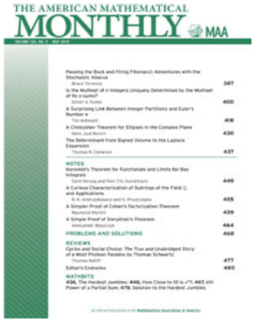
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Problems and Solutions

Edited by Gerald A. Edgar, Daniel H. Ullman, Douglas B. West & with the collaboration of Paul Bracken

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by September 30, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12111. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* A line segment AB can be oriented in two ways, which we denote (AB) and (BA). A square $ABCD$ can be oriented in two ways, which we denote $(ABCD)$ (the same as $(BCDA)$, $(CDAB)$, and $(DABC)$) and $(DCBA)$ (the same as $(CBAD)$, $(BADC)$, and $(ADCB)$). We say that the orientation $(ABCD)$ of a square *agrees with* the orientations (AB), (BC), (CD), and (DA) of its sides. Suppose that each edge and 2-dimensional face of an n-dimensional cube is given an orientation.
(a) What is the largest possible number of 2-dimensional faces whose orientation agrees with the orientations of its four sides?
(b) What is the largest possible number of edges whose orientation agrees with the orientations of all 2-dimensional faces containing the edge?

12112. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam.* Let ABC be a triangle with circumcenter O and nine-point center N. Let P be a point on its circumcircle and let D, E, and F be the circumcenters of triangles AOP, BOP, and COP, respectively. Let A', B', and C' be the feet of perpendiculars from D, E, and F onto the lines BC, CA, and AB, respectively. Prove that A', B', C', and N are collinear.

12113. *Proposed by Richard P. Stanley, University of Miami, Coral Gables, FL.* Define $f(n)$ and $g(n)$ for $n \geq 0$ by

$$\sum_{n\geq 0} f(n)x^n = \sum_{j\geq 0} x^{2^j} \prod_{k=0}^{j-1}\left(1 + x^{2^k} + x^{3\cdot 2^k}\right)$$

and

$$\sum_{n\geq 0} g(n)x^n = \prod_{i\geq 0}\left(1 + x^{2^i} + x^{3\cdot 2^i}\right).$$

Find all values of n for which $f(n) = g(n)$, and find $f(n)$ for these values.

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12114. *Proposed by Zachary Franco, Houston, TX.* Let n be a positive integer, and let $A_n = \{1/n, 2/n, \dots, n/n\}$. Let a_n be the sum of the numerators in A_n when these fractions are expressed in lowest terms. For example, $A_6 = \{1/6, 1/3, 1/2, 2/3, 5/6, 1/1\}$, so $a_6 = 1+1+1+2+5+1 = 11$. Find $\sum_{n=1}^{\infty} a_n/n^4$.

12115. *Proposed by Marius Drăgan, Bucharest, Romania.* Let a, b, c, and d be positive real numbers. Prove

$$(a^3+b^3)(a^3+c^3)(a^3+d^3)(b^3+c^3)(b^3+d^3)(c^3+d^3) \geq (a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2)^3.$$

12116. *Proposed by Rishubh Thaper, Fleminton, NJ.* In a round-robin tournament with n players, each player plays every other player exactly once, and each match results in a win for one player and a loss for the other. When player A defeats player B, we call B the *victim* of A. At the end of the tournament, each player computes the total number of losses incurred by the player's victims. Let q be the average of this quantity over all players. Prove that there exists a player with at most $\lfloor\sqrt{q}\rfloor$ wins and a player with at most $\lfloor\sqrt{q}\rfloor$ losses.

12117. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Prove

$$\frac{\sin^{n+1}(4\pi/7)}{\sin^{n+2}(\pi/7)} - \frac{\sin^{n+1}(\pi/7)}{\sin^{n+2}(2\pi/7)} + (-1)^n \frac{\sin^{n+1}(2\pi/7)}{\sin^{n+2}(4\pi/7)} = 2\sqrt{7}\sum \frac{(i+j+k)!}{i!\,j!\,k!}(-1)^{n-i}2^i,$$

where the sum is taken over all triples (i, j, k) of nonnegative integers satisfying $i + 2j + 3k = n$.

