The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://www.tandfonline.com/loi/uamm20

Problems and Solutions

Edited by Gerald A. Edgar, Daniel H. Ullman, Douglas B. West & with the collaboration of Paul Bracken

To cite this article: Edited by Gerald A. Edgar, Daniel H. Ullman, Douglas B. West & with the collaboration of Paul Bracken (2019) Problems and Solutions, The American Mathematical Monthly, 126:4, 370-378

To link to this article: https://doi.org/10.1080/00029890.2019.1574184

Published online: 22 Apr 2019.

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by August 31, 2019 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12104. *Proposed by Joe Buhler, Larry Carter, and Richard Stong, Center for Communications Research, San Diego, CA.* Consider a standard clock, where the hour, minute, and second hands all have integer lengths and all point straight up at noon and midnight. Is it possible for the ends of the hands to form, at some time, the vertices of an equilateral triangle?

12105. *Proposed by Gary Brookfield, California State University, Los Angeles, CA.* Let r be a real number, and let $f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r$. Suppose that f has real roots a, b, and c. Prove $a, b, c \in [-1, 1]$ and $|\arcsin a| + |\arcsin b| + |\arcsin c| = \pi$.

12106. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For any positive integer n, prove

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{1}{ij} = \sum_{1 \le i \le j \le n} \frac{2^n - 2^{n-i}}{ij}.$$

12107. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{1+x^2}\sqrt{1+y^2}(1-x^2y^2)}\, dx\, dy = G,$$

where G is Catalan's constant $\sum_{n=1}^{\infty} (-1)^{n-1}/(2n-1)^2$.

12108. *Proposed by Yifei Pan and William D. Weakley, Purdue University Fort Wayne, Fort Wayne, IN.* Let n be a positive integer, and let $\beta_1, \dots, \beta_n$ be indeterminates over a field F. Let M be the n-by-n matrix whose i, j-entry m_{ij} is given by $m_{ij} = \beta_i$ when $i = j$ and $m_{ij} = 1$ when $i \neq j$. Show that the polynomial $\det(M)$ is irreducible over F.

12109. *Proposed by George Stoica, Saint John, NB, Canada.* Let f be a function on $[0, \infty)$ that is nonnegative, bounded, and continuous. For $a > 0$ and $x \ge 0$, let $g(x) = \exp\left(\int_0^a \log(1 + xf(s))\, ds\right)$. For $0 < p < 1$, prove

$$\int_0^a f^p(x)\, ds = \frac{p \sin(p\pi)}{\pi} \int_0^\infty \frac{\log g(x)}{x^{p+1}}\, dx.$$

doi.org/10.1080/00029890.2019.1574184

12110. *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* Let $\alpha_k = (k + \sqrt{k^2+4})/2$. Evaluate

$$\lim_{k\to\infty} \prod_{n=1}^{\infty} \left(1 - \frac{k}{\alpha_k^n + \alpha_k}\right).$$

SOLUTIONS

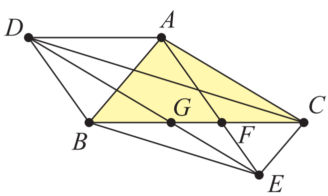
An Extremal Property of Affinely Regular Pentagons

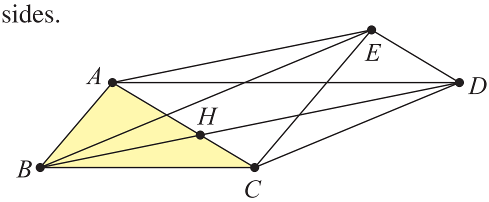
11988 [2017, 563]. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle. Find the extrema of

$$\frac{AC^2 + CE^2 + EB^2 + BD^2 + DA^2}{AB^2 + BC^2 + CD^2 + DE^2 + EA^2}$$

over all points D and E in the plane of ABC. At which points D and E are these extrema attained?

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $\phi = (1+\sqrt{5})/2$. We show that the given expression attains its minimum value ϕ^{-2} and its maximum value ϕ^2 at the left and right configurations shown below, respectively, where all the diagonals of the pentagons are parallel to the corresponding sides.





