler, Florida Atlantic University Problem Solving Group, and the proposer.

Integer Triangles

11979 [2017, 465]. *Proposed by Zachary Franco, Houston, Texas.* Let O and I denote the circumcenter and incenter, respectively, of a triangle. Are there infinitely many nonsimilar scalene triangles ABC for which the lengths AB, BC, CA, and OI are all integers?

Solution by Michael Reid, University of Central Florida, Orlando, FL. We exhibit an infinite family of such triangles in parametric form. For a positive integer t, let

$$\begin{aligned}
a &= 9t^4 + 48t^3 + 90t^2 + 68t + 16, \\
b &= 9t^4 + 54t^3 + 123t^2 + 126t + 49, \quad \text{and} \\
c &= 9t^4 + 60t^3 + 144t^2 + 148t + 55.
\end{aligned}$$

Comparing termwise, we see $a < b < c < a + b$, so a, b, c are indeed the side lengths of a scalene triangle. Write A, B, C for the angles of the triangle opposite sides a, b, c, respectively. Write s for the semiperimeter $(a + b + c)/2$. Using the law of cosines, we compute $\cos B = (a^2 + c^2 - b^2)/(2ac) = 1/2$, so $B = \pi/3$.

Let $d = OI$, and let r and R be the inradius and circumradius, respectively. Since $B = \pi/3$, we have $r = (s - b)\tan(B/2) = (a + c - b)/(2\sqrt{3})$. From the law of sines, $R = b/(2\sin B) = b/\sqrt{3}$. Therefore, by Euler's theorem,

$$d = \sqrt{R(R - 2r)} = \sqrt{\frac{b(2b - a - c)}{3}} = (2t + 3)(3t^2 + 9t + 7),$$

which is an integer.

Finally, we show that the triangles are pairwise dissimilar. The ratio R/d is equal to $(3t^2 + 9t + 7)((2t + 3)\sqrt{3})$ and hence is an increasing function of t for $t > 0$, so different values of t yield different values of R/d and thus dissimilar triangles.

Editorial comment. All solutions from all solvers used the angle $\pi/3$. Are there solutions that don't?

Also solved by R. Boukharfane (France), R. Chapman (U. K.) & R. Tauraso (Italy), M. E. Kidwell & M. D. Meyerson, J. C. Smith, Missouri State University Problem Solving Group, and the proposer.

Lower Bound for an L^3 Norm

11981 [2017, 465]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Suppose that $f : [0, 1] \to \mathbb{R}$ is a differentiable function with continuous derivative and with

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = 1.$$

Prove

$$\int_0^1 \left|f'(x)\right|^3 dx \ge \left(\frac{128}{3\pi}\right)^2.$$

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. Using integration by parts, we have

$$\begin{aligned}1 &= \int_0^1 (2x - 1)f(x)\,dx = (x^2 - x)f(x)\Big|_0^1 + \int_0^1 (x - x^2)f'(x)\,dx \\ &= \int_0^1 x(1 - x)f'(x)\,dx.\end{aligned}$$

Using Hölder's inequality with exponents $3/2$ and 3 yields

$$\begin{aligned}1 &\le \int_0^1 |x(1 - x)f'(x)|\,dx \le \left(\int_0^1 x^{3/2}(1 - x)^{3/2}\,dx\right)^{2/3} \left(\int_0^1 |f'(x)|^3\,dx\right)^{1/3} \\ &= B\left(\frac{5}{2}, \frac{5}{2}\right)^{2/3} \left(\int_0^1 |f'(x)|^3\,dx\right)^{1/3} = \left(\frac{\Gamma(5/2)^2}{\Gamma(5)}\right)^{2/3} \left(\int_0^1 |f'(x)|^3\,dx\right)^{1/3},\end{aligned}$$

where B is Euler's beta integral. Substituting the values $\Gamma(5) = 24$ and $\Gamma(5/2) = 3\sqrt{\pi}/4$, we obtain

$$\int_0^1 |f'(x)|^3\,dx \ge \left(\frac{24 \cdot 16}{9\pi}\right)^2 = \left(\frac{128}{3\pi}\right)^2.$$

The bound is achieved when $f'(x) = x^{3/2}(1-x)^{3/2}$.

Editorial comment. The AN-anduud Problem Solving Group provided the following extension: Let f be continuously differentiable, assume $\alpha > \beta \geq 0$, and assume $\int_0^1 x^\alpha f(x)\,dx = \int_0^1 x^\beta f(x)\,dx = 1$. If $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^1 |f'(x)|^q\,dx \geq \frac{(\alpha-\beta)^{2q-1}}{B\left(\dfrac{(\beta+1)p+1}{\alpha-\beta},\, p+1\right)^{q-1}},$$

where B is Euler's beta function.

Also solved by P. Acosta, K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, N. Grivaux (France), L. Han, E. A. Herman, K. T. L. Koo (China), O. Kouba (Syria), K. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), J. McHugh, V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), A. Pathak, S. Pathak (Canada), P. Perfetti (Italy), J. C. Smith, J. Steier, A. Stenger, R. Stong, R. Tauraso (Italy), E. I. Verriest, L. Zhou, AN-anduud Problem Solving Group (Mongolia), Florida Atlantic University Problem Solving Group, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

A Limit of a Power of a Sum

11982 [2017, 465]. *Proposed by Ovidiu Furdui, Mircea Ivan, and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\lim_{x\to\infty}\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n\right)^{1/x}.$$

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. It is easy to verify by induction that the known inequalities $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ imply $(n-1)!e^{n-1} \leq n^n \leq n!e^n$ for $n \geq 1$. Hence for $x > 0$,

$$\exp(x/e) - 1 = \sum_{n=1}^{\infty}\frac{x^n}{n!e^n} \leq \sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n \leq \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!e^{n-1}} = x\exp(x/e),$$

and therefore

$$\big(\exp(x/e)-1\big)^{1/x} \leq \left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n\right)^{1/x} \leq x^{1/x}\exp(1/e).$$

Since

$$\lim_{x\to\infty}(\exp(x/e)-1)^{1/x} = \exp(1/e) \quad \text{and} \quad \lim_{x\to\infty} x^{1/x}\exp(1/e) = \exp(1/e),$$

the required limit is $\exp(1/e)$.

Also solved by R. A. Agnew, K. F. Andersen (Canada), A. Berkane (Algeria), G. E. Bilodeau, R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), G. Fera (Italy), D. Fleischman, N. Ghosh, N. Grivaux (France), E. A. Herman, S. Kaczkowski, O. Kouba (Syria), K. Lau (China), O. P. Lossers (Netherlands), R. Martin (Germany), L. Matejíčka (Slovakia), C. Mendico (Italy), R. Molinari, M. Omarjee (France), S. Pathak (Canada), P. Perfetti (Italy), B. Ravan, M. Reid, P. K. Sharma (India), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, D. B. Tyler, C. I. Vălean (Romania), E. I. Verriest, M. Vowe (Switzerland), Florida Atlantic University Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposers.

Sum of Powers of the Sides of a Triangle

11984 [2017, 466]. *Proposed by Daniel Sitaru, Drobeta Turnu Severin, Romania.* Let a, b, and c be the lengths of the sides of a triangle with inradius r. Prove $a^6 + b^6 + c^6 \geq 5184r^6$.

Solution by Leonard Giugiuc, Drobeta Turnu Severin, Romania. We first prove the well-known inequality $a + b + c \geq 6\sqrt{3}r$. Writing $a + b + c = 2s$, where s is the semiperimeter and recalling that

$$\frac{s}{r} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}, \tag{$*$}$$

we see that this follows from Jensen's inequality in the form

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \geq 3\cot\frac{A+B+C}{6} = 3\cot\frac{\pi}{6} = 3\sqrt{3}.$$

The requested inequality follows from combining this with the power mean inequality $(a^6 + b^6 + c^6)/3 \geq ((a + b + c)/3)^6$.

Editorial comment. Motivated by the power mean inequality, we examine the inequalities

$$\left(\frac{a^p + b^p + c^p}{3}\right)^{1/p} \geq 2\sqrt{3}r.$$

The requested inequality is the case $p = 6$, so to solve the problem it suffices to prove this for any $p \leq 6$. When $p = 1$ we get the much stronger inequality (Question 1273 posed by M. E. Fauquembergue in 1878 in *Nouv. Ann. Math.* 37, p. 475, or item 5.11 in Bottema, O. et al. (1969), *Geometric Inequalities*, Groningen: Wolters-Noordhoff) that is proved above and was cited or reproved by many solvers. The strongest version that is in the literature seems to be the case $p = -1$, the inequality $\frac{\sqrt{3}}{2r} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. This appears (without proof) on page 342 of Posamentier, A. S., Lehmann, I. (2012), *The Secrets of Triangles*, New York: Prometheus Books. The $p = -2$ case is the strongest member of this family that holds. It asserts

$$\frac{1}{4r^2} \geq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

To prove it, we let $s = (a + b + c)/2$ as above and let $x = s - a$, $y = s - b$, and $z = s - c$. Using ($*$), we see that this inequality becomes

$$\frac{x + y + z}{4xyz} \geq \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2} + \frac{1}{(x + y)^2},$$

which follows by summing three inequalities

$$\frac{1}{4yz} \geq \frac{1}{(y + z)^2}, \quad \frac{1}{4xz} \geq \frac{1}{(x + z)^2}. \quad \text{and} \quad \frac{1}{4xy} \geq \frac{1}{(x + y)^2},$$

each a form of the AM–GM inequality.

Also solved by K. F. Andersen (Canada), D. Bailey & E. Campbell & C. Diminnie, H. Bailey, M. Bataille (France), R. Boukharfane (France), P. Bracken, D. Chakerian, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, A. Hannan (India), E. A. Herman, J. G. Heuver (Canada), A. Kadaveru, J. S. Kim (South Korea), K. T. L. Koo (China), O. Kouba (Syria), W. Lai, J. H. Lindsey II, O. P. Lossers (Netherlands), M. Lukarevski (Macedonia), D. Marinescu (Romania), J. McHugh, V. Mikayelyan (Armenia), R. Molinari, D. Moore, R. Nandan, T. Y. Noh (South Korea), A. Pathak, P. Perfetti (Italy), C. R. Pranesachar (India), M. Reid, J. C. Smith, A. Stadler (Switzerland) N. Stanciu & T. Zvonaru (Romania), R. Stong, M. Tang, R. Tauraso (Italy), V. Tibullo (Italy), M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), L. Zhou, AN-anduud Problem Solving Group (Mongolia), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by July 31, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12097. *Proposed by Zachary Franco, Houston, TX, and Richard P. Stanley, University of Miami, Coral Gables, FL.* The *Calkin–Wilf tree* T is a complete binary tree whose vertex set is the set of positive rational numbers. The root is $1/1$, and the children of p/q are $p/(p+q)$ and $(p+q)/q$. For $n \geq 0$, let T_n be the set of rational numbers at level n in T. For example $T_0 = \{1/1\}$, $T_1 = \{1/2, 2/1\}$, and $T_2 = \{1/3, 3/2, 2/3, 3/1\}$.
(a) Find the sum of the entries in T_n.
(b) Let $m_k(n)$ be the mean of the kth powers of the entries in T_n. Show that $\lim_{n\to\infty} m_k(n)$ exists and, denoting this limit by m_k, find $9m_2 - 2m_3$.
(c) Show that when k is odd, m_k is a rational linear combination of $m_0, m_1, \ldots, m_{k-1}$.

12098. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania, and Kadir Altintas, Emirdağ, Turkey.* Suppose that the centroid of a triangle with semiperimeter s and inradius r lies on its incircle. Prove $s \geq 3\sqrt{6}r$, and determine conditions for equality.

12099. *Proposed by Michel Bataille, Rouen, France.* Let m and n be integers with $0 \leq m \leq n-1$. Evaluate

$$\sum_{k=0,k\neq m}^{n-1} \cot^2\left(\frac{(m-k)\pi}{n}\right).$$

12100. *Proposed by Finbarr Holland, University College, Cork, Ireland, Thomas Laffey, University College, Dublin, Ireland, and Roger Smyth, Belfast, U. K.* For a positive integer n, let A_n be the n-by-n tridiagonal matrix whose i,j-entry is given by

$$a_{i,j} = \begin{cases} -2j(n-j+1) & \text{if } j = i; \\ j(n-j+1) & \text{if } j = i \pm 1; \text{ and} \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Determine the eigenvalues of A_n.

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12101. *Proposed by Hojoo Lee, Seoul National University, South Korea.* Find the least upper bound of

$$\sum_{n=1}^{\infty} \frac{\sqrt{x_{n+1}} - \sqrt{x_n}}{\sqrt{(1 + x_{n+1})(1 + x_n)}}$$

over all increasing sequences $x_1, x_2, \ldots$ of positive real numbers.

12102. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Prove

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{n} \right) = 2 - \zeta(2) - 2\zeta(3),$$

where H_n is the nth harmonic number $\sum_{k=1}^{n} 1/k$ and ζ is the Riemann zeta function, defined by $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$.

12103. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let a, b, and c be the side lengths of a triangle with inradius r and circumradius R. Let r_a, r_b, and r_c be the exradii opposite the sides of length a, b, and c, respectively. Prove

$$\frac{1}{2R^3} \leq \frac{r_a}{a^4} + \frac{r_b}{b^4} + \frac{r_c}{c^4} \leq \frac{1}{16r^3}.$$

SOLUTIONS

Greedy Partitioning

11980 [2017, 465]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $a_1, \ldots, a_n$ be a nonincreasing list of positive real numbers, and fix an integer k with $1 \leq k \leq n$. Prove that there exists a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$\min_{1 \leq j \leq k} \sum_{i \in B_j} a_i \geq \frac{1}{2} \min_{1 \leq j \leq k} \frac{1}{k + 1 - j} \sum_{i=j}^{n} a_i.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove the stronger inequality in which $1/(2k + 2 - 2j)$ in the lower bound is replaced by $1/(2k + 1 - 2j)$.

View $a_1, \ldots, a_n$ as weights and $B_1, \ldots, B_k$ as boxes. Put the weights into the boxes iteratively by the following greedy algorithm: for $1 \leq i \leq n$, put weight a_i into any box having least total weight up to that point.

Let x be the least resulting total weight among the boxes, and call one box with weight x the *lightest*. Let j be one more than the number of boxes that are not lightest and that contain only one weight; note that $j \in [k]$. We may assume that the single-weight boxes contain $a_1, \ldots, a_{j-1}$, since we can exchange the labels of a_{j-1} and a_j if $a_{j-1} = a_j = x$.

The total weight of the other boxes is $\sum_{i=j}^{n} a_i$. Consider any of these boxes other than the lightest. By the definition of the algorithm, it had least total weight before its last weight was added, and that weight must have been at most x, since there is a box with weight x at the end. Since the weights are nonincreasing, and this box had at least one weight already when its last weight was added, the last addition at most doubled its weight. Hence its final weight is at most $2x$.

We now have a lightest box of weight x and $k-j$ nonsingleton boxes with weight at most $2x$. Their total weight is $\sum_{i=j}^{n} a_i$, so $x+(k-j)2x \geq \sum_{i=j}^{n} a_i$. Thus

$$\min_{1\leq j\leq k} \sum_{i\in B_j} a_i = x \geq \frac{1}{2k+1-2j}\sum_{i=j}^{n} a_i \geq \min_{1\leq j\leq k} \frac{1}{2k+1-2j}\sum_{i=j}^{n} a_i.$$

Editorial comment. John H. Lindsey II also proved this stronger inequality, noting that it is sharp, since equality holds when $n=2k-1$ and the weights are equal. From the proof above, one can deduce that equality occurs when $n=2k-j$ for some $j\in[k]$ and the weights satisfy $a_{j-1}\geq 2a_j = 2a_{j+1}=\cdots=2a_n$.

Also solved by Y. J. Ionin, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Pathak, and the proposer.

A Doubly-Antisymmetric Determinant

11983 [2017, 466]. *Proposed by Askar Dzhumadil'daev, Kazakh–British Technical University, Almaty, Kazakhstan.* Given a positive integer n, let $x_1,\dots,x_{n-1}$ and $y_1,\dots,y_n$ be indeterminates. Let A be the $2n$-by-$2n$ matrix that is antisymmetric with respect to both main diagonals and whose i,j-entry is $\sinh(x_i+y_j)$ when $i<j\leq n$ and $\cosh(x_i+y_j)$ when $i<n<j\leq 2n-i$. For example, when $n=3$, the matrix A is

$$\begin{bmatrix} 0 & s(x_1+y_2) & s(x_1+y_3) & c(x_1+y_3) & c(x_1+y_2) & 0 \\ -s(x_1+y_2) & 0 & s(x_2+y_3) & c(x_2+y_3) & 0 & -c(x_1+y_2) \\ -s(x_1+y_3) & -s(x_2+y_3) & 0 & 0 & -c(x_2+y_3) & -c(x_1+y_3) \\ -c(x_1+y_3) & -c(x_2+y_3) & 0 & 0 & -s(x_2+y_3) & -s(x_1+y_3) \\ -c(x_1+y_2) & 0 & c(x_2+y_3) & s(x_2+y_3) & 0 & -s(x_1+y_2) \\ 0 & c(x_1+y_2) & c(x_1+y_3) & s(x_1+y_3) & s(x_1+y_2) & 0 \end{bmatrix},$$

where we have written $s(z)$ for $\sinh(z)$ and $c(z)$ for $\cosh(z)$. Prove $\det(A)=0$ when n is odd and $\det(A)=1$ when n is even.

Solution by Robin Chapman, University of Exeter, Exeter, U. K. Reversing the order of the last n rows and of the last n columns changes A into a new matrix B with the same determinant. This matrix B has block decomposition $B=\left(\begin{smallmatrix} S & C \\ C & S\end{smallmatrix}\right)$, where S and C are n-by-n skew-symmetric matrices whose entry in position (i,j) with $i<j$ is $\sinh(x_i+y_j)$ for S and $\cosh(x_i+y_j)$ for C. The matrix J defined by $J=\left(\begin{smallmatrix} I_n & I_n \\ I_n & -I_n\end{smallmatrix}\right)$ has inverse $J^{-1}=\frac{1}{2}J$. Also,

$$JBJ^{-1} = \begin{pmatrix} S+C & O \\ O & S-C \end{pmatrix} = \begin{pmatrix} E^+ & O \\ O & -E^- \end{pmatrix},$$

where E^+ and E^- are the n-by-n skew-symmetric matrices with (i,j)-entry for $i<j$ equal to $\exp(x_i+y_j)$ and $\exp(-x_i-y_j)$, respectively. Hence

$$\det(A)=\det(B)=\det(E^+)\det(-E^-)=(-1)^n\det(E^+)\det(E^-).$$

Now $\det(A)=0$ is evident for odd n, since the determinant of an odd-sized skew-symmetric matrix (such as E^+ here) is zero.

If n is even, then $\det(A)=\det(E^+)\det(E^-)$. Write $a_i=\exp(x_i)$ and $b_j=\exp(y_j)$. Note that a_i and b_j are nonzero. For $i<j$, entry (i,j) of E^+ is a_ib_j. We apply row operations to E^+. Subtracting $1/b_2$ times row 2 of E^+ from $a_2/(a_1b_2)$ times row 1 of E^+ yields the vector $(a_1,a_2,0,\dots,0)$. Adding b_j times this vector to row j of E^+ for $j\geq 3$

does not change the determinant of the matrix, which now has the form $\begin{pmatrix} F & * \\ O & G \end{pmatrix}$, where $F = \begin{pmatrix} 0 & a_1 b_2 \\ -a_1 b_2 & 0 \end{pmatrix}$ and G is the $(n-2)$-by-$(n-2)$ analogue of E^+ based on the variables $x_3, \dots, x_{n-1}$ and $y_4, \dots, y_n$. Thus $\det(E^+) = a_1^2 b_2^2 \det(G)$. Iterating this process gives

$$\det(E^+) = \prod_{k=1}^{n/2} a_{2k-1}^2 b_{2k}^2 = \prod_{k=1}^{n/2} \exp(2x_{2k-1} + 2y_{2k}).$$

Replacing each x_i and y_j by its negative gives

$$\det(E^-) = \prod_{k=1}^{n/2} \exp(-2x_{2k-1} - 2y_{2k}).$$

We conclude

$$\det(A) = \det(E^+)\det(E^-) = 1.$$

Also solved by P. P. Dályay (Hungary), D. Fleischman, E. A. Herman, O. Kouba (Syria), O. P. Lossers (Netherlands), R. Stong, and the proposer.

Log-Concavity of a Binomial Sum

11985 [2017, 563]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* For fixed $s, t \in \mathbb{N}$ with $s \le t$, let $a_n = \binom{n}{s} + \binom{n}{s+1} + \cdots + \binom{n}{t}$. Prove that this sequence is log-concave, namely that $a_n^2 \ge a_{n-1}a_{n+1}$ for $n \ge 1$.

Solution I by Li Zhou, Polk State College, Winter Haven, FL. After substituting for each binomial coefficient in $\binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j+1}$ in terms of $\binom{n+1}{i}$ or $\binom{n+1}{j+1}$, we obtain

$$\binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j+1} = \frac{j-i+1}{n+1}\binom{n+1}{i}\binom{n+1}{j+1}.$$

Hence, for $0 \le i \le j \le n$,

$$\binom{n}{i}\binom{n}{j} > \binom{n}{i-1}\binom{n}{j+1}, \tag{$*$}$$

where $\binom{n}{k} = 0$ when $k = -1$ or $k = n+1$. A special case of $(*)$ is $\binom{n}{s-1}\binom{n}{t-1} \ge \binom{n}{s-2}\binom{n}{t}$. Moreover, letting $a_n(s,t) = \sum_{i=s}^{t}\binom{n}{i}$, we have $\binom{n}{s-1}a_n(s-1,t-2) \ge \binom{n}{s-2}a_n(s,t-1)$ by summing $(*)$ over j, and $a_n(s,t-1)\binom{n}{t-1} \ge a_n(s-1,t-2)\binom{n}{t}$ by summing $(*)$ over i. These inequalities yield

$$\begin{aligned} a_n^2(s-1,t-1) &= \left(\binom{n}{s-1} + a_n(s,t-1)\right)\left(a_n(s-1,t-2) + \binom{n}{t-1}\right) \\ &\ge \left(\binom{n}{s-2} + a_n(s-1,t-2)\right)\left(a_n(s,t-1) + \binom{n}{t}\right) \\ &= a_n(s-2,t-2)a_n(s,t). \end{aligned}$$

Add $a_n(s-1,t-1)a_n(s,t)$ to both sides, factor out $a_n(s-1,t-1)$ on the left and $a_n(s,t)$ on the right, and on each side apply $a_n(i-1,j-1) + a_n(i,j) = a_{n+1}(i,j)$ (which follows by summing the binomial recurrence). The result is

$$a_n(s-1,t-1)a_{n+1}(s,t) \ge a_n(s,t)a_{n+1}(s-1,t-1).$$

Finally, add $a_n(s,t)a_{n+1}(s,t)$ to both sides, factor out $a_{n+1}(s,t)$ on the left and $a_n(s,t)$ on the right, and apply the recurrence again on each side. The result is the desired inequality

$$a_{n+1}^2(s,t) \geq a_n(s,t)a_{n+2}(s,t).$$

Solution II by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. Let $f_n(x) = \sum_{k=s}^{t} \binom{n}{k} x^k$. By the binomial recurrence,

$$f_n(x) = \sum_{k=s}^{t} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k = \binom{n-1}{s-1} x^s - \binom{n-1}{t} x^{t+1} + (x+1) f_{n-1}(x).$$

Let $g_n(x) = (f_n(x))^2 - f_{n-1}(x) f_{n+1}(x)$. We show that all coefficients in the polynomial g are nonnegative. We compute

$$\begin{aligned} g_n(x) = f_n(x)\left(\binom{n-1}{s-1} x^s - \binom{n-1}{t} x^{t+1} + (x+1) f_{n-1}(x) \right) \\ - f_{n-1}(x)\left(\binom{n}{s-1} x^s - \binom{n}{t} x^{t+1} + (x+1) f_n(x) \right) \end{aligned}$$

$$= \left(\binom{n-1}{s-1} f_n(x) - \binom{n}{s-1} f_{n-1}(x) \right) x^s + \left(\binom{n}{t} f_{n-1}(x) - \binom{n-1}{t} f_n(x) \right) x^{t+1}$$

$$= \sum_{k=s}^{t} \left(\binom{n-1}{s-1}\binom{n}{k} - \binom{n}{s-1}\binom{n-1}{k} \right) x^{k+s} + \sum_{k=s}^{t} \left(\binom{n}{t}\binom{n-1}{k} - \binom{n-1}{t}\binom{n}{k} \right) x^{k+t+1}.$$

Now note

$$\binom{n-1}{i}\binom{n}{j} = \frac{n-i}{n}\binom{n}{i}\binom{n}{j} \geq \frac{n-j}{n}\binom{n}{i}\binom{n}{j} = \binom{n}{i}\binom{n-1}{j}$$

when $i \leq j$. Applying this identity with $(i,j) = (s-1,k)$ and $(i,j) = (k,t)$ shows that the coefficients of g_n are indeed nonnegative.

Setting $x = 1$ then yields $a_n^2 - a_{n-1}a_{n+1} \geq 0$, as desired.

Also solved by O. Geupel (Germany), Y. J. Ionin, O. P. Lossers (Netherlands), M. Omarjee (France), J. C. Smith, R. Stong, and the proposer.

A Cyclic Square Root Inequality

11986 [2017, 563]. *Proposed by Martin Lukarevski, Goce Delčev University, Štip, Macedonia.* Let x, y, and z be positive real numbers. Prove

$$4(xy + yz + zx) \leq \left(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \right) \sqrt{(x+y)(y+z)(z+x)}.$$

Solution by Li Zhou, Polk State College, Winter Haven, FL. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \sqrt{x+y} \cdot \sqrt{(x+y)(y+z)(z+x)} &= \sqrt{x^2 + xy + yz + zx} \cdot \sqrt{y^2 + xy + yz + zx} \\ &= \| \langle x, \sqrt{xy+yz+zx} \rangle \| \cdot \| \langle y, \sqrt{xy+yz+zx} \rangle \| \\ &\geq \langle x, \sqrt{xy+yz+zx} \rangle \cdot \langle y, \sqrt{xy+yz+zx} \rangle \\ &= 2xy + yz + zx, \end{aligned}$$

where we have written $\langle u, v\rangle$ for the vector in $\mathbb{R}^2$ with components u and v. Similarly,

$$\sqrt{y+z}\cdot\sqrt{(x+y)(y+z)(z+x)} \geq xy+2yz+zx,$$
$$\sqrt{z+x}\cdot\sqrt{(x+y)(y+z)(z+x)} \geq xy+yz+2zx.$$

Summing these inequalities completes the proof.

Editorial comment. Neculai Stanciu pointed out that this problem appeared on the 2012 Balcan Mathematical Olympiad.

Also solved by A. Ali (India), H. I. Arshagi, D. Bailey & E. Campbell & C. Diminnie, M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (France), E. Braune (Austria), R. Chapman (U. K.), P. P. Dályay (Hungary), G. Fera (Italy), D. Fleischman, O. Geupel (Germany), L. Giugiuc (Romania), M. Goldenberg & M. Kaplan, J. Grivaux (France), A. Hannan (India), E. A. Herman, F. Holland (Ireland), S. Hwang (South Korea), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), P. Khalili, K. T. L. Koo (China), O. Kouba (Syria), W. Lai, O. P. Lossers (Netherlands), D. Marinescu (Romania), L. Matejíčka (Slovakia), V. Mikayelyan (Armenia), R. Nandan, T. Y. Noh (South Korea), A. Pathak, P. Perfetti (Italy), F. A. Rakhimjanovich (Uzbekistan), S. Reich (Israel), M. Reid, D. Smith, J. C. Smith, A. Stadler (Switzerland), N. Stanciu (Romania), A. Stenger, R. Stong, R. Tauraso (Italy), Z. Vörös (Hungary), T. Wiandt, M. R. Yegan (Iran), V. P. Yellambalse (India), J. Zacharias, B. Zhao (China), GCHQ Problem Solving Group (U. K.), and the proposer.

Maximal Antichains under Componentwise Strict Order

11987 [2017, 563]. *Proposed by Shen-Fu Tsai, Redmond, WA.* Let $n_1, \ldots, n_k$ be positive integers. Let $S = [n_1] \times \cdots \times [n_k]$, where we write $[n]$ for $\{1, \ldots, n\}$. Define a binary relation on S by putting $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ whenever $x_i < y_i$ for every $i \in [k]$. An *antichain* A is a subset of S such that, for all x and y in A, neither $x < y$ nor $y < x$. An antichain is *maximal* if it is not a proper subset of any other antichain. Show that all maximal antichains in S have the same size.

Solution by Richard Ehrenborg, University of Kentucky, Lexington, KY. Define an equivalence relation on S by making x and y equivalent if $x_i - y_i$ has the same value for all i. In each equivalence class, the elements form a chain in S, and these chains partition S. Let $\mathcal{D}$ be the resulting chain decomposition.

The bottom element of a chain in $\mathcal{D}$ has 1 in at least one coordinate. Hence these bottom elements are minimal in S and form an antichain of size $|\mathcal{D}|$. Since an antichain and a chain share at most one element, this is a largest antichain and $\mathcal{D}$ is a smallest chain cover. Furthermore, every element having 1 in at least one coordinate is the bottom of a chain in $\mathcal{D}$. The number of such elements is $\prod_{i=1}^{k} n_i - \prod_{i=1}^{k}(n_i - 1)$.

An antichain A contains elements from distinct chains in $\mathcal{D}$. If $|A| < |\mathcal{D}|$, then let C be a chain in $\mathcal{D}$ having no element of A. Each element of C must be comparable to some element of A, and no element above an element of A can lie below an element that is below an element of A. Hence C splits into two chains, with the portion C_1 whose elements are below some element of A lying below the remaining portion C_2. Since the top and bottom of C are maximal and minimal elements of S, both portions are nonempty.

Let x be the top element of C_1 and v be the bottom element of C_2. These elements are consecutive on C, so $v = (x_1 + 1, \ldots, x_d + 1)$. We also have $y, z \in A$ such that $x < y$ and $z < v$. However, this requires $z_i < v_i = x_i + 1 \leq y_i$ for all i, which implies $z < y$. This contradiction implies that some element of C can be added to A. Hence every maximal antichain intersects every chain in $\mathcal{D}$.

Also solved by W. J. Cowieson, S. Datta (India), Y. J. Ionin, O. P. Lossers (Netherlands), A. Pathak, R. Stong, L. Zhou, and the proposer.

A Lower Bound for an Infinite Product

11989 [2017, 563]. *Proposed by Spiros P. Andriopoulos, Third High School of Amaliada, Eleia, Greece.* Let x be a number between 0 and 1. Prove

$$\prod_{n=1}^{\infty}\left(1-x^n\right) \geq \exp\left(\frac{1}{2}-\frac{1}{2(1-x)^2}\right).$$

Solution I by Omran Kouba, Higher Institute for Applied Science and Technology, Damascas, Syria. For $0 < x < 1$,

$$\begin{aligned}
-\sum_{n=1}^{\infty}\ln(1-x^n) &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{x^{nk}}{k}\right) = \sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{n=1}^{\infty}(x^k)^n\right) = \sum_{k=1}^{\infty}\frac{1}{k}\,\frac{x^k}{1-x^k} \\
&= \frac{x}{1-x}\left(1+\sum_{k=2}^{\infty}\frac{x^{k-1}}{k(1+x+\cdots+x^{k-1})}\right) \leq \frac{x}{1-x}\left(1+\sum_{k=2}^{\infty}\frac{x^{k-1}}{2}\right) \\
&= \frac{x}{1-x}\left(1+\frac{x}{2(1-x)}\right) = -\frac{1}{2}+\frac{1}{2(1-x)^2}.
\end{aligned}$$

Change sign and exponentiate to get the desired inequality.

