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第 1 章 Integrals 积分

1.1 Stack Exchange

✎ *

Exercise 1.1.1: Prove that the following dilogarithmic integral has the indicated value:

$$\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_2(x)}{1-x}\mathrm{d}x = -11\zeta(5) + 6\zeta(3)\,\zeta(2)$$

Proof:

$$\begin{aligned}\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_2(x)}{1-x}\mathrm{d}x &= \int_0^1 \frac{\ln^2(x)}{1-x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\mathrm{d}x \\ &= \int_0^1 \frac{\ln^2(x)}{1-x}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1-x^n}{n^2}\right]\mathrm{d}x \\ &= \zeta(2)\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x - \int_0^1 \frac{\ln^2(x)}{1-x}\sum_{n=1}^{\infty}\frac{1-x^n}{n^2}\mathrm{d}x\end{aligned}$$

However,

$$\begin{aligned}\int_0^1 \frac{\ln^2(x)}{1-x}\mathrm{d}x &= \int_0^1 \ln(1-x)\left[2\ln(x)\,\frac{1}{x}\right]\mathrm{d}x = -2\int_0^1 \mathrm{Li}_2'(x)\ln(x)\,\mathrm{d}x \\ &= 2\int_0^1 \mathrm{Li}_2(x)\,\frac{1}{x}\mathrm{d}x = 2\int_0^1 \mathrm{Li}_3'(x)\,\mathrm{d}x = 2\mathrm{Li}_3(1) = 2\zeta(3)\end{aligned}$$

Such that

$$\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_2(x)}{1-x}\mathrm{d}x = 2\zeta(2)\,\zeta(3) - \int_0^1 \frac{\ln^2(x)}{1-x}\sum_{n=1}^{\infty}\frac{1-x^n}{n^2}\mathrm{d}x$$

Also

$$\begin{aligned}\int_0^1 \frac{\ln^2(x)}{1-x}\sum_{n=1}^{\infty}\frac{1-x^n}{n^2}\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 \ln^2(x)\,\frac{1-x^n}{1-x}\mathrm{d}x \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 \ln^2(x)\sum_{k=1}^{n}x^{k-1}\mathrm{d}x \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\overbrace{\int_0^1 \ln^2(x)\,x^{k-1}\mathrm{d}x}^{=\frac{2}{k^3}} = 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}\end{aligned}$$

The last sum can be evaluated with the generating function $\sum_{n=1}^{\infty} x^n H_n^{(3)} = \frac{\mathrm{Li}_3(x)}{1-x}$.Namely

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^n}{n}H_n^{(3)} &= \int_0^x \frac{\mathrm{Li}_3(t)}{t}\mathrm{d}t + \int_0^x \frac{\mathrm{Li}_3(t)}{1-t}\mathrm{d}t \\ &= \mathrm{Li}_4(x) - \ln(1-x)\,\mathrm{Li}_3(x) + \int_0^x \ln(1-t)\,\mathrm{Li}_3'(t)\,\mathrm{d}t \\ &= \mathrm{Li}_4(x) - \ln(1-x)\,\mathrm{Li}_3(x) + \int_0^x \ln(1-t)\frac{\mathrm{Li}_2(t)}{t}\mathrm{d}t \\ &= \mathrm{Li}_4(x) - \ln(1-x)\,\mathrm{Li}_3(x) + \int_0^x \mathrm{Li}_2(t)\,\mathrm{Li}_2'(t)\,\mathrm{d}t \\ &= \mathrm{Li}_4(x) - \ln(1-x)\,\mathrm{Li}_3(x) - \frac{1}{2}\mathrm{Li}_2^2(x)\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} &= \int_0^1 \frac{\mathrm{Li}_4(t)}{t}\mathrm{d}t - \int_0^1 \frac{\ln(1-t)\,\mathrm{Li}_3(t)}{t}\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2^2(t)}{t}\mathrm{d}t \\ &= \zeta(5) + \mathrm{Li}_2(1)\,\mathrm{Li}_3(1) - \int_0^1 \mathrm{Li}_2(t)\frac{\mathrm{Li}_2(t)}{t}\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2^2(t)}{t}\mathrm{d}t \\ &= \zeta(5) + \zeta(2)\,\zeta(3) - \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(t)}{t}\mathrm{d}t\end{aligned}$$

The last integral integrate once by parts,we find

$$\begin{aligned}I &= \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\mathrm{d}x = \int_0^1 \mathrm{Li}_2(x)\,\mathrm{d}\left(\mathrm{Li}_3(x)\right) \\ &= \mathrm{Li}_2(1)\,\mathrm{Li}_3(1) + \int_0^1 \frac{\mathrm{Li}_3(x)\ln(1-x)\,\mathrm{d}x}{x} \\ &= \zeta(2)\,\zeta(3) - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^3k(n+k)}\end{aligned}$$

where the last line is obtained by expanding $\mathrm{Li}_3(x)$ and $\ln(1-x)$ into series and integrating. Now if we donote$H_n = \sum_{k=1}^{n}\frac{1}{k}$ the nth harmonic number, the sum with respect to k can be written as

$$\sum_{k=1}^{\infty}\frac{1}{k(n+k)} = \frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{n+k}\right) = \frac{H_n}{n}$$

so that

$$I = \zeta(2)\,\zeta(3) - \sum_{n=1}^{\infty}\frac{H_n}{n^4} = 2\zeta(2)\,\zeta(3) - 3\zeta(5)$$

so we have

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} = \frac{11}{2}\zeta(5) - 2(2)\,\zeta(3)$$

$$\Rightarrow \int_0^1 \frac{\ln^2(x)}{1-x}\sum_{n=1}^{\infty}\frac{1-x^n}{n^2}\mathrm{d}x = 11\zeta(5) - 4\zeta(2)\,\zeta(3)$$

Now we have the final result:

$$\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_2(x)}{1-x}\mathrm{d}x = -11\zeta(5) + 6\zeta(3)\,\zeta(2)$$

□

Exercise 1.1.2: Find the value of

$$I=\int_0^{\frac{\pi}{2}}x^2\ln\left(\sin x\right)\ln\left(\cos x\right)\mathrm{d}x$$

Solution:

Tools Needed

$$\frac{1}{k\left(j-k\right)^2}=\frac{1}{j^2k}-\frac{1}{j^2\left(k-j\right)}+\frac{1}{j\left(k-j\right)^2}$$

$$\frac{1}{k\left(j+k\right)^2}=\frac{1}{j^2k}-\frac{1}{j^2\left(k+j\right)}-\frac{1}{j\left(k+j\right)^2}$$

$$\ln\left(\sin x\right)=-\ln 2-\sum_{k=1}^{\infty}\frac{\cos\left(2kx\right)}{k}$$

$$\ln\left(\cos x\right)=-\ln 2-\sum_{k=1}^{\infty}\left(-1\right)^k\frac{\cos\left(2kx\right)}{k}$$

$$\cos\left(2jx\right)\cos\left(2kx\right)=\frac{1}{2}\left[\cos\left(2\left(j-k\right)x\right)+\cos\left(2\left(j+k\right)x\right)\right]$$

$$\int_0^{\frac{\pi}{2}}x^2\cos\left(2kx\right)\mathrm{d}x=\begin{cases}\left(-1\right)^k\dfrac{\pi}{4k^2} & \text{if } k\neq 0\\ \dfrac{\pi^3}{24} & \text{if } k=0\end{cases}$$

Tool Use

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}x^2\ln\left(\sin x\right)\ln\left(\cos x\right)\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}x^2\left(\ln 2+\sum_{k=1}^{\infty}\frac{\cos\left(2kx\right)}{k}\right)\left(\ln 2+\sum_{k=1}^{\infty}\left(-1\right)^k\frac{\cos\left(2kx\right)}{k}\right)\mathrm{d}x\\
&=\ln^2 2\int_0^{\frac{\pi}{2}}x^2\mathrm{d}x+\ln 2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}}x^2\cos\left(4kx\right)\mathrm{d}x\\
&+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{2jk}\int_0^{\frac{\pi}{2}}x^2\left[\cos\left(2\left(j-k\right)x\right)+\cos\left(2\left(j+k\right)x\right)\right]\mathrm{d}x\\
&=\frac{\pi^3}{24}\ln^2 2+\ln 2\frac{\pi}{16}\zeta\left(3\right)\\
&+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j}\sum_{k=1}^{j-1}\frac{1}{k\left(j-k\right)^2}+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j^2}\frac{\pi^2}{6}+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j}\sum_{k=j+1}^{\infty}\frac{1}{k\left(j-k\right)^2}\\
&+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j}\sum_{k=1}^{\infty}\frac{1}{k\left(j+k\right)^2}\\
&=\frac{\pi^3}{24}\ln^2 2+\ln 2\frac{\pi}{16}\zeta\left(3\right)\\
&+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j}\left(\frac{2}{j^2}H_{j-1}+\frac{1}{j}H_{j-1}^{(2)}\right)-\frac{\pi^5}{576}+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j}\left(-\frac{1}{j^2}H_j+\frac{1}{j}\frac{\pi^2}{6}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{1}{j^2}H_j-\frac{1}{j}\frac{\pi^2}{6}+\frac{1}{j}H_j^{(2)}\right)\\
&=\frac{\pi^3}{24}\ln^2 2+\ln 2\frac{\pi}{16}\zeta(3)\\
&+\frac{\pi}{8}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{2}{j^2}H_j+\frac{2}{j}H_j^{(2)}-\frac{3}{j^3}\right)-\frac{\pi^5}{576}\\
&=\frac{\pi^3}{24}\ln^2 2+\ln 2\frac{\pi}{16}\zeta(3)+\frac{11\pi^5}{5760}+\frac{\pi}{4}\sum(-1)^j\left(\frac{1}{j^3}H_j+\frac{1}{j^2}H_j^{(2)}\right)\\
&=\frac{\pi^3}{24}\ln^2 2+\ln 2\frac{\pi}{16}\zeta(3)-\frac{\pi^5}{960}-\frac{\pi}{16}\sum_{j=1}^{\infty}\frac{H_{2j}}{j^3}
\end{aligned}
$$

Corollary 1.1.1

If H_n are the harmonic numbers, we have

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{n^3}=-\frac{\pi^4}{15}-\frac{1}{3}\pi^2\ln^2 2+\frac{\ln^4 2}{3}+8\mathrm{Li}_4\left(\frac{1}{2}\right)+7\ln 2\zeta(3)$$

So here is the final result:

$$I=\frac{\pi^3}{16}\ln^2 2+\frac{\pi^5}{320}-\frac{3}{8}\ln 2\zeta(3)-\frac{\pi}{48}\ln^4 2-\frac{1}{2}\mathrm{Li}_4\left(\frac{1}{2}\right)$$

□

Exercise 1.1.3: Evaluate this integral:

$$\int_0^{\infty}\log_2(1+ax)\cdot e^{-bx}\mathrm{d}x\qquad(a,b>0)$$

Solution:

Definition 1.1.1 Incomplete gamma function

$$\Gamma(\alpha,z)=\int_z^{\infty}t^{\alpha-1}e^{-t}\mathrm{d}t$$

So,we have

$$
\begin{aligned}
\int_0^{\infty}\log_2(1+ax)\cdot e^{-bx}\mathrm{d}x&=\frac{e^{b/a}}{a\ln 2}\int_1^{\infty}\ln v\cdot e^{-bv/a}\mathrm{d}v\qquad(v=1+ax)\\
&=\left[\frac{e^{b/a}}{a\ln 2}\ln v\left(-\frac{a}{b}e^{-bv/a}\right)\right]_1^{\infty}+\frac{e^{b/a}}{a\ln 2}\int_1^{\infty}v^{-1}\cdot e^{-bv/a}\mathrm{d}v\\
&=\frac{e^{b/a}}{a\ln 2}\int_{b/a}^{\infty}u^{0-1}\cdot e^{-u}\mathrm{d}u\qquad(u=bv/a)\\
&=\frac{e^{b/a}}{a\ln 2}\Gamma\left(0,\frac{b}{a}\right)
\end{aligned}
$$