First note that the law of cosines gives $CD^2 = DA^2 + AC^2 + 2\overrightarrow{DA}\cdot\overrightarrow{AC}$ and $EA^2 = AC^2 + CE^2 + 2\overrightarrow{AC}\cdot\overrightarrow{CE}$. Also, $\overrightarrow{DA} + \overrightarrow{CE} = -\overrightarrow{AC} - \overrightarrow{ED}$. Therefore, $CD^2 + EA^2 = DA^2 + CE^2 - 2\overrightarrow{AC}\cdot\overrightarrow{ED}$. Moreover, $-(AC)(DE) \le \overrightarrow{AC}\cdot\overrightarrow{ED} \le (AC)(DE)$. Equality holds in the left inequality if and only if $\overrightarrow{AC}$ and $\overrightarrow{ED}$ have opposite direction, while the equality holds in the right inequality if and only if they have the same direction. Hence

$$-2(AC)(DE) \le CD^2 + EA^2 - DA^2 - CE^2 \le 2(AC)(DE).$$

Adding to this the other four analogous inequalities, we get $-r^2 \le q^2 - p^2 \le r^2$, where p^2 and q^2 are respectively the numerator and denominator of the given expression, and

$$r^2 = (AC)(DE) + (CE)(AB) + (EB)(CD) + (BD)(EA) + (DA)(BC).$$

By the Cauchy–Schwarz inequality, we have $r^2 \le pq$, with equality if and only if

$$\frac{AC}{DE} = \frac{CE}{AB} = \frac{EB}{CD} = \frac{BD}{EA} = \frac{DA}{BC} = \lambda \qquad (*)$$

for some λ. Thus, $-pq \le q^2 - p^2 \le pq$, and so $\phi^{-1} \le p/q \le \phi$.

By $(*)$, $p/q = \phi^{-1}$ only if $\lambda = \phi^{-1}$, which leads to a construction by ruler and compass of D and E for minimal p/q: Locate F on BC such that $BF/FC = \phi$, and then construct D so that $BFAD$ is a parallelogram, and draw the line through D and parallel to AC to intersect AF at E. Let G be the intersection of DE and BC. Since $\triangle BDG$ is congruent to $\triangle FAC$, we have $BG = FC$. We have

$$\frac{GF}{FC} = \frac{BF}{FC} - \frac{BG}{FC} = \phi - 1 = \phi^{-1},$$

from which it follows that CE is parallel to AB, similarly EB is parallel to CD, and $(*)$ is satisfied.

Likewise, $p/q = \phi$ only if $\lambda = \phi$, which leads to a construction of D and E for maximal p/q: Locate H on AC such that $AH/HC = \phi$, and then draw the line through A and parallel BC to intersect line BH at D. Construct E so that $AHDE$ is a parallelogram. Arguing as before, CE is parallel to BA, BE is parallel to CD, and $(*)$ is satisfied.

Editorial comment. A pentagon of this type, where each diagonal is parallel to one of the sides, is affinely equivalent to a regular pentagon.

Also solved by G. Fera (Italy), O. Kouba (Syria), R. Stong, and the proposer.

Divisibility by an Arbitrary Power of Seven

11992 [2017, 659]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Prove that, for every positive integer n, there is a positive integer m such that $3^m + 5^m - 1$ is divisible by 7^n.

Solution by Sandeep Silwal, Brookline, MA. We show that $m = 7^{n-1}$ works. Let $v_p(k)$ denote the largest integer e such that p^e divides k. The lifting-the-exponent lemma states that if p is an odd prime dividing $x + y$ but dividing neither x nor y, then $v_p(x^n + y^n) = v_p(x + y) + v_p(n)$. Hence $v_7(5^m + 2^m) = v_7(5 + 2) + v_7(m) = n$ and similarly $v_7(3^m + 4^m) = n$. We conclude $5^m \equiv -2^m \pmod{7^n}$ and $3^m \equiv -2^{2m} \pmod{7^n}$, and therefore