Composite solution II by Robin Chapman, University of Exeter, Exeter, U. K., and Aritro Pathak, student, Brandeis University, Waltham, MA. Taking the logarithm of each side of the desired inequality yields

$$\sum_{n=1}^{\infty}\log(1-x^n) \geq \frac{1}{2}-\frac{1}{2(1-x)^2}.$$

Since $1/(1-x)^2 = 1+2x+3x^2+\cdots$, this is equivalent to

$$-\sum_{n=1}^{\infty}\log(1-x^n) \leq \frac{1}{2}\sum_{m=1}^{\infty}(m+1)x^m.$$

Now

$$\sum_{n=1}^{\infty}\log(1-x^n) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{nk}}{k} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{nx^{nk}}{nk} = -\sum_{m=1}^{\infty}\frac{\sigma(m)}{m}x^m,$$

where $\sigma(m)$ is the sum of all the positive integer divisors of m. Certainly, $\sigma(m)$ is at most the sum of all the numbers from 1 to m, which is $m(m+1)/2$. We conclude that

$$-\sum_{n=1}^{\infty}\log(1-x^n) = \sum_{m=1}^{\infty}\frac{\sigma(m)x^m}{m} \leq \frac{1}{2}\sum_{m=1}^{\infty}(m+1)x^m,$$

as required.

Also solved by R. A. Agnew, M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, S. Chakravarty, W. J. Cowieson, P. P. Dályay (Hungary), H. Y. Far, G. Fera (Italy), P. J. Fitzsimmons, D. Fleischman, F. Franco (Italy), M. Goldenberg & M. Kaplan, L. Han, A. Hannan (India), S. Hwang (South Korea), S. Kaczkowski, K. T. L. Koo (China), W. Lai, K. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), L. Matejíčka (Slovakia), V. Mikayelyan (Armenia), R. Molinari, M. Omarjee (France), S. Pathak (Canada), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. Terr, A. V. Vaze (India), E. I. Verriest, M. Wildon (U. K.), V. P. Yellambalse (India), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Weitzenböck Revisited

11990 [2017, 564]. *Proposed by Nicuşor Minculete, Transilvania University of Braşov, Romania.* Let a, b, and c be the lengths of the sides of a triangle of area S. Weitzenböck's inequality states that $a^2+b^2+c^2 \geq 4\sqrt{3}S$. Prove the following stronger inequality:

$$a^2+b^2+c^2 \geq \sqrt{3}(4S+(c-a)^2).$$

Solution by Parviz Khalili, Newport News, VA. We prove the still stronger inequality

$$a^2+b^2+c^2 \geq 4\sqrt{3}S+2(a-c)^2,$$

which rearranges to $b^2-a^2-c^2+4ac \geq 4\sqrt{3}S$. From Heron's formula

$$S=\sqrt{\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{16}}=\sqrt{\frac{(a+c)^2-b^2}{4}\cdot\frac{b^2-(c-a)^2}{4}},$$

we obtain

$$4\sqrt{3}S=\sqrt{\big((a+c)^2-b^2\big)\big(3b^2-3(c-a)^2\big)}.$$

Applying the AM–GM inequality, we therefore have

$$4\sqrt{3}S \leq \frac{\big((a+c)^2-b^2\big)+\big(3b^2-3(c-a)^2\big)}{2}=b^2-a^2-c^2+4ac,$$

as desired.

Editorial comment. A proof of Weitzenböck's inequality that yields the stronger result given in this solution (and by most solvers) appears as item 4.4 on page 43 of Bottema et. al. (1968), *Geometric Inequalities*, Groningen: Walters–Noordhoff.

Also solved by A. Ali (India), D. Bailey & E. Campbell & C. Diminnie, H. Bailey, M. Bataille (France), R. Boukharfane (France), P. Bracken, D. Chakerian, R. Chapman (U. K.), O. Geupel (Germany), A. Hannan (India), F. Holland (Ireland), S. Kaczkowski, K. T. L. Koo (China), O. Kouba (Syria), G. Lord, O. P. Lossers (Netherlands), J. F. Loverde, D. Marinescu (Romania), J. Minkus, D. Moore, M. Reid, J. C. Smith, R. Stong, R. Tauraso (Italy), E. I. Verriest, M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), J. Zacharias, L. Zhou, and the proposer.

Matrices with the Same Range

11991 [2017, 564]. *Proposed by Yongge Tian, Central University of Finance and Economics, Beijing, China.* Given two complex n-by-n positive definite matrices A and B, let $C=(A+B)/2$ and $D=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$; the matrices C and D are the arithmetic mean and geometric mean of A and B. Prove $\operatorname{range}(C-D)=\operatorname{range}(A-B)$ and

$$\operatorname{range}\begin{bmatrix} C & D \\ D & C \end{bmatrix}=\operatorname{range}\begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

Solution by BSI Problem Solving Group, Bonn, Germany. Let R and S be n-by-n matrices. First note that if $\operatorname{range}(R)=\operatorname{range}(S)$, then $\operatorname{range}(PRQ)=\operatorname{range}(PSQ)$ for all n-by-n invertible matrices P and Q. Also, if $\operatorname{range}(PRQ)=\operatorname{range}(PSQ)$ for some n-by-n invertible matrices P and Q, then $\operatorname{range}(R)=\operatorname{range}(S)$. In view of this, to show $\operatorname{range}(C-D)=\operatorname{range}(A-B)$ we consider the equivalent statement $\operatorname{range}(P(C-D)Q)=\operatorname{range}(P(A-B)Q)$, where $P=Q=A^{-1/2}$. That is, the desired range identity will follow if $(I+E)/2-E^{1/2}$ and $I-E$ have the same range, where $E=A^{-1/2}BA^{-1/2}$.

Through unitary diagonalization, we may assume that E is a diagonal matrix with positive eigenvalues $\mu_1, \dots, \mu_n$; that is, $E = \operatorname{diag}(\mu_1, \dots, \mu_n)$. Thus,

$$\frac{I+E}{2} - E^{1/2} = \operatorname{diag}\left(\frac{1+\mu_1}{2} - \sqrt{\mu_1}, \dots, \frac{1+\mu_n}{2} - \sqrt{\mu_n}\right)$$

and $I - E = \operatorname{diag}(1-\mu_1, \dots, 1-\mu_n)$. For $1 \le i \le n$, we have $\frac{1+\mu_i}{2} - \sqrt{\mu_i} \neq 0$ if and only if $1 - \mu_i \neq 0$ (since $\mu_i > 0$), so the ranges of $C - D$ and $A - B$ are identical.

For the block matrices, we show that $P \begin{bmatrix} C & D \\ D & C \end{bmatrix} Q$ and $P \begin{bmatrix} A & B \\ B & A \end{bmatrix} Q$ have the same range, where $P = Q = \begin{bmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{bmatrix}$, which is equivalent to saying

$$\operatorname{range}\begin{bmatrix} (I+E)/2 & E^{1/2} \\ E^{1/2} & (I+E)/2 \end{bmatrix} = \operatorname{range}\begin{bmatrix} I & E \\ E & I \end{bmatrix}.$$

Again we may assume E to be the diagonal matrix $\operatorname{diag}(\mu_1, \dots, \mu_n)$. Applying simultaneous row and column permutations to the block matrices yields two block-diagonal matrices consisting respectively of n 2-by-2 blocks of the form $\begin{bmatrix} (1+\mu_i)/2 & \sqrt{\mu_i} \\ \sqrt{\mu_i} & (1+\mu_i)/2 \end{bmatrix}$ and $\begin{bmatrix} 1 & \mu_i \\ \mu_i & 1 \end{bmatrix}$, for $1 \le i \le n$. Since the ranges of these blocks are the same, the claim follows.

Also solved by R. Chapman (U. K.), E. A. Herman, O. P. Lossers (Netherlands), R. Stong, and the proposer.

An Entire Function with No Real Zeros

12000 [2017, 754]. *Proposed by Mehtaab Sawhney, student, Massachusetts Institute of Technology, Cambridge, MA.* Let $H_k = \sum_{i=1}^{k} 1/i$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n} H_k}$$

has no real zeroes.

Solution by Moubinool Omarjee, Lycée Herni IV, Paris, France. Let $u_n = 1/\prod_{k=1}^{n} H_k$. Since $\lim_{n\to\infty} |u_n/u_{n+1}| = \lim_n H_{n+1} = \infty$, the radius of convergence of the power series is infinite. Thus f is defined on $\mathbb{R}$ and is continuous. Suppose to the contrary that f has at least one real zero. Writing $a = \sup\{x \in \mathbb{R} \colon f(x) = 0\}$ and noting that $f(x) \ge 1$ for $x \ge 0$, we have $a < 0$, $f(a) = 0$, and $f(x) > 0$ for $a < x \le 0$. Therefore

$$\int_a^0 \frac{f(x) - f(a)}{x - a}\, dx = \int_a^0 \frac{f(x)}{x - a}\, dx > 0.$$

On the other hand, since a power series with infinite radius of convergence may be integrated term by term,

$$\begin{aligned} \int_a^0 \frac{f(x) - f(a)}{x - a}\, dx &= \int_a^0 \sum_{n=1}^{\infty} \frac{x^n - a^n}{x - a} \frac{dx}{\prod_{k=1}^{n} H_k} \\ &= \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} H_k} \int_a^0 (x^{n-1} + x^{n-2}a + \cdots + a^{n-1})\, dx \\ &= -\sum_{n=1}^{\infty} \frac{a^n H_n}{\prod_{k=1}^{n} H_k} = -af(a) = 0. \end{aligned}$$

This contradiction completes the proof that f has no real zeros.

Also solved by O. Kouba (Syria), R. Tauraso (Italy), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by August 31, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12104. *Proposed by Joe Buhler, Larry Carter, and Richard Stong, Center for Communications Research, San Diego, CA.* Consider a standard clock, where the hour, minute, and second hands all have integer lengths and all point straight up at noon and midnight. Is it possible for the ends of the hands to form, at some time, the vertices of an equilateral triangle?

12105. *Proposed by Gary Brookfield, California State University, Los Angeles, CA.* Let r be a real number, and let $f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r$. Suppose that f has real roots a, b, and c. Prove $a, b, c \in [-1, 1]$ and $|\arcsin a| + |\arcsin b| + |\arcsin c| = \pi$.

12106. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For any positive integer n, prove

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{1\le i\le j\le k} \frac{1}{ij} = \sum_{1\le i\le j\le n} \frac{2^n - 2^{n-i}}{ij}.$$

12107. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{1+x^2}\sqrt{1+y^2}(1-x^2y^2)}\,dx\,dy = G,$$

where G is Catalan's constant $\sum_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^2$.

12108. *Proposed by Yifei Pan and William D. Weakley, Purdue University Fort Wayne, Fort Wayne, IN.* Let n be a positive integer, and let $\beta_1, \dots, \beta_n$ be indeterminates over a field F. Let M be the n-by-n matrix whose i, j-entry m_{ij} is given by $m_{ij} = \beta_i$ when $i = j$ and $m_{ij} = 1$ when $i \ne j$. Show that the polynomial $\det(M)$ is irreducible over F.

12109. *Proposed by George Stoica, Saint John, NB, Canada.* Let f be a function on $[0, \infty)$ that is nonnegative, bounded, and continuous. For $a > 0$ and $x \ge 0$, let $g(x) = \exp\left(\int_0^a \log(1 + xf(s))\,ds\right)$. For $0 < p < 1$, prove

$$\int_0^a f^p(x)\,ds = \frac{p\sin(p\pi)}{\pi} \int_0^\infty \frac{\log g(x)}{x^{p+1}}\,dx.$$

doi.org/10.1080/00029890.2019.1574184

12110. *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* Let $\alpha_k = (k + \sqrt{k^2+4})/2$. Evaluate

$$\lim_{k\to\infty} \prod_{n=1}^{\infty} \left(1 - \frac{k}{\alpha_k^n + \alpha_k}\right).$$

SOLUTIONS

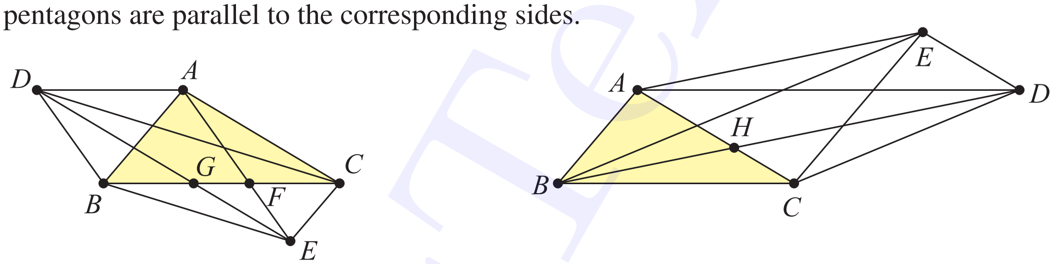
An Extremal Property of Affinely Regular Pentagons

11988 [2017, 563]. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle. Find the extrema of

$$\frac{AC^2 + CE^2 + EB^2 + BD^2 + DA^2}{AB^2 + BC^2 + CD^2 + DE^2 + EA^2}$$

over all points D and E in the plane of ABC. At which points D and E are these extrema attained?

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $\phi = (1+\sqrt{5})/2$. We show that the given expression attains its minimum value ϕ^{-2} and its maximum value ϕ^2 at the left and right configurations shown below, respectively, where all the diagonals of the pentagons are parallel to the corresponding sides.



First note that the law of cosines gives $CD^2 = DA^2 + AC^2 + 2\overrightarrow{DA}\cdot\overrightarrow{AC}$ and $EA^2 = AC^2 + CE^2 + 2\overrightarrow{AC}\cdot\overrightarrow{CE}$. Also, $\overrightarrow{DA} + \overrightarrow{CE} = -\overrightarrow{AC} - \overrightarrow{ED}$. Therefore, $CD^2 + EA^2 = DA^2 + CE^2 - 2\overrightarrow{AC}\cdot\overrightarrow{ED}$. Moreover, $-(AC)(DE) \le \overrightarrow{AC}\cdot\overrightarrow{ED} \le (AC)(DE)$. Equality holds in the left inequality if and only if $\overrightarrow{AC}$ and $\overrightarrow{ED}$ have opposite direction, while the equality holds in the right inequality if and only if they have the same direction. Hence

$$-2(AC)(DE) \le CD^2 + EA^2 - DA^2 - CE^2 \le 2(AC)(DE).$$

Adding to this the other four analogous inequalities, we get $-r^2 \le q^2 - p^2 \le r^2$, where p^2 and q^2 are respectively the numerator and denominator of the given expression, and

$$r^2 = (AC)(DE) + (CE)(AB) + (EB)(CD) + (BD)(EA) + (DA)(BC).$$

By the Cauchy–Schwarz inequality, we have $r^2 \le pq$, with equality if and only if

$$\frac{AC}{DE} = \frac{CE}{AB} = \frac{EB}{CD} = \frac{BD}{EA} = \frac{DA}{BC} = \lambda \qquad (*)$$

for some λ. Thus, $-pq \le q^2 - p^2 \le pq$, and so $\phi^{-1} \le p/q \le \phi$.

By $(*)$, $p/q = \phi^{-1}$ only if $\lambda = \phi^{-1}$, which leads to a construction by ruler and compass of D and E for minimal p/q: Locate F on BC such that $BF/FC = \phi$, and then construct D so that $BFAD$ is a parallelogram, and draw the line through D and parallel to AC to intersect AF at E. Let G be the intersection of DE and BC. Since $\triangle BDG$ is congruent to $\triangle FAC$, we have $BG = FC$. We have

$$\frac{GF}{FC} = \frac{BF}{FC} - \frac{BG}{FC} = \phi - 1 = \phi^{-1},$$

from which it follows that CE is parallel to AB, similarly EB is parallel to CD, and $(*)$ is satisfied.

Likewise, $p/q = \phi$ only if $\lambda = \phi$, which leads to a construction of D and E for maximal p/q: Locate H on AC such that $AH/HC = \phi$, and then draw the line through A and parallel BC to intersect line BH at D. Construct E so that $AHDE$ is a parallelogram. Arguing as before, CE is parallel to BA, BE is parallel to CD, and $(*)$ is satisfied.

Editorial comment. A pentagon of this type, where each diagonal is parallel to one of the sides, is affinely equivalent to a regular pentagon.

Also solved by G. Fera (Italy), O. Kouba (Syria), R. Stong, and the proposer.

Divisibility by an Arbitrary Power of Seven

11992 [2017, 659]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Prove that, for every positive integer n, there is a positive integer m such that $3^m + 5^m - 1$ is divisible by 7^n.

Solution by Sandeep Silwal, Brookline, MA. We show that $m = 7^{n-1}$ works. Let $v_p(k)$ denote the largest integer e such that p^e divides k. The lifting-the-exponent lemma states that if p is an odd prime dividing $x + y$ but dividing neither x nor y, then $v_p(x^n + y^n) = v_p(x + y) + v_p(n)$. Hence $v_7(5^m + 2^m) = v_7(5 + 2) + v_7(m) = n$ and similarly $v_7(3^m + 4^m) = n$. We conclude $5^m \equiv -2^m \pmod{7^n}$ and $3^m \equiv -2^{2m} \pmod{7^n}$, and therefore

$$3^m + 5^m - 1 \equiv -(2^{2m} + 2^m + 1) \pmod{7^n}.$$

Note that $(2^m - 1)(2^{2m} + 2^m + 1) = 8^m - 1$. By another application of the lifting-the-exponent lemma, $v_7(8^m - 1) = v_7(8 - 1) + v_7(m) = n$, and thus $8^m - 1 \equiv 0 \pmod{7^n}$. Because $m \equiv 1 \pmod 6$, Fermat's little theorem implies $2^m \equiv 2^1 = 2 \pmod 7$, so $2^m - 1$ is not divisible by 7. We conclude $2^{2m} + 2^m + 1 \equiv 0 \pmod{7^n}$, and hence $3^m + 5^m - 1 \equiv 0 \pmod{7^n}$, as desired.

Editorial comment. The lifting-the-exponent lemma can be found at brilliant.org/wiki/lifting-the-exponent/. Peter Lindstrom, O. P. Lossers, H. F. Mattson, and Michael Reid showed that setting $m = 5 \cdot 7^{n-1}$ also works. Boris Bekker & Yury Ionin, Stephen Gagola, and the BSI Problems Group showed that if a and b are the primitive 6th roots of unity modulo p, then $a^{p^{n-1}} + b^{p^{n-1}} - 1$ is divisible by p^n. Allen Stenger proved that if $p > 3$ is a prime, $n \geq 1$, $r = (p-1)/2$, $m = p^{n-1}$, and $a_1, \dots, a_r$ is the complete list of quadratic residues modulo p, then $\sum_{k=1}^{r} a_k^m \equiv 0 \pmod{p^n}$. Marian Tetiva showed that if $p > 3$ is a prime and a, b, c are integers such that both $a + b + c$ and $ab + ac + bc$ are divisible by p, then both $a^{p^n} + b^{p^n} + c^{p^n}$ and $a^{p^n}b^{p^n} + a^{p^n}c^{p^n} + b^{p^n}c^{p^n}$ are divisible by p^{n+1}.

Also solved by B. M. Bekker & Y. J. Ionin, R. Boukharfane (France), R. Chapman (U. K.), J. Christopher, S. M. Gagola, Jr., M. Goldenberg & M. Kaplan, R. A. Gordon, J. Iiams, E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), H. F. Mattson, U. Milutinović (Slovenia), V. I. Murashka (Belarus), M. Omarjee (France), C. R. Pranesachar (India), M. Reid, N. C. Singer, J. Singh (India), O. Sonebi (France), A. Stadler (Switzerland), A. Stenger, R. Stong, M. Tang, R. Tauraso (Italy), M. Tetiva (Romania), J. Van hamme (Belgium), Z. Vörös (Hungary), L. Wimmer, L. Zhou, BSI Problem Solving Group (Germany), GCHQ Problem Solving Group (U. K.) Northwestern University Problem Solving Group, and the proposer.

An Integral Related to Euler Sums

11993 [2017, 659]. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Prove

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = -\frac{\pi^4}{240}.$$

Solution by Abdelhak Berkane, University of Mentouri Brothers, Constantine, Algeria. We use the equality $ab^2 = \left((a+b)^3 + (a-b)^3 - 2a^3\right)/6$ with $a = \log(1-x)$ and $b = \log(1+x)$. From this we obtain

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = \frac{1}{6}I_1 + \frac{1}{6}I_2 - \frac{1}{3}I_3,$$

where

$$\begin{aligned} I_1 &= \int_0^1 \frac{(\log(1-x)+\log(1+x))^3}{x}dx = \int_0^1 \frac{(\log(1-x^2))^3}{x}dx \\ &= \frac{1}{2}\int_0^1 \frac{(\log(1-u))^3}{u}du = \frac{1}{2}\int_0^1 \frac{(\log t)^3}{1-t}dt, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 \frac{(\log(1-x)-\log(1+x))^3}{x}dx = \int_0^1 \frac{\left(\log\left(\frac{1-x}{1+x}\right)\right)^3}{x}dx \\ &= 2\int_0^1 \frac{(\log t)^3}{(1-t)(1+t)}dt = \int_0^1 \frac{(\log t)^3}{1-t}dt + \int_0^1 \frac{(\log t)^3}{1+t}dt, \end{aligned}$$

and

$$I_3 = \int_0^1 \frac{(\log(1-x))^3}{x}dx = \int_0^1 \frac{(\log t)^3}{1-t}dt.$$

Combining these yields

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = -\frac{1}{12}\int_0^1 \frac{(\log t)^3}{1-t}dt + \frac{1}{6}\int_0^1 \frac{(\log t)^3}{1+t}dt.$$

It is known (see, for example, entries 4.626.1 and 4.626.2 in I. S. Gradshteyn, I. M. Ryzhik, et al. (2015), *Tables of Integrals, Series, and Products*, 8th ed., San Diego, CA: Academic Press) that

$$\int_0^1 \frac{(\log t)^3}{1-t}dt = -\frac{\pi^4}{15} \qquad \text{and} \qquad \int_0^1 \frac{(\log t)^3}{1+t}dt = -\frac{7\pi^4}{120}.$$

We conclude

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = \frac{\pi^4}{180} - \frac{7\pi^4}{720} = -\frac{\pi^4}{240}.$$

Editorial comment. This integral was previously given by P. J. de Doelder (1991), On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y, *J. Comput. Appl. Math.* 37(1–3): 125–141.

As many solvers noted, this integral is closely related to Euler sums. Expanding the logarithms in power series, one sees that the requested integral is

$$I = \sum_{n=0}^{\infty}(-1)^n \frac{H_n H_{n+1}}{(n+1)^2} = \sum_{n=0}^{\infty}(-1)^n \frac{H_n^2}{(n+1)^2} + \sum_{n=0}^{\infty}(-1)^n \frac{H_n}{(n+1)^3},$$

where $H_n = \sum_{k=1}^{n} 1/k$. The sum in the first expression was evaluated in W. Chu (1997), Hypergeometric series and the Riemann zeta function, *Acta Arith.* 82(2): 103–118. The two sums in the final expression are evaluated in D. H. Bailey, J. M. Borwein, and R. Girgensohn, Experimental evaluation of Euler sums (1994), *Exper. Math.* 3(1): 17–30, which gives the first sum as

$$a_h(2,2) = -2\mathrm{Li}_4(1/2) - \frac{1}{12}\log^4 2 + \frac{99}{48}\zeta(4) - \frac{7}{4}\zeta(3)\log 2 + \frac{1}{2}\zeta(2)\log^2 2,$$

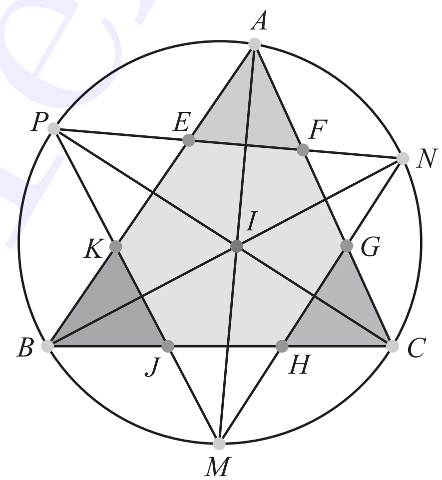
and in D. Borwein, J. M. Borwein, and R. Girgensohn (1995), Explicit evaluation of Euler sums, *Proc. Edin. Math. Soc. (2)*. 38(2): 277–294, which gives a nearly cancelling formula for the second sum $\alpha_h(1,3)$.

Also solved by P. Acosta, K. F. Andersen (Canada), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), R. Boukharfane (France), K. N. Boyadzhiev, P. Bracken, H. Chen, V. Dassios (Greece), B. E. Davis, G. Fera (Italy), M. L. Glasser, A. Hannan (India), O. Kouba (Syria), K. Lau (China), L. Matejíčka (Slovakia), V. Mikayelyan (Armenia), M. Omarjee (France), P. Perfetti (Italy), R. Schumacher (Switzerland), S. Sharma (India), S. Silwal, J. Singh (India), J. C. Smith, A. Stadler (Switzerland), M. Stofka (Slovakia), R. Tauraso (Italy), J. Van Casteren & L. Kempeneers (Belgium), M. R. Yegan (Iran), and the proposer.

A Hexagram Inequality

11994 [2017, 659]. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with incenter I and circumcircle ω. Let M, N, and P be the second points of intersection of ω with lines AI, BI, and CI, respectively. Let E and F be the points of intersection of NP with AB and AC, respectively. Similarly, let G and H be the points of intersection of MN with AC and BC, respectively, and let J and K be the points of intersection of MP with BC and AB, respectively. Prove

$$EF + GH + JK \le KE + FG + HJ.$$



Solution by Li Zhou, Polk State College, Winter Haven, FL. As usual, we let A, B, and C denote the angles of $\triangle ABC$. Since $\angle NPA$ and $\angle NBA$ are subtended by the same arc of ω and BN bisects $\angle ABC$, we have $\angle NPA = \angle NBA = B/2$. Similarly, $\angle PAB = C/2$. Since $\angle FEA$ is an exterior angle of $\triangle APE$, we have

$$\angle FEA = \angle NPA + \angle PAB = (B+C)/2,$$

and a similar argument shows that $\angle EFA = (B+C)/2$. Therefore $\triangle AEF$ is isosceles and AI bisects EF perpendicularly. Let Q be the intersection point of AI and EF.

We have $\angle PEK = \angle FEA = (B+C)/2$, and similarly $\angle PKE = (A+C)/2$. Since these are both acute angles, the perpendicular from P to KE hits KE at a point R that is strictly between E and K. By the similarity of $\triangle AQE$ and $\triangle PRE$ and the law of sines in $\triangle APE$,

$$\frac{2RE}{EF} = \frac{RE}{EQ} = \frac{PE}{EA} = \frac{\sin(\angle PAB)}{\sin(\angle NPA)} = \frac{\sin(C/2)}{\sin(B/2)}.$$

Likewise, $2KR/JK = \sin(C/2)/\sin(A/2)$. Hence,

$$KE = KR + RE = \frac{JK\sin(C/2)}{2\sin(A/2)} + \frac{EF\sin(C/2)}{2\sin(B/2)}.$$

Similarly,

$$FG = \frac{EF\sin(B/2)}{2\sin(C/2)} + \frac{GH\sin(B/2)}{2\sin(A/2)} \quad \text{and} \quad HJ = \frac{GH\sin(A/2)}{2\sin(B/2)} + \frac{JK\sin(A/2)}{2\sin(C/2)}.$$

Adding these three equations and invoking the AM–GM inequality yields

$$\begin{aligned} KE + FG + HJ &= EF\left(\frac{\sin(C/2)}{2\sin(B/2)} + \frac{\sin(B/2)}{2\sin(C/2)}\right) \\ &\quad + GH\left(\frac{\sin(B/2)}{2\sin(A/2)} + \frac{\sin(A/2)}{2\sin(B/2)}\right) + JK\left(\frac{\sin(C/2)}{2\sin(A/2)} + \frac{\sin(A/2)}{2\sin(C/2)}\right) \\ &\geq EF + GH + JK. \end{aligned}$$

Also solved by M. Bataille (France), R. Boukharfane (France), N. G. Cripe, G. Fera (Italy) O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. G. Heuver (Canada), O. Kouba (Syria), P. McPolin (U. K.), P. Nüesch (Switzerland), M. Omarjee (France), C. R. Pranesachar (India), C. Schacht, R. Stong, R. Tauraso (Italy), T. Toyonari (Japan), T. Wiandt, T. Zvonaru & N. Stanciu (Romania), and the proposer.

A Sequence Generated by Averaging Sines

11995 [2017, 659]. *Proposed by Dan Ştefan Marinescu, National College "Iancu de Hunedoara," Hunedoara, Romania, and Mihai Monea, National College "Decebal," Deva, Romania.* Suppose $0 < x_0 < \pi$, and for $n \geq 1$ define $x_n = (1/n)\sum_{k=0}^{n-1}\sin x_k$. Find $\lim_{n\to\infty} x_n\sqrt{\ln n}$.

Solution by Florin Stanescu, Gaesti, Romania. Since $\sin x_k \leq 1$, we have $x_n \leq 1$. It follows by induction that $x_n > 0$ for all n. Thus x_n is bounded. From $nx_n = \sum_{k=0}^{n-1}\sin x_k$ and $(n+1)x_{n+1} = \sum_{k=0}^{n}\sin x_k$, we obtain

$$(n+1)x_{n+1} - nx_n = \sin x_n, \tag{$*$}$$

and hence

$$x_n - x_{n+1} = (x_n - \sin x_n)/(n+1) > 0.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is decreasing and hence convergent. Let $l = \lim_{n\to\infty} x_n$. Applying the Stolz–Cesaro theorem, we obtain

$$\begin{aligned} l = \lim_{n\to\infty} x_n &= \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\sin x_k}{n} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n}\sin x_k - \sum_{k=0}^{n-1}\sin x_k}{n+1-n} \\ &= \lim_{n\to\infty}\sin x_n = \sin l. \end{aligned}$$

Thus $l = 0$, since this is the only solution to $l = \sin l$. The recurrence $(*)$ may be rewritten

$$\frac{x_{n+1}}{x_n} = \frac{n}{n+1} + \frac{\sin x_n}{(n+1)x_n}.$$

Noting that $\lim_{n\to\infty}(\sin x_n)/x_n = 1$, we see that $\lim_{n\to\infty} x_{n+1}/x_n = 1$. Using the Stolz–Cesaro theorem again we calculate

$$\lim_{n\to\infty} x_n^2 \ln n = \lim_{n\to\infty} \frac{\ln n}{1/x_n^2} = \lim_{n\to\infty} \frac{\ln(n+1) - \ln n}{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right) x_{n+1}^2 x_n^2}{(x_n - x_{n+1})(x_n + x_{n+1})}$$

$$= \lim_{n\to\infty} \frac{(n+1)\ln\left(1+\frac{1}{n}\right) x_{n+1}^2 x_n^2}{(x_n - \sin x_n)(x_n + x_{n+1})}$$

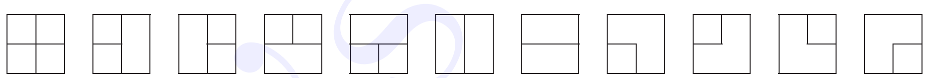
$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{x_n^3}{x_n - \sin x_n} \cdot \frac{\ln\left(1+\frac{1}{n}\right)^n}{\frac{x_n}{x_{n+1}} + \left(\frac{x_n}{x_{n+1}}\right)^2} = 1 \cdot 6 \cdot \frac{1}{2} = 3,$$

where we have used $\lim_{x\to 0}(x - \sin x)/x^3 = 1/6$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Hence $\lim_{n\to\infty} x_n\sqrt{\ln n} = \sqrt{3}$.