□

Exercise 1.1.4: Find a closed form for this integral:

$$I=\int_0^1 \frac{\ln^3(1+x)\ln x}{x}\mathrm{d}x$$

Solution: Start with integration by parts (IBP) by setting $u=\ln^3(1+x)$ and $\mathrm{d}v=\frac{\ln x}{x}\,\mathrm{d}x$ yields

$$\begin{aligned}
I&=-\frac{3}{2}\int_0^1 \frac{\ln^2(1+x)\ln^2 x}{1+x}\,\mathrm{d}x\\
&=-\frac{3}{2}\int_1^2 \frac{\ln^2 x\ln^2(x-1)}{x}\,\mathrm{d}x \quad\Rightarrow\quad x\mapsto 1+x\\
&=-\frac{3}{2}\int_{\frac12}^1 \left[\frac{\ln^2 x\ln^2(1-x)}{x}-\frac{2\ln^3 x\ln(1-x)}{x}+\frac{\ln^4 x}{x}\right]\mathrm{d}x \quad\Rightarrow\quad x\mapsto \frac{1}{x}\\
&=-\frac{3}{2}\int_{\frac12}^1 \frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x+3\int_{\frac12}^1 \frac{\ln^3 x\ln(1-x)}{x}\,\mathrm{d}x-\frac{3}{10}\ln^5 x\Big|_{\frac12}^1\\
&=-\frac{3}{2}\int_{\frac12}^1 \frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x+3\int_{\frac12}^1 \frac{\ln^3 x\ln(1-x)}{x}\,\mathrm{d}x-\frac{3}{10}\ln^5 2.
\end{aligned}$$

Applying IBP again to evaluate the red integral by setting $u=\ln^2(1-x)$ and $\mathrm{d}v=\frac{\ln^2 x}{x}\,\mathrm{d}x$ yields

$$\int_{\frac12}^1 \frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x=\frac{1}{3}\ln^5 2+\frac{2}{3}\int_{\frac12}^1 \frac{\ln^3 x\ln(1-x)}{1-x}\,\mathrm{d}x.$$

For the simplicity, let

$$\mathbf{H}_m^{(k)}(x)=\sum_{n=1}^\infty \frac{H_n^{(k)}x^n}{n^m} \qquad\Rightarrow\qquad \mathbf{H}(x)=\sum_{n=1}^\infty H_n x^n,$$

Introduce a generating function for the generalized harmonic numbers for $|x|<1$

$$\mathbf{H}^{(k)}(x)=\sum_{n=1}^\infty H_n^{(k)}x^n=\frac{\mathrm{Li}_k(x)}{1-x} \qquad\Rightarrow\qquad \mathbf{H}(x)=-\frac{\ln(1-x)}{1-x}$$

and the following identity

$$H_{n+1}^{(k)}-H_n^{(k)}=\frac{1}{(n+1)^k} \qquad\Rightarrow\qquad H_{n+1}-H_n=\frac{1}{n+1}$$

Let us integrating the indefinite form of the blue integral.

$$\begin{aligned}
\int \frac{\ln^3 x\ln(1-x)}{1-x}\,\mathrm{d}x&=-\int\sum_{n=1}^\infty H_n x^n\ln^3 x\,\mathrm{d}x\\
&=-\sum_{n=1}^\infty H_n\int x^n\ln^3 x\,\mathrm{d}x\\
&=-\sum_{n=1}^\infty H_n\frac{\partial^3}{\partial n^3}\left[\int x^n\,\mathrm{d}x\right]\\
&=-\sum_{n=1}^\infty H_n\frac{\partial^3}{\partial n^3}\left[\frac{x^{n+1}}{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{n=1}^{\infty} H_n \left[\frac{x^{n+1}\ln^3 x}{n+1} - \frac{3x^{n+1}\ln^2 x}{(n+1)^2} + \frac{6x^{n+1}\ln x}{(n+1)^3} - \frac{6x^{n+1}}{(n+1)^4}\right] \\
&= -\ln^3 x \sum_{n=1}^{\infty} \frac{H_{n+1}x^{n+1}}{n+1} + \ln^3 x \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^2} + 3\ln^2 x \sum_{n=1}^{\infty} \frac{H_{n+1}x^{n+1}}{(n+1)^2} \\
&\quad - 3\ln^2 x \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^3} - 6\ln x \sum_{n=1}^{\infty} \frac{H_{n+1}x^{n+1}}{(n+1)^3} + 6\ln x \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^4} \\
&\quad + 6\sum_{n=1}^{\infty} \frac{H_{n+1}x^{n+1}}{(n+1)^4} - 6\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^5} \\
&= -\sum_{n=1}^{\infty} \left[\frac{H_n x^n \ln^3 x}{n} - \frac{x^n \ln^3 x}{n^2} - \frac{3H_n x^n \ln^2 x}{n^2} + \frac{3x^n \ln^2 x}{n^3}\right. \\
&\quad \left. + \frac{6H_n x^n \ln x}{n^3} - \frac{6x^n \ln x}{n^4} - \frac{6H_n x^n}{n^4} + \frac{6x^n}{n^5}\right] \\
&= -\mathbf{H}_1(x)\ln^3 x + \mathrm{Li}_2(x)\ln^3 x + 3\mathbf{H}_2(x)\ln^2 x - 3\,\mathrm{Li}_3(x)\ln^2 x \\
&\quad - 6\mathbf{H}_3(x)\ln x + 6\,\mathrm{Li}_4(x)\ln x + 6\mathbf{H}_4(x) - 6\,\mathrm{Li}_5(x).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_{\frac{1}{2}}^{1} \frac{\ln^3 x \ln(1-x)}{1-x}\,\mathrm{d}x &= 6\mathbf{H}_4(1) - 6\,\mathrm{Li}_5(1) - \left[\mathbf{H}_1\left(\frac{1}{2}\right)\ln^3 2 - \mathrm{Li}_2\left(\frac{1}{2}\right)\ln^3 2\right. \\
&\quad + 3\mathbf{H}_2\left(\frac{1}{2}\right)\ln^2 2 - 3\,\mathrm{Li}_3\left(\frac{1}{2}\right)\ln^2 2 + 6\mathbf{H}_3\left(\frac{1}{2}\right)\ln 2 \\
&\quad \left. - 6\,\mathrm{Li}_4(x)\ln 2 + 6\mathbf{H}_4(x) - 6\,\mathrm{Li}_5(x)\right] \\
&= 12\zeta(5) - \pi^2\zeta(3) + \frac{3}{8}\zeta(3)\ln^2 2 - \frac{\pi^4}{120}\ln 2 - \frac{1}{4}\ln^5 2 \\
&\quad - 6\mathbf{H}_4\left(\frac{1}{2}\right) + 6\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 + 6\,\mathrm{Li}_5\left(\frac{1}{2}\right).
\end{aligned}
$$

Using the similar approach as calculating the blue integral, then

$$
\begin{aligned}
\int \frac{\ln^3 x \ln(1-x)}{x}\,\mathrm{d}x &= -\int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\ln^3 x\,\mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}\int x^{n-1}\ln^3 x\,\mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}\frac{\partial^3}{\partial n^3}\left[\int x^{n-1}\,\mathrm{d}x\right] \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}\frac{\partial^3}{\partial n^3}\left[\frac{x^n}{n}\right] \\
&= -\sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{x^n \ln^3 x}{n} - \frac{3x^n \ln^2 x}{n^2} + \frac{6x^n \ln x}{n^3} - \frac{6x^n}{n^4}\right] \\
&= \sum_{n=1}^{\infty} \left[-\frac{x^n \ln^3 x}{n^2} + \frac{3x^n \ln^2 x}{n^3} - \frac{6x^n \ln x}{n^4} + \frac{6x^n}{n^5}\right] \\
&= 6\,\mathrm{Li}_5(x) - 6\,\mathrm{Li}_4(x)\ln x + 3\,\mathrm{Li}_3(x)\ln^2 x - \mathrm{Li}_2(x)\ln^3 x.
\end{aligned}
$$

Hence

$$\int_{\frac{1}{2}}^{1} \frac{\ln^3 x \ln(1-x)}{x}\,\mathrm{d}x = \frac{\pi^2}{6}\ln^3 2 - \frac{21}{8}\zeta(3)\ln^2 2 - 6\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 - 6\,\mathrm{Li}_5\left(\frac{1}{2}\right) + 6\zeta(5).$$

Combining altogether, we have

$$\begin{aligned} I = &\frac{\pi^4}{120}\ln 2 - \frac{33}{4}\zeta(3)\ln^2 2 + \frac{\pi^2}{2}\ln^3 2 - \frac{11}{20}\ln^5 2 + 6\zeta(5) + \pi^2\zeta(3) \\ &+ 6\mathbf{H}_4\left(\frac{1}{2}\right) - 18\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 - 24\,\mathrm{Li}_5\left(\frac{1}{2}\right). \end{aligned}$$

Continuing with an answer: $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^3\,2^n}$

http://math.stackexchange.com/questions/909228/

we have

$$\begin{aligned} \mathbf{H}_3(x) =& \frac{1}{2}\zeta(3)\ln x - \frac{1}{8}\ln^2 x\ln^2(1-x) + \frac{1}{2}\ln x\left[\mathbf{H}_2(x) - \mathrm{Li}_3(x)\right] \\ &+ \mathrm{Li}_4(x) - \frac{\pi^2}{12}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_3(1-x)\ln x + \frac{\pi^4}{60}. \end{aligned}$$

Dividing by x and then integrating yields

$$\begin{aligned} \mathbf{H}_4(x) =& \frac{1}{4}\zeta(3)\ln^2 x - \frac{1}{8}\int\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x + \frac{1}{2}\int\frac{\ln x}{x}\left[\mathbf{H}_2(x) - \mathrm{Li}_3(x)\right]\mathrm{d}x \\ &+ \mathrm{Li}_5(x) - \frac{\pi^2}{12}\mathrm{Li}_3(x) - \frac{1}{2}\int\frac{\mathrm{Li}_3(1-x)\ln x}{x}\,\mathrm{d}x + \frac{\pi^4}{60}\ln x \\ =& \frac{1}{4}\zeta(3)\ln^2 x + \frac{\pi^4}{60}\ln x + \mathrm{Li}_5(x) - \frac{\pi^2}{12}\mathrm{Li}_3(x) - \frac{1}{8}\int\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x \\ &+ \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{H_n}{n^2}\int x^{n-1}\ln x\,\mathrm{d}x - \int\frac{\mathrm{Li}_3(x)\ln x}{x}\,\mathrm{d}x - \int\frac{\mathrm{Li}_3(1-x)\ln x}{x}\,\mathrm{d}x\right]. \end{aligned}$$

Evaluating the red integral using the same technique as the previous one yields

$$\int\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x = \frac{1}{3}\ln^3 x\ln^2(1-x) - \frac{2}{3}\int\frac{\ln(1-x)\ln^3 x}{1-x}\,\mathrm{d}x.$$