$$3^m + 5^m - 1 \equiv -(2^{2m} + 2^m + 1) \pmod{7^n}.$$

Note that $(2^m - 1)(2^{2m} + 2^m + 1) = 8^m - 1$. By another application of the lifting-the-exponent lemma, $v_7(8^m - 1) = v_7(8 - 1) + v_7(m) = n$, and thus $8^m - 1 \equiv 0 \pmod{7^n}$. Because $m \equiv 1 \pmod 6$, Fermat's little theorem implies $2^m \equiv 2^1 = 2 \pmod 7$, so $2^m - 1$ is not divisible by 7. We conclude $2^{2m} + 2^m + 1 \equiv 0 \pmod{7^n}$, and hence $3^m + 5^m - 1 \equiv 0 \pmod{7^n}$, as desired.

Editorial comment. The lifting-the-exponent lemma can be found at brilliant.org/wiki/lifting-the-exponent/. Peter Lindstrom, O. P. Lossers, H. F. Mattson, and Michael Reid showed that setting $m = 5 \cdot 7^{n-1}$ also works. Boris Bekker & Yury Ionin, Stephen Gagola, and the BSI Problems Group showed that if a and b are the primitive 6th roots of unity modulo p, then $a^{p^{n-1}} + b^{p^{n-1}} - 1$ is divisible by p^n. Allen Stenger proved that if $p > 3$ is a prime, $n \geq 1$, $r = (p-1)/2$, $m = p^{n-1}$, and $a_1, \ldots, a_r$ is the complete list of quadratic residues modulo p, then $\sum_{k=1}^{r} a_k^m \equiv 0 \pmod{p^n}$. Marian Tetiva showed that if $p > 3$ is a prime and a, b, c are integers such that both $a + b + c$ and $ab + ac + bc$ are divisible by p, then both $a^{p^n} + b^{p^n} + c^{p^n}$ and $a^{p^n}b^{p^n} + a^{p^n}c^{p^n} + b^{p^n}c^{p^n}$ are divisible by p^{n+1}.

Also solved by B. M. Bekker & Y. J. Ionin, R. Boukharfane (France), R. Chapman (U. K.), J. Christopher, S. M. Gagola, Jr., M. Goldenberg & M. Kaplan, R. A. Gordon, J. Iiams, E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), H. F. Mattson, U. Milutinović (Slovenia), V. I. Murashka (Belarus), M. Omarjee (France), C. R. Pranesachar (India), M. Reid, N. C. Singer, J. Singh (India), O. Sonebi (France), A. Stadler (Switzerland), A. Stenger, R. Stong, M. Tang, R. Tauraso (Italy), M. Tetiva (Romania), J. Van hamme (Belgium), Z. Vörös (Hungary), L. Wimmer, L. Zhou, BSI Problem Solving Group (Germany), GCHQ Problem Solving Group (U. K.) Northwestern University Problem Solving Group, and the proposer.

An Integral Related to Euler Sums

11993 [2017, 659]. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Prove

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = -\frac{\pi^4}{240}.$$

Solution by Abdelhak Berkane, University of Mentouri Brothers, Constantine, Algeria. We use the equality $ab^2 = \left((a+b)^3 + (a-b)^3 - 2a^3\right)/6$ with $a = \log(1-x)$ and $b = \log(1+x)$. From this we obtain

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = \frac{1}{6}I_1 + \frac{1}{6}I_2 - \frac{1}{3}I_3,$$

where

$$I_1 = \int_0^1 \frac{(\log(1-x)+\log(1+x))^3}{x}dx = \int_0^1 \frac{(\log(1-x^2))^3}{x}dx$$
$$= \frac{1}{2}\int_0^1 \frac{(\log(1-u))^3}{u}du = \frac{1}{2}\int_0^1 \frac{(\log t)^3}{1-t}dt,$$

$$I_2 = \int_0^1 \frac{(\log(1-x)-\log(1+x))^3}{x}dx = \int_0^1 \frac{\left(\log\left(\frac{1-x}{1+x}\right)\right)^3}{x}dx$$
$$= 2\int_0^1 \frac{(\log t)^3}{(1-t)(1+t)}dt = \int_0^1 \frac{(\log t)^3}{1-t}dt + \int_0^1 \frac{(\log t)^3}{1+t}dt,$$

and

$$I_3 = \int_0^1 \frac{(\log(1-x))^3}{x}dx = \int_0^1 \frac{(\log t)^3}{1-t}dt.$$