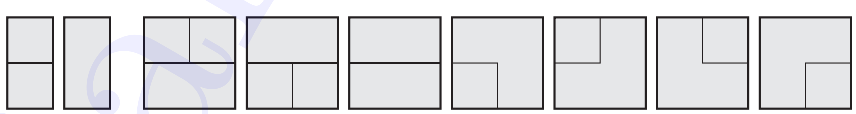
Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), R. Chapman (U. K.), G. Fera (Italy), E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy) & M. Omarjee (France), M. Tetiva (Romania), D. B. Tyler, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Tilings of a Strip

11996 [2017, 659]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Consider all the tilings of a 2-by-n rectangle comprised of tiles that are either a unit square, a domino, or a right tromino. Let f_n be the fraction of tiles among all such tilings that are unit squares. For example, $f_2 = 4/7$, because 16 out of the 28 tiles in the 11 tilings of a 2-by-2 rectangle are squares. What is $\lim_{n\to\infty} f_n$?



Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. The answer is $(30 - 4\sqrt{5})/41$, which is approximately $0.513554\ldots$. A *primitive* tiling of a strip is a tiling that cannot be split into tilings of two shorter strips. Let s_n be the number of primitive tilings of a 2-by-n strip. We have $s_1 = 2$, $s_2 = 7$, and $s_n = 8$ for $n \ge 3$. The last is because there are two cases of trominoes at both ends, four cases of a tromino at only one end, and two cases of no trominoes.



Let a_n be the number of all tilings of a 2-by-n strip. We have $a_0 = 1$ and $a_n = \sum_{i=1}^{n} s_i a_{n-i}$ for $n \ge 1$. Subtracting the expression for a_n from that for a_{n+1}, we obtain

$$a_{n+1} - 3a_n - 5a_{n-1} - a_{n-2} = 0.$$

Let x_n be the total number of squares in all of the 2-by-n tilings. Let p_n be the number of squares in the primitive tilings, so $p_1 = 2$ and $p_n = 8$ for $n \ge 2$. We obtain $x_n =$

$\sum_{i=1}^{n}(p_i a_{n-i} + s_i x_{n-i})$, which arises by letting i be the least index so that the initial 2-by-i subtiling is primitive, with the first term counting the squares in the first i positions and the second term counting the squares in the last $n-i$ positions. Subtracting the expression for x_n from that for x_{n+1} yields

$$x_{n+1} - 3x_n - 5x_{n-1} - x_{n-2} = 2a_n + 6a_{n-1}.$$

Let z_n be the total number of trominoes in 2-by-n tilings. We similarly obtain

$$z_{n+1} - 3z_n - 5z_{n-1} - z_{n-2} = 4a_{n-1} + 4a_{n-2}.$$

Let y_n be the total number of dominoes. Using $x_n + 2y_n + 3z_n = 2na_n$, we get

$$y_{n+1} - 3y_n - 5y_{n-1} - y_{n-2} = 2a_n + a_{n-1} - 3a_{n-2}.$$

Let t_n be the total number of tiles in all the tilings. Since $t_n = x_n + y_n + z_n$,

$$t_{n+1} - 3t_n - 5t_{n-1} - t_{n-2} = 4a_n + 11a_{n-1} + a_{n-2}.$$

The general solution of the recurrence for a_n is

$$A\lambda^n + B\mu^n + C\nu^n,$$

where λ, μ, and ν are the zeros of $x^3 - 3x^2 - 5x - 1$. Take λ to be $2+\sqrt{5}$, the largest root. Since the characteristic polynomials in the recurrences for x_n and t_n are the same as for a_n, and since their nonhomogeneous parts satisfy the same homogeneous recurrence (by definition), the general solutions for x_n and t_n have the form

$$(nA_1 + A_0)\,\lambda^n + (nB_1 + B_0)\,\mu^n + (nC_1 + C_0)\,\nu^n.$$

One can generate six initial values for a_n, x_n, and t_n using the recurrences. They are $(1, 2, 11, 44, 189, 798)$, $(0, 2, 16, 92, 512, 2654)$, and $(0, 3, 28, 66, 940, 4929)$, respectively. Solving 6×6 systems of linear equations then gives $A_1 = (5+\sqrt{5})/20$ in the solution for x_n and $A_1 = (17+5\sqrt{5})/40$ in the solution for t_n. The desired limiting ratio is the ratio of these two coefficients, which is $(30-4\sqrt{5})/41$.

Editorial comment. The proposer found the following closed-form expressions, with F_n being the nth Fibonacci number:

$$\begin{aligned} a_n &= \left(F_{3n+2} + (-1)^n\right)\big/\,2 \qquad \text{(see oeis.org/A110679)};\\ x_n &= F_{3n-1} + (-1)^n(n-1);\\ t_n &= \tfrac{1}{20}\left((17n+10)F_{3n+1} + (4n-12)F_{3n} + (15n-10)(-1)^n\right). \end{aligned}$$

Also solved by S. B. Ekhad, G. Fera (Italy), P. Lalonde (Canada), P. McPolin (U. K.), R. Molinari, R. Nandan, R. Pratt, R. Stong, and the proposer.

A Vanishing Sum

11997 [2017, 660]. *Proposed by Michael Drmota, Technical University of Vienna, Vienna, Austria; Christian Krattenthaler, University of Vienna, Vienna, Austria; and Gleb Pogudin, Johannes Kepler University, Linz, Austria.* Assume $|p| < 1$ and $pz \neq 0$. With $f(z) = \left(e^{(p-1)z} - e^{-z}\right)/(pz)$, define $f^*(z) = \prod_{k=0}^{\infty} f(p^k z)$, and then define $F_n(p)$ so that $f^*(z) = \sum_{n=0}^{\infty} F_n(p) z^n$. Prove the identity

$$\sum_{n=0}^{\infty} F_n(p)\, p^{\binom{n}{2}} = 0.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands). More generally, we claim that the identity holds for functions $f(z)$ with $f(0) = 1$ that have the form

$$f(z) = \frac{b(pz) - 1}{pz\, b(z)}$$

with $b(z) = \sum_{k=0}^{\infty} b_k z^k$ and $b_0 = b_1 = 1$, provided that all infinite sums and products converge for $|z|$ and $|p|$ sufficiently small. Here the numbers b_k may depend on p.

Consider $f(z) = a(z)/b(z)$ with $a(z) = \sum_{k=0}^{\infty} a_k z^k$ and $b(z) = \sum_{k=0}^{\infty} b_k z^k$, where $a_0 = b_0 = 1$ and in general a_k and b_k may be functions of p. With $f^*(z) = \prod_{k=0}^{\infty} f(p^k z)$, define $F_n(p)$ by $f^*(z) = \sum_{n=0}^{\infty} F_n(p) z^n$. Since $f^*(z) = f(z) f^*(pz)$, we have $b(z) f^*(z) = a(z) f^*(pz)$, and hence

$$\left(\sum_{k=0}^{\infty} b_k z^k\right)\left(\sum_{n=0}^{\infty} F_n(p) z^n\right) = \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{n=0}^{\infty} F_n(p) p^n z^n\right).$$

Comparing the coefficients of z^m on both sides of the above expression shows

$$\sum_{n=0}^{m} F_n(p)(b_{m-n} - a_{m-n} p^n) = 0 \tag{1}$$

for $m \geq 0$. Let $(c_m(p))_{m=0}^{\infty}$ be any sequence. Multiply (1) by c_m and sum over m, then interchange the summation order and rescale to obtain $\sum_{n=0}^{\infty} F_n(p) C_n(p) = 0$, where

$$C_n(p) = \sum_{k=0}^{\infty} c_{n+k}(b_k - a_k p^n). \tag{2}$$

If $c_{n+k+1} b_{k+1} = c_{n+k} a_k p^n$ for all n and k with $n, k \geq 0$, then the sum in (2) telescopes to yield $C_n(p) = c_n$. This happens if and only if

$$\frac{a_k}{b_{k+1}} = \frac{a_{k-1} p}{b_k} = \frac{c_{n+k+1}}{c_{n+k}} p^{-n}$$

for all n and k with $n \geq 0$ and $k \geq 1$, in which case

$$a_n = p^n b_{n+1}/b_1 \qquad \text{and} \qquad C_n(p) = c_n = p^{\binom{n}{2}}/b_1^n. \tag{3}$$

For the convergence of the telescoping sums we require $\lim_{k\to\infty} c_{n+k} a_k p^n = 0$. Using (3) above (and recalling $b_1 = 1$), we obtain

$$\lim_{k\to\infty} c_{n+k} a_k p^n = \lim_{k\to\infty} p^{\binom{n+k}{2}} p^k b_{k+1} p^n.$$

Since we assumed in defining $b(z)$ that its sum converges for suitably small z and p, it follows that $\lim_{k\to\infty} b_{k+1} z^{k+1} = 0$ and hence also $\lim_{k\to\infty} b_{k+1} p^{k+1} = 0$. Therefore $\lim_{k\to\infty} c_{n+k} a_k p^n = 0$ and the telescoping sum for $C_n(p)$ converges.

Finally, note that the first part of (3) is equivalent to $a(z) = (b(pz) - 1)(pzb_1)$. The identity in the problem results from the case $b(z) = e^z = \sum_{k=0}^{\infty} z^n/n!$, where $b_0 = b_1 = 1$ and $a(z) = (e^{pz} - 1)/(pz)$.

Also solved by P. Lalonde (Canada) and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit
Proposed solutions to the problems below should be submitted by September 30, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12111. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* A line segment AB can be oriented in two ways, which we denote (AB) and (BA). A square $ABCD$ can be oriented in two ways, which we denote $(ABCD)$ (the same as $(BCDA)$, $(CDAB)$, and $(DABC)$) and $(DCBA)$ (the same as $(CBAD)$, $(BADC)$, and $(ADCB)$). We say that the orientation $(ABCD)$ of a square *agrees with* the orientations (AB), (BC), (CD), and (DA) of its sides. Suppose that each edge and 2-dimensional face of an n-dimensional cube is given an orientation.
(a) What is the largest possible number of 2-dimensional faces whose orientation agrees with the orientations of its four sides?
(b) What is the largest possible number of edges whose orientation agrees with the orientations of all 2-dimensional faces containing the edge?

12112. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam.* Let ABC be a triangle with circumcenter O and nine-point center N. Let P be a point on its circumcircle and let D, E, and F be the circumcenters of triangles AOP, BOP, and COP, respectively. Let A', B', and C' be the feet of perpendiculars from D, E, and F onto the lines BC, CA, and AB, respectively. Prove that A', B', C', and N are collinear.

12113. *Proposed by Richard P. Stanley, University of Miami, Coral Gables, FL.* Define $f(n)$ and $g(n)$ for $n \geq 0$ by

$$\sum_{n\geq 0} f(n)x^n = \sum_{j\geq 0} x^{2^j} \prod_{k=0}^{j-1}\left(1 + x^{2^k} + x^{3\cdot 2^k}\right)$$

and

$$\sum_{n\geq 0} g(n)x^n = \prod_{i\geq 0}\left(1 + x^{2^i} + x^{3\cdot 2^i}\right).$$

Find all values of n for which $f(n) = g(n)$, and find $f(n)$ for these values.

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12114. *Proposed by Zachary Franco, Houston, TX.* Let n be a positive integer, and let $A_n = \{1/n, 2/n, \ldots, n/n\}$. Let a_n be the sum of the numerators in A_n when these fractions are expressed in lowest terms. For example, $A_6 = \{1/6, 1/3, 1/2, 2/3, 5/6, 1/1\}$, so $a_6 = 1 + 1 + 1 + 2 + 5 + 1 = 11$. Find $\sum_{n=1}^{\infty} a_n/n^4$.

12115. *Proposed by Marius Drăgan, Bucharest, Romania.* Let a, b, c, and d be positive real numbers. Prove

$$(a^3 + b^3)(a^3 + c^3)(a^3 + d^3)(b^3 + c^3)(b^3 + d^3)(c^3 + d^3) \\ \geq (a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2)^3.$$

12116. *Proposed by Rishubh Thaper, Fleminton, NJ.* In a round-robin tournament with n players, each player plays every other player exactly once, and each match results in a win for one player and a loss for the other. When player A defeats player B, we call B the *victim* of A. At the end of the tournament, each player computes the total number of losses incurred by the player's victims. Let q be the average of this quantity over all players. Prove that there exists a player with at most $\lfloor \sqrt{q} \rfloor$ wins and a player with at most $\lfloor \sqrt{q} \rfloor$ losses.

12117. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Prove

$$\frac{\sin^{n+1}(4\pi/7)}{\sin^{n+2}(\pi/7)} - \frac{\sin^{n+1}(\pi/7)}{\sin^{n+2}(2\pi/7)} + (-1)^n \frac{\sin^{n+1}(2\pi/7)}{\sin^{n+2}(4\pi/7)} \\ = 2\sqrt{7} \sum \frac{(i + j + k)!}{i!\, j!\, k!} (-1)^{n-i} 2^i,$$

where the sum is taken over all triples (i, j, k) of nonnegative integers satisfying $i + 2j + 3k = n$.

SOLUTIONS

A Trigonometric Functional Equation

11998 [2017, 660]. *Proposed by Roger Cuculière, Lycée Pasteur, Neuilly-sur-Seine, France.* Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy $f(z) \leq 1$ for some nonzero real number z and

$$f(x)^2 + f(y)^2 + f(x + y)^2 - 2f(x)f(y)f(x + y) = 1$$

for all real numbers x and y.

Solution by FAU Problem Solving Group, Florida Atlantic University, Boca Raton, FL. The solutions are the constant functions $f(x) = 1$ and $f(x) = -1/2$ and the functions $f(x) = \cos \alpha x$ for $\alpha > 0$. It is easy to see that all these functions satisfy the requirements.

Conversely, suppose that f is a continuous function satisfying the functional equation, $f(z) \leq 1$ for some nonzero z, and $f(x)$ is not identically 1 or $-1/2$. We first prove $f(0) = 1$. Let $c = f(0)$. Setting $x = y = 0$ in the functional equation yields $3c^2 - 2c^3 = 1$, an equation with a double root of 1 and a simple root of $-1/2$. Setting $y = 0$ in the functional equation yields $2(1 - c)f(x)^2 = 1 - c^2$ for all x. If $c = -1/2$, then this equation implies $f(x)^2 = 1/4$ for all x; by continuity and since $f(0) = c = -1/2$, we conclude $f(x) = -1/2$ for all x, a contradiction. Thus, $c = 1$; that is, $f(0) = 1$.

Next, setting $x = y$ yields

$$2f(x)^2 + f(2x)^2 - 2f(x)^2 f(2x) = 1,$$

or

$$(f(2x) - 1)(f(2x) - 2f(x)^2 + 1) = 0.$$

Hence

$$f(2x) = 1 \quad \text{or} \quad f(2x) = 2f(x)^2 - 1 \tag{1}$$

for all $x \in \mathbb{R}$.

We claim $f(y) = 1$ for some nonzero y. Suppose otherwise. By (1), $f(2x) = 2f(x)^2 - 1$, when $x \neq 0$. If $f(x) = 0$ for some nonzero x, then $f(2x) = -1$ and $f(4x) = 1$, which is a contradiction. Since $f(0) = 1$, we conclude $f(x) > 0$ for all x. By assumption there is some nonzero z such that $f(z) \leq 1$, and therefore $0 < f(z) < 1$. Letting $\epsilon = 1 - f(z)$, we have $0 < \epsilon < 1$ and

$$f(2z) = 2(1-\epsilon)^2 - 1 = 1 - 2\epsilon(2-\epsilon) \leq 1 - 2\epsilon.$$

By induction, $f(2^n z) \leq 1 - 2^n\epsilon$, implying $f(2^n z) \leq 0$ if n is large enough, a contradiction that establishes the claim.

Assume now $y \neq 0$ and $f(y) = 1$. Applying the functional equation, we get

$$f(x)^2 + f(x+y)^2 - 2f(x)f(x+y) = 0.$$

Thus $f(x+y) = f(x)$ for all real x, so y is a period of f. Since f is not identically 1, it has a minimum period T. Similarly, if $f(x) = 1$, then x is a period of f, and hence $x = kT$ for some $k \in \mathbb{Z}$. Therefore the second alternative of (1) holds for all $x \notin (T/2)\mathbb{Z}$. It follows by continuity that it holds for all x. Thus

$$f(2x) = 2f(x)^2 - 1 \tag{2}$$

for all $x \in \mathbb{R}$.

To conclude, we prove $f(x) = \cos(2\pi x/T)$. Since it is clear that f satisfies the functional equation if and only if $x \mapsto f(\alpha x)$ also satisfies it (where $\alpha > 0$ is a constant), it suffices to prove $f(x) = \cos x$ when $T = 2\pi$. Using (2), from $f(2\pi) = 1$ we get $f(\pi)^2 = 1$, and thus $f(\pi) = -1$ (since π is not a period). Using (2) again we get $f(\pi/2) = 0$. Moreover, if $0 < x < \pi/2$, then $f(x) \neq 1$, and if $f(x) = 0$, then using (2) yields $f(4x) = 1$ and $0 < 4x < 2\pi$, a contradiction. Since $f(0) = 1$ and $f(\pi/2) = 0$, we conclude $0 < f(x) < 1$ for $0 < x < \pi/2$. Since we have proved $0 \leq f(x) \leq 1$ when $x \in [0, \pi/2]$, equation (2) implies $|f(x)| \leq 1$ first for all $x \in [0, \pi]$, then for all $x \in [0, 2\pi]$, and finally by periodicity for all $x \in \mathbb{R}$.

From (2) and induction on n, since both $f(x)$ and $\cos x$ are nonnegative in $[0, \pi/2]$, we see that $f(x) = \cos x$ when $x = \pi/2^n$ for $n \in \mathbb{N}$. Next, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = \sqrt{1 - f(x)^2}$. Solving the functional equation for $f(x+y)$, we find

$$f(x+y) = f(x)f(y) \pm g(x)g(y).$$

To decide which sign applies when $0 < x, y < \pi/2$, let $Q = (0, \pi/2) \times (0, \pi/2)$, and let

$$A = \{(x, y) \in Q\colon\ f(x+y) = f(x)f(y) - g(x)g(y)\}.$$

The set A is clearly closed in Q. It is also open in Q; in fact, its complement is the closed set $\{(x, y) \in Q : f(x+y) = f(x)f(y) + g(x)g(y)\}$. Since Q is connected, either $A = Q$ or A is empty. Now $f(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, and hence $g(\pi/4) = 1/\sqrt{2}$, so

$$f(\pi/2) = 0 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = f(\pi/4)f(\pi/4) - g(\pi/4)g(\pi/4).$$

Thus $(\pi/4, \pi/4) \in A$, implying A is not empty, so $A = Q$ and

$$f(x+y) = f(x)f(y) - g(x)g(y) \tag{3}$$

for all $(x, y) \in Q$. Since f, g, cosine, and sine are all positive in $(0, \pi/2)$, it follows that at points $(x, y) \in Q$ such that $f(x) = \cos x$ and $f(y) = \cos y$, we also have $g(x) = \sin x$, $g(y) = \sin y$, and $f(x+y) = \cos(x+y)$. Having proved that $f(\pi/2^n) = \cos(\pi/2^n)$ for $n \in \mathbb{N}$, we can thus conclude that $f(x) = \cos x$ for all points $x = m\pi/2^n$, where $n \in \mathbb{N}$ and $m = 0, \ldots, 2^{n-1}$. Since these points are dense in $[0, \pi/2]$ and since f is continuous, we have established that $f(x) = \cos x$ for $0 \le x \le \pi/2$. It follows from (2) that we also have $f(x) = \cos x$ in $[0, \pi]$, then in $[0, 2\pi]$, and finally for all $x \in \mathbb{R}$.

Editorial comment. Several solvers pointed out that if we drop the condition that $f(z) \le 1$ for some nonzero z, then we get the additional solutions $f(x) = \cosh \alpha x$ for $\alpha > 0$.

Also solved by R. Chapman (U. K.), R. Ger (Poland), J. W. Hagood, E. A. Herman, E. J. Ionaşcu, Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), O. P. Lossers (Netherlands), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

A Variation on Euler's Formula for Pi

11999 [2017, 754]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)}.$$

Solution by Russell A. Gordon, Whitman College, Walla Walla, WA. The value is $\pi^2/3 - 3$.

First, we compute $\lfloor \sqrt{k} + \sqrt{k+1} \rfloor$ for $k \in \mathbb{N}$. Let $n = \lfloor \sqrt{k} \rfloor$, so $n^2 \le k < (n+1)^2$. We prove

$$\left\lfloor \sqrt{k} + \sqrt{k+1} \right\rfloor = \begin{cases} 2n, & \text{when } n^2 \le k \le n^2 + n - 1; \\ 2n+1, & \text{when } n^2 + n \le k \le n^2 + 2n. \end{cases} \tag{$*$}$$

It is immediate that $\sqrt{k} + \sqrt{k+1}$ is either $2n$ or $2n+1$. If $k \le n^2 + n - 1$, then

$$\sqrt{k} + \sqrt{k+1} \le \sqrt{n^2+n-1} + \sqrt{n^2+n} < 2\sqrt{n^2+n} < \sqrt{4n^2+4n+1} = 2n+1.$$

If $k \ge n^2 + n$, then

$$\begin{aligned} \sqrt{k} + \sqrt{k+1} &\ge \sqrt{n^2+n} + \sqrt{n^2+n+1} \\ &= \sqrt{n^2+n+2\sqrt{(n^2+n)(n^2+n+1)} + n^2+n+1} \\ &> \sqrt{4(n^2+n)+1} = 2n+1. \end{aligned}$$

This yields $(*)$.

The given series is absolutely convergent, since the series comprised of the absolute values of its terms is dominated by $\sum_{k=1}^{\infty} 1/k^2$. Hence rearrangements and regroupings do not affect the sum. Also, we note the simplifying presence of telescoping sums:

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \sum_{k=m}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=m}^{n} \frac{1}{k} - \sum_{k=m+1}^{n+1} \frac{1}{k} = \frac{1}{m} - \frac{1}{n+1}.$$

Finally, recall Euler's famous formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. Putting these facts together, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)} &= \sum_{n=1}^{\infty} \left(\sum_{k=n^2}^{n^2+n-1} \frac{1}{k(k+1)} - \sum_{k=n^2+n}^{n^2+2n} \frac{1}{k(k+1)} \right) \\
&= \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} - \frac{1}{n^2+n} \right) - \left(\frac{1}{n^2+n} - \frac{1}{(n+1)^2} \right) \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \; = \; 2 \cdot \frac{\pi^2}{6} - 1 - 2 \cdot 1 \; = \; \frac{\pi^2}{3} - 3.
\end{aligned}$$

Also solved by U. Abel (Germany), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), A. Berkane (Algeria), R. Bittencourt (Brazil), R. Boukharfane (France), R. Brase, R. Chapman (U. K.), H. Chen, W. J. Cowieson, R. Cuculière (France), P. P. Dályay (Hungary), V. Dassios (Greece), B. E. Davis, T. de Souza Leão (Brazil), S. Dzhatdoyev & Q. Liu, G. Fera (Italy), K. Gatesman, C. Georghiou (Greece), O. Geupel (Greece), M. L. Glasser, N. Grivaux (France), A. Habil (Syria), E. A. Herman, Y. J. Ionin, W. P. Johnson, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada) & L. Cooper (Canada) & E. Drake (Canada) & L. Kenney (Canada), M. Lafond (France), P. Lalonde (Canada), J. H. Lindsey II, L. Lipták, O. P. Lossers (Netherlands), J. Magliano, R. Martin (Germany), P. McPolin (U. K.), N. Merz, M. D. Meyerson, V. Mikayelyan (Armenia), R. Molinari, R. Nandan, M. Omarjee (France), A. Pathak, Á. Plaza & F. Perdomo (Spain), M. A. Prasad (India), F. A. Rakhimjanovich (Uzbekistan), H. Ricardo, C. Schacht, V. Schindler (Germany), E. Schmeichel, R. Schumacher (Switzerland), N. C. Singer, J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, M. Tang, R. Tauraso (Italy), D. B. Tyler, J. Vinuesa (Spain), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), Y. Xiang (China), L. Zhou, GCHQ Problem Solving Group (U. K.), Lafayette Problem Solving Group, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Unlucky Thirteen

12001 [2017, 754]. *Proposed by Marius Coman, Bucharest, Romania, and Florian Luca, Johannesburg, South Africa.* A *base-*2 *pseudoprime* is an odd composite number n that divides $2^n - 2$. Prove that if p is a prime number greater than 13, then there is a base-2 pseudoprime that divides $2^{p-1} - 1$.

Solution by Michael Tang, student, Massachusetts Institute of Technology, Cambridge, MA. First suppose that $p - 1$ has a prime factor q with $q \geq 5$. We claim that $n = (2^{2q} - 1)/3$ is a base-2 pseudoprime that divides $2^{p-1} - 1$. To see this, first note $n = (2^q - 1)(2^q + 1)/3$. Both factors in the numerator are larger than 3 and odd, so n is also odd and composite. Since $q > 3$ and $\varphi(2q) = q - 1$, where φ is Euler's totient function, by Euler's theorem $n \equiv (2^2 - 1)/3 = 1 \pmod{2q}$, so $2q \mid (n - 1)$. Also $2q \mid (p - 1)$, because both 2 and q divide $p - 1$. Hence, $2^{2q} - 1$ divides both $2^{n-1} - 1$ and $2^{p-1} - 1$, so n divides both $2^n - 2$ and $2^{p-1} - 1$ as claimed.

It remains to consider primes p with $p > 13$ such that $p - 1 = 2^a \cdot 3^b$ for some integers $a, b \geq 0$. Since $p - 1$ is even, $a \geq 1$. Also, $p \geq 17$, so $p - 1 \geq 16$. Hence either $b = 0$ and $a \geq 4$, or $b = 1$ and $a \geq 3$, or $b \geq 2$ and $a \geq 1$. We conclude that 16, 24, or 18, respectively, must divide $p - 1$. It is easy to verify that 4369 (equal to $17 \cdot 257$), 1105 (equal to $5 \cdot 13 \cdot 17$), and 1387 (equal to $19 \cdot 73$) are base-2 pseudoprimes that divide $2^{16} - 1$, $2^{24} - 1$, and $2^{18} - 1$, respectively. Hence at least one of these integers divides $2^{p-1} - 1$, completing the proof.

Editorial comment. Yury J. Ionin noted (as can be also seen from the above proof) that p does not need to be prime, only odd, and that the result also holds for $p = 11$ in addition to every odd number larger than 13. Stephen M. Gagola Jr. showed that for any prime p and any $a \geq 2$ with $\gcd(p, a) = 1$, there is a base-a pseudoprime that divides $a^{p-1} - 1$ with the exceptions given in the problem ($a = 2$, $p = 3, 5, 7, 13$) and when (a, p) is either $(3, 2)$ or $(3, 5)$.

Also solved by R. Brase, S. M. Gagola Jr., Y. J. Ionin, P. Komjáth (Hungary), P. W. Lindstrom, O. P. Lossers (Netherlands), M. A. Prasad (India), J. P. Robertson, A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Geometric Realization of Hlawka's Inequality

12002 [2017, 754]. *Proposed by Florin Stanescu, Gaesti, Romania.* Let ABC be a triangle with area S, circumradius R, circumcenter O, and orthocenter H. Let D be the point of intersection of lines AO and BC. Similarly, let E be the point of intersection of lines BO and CA, and let F be the point of intersection of lines CO and AB. Let $T = \sqrt{(3R^2 - OH^2)^2 + 16S^2}\big/R^2$. Prove

$$T \leq \frac{AH}{OD} + \frac{BH}{OE} + \frac{CH}{OF} \leq 3 + \frac{T}{2}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Without loss of generality, we assume that the circumcircle of ΔABC is the unit circle in the complex plane, and the vertices are represented by the complex numbers α, β, and γ. The radian measures of the angles at A, B, and C are also denoted A, B, and C. Recalling that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, we see that

$$OH^2 = 3 + 2(\cos 2A + \cos 2B + \cos 2C). \tag{1}$$

On the other hand, since $A + B = \pi - C$,

$$\begin{aligned} S &= \frac{1}{2}ab\sin C = 2\sin A\sin B\sin C = \big(\cos(A-B) - \cos(A+B)\big)\sin C \\ &= \cos(A-B)\sin(A+B) - \cos(A+B)\sin(A+B) \\ &= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C). \end{aligned} \tag{2}$$

It follows from (1) and (2) that

$$T^2 = 4\left(\cos 2A + \cos 2B + \cos 2C\right)^2 + 4\left(\sin 2A + \sin 2B + \sin 2C\right)^2,$$

which can be put in the form

$$T = 2|e^{2iA} + e^{2iB} + e^{2iC}|. \tag{3}$$

Since D is the point of intersection of the lines OA and BC, there exist two real numbers t and s such that $\overrightarrow{OD} = t\overrightarrow{OA} = \overrightarrow{OB} + s\overrightarrow{BC}$. This condition is exactly $t\alpha = \beta + s(\gamma - \beta)$. Taking complex conjugates and then multiplying both sides by $\alpha\beta\gamma$, we obtain

$$t\beta\gamma = \alpha(\gamma - s(\gamma - \beta)) = \alpha(\gamma + \beta - t\alpha).$$

Thus $t(\beta\gamma + \alpha^2) = \alpha(\beta + \gamma)$, so $OD = |t| = |\beta + \gamma|/|\beta\gamma + \alpha^2|$. Also $\overrightarrow{AH} = \overrightarrow{OB} + \overrightarrow{OC}$, and hence $AH = |\beta + \gamma|$. Therefore

$$\frac{AH}{OD} = |\beta\gamma + \alpha^2| = \left|\frac{\beta}{\alpha} + \frac{\alpha}{\gamma}\right| = |e^{2iC} + e^{2iB}|. \tag{4}$$

Similarly,

$$\frac{BH}{OE} = |e^{2iA} + e^{2iC}| \quad \text{and} \quad \frac{CH}{OF} = |e^{2iB} + e^{2iA}|. \tag{5}$$

By the triangle inequality,

$$\begin{aligned} &\left|e^{2iC} + e^{2iB} + e^{2iA} + e^{2iC} + e^{2iB} + e^{2iA}\right| \\ &\qquad\qquad \le \left|e^{2iC} + e^{2iB}\right| + \left|e^{2iA} + e^{2iC}\right| + \left|e^{2iB} + e^{2iA}\right|. \end{aligned}$$

The left side of this is T by (3), so using (4) and (5) on the right side, we get the first inequality in the problem statement. Hlawka's inequality yields

$$\begin{aligned} &|e^{2iC} + e^{2iB}| + |e^{2iA} + e^{2iC}| + |e^{2iB} + e^{2iA}| \\ &\qquad\qquad \le |e^{2iA}| + |e^{2iB}| + |e^{2iC}| + |e^{2iA} + e^{2iB} + e^{2iC}|, \end{aligned}$$

and the right side of this inequality is equal to $3 + T/2$, yielding the desired second inequality.

Also solved by P. P. Dályay (Hungary), D. Fleischman, R. Stong, and the proposer.

A GCD-weighted Trigonometric Sum

12003 [2017, 754]. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Given an odd positive integer n, compute

$$\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2(\pi k/n)}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Letting the prime factorization of n be $\prod_{i=1}^{s} p_i^{r_i}$, we prove

$$\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2(\pi k/n)} = \prod_{i=1}^{s} \left(p_i^{2r_i} + p_i^{2r_i-1} - p_i^{r_i-1} \right).$$

We first show $\sum_{k=1}^{n} (\cos(\pi k/n))^{-2} = n^2$. Let T_n be the nth Chebyshev polynomial of the first kind, defined by the recurrence $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$ for $n \ge 1$, with $T_0(z) = 1$ and $T_1(z) = z$. From this recurrence, one can show by induction on n that $T_n(\cos\theta) = \cos n\theta$, $T_n(-1) = (-1)^n$, and $T_n'(-1) = (-1)^{n-1}n^2$. Now let $P_n(x) = T_n(2x-1) - 1$. The n roots of the polynomial P_n, with the correct multiplicities, are $(\cos(2\pi k/n) + 1)/2$ for $1 \le k \le n$. The constant term c_0 of P_n is $T_n(-1) - 1$, which is -2, since n is odd. The linear coefficient c_1 of P_n is $2T_n'(-1)$, which is $2(-1)^{n-1}n^2$, or $2n^2$. Since the sum of the reciprocals of the roots of a polynomial $\sum_{i=0}^{n} c_i x^i$ is $-c_1/c_0$, we obtain $\sum_{k=1}^{n} \big(\cos(\pi k/n)\big)^{-2} = \sum_{k=1}^{n} \big((\cos(2\pi k/n) + 1)/2\big)^{-1} = n^2$.