Evaluating the purple integral yields

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{H_n}{n^2}\int x^{n-1}\ln x\,\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{H_n}{n^2}\frac{\partial}{\partial n}\left[\int x^{n-1}\,\mathrm{d}x\right] \\ &= \sum_{n=1}^{\infty}\frac{H_n}{n^2}\left[\frac{x^n\ln x}{n} - \frac{x^n}{n^2}\right] \\ &= \mathbf{H}_3(x)\ln x - \mathbf{H}_4(x). \end{aligned}$$

Evaluating the green integral using IBP by setting $u = \ln x$ and $\mathrm{d}v = \dfrac{\mathrm{Li}_3(x)}{x}\,\mathrm{d}x$ yields

$$\begin{aligned} \int\frac{\mathrm{Li}_3(x)\ln x}{x}\,\mathrm{d}x &= \mathrm{Li}_4(x)\ln x - \int\frac{\mathrm{Li}_4(x)}{x}\,\mathrm{d}x \\ &= \mathrm{Li}_4(x)\ln x - \mathrm{Li}_5(x). \end{aligned}$$

Evaluating the orange integral using IBP by setting $u = \mathrm{Li}_3(1-x)$ and $\mathrm{d}v = \dfrac{\ln x}{x}\,\mathrm{d}x$ yields

$$\int \frac{\mathrm{Li}_3(1-x)\ln x}{x}\,\mathrm{d}x = \frac{1}{2}\mathrm{Li}_3(1-x)\ln^2 x + \frac{1}{2}\int \frac{\mathrm{Li}_2(1-x)\ln^2 x}{1-x}\,\mathrm{d}x.$$

Applying IBP again to evaluate the maroon integral by setting $u = \mathrm{Li}_2(1-x)$ and

$$\mathrm{d}v = \frac{\ln^2 x}{1-x}\,\mathrm{d}x \quad \Rightarrow \quad v = 2\,\mathrm{Li}_3(x) - 2\,\mathrm{Li}_2(x)\ln x - \ln(1-x)\ln^2 x,$$

we have

$$\begin{aligned}\int \frac{\mathrm{Li}_2(1-x)\ln^2 x}{1-x}\,\mathrm{d}x = &\left[2\,\mathrm{Li}_3(x) - 2\,\mathrm{Li}_2(x)\ln x - \ln(1-x)\ln^2 x\right]\mathrm{Li}_2(1-x)\\ &- 2\int \frac{\mathrm{Li}_3(x)\ln x}{1-x}\,\mathrm{d}x + 2\int \frac{\mathrm{Li}_2(x)\ln x}{1-x}\,\mathrm{d}x + \int \frac{\ln(1-x)\ln^3 x}{1-x}\,\mathrm{d}x.\end{aligned}$$

We use the generating function for the generalized harmonic numbers evaluate the above integrals involving polylogarithm.

$$\begin{aligned}\int \frac{\mathrm{Li}_k(x)\ln x}{1-x}\,\mathrm{d}x &= \sum_{n=1}^{\infty} H_n^{(k)} \int x^n \ln x\,\mathrm{d}x\\ &= \sum_{n=1}^{\infty} H_n^{(k)} \frac{\partial}{\partial n}\left[\int x^n\,\mathrm{d}x\right]\\ &= \sum_{n=1}^{\infty} H_n^{(k)} \left[\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2}\right]\\ &= \sum_{n=1}^{\infty} \left[\frac{H_{n+1}^{(k)} x^{n+1}\ln x}{n+1} - \frac{x^{n+1}\ln x}{(n+1)^{k+1}} - \frac{H_{n+1}^{(k)} x^{n+1}}{(n+1)^2} + \frac{x^{n+1}}{(n+1)^{k+2}}\right]\\ &= \sum_{n=1}^{\infty} \left[\frac{H_n^{(k)} x^n \ln x}{n} - \frac{x^n \ln x}{n^{k+1}} - \frac{H_n^{(k)} x^n}{n^2} + \frac{x^n}{n^{k+2}}\right]\\ &= \mathbf{H}_1^{(k)}(x)\ln x - \mathrm{Li}_{k+1}(x)\ln x - \mathbf{H}_2^{(k)}(x) + \mathrm{Li}_{k+2}(x).\end{aligned}$$

Dividing generating function of $\mathbf{H}^{(k)}(x)$ by x and then integrating yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(k)} x^n}{n} &= \int \frac{\mathrm{Li}_k(x)}{x(1-x)}\,\mathrm{d}x\\ \mathbf{H}_1^{(k)}(x) &= \int \frac{\mathrm{Li}_k(x)}{x}\,\mathrm{d}x + \int \frac{\mathrm{Li}_k(x)}{1-x}\,\mathrm{d}x\\ &= \mathrm{Li}_{k+1}(x) + \int \frac{\mathrm{Li}_k(x)}{1-x}\,\mathrm{d}x.\end{aligned}$$

Repeating the process above yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(k)} x^n}{n^2} &= \int \frac{\mathrm{Li}_{k+1}(x)}{x}\,\mathrm{d}x + \int \frac{\mathrm{Li}_k(x)}{x(1-x)}\,\mathrm{d}x\\ \mathbf{H}_2^{(k)}(x) &= \mathrm{Li}_{k+2}(x) + \mathrm{Li}_{k+1}(x) + \int \frac{\mathrm{Li}_k(x)}{1-x}\,\mathrm{d}x,\end{aligned}$$

where it is easy to show by using IBP that

$$\begin{aligned}\int \frac{\mathrm{Li}_2(x)}{1-x}\,\mathrm{d}x &= -\int \frac{\mathrm{Li}_2(1-x)}{x}\,\mathrm{d}x\\ &= 2\,\mathrm{Li}_3(x) - 2\,\mathrm{Li}_2(x)\ln(x) - \mathrm{Li}_2(1-x)\ln x - \ln(1-x)\ln^2 x\end{aligned}$$

and

$$\int \frac{\mathrm{Li}_3(x)}{1-x}\,\mathrm{d}x = -\int \frac{\mathrm{Li}_3(1-x)}{x}\,\mathrm{d}x = -\frac{1}{2}\,\mathrm{Li}_2^2(1-x) - \mathrm{Li}_3(1-x)\ln x.$$

Now,all unknown terms have been obtained. Putting altogether to $\mathbf{H}_4(x)$, we have

$$\begin{aligned}\mathbf{H}_4(x) = {}& \frac{1}{10}\zeta(3)\ln^2 x + \frac{\pi^4}{150}\ln x - \frac{\pi^2}{30}\,\mathrm{Li}_3(x) - \frac{1}{60}\ln^3 x\ln^2(1-x) + \frac{6}{5}\,\mathrm{Li}_5(x)\\ & - \frac{1}{5}\left[\mathrm{Li}_3(x) - \mathrm{Li}_2(x)\ln x - \frac{1}{2}\ln(1-x)\ln^2 x\right]\mathrm{Li}_2(1-x) - \frac{1}{5}\,\mathrm{Li}_4(x)\\ & - \frac{3}{5}\,\mathrm{Li}_4(x)\ln x + \frac{1}{5}\,\mathrm{Li}_3(x)\ln x + \frac{1}{5}\,\mathrm{Li}_3(x)\ln^2 x - \frac{1}{10}\,\mathrm{Li}_3(1-x)\ln^2 x\\ & - \frac{1}{15}\,\mathrm{Li}_2(x)\ln^3 x - \frac{1}{5}\mathbf{H}_2^{(3)}(x) + \frac{1}{5}\mathbf{H}_2^{(2)}(x) + \frac{1}{5}\mathbf{H}_1^{(3)}(x)\ln x\\ & - \frac{1}{5}\mathbf{H}_1^{(2)}(x)\ln x + \frac{2}{5}\mathbf{H}_3(x)\ln x - \frac{1}{5}\mathbf{H}_2(x)\ln^2 x + \frac{1}{15}\mathbf{H}_1(x)\ln^3 x + C.\end{aligned}$$

The next step is finding the constant of integration. Setting $x = 1$

$$\begin{aligned}\mathbf{H}_4(1) &= -\frac{\pi^2}{30}\,\mathrm{Li}_3(1) + \frac{6}{5}\,\mathrm{Li}_5(1) - \frac{1}{5}\,\mathrm{Li}_4(1) - \frac{1}{5}\mathbf{H}_2^{(3)}(1) + \frac{1}{5}\mathbf{H}_2^{(2)}(1) + C\\ 3\zeta(5) + \zeta(2)\zeta(3) &= -\frac{\pi^2}{30}\,\mathrm{Li}_3(1) + \frac{19}{30}\,\mathrm{Li}_5(1) + \frac{3}{5}\,\mathrm{Li}_3(1) + C\\ C &= \frac{\pi^4}{450} + \frac{\pi^2}{5}\zeta(3) - \frac{3}{5}\zeta(3) + 3\zeta(5).\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{H}_4(x) = {}& \frac{1}{10}\zeta(3)\ln^2 x + \frac{\pi^4}{150}\ln x - \frac{\pi^2}{30}\,\mathrm{Li}_3(x) - \frac{1}{60}\ln^3 x\ln^2(1-x) + \frac{6}{5}\,\mathrm{Li}_5(x)\\ & - \frac{1}{5}\left[\mathrm{Li}_3(x) - \mathrm{Li}_2(x)\ln x - \frac{1}{2}\ln(1-x)\ln^2 x\right]\mathrm{Li}_2(1-x) - \frac{1}{5}\,\mathrm{Li}_4(x)\\ & - \frac{3}{5}\,\mathrm{Li}_4(x)\ln x + \frac{1}{5}\,\mathrm{Li}_3(x)\ln x + \frac{1}{5}\,\mathrm{Li}_3(x)\ln^2 x - \frac{1}{10}\,\mathrm{Li}_3(1-x)\ln^2 x\\ & - \frac{1}{15}\,\mathrm{Li}_2(x)\ln^3 x - \frac{1}{5}\mathbf{H}_2^{(3)}(x) + \frac{1}{5}\mathbf{H}_2^{(2)}(x) + \frac{1}{5}\mathbf{H}_1^{(3)}(x)\ln x\\ & - \frac{1}{5}\mathbf{H}_1^{(2)}(x)\ln x + \frac{2}{5}\mathbf{H}_3(x)\ln x - \frac{1}{5}\mathbf{H}_2(x)\ln^2 x + \frac{1}{15}\mathbf{H}_1(x)\ln^3 x\\ & + \frac{\pi^4}{450} + \frac{\pi^2}{5}\zeta(3) - \frac{3}{5}\zeta(3) + 3\zeta(5)\end{aligned}$$

and setting $x = \dfrac{1}{2}$

$$\begin{aligned}\mathbf{H}_4\left(\frac{1}{2}\right) = {}& \frac{\ln^5 2}{40} - \frac{\pi^2}{36}\ln^3 2 + \frac{\zeta(3)}{2}\ln^2 2 - \frac{\pi^2}{12}\zeta(3)\\ & + \frac{\zeta(5)}{32} - \frac{\pi^4}{720}\ln 2 + \mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 + 2\,\mathrm{Li}_5\left(\frac{1}{2}\right).\end{aligned}$$

Finally, we obtain

$$\int_0^1 \frac{\ln^3(1+x)\ln x}{x}\mathrm{d}x = \frac{\pi^2}{2}\zeta(3) + \frac{99}{16}\zeta(5) - \frac{2}{5}\ln^5 2 + \frac{\pi^2}{3}\ln^3 2 - \frac{21}{4}\zeta(3)\ln^2 2 \\ -12\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 - 12\,\mathrm{Li}_5\left(\frac{1}{2}\right)$$