Combining these yields

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = -\frac{1}{12}\int_0^1 \frac{(\log t)^3}{1-t}dt + \frac{1}{6}\int_0^1 \frac{(\log t)^3}{1+t}dt.$$

It is known (see, for example, entries 4.626.1 and 4.626.2 in I. S. Gradshteyn, I. M. Ryzhik, et al. (2015), *Tables of Integrals, Series, and Products*, 8th ed., San Diego, CA: Academic Press) that

$$\int_0^1 \frac{(\log t)^3}{1-t}dt = -\frac{\pi^4}{15} \qquad \text{and} \qquad \int_0^1 \frac{(\log t)^3}{1+t}dt = -\frac{7\pi^4}{120}.$$

We conclude

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}dx = \frac{\pi^4}{180} - \frac{7\pi^4}{720} = -\frac{\pi^4}{240}.$$

Editorial comment. This integral was previously given by P. J. de Doelder (1991), On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y, *J. Comput. Appl. Math.* 37(1–3): 125–141.

As many solvers noted, this integral is closely related to Euler sums. Expanding the logarithms in power series, one sees that the requested integral is

$$I = \sum_{n=0}^{\infty}(-1)^n \frac{H_n H_{n+1}}{(n+1)^2} = \sum_{n=0}^{\infty}(-1)^n \frac{H_n^2}{(n+1)^2} + \sum_{n=0}^{\infty}(-1)^n \frac{H_n}{(n+1)^3},$$

where $H_n = \sum_{k=1}^{n} 1/k$. The sum in the first expression was evaluated in W. Chu (1997), Hypergeometric series and the Riemann zeta function, *Acta Arith.* 82(2): 103–118. The two sums in the final expression are evaluated in D. H. Bailey, J. M. Borwein, and R. Girgensohn, Experimental evaluation of Euler sums (1994), *Exper. Math.* 3(1): 17–30, which gives the first sum as

$$a_h(2,2) = -2\mathrm{Li}_4(1/2) - \frac{1}{12}\log^4 2 + \frac{99}{48}\zeta(4) - \frac{7}{4}\zeta(3)\log 2 + \frac{1}{2}\zeta(2)\log^2 2,$$

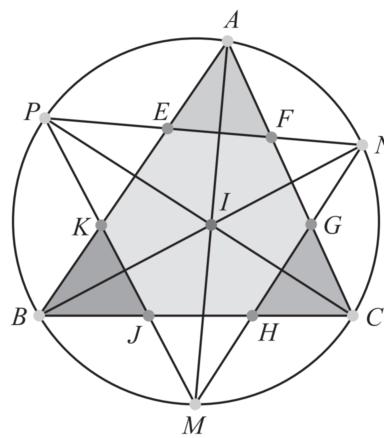
and in D. Borwein, J. M. Borwein, and R. Girgensohn (1995), Explicit evaluation of Euler sums, *Proc. Edin. Math. Soc. (2)*. 38(2): 277–294, which gives a nearly cancelling formula for the second sum $\alpha_h(1,3)$.

Also solved by P. Acosta, K. F. Andersen (Canada), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), R. Boukharfane (France), K. N. Boyadzhiev, P. Bracken, H. Chen, V. Dassios (Greece), B. E. Davis, G. Fera (Italy), M. L. Glasser, A. Hannan (India), O. Kouba (Syria), K. Lau (China), L. Matejíčka (Slovakia), V. Mikayelyan (Armenia), M. Omarjee (France), P. Perfetti (Italy), R. Schumacher (Switzerland), S. Sharma (India), S. Silwal, J. Singh (India), J. C. Smith, A. Stadler (Switzerland), M. Stofka (Slovakia), R. Tauraso (Italy), J. Van Casteren & L. Kempeneers (Belgium), M. R. Yegan (Iran), and the proposer.