The Euler totient $\phi(m)$ is the number of values in $[m]$ that are relatively prime to m; it satisfies $m = \sum_{d|m} \phi(d)$ for all $m \in \mathbb{N}$. Applying this with $m = \gcd(k,n)$, interchanging the order of summation, and letting $r = k/d$, we obtain

$$\sum_{k=1}^{n} \sum_{d|\gcd(k,n)} \frac{\phi(d)}{\cos^2(\pi k/n)} = \sum_{d|n} \sum_{r=1}^{n/d} \frac{\phi(d)}{\cos^2(\pi r/(n/d))} = \sum_{d|n} \frac{\phi(d) n^2}{d^2}.$$

When n_1 and n_2 are relatively prime, the divisors of $n_1 n_2$ are the products of the divisors of n_1 and n_2, hence the sum we have obtained is a multiplicative function of n. When n is a prime power, say $n = p^r$, we use $\phi(p^j) = p^j - p^{j-1}$ for $j \geq 1$ to evaluate the sum as

$$\sum_{d|n} \frac{\phi(d)n^2}{d^2} = p^{2r} + \sum_{j=1}^{r}(p^{2r-j} - p^{2r-j-1}) = p^{2r} + p^{2r-1} - p^{r-1}.$$

The result follows.

Also solved by R. Bittencourt (Brazil), R. Brase, R. Chapman (U. K.), K. Gatesman, Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), M. A. Prasad (India), I. Sfikas, N. C. Singer, A. Stadler (Switzerland), M. Tang, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Divergence of a Series

12004 [2017, 755]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_1, a_2, \ldots$ be a strictly increasing sequence of real numbers satisfying $a_n \leq n^2 \ln n$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} 1/(a_{n+1} - a_n)$ diverges.

Solution by Nicholas C. Singer, Annandale, VA. For $k \geq 1$, apply the Harmonic-Mean–Arithmetic-Mean inequality to the positive numbers in $\{a_{2^k+j} - a_{2^k+j-1} \colon\ 1 \leq j \leq 2^k\}$ to obtain

$$\frac{1}{a_{2^k+1} - a_{2^k}} + \frac{1}{a_{2^k+2} - a_{2^k+1}} + \cdots + \frac{1}{a_{2^{k+1}} - a_{2^{k+1}-1}} \geq \frac{4^k}{a_{2^{k+1}} - a_{2^k}} \geq \frac{4^k}{a_{2^{k+1}} - a_1}.$$

Since $a_1 \leq 0$,

$$\frac{4^k}{a_{2^{k+1}} - a_1} = \frac{4^k}{a_{2^{k+1}} + |a_1|} \geq \frac{4^k}{2^{2k+2}(k+1)\ln 2 + |a_1|} = \frac{1}{4(k+1)\ln 2 + |a_1|/4^k}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1} - a_n} = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \frac{1}{a_{2^k+j} - a_{2^k+j-1}} \geq \sum_{k=0}^{\infty} \frac{1}{4(k+1)\ln 2 + |a_1|/4^k} = \infty.$$

Editorial comment. Several solvers overlooked the possibility that a_n might be negative for some (or all) n.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Brase, H. Chen, P. J. Fitzsimmons, D. Fleischman, E. J. Ionaşcu, M. Javaheri, P. Komjáth (Hungary), O. Kouba (Syria), K. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), V. Mikayelyan (Armenia), P. Perfetti (Italy), Á. Plaza & K. Sadarangani (Spain), M. A. Prasad (India), J. C. Smith, O. Sonebi (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), GCHQ Problem Solving Group (U. K.), and the proposer.

A Suspicious Formula Involving Pi

12006 [2017, 970]. *Proposed by Jonathan D. Lee, Merton College, Oxford, U. K., and Stan Wagon, Macalester College, St. Paul, MN.* When n is an integer and $n \geq 2$, let $a_n = \lceil n/\pi \rceil$ and $b_n = \lceil \csc(\pi/n) \rceil$. The sequences $a_2, a_3, \ldots$ and $b_2, b_3, \ldots$ are, respectively,

$$1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots$$

and

$$1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots.$$

They differ when $n = 3$. Are they equal for all larger n?

Solution by Albert Stadler, Herrliberg, Switzerland. The answer is no, as can be checked by direct calculation for $n = 80143857$. As motivation for this answer, the Laurent expansion of $\csc(\pi x)$ is $1/(\pi x) + \pi x/6 + \cdots$ with all coefficients positive. Thus when $n \geq 2$ we have $0 < \csc(\pi/n) - n/\pi \leq \csc(\pi/2) - 2/\pi < 1$. It follows that $b_n - 1 \leq a_n \leq b_n$, and furthermore that $b_n = a_n + 1$ when there exists an integer m such that

$$0 < \frac{m}{n} - \frac{1}{\pi} < \frac{\pi}{6n^2}. \tag{$*$}$$

Good candidates for m/n are given by the continued fraction convergents of $1/\pi$, every second one of which is greater than $1/\pi$. The continued fraction representation of $1/\pi$ is $[0; 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots]$, and so one may compute that the first two convergents that satisfy ($*$) are the second and 14th. These are $1/3$ and $25510582/80143857$, leading to $a_n \neq b_n$ for $n = 3$ and $n = 80143857$.

Editorial comment. Direct computation shows that $a_n = b_n$ when $4 \leq n \leq 80143856$.

It is natural to wonder whether the sequences differ infinitely often. The proposers noted that by Hurwitz's theorem there are infinitely many convergents to $1/\pi$ such that $|\frac{1}{\pi} - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}$, which implies $|\frac{1}{\pi} - \frac{m}{n}| < \frac{\pi}{6n^2}$. However, only even-numbered convergents will be greater than $1/\pi$, as needed for ($*$). It seems likely, given how the continued fraction of π is expected to behave, that there are infinitely many even-numbered convergents among the ones that satisfy the condition of Hurwitz's theorem, but this is currently unresolved.

Also solved by A. Berele, R. Chapman (U. K.), S. Demers (Canada), G. Fera (Italy), O. P. Lossers (Netherlands), M. D. Meyerson, V. Mikayelyan (Armenia), M. Reid, C. Schacht, V. Schindler (Germany), J. C. Smith, A. Stenger, A. Stewart, R. Stong, W. Stromquist, R. Tauraso (Italy), D. Terr, H. Widmer (Switzerland), L. Zhou, Armstrong Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposers.

An Application of the Phragmén–Lindelöf Principle

12009 [2017, 970]. *Proposed by George Stoica, Saint John, NB, Canada.* Find all continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying $\left|\int_0^1 e^{xy} f(x)\,dx\right| < 1/y$ for all positive real numbers y.

Solution by James Christopher Smith, Knoxville, TN. We claim that the only such function is the constant 0. Let $g(z) = \int_0^1 e^{xz} f(x)\,dx$ for all $z \in \mathbb{C}$. Because f is continuous on $[0, 1]$, it is bounded and measurable, so g is an entire function.

We apply the Phragmén–Lindelöf principle to $g(z)$ on the first quadrant D in the complex plane. First, we note the estimate

$$|g(z)| \leq \int_0^1 \left|e^{xz} f(x)\right|\,dx \leq Me^{|z|},$$

where $M = \int_0^1 |f(x)|\,dx$. Second, we claim that g is bounded on the real axis. Indeed, when $-\infty < y \leq 1$ we have $|g(y)| \leq Me$ and for $y \geq 1$ we have $|g(y)| \leq 1/y \leq 1$. And third, we claim that g is bounded on the imaginary axis. Indeed, for $y \in \mathbb{R}$ we have $|g(iy)| \leq \int_0^1 \left|e^{ixy} f(x)\right|\,dx \leq M$. Therefore, by the Phragmén–Lindelöf principle, $g(z)$ is bounded in the quadrant D. Similarly, $g(z)$ is bounded in each of the other three quadrants as well.

Thus $g(z)$ is a bounded entire function, so by Liouville's theorem $g(z)$ is constant. Hence, for all $n \geq 1$, we have $0 = g^{(n)}(0) = \int_0^1 x^n f(x)\,dx$. By the Weierstrass approximation theorem applied to $xf(x)$, we conclude that f is the constant function 0.

Also solved by K. F. Andersen (Canada), A. Stadler (Switzerland), G. Vidiani (France), GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by October 31, 2019, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12118. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ when $n \geq 2$. Compute

$$\sum_{n=0}^{\infty} \frac{1}{F_{2mn} + F_m i},$$

where m is an odd integer and $i = \sqrt{-1}$.

12119. *Proposed by Vu Thanh Tung, Nam Dinh, Vietnam.* Let I be a real interval, and let $F\colon I \times I \to \mathbb{R}$ be a function such that

$$\frac{\partial^3 F}{\partial x\, \partial y^2} \geq 0 \geq \frac{\partial^3 F}{\partial x^2\, \partial y}.$$

For a positive integer n, suppose that $a_1, \ldots, a_n$ are real numbers in I satisfying $a_1 \geq a_2 \geq \cdots \geq a_n$, and let $a_{n+1} = a_1$. Prove

$$\sum_{i=1}^{n} F(a_i, a_{i+1}) \geq \sum_{i=1}^{n} F(a_{i+1}, a_i).$$

12120. *Proposed by Michel Bataille, Rouen, France.* For positive integers n and k with $n \geq k$, let $a(n,k) = \sum_{j=0}^{k-1} \binom{n}{j} 3^j$.
(a) Evaluate

$$\lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{n} \frac{a(n,k)}{k}.$$

(b) Evaluate

$$\lim_{n\to\infty} n \left(4^n L - \sum_{k=1}^{n} \frac{a(n,k)}{k} \right),$$

where L is the limit in part **(a)**.

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12121. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania, and Kunihiko Chikaya, Tokyo, Japan.* For what values of k does

$$k(a^x + b^x + c^x) + \sqrt{a(1-b) + b(1-c) + c(1-a)} \geq 3k$$

hold for all $x \in (0, 1)$ and all positive real numbers a, b, and c satisfying $a + b + c = 3$?

12122. *Proposed by Marius Munteanu, State University of New York, Oneonta, NY.* Let f be a real-valued function on an abelian group G such that $f(a+a) = 2f(a)$ and $f(-a) = f(a)$ for all $a \in G$. Prove that if

$$f(a) + f(b) + f(c) + f(a+b+c) \geq f(a+b) + f(b+c) + f(c+a)$$

for all a, b, and c in G, then

$$f(a+b) + f(b+c) + f(c+a) + f(a+b+c) \geq f(a) + f(b) + f(c)$$

for all a, b, and c in G.

12123. *Proposed by Andrew Wu, student, St. Albans School, Washington, DC.* Let ABC be a triangle with $AB \neq AC$ and with incenter I. Let M be the midpoint of BC, and let L be the midpoint of the circular arc BAC. Lines through M parallel to BI and CI meet AB and AC at E and F, respectively, and meet LB and LC at P and Q, respectively. Show that I lies on the radical axis of the circumcircles of triangles EMF and PMQ.

12124. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let p be a real number greater than 1, and let $a_1, a_2, \ldots$ be a sequence of positive real numbers. Prove that if

$$\sum_{k=1}^{n} \frac{1}{1 + a_k^p} = O\left(\frac{1}{1 + a_n^p}\right),$$

then

$$\sum_{k=1}^{n} \frac{1}{1 + a_k} = O\left(\frac{1}{(1 + a_n^p)^{1/p}}\right).$$

(Here, as usual, $f(n) = O(g(n))$ means that there exist M and N so that $|f(n)| < Mg(n)$ for all $n \geq N$.)

SOLUTIONS

A Bird's Eye View

12007 [2017, 970]. *Proposed by Kadir Altintas, Afyon, Turkey, and Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let G be the centroid of triangle ABC, and let M be an interior point of ABC. Let D, E, and F be the centroids of sub-triangles CMB, AMC, and BMA, respectively.
(a) Prove that the lines AD, BE, and CF are concurrent.
(b) Suppose that $M \neq G$ and that P is the point of concurrency in part **(a)**. Prove that G, P, and M are collinear, with P between G and M, and $PM = 3PG$.

Solution I by Don Chakerian, Davis, CA. The medians of a tetrahedron intersect at its centroid, and each median is divided in the ratio 3:1. If M is any point not in the plane of triangle ABC, then AD, BE, CF, and MG are medians of tetrahedron $ABCM$

and therefore intersect at its centroid. This point P divides each median in the ratio 3:1. The proposed problem is the limiting case where M lies inside ABC. Note that M need not be interior to ABC. This problem generalizes to an n-dimensional simplex divided into sub-simplices by a point M. The lines connecting vertices to the centroids of their opposite sub-simplices concur at P with $PM = nPG$.

Solution II by Giuseppe Fera, Vicenza, Italy. Let Q be the center of the homethety $\mathcal{H}$ with ratio $-1/3$ that sends M to G. Observe that G, Q, and M are collinear with $PM = 3PG$. Take Q to be the origin, so the centroid D of triangle CMB satisfies

$$D = \frac{C + M + B}{3} = \frac{C - 3G + B}{3} = \frac{C - (A + B + C) + B}{3} = -\frac{A}{3}.$$

Thus $D = \mathcal{H}(A)$, and likewise $E = \mathcal{H}(B)$ and $F = \mathcal{H}(C)$. Since $\mathcal{H}$ is a homothety, AD, BE, and CF are concurrent at its center, and so $P = Q$.

Also solved by H. Bailey, R. Barraso Campos (Spain), M. Bataille (France), A. Berele, Z. Bingsong, J. Cade, R. Chapman (U. K.), H. Chen, I. Dimitrić, A. Fanchini (Italy), D. Fleischman, O. Geupel (Greece), E. P. Goldenberg, M. Goldenberg & M. Kaplan, J. Grivaux (France), M. Hajja (Jordan), Y. J. Ionin, K. T. L. Koo (China), O. Kouba (Syria), S. S. Kumar, J. H. Lindsey II, G. Lord, O. P. Lossers (Netherlands), T. McDevitt, M. D. Meyerson, J. Minkus, C. G. Petalas (Greece), C. R. Pranesachar (India), M. Reid, C. Schacht, V. Schindler (Germany), R. A. Simón, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), L. Wimmer (Germany), S. Witt, L. Zhou, Armstrong Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

Special Commutative Rings

12010 [2017, 971]. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO, and Adam Salminen, University of Evansville, Evansville, IN.* Given a ring with multiplicative identity 1, we say that a subring is *unital* if it contains 1. Find all commutative rings R (up to isomorphism) such that R has a multiplicative identity, R is not a field, R has a proper unital subring, and every proper unital subring of R is a field.

Solution by Allan Berele, DePaul University, Chicago, IL. The ring R must have the form $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_p[x]/(x^2)$.

Each of these contains the unital subfield $\mathbb{Z}_p$. Any other unital subring must be a vector space over $\mathbb{Z}_p$ of dimension at least 2 and thus is all of R. Hence, these examples satisfy the conditions.

Since the subring generated by 1 must be a field, its characteristic p must be nonzero. Otherwise, R would contain a subring isomorphic to $\mathbb{Z}$. If this subring is all of R, then R does not have a proper unital subring, while if it is not all of R, then not every proper unital subring is a field.

Since R is not a field, it contains a noninvertible element a. Since a is not invertible and the subring generated by 1 and a is unital but not a field, this subring must be all of R. Hence, the homomorphism from $\mathbb{Z}_p[x]$ to R defined by mapping $f(x)$ to $f(a)$ for each $f(x)$ in the polynomial ring $\mathbb{Z}_p[x]$ must be onto, so R is an image of $\mathbb{Z}_p[x]$.

Since $\mathbb{Z}_p[x]$ contains many subrings that are not fields, R must be isomorphic to $\mathbb{Z}_p[x]/I$ for a nonzero ideal I. Since $\mathbb{Z}_p[x]$ is a principal ideal domain, $I = (f(x))$ for some polynomial $f(x)$. Let $f(x)$ factor as $\Pi g_i(x)^{n_i}$, where each $g_i(x)$ is irreducible, so $R \cong \oplus \mathbb{Z}_p[x]/(g_i(x)^{n_i})$. We now deduce that R must be one of the claimed rings from the following observations.

- *R cannot be a product of more than two rings.* If $R = R_1 \oplus R_2 \oplus R_3$, then the subring generated by the unit $(1, 1, 1)$ and the element $(1, 0, 0)$ cannot be a field (it contains the zero divisors $(1, 0, 0)$ and $(0, 1, 1)$) and cannot be all of R (it does not contain $(0, 1, 0)$).

- *If $R = R_1 \oplus R_2$, then $R = \mathbb{Z}_p \times \mathbb{Z}_p$.* The subring generated by $(1, 1)$ and $(1, 0)$, which equals the subring generated by $(1, 0)$ and $(0, 1)$, must be all of R.
- *If $R = \mathbb{Z}_p[x]/(g(x)^n)$, then $n = 2$.* We have $n \neq 1$ since R is not a field. If $n \geq 3$, then R contains the proper unital subring generated by 1 and the nilpotent element $g(x)^{n-1}$.
- *If $R = \mathbb{Z}_p[x]/(g(x)^2)$, then $R = \mathbb{Z}_p[x]/(x-a)^2$, and every such ring is isomorphic to* $\mathbb{Z}_p[x]/(x)^2$. The set $\{a + bg(x)\}$ is a unital subring and is not a field, so it must be all of R. Hence it contains x, and this can only happen if $g(x)$ has degree 1.

Also solved by A. J. Bevelacqua, R. Chapman (U. K.), S. M. Gagola, Jr., O. P. Lossers (Netherlands), M. Reid, J. H. Smith, R. Stong, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Asymptotics of a Double Integral

12011 [2017, 971]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Calculate

$$\lim_{n\to\infty}\left(\frac{1}{n!}\int_0^\infty\int_0^\infty \frac{x^n - y^n}{e^x - e^y}\,dx\,dy \;-\; 2n\right).$$

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. The limit equals 2. Note that although the integrand is undefined on the line $y = x$, it extends to a function that is continuous on $(0, \infty) \times (0, \infty)$. Since the integrand is symmetric with respect to the line $y = x$,

$$I_n = \int_0^\infty\int_0^\infty \frac{x^n - y^n}{e^x - e^y}\,dx\,dy = 2\int_0^\infty\int_y^\infty \frac{x^n - y^n}{e^x - e^y}\,dx\,dy.$$

With $t = x - y$, we have

$$e^x - e^y = e^y(e^t - 1)$$

and

$$x^n - y^n = (t + y)^n - y^n = \sum_{j=1}^{n}\binom{n}{j} t^j y^{n-j},$$

and so

$$\begin{aligned} I_n &= 2\sum_{j=1}^{n}\binom{n}{j}\int_0^\infty e^{-y}y^{n-j}\,dy\int_0^\infty t^j\,\frac{e^{-t}}{1-e^{-t}}\,dt \\ &= 2\sum_{j=1}^{n}\binom{n}{j}\int_0^\infty e^{-y}y^{n-j}\,dy\int_0^\infty t^j\sum_{k=1}^{\infty}e^{-kt}\,dt \\ &= 2\sum_{j=1}^{n}\binom{n}{j}\int_0^\infty e^{-y}y^{n-j}\,dy\sum_{k=1}^{\infty}\int_0^\infty t^j e^{-kt}\,dt. \end{aligned}$$

Since $t^j e^{-kt} \geq 0$ for $t \geq 0$, the interchange in summation and integration is justified by Fubini's theorem. The elementary integration formula

$$\int_0^\infty t^m e^{-at}\,dt = \frac{m!}{a^{m+1}}$$

for $a > 0$ and nonnegative integer m then shows

$$I_n = 2n!\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k^{j+1}} = 2n!\sum_{k=1}^{\infty}\sum_{j=1}^{n}\frac{1}{k^{j+1}} = 2n!\left(n + \sum_{k=2}^{\infty}\sum_{j=1}^{n}\frac{1}{k^{j+1}}\right)$$

$$= 2n!\left(n + \sum_{k=2}^{\infty}\left(\frac{1}{k(k-1)} - \frac{1}{k^{n+1}(k-1)}\right)\right) = 2n!\left(n + 1 - \sum_{k=2}^{\infty}\frac{1}{k^{n+1}(k-1)}\right).$$

Thus, we have

$$\frac{1}{n!}\int_0^{\infty}\int_0^{\infty}\frac{x^n - y^n}{e^x - e^y}\,dx\,dy - 2n = 2 - 2\sum_{k=2}^{\infty}\frac{1}{k^{n+1}(k-1)}.$$

Since

$$\sum_{k=2}^{\infty}\frac{1}{k^{n+1}(k-1)} \le \frac{1}{2^n}\sum_{k=2}^{\infty}\frac{1}{k(k-1)} = \frac{1}{2^n},$$

the required limit is equal to 2, as claimed.

Also solved by A. Berkane (Algeria), R. Chapman (U. K.), H. Chen, G. Fera (Italy), J. A. Grzesik, E. A. Herman, J. H. Lindsey II, O. P. Lossers (Netherlands), I. Mező (China), V. Mikayelyan (Armenia), M. Omarjee (France), J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), Y. Xiang (China), and the proposer.

A Power Series Involving Tails of e

12012 [2017, 971]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let k be a nonnegative integer. Find the set of real numbers x for which the power series

$$\sum_{n=k}^{\infty}\binom{n}{k}\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right)x^n$$

converges, and determine the sum.

Solution by Vazgen Mikayelyan, Yerevan State University, Yerevan, Armenia. Let

$$S_k(x) = \sum_{n=k}^{\infty}\binom{n}{k}\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right)x^n,$$

and let $I_n = (1/n!)\int_0^1 (1-t)^n e^t\,dt$. Since $I_0 = e - 1$ and integration by parts gives $I_n = I_{n-1} - 1/n!$, we see that

$$I_n = e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}.$$

In particular, since $0 < I_n < (1/n!)\int_0^1 e^t\,dt = (e-1)/n!$, we get

$$|S_k(x)| \le \sum_{n=k}^{\infty}\binom{n}{k}\frac{e-1}{n!}|x|^n = \frac{|x|^k(e-1)e^{|x|}}{k!}$$

for all real x. Thus the series $S_k(x)$ converges for all x.

We now compute

$$S_k(x) = \sum_{n=k}^{\infty} \binom{n}{k} I_n x^n = \int_0^1 \sum_{n=k}^{\infty} \frac{(1-t)^n x^n}{k!(n-k)!} e^t \, dt$$
$$= \frac{x^k}{k!} \int_0^1 (1-t)^k e^{t+(1-t)x} \, dt = \frac{x^k e}{k!} \int_0^1 s^k e^{-(1-x)s} \, ds.$$

This integral can be done by parts, giving

$$S_k(x) = \begin{cases} \dfrac{e}{(k+1)!}, & \text{if } x = 1; \\ \dfrac{x^k e}{(1-x)^{k+1}} \left(1 - e^{x-1} \displaystyle\sum_{l=0}^{k} \frac{(1-x)^l}{l!}\right), & \text{if } x \neq 1. \end{cases}$$

Also solved by K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (France), K. N. Boyadzhiev, R. Chapman (U. K.), H. Chen, B. E. Davis, G. Fera (Italy), M. L. Glasser, N. Grivaux (France), E. A. Herman, O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), A. Pathak, P. Perfetti (Italy), J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), Armstrong Problem Solving Group, and the proposer.

An Inequality in Symmetric Functions

12013 [2018, 81]. *Proposed by David Stoner, student, Harvard University, Cambridge, MA.* Suppose that a, b, c, d, e, and f are nonnegative real numbers that satisfy $a + b + c = d + e + f$. Let t be a real number greater than 1. Prove that at least one of the inequalities

$$a^t + b^t + c^t > d^t + e^t + f^t,$$
$$(ab)^t + (bc)^t + (ca)^t > (de)^t + (ef)^t + (fd)^t, \text{ and}$$
$$(abc)^t > (def)^t$$

is false.

Composite solution by the proposer and the editors. We prove a slightly strengthened form of the contrapositive. Specifically, let $x = a^t$, $y = b^t$, $z = c^t$, $u = d^t$, $v = e^t$, $w = f^t$, and $s = 1/t$. We show that if $x + y + z \geq u + v + w$, $xy + yz + zx \geq uv + vw + wu$, and $xyz \geq uvw$, then $x^s + y^s + z^s \geq u^s + v^s + w^s$ for any $s \in (0, 1)$, with equality if and only if u, v, w is a permutation of x, y, z.

Let F be a symmetric function of x, y, z. We can view F as a function of the elementary symmetric polynomials $\sigma_1 = x + y + z$, $\sigma_2 = xy + yz + zx$, and $\sigma_3 = xyz$. The desired result will follow if we show that for $F(x, y, z) = x^s + y^s + z^s$, we have $\frac{\partial F}{\partial \sigma_1}$, $\frac{\partial F}{\partial \sigma_2}$, and $\frac{\partial F}{\partial \sigma_3}$ all positive. This uses the fact, proved in the lemma below, that we can join $\{u, v, w\}$ to $\{x, y, z\}$ by a path along which each of the σ_k is nondecreasing.

Let us compute the signs of these partial derivatives. From the chain rule we have

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \sigma_1} + (y+z)\frac{\partial F}{\partial \sigma_2} + yz\frac{\partial F}{\partial \sigma_3},$$

and symmetrically for the other two variables. Hence inverting this linear system gives formulas for the partial derivatives of F with respect to the σ_k that can be summarized by the formula

$$\frac{\partial F}{\partial \sigma_k} = \sum_{\text{cyc}} \frac{(-1)^{k-1} x^{3-k}}{(x-y)(x-z)} \frac{\partial F}{\partial x},$$

where the notation $\sum_{\text{cyc}}$ indicates that a sum is taken over all cyclic permutations of x, y, z. (When two of x, y, z are equal, one can make sense of this formula by taking a limit.) In particular for $F(x, y, z) = x^s + y^s + z^s$ we find

$$\frac{\partial F}{\partial \sigma_k} = \sum_{\text{cyc}} \frac{(-1)^{k-1} s x^{2+s-k}}{(x-y)(x-z)}.$$

To determine the signs of these sums, we note that for any twice continuously differentiable function g we have

$$\sum_{\text{cyc}} \frac{g(x)}{(x-y)(x-z)} = \int_0^1 \int_0^{1-r} g''(rx + sy + (1-r-s)z)\, ds\, dr = \frac{1}{2} g''(\xi)$$

for some $\xi \in [\min(x, y, z), \max(x, y, z)]$. Hence

$$\frac{\partial F}{\partial \sigma_1} = \frac{s^2(s+1)}{2} \xi_1^{s-1} > 0,$$

$$\frac{\partial F}{\partial \sigma_2} = \frac{s^2(1-s)}{2} \xi_2^{s-2} > 0, \quad \text{and}$$

$$\frac{\partial F}{\partial \sigma_3} = \frac{s(1-s)(2-s)}{2} \xi_3^{s-3} > 0.$$

Thus all three partial derivatives are positive as required.

Lemma. *If $\{u, v, w\}$ and $\{x, y, z\}$ are unordered triples of nonnegative reals such that $x + y + z \geq u + v + w$, $xy + yz + zx \geq uv + vw + wu$, and $xyz \geq uvw$, then there is a path from $\{u, v, w\}$ to $\{x, y, z\}$ along which all three elementary symmetric functions are nondecreasing.*

Proof. Associated to an unordered triple $\{x, y, z\}$ we have a monic cubic polynomial $(X - x)(X - y)(X - z) = X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3$ with three nonnegative real roots and, conversely, given such a polynomial we get an unordered triple. We find it more convenient to work with paths between polynomials rather than paths between triples.

Let $0 \leq x \leq y \leq z$ and $0 \leq u \leq v \leq w$. We first observe that if the graph of $Y = (X - u)(X - v)(X - w)$ lies entirely above the graph of $Y = (X - x)(X - y)(X - z)$ (for nonnegative X) — that is, $(X - u)(X - v)(X - w) \geq (X - x)(X - y)(X - z)$ for all $X \geq 0$—then we have one of the two orderings:

$$0 \leq u \leq v \leq w \leq x \leq y \leq z, \tag{1}$$

$$0 \leq u \leq x \leq y \leq v \leq w \leq z. \tag{2}$$

When (2) holds, any other cubic with graph between these two graphs (for nonnegative X) also has three nonnegative real zeros (counted with multiplicity).

We highlight two special cases:

Case 1: Suppose that the triples $\{u, v, w\}$ and $\{x, y, z\}$ have the same value for σ_2. For nonnegative X, the graph of the cubic $Y = (X - u)(X - v)(X - w)$ lies entirely above the graph of $Y = (X - x)(X - y)(X - z)$. Since the two cubics have the same value for σ_2, the zeros are ordered as in (2), so any cubic whose graph (for nonnegative X) lies entirely between will also have three nonnegative real roots. Thus linearly interpolating σ_1 and σ_3 between them (keeping σ_2 fixed) gives the required path.

Case 2: Suppose that the triples have the same values for both σ_1 and σ_3. The graph of the cubic $Y = (X-u)(X-v)(X-w)$ lies entirely below the graph of the cubic $Y = (X-x)(X-y)(X-z)$ and again this ensures ordering in (2), so any interpolation has three nonnegative real roots. Thus increasing σ_2 keeping σ_1 and σ_3 fixed gives the desired path.

Starting from an arbitrary triple $\{u, v, w\}$ (and setting notation so that $u \le v \le w$), if we increase w keeping u and v fixed, then all three σ_k can only increase. Thus we may assume $\{u, v, w\}$ and $\{x, y, z\}$ have at least one σ_k equal. If this is σ_2, then we are done by Case 1. If it is σ_3, then we switch to the path $\{u, v/t, tw\}$ for $t \ge 1$ (or the path $\{0, 0, t-1\}$ if $\{u, v, w\} = \{0, 0, 0\}$). On this path σ_3 is fixed but σ_1 and σ_2 only increase. Thus we may follow this path until at least two of the σ_k agree with the values for $\{x, y, z\}$. At that point, either Case 1 or Case 2 applies and we are done. Finally, if the triples have the same value for σ_1, then we follow the path

$$(u_t, v_t, w_t) = (u + t(v + w - 2u), v + t(w + u - 2v), w + t(u + v - 2w))$$

for $0 \le t \le 1/3$. On this path σ_1 is fixed and σ_2 and σ_3 only increase. To see this, note that

$$\begin{aligned}\frac{d(u_t v_t + v_t w_t + w_t u_t)}{dt} &= u_t(2u - v - w) + v_t(2v - w - u) + w_t(2w - u - v)\\ &\ge \frac{(u_t + v_t + w_t)((2u - v - w) + (2v - w - u) + (2w - u - v))}{3} = 0\end{aligned}$$

and

$$\begin{aligned}\frac{du_t v_t w_t}{dt} &= u_t v_t(v + w - 2u) + v_t w_t(w + u - 2v) + w_t u_t(u + v - 2w)\\ &\ge \frac{(u_t v_t + v_t w_t + w_t u_t)((v + w - 2u) + (w + u - 2v) + (u + v - 2w))}{3} = 0,\end{aligned}$$

where the inequalities follow from the rearrangement inequality, since (u, v, w) and (u_t, v_t, w_t) are ordered in the same way. Thus again we can follow this path until two of $\sigma_1, \sigma_2, \sigma_3$ match with $\{x, y, z\}$ and then apply one of Case 1 or Case 2. ■

No other correct solutions were received.

A Consequence of Hlawka's Inequality and Levi Reduction

12015 [2018, 81]. *Proposed by Dao Thanh Oai, Kien Xuong, Vietnam.* Let ABC be a triangle, let G be its centroid, and let D, E, and F be the midpoints of BC, CA, and AB, respectively. For any point P in the plane of ABC, prove

$$PA + PB + PC \le 2(PD + PE + PF) + 3PG,$$

and determine when equality holds.