□

Exercise 1.1.5: Prove:

$$\int_0^{\frac{\pi}{2}} \frac{\ln^2\left(\cos\left(x\right)\right)\ln^2\left(\sin\left(x\right)\right)}{\cos\left(x\right)\sin\left(x\right)}\,\mathrm{d}x = \frac{1}{4}\left[2\,\zeta\left(5\right) - \zeta\left(2\right)\zeta\left(3\right)\right]$$

Proof: Use the change of variables $\ln(\cos(x)) = t$ to transform the integral to

$$I = \int_0^{\frac{\pi}{2}} \frac{\ln^2\cos x \ln^2\sin x}{\cos x \sin x}\mathrm{d}x = \frac{1}{4}\int_{-\infty}^0 \frac{t^2\left(\ln\left(1-\mathrm{e}^{2\,t}\right)\right)^2}{1-\mathrm{e}^{2t}}\mathrm{d}t.$$

Follow it by another change of variables $1 - e^{2t} = z$ gives

$$\frac{1}{4}\int_{-\infty}^0 \frac{t^2\left(\ln\left(1-\mathrm{e}^{2\,t}\right)\right)^2}{1-\mathrm{e}^{2t}}\mathrm{d}t = \frac{1}{32}\int_0^1 \frac{\left(\ln\left(1-z\right)\right)^2\left(\ln\left(z\right)\right)^2}{z\left(1-z\right)}\mathrm{d}z$$

$$= \frac{1}{32}\int_0^1 \frac{\left(\ln\left(1-z\right)\right)^2\left(\ln\left(z\right)\right)^2}{z}\mathrm{d}z + \frac{1}{32}\int_0^1 \frac{\left(\ln\left(1-z\right)\right)^2\left(\ln\left(z\right)\right)^2}{(1-z)}\mathrm{d}z$$

$$\Longrightarrow I = \frac{1}{16}\int_0^1 \frac{\left(\ln\left(1-z\right)\right)^2\left(\ln\left(z\right)\right)^2}{z}\mathrm{d}z.$$

Getting the exact result:

$$\frac{1}{16}\int_0^1 \frac{\left(\ln\left(1-z\right)\right)^2\left(\ln\left(z\right)\right)^2}{z}\mathrm{d}z = \frac{1}{16}\lim_{w\to 0}\lim_{s\to 0^+}\frac{\mathrm{d}^2}{\mathrm{d}w^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\int_0^1 (1-z)^w z^{s-1}\mathrm{d}z$$

$$= \frac{1}{16}\lim_{w\to 0}\lim_{s\to 0^+}\frac{\mathrm{d}^2}{\mathrm{d}w^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\beta(s,w+1) = \frac{1}{16}\lim_{w\to 0}\lim_{s\to 0^+}\frac{\mathrm{d}^2}{\mathrm{d}w^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\Gamma(s)\Gamma(w+1)}{\Gamma(s+w+1)}$$

$$I = \frac{1}{4}\left(2\zeta\left(5\right) - \zeta\left(2\right)\zeta\left(3\right)\right)$$

□

Exercise 1.1.6: Find the value of this integral:

$$J = \int_0^{\pi/2} \ln^2(\sin x)\ln(\cos x)\tan x\,\mathrm{d}x$$

Solution: It is easy to see that

$$J = \int_0^{\pi/4} \ln^2(\sin x)\ln(\cos x)\tan x\,\mathrm{d}x + \int_{\pi/4}^{\pi/2} \ln^2(\sin x)\ln(\cos x)\tan x\,\mathrm{d}x$$
$$= \int_0^{\pi/4} \ln^2(\sin x)\ln(\cos x)\tan x\,\mathrm{d}x + \int_0^{\pi/4} \ln^2(\cos x)\ln(\sin x)\cot x\,\mathrm{d}x$$

Now, to calculate J we make the substitution $t \leftarrow \sin^2 x$:

$$J = \frac{1}{16}\int_0^1 \frac{\ln(1-u)}{1-u}\ln^2(u)\,\mathrm{d}u$$

But

$$\frac{\ln(1-u)}{1-u} = -\left(\sum_{n=0}^{\infty} u^n\right)\left(\sum_{n=1}^{\infty} \frac{u^n}{n}\right) = -\sum_{n=1}^{\infty} H_n u^n$$

where $H_n = \sum_{k=1}^{n} 1/k$.Hence

$$J = -\frac{1}{16}\sum_{n=1}^{\infty} H_n \int_0^1 u^n \ln^2(u)\mathrm{d}u = -\frac{1}{8}\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3}$$

The sum $\sum_{n=1}^{\infty} \frac{H_n}{n^3}$ is known, it can be evaluated as follows, first we have

$$H_n = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+n}\right) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)}$$

Thus

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \sum_{k,n\geq 1} \frac{1}{n^2 k(n+k)} = \sum_{k,n\geq 1} \frac{1}{k^2 n(n+k)}$$

Taking the half sum we find

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{1}{2}\sum_{k,n\geq 1} \frac{1}{kn(k+n)}\left(\frac{1}{k}+\frac{1}{n}\right) = \frac{1}{2}\sum_{k,n\geq 1} \frac{1}{k^2 n^2} = \frac{1}{2}\zeta^2(2)$$

then we obtain

$$J = \frac{1}{8}\zeta(4) - \frac{1}{16}\zeta^2(2) = -\frac{\pi^4}{2880}$$

□

Exercise 1.1.7: Find a closed form for this integral:

$$\int_0^{\pi/2} \frac{\sin\left(x - a\ln 2\cdot\tan x\right)}{\left(e^{2\pi a\tan x}-1\right)\cdot\cos x}\,\mathrm{d}x$$

Solution: Preliminary Manipulations:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x - a\ln 2\tan x)}{(e^{2\pi a\tan x}-1)\cos x}\,\mathrm{d}x = \Im\int_0^{\infty} \frac{e^{-i(a\ln 2)x}}{(e^{2\pi a x}-1)(1-ix)}\,\mathrm{d}x \tag{1}$$

$$= \Im\int_{-\infty}^{0} \frac{e^{-i(a\ln 2)x}}{1-ix}\left(1+\frac{1}{e^{2\pi a x}-1}\right)\,\mathrm{d}x \tag{2}$$

$$= \frac{1}{2}\Im\int_0^{\infty} \frac{e^{i(a\ln 2)x}}{1+ix}\,\mathrm{d}x + \frac{1}{2}\Im\int_{-\infty}^{\infty} \frac{e^{-i(a\ln 2)x}}{(e^{2\pi a x}-1)(1-ix)}\,\mathrm{d}x \tag{3}$$

Explanation:

(1): Substituted $x \mapsto \arctan x$.

(2): Substituted $x \mapsto -x$.

(3): Averaged (1) and (2).

Contour Integration: The First Integral Integrating $f(z) = \frac{1}{2}\frac{e^{i\alpha z}}{1+iz}$ along the contour $[0,R] \cup Re^{i[0,\pi/2]} \cup i[R,0]$,

$$\begin{aligned}\frac{1}{2}\Im\int_0^\infty \frac{e^{i(a\ln 2)x}}{1+ix}\,\mathrm{d}x &= \Im\left(\pi i \operatorname*{Res}_{z=i} f(z) + \frac{i}{2}PV\int_0^\infty \frac{e^{-(a\ln 2)y}}{1-y}\,\mathrm{d}y\right)\\ &= -\frac{1}{2}e^{-a\ln 2}\operatorname{Ei}(a\ln 2(1-y))\Big|_{y=0}^{\infty} = \frac{1}{2}\frac{\operatorname{Ei}(a\ln 2)}{e^{a\ln 2}}\\ &= \frac{\operatorname{li}(2^a)}{2^{a+1}}\end{aligned}$$

Contour Integration: The Second Integral

Case 1:$a \in \mathbb{R}^+\backslash\mathbb{Z}^+$

Integrate $g(z) = \frac{1}{2}\frac{e^{-i(a\ln 2)z}}{(e^{2\pi az}-1)(1-iz)}$ along the contour $[-R,R] \cup Re^{i[2\pi,\pi]}$.This (clockwise) contour encloses only simple poles at $z=-i$ and $z=-\frac{in}{a}$. Summing up the residues,

$$\begin{aligned}&\frac{1}{2}\Im\int_{-\infty}^{\infty}\frac{e^{-i(a\ln 2)x}}{(e^{2\pi ax}-1)(1-ix)}\,\mathrm{d}x\\ &= -\frac{1}{2}\Im\left(\pi i\operatorname*{Res}_{z=0} g(z) + 2\pi i\operatorname*{Res}_{z=-i} g(z) + 2\pi i\sum_{n=1}^{\infty}\operatorname*{Res}_{z=-in/a} g(z)\right)\\ &= -\frac{1}{2}\Im\left(\pi i\cdot\frac{1}{2\pi a} + 2\pi i\cdot\frac{i}{2^a}\frac{1}{e^{-2\pi i a}-1} + 2\pi i\sum_{n=1}^{\infty}\frac{1}{2^{n+1}\pi(a-n)}\right)\\ &= -\frac{1}{4a} + \frac{\pi\cot(\pi a)}{2^{a+1}} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n(n-a)}\\ &= \frac{1}{4a} + \frac{\pi\cot(\pi a)}{2^{a+1}} + \frac{1}{2}\Phi\left(\frac{1}{2},1,-a\right)\end{aligned}$$

where Φ is the lerch transcendent defined as $\Phi(z,s,\alpha) = \sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s}$ for $|z|<1$ $\alpha \notin \mathbb{Z}^-$. (It is implemented in Mathematica or Wolfram as HurwitzLerchPhi[z,s,a], not LerchPhi[z,s,a].

Case 2: $a \in \mathbb{Z}^+$ In this case, the pole at $z=-i$ is order 2.Applying the residue theorem once again,

$$\begin{aligned}&\frac{1}{2}\Im\int_{-\infty}^{\infty}\frac{e^{-i(a\ln 2)x}}{(e^{2\pi ax}-1)(1-ix)}\,\mathrm{d}x\\ &= -\frac{1}{2}\Im\left(\pi i\cdot\frac{1}{2\pi a} + 2\pi i\cdot\frac{\ln 2-\pi i}{2^{a+1}\pi} + 2\pi i\sum_{\substack{n=1\\ n\neq a}}^{\infty}\frac{1}{2^{n+1}\pi(a-n)}\right)\\ &= \frac{1}{4a} - \frac{\ln 2}{2^{a+1}} + \frac{1}{2}\sum_{\substack{n=0\\ n\neq a}}^{\infty}\frac{1}{2^n(n-a)} = \frac{1}{4a} - \frac{\ln 2}{2^{a+1}} + \frac{1}{2}\sum_{n=0}^{a-1}\frac{1}{2^n(n-a)}\\ &+ \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^{n+(a+1)}(n+(a+1)-a)} = \frac{1}{4a} + \frac{1}{2}\sum_{n=0}^{a-1}\frac{1}{2^n(n-a)}\end{aligned}$$

The Final Result:

Combining both blue and red integrals gives

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x - a\ln 2\tan x)}{(e^{2\pi a\tan x} - 1)\cos x}\,\mathrm{d}x = \begin{cases} \dfrac{\mathrm{li}(2^a)}{2^{a+1}} + \dfrac{1}{4a} + \dfrac{\pi\cot(\pi a)}{2^{a+1}} + \dfrac{1}{2}\Phi\left(\dfrac{1}{2}, 1, -a\right), & \text{if } a \in \mathbb{R}^+\backslash\mathbb{Z}^+ \\ \dfrac{\mathrm{li}(2^a)}{2^{a+1}} + \dfrac{1}{4a} + \dfrac{1}{2}\displaystyle\sum_{n=0}^{a-1} \dfrac{1}{2^n(n-a)}, & \text{if } a \in \mathbb{Z}^+ \end{cases}$$