A Hexagram Inequality

11994 [2017, 659]. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with incenter I and circumcircle ω. Let M, N, and P be the second points of intersection of ω with lines AI, BI, and CI, respectively. Let E and F be the points of intersection of NP with AB and AC, respectively. Similarly, let G and H be the points of intersection of MN with AC and BC, respectively, and let J and K be the points of intersection of MP with BC and AB, respectively. Prove

$$EF + GH + JK \leq KE + FG + HJ.$$



Solution by Li Zhou, Polk State College, Winter Haven, FL. As usual, we let A, B, and C denote the angles of $\triangle ABC$. Since $\angle NPA$ and $\angle NBA$ are subtended by the same arc of ω and BN bisects $\angle ABC$, we have $\angle NPA = \angle NBA = B/2$. Similarly, $\angle PAB = C/2$. Since $\angle FEA$ is an exterior angle of $\triangle APE$, we have

$$\angle FEA = \angle NPA + \angle PAB = (B+C)/2,$$

and a similar argument shows that $\angle EFA = (B+C)/2$. Therefore $\triangle AEF$ is isosceles and AI bisects EF perpendicularly. Let Q be the intersection point of AI and EF.

We have $\angle PEK = \angle FEA = (B+C)/2$, and similarly $\angle PKE = (A+C)/2$. Since these are both acute angles, the perpendicular from P to KE hits KE at a point R that is strictly between E and K. By the similarity of $\triangle AQE$ and $\triangle PRE$ and the law of sines in $\triangle APE$,

$$\frac{2RE}{EF} = \frac{RE}{EQ} = \frac{PE}{EA} = \frac{\sin(\angle PAB)}{\sin(\angle NPA)} = \frac{\sin(C/2)}{\sin(B/2)}.$$

Likewise, $2KR/JK = \sin(C/2)/\sin(A/2)$. Hence,

$$KE = KR + RE = \frac{JK \sin(C/2)}{2\sin(A/2)} + \frac{EF \sin(C/2)}{2\sin(B/2)}.$$

Similarly,

$$FG = \frac{EF \sin(B/2)}{2\sin(C/2)} + \frac{GH \sin(B/2)}{2\sin(A/2)} \quad \text{and} \quad HJ = \frac{GH \sin(A/2)}{2\sin(B/2)} + \frac{JK \sin(A/2)}{2\sin(C/2)}.$$

Adding these three equations and invoking the AM–GM inequality yields

$$\begin{aligned} KE + FG + HJ &= EF\left(\frac{\sin(C/2)}{2\sin(B/2)} + \frac{\sin(B/2)}{2\sin(C/2)}\right) \\ &\quad + GH\left(\frac{\sin(B/2)}{2\sin(A/2)} + \frac{\sin(A/2)}{2\sin(B/2)}\right) + JK\left(\frac{\sin(C/2)}{2\sin(A/2)} + \frac{\sin(A/2)}{2\sin(C/2)}\right) \\ &\geq EF + GH + JK. \end{aligned}$$

Also solved by M. Bataille (France), R. Boukharfane (France), N. G. Cripe, G. Fera (Italy) O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. G. Heuver (Canada), O. Kouba (Syria), P. McPolin (U. K.), P. Nüesch (Switzerland), M. Omarjee (France), C. R. Pranesachar (India), C. Schacht, R. Stong, R. Tauraso (Italy), T. Toyonari (Japan), T. Wiandt, T. Zvonaru & N. Stanciu (Romania), and the proposer.

A Sequence Generated by Averaging Sines

11995 [2017, 659]. *Proposed by Dan Ştefan Marinescu, National College "Iancu de Hunedoara," Hunedoara, Romania, and Mihai Monea, National College "Decebal," Deva, Romania.* Suppose $0 < x_0 < \pi$, and for $n \geq 1$ define $x_n = (1/n)\sum_{k=0}^{n-1} \sin x_k$. Find $\lim_{n\to\infty} x_n\sqrt{\ln n}$.