Solution I by Giuseppe Fera and Giorgio Tescaro, Vicenza, Italy. For each of the lines AB, BC, and CA, define its *inside* to be the half-plane containing the interior of $\triangle ABC$. By symmetry, it suffices to consider three regions for the location of P: $\triangle BCG$, Ω_1, and Ω_2, where Ω_1 is the intersection of the outsides of AB and AC, and Ω_2 is the intersection of the inside of AB, the inside of AC, and the outside of BC. All three regions include their boundaries.

Since $\overrightarrow{PA} = 3\overrightarrow{PG} - 2\overrightarrow{PD}$, we have $PA \le 2PD + 3PG$, with equality if and only if P lies on the line segment GD. Thus it suffices to prove $PB + PC \le 2(PE + PF)$ for P in the three regions above. If P lies in $\triangle BCG$, then G lies in $\triangle EFP$, so

$$PB + PC \le GB + GC = 2(GE + GF) \le 2(PE + PF),$$

where both equalities hold if and only if $P = G$. Now suppose that P is in Ω_1 or Ω_2. Define point Q such that $\overrightarrow{AQ} = 2\overrightarrow{AP}$. Since P lies in $\triangle BCQ$, we have $PB + PC \le QB + QC = 2(PF + PE)$, with equality if and only if $P = Q = A$. This completes the proof and shows that equality holds if and only if $P = G$.

Solution II by Celia Schacht, student, Arizona State University, Glendale, AZ. Given a point X, use the corresponding lower case letter $\mathbf{x}$ to denote $\overrightarrow{PX}$. Let

$$\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \|\mathbf{b} + \mathbf{c}\| + \|\mathbf{c} + \mathbf{a}\| + \|\mathbf{a} + \mathbf{b}\| + \|\mathbf{a} + \mathbf{b} + \mathbf{c}\| - \|\mathbf{a}\| - \|\mathbf{b}\| - \|\mathbf{c}\|.$$

The desired inequality is $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}) \ge 0$.

We appeal to the following reduction theorem of F. W. Levi (see page 175 in D. S. Mitrinović (1970), *Analytic Inequalities*, Berlin: Springer). Let k_i and p_{ij} be real constants for $1 \le i \le m$ and $1 \le j \le r$. If $\sum_{i=1}^m k_i |p_{i1}x_1 + \cdots + p_{ir}x_r| \ge 0$ for all real numbers $x_1, \ldots, x_r$, then $\sum_{i=1}^m k_i \|p_{i1}\mathbf{v}_1 + \cdots + p_{ir}\mathbf{v}_r\| \ge 0$ for all vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{E}^n$ and any positive integer n. A necessary condition for equality in the conclusion is that $\sum_{i=1}^m k_i |(p_{i1}\mathbf{v}_1 + \cdots + p_{ir}\mathbf{v}_r) \cdot \mathbf{u}| = 0$ for all unit vectors $\mathbf{u}$ in $\mathbb{E}^n$, where $\mathbf{v} \cdot \mathbf{u}$ stands for the inner product.

Let

$$\delta(a, b, c) = |b + c| + |c + a| + |a + b| + |a + b + c| - |a| - |b| - |c|$$

for all real numbers a, b, and c. By Levi's theorem, it is sufficient to show $\delta(a, b, c) \ge 0$, the one-dimensional analogue of the desired result. To this end, fix real numbers b and c, and let $f(x) = \delta(x, b, c)$. Notice that $\lim_{x\to-\infty} f(x) = \infty = \lim_{x\to\infty} f(x)$. Also, f is a piecewise linear function of x and its critical points are 0, $-b - c$, $-b$, and $-c$. It suffices to show that f is nonnegative at these critical points. Indeed, $f(0) = 2|b + c| \ge 0$, $f(-b - c) = 0$, and

$$f(-b) = |b - c| + |b + c| - 2|b| \ge |(b - c) + (b + c)| - 2|b| = 0.$$

Similarly, $f(-c) \ge 0$. This proves the inequality for $\delta(a, b, c)$.

Finally, we prove that equality holds if and only if $P = G$. If $P = G$, then $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, so $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$. Now suppose that P is not on the line AD. We may assume that P and B are on the same side of AD. Let $\mathbf{u}$ be the unit vector perpendicular to AD, oriented so that $\mathbf{c} \cdot \mathbf{u} > 0$. Let $a = \mathbf{a} \cdot \mathbf{u}$, $b = \mathbf{b} \cdot \mathbf{u}$, $c = \mathbf{c} \cdot \mathbf{u}$, and $d = \mathbf{d} \cdot \mathbf{u}$. We have $c > a = d > 0$, so $a + c > 0$. Also, $d = (b + c)/2$, so $b + c > 0$ and thus $a + b + c > 0$. Hence

$$\delta(a, b, c) = a + 2(b + c) + |a + b| - |b| > a + |a + b| - |b| \ge 0.$$

Thus $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}) > 0$ by Levi's theorem. Likewise, equality cannot hold if P is not on BE or CF. Thus equality requires $P = G$.

Editorial comment. Hlawka's inequality asserts that

$$\|\mathbf{y} + \mathbf{z}\| + \|\mathbf{z} + \mathbf{x}\| + \|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{z}\| + \|\mathbf{x} + \mathbf{y} + \mathbf{z}\|$$

for all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. It can be proved via Levi's theorem as in the solution by Schacht. Most solvers noticed that the stated inequality follows immediately from Hlawka's inequality by substituting $\mathbf{x} = \mathbf{b} + \mathbf{c} - \mathbf{a}$, $\mathbf{y} = \mathbf{c} + \mathbf{a} - \mathbf{b}$, and $\mathbf{z} = \mathbf{a} + \mathbf{b} - \mathbf{c}$.

Levi's theorem can be proved by integrating over the unit sphere. In the special case of this problem, the integration is over the unit circle, which was carried out by solvers Yury J. Ionin and Richard Stong.

Also solved by M. Bataille (France), H. Chen (China), M. Dincă (Romania), Y. J. Ionin, J. H. Lindsey II, O. P. Lossers (Netherlands), J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, J. Zacharias, L. Zhou, Davis Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at americanmathematicalmonthly.submittable.com/submit. *Proposed solutions to the problems below should be submitted by November 30, 2019, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

12125. *Proposed by James Propp, University of Massachusetts, Lowell, MA.*

(a) In the picture at right, nine equally spaced points on a circle are joined by nine chords, forming seven triangles. Show that the sum of the areas of the three outermost black triangles plus the area of the innermost (equilateral) black triangle equals the sum of the areas of the other three triangles.

(b) Part **(a)** can be phrased as the assertion that a certain self-intersecting 9-gon has signed area zero. For what values of n does there exist a self-intersecting n-gon of signed area zero whose vertices coincide with the vertices of a regular n-gon?

12126. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let $P(n)$ be the greatest prime divisor of the positive integer n. Prove that $P(n^2 - n + 1) < P(n^2 + n + 1)$ and $P(n^2 - n + 1) > P(n^2 + n + 1)$ each hold for infinitely many positive integers n.

12127. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\int_0^1 \left(\frac{\mathrm{Li}_2(1) - \mathrm{Li}_2(x)}{1 - x} \right)^2 dx,$$

where Li_2 denotes the dilogarithm function, defined by $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k / k^2$.

12128. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Find, in terms of n, the number of trailing zeros in the decimal representation of F_n.

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12129. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Compute

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{2+\cdots}}}}},$$

where the sequence of signs consists of $n-1$ plus signs followed by a minus sign and repeats with period n.

12130. *Proposed by Dan Ştefan Marinescu, Hunedoara, Romania, and Mihai Monea, Deva, Romania.* Let P be a point in the interior of triangle ABC. Suppose that the lines AP, BP, and CP intersect the circumcircle of ABC again at A', B', and C', respectively. Prove

$$\frac{\mathrm{S}(BPC)}{AP}+\frac{\mathrm{S}(APC)}{BP}+\frac{\mathrm{S}(APB)}{CP}\geq\frac{\mathrm{S}(BPC)}{A'P}+\frac{\mathrm{S}(APC)}{B'P}+\frac{\mathrm{S}(APB)}{C'P},$$

where $S(XYZ)$ denotes the area of triangle XYZ.

12131. *Proposed by Michael Maltenfort, Northwestern University, Evanston, IL.* Let m and n be positive integers with $n\geq 2$. Suppose that U is an open subset of $\mathbb{R}^m$ and $f:U\to\mathbb{R}^n$ is continuously differentiable. Let K be the set of all $x\in U$ such that the derivative $Df(x)$, as a linear transformation, has rank less than n. Prove that if $f(K)$ is countable, $U\setminus K\neq\emptyset$, and $f(U)$ is closed, then $f(U)=\mathbb{R}^n$.

SOLUTIONS

Cycle of Powers

11665 [2012, 669]. *Proposed by Raitis Ozols, student, University of Latvia, Riga, Latvia.* Let $a=(a_1,\ldots,a_n)$, where $n\geq 2$ and each a_j is a positive real number. Let $S(a)=a_1^{a_2}+\cdots+a_{n-1}^{a_n}+a_n^{a_1}$.
(a) Prove that $S(a)>1$.
(b) Prove that for all $\epsilon>0$ and $n\geq 2$ there exists a of length n with $S(a)<1+\epsilon$.

Solution by Traian Viteam, Punta Arenas, Chile. First, we prove the result for $n=2$. We show that if $a,b>0$, then $a^b+b^a>1$. If one of a and b is at least 1, this is clear, so we henceforth assume $0<a,b<1$. From Bernoulli's inequality, we have

$$a^{1-b}=(1+(a-1))^{1-b}<1+(1-b)(a-1)=a+b-ab.$$

Hence $a^b>\frac{a}{a+b-ab}$. Similarly, $b^a>\frac{b}{a+b-ab}$, so

$$a^b+b^a>\frac{a}{a+b-ab}+\frac{b}{a+b-ab}=\frac{a+b}{a+b-ab}>1.$$

For $n\geq 3$, we may assume by cyclic symmetry that $a_1=\max\{a_1,\ldots,a_n\}$. Again, when $a_1\geq 1$ we are obviously done, so we may assume that a_i is in $(0,1)$ for all i. We then have

$$S(a)>a_1^{a_2}+a_2^{a_3}\geq a_1^{a_2}+a_2^{a_1}>1,$$

where the final step is the case $n=2$.

For part (b), let ϵ be an arbitrary positive constant. Choose $a_n=1$. We define $a_{n-1},\ldots,a_1$ inductively. Assume that we have defined positive reals $a_{n-k},\ldots,a_n$. Since

$\lim_{x\to 0} x^{a_{n-k}} = 0$, we can choose a_{n-k-1} small enough so $a_{n-k-1}^{a_{n-k}} < \epsilon/(n-1)$. Once we have defined $a_1, \ldots, a_n$ in this way,

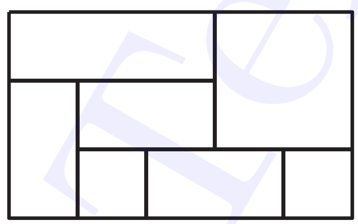
$$S(a) < (n-1)\frac{\epsilon}{n-1} + 1 = 1 + \epsilon.$$

Editorial comment. The editors regret the delay in the appearance of this solution. The case $n = 2$ of this inequality, from which the general case easily follows as shown above, has appeared before. For example, it is inequality 3.6.38 on page 281 in D. S. Mitrinović, (1970), *Analytic Inequalities*, Berlin: Springer-Verlag.

Also solved by K. F. Andersen (Canada), G. Apostolopoulos (Greece), R. Boukharfane (France), N. Caro (Brazil) and O. López (Colombia), H. Chen, J. Chun (South Korea), P. P. Dályay (Hungary), V. De Angelis, A. Ercan (Turkey), D. Fleischman, A. Habil (Syria), E. A. Herman, Y. J. Ionin, H. Katsuura & E. Schmeichel, O. Kouba (Syria), J. Li, M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), R. Stong, M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

Tight Pavings by Integer Rectangles

12005 [2018, 755]. *Proposed by Donald E. Knuth, Stanford, CA.* A *tight m-by-n paving* is a decomposition of an m-by-n rectangle into $m + n - 1$ rectangular tiles with integer sides such that each of the $m - 1$ horizontal lines and $n - 1$ vertical lines within the rectangle is part of the boundary of at least one tile. For example, one of the 1071 tight 3-by-5 pavings is pictured here:



Let $a_{m,n}$ denote the number of tight m-by-n pavings.
(a) Determine $a_{3,n}$ as a function of n.
(b) Show for $m \geq 3$ that $\lim_{n\to\infty} a_{m,n}/m^n$ exists, and compute its value.

Composite solution by Richard Stong, Center for Communications Research, San Diego, CA, Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy, and O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. The answers are **(a)** $a_{3,n} = \frac{27}{4}3^n - 20 \cdot 2^n + n^2 + \frac{13}{2}n + \frac{53}{4}$ and **(b)** $\lim_{n\to\infty} a_{m,n}/m^n = m^{2m-1}/(m!)^2$.

A *paving* is any decomposition as described in the problem statement, except for dropping the requirement that the number of tiles is $m + n - 1$. We show that the minimum number of tiles in a paving is $m + n - 1$. The pavings achieving this minimum number of tiles are called *tight*. For convenience, we use *gridline* to mean one of the $m + n - 2$ horizontal or vertical lines that cross the rectangle internally at a positive integer distance from the sides. An *edge* is a side of any rectangle in the paving. A *segment* is a maximal connected union of edges along a single gridline. The condition for a paving is that every gridline contains at least one edge.

Lemma. *In a tight paving, no vertical segment crosses a horizontal segment (at an internal point of both), and the edges on any gridline form a single segment.*

Proof. In any paving, say that a tile T *witnesses* a horizontal gridline h if it is the leftmost tile whose top is on h and *witnesses* a vertical gridline v if it is the highest tile whose left side is on v. Note that (1) the tile U at the upper left corner witnesses no gridline, (2) each gridline is witnessed by exactly one tile, and (3) no tile witnesses more than one gridline (the segments at the top and left of a tile T witnessing horizontal and vertical gridlines would not continue leftward or upward, preventing the tiling from being completed).

These three observations imply that every paving has at least $m + n - 1$ tiles, so tight pavings are those with the fewest tiles, and every tile other than U in such a paving witnesses exactly one gridline. If two segments cross, then the crossing point is a corner of four tiles, and the one on the lower right of these four would witness no gridline.

For the second statement, suppose by symmetry that a horizontal gridline h contains more than one segment. Let T_1 be the tile witnessing h, and let edge E be a leftmost edge on the next segment along h. Since the segment containing E does not extend leftward, the portion of h to the left of E is internal to some tile T_2. Now the left endpoint of E is the upper left corner of a tile T_3 that does not witness the gridline for its top or left edge, contradicting that every tile other than U witnesses a gridline. ∎

(a) An m-by-n rectangle has $m - 1$ horizontal gridlines. By the lemma, every tight paving contains exactly one segment on each horizontal gridline. Let H_j denote the interval obtained by projecting the segment from the gridline at height j onto the horizontal axis.

For $m = 3$, consider first the case where $H_1 = H_2$ (as in Figure 1, where $x_2 = 3$). Since neither horizontal segment extends and each gridline contains a single segment, there are no horizontal edges not on these segments, so all the tiles to the left and right of these horizontal segments have width 1 and height 3.

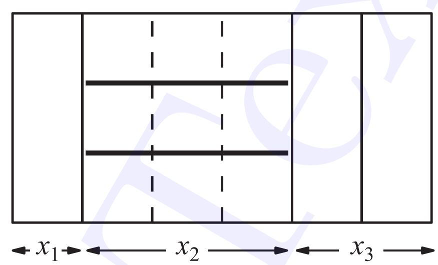


Figure 1. Horizontal segments of equal extent.

Now consider the vertical segments between the endpoints of the two horizontal segments. Since segments cannot cross, each of these $x_2 - 1$ vertical gridlines contains a segment of length one in one of three possible places, and all such choices yield pavings. Each insertion of a vertical segment increases the number of tiles by 1, so there are $3 + x_2 - 1$ tiles along the horizontal segments and $n - x_2$ tiles outside them, totaling $n + 2$.

Letting N be the number of tight pavings in this case, we have $N = \sum_{x \in P_1} 3^{x_2 - 1}$, where P_1 is the set of nonnegative integer triples (x_1, x_2, x_3) with sum n such that $x_2 \geq 1$. Using $[z^n] f(z)$ to mean the coefficient of z^n in $f(z)$, we have

$$N = [z^n] \sum_{x_1 \geq 0} z^{x_1} \sum_{x_2 \geq 1} \frac{1}{3}(3z)^{x_2} \sum_{x_3 \geq 0} z^{x_3} = [z^n] \frac{1}{1-z} \frac{z}{1-3z} \frac{1}{1-z}.$$

There are four other cases, illustrated in Figure 2. The intervals H_1 and H_2 may have no positive overlap, have overlap without containment, exhibit strict containment at both ends, or be equal at one end. Due to reflections, the first three of these cases may occur in two ways, the last in four ways.

These cases lead, in the same way as above, to four generating functions. For each case, the contribution to $a_{3,n}$ will be a sum over nonnegative choices of the variables summing to n, where variables giving lengths of portions of the horizontal segments must be positive. For a variable x measuring a portion covered by both horizontal segments, the factor in the

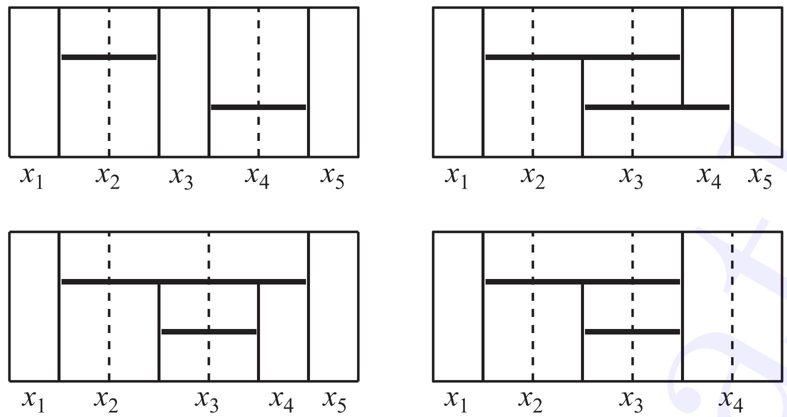


Figure 2. The remaining four cases.

number of choices is 3^{x-1}; for a portion covered by only one of the horizontal segments, it is 2^{x-1} (again because no two segments cross). We obtain the following contributions.

Case	#Tilings	Generating Function
0 (Figure 1)	$\sum 3^{x_2-1}$	$\frac{z}{(1-z)^2(1-3z)}$
1 (Figure 2)	$2\sum 2^{x_2-1}2^{x_4-1}$	$\frac{2z^2}{(1-z)^3(1-2z)^2}$
2 (Figure 2)	$2\sum 2^{x_2-1}3^{x_3-1}2^{x_4-1}$	$\frac{2z^3}{(1-z)^2(1-2z)^2(1-3z)}$
3 (Figure 2)	$2\sum 2^{x_2-1}3^{x_3-1}2^{x_4-1}$	$\frac{2z^3}{(1-z)^2(1-2z)^2(1-3z)}$
4 (Figure 2)	$4\sum 2^{x_2-1}3^{x_3-1}$	$\frac{4z^2}{(1-z)^2(1-2z)(1-3z)}$

The sum of the five rational functions is $\frac{z(1+3z)}{(1-z)^3(1-2z)(1-3z)}$, which has partial fraction expansion

$$\frac{27/4}{1-3z} - \frac{20}{1-2z} + \frac{2}{(1-z)^3} + \frac{7/2}{(1-z)^2} + \frac{31/4}{1-z}.$$

Thus

$$\begin{aligned} a_{3,n} &= \frac{27}{4}3^n - 20\cdot 2^n + 2\binom{n+2}{2} + \frac{7}{2}(n+1) + \frac{31}{4} \\ &= \frac{27}{4}3^n - 20\cdot 2^n + n^2 + \frac{13}{2}n + \frac{53}{4}. \end{aligned}$$

(b) Let $\lambda_m = \lim_{n\to\infty} a_{m,n}/m^n$. Asymptotically, we can restrict to tight pavings where $H_1, \dots, H_{m-1}$ have a common subinterval of positive length. The reason is that the number of tight pavings yielding no such overlap is less than $n^{2(m-1)}(m-1)^{n-1}$ (and the ratio of this to m^n tends to 0 as $n \to \infty$). To see this, note first that each of $H_1, \dots, H_{m-1}$ can be specified in fewer than n^2 ways. For the vertical segments, since each gridline has one segment and they don't cross, the lack of a common horizontal overlap implies that there are at most $m-1$ ways to place each vertical segment (extending part **(a)**). Let $\hat{a}_{m,n}$ be the number of tight pavings of the m-by-n rectangle where $H_1, \dots, H_{m-1}$ have a common overlap.

For any paving counted by $\hat{a}_{m,n}$, we partition the interval $[0, n]$ into three subintervals of lengths k, d, and l, where d is the positive length of $\bigcap H_i$, k is the length of the part of

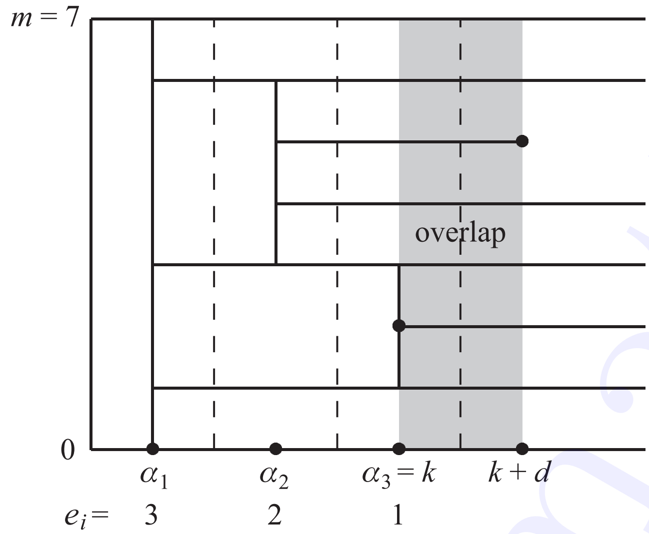


Figure 3. Part of a tight paving with $(m, k) = (7, 5)$ and multiset $[1^3, 3^2, 5^1]$.

the gridlines to its left, and l is the remaining length to the right. Some H_i starts at k, and some H_i ends at $k + d$.

The left ends of $H_1, \dots, H_{m-1}$ form a multiset of size $m - 1$ from $\{0, \dots, k\}$, using k at least once. With $H_i = [a_i, b_i]$, let $\alpha_1, \dots, \alpha_r$ in increasing order be the values occurring as some a_i, having multiplicities $e_1, \dots, e_r$. Write the multiset as $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$.

The key restriction on the list $a_1, \dots, a_{m-1}$ is that if $a_i = a_j = \beta$ with $i < j$, then $a_t \ge \beta$ for all t with $i < t < j$. Since H_i and H_j do not extend leftward of a_i, the points (β, i) and (β, j) lie on vertical edges. Since each vertical gridline contains only one segment, (β, t) is internal to the vertical segment at horizontal position β. Since $b_t \ge k + d > \beta$ and segments cannot cross, $a_t \ge \beta$.

With this restriction, we count the ways to form the list $a_1, \dots, a_{m-1}$ using the multiset $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$. The restriction implies that the copies of α_j in $a_1, \dots, a_{m-1}$ occupy e_j consecutive blank positions among the $m - 1 - \sum_{i=j+1}^{r} e_i$ blank positions left by placing the copies of all α_i with $i > j$. Since e_j copies of α_j must be placed, there are $m - \sum_{i=j}^{r} e_i$ possible places to start the copies of α_j, regardless of how the larger values were placed. Since $\sum_{i=1}^{r} e_i = m - 1$, the number of configurations of the left endpoints corresponding to the given multiset is $\prod_{j=1}^{r-1} \left(1 + \sum_{i=1}^{j} e_i\right)$.

Between the horizontal positions α_j and α_{j+1} are $\alpha_{j+1} - \alpha_j - 1$ vertical gridlines. No horizontal segments end at these gridlines. Hence the segment on each such vertical gridline is a single edge joining two of the horizontal segments (including the top and bottom edges) that start at position α_j or earlier. That gives $1 + \sum_{i=1}^{j} e_i$ choices for the vertical segment.

After forming the list $a_1, \dots, a_{m-1}$ and placing the vertical segments, we have $\prod_{j=1}^{r-1} \left(1 + \sum_{i=1}^{j} e_i\right)^{\alpha_{j+1}-\alpha_j}$ ways to form the left part of the paving from the given multiset. Let $s_{m,k}$ denote the sum of these quantities over all multisets of size $m - 1$ chosen from $\{0, \dots, k\}$.

We can write a multiset $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$ as $[0^{f_0}, \dots, k^{f_k}]$ by including the multiplicities of the unused elements, which equal 0. We then have

$$\prod_{j=1}^{r-1} \left(1 + \sum_{i=1}^{j} e_i\right)^{\alpha_{j+1}-\alpha_j} = \prod_{l=0}^{k-1} \left(1 + \sum_{i=0}^{l} f_i\right) = \prod_{l=0}^{k-1} c_l,$$

where $c_l = 1 + \sum_{i=0}^{l} f_i$. The list $c_0, \dots, c_{k-1}$ is a weakly increasing integer list with values between 1 and $m-1$. Over all choices of the multiset $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$ from $\{0, \dots, k\}$, we obtain all such lists. That is, $s_{m,k} = \sum_{c \in L_{m,k}} \prod_{l=0}^{k-1} c_l$, where $L_{m,k}$ is the set of all k-element nonnegative integer lists c such that $1 \le c_0 \le \cdots \le c_{k-1} \le m-1$.

Within the central overlap portion, each vertical gridline must have a single edge of length 1; there are m^{d-1} ways to place these. The right portion of the paving is constructed symmetrically to the left portion, over an interval of length $n-d-k$. Thus

$$\hat{a}_{m,n} = \sum_{d=1}^{n} \sum_{k=0}^{n-d} s_{m,k} s_{m,n-d-k} m^{d-1}.$$

Replacing d with $n-k-l$, we write

$$\lambda_m = \lim_{n\to\infty} \frac{\hat{a}_{m,n}}{m^n} = \lim_{n\to\infty} \frac{1}{m} \sum_{k+l<n} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l}.$$

The key now is to replace the sum over a triangle of values with a sum over a square of values, separating the sums over k and l. We have

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l} \le \sum_{k+l<n} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l} \le \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l}.$$

As $n \to \infty$, the upper and lower bounds are the same; hence the limit of the middle expression must be the same as the limit of the outer expressions.

Thus $\lambda_m = \frac{1}{m} \left(\sum_{k=0}^{\infty} s_{m,k}/m^k \right)^2$. To turn this into the desired limit $m^{2m-1}/(m!)^2$, it suffices to prove $\sum_{k=0}^{\infty} s_{m,k}/m^k = m^{m-1}/(m-1)!$. To do this, we compute

$$\sum_{k=0}^{\infty} \frac{s_{m,k}}{m^k} = \sum_{k=0}^{\infty} \sum_{c \in L_{m,k}} \prod_{i=0}^{k-1} \frac{c_i}{m} = \prod_{q=0}^{m-1} \sum_{t=0}^{\infty} \left(\frac{q}{m} \right)^t = \prod_{q=0}^{m-1} \frac{1}{1-q/m} = \frac{m^{m-1}}{(m-1)!}.$$

To justify the second equality here, note that the double sum $\sum_{k=0}^{\infty} \sum_{c \in L_{m,k}}$ encounters every multiset of values chosen from $\{0, \dots, m-1\}$. Over the full sum, any multiplicity of a given value q is grouped with all possible multiplicities of other values. Hence we can regroup the terms by the values, leading to the product of infinite sums for each of the values.

Editorial comment. The sequence in part **(a)** appears as sequence A285361 at oeis.org.

Also solved by H. K. Pillai (India) and M. A. Prasad (India; part (a) only).

A Hyperbolic Limit of Trigonometric Matrices

12014 [2018, 81]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let a, b, c, and d be real numbers with $bc > 0$. Calculate

$$\lim_{n\to\infty} \begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix}^n.$$

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. The limit is

$$\begin{bmatrix} \cosh\sqrt{bc} & \sqrt{b/c}\,\sinh\sqrt{bc} \\ \sqrt{c/b}\,\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{bmatrix}.$$

Letting

$$A_n = \begin{bmatrix} \cos(a/n) - 1 & \sin(b/n) \\ \sin(c/n) & \cos(d/n) - 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix} = I + A_n,$$

where I is the 2×2 identity matrix. When n is large enough, $\|A_n\| < 1$ and

$$\log(I + A_n) = A_n - \frac{1}{2}A_n^2 + \frac{1}{3}A_n^3 - \frac{1}{4}A_n^4 + \cdots .$$

Since

$$A_n = \begin{bmatrix} O(1/n^2) & b/n + O(1/n^3) \\ c/n + O(1/n^3) & O(1/n^2) \end{bmatrix},$$

we have $\log(I + A_n) = A_n + O(1/n^2)$ and $n \log(I + A_n) = nA_n + O(1/n)$. Since

$$\lim_{n\to\infty} n \log(I + A_n) = \lim_{n\to\infty} nA_n = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix},$$

we obtain

$$\lim_{n\to\infty} \begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix}^n = \lim_{n\to\infty} \exp(n \log(I + A_n)) = \exp\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right).$$

If $bc > 0$, then the matrix $\left[\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right]$ has distinct eigenvalues $\sqrt{bc}$ and $-\sqrt{bc}$, and

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{b} & \sqrt{b} \\ \sqrt{c} & -\sqrt{c} \end{bmatrix} \begin{bmatrix} \sqrt{bc} & 0 \\ 0 & -\sqrt{bc} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{b}} & \frac{1}{2\sqrt{c}} \\ \frac{1}{2\sqrt{b}} & -\frac{1}{2\sqrt{c}} \end{bmatrix},$$

where $b, c > 0$. Thus

$$\begin{aligned} \exp\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right) &= \begin{bmatrix} \sqrt{b} & \sqrt{b} \\ \sqrt{c} & -\sqrt{c} \end{bmatrix} \begin{bmatrix} e^{\sqrt{bc}} & 0 \\ 0 & e^{-\sqrt{bc}} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{b}} & \frac{1}{2\sqrt{c}} \\ \frac{1}{2\sqrt{b}} & -\frac{1}{2\sqrt{c}} \end{bmatrix} \\ &= \begin{bmatrix} \cosh\sqrt{bc} & \sqrt{b/c}\,\sinh\sqrt{bc} \\ \sqrt{c/b}\,\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{bmatrix}. \end{aligned}$$

The case where $b, c < 0$ is similar.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Chapman (U. K.), H. Chen, G. Fera (Italy), D. Fleischman, C. Georghiou (Greece), J. Grivaux (France), A. Goel, E. A. Herman, Y. Hu (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Minasyan (Russia), R. Nandan, M. Omarjee, F. Perdomo & Á. Plaza (Spain), K. Schilling, J. Singh (India), J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), N. Thornber, E. I. Verriest, Z. Vörös (Hungary), A. Wentworth, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

A Symmetric Sum

12016 [2018, 81]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For nonnegative integers m, n, r, and s, prove

$$\sum_{k=0}^{s} \binom{m+r}{n-k}\binom{r+k}{k}\binom{s}{k} = \sum_{k=0}^{r} \binom{m+s}{n-k}\binom{s+k}{k}\binom{r}{k}.$$

Solution by Nicole Grivaux, Paris, France. Let $A(r,s)$ be the left side of the equation to be proved. Throughout, we use the convention that $\binom{a}{b}=0$ whenever $b>a\geq 0$. By the Vandermonde identity and symmetry,

$$\binom{r+k}{k}=\sum_{i=0}^{r}\binom{r}{i}\binom{k}{k-i}=\sum_{i=0}^{r}\binom{r}{i}\binom{k}{i}.$$

Hence

$$\begin{aligned}A(r,s)&=\sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{s}{k}\binom{k}{i}\\&=\sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{s}{i}\binom{s-i}{k-i}\\&=\sum_{i=0}^{\min(r,s)}\binom{r}{i}\binom{s}{i}\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{s-i}{k-i}\\&=\sum_{i=0}^{\min(r,s)}\binom{r}{i}\binom{s}{i}\binom{m+r+s-i}{n-i}.\end{aligned}$$

The second equality follows from $\binom{s}{k}\binom{k}{i}=\binom{s}{i}\binom{s-i}{k-i}$, while the fourth is another application of the Vandermonde identity. Since the final form is symmetric in r and s, we conclude $A(r,s)=A(s,r)$, which is the desired equality.