□

Exercise 1.1.8: Prove that the following conjectured value for the definite integral $\mathcal{I}$ is correct:

$$\mathcal{I} = \int_0^1 \frac{\ln(1+8x)}{x^{2/3}(1-x)^{2/3}(1+8x)^{1/3}}\,\mathrm{d}x \stackrel{?}{=} \frac{\ln(3)}{\sqrt{3}\,\pi}\left[\Gamma\left(\frac{1}{3}\right)\right]^3$$

Proof: Elimination of logarithmic factor from the integrand:

Suppose we have a substitution relation of the form $1 + 8x = \dfrac{k}{1+8t}$, with k being some positive real constant greater than 1. First of all, the symmetry of the relation with respect to the variables x and t implies that t solved for as a function of x will have the same functional form as x solved for as a function of t:

$$1 + 8x = \frac{k}{1+8t} \implies t = \frac{k-(1+8x)}{8(1+8x)}, \quad x = \frac{k-(1+8t)}{8(1+8t)}.$$

Transforming the integral $\mathcal{I}$ via this substitution, we find:

$$\begin{aligned} \mathcal{I} &= \int_0^1 x^{-2/3}(1-x)^{-2/3}(1+8x)^{-1/3}\ln(1+8x)\,\mathrm{d}x \\ &= \int_0^1 \frac{\ln(1+8x)}{\sqrt[3]{x^2(1-x)^2(1+8x)}}\,\mathrm{d}x \\ &= \int_{\frac{k-1}{8}}^{\frac{k-9}{72}} \sqrt[3]{\frac{2^{12}(1+8t)^5}{k(9-k+72t)^2(k-1-8t)^2}}\ln\left(\frac{k}{1+8t}\right)\cdot\frac{(-k)}{(1+8t)^2}\,\mathrm{d}t \\ &= \left(\frac{k}{9}\right)^{2/3}\int_{\frac{k-9}{72}}^{\frac{k-1}{8}} \frac{\ln\left(\dfrac{k}{1+8t}\right)}{\left(\dfrac{9-k}{72}+t\right)^{2/3}\left(\dfrac{k-1}{8}-t\right)^{2/3}(1+8t)^{1/3}}\,\mathrm{d}t, \end{aligned}$$

which clearly suggests the choice $k = 9$ as being the simplest, in which case:

$$\int_0^1 \frac{\ln\left(\dfrac{9}{1+8t}\right)\mathrm{d}t}{t^{2/3}(1-t)^{2/3}(1+8t)^{1/3}} = \int_0^1 \frac{\ln(9)\mathrm{d}t}{t^{2/3}(1-t)^{2/3}(1+8t)^{1/3}} - \int_0^1 \frac{\ln(1+8t)\mathrm{d}t}{t^{2/3}(1-t)^{2/3}(1+8t)^{1/3}}$$

$$\begin{aligned} &= 2\ln(3)\int_0^1 \frac{\mathrm{d}t}{t^{2/3}(1-t)^{2/3}(1+8t)^{1/3}} - \mathcal{I} \\ \implies 2\mathcal{I} &= 2\ln(3)\int_0^1 \frac{\mathrm{d}t}{t^{2/3}(1-t)^{2/3}(1+8t)^{1/3}} \\ \implies \mathcal{I} &= \ln(3)\int_0^1 \frac{\mathrm{d}t}{t^{2/3}(1-t)^{2/3}(1+8t)^{1/3}} \\ &= \ln(3)\,\mathcal{J}, \end{aligned}$$

where in the last line we've simply introduced the symbol $\mathcal{J}$ to denote the last integral for convenience. It's approximate value is $\mathcal{J} \approx 3.53328$.

Thus, to prove that the conjectured value is indeed correct, it suffices to prove the following equivalent conjecture:

$$\mathcal{J} = \int_0^1 \frac{\mathrm{d}x}{x^{2/3}(1-x)^{2/3}(1+8x)^{1/3}} = \frac{1}{\sqrt{3}\,\pi}\left[\Gamma\left(\frac{1}{3}\right)\right]^3.$$

□

Exercise 1.1.9: Find a possible closed form:

$$I = \int_0^\infty \frac{\ln x}{\sqrt{x}\,\sqrt{x+1}\,\sqrt{2x+1}}dx$$

Solution: After first multiplying and dividing the integrand by 2, substitute $x = \dfrac{t}{2}$

$$I = \int_0^\infty \frac{2\ln(x)}{\sqrt{2x}\,\sqrt{2x+2}\,\sqrt{2x+1}}\mathrm{d}x = \int_0^\infty \frac{\ln\left(\frac{t}{2}\right)}{\sqrt{t}\,\sqrt{t+2}\,\sqrt{t+1}}\mathrm{d}t.$$

Next, substituting $t = \dfrac{1}{u}$ yields:

$$\begin{aligned} I &= -\int_0^\infty \frac{\ln(2u)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u \\ &= -\int_0^\infty \frac{\ln(2)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u - \int_0^\infty \frac{\ln(u)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u \\ &= -\int_0^\infty \frac{\ln(2)}{\sqrt{u}\sqrt{u+1}\sqrt{2u+1}}\mathrm{d}u - I \\ \implies I &= -\frac{\ln(2)}{2}\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\sqrt{x+1}\sqrt{2x+1}}. \end{aligned}$$

Making the sequence of substitutions $x = \dfrac{u-1}{2}$, then $u = \dfrac{1}{t}$, and finally $t = \sqrt{w}$, puts this integral into the form of a beta function:

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\sqrt{x+1}\sqrt{2x+1}} = \int_1^\infty \frac{\mathrm{d}u}{\sqrt{u-1}\sqrt{u+1}\sqrt{u}} = \int_1^\infty \frac{\mathrm{d}u}{\sqrt{u^2-1}\sqrt{u}} \\ &= \int_1^0 \frac{t^{3/2}}{\sqrt{1-t^2}}\frac{(-1)}{t^2}\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t}\,\sqrt{1-t^2}} = \frac{1}{2}\int_0^1 \frac{\mathrm{d}w}{w^{3/4}\sqrt{1-w}} \\ &= \frac{1}{2}\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\pi^{3/2}}{2^{1/2}\Gamma^2\left(\frac{3}{4}\right)} \end{aligned}$$

Hence,

$$I = -\frac{\ln(2)}{2}\frac{\pi^{3/2}}{2^{1/2}\Gamma^2\left(\frac{3}{4}\right)} = -\frac{\pi^{3/2}\ln(2)}{2^{3/2}\,\Gamma^2\left(\frac{3}{4}\right)}.$$

Possible Alternative: You could also derive the answer from the complete elliptic integral of the first kind instead of from the beta function by making the substitution $t = z^2$ instead of

$t=\sqrt{w}$.

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\sqrt{x+1}\sqrt{2x+1}} = \int_1^\infty \frac{\mathrm{d}u}{\sqrt{u-1}\sqrt{u+1}\sqrt{u}} = \int_1^\infty \frac{\mathrm{d}u}{\sqrt{u^2-1}\sqrt{u}}$$
$$= \int_1^0 \frac{t^{3/2}}{\sqrt{1-t^2}}\frac{(-1)}{t^2}\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t}\,\sqrt{1-t^2}} = 2\int_0^1 \frac{\mathrm{d}z}{\sqrt{1-z^4}}$$
$$= 2\,\mathbf{K}(-1) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} = \frac{\pi^{3/2}}{2^{1/2}\Gamma^2\left(\frac{3}{4}\right)}$$

□

Exercise 1.1.10: Prove the following closed-form:

$$I=\int_0^1 \left(\frac{(x^2+1)\arcsin(x)}{\sqrt{1-x^2}}+2x\ln\left(x^2+1\right)\right)\frac{\ln x}{x^3+x}\,dx=-2\mathbf{G}\ln 2$$

Proof: Apply the obvious substitution $x\mapsto \sin x$ to the first integral

$$I=\int_0^{\frac{\pi}{2}} \frac{x\ln(\sin x)}{\sin x}\,\mathrm{d}x+\int_0^1 \frac{2\ln x\ln(1+x^2)}{1+x^2}\,\mathrm{d}x$$

The latter integral has been addressed here and is equivalent to

$$\int_0^1 \frac{2\ln x\ln(1+x^2)}{1+x^2}\,\mathrm{d}x = 4\Im\mathrm{Li}_3(1-i)-2\mathbf{G}\ln 2+\frac{3\pi^3}{16}+\frac{\pi}{4}\ln^2 2$$

As for the first integral,

$$\int_0^{\frac{\pi}{2}} \frac{x\ln(\sin x)}{\sin x}\,\mathrm{d}x$$
$$= 2\int_0^1 \frac{\arctan x\ln\left(\frac{2x}{1+x^2}\right)}{x}\,\mathrm{d}x$$
$$= 2\mathrm{Ti}_2(1)\ln 2+2\int_0^1 \frac{\arctan x\ln x}{x}\,\mathrm{d}x-2\int_0^1 \frac{\arctan x\ln(1+x^2)}{x}\,\mathrm{d}x$$
$$= 2\mathbf{G}\ln 2-2\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3}+2\int_0^1 \frac{\ln x\ln(1+x^2)}{1+x^2}\,\mathrm{d}x+4\int_0^1 \frac{x\arctan x\ln x}{1+x^2}\,\mathrm{d}x$$

and the integral

$$4\int_0^1 \frac{x\arctan x\ln x}{1+x^2}\,\mathrm{d}x=-8\Im\mathrm{Li}_3(1-i)-\frac{5\pi^3}{16}-\frac{\pi}{2}\ln^2 2$$

has also been established in the link above. (Both integrals were covered in the evaluation of $\mathscr{J}_2$.) Hence the closed form is indeed

$$I=-2\mathbf{G}\ln 2$$

□

Exercise 1.1.11: Evaluate:

$$\int_0^1 \arctan^2(x)\ln\left(\frac{1+x^2}{2x^2}\right)\mathrm{d}x$$

✎ **Solution:** Results used

I will just state the following result as I do not wish to replicate Random Variable's brilliant work in this answer.

$$\int_0^{\frac{\pi}{4}} \ln^2(\cos x)\ \mathrm{d}x = \Im\mathrm{Li}_3(1-i) - \frac{\mathbf{G}}{2}\ln 2 + \frac{7\pi^3}{192} + \frac{5\pi}{16}\ln^2 2$$

It is also quite easy to show that

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}z^n = \mathrm{Li}_3(z) - \mathrm{Li}_3(1-z) + \mathrm{Li}_2(1-z)\ln(1-z) + \frac{1}{2}\ln z\ln^2(1-z) + \zeta(3)$$

Splitting up the integral

We may split up the integral into 3 simpler integrals.

$$\mathscr{I} = -\ln 2\underbrace{\int_0^1 \arctan^2 x\ \mathrm{d}x}_{\mathscr{I}_1} - 2\underbrace{\int_0^1 \arctan^2 x\ln x\ \mathrm{d}x}_{\mathscr{I}_2} + \underbrace{\int_0^1 \arctan^2 x\ln(1+x^2)\ \mathrm{d}x}_{\mathscr{I}_3}$$

Evaluation of $\mathscr{I}_1$

Integrate by parts.