Solution by Florin Stanescu, Gaesti, Romania. Since $\sin x_k \leq 1$, we have $x_n \leq 1$. It follows by induction that $x_n > 0$ for all n. Thus x_n is bounded. From $nx_n = \sum_{k=0}^{n-1} \sin x_k$ and $(n+1)x_{n+1} = \sum_{k=0}^{n} \sin x_k$, we obtain

$$(n+1)x_{n+1} - nx_n = \sin x_n, \tag{$*$}$$

and hence

$$x_n - x_{n+1} = (x_n - \sin x_n)/(n+1) > 0.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is decreasing and hence convergent. Let $l = \lim_{n\to\infty} x_n$. Applying the Stolz–Cesaro theorem, we obtain

$$\begin{aligned} l = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \sin x_k}{n} &= \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \sin x_k - \sum_{k=0}^{n-1} \sin x_k}{n+1-n} \\ = \lim_{n\to\infty} \sin x_n = \sin l. \end{aligned}$$

Thus $l = 0$, since this is the only solution to $l = \sin l$. The recurrence $(*)$ may be rewritten

$$\frac{x_{n+1}}{x_n} = \frac{n}{n+1} + \frac{\sin x_n}{(n+1)x_n}.$$

Noting that $\lim_{n\to\infty}(\sin x_n)/x_n = 1$, we see that $\lim_{n\to\infty} x_{n+1}/x_n = 1$. Using the Stolz–Cesaro theorem again we calculate

$$\lim_{n\to\infty} x_n^2 \ln n = \lim_{n\to\infty} \frac{\ln n}{1/x_n^2} = \lim_{n\to\infty} \frac{\ln(n+1) - \ln n}{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right) x_{n+1}^2 x_n^2}{(x_n - x_{n+1})(x_n + x_{n+1})}$$

$$= \lim_{n\to\infty} \frac{(n+1)\ln\left(1+\frac{1}{n}\right) x_{n+1}^2 x_n^2}{(x_n - \sin x_n)(x_n + x_{n+1})}$$

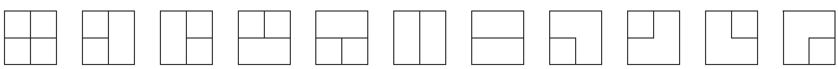
$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{x_n^3}{x_n - \sin x_n} \cdot \frac{\ln\left(1+\frac{1}{n}\right)^n}{\frac{x_n}{x_{n+1}} + \left(\frac{x_n}{x_{n+1}}\right)^2} = 1 \cdot 6 \cdot \frac{1}{2} = 3,$$

where we have used $\lim_{x\to 0}(x - \sin x)/x^3 = 1/6$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Hence $\lim_{n\to\infty} x_n\sqrt{\ln n} = \sqrt{3}$.

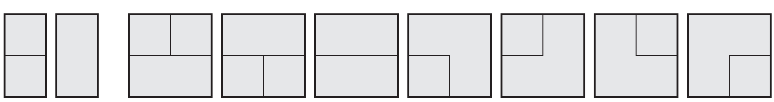
Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), R. Chapman (U. K.), G. Fera (Italy), E. J. Ionaşcu, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy) & M. Omarjee (France), M. Tetiva (Romania), D. B. Tyler, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Tilings of a Strip

11996 [2017, 659]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Consider all the tilings of a 2-by-n rectangle comprised of tiles that are either a unit square, a domino, or a right tromino. Let f_n be the fraction of tiles among all such tilings that are unit squares. For example, $f_2 = 4/7$, because 16 out of the 28 tiles in the 11 tilings of a 2-by-2 rectangle are squares. What is $\lim_{n\to\infty} f_n$?



Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. The answer is $(30 - 4\sqrt{5})/41$, which is approximately $0.513554\ldots$. A *primitive* tiling of a strip is a tiling that cannot be split into tilings of two shorter strips. Let s_n be the number of primitive tilings of a 2-by-n strip. We have $s_1 = 2$, $s_2 = 7$, and $s_n = 8$ for $n \geq 3$. The last is because there are two cases of trominoes at both ends, four cases of a tromino at only one end, and two cases of no trominoes.