SOLUTIONS

A Trigonometric Functional Equation

11998 [2017, 660]. *Proposed by Roger Cuculière, Lycée Pasteur, Neuilly-sur-Seine, France.* Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy $f(z) \leq 1$ for some nonzero real number z and

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

for all real numbers x and y.

Solution by FAU Problem Solving Group, Florida Atlantic University, Boca Raton, FL. The solutions are the constant functions $f(x) = 1$ and $f(x) = -1/2$ and the functions $f(x) = \cos\alpha x$ for $\alpha > 0$. It is easy to see that all these functions satisfy the requirements.

Conversely, suppose that f is a continuous function satisfying the functional equation, $f(z) \leq 1$ for some nonzero z, and $f(x)$ is not identically 1 or $-1/2$. We first prove $f(0) = 1$. Let $c = f(0)$. Setting $x = y = 0$ in the functional equation yields $3c^2 - 2c^3 = 1$, an equation with a double root of 1 and a simple root of $-1/2$. Setting $y = 0$ in the functional equation yields $2(1-c)f(x)^2 = 1 - c^2$ for all x. If $c = -1/2$, then this equation implies $f(x)^2 = 1/4$ for all x; by continuity and since $f(0) = c = -1/2$, we conclude $f(x) = -1/2$ for all x, a contradiction. Thus, $c = 1$; that is, $f(0) = 1$.

Next, setting $x = y$ yields

$$2f(x)^2 + f(2x)^2 - 2f(x)^2 f(2x) = 1,$$

or

$$(f(2x) - 1)(f(2x) - 2f(x)^2 + 1) = 0.$$

Hence

$$f(2x) = 1 \quad \text{or} \quad f(2x) = 2f(x)^2 - 1 \tag{1}$$

for all $x \in \mathbb{R}$.

We claim $f(y) = 1$ for some nonzero y. Suppose otherwise. By (1), $f(2x) = 2f(x)^2 - 1$, when $x \neq 0$. If $f(x) = 0$ for some nonzero x, then $f(2x) = -1$ and $f(4x) = 1$, which is a contradiction. Since $f(0) = 1$, we conclude $f(x) > 0$ for all x. By assumption there is some nonzero z such that $f(z) \leq 1$, and therefore $0 < f(z) < 1$. Letting $\epsilon = 1 - f(z)$, we have $0 < \epsilon < 1$ and

$$f(2z) = 2(1-\epsilon)^2 - 1 = 1 - 2\epsilon(2-\epsilon) \leq 1 - 2\epsilon.$$

By induction, $f(2^n z) \leq 1 - 2^n\epsilon$, implying $f(2^n z) \leq 0$ if n is large enough, a contradiction that establishes the claim.

Assume now $y \neq 0$ and $f(y) = 1$. Applying the functional equation, we get

$$f(x)^2 + f(x+y)^2 - 2f(x)f(x+y) = 0.$$

Thus $f(x+y) = f(x)$ for all real x, so y is a period of f. Since f is not identically 1, it has a minimum period T. Similarly, if $f(x) = 1$, then x is a period of f, and hence $x = kT$ for some $k \in \mathbb{Z}$. Therefore the second alternative of (1) holds for all $x \notin (T/2)\mathbb{Z}$. It follows by continuity that it holds for all x. Thus

$$f(2x) = 2f(x)^2 - 1 \tag{2}$$

for all $x \in \mathbb{R}$.

To conclude, we prove $f(x) = \cos(2\pi x/T)$. Since it is clear that f satisfies the functional equation if and only if $x \mapsto f(\alpha x)$ also satisfies it (where $\alpha > 0$ is a constant), it suffices to prove $f(x) = \cos x$ when $T = 2\pi$. Using (2), from $f(2\pi) = 1$ we get $f(\pi)^2 = 1$, and thus $f(\pi) = -1$ (since π is not a period). Using (2) again we get $f(\pi/2) = 0$. Moreover, if $0 < x < \pi/2$, then $f(x) \neq 1$, and if $f(x) = 0$, then using (2) yields $f(4x) = 1$ and $0 < 4x < 2\pi$, a contradiction. Since $f(0) = 1$ and $f(\pi/2) = 0$, we conclude $0 < f(x) < 1$ for $0 < x < \pi/2$. Since we have proved $0 \leq f(x) \leq 1$ when $x \in [0, \pi/2]$, equation (2) implies $|f(x)| \leq 1$ first for all $x \in [0, \pi]$, then for all $x \in [0, 2\pi]$, and finally by periodicity for all $x \in \mathbb{R}$.