$$\begin{aligned}
\mathscr{I}_1 =& x\arctan^2 x\Big|_0^1 - \int_0^1 \frac{2x\arctan x}{1+x^2}\mathrm{d}x\\
=& \frac{\pi^2}{16} - \Big[\ln(1+x^2)\arctan x\Big]_0^1 + \int_0^1 \frac{\ln(1+x^2)}{1+x^2}\mathrm{d}x\\
=& \frac{\pi^2}{16} - \frac{\pi}{4}\ln 2 - 2\int_0^{\frac{\pi}{4}} \ln(\cos x)\ \mathrm{d}x\\
=& \frac{\pi^2}{16} - \frac{\pi}{4}\ln 2 + \frac{\pi}{2}\ln 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\ \mathrm{d}x\\
=& \frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi/2)}{n^2}\\
=& \frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 + \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}(-1)^n}{(2n+1)^2}\\
=& \frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 - \mathbf{G}
\end{aligned}$$

Evaluation of $\mathscr{I}_2$

$$\begin{aligned}
\mathscr{I}_2 =& \int_0^1 \arctan^2 x\ \mathrm{d}x - \int_0^1 \frac{2x\arctan x\ln x}{1+x^2}\mathrm{d}x\\
=& -\frac{\pi^2}{16} - \frac{\pi}{4}\ln 2 + \mathbf{G} + 2\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+3)^2} - \sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(2n+3)^2}\\
=& \frac{\pi^3}{16} - \frac{\pi^2}{16} - \frac{\pi}{4}\ln 2 + \mathbf{G} - 2\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^2}
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2} =& \Im\sum_{n=1}^{\infty}\frac{H_n}{n^2}i^n\\
=& -\Im\mathrm{Li}_3(1-i) - \frac{\mathbf{G}}{2}\ln 2 - \frac{\pi}{16}\ln^2 2
\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^2}&=\int_0^1\frac{\ln x\ln(1+x^2)}{1+x^2}\mathrm{d}x\\&=-2\int_0^{\frac{\pi}{4}}(\ln(\sin x)-\ln(\cos x))\ln(\cos x)\,\mathrm{d}x\\&=-\frac{1}{8}\frac{\partial^2\mathrm{B}}{\partial a\partial b}\left(\frac{1}{2},\frac{1}{2}\right)+2\int_0^{\frac{\pi}{4}}\ln^2(\cos x)\,\mathrm{d}x\\&=2\Im\mathrm{Li}_3(1-i)-\mathbf{G}\ln 2+\frac{3\pi^3}{32}+\frac{\pi}{8}\ln^2 2\end{aligned}$$

We have

$$\mathscr{I}_2=4\Im\mathrm{Li}_3(1-i)+\mathbf{G}+\frac{5\pi^3}{32}+\frac{\pi}{4}\ln^2 2-\frac{\pi^2}{16}-\frac{\pi}{4}\ln 2$$

Evaluation of $\mathscr{I}_3$

$\mathscr{I}_3$ is rather straightforward to evaluate.

$$\begin{aligned}\mathscr{I}_3=&x\arctan^2 x\ln(1+x^2)\Big|_0^1-\int_0^1\frac{2x\arctan x\ln(1+x^2)}{1+x^2}\mathrm{d}x-\int_0^1\frac{2x^2\arctan^2 x}{1+x^2}\mathrm{d}x\\=&\frac{\pi^2}{16}\ln 2-\frac{1}{2}\ln^2(1+x^2)\arctan x\Big|_0^1+\frac{1}{2}\int_0^1\frac{\ln^2(1+x^2)}{1+x^2}\mathrm{d}x-2\int_0^1\arctan^2 x\,\mathrm{d}x\\&+2\int_0^1\frac{\arctan^2 x}{1+x^2}\mathrm{d}x\\=&2\mathbf{G}+\frac{\pi^3}{96}+\frac{\pi^2}{16}\ln 2-\frac{\pi}{8}\ln^2 2-\frac{\pi^2}{8}-\frac{\pi}{2}\ln 2+2\int_0^{\frac{\pi}{4}}\ln^2(\cos x)\,\mathrm{d}x\\=&2\Im\mathrm{Li}_3(1-i)-\mathbf{G}\ln 2+2\mathbf{G}+\frac{\pi^3}{12}+\frac{\pi^2}{16}\ln 2+\frac{\pi}{2}\ln^2 2-\frac{\pi^2}{8}-\frac{\pi}{2}\ln 2\end{aligned}$$

The closed form

Combining these results, we get

$$\mathscr{I}=-6\Im\mathrm{Li}_3(1-i)-\frac{11\pi^3}{48}-\frac{\pi}{4}\ln^2 2$$

□

Exercise 1.1.12: Find a closed-form for

$$\int_0^{\infty}\frac{\mathrm{Li}_2^3(-x)}{x^3}\,\mathrm{d}x$$

Solution: As tired indicates, the proposed closed form isn't correct. The actual value is actually a little more compact and can be found fairly directly without resorting to series

methods. I find:

$$\mathcal{I}=\int_0^\infty\frac{\operatorname{Li}_2\left(-x\right)^3}{x^3}\,\mathrm{d}x=\int_1^\infty\frac{\operatorname{Li}_2\left(1-y\right)^3}{\left(y-1\right)^3}\,\mathrm{d}y=\int_0^1\frac{\operatorname{Li}_2\left(1-\frac{1}{z}\right)^3}{z^2\left(\frac{1}{z}-1\right)^3}\,\mathrm{d}z$$

$$=\int_0^1\frac{z\operatorname{Li}_2\left(-\frac{1-z}{z}\right)^3}{\left(1-z\right)^3}\,\mathrm{d}z=\int_0^1\frac{\left(1-w\right)\operatorname{Li}_2\left(\frac{w}{w-1}\right)^3}{w^3}\,\mathrm{d}w;\quad[1-z=w]$$

$$=\frac{3}{2}\int_0^1\frac{\left(1-w\right)\ln\left(1-w\right)\operatorname{Li}_2\left(\frac{w}{w-1}\right)^2}{w^3}\,\mathrm{d}w;\quad I.B.P.$$

$$=\frac{3}{2}\int_0^1\frac{\left[\left(\frac{1-w}{w}\right)^2\ln\left(1-w\right)-\ln\left(w\right)-\left(\frac{1-w}{w}\right)\right]\ln\left(1-w\right)\operatorname{Li}_2\left(\frac{w}{w-1}\right)}{w(1-w)}\,\mathrm{d}w;\ I.B.P.$$

$$=\frac{3}{2}\int_0^1\frac{\left(1-w\right)\ln^2\left(1-w\right)\operatorname{Li}_2\left(\frac{w}{w-1}\right)}{w^3}\,\mathrm{d}w$$

$$-\frac{3}{2}\int_0^1\frac{\left[\ln\left(w\right)+\left(\frac{1-w}{w}\right)\right]\ln\left(1-w\right)\operatorname{Li}_2\left(\frac{w}{w-1}\right)}{w(1-w)}\,\mathrm{d}w$$

$$=-\frac{3}{2}\int_0^1\frac{\left(1-w\right)\ln^2\left(1-w\right)\left[\operatorname{Li}_2\left(w\right)+\frac{1}{2}\ln^2\left(1-w\right)\right]}{w^3}\,\mathrm{d}w$$

$$+\frac{3}{2}\int_0^1\frac{\left[\ln\left(w\right)+\left(\frac{1-w}{w}\right)\right]\ln\left(1-w\right)\left[\operatorname{Li}_2\left(w\right)+\frac{1}{2}\ln^2\left(1-w\right)\right]}{w(1-w)}\,\mathrm{d}w$$

$$=-\frac{3}{2}\int_0^1\frac{\left(1-w\right)\ln^2\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w^3}\,\mathrm{d}w-\frac{3}{4}\int_0^1\frac{\left(1-w\right)\ln^4\left(1-w\right)}{w^3}\,\mathrm{d}w$$

$$+\frac{3}{2}\int_0^1\frac{\left[\ln\left(w\right)+\left(\frac{1-w}{w}\right)\right]\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w(1-w)}\,\mathrm{d}w$$

$$+\frac{3}{4}\int_0^1\frac{\left[\ln\left(w\right)+\left(\frac{1-w}{w}\right)\right]\ln^3\left(1-w\right)}{w(1-w)}\,\mathrm{d}w$$

$$=\frac{3}{2}\int_0^1\frac{\left(1-w\right)\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w^2}\,\mathrm{d}w+\frac{3}{2}\int_0^1\frac{\left(\frac{1-w}{w}\right)^2\ln^3\left(1-w\right)}{2w}\,\mathrm{d}w$$

$$+\frac{3}{2}\int_0^1\frac{\left(1-w\right)\ln^3\left(1-w\right)}{w^2}\,\mathrm{d}w$$

$$+\frac{3}{2}\int_0^1\frac{\ln\left(w\right)\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w(1-w)}\,\mathrm{d}w+\frac{3}{2}\int_0^1\frac{\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w^2}\,\mathrm{d}w$$

$$+\frac{3}{4}\int_0^1\frac{\ln\left(w\right)\ln^3\left(1-w\right)}{w(1-w)}\,\mathrm{d}w+\frac{3}{4}\int_0^1\frac{\ln^3\left(1-w\right)}{w^2}\,\mathrm{d}w$$

$$=\frac{3}{2}\int_0^1\frac{\left(2-w\right)\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w^2}\,\mathrm{d}w+\frac{3}{4}\int_0^1\frac{\left(1+w-w^2\right)\ln^3\left(1-w\right)}{w^3}\,\mathrm{d}w$$

$$+\frac{3}{2}\int_0^1\frac{\ln\left(w\right)\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{w}\,\mathrm{d}w+\frac{3}{2}\int_0^1\frac{\ln\left(w\right)\ln\left(1-w\right)\operatorname{Li}_2\left(w\right)}{1-w}\,\mathrm{d}w$$

$$+\frac{3}{4}\int_0^1\frac{\ln\left(w\right)\ln^3\left(1-w\right)}{w}\,\mathrm{d}w+\frac{3}{4}\int_0^1\frac{\ln\left(w\right)\ln^3\left(1-w\right)}{1-w}\,\mathrm{d}w$$

$$
\begin{aligned}
&= \frac{3}{2}\int_0^1 \frac{(2-w)\ln(1-w)\operatorname{Li}_2(w)}{w^2}\,\mathrm{d}w + \frac{3}{4}\int_0^1 \frac{(1+w-w^2)\ln^3(1-w)}{w^3}\,\mathrm{d}w \\
&\quad + \frac{3}{2}\int_0^1 \frac{\ln(w)\ln(1-w)\operatorname{Li}_2(w)}{w}\,\mathrm{d}w + \frac{3}{2}\int_0^1 \frac{\ln(v)\ln(1-v)\operatorname{Li}_2(1-v)}{v}\,\mathrm{d}v \\
&\quad + \frac{3}{4}\int_0^1 \frac{\ln(w)\ln^3(1-w)}{w}\,\mathrm{d}w + \frac{3}{4}\int_0^1 \frac{\ln^3(v)\ln(1-v)}{v}\,\mathrm{d}v; \quad [1-w=v]
\end{aligned}
$$