Let a_n be the number of all tilings of a 2-by-n strip. We have $a_0 = 1$ and $a_n = \sum_{i=1}^{n} s_i a_{n-i}$ for $n \geq 1$. Subtracting the expression for a_n from that for a_{n+1}, we obtain

$$a_{n+1} - 3a_n - 5a_{n-1} - a_{n-2} = 0.$$

Let x_n be the total number of squares in all of the 2-by-n tilings. Let p_n be the number of squares in the primitive tilings, so $p_1 = 2$ and $p_n = 8$ for $n \geq 2$. We obtain $x_n =$

$\sum_{i=1}^{n}(p_i a_{n-i} + s_i x_{n-i})$, which arises by letting i be the least index so that the initial 2-by-i subtiling is primitive, with the first term counting the squares in the first i positions and the second term counting the squares in the last $n - i$ positions. Subtracting the expression for x_n from that for x_{n+1} yields

$$x_{n+1} - 3x_n - 5x_{n-1} - x_{n-2} = 2a_n + 6a_{n-1}.$$

Let z_n be the total number of trominoes in 2-by-n tilings. We similarly obtain

$$z_{n+1} - 3z_n - 5z_{n-1} - z_{n-2} = 4a_{n-1} + 4a_{n-2}.$$

Let y_n be the total number of dominoes. Using $x_n + 2y_n + 3z_n = 2na_n$, we get

$$y_{n+1} - 3y_n - 5y_{n-1} - y_{n-2} = 2a_n + a_{n-1} - 3a_{n-2}.$$

Let t_n be the total number of tiles in all the tilings. Since $t_n = x_n + y_n + z_n$,

$$t_{n+1} - 3t_n - 5t_{n-1} - t_{n-2} = 4a_n + 11a_{n-1} + a_{n-2}.$$

The general solution of the recurrence for a_n is

$$A\lambda^n + B\mu^n + C\nu^n,$$

where λ, μ, and ν are the zeros of $x^3 - 3x^2 - 5x - 1$. Take λ to be $2 + \sqrt{5}$, the largest root. Since the characteristic polynomials in the recurrences for x_n and t_n are the same as for a_n, and since their nonhomogeneous parts satisfy the same homogeneous recurrence (by definition), the general solutions for x_n and t_n have the form

$$(nA_1 + A_0)\,\lambda^n + (nB_1 + B_0)\,\mu^n + (nC_1 + C_0)\,\nu^n.$$

One can generate six initial values for a_n, x_n, and t_n using the recurrences. They are $(1, 2, 11, 44, 189, 798)$, $(0, 2, 16, 92, 512, 2654)$, and $(0, 3, 28, 66, 940, 4929)$, respectively. Solving 6×6 systems of linear equations then gives $A_1 = (5 + \sqrt{5})/20$ in the solution for x_n and $A_1 = (17 + 5\sqrt{5})/40$ in the solution for t_n. The desired limiting ratio is the ratio of these two coefficients, which is $(30 - 4\sqrt{5})/41$.

Editorial comment. The proposer found the following closed-form expressions, with F_n being the nth Fibonacci number:

$$\begin{aligned} a_n &= \left(F_{3n+2} + (-1)^n\right)\big/ 2 \qquad \text{(see oeis.org/A110679)};\\ x_n &= F_{3n-1} + (-1)^n(n-1);\\ t_n &= \tfrac{1}{20}\left((17n+10)F_{3n+1} + (4n-12)F_{3n} + (15n-10)(-1)^n\right). \end{aligned}$$

Also solved by S. B. Ekhad, G. Fera (Italy), P. Lalonde (Canada), P. McPolin (U. K.), R. Molinari, R. Nandan, R. Pratt, R. Stong, and the proposer.