From (2) and induction on n, since both $f(x)$ and $\cos x$ are nonnegative in $[0, \pi/2]$, we see that $f(x) = \cos x$ when $x = \pi/2^n$ for $n \in \mathbb{N}$. Next, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = \sqrt{1 - f(x)^2}$. Solving the functional equation for $f(x+y)$, we find

$$f(x+y) = f(x)f(y) \pm g(x)g(y).$$

To decide which sign applies when $0 < x, y < \pi/2$, let $Q = (0, \pi/2) \times (0, \pi/2)$, and let

$$A = \{(x, y) \in Q\colon\ f(x+y) = f(x)f(y) - g(x)g(y)\}.$$

The set A is clearly closed in Q. It is also open in Q; in fact, its complement is the closed set $\{(x, y) \in Q : f(x+y) = f(x)f(y) + g(x)g(y)\}$. Since Q is connected, either $A = Q$ or A is empty. Now $f(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, and hence $g(\pi/4) = 1/\sqrt{2}$, so

$$f(\pi/2) = 0 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = f(\pi/4)f(\pi/4) - g(\pi/4)g(\pi/4).$$

Thus $(\pi/4, \pi/4) \in A$, implying A is not empty, so $A = Q$ and

$$f(x+y) = f(x)f(y) - g(x)g(y) \tag{3}$$

for all $(x, y) \in Q$. Since f, g, cosine, and sine are all positive in $(0, \pi/2)$, it follows that at points $(x, y) \in Q$ such that $f(x) = \cos x$ and $f(y) = \cos y$, we also have $g(x) = \sin x$, $g(y) = \sin y$, and $f(x+y) = \cos(x+y)$. Having proved that $f(\pi/2^n) = \cos(\pi/2^n)$ for $n \in \mathbb{N}$, we can thus conclude that $f(x) = \cos x$ for all points $x = m\pi/2^n$, where $n \in \mathbb{N}$ and $m = 0, \dots, 2^{n-1}$. Since these points are dense in $[0, \pi/2]$ and since f is continuous, we have established that $f(x) = \cos x$ for $0 \le x \le \pi/2$. It follows from (2) that we also have $f(x) = \cos x$ in $[0, \pi]$, then in $[0, 2\pi]$, and finally for all $x \in \mathbb{R}$.

Editorial comment. Several solvers pointed out that if we drop the condition that $f(z) \le 1$ for some nonzero z, then we get the additional solutions $f(x) = \cosh \alpha x$ for $\alpha > 0$.

Also solved by R. Chapman (U. K.), R. Ger (Poland), J. W. Hagood, E. A. Herman, E. J. Ionaşcu, Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), O. P. Lossers (Netherlands), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

A Variation on Euler's Formula for Pi

11999 [2017, 754]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k} + \sqrt{k+1} \rfloor}}{k(k+1)}.$$

Solution by Russell A. Gordon, Whitman College, Walla Walla, WA. The value is $\pi^2/3 - 3$.

First, we compute $\lfloor \sqrt{k} + \sqrt{k+1} \rfloor$ for $k \in \mathbb{N}$. Let $n = \lfloor \sqrt{k} \rfloor$, so $n^2 \le k < (n+1)^2$. We prove

$$\left\lfloor \sqrt{k} + \sqrt{k+1} \right\rfloor = \begin{cases} 2n, & \text{when } n^2 \le k \le n^2 + n - 1; \\ 2n+1, & \text{when } n^2 + n \le k \le n^2 + 2n. \end{cases} \tag{$*$}$$