Continuing,

$$
\begin{aligned}
\mathcal{I} &= 3\int_0^1 \frac{\ln(1-w)\operatorname{Li}_2(w)}{w^2}\,\mathrm{d}w + \frac{3}{4}\int_0^1 \frac{(1+w)\ln^3(1-w)}{w^3}\,\mathrm{d}w \\
&\quad - \frac{3}{2}\int_0^1 \frac{\ln(1-w)\operatorname{Li}_2(w)}{w}\,\mathrm{d}w - \frac{3}{4}\int_0^1 \frac{\ln^3(1-w)}{w}\,\mathrm{d}w \\
&\quad + \frac{3}{2}\int_0^1 \frac{\ln(w)\ln(1-w)\left[\operatorname{Li}_2(w)+\operatorname{Li}_2(1-w)\right]}{w}\,\mathrm{d}w \\
&\quad + \frac{9}{2}S_{2,3}(1) + \frac{9}{2}S_{4,1}(1) \\
&= -\int_0^1 \left[-\frac{1-w}{w}\ln(1-w) - \ln(w)\right]\frac{(-3)\ln(1-w)}{w}\,\mathrm{d}w \\
&\quad - \frac{3}{4}\int_0^1 \left(\frac{3}{2} - \frac{2w+1}{2w^2}\right)\frac{(-3)\ln^2(1-w)}{1-w}\,\mathrm{d}w \\
&\quad + \frac{3}{4}\operatorname{Li}_2(1)^2 + \frac{9}{2}S_{1,3}(1) \\
&\quad + \frac{3}{2}\int_0^1 \frac{\ln(w)\ln(1-w)\left[\zeta(2)-\ln(w)\ln(1-w)\right]}{w}\,\mathrm{d}w \\
&\quad + \frac{9}{2}S_{2,3}(1) + \frac{9}{2}S_{4,1}(1) \\
&= -3\int_0^1 \frac{(1-w)\ln^2(1-w)}{w^2}\,\mathrm{d}w - \frac{9}{4}\int_0^1 \frac{(1+3w)\ln^2(1-w)}{2w^2}\,\mathrm{d}w \\
&\quad + \frac{3}{4}\zeta(2)^2 + \frac{9}{2}S_{1,3}(1) + \left(\frac{3}{2}\zeta(2)-3\right)\int_0^1 \frac{\ln(w)\ln(1-w)}{w}\,\mathrm{d}w \\
&\quad - \frac{3}{2}\int_0^1 \frac{\ln^2(w)\ln^2(1-w)}{w}\,\mathrm{d}w + \frac{9}{2}S_{2,3}(1) + \frac{9}{2}S_{4,1}(1) \\
&= -\frac{33}{4}\int_0^1 \frac{\ln^2(1-w)}{2w^2}\,\mathrm{d}w - \frac{3}{4}S_{1,2}(1) \\
&\quad + \frac{3}{4}\zeta(2)^2 + \frac{9}{2}S_{1,3}(1) + \left(\frac{3}{2}\zeta(2)-3\right)S_{2,1}(1) \\
&\quad - 6\,S_{3,2}(1) + \frac{9}{2}S_{2,3}(1) + \frac{9}{2}S_{4,1}(1)
\end{aligned}
$$

So the final result is:

$$
\int_0^\infty \frac{\operatorname{Li}_2^3(-x)}{x^3}\,\mathrm{d}x = \frac{3}{2}\zeta(5) + \left(\frac{\pi^2}{2} - \frac{15}{4}\right)\zeta(3) + \frac{17\pi^4}{240} - \frac{11\pi^2}{8}
$$

□

Exercise 1.1.13: What is a closed form for the next integral?

$$
I = \int_0^1 \frac{\ln^3(1+x)\,\ln^2 x}{x}\mathrm{d}x
$$

Solution: I will be using the following results:

$$
2\sum_{n=1}^{\infty}\frac{H_n}{n^q} = (q+2)\zeta(q+1) - \sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j)
$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n}=\zeta(3)-\frac{\pi^2}{12}\ln 2$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}=\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{\pi^4}{720}-\frac{1}{8}\zeta(3)\ln 2+\frac{1}{24}\ln^4 2$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}=&2\operatorname{Li}_5\left(\frac{1}{2}\right)+\frac{1}{32}\zeta(5)+\operatorname{Li}_4\left(\frac{1}{2}\right)\ln 2-\frac{\pi^4}{720}\ln 2+\frac{1}{2}\zeta(3)\ln^2 2\\&-\frac{\pi^2}{12}\zeta(3)-\frac{\pi^2}{36}\ln^3 2+\frac{1}{40}\ln^5 2\end{aligned}$$

Using I to denote the integral in question,

$$\begin{aligned}\mathcal{I}&=-\int_0^1\frac{\ln^3 x\ln^2(1+x)}{1+x}\mathrm{d}x=-\int_1^2\frac{\ln^2 x\ln^3(x-1)}{x}\mathrm{d}x\\&=\underbrace{-\int_{\frac{1}{2}}^1\frac{\ln^2 x\ln^3(1-x)}{x}\mathrm{d}x}_{\mathcal{I}_1}\underbrace{+3\int_{\frac{1}{2}}^1\frac{\ln^3 x\ln^2(1-x)}{x}}_{\mathcal{I}_2}\underbrace{-3\int_{\frac{1}{2}}^1\frac{\ln^4 x\ln(1-x)}{x}\mathrm{d}x}_{\mathcal{I}_3}-\frac{1}{6}\ln^6 2\end{aligned}$$

For $\mathcal{I}_1$, integration by parts gives

$$\mathcal{I}_1=\frac{1}{3}\ln^6 2-\int_{\frac{1}{2}}^1\frac{\ln^3 x\ln^2(1-x)}{1-x}\mathrm{d}x$$

On the other hand, $x\mapsto 1-x$ yields

$$\mathcal{I}_1=-\int_0^{\frac{1}{2}}\frac{\ln^3 x\ln^2(1-x)}{1-x}\mathrm{d}x$$

Combining these two equalities, we have

$$\begin{aligned}\mathcal{I}_1&=\frac{1}{6}\ln^6 2-\frac{1}{2}\int_0^1\frac{\ln^3 x\ln^2(1-x)}{1-x}\mathrm{d}x=\frac{1}{6}\ln^6 2-\frac{1}{2}\frac{\partial^5\beta}{\partial a^3\partial b^2}(1,0^+)\\&=\frac{1}{6}\ln^6 2-\frac{1}{2}\left[\frac{1}{b}+\mathcal{O}(1)\right]\left[\left(12\zeta^2(3)-\frac{23\pi^6}{1260}\right)b+\mathcal{O}(b^2)\right]_{b=0}\\&=\frac{23\pi^6}{2520}-6\zeta^2(3)+\frac{1}{6}\ln^6 2\end{aligned}$$

Even with the help of Wolfram Alpha, evaluating that fifth derivative was horribly unpleasant to say the least. As for $\mathcal{I}_2$,

$$\begin{aligned}\mathcal{I}_2&=6\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_{\frac{1}{2}}^1 x^n\ln^3 x\,\mathrm{d}x=6\sum_{n=1}^{\infty}\frac{H_n}{n+1}\frac{\partial^3}{\partial n^3}\left(\frac{1}{n+1}-\frac{1}{(n+1)2^{n+1}}\right)\\&=-\sum_{n=1}^{\infty}\frac{36H_n}{(n+1)^5}+\sum_{n=1}^{\infty}\frac{36H_n}{(n+1)^5 2^{n+1}}+\sum_{n=1}^{\infty}\frac{36\ln 2H_n}{(n+1)^4 2^{n+1}}+\sum_{n=1}^{\infty}\frac{18\ln^2 2H_n}{(n+1)^3 2^{n+1}}\\&\quad+\sum_{n=1}^{\infty}\frac{6\ln^3 2H_n}{(n+1)^2 2^{n+1}}\end{aligned}$$

$$
\begin{aligned}
=&-\frac{\pi^6}{35}+18\zeta^2(3)+\sum_{n=1}^{\infty}\frac{36H_n}{n^5 2^n}-36\mathrm{Li}_6\left(\frac{1}{2}\right)+36\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2+\frac{9}{8}\zeta(5)\ln 2\\
&+36\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2 2-\frac{\pi^4}{20}\ln^2 2+18\zeta(3)\ln^3 2-3\pi^2\zeta(3)\ln 2-\pi^2\ln^4 2+\frac{9}{10}\ln^6 2\\
&+\frac{\pi^4}{40}\ln^2 2-\frac{9}{4}\zeta(3)\ln^3 2+\frac{3}{4}\ln^6 2+\frac{3}{4}\zeta(3)\ln^3 2-\ln^6 2\\
=&\sum_{n=1}^{\infty}\frac{36H_n}{n^5 2^n}-36\mathrm{Li}_6\left(\frac{1}{2}\right)-\frac{\pi^6}{35}+36\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2+\frac{9}{8}\zeta(5)\ln 2+36\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2 2\\
&-\frac{\pi^4}{40}\ln^2 2+18\zeta^2(3)-3\pi^2\zeta(3)\ln 2+\frac{33}{2}\zeta(3)\ln^3 2-\pi^2\ln^4 2+\frac{13}{20}\ln^6 2
\end{aligned}
$$

For $\mathcal{I}_3$,

$$
\begin{aligned}
\mathcal{I}_3=&3\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^{1}x^{n-1}\ln^4 x\,\mathrm{d}x\\
=&3\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial^4}{\partial n^4}\left(\frac{1}{n}-\frac{1}{n2^n}\right)\\
=&\sum_{n=1}^{\infty}\left(\frac{72}{n^6}-\frac{72}{n^6 2^n}-\frac{72\ln 2}{n^5 2^n}-\frac{36\ln^2 2}{n^4 2^n}-\frac{12\ln^3 2}{n^3 2^n}-\frac{3\ln^4 2}{n^2 2^n}\right)\\
=&-72\mathrm{Li}_6\left(\frac{1}{2}\right)+\frac{8\pi^6}{105}-72\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2-36\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2 2\\
&-\frac{21}{2}\zeta(3)\ln^3 2+\frac{3\pi^2}{4}\ln^4 2-\frac{1}{2}\ln^6 2
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{I}=&36\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}-108\mathrm{Li}_6\left(\frac{1}{2}\right)+\frac{143\pi^6}{2520}-36\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2+\frac{9}{8}\zeta(5)\ln 2-\frac{\pi^4}{40}\ln^2 2\\
&+12\zeta^2(3)-3\pi^2\zeta(3)\ln 2+6\zeta(3)\ln^3 2-\frac{\pi^2}{4}\ln^4 2+\frac{3}{20}\ln^6 2
\end{aligned}
$$