Also solved by U. Abel (Germany), H. Almusawa & N. Alobaidan & R. Jacobs & D. Nuraliyev & J. Shive & M. Apagodu, T. Amdeberhan & V. H. Moll, R. Chapman (U. K.), S. B. Ekhad, R. Evans, G. Fera (Italy), D. Fleischman, O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), J. C. Smith, A. Stadler (Switzerland), R. Stong, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposers.

Euler's Totient is Sparse

12021 [2018, 179]. *Proposed by Omar Sonebi, Lycée Technique, Settat, Morocco.* Let ϕ be the Euler totient function. Given $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}^+$, show that there exists $n\in\mathbb{Z}^+$ such that $an+b$ is not in the range of ϕ.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $d=\gcd(a,b)$, with $a=dr$ and $b=ds$. Set $t=r\prod_{i=1}^{d}(is+1)$; note that t is relatively prime to s. By Dirichlet's theorem, there is a prime p of the form $tm+s$ for some $m\in\mathbb{Z}^+$. Let $n=tm/r$. We claim that $an+b$, which equals dp, is not in the range of ϕ. If $dp=\phi(N)$ for some $N\in\mathbb{Z}^+$ having prime factorization $\prod_{j=1}^{k}p_j^{e_j}$, then $dp=\prod_{j=1}^{k}p_j^{e_j-1}(p_j-1)$. Since $p-1>d$, we conclude that p is a factor of p_i-1 for some i. Now $p_i=qp+1$ for some q with $1\leq q\leq d$. Since $qp+1=q(tm+s)+1=(qs+1)+qmr\prod_{i=1}^{d}(is+1)$, this requires $qp+1$ to have $qs+1$ as a proper factor, so $qp+1$ cannot be prime. This contradiction completes the proof of the claim.

Editorial comment. Souvik Dey and Celia Schacht noted that the claim immediately follows from the more general result of S. S. Pillai (1929), On some functions connected with $\phi(n)$, *Bull. Amer. Math. Soc.* 35: 832–836, which implies that if $N(n)$ is the number of positive integers up to n that are in the range of ϕ, then $\lim_{n\to\infty}N(n)/n=0$.

Also solved by S. Chandrasekhar, A. Cheraghi (Canada), S. Dey (India), G. Fera (Italy), D. Fleischman, K. Gatesman, Y. J. Ionin, J. Kim (South Korea), O. P. Lossers (Netherlands), M. Omarjee (France), M. Reid, C. Schacht, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), AN-anduud Problem Solving Group (Mongolia), GCHQ Problems Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

PROBLEMS AND SOLUTIONS

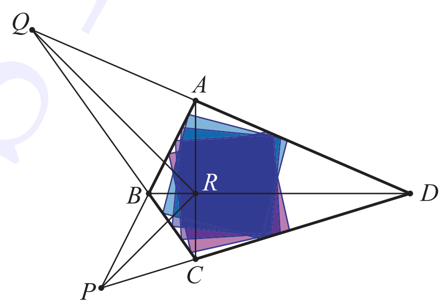
Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by February 29, 2020, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12132. *Proposed by K. S. Bhanu and Mukta Deshpande, Institute of Science, Nagpur, India, and P. G. Dixit, Modern College, Pune, India.* Let n be a positive integer, and let $X_0 = n+1$. Repeatedly choose the integer X_k uniformly at random among the integers j with $1 \le j < X_{k-1}$, stopping when $X_m = 1$.
(a) What is the expected value of m?
(b) What is the expected value of X_{m-1}?

12133. *Proposed by Daniel Hu, Los Altos High School, Los Altos, CA.* Let $ABCD$ be a convex quadrilateral. Suppose that lines AB and CD meet at P, lines AD and BC meet at Q, and AC and BD meet at R. Prove that there are infinitely many squares with one vertex on each side of $ABCD$ if and only if $AC \perp BD$ and $PR \perp QR$.



12134. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Evaluate the series

$$\sum_{n=1}^{\infty}\left(n\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)-1-\frac{1}{2n}\right).$$

12135. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Suppose that a triangle in the plane has inradius r, circumradius R, angles A, B, and C, and corresponding medians m_A, m_B, and m_C. Prove

$$3\sqrt{3}\frac{r^2}{R^3} \le \frac{\sin^3 A}{m_A}+\frac{\sin^3 B}{m_B}+\frac{\sin^3 C}{m_C} \le \frac{3\sqrt{3}(R-r)}{8r^2}.$$

doi.org/10.1080/00029890.2019.1642074
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uamm.

12136. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Prove

$$a^2+b^2+c^2 \geq a\sqrt[4]{\frac{b^4+c^4}{2}}+b\sqrt[4]{\frac{c^4+a^4}{2}}+c\sqrt[4]{\frac{a^4+b^4}{2}}$$

for all positive real numbers a, b, and c.

12137. *Proposed by Nikolai Beluhov, Stara Zagora, Bulgaria.* A *polyomino* is a region with connected interior that is a union of a finite number of squares from a grid of unit squares. Do there exist a positive integer n with $n \geq 5$ and a polyomino P contained entirely within an n-by-n grid such that P contains exactly 3 unit squares in every row and every column of the grid?

12138. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let P be a nonconstant polynomial with complex coefficients, and let $Q(x, y) = P(x) - P(y)$. Let k be the number of linear factors of $Q(x, y)$, and let $R(x, y)$ be a nonconstant factor of $Q(x, y)$ whose degree is less than k. Prove that $R(x, y)$ is a product of linear polynomials with complex coefficients.

SOLUTIONS

A Slow Shuffle

12008 [2017, 970]. *Proposed by P. Kórus, University of Szeged, Szeged, Hungary.* You hold in your hand a deck of n cards, numbered 1 to n from top to bottom. Shuffle them as follows. Put the top card in the deck on the bottom and the second card on the table. Repeat this step until all the cards are on the table.

(a) For which n does card number 1 end up at the top of the deck of cards on the table?

(b) Shuffle the deck a second time in the same way. For which n does card number 1 end up at the top of the cards on the table?

(c)* Shuffle the deck a third time in the same way. For which n does card number 1 end up at the top of the cards on the table?

(d)* For which n does this shuffle amount to a permutation consisting of a single cycle?

Solution to **(a)**, **(b)**, *and* **(c)** *by Yury J. Ionin, Central Michigan University, Mt. Pleasant, MI.* Let $\tau_n(i)$ denote the final position of card i resulting from one shuffle. We express τ_n as a composition of $n-1$ permutations on the positive integers. For $n \geq 2$, define a permutation σ_n on the positive integers by

$$\sigma_n(i) = \begin{cases} n-1 & \text{if } i = 1, \\ n & \text{if } i = 2, \\ i-2 & \text{if } 3 \leq i \leq n, \\ i & \text{if } i > n. \end{cases}$$

Here $1, \ldots, n$ represent the initial deck in the hand, values starting with $n+1$ represent the initial deck on the table, and σ_n moves the top element to the bottom of the first deck and the second element to the top of the second deck. Note that σ_2 is the identity. Letting $\tau_n = \sigma_2 \cdots \sigma_n$, the first three parts of the problem ask for the values of n such that $\tau_n(1)$, $\tau_n^2(1)$, or $\tau_n^3(1)$ equal 1.

We begin with a formula for $\tau_n(i)$. For any positive integer n, let $f(n)$ be the largest odd divisor of n; note that $f(n) = n$ when n is odd. For $n \geq 2$ and any positive integer i, we claim

$$\tau_n(i) = \begin{cases} \frac{1+f(2n+1-i)}{2} & \text{if } i \leq n, \\ i & \text{if } i > n\,. \end{cases}$$

We prove this by induction on n. The case $n = 2$ holds by inspection, so consider $n \geq 3$. If $i > n$, then $\sigma_k(i) = i$ for $2 \leq k \leq n$, so $\tau_n(i) = i$. When $3 \leq i \leq n$, the induction hypothesis yields

$$\tau_n(i) = \tau_{n-1}(i-2) = \frac{1 + f(2n-1-(i-2))}{2} = \frac{1 + f(2n+1-i)}{2}.$$

Note that $\tau_n(2) = n = \frac{1}{2}(1 + f(2n-1))$, as desired. Finally, for $i = 1$, the induction hypothesis yields

$$\tau_n(1) = \tau_{n-1}(n-1) = \frac{1 + f(n)}{2} = \frac{1 + f(2n+1-1)}{2}.$$

(a) The formula for τ yields $\tau_n(1) = (1 + f(n))/2$, so $\tau_n(1) = 1$ if and only if $f(n) = 1$. This occurs precisely when n is a power of 2.

(b) Since $f(n) \leq n$, we have $\tau_n(1) \leq (1+n)/2 < n$. Thus

$$\tau_n^2(1) = \tau_n\left(\frac{1 + f(n)}{2}\right) = \frac{1 + f\left(2n + 1 - \frac{1}{2}(1 + f(n))\right)}{2},$$

so $\tau_n^2(1) = 1$ if and only if $f(2n + 1 - (1 + f(n))/2) = 1$.

We prove first that this cannot happen when $f(n) \equiv -1 \bmod 4$. If $f(n) = 4m - 1$, then $\tau_n^2(1) = 1$ if and only if $f(2n + 1 - 2m) = 1$, which cannot occur since $2n + 1 - 2m$ is not a power of 2.

Hence we may assume $f(n) = 4m + 1$, where $m \in \mathbb{N}$ and $n = 2^k(4m+1)$. Now $\tau_n^2(1) = 1$ reduces to $f(2^{k+1}(4m+1) - 2m) = 1$, requiring $2^k(4m+1) - m$ to be a power of 2, say 2^s. That is, $(2^{k+2} - 1)m = 2^s - 2^k$ with $s \geq k$.

Since $2^{k+2} - 1$ and 2^k are relatively prime, $(2^{k+2} - 1) \mid (2^{s-k} - 1)$. It is an exercise in elementary number theory that $(2^a - 1) \mid (2^b - 1)$ requires $a \mid b$. To see this, write $b = aq + r$ with $0 \leq r < a$. From the formula for a geometric series, $2^a - 1$ divides $2^{aq} - 1$, so $2^a - 1$ divides $2^r(2^{aq} - 1)$, which equals $2^b - 2^r$. Now $2^a - 1$ divides the difference $(2^b - 1) - (2^b - 2^r)$, which equals $2^r - 1$. Since $r < a$, this requires $r = 0$, so $a \mid b$. Thus $(k+2) \mid (s-k)$, which implies $(k+2) \mid (s+2)$.

From $m = (2^s - 2^k)/(2^{k+2} - 1)$, we have $4m + 1 = (2^{s+2} - 1)/(2^{k+2} - 1)$. Since $n = 2^k(4m+1)$, we thus have the following answer: $\tau_n^2(1) = 1$ if and only if $n = 2^k(2^{s+2} - 1)/(2^{k+2} - 1)$ with $s \geq k \geq 0$ and $k + 2$ dividing $s + 2$. The values under 1000 are the powers of 2 together with 5, 18, 21, 68, 85, 146, 264, and 341.

(c) Now consider the equation $\tau_n^3(1) = 1$. By the formula for τ_n, we have $\tau_n(i) = 1$ if and only if $2n + 1 - i$ is a power of 2; that is, if and only if $i = 2n + 1 - 2^k$ for some k with $n < 2^k \leq 2n$. (There is exactly one such k for each n.) Note also that $\tau_n(\tau_n(1)) = \frac{1}{2}\left(1 + f(2n + 1 - \tau_n(1))\right)$. Writing the condition $\tau_n^3(1) = 1$ as $\tau_n(\tau_n(1)) = \tau_n^{-1}(1)$, the requirement reduces to

$$\frac{1}{2}\left(1 + f(2n + 1 - \tau_n(1))\right) = 2n + 1 - 2^k,$$

or $f(2n + 1 - \tau_n(1)) = 4n + 1 - 2^{k+1}$. This requires that $2n + 1 - \tau_n(1) = 2^l(4n + 1 - 2^{k+1})$ for some nonnegative l, or $\tau_n(1) = 2^{k+l+1} - 2^l + 1 - (2^{l+2} - 2)n$. Since $\tau_n(1) = (1 + f(n))/2$, we require $f(n) = 2^{k+l+2} - 2^{l+1} + 1 - (2^{l+3} - 4)n$, which implies $n = 2^m\left(2^{k+l+2} - 2^{l+1} + 1 - (2^{l+3} - 4)n\right)$ for some nonnegative m.

Thus the problem reduces to finding solutions of

$$(2^{l+m+3} - 2^{m+2} + 1)n = 2^m(2^{k+l+2} - 2^{l+1} + 1), \qquad (*)$$

where k, l, m are nonnegative integers and $n < 2^k \leq 2n$.

If n is a solution, then $2^{l+m+3} - 2^{m+2} + 1$ divides $2^{k+l+2} - 2^{l+1} + 1$, so $2^{l+m+3} + 2^{l+1} + 1 \le 2^{k+l+2} + 2^{m+2} + 1$. If $m \ge k$, then $2^{l+m+3} = 1 + \sum_{i=0}^{l+m+2} 2^i > 2^{k+l+2} + 2^{m+2} + 1$. Hence we require $m \le k - 1$.

It remains to ensure $n < 2^k \le 2n$. By replacing n with 2^k in $(*)$ and rearranging terms, we have $n < 2^k$ if and only if $2^{k+l+m+2} + 2^k + 2^{l+m+1} > 2^{k+m+2} + 2^m$, which is true for $l \ge 0$.

Similarly, by replacing n by 2^{k-1} in $(*)$, we have $2n \ge 2^k$ if and only if $2^{l+m+2} + 2^k \le 2^{k+m+2} + 2^{m+1}$. This inequality holds if and only if $l < k$ or $l = k$ and $m + 1 \ge k$. We thus have the following answer: $\tau_n^3(1) = 1$ if and only if

$$n = 2^m(2^{k+l+2} - 2^{l+1} + 1)/(2^{l+m+3} - 2^{m+2} + 1),$$

where $2^{l+m+3} - 2^{m+2} + 1$ divides $2^{k+l+2} - 2^{l+1} + 1$ and either $l < k$ and $m < k$ or $l = k = m + 1$ (which gives $n = 2^m$). The values under 1000 are the powers of 2 together with 3, 10, 14, 36, 51, 60, 136, 141, 248, 528, 819, and 910. To obtain infinitely many examples, let $(k, l, m) = (20t + 4, 1, 1)$ for $t \ge 0$. The resulting value is $(2^{20t+8} - 6)/25$.

Solution to **(d)** *by Richard Stong, Center for Communications Research, San Diego, CA.* The values of n for which the shuffle is a full cycle are those n such that $4n + 1$ is prime and 2 is a primitive root modulo $4n + 1$. In particular, the values of n under 100 are 1, 3, 7, 9, 13, 15, 25, 37, 43, 45, 49, 67, 73, 79, 87, 93, and 97 (see oeis.org/A137310).

To prove the result, we use an alternative description of the shuffle as an iterative process on a pile of n cards. For $n - 1$ steps, indexed from $j = 0$ to $j = n - 2$, at step j take the top two cards and reinsert them with j cards below them. Steps j through $n - 2$ do not change the bottom j cards; these are the cards "on the table" during that time. The remaining $n - j$ cards are still "in the hand." Putting the top two cards between these sets (and incrementing j) moves the top card to the bottom of the deck in hand and puts the next card on the table. The jth step is an even permutation (two steps of rotating the top $n - j$ cards up by one step). Thus the permutation induced by the shuffle is even. It follows that the permutation can be a full cycle only when n is odd.

We now express the shuffle as the permutation π_n that maps each position to the index of the card that occupies it. This is the inverse of τ_n, and it is a cycle if and only if τ_n is a cycle. It is also convenient to index the cards and the positions by the set S of odd integers from 1 to $2n - 1$, treating π_n as a permutation of S. That is, assign the card at position a the value $2a - 1$, which we call a'.

We use the formula for $\tau_n(i)$ to give a formula for $\pi_n(a')$, the modified value of the card ending in position a. We claim $\pi_n(a') = 4n + 1 - 2^{u(a')}a'$, where $u(a')$ is the unique positive integer such that $2^{u(a')}a' \in [2n + 2, 4n]$.

To see this, let $i = 2n + 1 - 2^{u(a')-1}a'$. We have $2n + 1 - i = 2^{u(a')-1}a'$. Since a' is odd, it is the largest odd divisor of $2n + 1 - i$; this is why we express π_n as a permutation of odd values. With $f(2n + 1 - i) = 2a - 1$, the formula $2\tau_n(i) - 1 = f(2n + 1 - i)$ yields $\tau_n(i) = a$. Thus $\pi_n(a') = 2i - 1$, as claimed.

We now show that the condition on n is necessary and sufficient for the shuffle to be a full cycle.

Necessity. Suppose that π_n is a cycle. Let $p = 4n + 1$. We have $\pi_n(a') \equiv -2^{u(a')}a' \bmod p$. Hence any value we can reach starting from $a' = 1$ by iterating π_n has the form $\pm 2^v \bmod p$. If q is a proper odd prime factor of p, then we cannot reach q; thus p must be prime. Since we have shown that n is odd and defined $p = 4n + 1$, we have $p \equiv 5 \bmod 8$. By the law of quadratic reciprocity, 2 is a square modulo an odd prime p if and only if p is congruent to 1 or 7 modulo 8. Hence 2 is not a square.

In addition, Fermat's little theorem implies that $2^{(p-1)/2}$ is congruent to ± 1 modulo p. The value is $+1$ if and only if 2 is a square. Hence -1 is a power of 2, modulo p. This means that all values we can reach have the form $2^v \bmod p$. For π to be a cycle, these powers must include all odd numbers from 1 to $2n-1$. Since -1 and 2 are also powers of 2 modulo p, we conclude that every nonzero value is a power of 2 modulo p, so 2 is a primitive root.

Sufficiency. Again letting $p = 4n+1$, suppose that p is prime and that 2 is a primitive root modulo p. Since 2 is a primitive root, 2 cannot be a square modulo p, so p is congruent to 3 or 5 modulo 8, again by quadratic reciprocity. Since p has the form $4n+1$, it follows that $p \equiv 5 \bmod 8$ and n is odd.

We prove that π_n^2, the result of shuffling twice, is a cycle; this implies that π_n itself is a cycle. Consider the application of π_n in terms of the cycle of powers of 2 modulo p. Suppose that $\pi_n(a') = b'$, meaning that the card in position a after the shuffle is b. Since $\pi_n(a') = 4n+1-2^{u(a')}a'$, we obtain b' from a' by multiplying by 2 successively to reach the interval $[2n+2, 4n]$ and then subtracting from $4n+1$. Since $a' \le 2n-1$, the result is odd and lies in S.

Modulo p, we have $b' \equiv -2^{u(a')}a'$. Thus b' is the negative of the value that is $u(a')$ steps beyond a' in the cycle of powers of 2. However, $2^{u(a')}a'$ is not in S. Applying the shuffle to obtain c' from b', we have $c' \equiv 2^{u(a')+u(b')}a'$. Thus π_n^2 moves each value some distance along the cycle of powers of 2 and returns each element of S to another value that when reduced modulo p lies in S.

Furthermore, $\pi_n^2(a')$ is the first value after a' in the cycle of powers of 2 that lies in S. Since 2 is a primitive root modulo p, that cycle visits all of S. Hence π_n^2 is a cycle through S, as desired.

Editorial comment. Because it is not known whether there are infinitely many primes for which 2 is a primitive root (this is the Artin conjecture), it is not known whether there are infinitely many examples for part (d). Several solvers observed that, given n, the card atop the shuffled deck is the number that solves the Josephus problem for a circle of n soldiers (see oeis.org/A006257).

Part (a) also solved by D. Fleischman, O. Geupel (Germany), and R. Prather. Parts (a) and (b) also solved by T. Ayton & A. Lopez & R. Tuminello, N. Grivaux (France), J. H. Lindsey II, O. P. Lossers (Netherlands), P. McPolin (UK), L. Meissner & E. Newman & R. Toth & S. Weigel, and the proposer. Parts (a), (b), and (c) also solved by GCHQ Problem Solving Group (UK) and R. Stong. All four parts solved by Armstrong Problem Solving Group.

Reducible Combinations of Elementary Symmetric Polynomials

12017 [2018, 82]. *Proposed by Mowaffaq Hajja, Philadelphia University, Amman, Jordan.* For $n \ge 2$, let R be the ring $F[t_1, \dots, t_n]$ of polynomials in n variables over a field F. For j with $1 \le j \le n$, let $s_j = \sum \prod_{i=1}^{j} t_{m_i}$, where the sum is taken over all j-element subsets $\{m_1, \dots, m_j\}$ of $\{1, \dots, n\}$. This is the *elementary symmetric polynomial* of degree j in the variables $t_1, \dots, t_n$. Let $f = \sum_{i=0}^{n} c_i s_i$ for some $c_0, \dots, c_n$ in F with $c_1, \dots, c_n$ not all 0. Show that f is reducible in R if and only if either $c_0 = \cdots = c_{n-1} = 0$ or $(c_0, \dots, c_n)$ is a geometric progression, meaning that there is $r \in F$ such that $c_i = rc_{i-1}$ for all i with $1 \le i \le n$.

Solution by Michael Reid, University of Central Florida, Orlando, FL. For sufficiency, f factors as $c_n \prod_{i=1}^{n} t_i$ if $c_0 = \cdots = c_{n-1} = 0$ and as $c \prod_{i=1}^{n}(1 + rt_i)$ if $(c_0, c_1, \dots, c_n) = (c, cr, \dots, cr^n)$ with $c, r \ne 0$.

For necessity, suppose that f is reducible, and let g be an irreducible factor. For each t_i, since f has degree 1 in t_i, g has degree 0 or 1 in t_i. Moreover, since g is nonconstant, it has degree 1 in at least one variable. We claim that g has degree 0 in all of the other variables.

To prove the claim, suppose otherwise. By symmetry, we may assume that g has degree 1 in both t_1 and t_2. Since f/g is not constant, g has degree 0 in t_k for some k. Note that f is fixed by the automorphism of R that interchanges t_2 and t_k and fixes all other variables. The image g' of g under this automorphism also divides f. Note that R is a unique factorization domain, and g and g' are not constant multiples of each other since they have different degrees in t_2. Since f is divisible by both g and g', it follows that f is divisible by gg'. This is a contradiction, since gg' has degree 2 in t_1.

By the claim, $g = a + bt_j$ for some j and some $a, b \in F$ with $b \neq 0$. Let $g_i = a + bt_i$ for $1 \leq i \leq n$. By symmetry, f is divisible by each g_i. It follows that $f = hg_1 \cdots g_n$ with $h \in R$. Since $g_1 \cdots g_n$ has degree 1 in each variable, h has degree 0 in each variable; thus h is a nonzero constant. If $a = 0$, then f has the required form, with $(c_0, \dots, c_n) = (0, \dots, 0, hb^n)$. If $a \neq 0$, then f also has the required form, with $c_i = ha^n r^i$, where $r = b/a \neq 0$.

Also solved by A. J. Bevelacqua, D. Fleischman, O. P. Lossers (Netherlands), J. C. Smith, R. Stong, Missouri State University Problem Solving Group, and the proposer.

Exponentiating Until a Triangle Vanishes

12018 [2018, 82]. *Proposed by Zachary Franco, Houston, TX.* For $n > 1$, let $k(n)$ be the largest integer k for which there exists a triangle with sides of length n^k, $(n+4)^k$, and $(n+5)^k$. Find $\lim_{n\to\infty} k(n)/n$.

Solution by GCHQ Problem Solving Group, Cheltenham, UK. Since $n < n+4 < n+5$, a triangle exists if and only if $n^k + (n+4)^k > (n+5)^k$. For fixed n, consider

$$f(x) = (n+5)^x - (n+4)^x - n^x,$$

whose derivative $f'(x)$ is

$$\begin{aligned}&(n+5)^x \log(n+5) - (n+4)^x \log(n+4) - n^x \log(n)\\ &\quad= \log(n+4)((n+5)^x - (n+4)^x - n^x) + (n+5)^x \log\frac{n+5}{n+4} + n^x \log\frac{n+4}{n}.\end{aligned}$$

At any point where $f(x) \geq 0$, we have $f'(x) > 0$. Therefore $f(x)$ has at most one positive zero. Now $f(1) < 0$, and $f(x) \to \infty$ as $x \to \infty$. Thus for each n, $f(x)$ has a unique positive zero $K(n)$. For $k < K(n)$ it follows that $f(k) < 0$, and a triangle exists. For $k > K(n)$ we have $f(k) > 0$, and no triangle exists. Thus $K(n) - 1 \leq k(n) \leq K(n)$. Let $\lambda = \lambda(n)$ be the root of the equation

$$\left(1 + \frac{5}{n}\right)^{n\lambda} - \left(1 + \frac{4}{n}\right)^{n\lambda} = 1. \qquad (*)$$

For large y, let $g(y) = (1 + \frac{a}{y})^y$. Note that

$$\log\big(g(y)\big) = y \log\left(1 + \frac{a}{y}\right) = y\left(\frac{a}{y} - \frac{a^2}{2y^2} + \frac{a^3}{3y^3} + O(\frac{1}{y^4})\right) > a + \log\left(1 - \frac{a^2}{y}\right).$$

Use this with $a \in \{5\lambda, 4\lambda\}$ and $y = n\lambda$ to deduce that for sufficiently large n and positive λ,

$$e^{5\lambda} > \left(1 + \frac{5}{n}\right)^{n\lambda} > e^{5\lambda}\left(1 - \frac{25\lambda}{n}\right) \quad \text{and} \quad e^{4\lambda} > \left(1 + \frac{4}{n}\right)^{n\lambda} > e^{4\lambda}\left(1 - \frac{16\lambda}{n}\right).$$

These inequalities imply

$$e^{5\lambda}\left(1 - \frac{25\lambda}{n}\right) - e^{4\lambda} < \left(1 + \frac{5}{n}\right)^{n\lambda} - \left(1 + \frac{4}{n}\right)^{n\lambda} < e^{5\lambda} - e^{4\lambda}\left(1 - \frac{16\lambda}{n}\right).$$

The middle term of the inequalities is 1 owing to $(*)$. Thus $\lambda \to \mu$ as $n \to \infty$, where μ is the root of the equation $e^{5\mu} - e^{4\mu} = 1$, Setting $X = e^{\mu}$, we see that X is the unique positive root of the quintic

$$X^5 - X^4 - 1 = (X^2 - X + 1)(X^3 - X - 1) = 0,$$

which is

$$X = \frac{\sqrt[3]{9 - \sqrt{69}} + \sqrt[3]{9 + \sqrt{69}}}{\sqrt[3]{18}}.$$

Since $n\lambda - 1 \le k(n) \le n\lambda$, the required limit must be μ, which is $\log X$.

For $0 \le \alpha < \beta$, the method clearly generalizes to lengths n^k, $(n+\alpha)^k$, and $(n+\beta)^k$.

Also solved by K. F. Andersen (Canada), R. Chapman (UK), E. Donelson, G. Fera (Italy), D. Fleischman, C. Gabor & J. Zacharias, O. Geupel (Germany), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), K. Park (South Korea), F. Perdomo & Á. Plaza (Spain), M. Reid, N. C. Singer, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Terr, Z. Vörös (Hungary), and the proposer.

An Exponential Diophantine Equation

12019 [2018, 82]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Find all positive integers n such that $(2^n - 1)(5^n - 1)$ is a perfect square.

Solution by Sae Ho Jung, Seoul Science High School, Seoul, South Korea. The only such integer is 1.

Let $f(n) = (2^n - 1)(5^n - 1)$. Note that $f(1)$ is a perfect square and that $f(2)$ and $f(3)$ are not. Suppose that $n \ge 4$ and that $f(n)$ is a perfect square.

If n is odd, then $(2^n - 1)(5^n - 1) \equiv 1 - 2^n \pmod 5$, which is a quadratic residue modulo 5 only if $n \equiv 1 \pmod 4$. In that case

$$(2^n - 1)(5^n - 1) \equiv (-1)(5 \times 625^{(n-1)/4} - 1) \equiv 12 \pmod{16},$$

since $2^n \equiv 0$ and $625 \equiv 1 \pmod{16}$. Since 12 is not a quadratic residue modulo 16, we conclude that $f(n)$ is not a perfect square when n is odd and exceeds 4.

If n is even, then let $n = 2k$. Write $2^{2k} - 1 = dp^2$ and $5^{2k} - 1 = dr^2$, where $d = \gcd(2^n - 1, 5^n - 1)$; we have $(2^n - 1)(5^n - 1) = (dpr)^2$. However, 2^k and 5^k are both now solutions for x in $x^2 - dy^2 = 1$. If d is a perfect square, then $(2^k)^2$ and dp^2 are squares differing by 1, which cannot happen. For nonsquare d, the equation $x^2 - dy^2 = 1$ is a Pell equation with solution set $\{(x_i, y_i) \colon i \ge 0\}$, where $(x_0, y_0) = (1, 0)$, (x_1, y_1) is a so-called fundamental solution, and $x_i + y_i\sqrt{d} = (x_1 + y_1\sqrt{d})^i$ for $i \ge 2$. It is known that these solutions satisfy the recurrence $x_{i+1} = 2x_1x_i - x_{i-1}$ for $i \ge 1$.

If $x_1 \equiv 0 \pmod 5$, then $x_{2a} \equiv x_{2a-2} \equiv \cdots \equiv x_0 \equiv 1 \pmod 2$ and $x_{2a+1} \equiv x_{2a-1} \equiv \cdots \equiv x_1 \equiv 0 \pmod 5$. Neither can be satisfied when $x_i = 2^k$.

In the remaining cases, we show that x_i can never be a multiple of 5 and hence cannot be a power of 5. If $x_1 \equiv 1 \pmod 5$, then $x_i \equiv 1 \pmod 5$ for all i. If $x_1 \equiv 2 \pmod 5$, then $(x_0, x_1, x_2) \equiv (1, 2, 2) \pmod 5$, and the pattern $1, 2, 2 \pmod 5$ repeats. If $x_1 \equiv 3 \pmod 5$, then $(x_0, x_1, x_2, x_3, x_4, x_5) \equiv (1, 3, 2, 4, 2, 3) \pmod 5$, and that pattern repeats. If $x_1 \equiv 4 \pmod 5$, then the values modulo 5 alternate between 1 and 4.

Thus $f(n)$ is a perfect square if and only if $n = 1$.

Editorial comment. Many solvers noted that this result appears in László Szalay (2000), On the Diophantine equations $(2^n - 1)(3^n - 1) = x^2$, *Publ. Math. Debrecen*, 57(1–2): 1–9.

Also solved by M. Aassila (France), C. Boggs & E. J. Ionaşcu, S. Dey (India), K. T. L. Koo (China), O. P. Lossers (Netherlands), M. Reid, J. P. Robertson, C. Schacht, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), Z. Vörös (Hungary), and the proposer.

An Inequality for the Brocard Angle

12020 [2018, 179]. *Proposed by Erhard Braune, Linz, Austria.* Let α, β, and γ be the radian measures of the three angles of a triangle, and let ω be the radian measure of its Brocard angle. (The *Brocard angle* of triangle ABC is the angle TAB, where T is the unique point such that $\angle TAB$, $\angle TBC$, and $\angle TCA$ are congruent.) Yff's inequality asserts that $8\omega^3$ is a lower bound for $\alpha\beta\gamma$. Show that $\omega\pi^3/4$ is an upper bound for the same product.