It is immediate that $\sqrt{k} + \sqrt{k+1}$ is either $2n$ or $2n+1$. If $k \le n^2 + n - 1$, then

$$\sqrt{k} + \sqrt{k+1} \le \sqrt{n^2+n-1} + \sqrt{n^2+n} < 2\sqrt{n^2+n} < \sqrt{4n^2+4n+1} = 2n+1.$$

If $k \ge n^2 + n$, then

$$\begin{aligned} \sqrt{k} + \sqrt{k+1} &\ge \sqrt{n^2+n} + \sqrt{n^2+n+1} \\ &= \sqrt{n^2+n+2\sqrt{(n^2+n)(n^2+n+1)} + n^2+n+1} \\ &> \sqrt{4(n^2+n)+1} = 2n+1. \end{aligned}$$

This yields $(*)$.

The given series is absolutely convergent, since the series comprised of the absolute values of its terms is dominated by $\sum_{k=1}^{\infty} 1/k^2$. Hence rearrangements and regroupings do not affect the sum. Also, we note the simplifying presence of telescoping sums:

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \sum_{k=m}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=m}^{n} \frac{1}{k} - \sum_{k=m+1}^{n+1} \frac{1}{k} = \frac{1}{m} - \frac{1}{n+1}.$$

Finally, recall Euler's famous formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. Putting these facts together, we obtain

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)} = \sum_{n=1}^{\infty} \left(\sum_{k=n^2}^{n^2+n-1} \frac{1}{k(k+1)} - \sum_{k=n^2+n}^{n^2+2n} \frac{1}{k(k+1)} \right)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} - \frac{1}{n^2+n} \right) - \left(\frac{1}{n^2+n} - \frac{1}{(n+1)^2} \right) \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 2 \cdot \frac{\pi^2}{6} - 1 - 2 \cdot 1 = \frac{\pi^2}{3} - 3.$$

Also solved by U. Abel (Germany), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), A. Berkane (Algeria), R. Bittencourt (Brazil), R. Boukharfane (France), R. Brase, R. Chapman (U. K.), H. Chen, W. J. Cowieson, R. Cuculière (France), P. P. Dályay (Hungary), V. Dassios (Greece), B. E. Davis, T. de Souza Leão (Brazil), S. Dzhatdoyev & Q. Liu, G. Fera (Italy), K. Gatesman, C. Georghiou (Greece), O. Geupel (Greece), M. L. Glasser, N. Grivaux (France), A. Habil (Syria), E. A. Herman, Y. J. Ionin, W. P. Johnson, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada) & L. Cooper (Canada) & E. Drake (Canada) & L. Kenney (Canada), M. Lafond (France), P. Lalonde (Canada), J. H. Lindsey II, L. Lipták, O. P. Lossers (Netherlands), J. Magliano, R. Martin (Germany), P. McPolin (U. K.), N. Merz, M. D. Meyerson, V. Mikayelyan (Armenia), R. Molinari, R. Nandan, M. Omarjee (France), A. Pathak, Á. Plaza & F. Perdomo (Spain), M. A. Prasad (India), F. A. Rakhimjanovich (Uzbekistan), H. Ricardo, C. Schacht, V. Schindler (Germany), E. Schmeichel, R. Schumacher (Switzerland), N. C. Singer, J. C. Smith, A. Stadler (Switzerland), A. Stenger, R. Stong, M. Tang, R. Tauraso (Italy), D. B. Tyler, J. Vinuesa (Spain), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), Y. Xiang (China), L. Zhou, GCHQ Problem Solving Group (U. K.), Lafayette Problem Solving Group, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Unlucky Thirteen

12001 [2017, 754]. *Proposed by Marius Coman, Bucharest, Romania, and Florian Luca, Johannesburg, South Africa.* A *base*-2 *pseudoprime* is an odd composite number n that divides $2^n - 2$. Prove that if p is a prime number greater than 13, then there is a base-2 pseudoprime that divides $2^{p-1} - 1$.