A Vanishing Sum

11997 [2017, 660]. *Proposed by Michael Drmota, Technical University of Vienna, Vienna, Austria; Christian Krattenthaler, University of Vienna, Vienna, Austria; and Gleb Pogudin, Johannes Kepler University, Linz, Austria.* Assume $|p| < 1$ and $pz \neq 0$. With $f(z) = \left(e^{(p-1)z} - e^{-z}\right)/(pz)$, define $f^*(z) = \prod_{k=0}^{\infty} f(p^k z)$, and then define $F_n(p)$ so that $f^*(z) = \sum_{n=0}^{\infty} F_n(p) z^n$. Prove the identity

$$\sum_{n=0}^{\infty} F_n(p)\, p^{\binom{n}{2}} = 0.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands). More generally, we claim that the identity holds for functions $f(z)$ with $f(0) = 1$ that have the form

$$f(z) = \frac{b(pz) - 1}{pz\, b(z)}$$

with $b(z) = \sum_{k=0}^{\infty} b_k z^k$ and $b_0 = b_1 = 1$, provided that all infinite sums and products converge for $|z|$ and $|p|$ sufficiently small. Here the numbers b_k may depend on p.

Consider $f(z) = a(z)/b(z)$ with $a(z) = \sum_{k=0}^{\infty} a_k z^k$ and $b(z) = \sum_{k=0}^{\infty} b_k z^k$, where $a_0 = b_0 = 1$ and in general a_k and b_k may be functions of p. With $f^*(z) = \prod_{k=0}^{\infty} f(p^k z)$, define $F_n(p)$ by $f^*(z) = \sum_{n=0}^{\infty} F_n(p) z^n$. Since $f^*(z) = f(z) f^*(pz)$, we have $b(z) f^*(z) = a(z) f^*(pz)$, and hence

$$\left(\sum_{k=0}^{\infty} b_k z^k\right)\left(\sum_{n=0}^{\infty} F_n(p) z^n\right) = \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{n=0}^{\infty} F_n(p) p^n z^n\right).$$

Comparing the coefficients of z^m on both sides of the above expression shows

$$\sum_{n=0}^{m} F_n(p)(b_{m-n} - a_{m-n} p^n) = 0 \qquad (1)$$

for $m \geq 0$. Let $(c_m(p))_{m=0}^{\infty}$ be any sequence. Multiply (1) by c_m and sum over m, then interchange the summation order and rescale to obtain $\sum_{n=0}^{\infty} F_n(p) C_n(p) = 0$, where

$$C_n(p) = \sum_{k=0}^{\infty} c_{n+k}(b_k - a_k p^n). \qquad (2)$$

If $c_{n+k+1} b_{k+1} = c_{n+k} a_k p^n$ for all n and k with $n, k \geq 0$, then the sum in (2) telescopes to yield $C_n(p) = c_n$. This happens if and only if

$$\frac{a_k}{b_{k+1}} = \frac{a_{k-1} p}{b_k} = \frac{c_{n+k+1}}{c_{n+k}} p^{-n}$$

for all n and k with $n \geq 0$ and $k \geq 1$, in which case

$$a_n = p^n b_{n+1}/b_1 \qquad \text{and} \qquad C_n(p) = c_n = p^{\binom{n}{2}}/b_1^n. \qquad (3)$$

For the convergence of the telescoping sums we require $\lim_{k\to\infty} c_{n+k} a_k p^n = 0$. Using (3) above (and recalling $b_1 = 1$), we obtain

$$\lim_{k\to\infty} c_{n+k} a_k p^n = \lim_{k\to\infty} p^{\binom{n+k}{2}} p^k b_{k+1} p^n.$$

Since we assumed in defining $b(z)$ that its sum converges for suitably small z and p, it follows that $\lim_{k\to\infty} b_{k+1} z^{k+1} = 0$ and hence also $\lim_{k\to\infty} b_{k+1} p^{k+1} = 0$. Therefore $\lim_{k\to\infty} c_{n+k} a_k p^n = 0$ and the telescoping sum for $C_n(p)$ converges.

Finally, note that the first part of (3) is equivalent to $a(z) = (b(pz) - 1)(pzb_1)$. The identity in the problem results from the case $b(z) = e^z = \sum_{k=0}^{\infty} z^n/n!$, where $b_0 = b_1 = 1$ and $a(z) = (e^{pz} - 1)/(pz)$.

Also solved by P. Lalonde (Canada) and the proposer.