Solution by Kyle Gatesman, student, Thomas Jefferson High School, Alexandria, VA, and the editors. We prove the stronger inequality $\alpha\beta\gamma \le \omega\gamma(\pi-\gamma)$, in which γ is the largest vertex angle of the triangle. Note that $\omega\gamma(\pi-\gamma)$ reaches a maximum of $\omega\pi^2/4$, which is a smaller upper bound than the proposed $\omega\pi^3/4$. Since $\pi-\gamma=\alpha+\beta$, the stronger inequality is equivalent to

$$\frac{1}{\omega} \le \frac{1}{\alpha}+\frac{1}{\beta}.$$

To prove this, let $f(x)=1/x-\cot x$ for $x\in(0,\pi)$. Since $f'(x)>0$, we see that $f(x)$ is increasing. Furthermore,

$$f''(x)=\frac{2\left(\sin^3 x - x^3\cos x\right)}{x^3\sin^3 x}.$$

This is clearly positive for $x\in[\pi/2,\pi)$. It is also positive for $x\in(0,\pi/2)$, since

$$\sin x > x-\frac{x^3}{6} \quad \text{and} \quad \cos x < 1-\frac{x^2}{2}+\frac{x^4}{24}$$

imply

$$\sin^3 x - x^3\cos x > x^3\left(\left(1-\frac{x^2}{6}\right)^3 - 1 + \frac{x^2}{2} - \frac{x^4}{24}\right) = \frac{x^7}{24}\left(1-\frac{x^2}{9}\right) > 0.$$

Thus f is convex on $(0,\pi)$. By Jensen's inequality,

$$f(\alpha)+f(\beta)+f(\gamma) \ge 3f(\pi/3) = 9/\pi-\sqrt{3}.$$

By the well-known fact that $\omega\le\pi/6$, we have $6/\pi-\sqrt{3}=f(\pi/6)\ge f(\omega)$. Adding these two inequalities and invoking the well-known fact that $\cot\alpha+\cot\beta+\cot\gamma=\cot\omega$, we obtain

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} \ge \frac{1}{\omega}+\frac{3}{\pi}.$$

Since $\gamma\ge\pi/3$, this immediately implies that $\frac{1}{\omega}\le\frac{1}{\alpha}+\frac{1}{\beta}$.

Editorial comment. The elegant symmetrical inequality

$$\frac{1}{\omega}+\frac{3}{\pi} \le \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}.$$

appears in Abi-Khuzam, F. F., Boghossian, A. B. (1989), Some recent geometric inequalities, *Amer. Math. Monthly* 96(7): 576–589, although the proof above is more direct in establishing the convexity of $f(x)$.

Also solved by P. P. Dályay (Hungary), G. Fera (Italy), L. Peterson, R. Stong, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

An Identity of Euler, Revisited

12022 [2018, 179]. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let n be a positive integer, and let x be a real number not equal to -1 or 1. Prove

$$\sum_{k=0}^{n-1} \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} = n$$

and

$$\sum_{k=0}^{n-1}(-1)^k \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} x^{\binom{n-1-k}{2}} = n x^{\binom{n}{2}}.$$

Solution by Warren P. Johnson, Connecticut College, New London, CT. For such real x, let $S_n(x)$ denote the left side of the first identity. In the sum $S_{n+1}(x)$, write $1-x^{n+1}$ as $1-x^{n-k}+x^{n-k}\left(1-x^{k+1}\right)$, so

$$\frac{1-x^{n+1}}{1-x^{k+1}} = \frac{1-x^{n-k}}{1-x^{k+1}} + 1 - \left(1-x^{n-k}\right).$$

Now

$$S_{n+1}(x) = \sum_{k=0}^{n} \frac{\prod_{i=0}^{k}(1-x^{n-i})}{1-x^{k+1}} + \sum_{k=0}^{n}\prod_{i=0}^{k-1}(1-x^{n-i}) - \sum_{k=0}^{n}\prod_{i=0}^{k}(1-x^{n-i}).$$

Almost all the summands of the last two sums cancel, leaving only 1 from the first term of the middle sum and 0 from the last term of the last sum. Also, the upper index in the first sum can be reduced by 1 because the last term is 0. Thus $S_{n+1}(x) = S_n(x)+1$. Since $S_1(x)=1$, the first identity follows by induction on n.

Replacing x by $1/x$ in the first identity gives

$$\begin{aligned} n &= \sum_{k=0}^{n-1} \frac{\prod_{i=0}^{k}(1-x^{i-n})}{1-x^{-k-1}} \frac{\prod_{i=0}^{k} x^{n-i}}{x^{k+1}} \frac{x^{k+1}}{\prod_{i=0}^{k} x^{n-i}} \\ &= \sum_{k=0}^{n-1} \frac{\prod_{i=0}^{k}(x^{n-i}-1)}{x^{k+1}-1} x^{-n(k+1)} x^{\sum_{i=1}^{k+1} i} \\ &= \sum_{k=0}^{n-1} (-1)^k \frac{\prod_{i=0}^{k}(1-x^{n-i})}{1-x^{k+1}} x^{\binom{k+2}{2}-n(k+1)}. \end{aligned}$$

Applying $\binom{k+2}{2} - n(k+1) = \binom{n-1-k}{2} - \binom{n}{2}$ and multiplying by $x^{\binom{n}{2}}$ yields the second identity.

Editorial comment. Johnson noted that the first identity was published by Euler in 1753. See paper E190, *Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae*, in *Opera Omnia*, 1st series, vol. 14, pp. 516–541; a translation is available at eulerarchive.maa.org.

Also solved by K. F. Andersen (Canada), M. J. S. Belaghi (Turkey), A. Berkane (Romania), R. Chapman (UK), P. P. Dályay (Hungary), S. B. Ekhad, D. Fleischman, N. Grivaux (France), B. Karaivanov (USA) & T. S. Vassilev (Canada), K. T. L. Koo (China), O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), J. Minkus, V. Moll & T. Amdeberhan, M. Omarjee (France), M. Reid, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Van hamme (Belgium), S. H. Yu (South Korea), L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by April 30, 2020, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12146. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let n be an integer greater than 1, and let $[n]$ denote $\{1, \ldots, n\}$ as usual. Let $\pi_1, \pi_2, \ldots, \pi_{n!}$ be a list of the $n!$ permutations of $[n]$, ordered lexicographically with respect to the word $\pi_k(1)\pi_k(2)\cdots\pi_k(n)$. For example, with $n = 3$, the 6 words in order are 123, 132, 213, 231, 312, and 321.
(a) For $1 \leq k < n!$, let ψ_k be the permutation of $[n]$ defined by $\psi_k(i) = j$ if and only if $\pi_k(i) = \pi_{k+1}(j)$. What is the cardinality of $\{\psi_k : 1 \leq k < n!\}$?
(b) For $1 \leq k < n!$, let φ_k be the permutation of $[n]$ defined by $\varphi_k(\pi_k(j)) = \pi_{k+1}(j)$. What is the cardinality of $\{\varphi_k : 1 \leq k < n!\}$?

12147. *Proposed by Luis González, Houston, TX, and Tran Quang Hung, Hanoi National University, Hanoi, Vietnam.* Let $ABCD$ be a quadrilateral that is not a parallelogram. The *Newton line* of $ABCD$ is the line that connects the midpoints of the diagonals AC and BD. Let X be the intersection of the perpendicular bisectors of AB and CD, and let Y be the intersection of the perpendicular bisectors of BC and DA. Prove that XY is perpendicular to the Newton line of $ABCD$.

12148. *Proposed by Tibor Beke, University of Massachusetts, Lowell, MA.* Let p be a prime number, and let f be a symmetric polynomial in $p-1$ variables with integer coefficients. Suppose that f is homogeneous of degree d and that $p-1$ does not divide d. Prove that p divides $f(1, 2, \ldots, p-1)$.

12149. *Proposed by Mohammadhossein Mehrabi, Sala, Sweden.* Let Γ be the gamma function, defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$. Prove

$$x^x y^y\, \Gamma\left(\frac{x+y}{2}\right)^2 \leq \left(\frac{x+y}{2}\right)^2 \Gamma(x)\Gamma(y)$$

for all positive real numbers x and y.

12150. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.* Let $X_0, \ldots, X_n$ be independent random variables, each distributed uniformly on $[0, 1]$. Calculate the expected value of $\min_{1 \leq k \leq n} |X_0 - X_k|$.

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12151. *Proposed by Leonard Giugiuc and Cezar Alexandru Trancanau, Drobeta Turnu Severin, Romania, and Michael Rozenberg, Tel Aviv, Israel.* Let A, B, C, and M be points in the plane with A, B, and C distinct. Let A', B', and C' be the reflections through M of A, B, and C, respectively. Determine the minimum value of $AB'/AB + BC'/BC + CA'/CA$ under the constraint that
(a) A, B, C, and M are collinear.
(b) A, B, and C are not collinear.

12152. *Proposed by George Stoica, Saint John, NB, Canada.* Let f be a twice differentiable real-valued function on $[0, \infty)$ such that $f(0) = 1$, $f'(0) = 0$, and $f(x) f''(x) = 1$ for all positive x. Find $\lim_{x\to\infty} f(x)/(x\sqrt{\ln x})$.

SOLUTIONS

An Integral Involving Fractional Parts

12031 [2018, 277]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* **(a)** Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\} dx\, dy = 1 - \gamma,$$

where $\{a\}$ denotes the fractional part of a, and γ is Euler's constant.
(b) Let k be a nonnegative integer. Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\}^k dx\, dy = \int_0^1 \left\{ \frac{1}{x} \right\}^k dx.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. We begin with (**b**). More generally, we prove

$$\int_0^1 \int_0^1 f\left(\left\{ \frac{x}{1-xy} \right\} \right) dx\, dy = \int_0^1 f\left(\left\{ \frac{1}{x} \right\} \right) dx$$

for any bounded measurable function f on $[0, 1]$. To prove this, we first change variables to $u = 1/x - y$ and $v = y$. Thus $x = (u+v)^{-1}$ and $y = v$, and so we have $dx\, dy = (u+v)^{-2}\, dv\, du$. Since $u + v = 1/x \ge 1$, the new domain of integration consists of the two regions $\{(u, v) : 1 \le u < \infty, 0 \le v \le 1\}$ and $\{(u, v) : 0 \le u \le 1, 1 - u \le v \le 1\}$. Therefore

$$\begin{aligned}
&\int_0^1 \int_0^1 f\left(\left\{ \frac{x}{1-xy} \right\} \right) dx\, dy \\
&\quad = \int_1^\infty \int_0^1 f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{(u+v)^2}\, dv\, du + \int_0^1 \int_{1-u}^1 f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{(u+v)^2}\, dv\, du \\
&\quad = \int_1^\infty f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{u(u+1)}\, du + \int_0^1 f\left(\left\{ \frac{1}{u} \right\} \right) \left(1 - \frac{1}{u+1} \right) du.
\end{aligned}$$

Since $\{1/u\} = 1/u$ when $u > 1$, it remains to show

$$\int_0^1 f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{u+1}\, du = \int_1^\infty f\left(\frac{1}{u} \right) \frac{1}{u(u+1)}\, du.$$

To prove this, we substitute $w = 1/u$ in the integral on the left side, and then, later, $u = 1/(w - j)$:

$$\int_0^1 f\left(\left\{\frac{1}{u}\right\}\right)\frac{1}{u+1}\,du = \int_1^\infty f(\{w\})\frac{1}{w(w+1)}\,dw$$

$$= \sum_{j=1}^{\infty}\int_j^{j+1} f(w-j)\frac{1}{w(w+1)}\,dv$$

$$= \sum_{j=1}^{\infty}\int_1^\infty f\left(\frac{1}{u}\right)\frac{du}{(1+ju)(1+(j+1)u)}$$

$$= \int_1^\infty f\left(\frac{1}{u}\right)\frac{1}{u}\sum_{j=1}^{\infty}\left(\frac{1}{1+ju} - \frac{1}{1+(j+1)u}\right)du$$

$$= \int_1^\infty f\left(\frac{1}{u}\right)\frac{1}{u(u+1)}\,du.$$

(**a**) By (**b**) and the asymptotic formula $H_n = \log n + \gamma + O(1/n)$ for the harmonic numbers H_n,

$$\int_0^1\int_0^1\left\{\frac{x}{1-xy}\right\}dx\,dy = \int_0^1\left\{\frac{1}{x}\right\}dx = \sum_{j=1}^{\infty}\int_{1/(j+1)}^{1/j}\left(\frac{1}{x} - j\right)dx$$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}\left(-\log j + \log(j+1) - \frac{1}{j+1}\right)$$

$$= \lim_{n\to\infty}\left(\log(n+1) - (H_{n+1} - 1)\right)$$

$$= \lim_{n\to\infty}\left(\log(n+1) - (\log(n+1) + \gamma - 1)\right) = 1 - \gamma.$$

Editorial comment. The proposer and the GCHQ Problem Solving Group noted that when $k = 2$, the value of the integral in (**b**) is $\log(2\pi) - \gamma - 1$.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), H. Chen, G. Fera, K. Gatesman, M. L. Glasser, J. A. Grzesik, O. Kouba (Syria), J. H. Lindsey II, Y. Mikayelyan (Armenia), T. Amdeberhan & V. H. Moll, P. Perfetti (Italy), N. C. Singer, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

An Oscillating Binomial Sum

12032 [2018, 277]. *Proposed by David Galante (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* For a positive integer n, compute

$$\sum_{p=0}^{n}\sum_{k=p}^{n}(-1)^{k-p}\binom{k}{2p}\binom{n}{k}2^{n-k}.$$

Solution by Pierre Lalonde, Kingsey Falls, QC, Canada. The value is $2^{n/2}\cos(n\pi/4)$. Interchanging the order of summation converts the sum to

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}\sum_{p=0}^{k}(-1)^p\binom{k}{2p}.$$

Since $(1 \pm i)^k = \sum_{p=0}^{k} (\pm 1)^p \binom{k}{p} i^p$, where $i = \sqrt{-1}$, cancellation in the binomial expansions yields

$$\frac{1}{2}\left((1+i)^k + (1-i)^k\right) = \sum_{p=0}^{k} i^{2p}\binom{k}{2p} = \sum_{p=0}^{k}(-1)^p\binom{k}{2p},$$

so the sum equals

$$\frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}\left((1+i)^k + (1-i)^k\right).$$

This sum contains the binomial expansions of $(2-(1+i))^n$ and $(2-(1-i))^n$, so the value is $\frac{1}{2}\left((1-i)^n + (1+i)^n\right)$. Finally, we compute

$$\begin{aligned}\frac{(1+i)^n + (1-i)^n}{2} &= \frac{\left(\sqrt{2}e^{i\pi/4}\right)^n + \left(\sqrt{2}e^{-i\pi/4}\right)^n}{2}\\ &= 2^{n/2}\,\frac{e^{n\pi i/4} + e^{-n\pi i/4}}{2} = 2^{n/2}\cos(n\pi/4).\end{aligned}$$

Also solved by U. Abel (Germany), T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), M. A. Carlton, R. Chapman (UK), P. P. Dályay (Hungary), G. Fera (Italy), D. Fleischman, K. Gatesman, M. Jones, O. Kouba (Syria), K. T. L. Koo (China), O. P. Lossers (Netherlands), B. Lu, M. Omarjee (France), L. J. Peterson, R. Pratt, N. C. Singer, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Wildon, L. Zhou, GCHQ Problem Solving Group (UK), and the proposers.

A Quadrilateral Inequality

12033 [2018, 277]. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let $ABCD$ be a convex quadrilateral with area S. Prove

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \geq 8S + AB\cdot CD + BC\cdot AD - AC\cdot BD.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Ptolemy's inequality is $AB\cdot CD + BC\cdot AD \geq AC\cdot BD$. The AM-GM inequality then gives

$$2\,AC\cdot BD \leq 2(AB\cdot CD + BC\cdot AD) \leq AB^2 + CD^2 + BC^2 + AD^2 \tag{1}$$

and

$$2\,AC\cdot BD \leq AC^2 + BD^2. \tag{2}$$

Also,

$$0 \leq (AB - CD)^2 + (BC - AD)^2 + (AC - BD)^2. \tag{3}$$

Adding (1), (2), and (3) and dividing through by 2 yields

$$\begin{aligned}2\,AC\cdot BD \leq{} & AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2\\ & - AB\cdot CD - BC\cdot AD - AC\cdot BD,\end{aligned}$$

which is equivalent to

$$\begin{aligned}AB\cdot CD &+ BC\cdot AD - AC\cdot BD + 4AC\cdot BD\\ &\leq AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2.\end{aligned} \tag{4}$$

The final step is to note that if θ is the angle between the diagonals AC and BD, then

$$S = \frac{1}{2} AC \cdot BD \cdot \sin\theta \leq \frac{1}{2} AC \cdot BD. \tag{5}$$

The desired result follows from (4) and (5).

Equality holds when θ in (5) is a right angle and the right side of (3) is 0. These happen only when the quadrilateral is a square.

Editorial comment. Solvers Richard Stong and Li Zhou noted the stronger inequality

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \geq 8S + 2\big(AB \cdot CD + BC \cdot AD - AC \cdot BD\big)$$

Also solved by E.. Bojaxhiu & E. Hysnelaj, P. P. Dályay (Hungary), D. Fleischman, K. Gatesman, H. Hyun (South Korea), K. T. L. Koo (China), V. Mikayelyan (Armenia), Davis Problem Solving Group, J. C. Smith, A. Stadler (Switzerland), R. Stong, B. Karaivanov (USA) & T. S. Vassilev (Canada), E. A. Weinstein, M. R. Yegan (Iran), L. Zhou, Davis Problem Solving Group, GCHQ Problem Solving Group (UK), and the proposer.

Multiples Without Large Digits

12034 [2018, 370]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let N be any natural number that is not a multiple of 10. Prove that there is a multiple of N each of whose digits in base 10 is 1, 2, 3, 4, or 5.

Solution by Michael Reid, University of Central Florida, Orlando, FL. Let M be a natural number greater than 1, and let $L = M/q$, where q is the smallest prime divisor of M. As usual, let $[n] = \{1, \dots, n\}$. We prove the more general statement that every natural number N that is not a multiple of M has a multiple whose base M expansion has entries only in $[L]$. (In the given problem, $(M, q, L) = (10, 2, 5)$.)

Lemma 1. *If* $\gcd(N, M) = 1$*, then N divides $\sum_{i=0}^{t} M^i$ for some nonnegative t.*

Proof. With $a_s = \sum_{i=0}^{s} M^i$, by the pigeonhole principle some two numbers among $a_0, \dots, a_N$ are congruent modulo N. Since N divides their difference, which has the form $M^j a_t$, we see that N also divides a_t. ∎

Lemma 2. *If A is a divisor of M such that* $\gcd(A, M/A) = 1$*, then the A^k numbers whose base-M expansions consist of k entries from $[A]$ are distinct modulo A^k. In particular, one of them is divisible by A^k.*

Proof. We use induction on k; the claim is trivial for $k = 1$. For $k \geq 1$, suppose that $\sum_{i=0}^{k} a_i M^i$ and $\sum_{i=0}^{k} b_i M^i$ are congruent modulo A^{k+1}. Since A divides M, the numbers $a_k M^k$ and $b_k M^k$ are divisible by A^k. Hence $\sum_{i=0}^{k-1} a^i M^i$ and $\sum_{i=0}^{k-1} b_i M_i$ are congruent modulo A^k. By the induction hypothesis, $a_i = b_i$ for $0 \leq i \leq k-1$. Subtracting the terms for $i < k$ from the assumed congruence leaves $a_k M^k \equiv b_k M^k \bmod A^{k+1}$. Thus A^{k+1} divides $(a_k - b_k)M^k$. Since A^k divides M^k, and M/A is relatively prime to A, we conclude that A divides $a_k - b_k$. Since $a_k, b_k \in [A]$, we have $a_k = b_k$. ∎

Now let N be a positive integer not a multiple of M. For some prime p, the largest power p^b dividing N is less than the largest power p^c dividing M. Write N as $p^b RS$, where S is the largest divisor of N relatively prime to M. Thus every prime dividing R divides M, and $p \nmid R$.

Let $A = M/p^c$. Thus R divides some power of A, say A^k. Also A and M/A are relatively prime. By Lemma 2, R divides some number B whose base-M expansion consists of k entries from $[A]$.

Since S is relatively prime to M and thus also to M^k, Lemma 1 implies that S divides a number C of the form $\sum_{i=0}^{t}(M^k)^i$. Now BC is a multiple of RS, and the base-M expansion of BC consists of the expansion of B repeated $t+1$ times. Hence all the entries of this expansion lie in $[A]$. Finally, p^bBC is a multiple of p^bRS, which equals N. The entries in the base-M expansion of p^bBC are in $\{p^b, 2p^b, \ldots, Ap^b\}$, which is contained in $[L]$ since $Ap^b \le M/p \le M/q = L$.

Editorial comment. The restriction of entries to $[L]$ is in some sense sharp. If s is not a multiple of q, then sL is not divisible by M, and the units position of every multiple of sL is divisible by L and hence not in $[L-1]$.

On the other hand, when M is not squarefree, the set $[L]$ can be reduced to a proper subset. Suppose that M has prime factorization $\prod_{i=1}^{r} p_i^{e_i}$, and let $A_i = M/p_i^{e_i}$ for $i \in [r]$. The proof shows that every N not divisible by M has a multiple whose base-M expansion has all entries in the set $\bigcup_{i=1}^{r}\{p_i^{e_i-1}, 2p_i^{e_i-1}, \ldots, A_i p_i^{e_i-1}\}$, which is a proper subset of $[L]$ when M has a repeated prime factor.

For the original problem, several readers employed the Euler phi-function. In particular, when $\gcd(n, q-1) = 1$, the summed geometric series $\sum_{i=0}^{\phi(n)-1} q^i$ (a q-analogue of $\phi(n)$) is divisible by n, by Euler's theorem. For example, when $q = 10$ and $n = 77$, we have $\phi(77) = 60$, and hence 77 divides $\sum_{i=0}^{59} 10^i$.

Some substantial papers have been written about the digit distribution of multiples of integers. An example is Schmidt, W. M. (1983), The joint distribution of the digits of certain integer s-tuples, in Erdős, P., et al., eds., *Studies in Pure Mathematics: To the Memory of Paul Turán*, Basel: Birkhäuser, pp. 605–622.

Also solved by R. Chapman (UK), P. P. Dályay (Hungary), D. Fleischman, K. Gatesman, O. Geupel (Germany), E. J. Ionaşcu, D. Kim (South Korea), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, Y. Sun, R. Tauraso (Italy), M. Tetiva (Romania), GCHQ Problem Solving Group (UK), and the proposers.

Solving a Cubic to Minimize a Rational Expression

12035 [2018, 370]. *Proposed by Dinh Thi Nguyen, Tuy Hòa, Vietnam.* Find the minimum value of

$$(a^2+b^2+c^2)\left(\frac{1}{(3a-b)^2}+\frac{1}{(3b-c)^2}+\frac{1}{(3c-a)^2}\right)$$

as a, b, and c vary over all real numbers with $3a \ne b$, $3b \ne c$, and $3c \ne a$.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $x = 3b-c$, $y = 3c-a$, and $z = 3a-b$. The hypothesis implies that x, y, and z are nonzero. The given expression becomes $F/52$ where

$$F = \left(4(x^2+y^2+z^2)+3(x+y+z)^2\right)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right).$$

To search for the minimum of F, it suffices to consider $x, y > 0$ and $z = -t < 0$. By the AM-GM inequality, $2(x^2+y^2) \ge (x+y)^2$ and

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{2}{xy} \ge \frac{8}{(x+y)^2},$$

with equality when $x = y$. Putting $x + y = s$, we then have

$$F \ge (2s^2+4t^2+3(s-t)^2)\left(\frac{8}{s^2}+\frac{1}{t^2}\right). \qquad (*)$$

Let r equal s/t, which is positive. The right side of $(*)$ becomes

$$5r^2 - 6r + 47 - \frac{48}{r} + \frac{56}{r^2},$$

which we denote $f(r)$. Notice that $\lim_{r\to 0} f(r) = \lim_{r\to\infty} f(r) = \infty$ and

$$f'(r) = \frac{10r^4 - 6r^3 + 48r - 112}{r^3} = \frac{2(r+2)(5r^3 - 13r^2 + 26r - 28)}{r^3}.$$

According to the Cardano formula, the only positive zero ξ of $f'(r)$ is

$$\frac{13 + \sqrt[3]{4042 + 15\sqrt{120585}} + \sqrt[3]{4042 - 15\sqrt{120585}}}{15},$$

which is approximately 1.56431. Hence the required minimum value is $f(\xi)/52$, which is

$$\frac{2062 + \sqrt[3]{4420439038 + 12661425\sqrt{120585}} + \sqrt[3]{4420439038 - 12661425\sqrt{120585}}}{5460},$$

or approximately 0.8086454638.

Also solved by H. Chen, G. Fera, K. Gatesman, L. Giugiuc (Romania), O. Kouba (Syria), W.-K. Lai & J. Risher, K.-W. Lau (China), L. J. Peterson, M. Reid, J. C. Smith, A. Stadler (Switzerland), R. Stong, D. B. Tyler, and the proposer.

Metric Spaces with Few Isometry Types

12036 [2018, 370]. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.* Two metric spaces (X, d) and (X', d') are said to be *isometric* if there is a bijection $\phi\colon X \to X'$ such that $d(a, b) = d'(\phi(a), \phi(b))$ for all $a, b \in X$. Let X be an infinite set. Find all metrics d on X such that (X, d) and (X', d') are isometric for every subset X' of X of the same cardinality as X. (Here, d' is the metric induced on X' by d.)

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. If $d(a, b)$ is independent of a and b when a and b differ, then (X, d) has the required property. We show that this is the only case. Suppose that at least two nonzero distances occur. Choose one of the distances, say δ, and define a coloring of the edges of the complete graph with vertex set X by letting xy be red if $d(x, y) = \delta$ and blue otherwise.

Given a point $p \in X$, let R be the set of neighbors of p via red edges, and let B be the set of neighbors of p via blue edges: $X = \{p\} \cup R \cup B$. Since X is infinite, R or B has the same cardinality as X; suppose it is B. Let $X' = X \setminus R = \{p\} \cup B$. Since X' has the same cardinality as X, by assumption the metric spaces (X, d) and (X', d') are isomorphic. Also the edge-colored complete graph on X and the induced one on X' are isomorphic. Since X' contains a vertex p incident only with blue edges, X also contains a vertex incident with only blue edges.

Let Y denote the subset of X consisting of all points incident with at least one red edge. The cardinality of Y must be smaller than the cardinality of X, because Y has no point incident only with blue edges. Finally, let $Y' = X \setminus Y$; the set Y' has the same cardinality as X. The graph induced by Y' has only blue edges, which implies that the original graph has only blue edges, contradicting our assumption.

The assumption that R has the same cardinality as X leads to a contradiction in the same way.

Editorial comment. Frederic Brulois and Gary Gruenhage provided a generalization: Let $\binom{X}{2}$ denote the family of 2-element subsets of X. Consider a function $f\colon \binom{X}{2} \to S$, where S is any set. If X is infinite and for any subset Y of X with the same cardinality as X there is a bijection $b\colon Y \to X$ such that $f(\{y_1, y_2\}) = f(\{b(y_1), b(y_2)\})$ for all $y_1, y_2 \in Y$, then f is a constant function.

Klaas Pieter Hart provided a different generalization: An infinite graph G that is isomorphic to all its induced subgraphs whose vertex sets have the same cardinality as G must be the complete graph or have no edge.

Also solved by F. Brulois, G. Gruenhage, J. W. Hagood, K. P. Hart (Netherlands), J. H. Lindsey II, A. Pathak, M. Reid, N. Sahoo, K. Schilling, R. Stong, and the proposer.

A Familiar Set Disguised

12037 [2018, 370]. *Proposed by José Manuel Rodríguez Caballero, Université du Québec, Montreal, QC, Canada.* For a positive integer n, let S_n be the set of pairs (a, k) of positive integers such that $\sum_{i=0}^{k-1}(a+i) = n$. Prove that the set

$$\left\{ n : \sum_{(a,k)\in S_n} (-1)^{a-k} \neq 0 \right\}$$

is closed under multiplication.

Solution by GCHQ Problem Solving Group, Cheltenham, UK. Let A be the set defined in the problem statement. Each $(a, k) \in S_n$ satisfies

$$n = ka + \sum_{i=0}^{k-1} i = ka + \frac{k(k-1)}{2},$$

and thus

$$2n = k(k + 2a - 1).$$

The factors k and $k + 2a - 1$ have opposite parity, and also $k + 2a - 1 > k$. Given n, we can generate a pair $(a, k) \in S_n$ by writing $2n = E \times O$, where E is even and O is odd, and setting $k = \min(E, O)$ and $a = (|O - E| + 1)/2$. The process is reversible, so we have a bijection from S_n to the set of even/odd factorizations $2n = E \times O$. We write these as $2n = (2^T u) \times v$, where u and v are both odd.

Note also that $a + k = (E + O + 1)/2$. If $(2^T u) + v \equiv 1 \pmod 4$, then $a + k$ is odd, while if $(2^T u) + v \equiv 3 \pmod 4$, then $a + k$ is even. Because $(-1)^{a-k} = (-1)^{a+k}$, we have $n \in A$ if and only if the number of even/odd factorizations resulting in $a + k$ even is different from the number resulting in $a + k$ odd.

Let p be a prime factor of n. Switching p from u to v or vice versa does not change the congruence class of $2^T u$ or v modulo 4 if $p \equiv 1 \pmod 4$. However, if $p \equiv 3 \pmod 4$, then the switch changes the sign of v and leaves the congruence class of $2^T u$ unchanged, so it changes the class of $(2^T u) + v$.

If some prime factor p congruent to 3 modulo 4 occurs in $2n$ with odd power, then for any fixed distribution of the other factors, there are the same number of factorizations in which p contributes an even number or an odd number of factors to v. Hence there are the same number of factorizations with $a + k$ even or odd, and $n \notin A$.

Conversely, suppose that all such prime factors occur with even power. When all the odd prime factors are in v, and $u = 1$, we have $(2^T u) + v \equiv 2^T + 1 \pmod 4$, and the class depends on whether $T > 1$. The class remains the same for any distribution of the prime factors congruent to 1 modulo 4. Thus we need only consider multisets of the prime factors congruent to 3 modulo 4, where the bound on the multiplicity of each is even. With an even bound, the number of choices for the multiplicity of each such factor is odd. Hence there are an odd number of multisets of the prime factors congruent to 3 modulo 4. With an odd number of choices, there cannot be the same number with even size as odd size. Hence there will not be the same number of factorizations with $a + k$ even and odd, and so $n \in A$.

Since the product of two numbers whose prime factorizations have each prime factor congruent to 3 modulo 4 occurring with even power also has the same property, A is closed under multiplication.

Editorial comment. Several solvers noted that A is the set of all positive integers that can be expressed as a sum of two squares.

Also solved by R. Chapman (UK), K. Gatesman, E. J. Ionaşcu, P. Lalonde (Canada), O. P. Lossers (Netherlands), J. C. Smith, and the proposer.

An Inequality with Medians

12038 [2018, 370]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let ABC be an acute triangle with sides of length a, b, and c opposite angles A, B, and C, respectively, and with medians of length m_a, m_b, and m_c emanating from A, B, and C, respectively. Prove

$$\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} \geq 9 \cos A \, \cos B \, \cos C.$$

Solution by Subhankar Gayen, Vivekananda Mission Mahavidyalaya, India. Let M be the midpoint of BC. Suppose that AM intersects the circumcircle of $\triangle ABC$ at D. By the power-of-the-point theorem, $m_a \cdot MD = a^2/4$, and two applications of the law of cosines yields $a^2/4 = (b^2+c^2)/2 - m_a^2$. Hence $b^2 + c^2 = 2m_a\,(m_a + MD)$. Since AD is a chord of the circumcircle, $m_a + MD \leq 2R$, where R is the circumradius of $\triangle ABC$. Hence $4Rm_a \geq b^2 + c^2$. Using this and the two other analogous inequalities yields

$$\begin{aligned}
\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} &\geq \frac{b^2+c^2}{16R^2} + \frac{c^2+a^2}{16R^2} + \frac{a^2+b^2}{16R^2} \\
&= \frac{a^2+b^2+c^2}{8R^2} \\
&= \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} \\
&= 1 + \cos A \cos B \cos C,
\end{aligned}$$

where we have used the generalized law of sines in the second-to-last step and $A + B + C = \pi$ to obtain the last equality.