Solution by Michael Tang, student, Massachusetts Institute of Technology, Cambridge, MA. First suppose that $p-1$ has a prime factor q with $q \geq 5$. We claim that $n = (2^{2q}-1)/3$ is a base-2 pseudoprime that divides $2^{p-1}-1$. To see this, first note $n = (2^q-1)(2^q+1)/3$. Both factors in the numerator are larger than 3 and odd, so n is also odd and composite. Since $q > 3$ and $\varphi(2q) = q-1$, where φ is Euler's totient function, by Euler's theorem $n \equiv (2^2-1)/3 = 1 \pmod{2q}$, so $2q \mid (n-1)$. Also $2q \mid (p-1)$, because both 2 and q divide $p-1$. Hence, $2^{2q}-1$ divides both $2^{n-1}-1$ and $2^{p-1}-1$, so n divides both 2^n-2 and $2^{p-1}-1$ as claimed.

It remains to consider primes p with $p > 13$ such that $p-1 = 2^a \cdot 3^b$ for some integers $a, b \geq 0$. Since $p-1$ is even, $a \geq 1$. Also, $p \geq 17$, so $p-1 \geq 16$. Hence either $b=0$ and $a \geq 4$, or $b=1$ and $a \geq 3$, or $b \geq 2$ and $a \geq 1$. We conclude that 16, 24, or 18, respectively, must divide $p-1$. It is easy to verify that 4369 (equal to $17 \cdot 257$), 1105 (equal to $5 \cdot 13 \cdot 17$), and 1387 (equal to $19 \cdot 73$) are base-2 pseudoprimes that divide $2^{16}-1$, $2^{24}-1$, and $2^{18}-1$, respectively. Hence at least one of these integers divides $2^{p-1}-1$, completing the proof.

Editorial comment. Yury J. Ionin noted (as can be also seen from the above proof) that p does not need to be prime, only odd, and that the result also holds for $p = 11$ in addition to every odd number larger than 13. Stephen M. Gagola Jr. showed that for any prime p and any $a \geq 2$ with $\gcd(p, a) = 1$, there is a base-a pseudoprime that divides $a^{p-1} - 1$ with the exceptions given in the problem ($a = 2$, $p = 3, 5, 7, 13$) and when (a, p) is either $(3, 2)$ or $(3, 5)$.

Also solved by R. Brase, S. M. Gagola Jr., Y. J. Ionin, P. Komjáth (Hungary), P. W. Lindstrom, O. P. Lossers (Netherlands), M. A. Prasad (India), J. P. Robertson, A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Geometric Realization of Hlawka's Inequality

12002 [2017, 754]. *Proposed by Florin Stanescu, Gaesti, Romania.* Let ABC be a triangle with area S, circumradius R, circumcenter O, and orthocenter H. Let D be the point of intersection of lines AO and BC. Similarly, let E be the point of intersection of lines BO and CA, and let F be the point of intersection of lines CO and AB. Let $T = \sqrt{(3R^2 - OH^2)^2 + 16S^2}\big/R^2$. Prove

$$T \leq \frac{AH}{OD} + \frac{BH}{OE} + \frac{CH}{OF} \leq 3 + \frac{T}{2}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Without loss of generality, we assume that the circumcircle of ΔABC is the unit circle in the complex plane, and the vertices are represented by the complex numbers α, β, and γ. The radian measures of the angles at A, B, and C are also denoted A, B, and C. Recalling that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, we see that

$$OH^2 = 3 + 2(\cos 2A + \cos 2B + \cos 2C). \tag{1}$$

On the other hand, since $A + B = \pi - C$,

$$\begin{aligned} S &= \frac{1}{2}ab \sin C = 2 \sin A \sin B \sin C = \big(\cos(A - B) - \cos(A + B)\big) \sin C \\ &= \cos(A - B)\sin(A + B) - \cos(A + B)\sin(A + B) \\ &= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C). \end{aligned} \tag{2}$$

It follows from (1) and (2) that

$$T^2 = 4\left(\cos 2A + \cos 2B + \cos 2C\right)^2 + 4\left(\sin 2A + \sin 2B + \sin 2C\right)^2,$$

which can be put in the form

$$T = 2|e^{2iA} + e^{2iB} + e^{2iC}|. \tag{3}$$

Since D is the point of intersection of the lines OA and BC, there exist two real numbers t and s such that $\overrightarrow{OD} = t\overrightarrow{OA} = \overrightarrow{OB} + s\overrightarrow{BC}$. This condition is exactly $t\alpha = \beta + s(\gamma - \beta)$. Taking complex conjugates and then multiplying both sides by $\alpha\beta\gamma$, we obtain

$$t\beta\gamma = \alpha(\gamma - s(\gamma - \beta)) = \alpha(\gamma + \beta - t\alpha).$$

Thus $t(\beta\gamma + \alpha^2) = \alpha(\beta + \gamma)$, so $OD = |t| = |\beta + \gamma|/|\beta\gamma + \alpha^2|$. Also $\overrightarrow{AH} = \overrightarrow{OB} + \overrightarrow{OC}$, and hence $AH = |\beta + \gamma|$. Therefore

$$\frac{AH}{OD} = |\beta\gamma + \alpha^2| = \left|\frac{\beta}{\alpha} + \frac{\alpha}{\gamma}\right| = |e^{2iC} + e^{2iB}|. \tag{4}$$

Similarly,

$$\frac{BH}{OE} = |e^{2iA} + e^{2iC}| \quad \text{and} \quad \frac{CH}{OF} = |e^{2iB} + e^{2iA}|. \tag{5}$$

By the triangle inequality,

$$\begin{aligned} &\left|e^{2iC} + e^{2iB} + e^{2iA} + e^{2iC} + e^{2iB} + e^{2iA}\right| \\ &\qquad \le \left|e^{2iC} + e^{2iB}\right| + \left|e^{2iA} + e^{2iC}\right| + \left|e^{2iB} + e^{2iA}\right|. \end{aligned}$$

The left side of this is T by (3), so using (4) and (5) on the right side, we get the first inequality in the problem statement. Hlawka's inequality yields

$$\begin{aligned} &|e^{2iC} + e^{2iB}| + |e^{2iA} + e^{2iC}| + |e^{2iB} + e^{2iA}| \\ &\qquad \le |e^{2iA}| + |e^{2iB}| + |e^{2iC}| + |e^{2iA} + e^{2iB} + e^{2iC}|, \end{aligned}$$

and the right side of this inequality is equal to $3 + T/2$, yielding the desired second inequality.

Also solved by P. P. Dályay (Hungary), D. Fleischman, R. Stong, and the proposer.

A GCD-weighted Trigonometric Sum

12003 [2017, 754]. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Given an odd positive integer n, compute

$$\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2(\pi k/n)}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Letting the prime factorization of n be $\prod_{i=1}^{s} p_i^{r_i}$, we prove

$$\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2(\pi k/n)} = \prod_{i=1}^{s} \left(p_i^{2r_i} + p_i^{2r_i-1} - p_i^{r_i-1}\right).$$

We first show $\sum_{k=1}^{n}(\cos(\pi k/n))^{-2} = n^2$. Let T_n be the nth Chebyshev polynomial of the first kind, defined by the recurrence $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$ for $n \ge 1$, with $T_0(z) = 1$ and $T_1(z) = z$. From this recurrence, one can show by induction on n that $T_n(\cos\theta) = \cos n\theta$, $T_n(-1) = (-1)^n$, and $T_n'(-1) = (-1)^{n-1}n^2$. Now let $P_n(x) = T_n(2x-1) - 1$. The n roots of the polynomial P_n, with the correct multiplicities, are $(\cos(2\pi k/n) + 1)/2$ for $1 \le k \le n$. The constant term c_0 of P_n is $T_n(-1) - 1$, which is -2, since n is odd. The linear coefficient c_1 of P_n is $2T_n'(-1)$, which is $2(-1)^{n-1}n^2$, or $2n^2$. Since the sum of the reciprocals of the roots of a polynomial $\sum_{i=0}^{n} c_i x^i$ is $-c_1/c_0$, we obtain $\sum_{k=1}^{n}\big(\cos(\pi k/n)\big)^{-2} = \sum_{k=1}^{n}\big((\cos(2\pi k/n) + 1)/2\big)^{-1} = n^2$.