We complete the proof by showing that $1 \geq 8 \cos A \cos B \cos C$. This follows from $\cos(x)\cos(y) < \cos^2((x+y)/2)$ when $x \neq y$, because this last inequality shows that $\cos A \cos B \cos C$ cannot take its maximum value on a triangle ABC unless $A = B = C = \pi/3$.

Note that the assumption that $\triangle ABC$ is acute is unnecessary and also that equality holds only when $\triangle ABC$ is equilateral.

Also solved by H. Bailey, M. Bataille (France), H. Chen, G. Fera, L. Giugiuc (Romania), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, J. F. Loverde, M. Lukarevski (Macedonia), P. Nüesch (Switzerland), P. Perfetti (Italy), C. R. Pranesachar (India), V. Schindler (Germany), D. Smith (Canada), J. C. Smith, A. Stadler (Switzerland), R. Stong, M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), L. Zhou, T. Zvonaru (Romania), GCHQ Problem Solving Group (UK), and the proposer.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by May 31, 2020, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12153. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* For a real number x whose fractional part is not 1/2, let $\langle x \rangle$ denote the nearest integer to x. For a positive integer n, let

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{\langle \sqrt{k} \rangle} \right) - 2\sqrt{n}.$$

(a) Prove that the sequence $a_1, a_2, \ldots$ is convergent, and find its limit L.
(b) Prove that the set $\{\sqrt{n}(a_n - L) : n \geq 1\}$ is a dense subset of $[0, 1/4]$.

12154. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, North Macedonia.* Let r_a, r_b, and r_c be the exradii of a triangle with circumradius R and inradius r. Prove

$$\frac{r_a}{r_b + r_c} + \frac{r_b}{r_c + r_a} + \frac{r_c}{r_a + r_b} \geq 2 - \frac{r}{R}.$$

12155. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let $f : [0, \infty) \to [0, 1]$ be the function that satisfies $f(0) = 1$, is differentiable on $(0, \infty)$, and has the following property: If A is a point on the graph of f and B is the x-intercept of the line tangent to the graph of f at A, then $AB = 1$.
(a) Prove $\int_0^\infty f(x)\, dx = \pi/4$.
(b) For $n \in \mathbb{N}$, prove that $\int_0^\infty x^{2n} f(x)\, dx$ is a rational polynomial of π.

12156. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For positive integers m and n and nonnegative integers r and s, prove

$$\sum_{0 \leq j_1 \leq \cdots \leq j_m \leq r} \frac{\binom{n+s}{n}\binom{n+j_1}{n}\binom{s+j_1}{s}}{\prod_{i=1}^{m}(n + j_i)} = \sum_{0 \leq j_1 \leq \cdots \leq j_m \leq s} \frac{\binom{n+r}{n}\binom{n+j_1}{n}\binom{r+j_1}{r}}{\prod_{i=1}^{m}(n + j_i)}.$$

12157. *Proposed by Nick MacKinnon, Winchester College, Winchester, UK.* Show that there are infinitely many positive integers that are neither the sum of a cube and a prime nor the difference of a cube and a prime (in either order).

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12158. *Proposed by Hervé Grandmontagne, Paris, France.* Prove

$$\int_0^1 \frac{(\ln x)^2 \arctan x}{1+x}\, dx = \frac{21}{64}\pi\zeta(3) - \frac{1}{24}\pi^2 G - \frac{1}{32}\pi^3 \ln 2,$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$ and G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

12159. *Proposed by Rudolf Avenhaus, Universität der Bundeswehr München, Neubiberg, Germany, and Thomas Krieger, Forschungszentrum Jülich, Jülich, Germany.* Let Φ denote the distribution function of a standard normal random variable, and let U denote its inverse function. Let n be a positive integer, and suppose $0 < \alpha < 1$ and $\mu \geq 0$. Prove

$$\Phi\left(U(\alpha) - \sqrt{n}\mu\right) \leq \left(\Phi\left(U(\sqrt[n]{\alpha}) - \mu\right)\right)^n.$$

SOLUTIONS

An Even Number of Common Neighbors

12039 [2018, 371]. *Proposed by Sandeep Silwal, Brookline, MA.* Let G be a graph with an even number of vertices. Show that there are two vertices in G with an even number of common neighbors.

Solution by Aritro Pathak, graduate student, Brandeis University, Waltham, MA. We write $N(v)$ for the set of neighbors of v, and we write $d(v)$ for the degree $|N(v)|$ of v. Let G have n vertices, and suppose that all pairs of vertices in G have an odd number of common neighbors. The number p of paths of length 2 starting at a vertex v is $\sum_{u \in N(v)}(d(u) - 1)$. On the other hand, p is also the sum over all vertices u other than v of the number of common neighbors of v and u. Since n is even, our assumption implies that p is odd.

If G has no vertex of odd degree, then in $\sum_{u \in N(v)}(d(u) - 1)$ every summand is odd, and the number of summands is even, so p is even, which is a contradiction.

Hence we may take v to be a vertex of odd degree. Now let q be the sum, over edges e incident to v, of the number of triangles containing e. When $e = uv$, the summand is the number of common neighbors of u and v. By hypothesis this is odd, and q is the sum of an odd number of these terms. Hence q is odd. On the other hand, every triangle containing v contributes 2 to the sum, so q is even. Again this is a contradiction, so the hypothesis that all pairs of vertices have an odd number of common neighbors is false.

Editorial comment. Several solvers observed that this is problem 10 in Chapter 14 of A. Engel (1998), *Problem-Solving Strategies*, New York: Springer. Gordon Royle raised the question of classifying the graphs for which every pair of distinct vertices has an odd number of common neighbors at mathoverflow.net/questions/17809/graphs-where-every-two-vertices-have-odd-number-of-mutual-neighbours. This problem indicates that the number of vertices must be odd, and various examples are known (starting with the complete graphs), but a complete description seems difficult.

Also solved by O. Geupel (Germany), A. Goel, E. J. Ionaşcu, Y. J. Ionin, S. C. Locke, O. P. Lossers (Netherlands), H. Mikaelian (Armenia), M. Reid, J. C. Smith, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), and the proposer.

A Series Condition

12040 [2018, 371]. *Proposed by George Stoica, Saint John, NB, Canada.* Find all convergent series $\sum_{n=1}^{\infty} x_n$ of positive terms such that $\sum_{n=1}^{\infty} x_n x_{n+k}/x_k$ is independent of the positive integer k.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. If $x_n = \alpha r^n$ with $\alpha > 0$ and $0 < r < 1$, then x_{n+k}/x_k is independent of k and the geometric series $\sum x_n$ satisfies the conditions in the statement. We claim that these are the only such series. Suppose that $\sum x_n$ satisfies the conditions in the statement. We show that the series is geometric.

Let $\sum_{n=1}^{\infty} x_n x_{n+k} = a x_k$, and define the complex function f by $f(z) = \sum_{n=1}^{\infty} x_n z^n$. This series converges uniformly on the closed unit disk in the complex plane, so f is continuous on the closed disk and analytic on the open disk.

When $z\bar{z} = 1$,

$$f(z)\overline{f(z)} = af(z) + a\overline{f(z)} + \sum_{n=1}^{\infty} x_n^2,$$

or $|f(z) - a|^2 = a^2 + \sum_{n=1}^{\infty} x_n^2$. Let $A = \left(a^2 + \sum_{n=1}^{\infty} x_n^2\right)^{1/2}$, and let $b = a/A$. Note that $0 < b < 1$. Define g by $g(z) = f(z)/A$, so $|g(z) - b| = 1$ when $|z| = 1$. The composition of the function $z \mapsto g(z) - b$ with the transformation $z \mapsto (z + b)/(bz + 1)$ maps the unit disk into itself, maps 0 to 0, and maps $|z| = 1$ onto itself. According to the Schwarz lemma, this composition is a rotation $z \mapsto cz$ where $|c| = 1$, so

$$\frac{(g(z) - b) + b}{b(g(z) - b) + 1} = cz.$$

It follows that $g(z) = (1 - b^2)cz/(1 - bcz)$, or $f(z) = A(1 - b^2)cz/(1 - bcz)$. This yields $x_n = \alpha r^n$ for $n \geq 1$, where $\alpha = A(1 - b^2)/b$ and $r = bc$. Since $x_n > 0$ for all n, we conclude $\alpha > 0$ and $0 < r < 1$.

Also solved by K. F. Andersen (Canada), A. Stadler (Switzerland), R. Stong, and the proposer.

Counting Factors in a Square of Binomial Coefficients

12041 [2018, 466]. *Proposed by Richard Stanley, University of Miami, Coral Gables, FL.* Let p be prime. For a positive integer c, let $\nu_p(c)$ denote the largest integer d such that p^d divides c. Let

$$H_m = \prod_{i=0}^{m} \prod_{j=0}^{m} \binom{i+j}{i}.$$

For $n \geq 1$, prove

$$\nu_p(H_{p^n-1}) = \frac{1}{2}\left(\left(n - \frac{1}{p-1}\right)p^{2n} + \frac{p^n}{p-1}\right).$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA. Let $s(k)$ denote the sum of the digits of k in base p, and let $m = p^n - 1$. Applying Legendre's formula $(p-1)\nu_p(k!) = k - s(k)$, we find

$$\begin{aligned}(p-1)\nu_p(H_m) &= \sum_{i=0}^{m}\sum_{j=0}^{m}\big(s(i) + s(j) - s(i+j)\big)\\ &= 2p^n \sum_{i=0}^{m} s(i) - \sum_{i=0}^{m}\sum_{j=0}^{m} s(i+j). \qquad (1)\end{aligned}$$

Observe that $0 \le k \le m$ implies $s(k + p^n) = s(k) + 1$. Reindexing with $k = i + j$ to simplify the double sum yields

$$\begin{aligned}\sum_{i=0}^{m}\sum_{j=0}^{m} s(i+j) &= \sum_{k=0}^{m}(k+1)s(k) + \sum_{k=0}^{m}(m-k)s(k+p^n)\\ &= \sum_{k=0}^{m}(k+1)s(k) + \sum_{k=0}^{m}(p^n-1-k)(s(k)+1) \qquad (2)\\ &= p^n\sum_{k=0}^{m}s(k) + \sum_{k=0}^{m}(p^n-k-1) = p^n\sum_{k=0}^{m}s(k) + \frac{p^n(p^n-1)}{2}.\end{aligned}$$

Since the base-p representation of k realizes every n-tuple modulo p as it runs from 0 to $p^n - 1$, we have $\sum_{k=0}^{m} s(k) = np^{n-1}p(p-1)/2 = np^n(p-1)/2$. Combining this with (1) and (2) yields

$$(p-1)\nu_p(H_{p^n-1}) = p^n\left(\frac{np^n(p-1)}{2}\right) - \frac{p^n(p^n-1)}{2} = \frac{p^{2n}\left(n(p-1)-1\right)+p^n}{2},$$

which gives the desired result.

Also solved by N. Caro (Brazil), R. Chapman (UK), P. P. Dályay (Hungary), K. Gatesman, Y. J. Ionin, A. Jorza, O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), J. H. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (UK), and the proposer.

A Generalization of Leuenberger's Inequality

12042 [2018, 466]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, Macedonia.* Let x, y, and z be positive real numbers. For a triangle with sides of lengths a, b, and c and circumradius R, prove

$$\frac{x+y}{cz} + \frac{y+z}{ax} + \frac{z+x}{by} \ge \frac{2\sqrt{3}}{R}.$$

Solution by Mohammad Reza Yegan, Tehran, Iran. Let A, B, and C be the angles opposite the sides a, b, and c. We use the AM-GM inequality, Jensen's inequality applied to the concave function $\sin x$ on the interval $[0, \pi]$, and the generalized law of sines, which says that $a/\sin A = b/\sin B = c/\sin C = 2R$, as follows:

$$\begin{aligned}&\frac{x+y}{cz} + \frac{y+z}{ax} + \frac{z+x}{by} = \frac{x}{cz} + \frac{y}{cz} + \frac{y}{ax} + \frac{z}{ax} + \frac{z}{by} + \frac{x}{by}\\ &\ge 6\sqrt[6]{\frac{x}{cz}\cdot\frac{y}{cz}\cdot\frac{y}{ax}\cdot\frac{z}{ax}\cdot\frac{z}{by}\cdot\frac{x}{by}} = \frac{6}{\sqrt[3]{abc}} \ge \frac{18}{a+b+c} = \frac{9}{R(\sin A + \sin B + \sin C)}\\ &\ge \frac{3}{R\sin((A+B+C)/3)} = \frac{3}{R\sin(\pi/3)} = \frac{2\sqrt{3}}{R}.\end{aligned}$$

Editorial comment. Several solvers noted that we have equality if and only if $x = y = z$ and $a = b = c$. The proposer pointed out that the problem generalizes Leuenberger's inequality, which says that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{\sqrt{3}}{R}.$$

Also solved by F. R. Ataev (Uzbekistan), M. Bataille (France), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (France), P. Bracken, P. P. Dályay (Hungary), D. Bailey, E. Campbell, C. Diminnie, & T. Smith, G. Fera (Italy), K. Gatesman, S. Gayen (India), O. Geupel (Germany) E. J. Ionaşcu, W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), D.-Ş. Marinescu (Romania), V. Mikayelyan (Armenia), A. Pathak, D. Ritter, V. Schindler (Germany), J. Sim (South Korea), J. C. Smith, A. Stadler (Switzerland), N. Stanciu (Romania), R. Stong, B. Karaivanov (USA) & T. S. Vassilev (Canada), M. Vowe (Switzerland), J. Zacharias, L. Zhou, An-Anduud Problem Solving Group (Mongolia), GCHQ Problem Solving Group (UK), and the proposer.

A Large Prime Factor

12043 [2018, 125]. *Proposed by Max A. Alekseyev, George Washington University, Washington, DC.* Let n and k be integers with $n \geq 3$ and $k \geq 2$. Prove that $n^k + 1$ has a prime factor greater than $2k$.

Solution by Peter W. Lindstrom, Saint Anselm College, Manchester, NH. We use the following theorem of Zsigmondy (K. Zsigmondy (1892), Zur Theorie der Potenzreste, *J. Monatshefte für Math.* 3(1): 265–284): For $a, b, m \in \mathbb{N}$ with $\gcd(a, b) = 1$ and $a > b$, there is a prime p such that p divides $a^m - b^m$ and p does not divide $a^j - b^j$ for $1 \leq j < m$, with the following exceptions: (i) $m = 1$ and $a - b = 1$, (ii) $m = 2$ and $a + b$ is a power of 2, and (iii) $m = 6$ and $(a, b) = (2, 1)$.

Now take $m = 2k$, $a = n$, and $b = 1$. The hypotheses of the theorem hold, and because $m \geq 4$ and $a \geq 3$, none of the exceptions apply. Hence there is a prime p that divides $n^{2k} - 1$ and does not divide $n^j - 1$ for $1 \leq j < 2k$. This implies that the multiplicative order of n mod p is $2k$. By Fermat's little theorem, $2k$ divides $p - 1$, and thus $p > 2k$. Also, since p divides $n^{2k} - 1$ and does not divide $n^k - 1$, it divides $n^k + 1$.

Also solved by R. Chapman (UK), S. Dey, S. M. Gagola, Jr., A. Goel, D. Kim (South Korea), O. Kouba (Syria), M. Reid, J. P. Robertson, A. Stadler (Switzerland), R. Stong, B. Sury (India), An-Anduud Problem Solving Group (Mongolia), NSA Problems Group, and the proposer.

Sums of Triples with One Pair Relatively Prime

12044 [2018, 466]. *Proposed by Freddy Barrera, Colombia Aprendiendo, Bogota, Colombia, Bernardo Recamán Santos, Universidad de los Andes, Bogota, Colombia, and Stan Wagon, Macalester College, St. Paul, MN.* Prove that any integer greater than 210 can be written as the sum of positive integers a, b, and c such that $\gcd(a, b) = 1$ but $\gcd(a, c)$ and $\gcd(b, c)$ are both greater than 1.

Solution by NSA Problems Group, Fort Meade, MD. Given $m \in \mathbb{N}$ with $m > 210$, let p and q be the smallest primes that do not divide m, with $p < q$. We find integers i and j such that $\gcd(a, b) = 1$ and $a + b + c = m$, where $a = pi$, $b = qj$, and $c = pq$.

We first prove $2pq \leq m$. If $q \leq 11$, then $2pq \leq 2 \cdot 7 \cdot 11 = 154 < m$. If $q = 13$, then $2pq \leq 2 \cdot 11 \cdot 13 = 286$, but m is divisible by $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11/p$, and the smallest such m greater than 210 is 330; hence $m \geq 330 \geq 2pq$. If $q = 17$, then $2pq \leq 2 \cdot 13 \cdot 17 = 442$ and m is divisible by $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13/p$, which is at least $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$; that is, m is divisible by 2310, so $m \geq 442 \geq 2pq$.

Finally, suppose $q \geq 19$. Let p_k be the kth-smallest prime. If $q \geq 19$, then $q = p_k$ with $k \geq 8$. Also $2pq \leq 2p_{k-1}p_k$, and m is divisible by $(1/p)\prod_{i=1}^{k-1} p_i$, which is at least $\prod_{i=1}^{k-2} p_i$. By Bertrand's theorem, $2p_{j-1} > p_j$, so $32p_{k-3}p_{k-2} > 2p_{k-1}p_k$. Since $k \geq 8$,

$$m \geq \prod_{i=1}^{k-2} p_i \geq p_1 p_2 p_3 p_4 p_{k-3} p_{k-2} = 2 \cdot 3 \cdot 5 \cdot 7 p_{k-3} p_{k-2} \geq 32 p_{k-3} p_{k-2} > 2 p_{k-1} p_k.$$

Since $p_{k-1}p_k \geq pq$, we conclude $m \geq 2pq$.

With $m \geq 2pq$, we have $m = pqs + r$, where $s \geq 2$ and $0 \leq r < pq$. Because neither p nor q divides m, neither divides r; so $r \geq 1$ and $\gcd(pq, r) = 1$. Let $m' = m - pq = pq(s-1) + r$. Since $s \geq 2$, we have $m' \geq pq + r > pq$ and $\gcd(m', pq) = \gcd(r, pq) = 1$. The integers $m' - qj$ for $1 \leq j \leq p$ are positive and distinct modulo p, so there is a unique j with $m' - qj \equiv 0 \bmod p$. Hence $m' = pi + qj$ for some positive i and some j with $1 \leq j \leq p$. Furthermore, since p does not divide m', we have $j < p$.

A prime divisor of pi and qj must divide m'. Since m' is not divisible by p or q, such a divisor t cannot equal p or q. Thus t divides j. Since $j < p$, we have $t < p$. Since p is the smallest prime not dividing m, it follows that t divides m. Hence t divides $m - m'$, which equals pq. This requires $t = p$ or $t = q$, a contradiction. Therefore, $\gcd(pi, qj) = 1$ and $m = pi + qj + pq$, as required.

Editorial comment. Stephen Gagola, Eugen Ionaşcu, the GCHQ group, and the proposers showed that the only numbers not admitting a representation as in the problem are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 22, 24, 30, 36, 42, 48, 60, 84, 90, and 210.

Also solved by D. Fleischman, S. M. Gagola Jr., K. Gatesman, E. J. Ionaşcu, Y. I. Ionin, M. Reid, J. P. Robertson, GCHQ Problem Solving Group (UK), and the proposer.

An Alternating Iterated Sum

12045 [2018, 467]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Prove that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(n \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) - 1 \right)$$

converges to $\frac{\pi^2}{16} - \frac{\ln 2}{2} - \frac{1}{2}$.

Solution I by Giuseppe Fera, Italy. Let

$$p_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}, \quad q_n = np_n - 1, \quad s = \sum_{n=1}^{\infty} (-1)^{n-1} p_n, \quad \text{and} \quad t = \sum_{n=1}^{\infty} (-1)^{n-1} q_n.$$

We seek the value of t. Note that $p_1 = \frac{\pi^2}{6} - 1$ and $p_{n+1} = p_n - \frac{1}{(n+1)^2}$. Thus

$$\begin{aligned} s &= \sum_{n=0}^{\infty} (-1)^n p_{n+1} = p_1 - \sum_{n=1}^{\infty} (-1)^{n-1} p_n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^2} \\ &= \frac{\pi^2}{6} - 1 - s + 1 - \frac{\pi^2}{12} = \frac{\pi^2}{12} - s, \end{aligned}$$

so $s = \pi^2/24$. Next, since $q_1 = \frac{\pi^2}{6} - 2$ and

$$q_{n+1} = (n+1)p_{n+1} - 1 = (n+1)p_n - \frac{1}{n+1} - 1 = q_n + p_n - \frac{1}{n+1},$$

we have

$$\begin{aligned} t &= \sum_{n=0}^{\infty} (-1)^n q_{n+1} = q_1 - \sum_{n=1}^{\infty} (-1)^{n-1} q_n - \sum_{n=1}^{\infty} (-1)^{n-1} p_n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1} \\ &= \frac{\pi^2}{6} - 2 - t - s + 1 - \ln 2 = \frac{\pi^2}{8} - \ln 2 - 1 - t. \end{aligned}$$

Therefore, $t = \frac{\pi^2}{16} - \frac{1}{2}\ln 2 - \frac{1}{2}$.

Solution II by Li Zhou, Polk State College, Winter Haven, FL. Let $a_n = (-1)^{n-1}n$ and

$$b_n = \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2}\right) - \frac{1}{n} = \sum_{k=n+1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{(k-1)k}\right) = \sum_{k=n+1}^{\infty} \frac{-1}{(k-1)k^2}.$$

By Abel's summation formula,

$$\sum_{n=1}^{N} a_n b_n = A_N b_{N+1} + \sum_{n=1}^{N} A_n(b_n - b_{n+1}),$$

where $A_n = \sum_{k=1}^{n} a_k = (-1)^{n-1}\lceil n/2\rceil$. As $N \to \infty$,

$$|A_N b_{N+1}| \le \frac{N+1}{2}\sum_{k=N+2}^{\infty}\frac{1}{(k-1)k^2} < \frac{1}{2}\sum_{k=N+2}^{\infty}\frac{1}{k^2} \to 0.$$

Thus the requested sum is

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} A_n(b_n - b_{n+1}) = \sum_{n=1}^{\infty}\left\lceil\frac{n}{2}\right\rceil\frac{(-1)^n}{n(n+1)^2}.$$

The odd terms of this final series sum to

$$\sum_{k=1}^{\infty}\frac{-k}{(2k-1)(2k)^2} = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) = -\frac{1}{2}\ln 2,$$

while the even terms sum to

$$\sum_{k=1}^{\infty}\frac{k}{2k(2k+1)^2} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), P. Bracken, R. Chapman (UK). H. Chen, Ó. Ciaurri & M. Bello & M. Benito & E. Fernández & L. Roncal (Italy), K. Gatesman, M. L. Glasser, J.-P. Grivaux (France), E. A. Herman, A. Jorza, K. T. Hun (South Korea), O. Kouba (Syria), P. Lalonde (Canada), K.-W. Lau (China), L. Lipták, O. P. Lossers (Netherlands), J. Magliano, L. Matejíčka (Romania), V. Mikayelyan (Armenia), R. Molinari, M. Omarjee (France), A. Pathak, P. Paolo (Italy), N. C. Singer, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, An-Anduud Problem Solving Group (Mongolia), GCHQ Problem Solving Group (UK), and the proposer.

An Inequality for Moments

12046 [2018,467]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous and nonnegative third derivative, and suppose $\int_0^1 f(x)\,dx = 0$. Prove

$$10\int_0^1 x^3 f(x)\,dx + 6\int_0^1 x f(x)\,dx \ge 15\int_0^1 x^2 f(x)\,dx.$$

Solution by Kee-Wai Lau, Hong Kong, China. Let $h(x) = 12(10x^3 - 15x^2 + 6x) - 6$. Since $\int_0^1 f(x)dx = 0$, the given inequality is equivalent to $\int_0^1 h(x)f(x)\,dx \ge 0$. Integrating by parts three times yields

$$\begin{aligned}\int_0^1 h(x)f(x)\,dx &= \int_0^1 \left(120x^3-180x^2+72x-6\right)f(x)\,dx\\ &= \int_0^1 \left(-30x^4+60x^3-36x^2+6x\right)f'(x)\,dx\\ &= \int_0^1 \left(6x^5-15x^4+12x^3-3x^2\right)f''(x)\,dx\\ &= \int_0^1 \left(-x^6+3x^5-3x^4+x^3\right)f'''(x)\,dx = \int_0^1 x^3(1-x)^3 f'''(x)\,dx,\end{aligned}$$

which is clearly nonnegative, since f''' is nonnegative by hypothesis.

Editorial comment. Most solutions used integration by parts in some form. Equality holds if and only if $f'''=0$ so $f(x)=a(x^2-1/3)+b(x-1/2)$. Erik Verriest's solution used calculus of variations. For $n \ge 4$, if $f^{(n)}(x)\ge 0$, a similar inequality follows from

$$\int_0^1 x^n(1-x)^n f^{(n)}(x)\,dx = (-1)^n \int_0^1 f(x)\frac{d^n}{dx^n}\left(x^n(1-x)^n\right)\,dx.$$

Also solved by K. F. Andersen (Canada), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal, A. Berkane (Algeria), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), P. Bracken, R. Chapman (UK), P. P. Dályay (Hungary), L. Di Giacomo (Italy), E. A. Herman, T. H. Kim (South Korea), K. T. L. Koo (China), O. Kouba (Syria), O. P. Lossers (Netherlands), D.-Ş. Marinescu (Romania), L. Matejíčka (Slovakia), L. Meykhanadzhyan (Russia), V. Mikayelyan (Armenia), A. Pathak, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Wiandt, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

Two Polygons Inscribed in Concentric Circles

12047 [2018, 467]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let C and D be concentric circles with radii r and R, respectively, with $r<R$. Let $A_1A_2\cdots A_n$ be a convex n-gon with perimeter p inscribed in C. For $1\le k\le n$, let B_k be the intersection of the ray A_kA_{k+1} with the circle D, where $A_{n+1}=A_1$. Let q be the perimeter of the n-gon $B_1B_2\cdots B_n$. Prove $p/q\le r/R$, and determine when equality holds.

Solution by the proposer. Let $B_{n+1}=B_1$. Applying Ptolemy's inequality to the quadrilateral $OA_{i+1}B_iB_{i+1}$, we obtain

$$OA_{i+1}\cdot B_iB_{i+1} + OB_{i+1}\cdot A_{i+1}B_i \ge OB_i\cdot A_{i+1}B_{i+1}. \tag{$*$}$$

Since A_{i+2} is between A_{i+1} and B_{i+1}, we have $A_{i+1}B_{i+1}=A_{i+1}A_{i+2}+A_{i+2}B_{i+1}$. Also, $OA_{i+1}=r$ and $OB_{i+1}=OB_i=R$. It follows that $(*)$ is equivalent to

$$r\cdot B_iB_{i+1} + R(A_{i+1}B_i - A_{i+2}B_{i+1}) \ge R\cdot A_{i+1}A_{i+2}.$$

Summing both sides as i varies from 1 to n, we obtain $rq\ge Rp$. This is the desired inequality.

We now show that equality holds if and only if the polygon $A_1A_2\cdots A_n$ is regular. Clearly, $rq=Rp$ if and only if there is equality in $(*)$ for every i, and this is equivalent to the fact that the quadrilateral $OA_{i+1}B_iB_{i+1}$ is cyclic for every i.

If the quadrilateral $OA_{i+1}B_iB_{i+1}$ is cyclic for all i, then we have $\angle OA_{i+1}A_{i+2} = \angle OA_{i+1}B_{i+1} = \angle OB_iB_{i+1}$ and $\angle OB_{i+1}B_i = \angle OA_{i+1}A_i$. Since $\triangle OB_iB_{i+1}$ is isosceles, $\angle OB_iB_{i+1}=\angle OB_{i+1}B_i$. It follows that $\angle OA_iA_{i+1}=\angle OA_{i+1}A_{i+2}$. Consequently,

$$\angle A_iOA_{i+1} = \pi - 2\angle OA_iA_{i+1} = \pi - 2\angle OA_{i+1}A_{i+2} = \angle A_{i+1}OA_{i+2}.$$

This shows that $\angle A_i O A_{i+1}$ is independent of i, and hence the polygon $A_1 A_2 \cdots A_n$ is regular.

Conversely, if $A_1 A_2 \cdots A_n$ is a regular n-gon, then $B_1 B_2 \cdots B_n$ is also a regular n-gon. In this case, the ratio of the perimeters is equal to the ratio of the radii of the circumscribed circles.

Also solved by D. Fleischman, R. Stong, L. Zhou, and An-Anduud Problem Solving Group (Mongolia).

Carmichael in a Taxicab

12048 [2018, 562]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Call an integer a *special Carmichael number* if it can be written as $(6k+1)(12k+1)(18k+1)$ for some integer k, with each of $6k+1$, $12k+1$, and $18k+1$ being prime. Call an integer a *taxicab number* if it can be written as the sum of two positive integer cubes in two different ways. Show that 1729 is the only positive integer that is both a special Carmichael number and a taxicab number.

Solution by Albert Stadler, Herrliberg, Switzerland. Let $p = 6k+1$, $q = 12k+1$, and $r = 18k+1$. If $k = 1$, then $(p, q, r) = (7, 13, 19)$ and $n = pqr = 1729 = 12^3 + 1^3 = 10^3 + 9^3$. Thus 1729 is both a taxicab number and a special Carmichael number. Suppose that n is a special Carmichael number of the form pqr with $k \geq 2$. Suppose also that n is representable as a sum of two positive integer cubes: $n = a^3 + b^3 = (a+b)(a^2 - ab + b^2)$. We prove that this representation is unique (up to the order of the summands). Begin with the inequalities

$$a^2 - ab + b^2 < (a+b)^2 \leq 4(a^2 - ab + b^2).$$

Multiply by $a+b$ to get

$$pqr = n = a^3 + b^3 < (a+b)^3 \leq 4(a^3 + b^3) = 4n = 4pqr.$$

Since $a+b$ divides n, also $a+b$ is divisible by p, q, or r. The inequality $(a+b)^3 \leq 4pqr$ implies that $a+b$ cannot be divisible by two of these primes, since $p^3 q^3 > 4pqr$. Therefore, $a+b \in \{p, q, r\}$. If $a+b = p$, then

$$(6k+1)(12k+1)(18k+1) < p^3 = (6k+1)^3,$$

which is a contradiction. If $a+b = r$, then

$$(18k+1)^3 = r^3 \leq 4(6k+1)(12k+1)(18k+1),$$

which is equivalent to $12(k^2 - k) \leq 1$, also a contradiction. Therefore, $a+b = q = 12k+1$ and $(a+b)^2 - 3ab = a^2 - ab + b^2 = (6k+1)(18k+1)$. This implies that both $a+b$ and ab are determined by k, so the set $\{a, b\}$ is determined uniquely by k. Therefore, n has at most one representation as a sum of two cubes and cannot be a taxicab number.

Editorial comment. By eliminating k in the equations for $a+b$ and $a^2 - ab + b^2$ and reducing to a Pell equation, O. P. Lossers and John P. Robertson showed that if a special Carmichael number other than 1729 is a sum of two positive integer cubes, then k must be very large; Robertson obtained $k > 10^{5000}$.

Also solved by R. Boukharfane (France), R. Chapman (UK), J. Christopher, S. Das Biswas (India), D. Fleischman, S. M. Gagola Jr., K. Gatesman, E. J. Ionaşcu, Y. J. Ionin, O. P. Lossers (Netherlands), A. Pathak, M. Reid, J. P. Robertson, C. Schacht, J. C. Smith, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (UK), and the proposer.