The Euler totient $\phi(m)$ is the number of values in $[m]$ that are relatively prime to m; it satisfies $m = \sum_{d|m} \phi(d)$ for all $m \in \mathbb{N}$. Applying this with $m = \gcd(k,n)$, interchanging the order of summation, and letting $r = k/d$, we obtain

$$\sum_{k=1}^{n} \sum_{d|\gcd(k,n)} \frac{\phi(d)}{\cos^2(\pi k/n)} = \sum_{d|n} \sum_{r=1}^{n/d} \frac{\phi(d)}{\cos^2(\pi r/(n/d))} = \sum_{d|n} \frac{\phi(d)n^2}{d^2}.$$

When n_1 and n_2 are relatively prime, the divisors of $n_1 n_2$ are the products of the divisors of n_1 and n_2, hence the sum we have obtained is a multiplicative function of n. When n is a prime power, say $n = p^r$, we use $\phi(p^j) = p^j - p^{j-1}$ for $j \geq 1$ to evaluate the sum as

$$\sum_{d|n} \frac{\phi(d)n^2}{d^2} = p^{2r} + \sum_{j=1}^{r} (p^{2r-j} - p^{2r-j-1}) = p^{2r} + p^{2r-1} - p^{r-1}.$$

The result follows.

Also solved by R. Bittencourt (Brazil), R. Brase, R. Chapman (U. K.), K. Gatesman, Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), M. A. Prasad (India), I. Sfikas, N. C. Singer, A. Stadler (Switzerland), M. Tang, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Divergence of a Series

12004 [2017, 755]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_1, a_2, \ldots$ be a strictly increasing sequence of real numbers satisfying $a_n \leq n^2 \ln n$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} 1/(a_{n+1} - a_n)$ diverges.

Solution by Nicholas C. Singer, Annandale, VA. For $k \geq 1$, apply the Harmonic-Mean–Arithmetic-Mean inequality to the positive numbers in $\{a_{2^k+j} - a_{2^k+j-1} \colon 1 \leq j \leq 2^k\}$ to obtain

$$\frac{1}{a_{2^k+1} - a_{2^k}} + \frac{1}{a_{2^k+2} - a_{2^k+1}} + \cdots + \frac{1}{a_{2^{k+1}} - a_{2^{k+1}-1}} \geq \frac{4^k}{a_{2^{k+1}} - a_{2^k}} \geq \frac{4^k}{a_{2^{k+1}} - a_1}.$$

Since $a_1 \leq 0$,

$$\frac{4^k}{a_{2^{k+1}} - a_1} = \frac{4^k}{a_{2^{k+1}} + |a_1|} \geq \frac{4^k}{2^{2k+2}(k+1)\ln 2 + |a_1|} = \frac{1}{4(k+1)\ln 2 + |a_1|/4^k}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1} - a_n} = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \frac{1}{a_{2^k+j} - a_{2^k+j-1}} \geq \sum_{k=0}^{\infty} \frac{1}{4(k+1)\ln 2 + |a_1|/4^k} = \infty.$$

Editorial comment. Several solvers overlooked the possibility that a_n might be negative for some (or all) n.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Brase, H. Chen, P. J. Fitzsimmons, D. Fleischman, E. J. Ionaşcu, M. Javaheri, P. Komjáth (Hungary), O. Kouba (Syria), K. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), V. Mikayelyan (Armenia), P. Perfetti (Italy), Á. Plaza & K. Sadarangani (Spain), M. A. Prasad (India), J. C. Smith, O. Sonebi (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), GCHQ Problem Solving Group (U. K.), and the proposer.

A Suspicious Formula Involving Pi

12006 [2017, 970]. *Proposed by Jonathan D. Lee, Merton College, Oxford, U. K., and Stan Wagon, Macalester College, St. Paul, MN.* When n is an integer and $n \geq 2$, let $a_n = \lceil n/\pi \rceil$ and $b_n = \lceil \csc(\pi/n) \rceil$. The sequences $a_2, a_3, \ldots$ and $b_2, b_3, \ldots$ are, respectively,

$$1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots$$

and

$$1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots.$$

They differ when $n = 3$. Are they equal for all larger n?

Solution by Albert Stadler, Herrliberg, Switzerland. The answer is no, as can be checked by direct calculation for $n = 80143857$. As motivation for this answer, the Laurent expansion of $\csc(\pi x)$ is $1/(\pi x) + \pi x/6 + \cdots$ with all coefficients positive. Thus when $n \geq 2$ we have $0 < \csc(\pi/n) - n/\pi \leq \csc(\pi/2) - 2/\pi < 1$. It follows that $b_n - 1 \leq a_n \leq b_n$, and furthermore that $b_n = a_n + 1$ when there exists an integer m such that

$$0 < \frac{m}{n} - \frac{1}{\pi} < \frac{\pi}{6n^2}. \qquad (*)$$

Good candidates for m/n are given by the continued fraction convergents of $1/\pi$, every second one of which is greater than $1/\pi$. The continued fraction representation of $1/\pi$ is $[0; 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots]$, and so one may compute that the first two convergents that satisfy $(*)$ are the second and 14th. These are $1/3$ and $25510582/80143857$, leading to $a_n \neq b_n$ for $n = 3$ and $n = 80143857$.

Editorial comment. Direct computation shows that $a_n = b_n$ when $4 \leq n \leq 80143856$.

It is natural to wonder whether the sequences differ infinitely often. The proposers noted that by Hurwitz's theorem there are infinitely many convergents to $1/\pi$ such that $|\frac{1}{\pi} - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}$, which implies $|\frac{1}{\pi} - \frac{m}{n}| < \frac{\pi}{6n^2}$. However, only even-numbered convergents will be greater than $1/\pi$, as needed for $(*)$. It seems likely, given how the continued fraction of π is expected to behave, that there are infinitely many even-numbered convergents among the ones that satisfy the condition of Hurwitz's theorem, but this is currently unresolved.

Also solved by A. Berele, R. Chapman (U. K.), S. Demers (Canada), G. Fera (Italy), O. P. Lossers (Netherlands), M. D. Meyerson, V. Mikayelyan (Armenia), M. Reid, C. Schacht, V. Schindler (Germany), J. C. Smith, A. Stenger, A. Stewart, R. Stong, W. Stromquist, R. Tauraso (Italy), D. Terr, H. Widmer (Switzerland), L. Zhou, Armstrong Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposers.

An Application of the Phragmén–Lindelöf Principle

12009 [2017, 970]. *Proposed by George Stoica, Saint John, NB, Canada.* Find all continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying $\left|\int_0^1 e^{xy} f(x)\, dx\right| < 1/y$ for all positive real numbers y.

Solution by James Christopher Smith, Knoxville, TN. We claim that the only such function is the constant 0. Let $g(z) = \int_0^1 e^{xz} f(x)\, dx$ for all $z \in \mathbb{C}$. Because f is continuous on $[0, 1]$, it is bounded and measurable, so g is an entire function.

We apply the Phragmén–Lindelöf principle to $g(z)$ on the first quadrant D in the complex plane. First, we note the estimate

$$|g(z)| \leq \int_0^1 \left|e^{xz} f(x)\right| dx \leq Me^{|z|},$$

where $M = \int_0^1 |f(x)|\, dx$. Second, we claim that g is bounded on the real axis. Indeed, when $-\infty < y \leq 1$ we have $|g(y)| \leq Me$ and for $y \geq 1$ we have $|g(y)| \leq 1/y \leq 1$. And third, we claim that g is bounded on the imaginary axis. Indeed, for $y \in \mathbb{R}$ we have $|g(iy)| \leq \int_0^1 \left|e^{ixy} f(x)\right| dx \leq M$. Therefore, by the Phragmén–Lindelöf principle, $g(z)$ is bounded in the quadrant D. Similarly, $g(z)$ is bounded in each of the other three quadrants as well.

Thus $g(z)$ is a bounded entire function, so by Liouville's theorem $g(z)$ is constant. Hence, for all $n \geq 1$, we have $0 = g^{(n)}(0) = \int_0^1 x^n f(x)\, dx$. By the Weierstrass approximation theorem applied to $xf(x)$, we conclude that f is the constant function 0.

Also solved by K. F. Andersen (Canada), A. Stadler (Switzerland), G. Vidiani (France), GCHQ Problem Solving Group (U. K.), and the proposer.