We note that

$$
\begin{aligned}
\zeta(\bar{5},1)=&\frac{1}{24}\int_0^1\frac{\ln^4 x\ln(1+x)}{1+x}\mathrm{d}x=\frac{1}{24}\int_1^2\frac{\ln x\ln^4(x-1)}{x}\mathrm{d}x\\
=&-\frac{1}{24}\int_{\frac{1}{2}}^1\frac{\ln x\ln^4(1-x)}{x}\mathrm{d}x+\frac{1}{6}\int_{\frac{1}{2}}^1\frac{\ln^2 x\ln^3(1-x)}{x}\mathrm{d}x-\frac{1}{4}\int_{\frac{1}{2}}^1\frac{\ln^3 x\ln^2(1-x)}{x}\mathrm{d}x\\
&+\frac{1}{6}\int_{\frac{1}{2}}^1\frac{\ln^4 x\ln(1-x)}{x}\mathrm{d}x+\frac{1}{144}\ln^6 2\\
=&\underbrace{-\frac{1}{24}\int_0^{\frac{1}{2}}\frac{\ln^4 x\ln(1-x)}{1-x}\mathrm{d}x}_{\mathcal{J}}-3\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}+7\mathrm{Li}_6\left(\frac{1}{2}\right)-\frac{17\pi^6}{5040}+\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2\\
&-\frac{3}{32}\zeta(5)\ln 2-\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2 2+\frac{\pi^4}{480}\ln^2 2-\frac{1}{2}\zeta^2(3)+\frac{\pi^2}{4}\zeta(3)\ln 2-\frac{19}{24}\zeta(3)\ln^3 2\\
&+\frac{\pi^2}{24}\ln^4 2-\frac{17}{360}\ln^6 2
\end{aligned}
$$

since we have already derived the values of the last three integrals. For the remaining integral,

$$
\mathcal{J}=\frac{1}{24}\sum_{n=1}^{\infty}H_n\frac{\partial^4}{\partial n^4}\left(\frac{1}{(n+1)2^{n+1}}\right)=\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^5 2^{n+1}}+\sum_{n=1}^{\infty}\frac{\ln 2H_n}{(n+1)^4 2^{n+1}}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\frac{\ln^2 2H_n}{2(n+1)^3 2^{n+1}}+\sum_{n=1}^{\infty}\frac{\ln^3 2H_n}{6(n+1)^2 2^{n+1}}+\sum_{n=1}^{\infty}\frac{\ln^4 2H_n}{24(n+1)2^{n+1}}\\
=&\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}-\mathrm{Li}_6\left(\frac{1}{2}\right)+\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2+\frac{1}{32}\zeta(5)\ln 2+\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2 2-\frac{\pi^4}{720}\ln^2 2\\
&+\frac{1}{2}\zeta(3)\ln^3 2-\frac{\pi^2}{12}\zeta(3)\ln 2-\frac{\pi^2}{36}\ln^4 2+\frac{1}{40}\ln^6 2+\frac{\pi^4}{1440}\ln^2 2-\frac{1}{16}\zeta(3)\ln^3 2\\
&+\frac{1}{48}\ln^6 2+\frac{1}{48}\zeta(3)\ln^3 2-\frac{1}{36}\ln^6 2+\frac{1}{48}\ln^6 2\\
=&\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}-\mathrm{Li}_6\left(\frac{1}{2}\right)+\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2+\frac{1}{32}\zeta(5)\ln 2+\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2 2-\frac{\pi^4}{1440}\ln^2 2\\
&+\frac{11}{24}\zeta(3)\ln^3 2-\frac{\pi^2}{12}\zeta(3)\ln 2-\frac{\pi^2}{36}\ln^4 2+\frac{7}{180}\ln^6 2
\end{aligned}
$$

Hence we can express $\zeta(\bar{5},1)$ as

$$
\begin{aligned}
\zeta(\bar{5},1)=&-2\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}+6\mathrm{Li}_6\left(\frac{1}{2}\right)-\frac{17\pi^6}{5040}+2\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2-\frac{1}{16}\zeta(5)\ln 2+\frac{\pi^4}{720}\ln^2 2\\
&-\frac{1}{2}\zeta^2(3)-\frac{1}{3}\zeta(3)\ln^3 2+\frac{\pi^2}{6}\zeta(3)\ln 2+\frac{\pi^2}{72}\ln^4 2-\frac{1}{120}\ln^6 2
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}=&3\mathrm{Li}_6\left(\frac{1}{2}\right)-\frac{1}{2}\zeta(\bar{5},1)-\frac{17\pi^6}{10080}+\mathrm{Li}_5\left(\frac{1}{2}\right)\ln 2-\frac{1}{32}\zeta(5)\ln 2+\frac{\pi^4}{1440}\ln^2 2\\
&-\frac{1}{4}\zeta^2(3)-\frac{1}{6}\zeta(3)\ln^3 2+\frac{\pi^2}{12}\zeta(3)\ln 2+\frac{\pi^2}{144}\ln^4 2-\frac{1}{240}\ln^6 2
\end{aligned}
$$

Plucking this back into the original integral, we get another form in terms of $\zeta(\bar{5},1)$

$$\mathcal{I}=-\frac{\pi^6}{252}-18\zeta(\bar{5},1)+3\zeta^2(3)$$

This is as close to a "closed form" as I can get. The sheer number of cancellations involved in the last step makes me think that my answer could be roundabout and inefficient. Note that no known simple closed form for $\zeta(\bar{5},1)$ exists, implying that closed forms for higher power integrals are unlikely to exist as well. □

Exercise 1.1.14: Find a closed-form of this integral:

$$\int_0^1\frac{\ln x\ln^2(1+x)}{x}\,\mathrm{d}x$$

Solution: Explanation :

(1)Use generating function $\sum_{k=1}^{\infty}(-1)^{k-1}H_k x^k=\frac{\ln(1+x)}{1+x}$

(2)Use formula $\int_0^1 x^k\ln^n x\,\mathrm{d}x=\frac{(-1)^n n!}{(k+1)^{n+1}}\quad,\ n\in\mathbb{Z}_{n\geq 0}$

(3)Use property $H_k=H_{k+1}-\frac{1}{k+1}$

(4)Use the result $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^3}=\frac{11\pi^4}{360}-2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7\zeta(3)\ln 2}{4}+\frac{\pi^2\ln^2 2}{12}-\frac{\ln^4 2}{12}$

(5)Use property of Dirichlet eta function $\eta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}=\left(1-2^{1-s}\right)\zeta(s)$

Performing integration by parts by taking $u=\ln^2(1+x)$ and $\mathrm{d}v=\frac{\ln x}{x}\,\mathrm{d}x$, then

$$\begin{aligned}
I&=\int_0^1\frac{\ln x\ln^2(1+x)}{x}\,\mathrm{d}x\\
&=\frac{1}{2}\ln^2 x\ln^2(1+x)\Big|_0^1-\int_0^1\frac{\ln^2 x\ln(1+x)}{1+x}\,\mathrm{d}x\\
&=-\int_0^1\sum_{k=1}^{\infty}(-1)^{k-1}H_k\,x^k\ln^2 x\,\mathrm{d}x\\
&=-\sum_{k=1}^{\infty}(-1)^{k-1}H_k\int_0^1 x^k\ln^2 x\,\mathrm{d}x\\
&=-2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{(k+1)^3}\\
&=-2\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{H_{k+1}}{(k+1)^3}-\frac{1}{(k+1)^4}\right]\\
&=2\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{H_k}{k^3}-\frac{1}{k^4}\right]\\
&=\frac{11\pi^4}{180}-4\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7\zeta(3)\ln 2}{2}+\frac{\pi^2\ln^2 2}{6}-\frac{\ln^4 2}{6}-2\eta(4)\\
&=\frac{11\pi^4}{180}-4\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7\zeta(3)\ln 2}{2}+\frac{\pi^2\ln^2 2}{6}-\frac{\ln^4 2}{6}-\frac{7\zeta(4)}{4}\\
&=\frac{\pi^4}{24}-4\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7\zeta(3)\ln 2}{2}+\frac{\pi^2\ln^2 2}{6}-\frac{\ln^4 2}{6}
\end{aligned}$$

□

Exercise 1.1.15: Prove:

$$\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x=\frac{\zeta(3)}{4}$$

Proof:

$$\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x=\int_1^2\frac{\ln^2(x)}{x-1}\mathrm{d}x=\int_1^{1/2}\frac{\ln^2\left(\frac{1}{x}\right)}{\frac{1}{x}-1}\left(-\frac{\mathrm{d}x}{x^2}\right)=\int_{1/2}^1\frac{\ln^2(x)}{x(1-x)}\mathrm{d}x$$

$$=\int_{1/2}^1\frac{\ln^2(x)}{x}\mathrm{d}x+\int_{1/2}^1\frac{\ln^2(x)}{1-x}\mathrm{d}x=\frac{1}{3}\ln^3(2)+\sum_{n=0}^{\infty}\int_{1/2}^1\ln^2(x)\,x^n\mathrm{d}x=\frac{\partial^2}{\partial\mu^2}\sum_{n=1}^{\infty}\int_{1/2}^1 x^{\mu-1}\mathrm{d}x\Big|_{\mu=n}$$

$$=\frac{\partial^2}{\partial\mu^2}\sum_{n=1}^{\infty}\int_{1/2}^1\frac{1-2^{-\mu}}{\mu}\Big|_{\mu=n}=2\sum_{n=1}^{\infty}\frac{1}{n^3}-2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{1}{2}\right)^n-2\ln(2)\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{2}\right)^n$$

$$-\ln^2(2)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n=2\zeta(3)-2\mathrm{Li}_3\left(\frac{1}{2}\right)-2\ln(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right)-\ln^2(2)\,\mathrm{Li}_1\left(\frac{1}{2}\right)$$

For

$$\mathrm{Li}_1\left(\frac{1}{2}\right)=\ln(2),\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2(2),\mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{1}{6}\ln^3(2)-\frac{\pi^2}{12}\ln(2)+\frac{7}{8}\zeta(3)$$

now we have

$$\int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x = \frac{\zeta(3)}{4}$$

□

Exercise 1.1.16: How to find this integral:

$$\int_0^\infty \frac{\mathrm{d}x}{(x^3+(1+x^2)^{3/2}+x)(1+\sqrt{1+x^2}\arctan x)}$$

Solution: Rewrite the given integral as

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)\left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+x^2}\arctan x+1\right)}$$

Let $y=\tan x \;\Rightarrow\; \mathrm{d}y=\sec^2 x\,\mathrm{d}x$ and $\sec x=\sqrt{1+x^2}$, the integral turns out to be

$$\int_0^{\pi/2} \frac{\mathrm{d}y}{(\sec y+\tan y)(y\sec y+1)}$$

Multiply the numerator and denominator by $\cos^2 y$ yields

$$\int_0^{\pi/2} \frac{\cos^2 y\,\mathrm{d}y}{(1+\sin y)(y+\cos y)}$$

The last step, let $u=y+\cos y \;\Rightarrow\; \mathrm{d}u=(1-\sin y)\,\mathrm{d}y$, the integral turns out to be

$$\int_0^\infty \frac{\mathrm{d}x}{(x^3+(1+x^2)^{3/2}+x)(1+\sqrt{1+x^2}\arctan x)} = \int_1^{\pi/2} \frac{1}{u}\,\mathrm{d}u = \ln\left(\frac{\pi}{2}\right)$$

□

Exercise 1.1.17: Prove the following integral:

$$I=\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\sin^2(\tan x)} = \frac{\pi}{2\sqrt{2}}\left(\frac{e^2+3-2\sqrt{2}}{e^2-3+2\sqrt{2}}\right)$$

Proof: First note that

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{1+\sin^2(\tan x)}\,\mathrm{d}x &= \int_0^\infty \frac{1}{(1+\sin^2 t)(1+t^2)}\,\mathrm{d}t \\ &= 2\int_0^\infty \frac{1}{(3-\cos 2t)(1+t^2)}\,\mathrm{d}t\end{aligned}$$

Then using the identity

$$\sum_{k=0}^\infty a^k\cos(kx) = \frac{1-a\cos x}{1-2a\cos x+a^2},\quad |a|<1$$

we have

$$1+2\sum_{k=1}^\infty a^k\cos(kx) = 1+2\left(\frac{1-a\cos x}{1-2a\cos x+a^2}-1\right) = \frac{1-a^2}{1-2a\cos x+a^2}$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{1}{(1-2a\cos 2x+a^2)(1+x^2)}\,\mathrm{d}x &= \frac{1}{1-a^2}\int_0^\infty \left(1+2\sum_{k=1}^\infty a^k\cos(2kx)\right)\frac{1}{1+x^2}\,\mathrm{d}x\\ &= \frac{\pi}{2}\frac{1}{1-a^2}+\frac{2}{1-a^2}\sum_{k=1}^\infty a^k\int_0^\infty \frac{\cos(2kx)}{1+x^2}\,\mathrm{d}x\\ &= \frac{\pi}{2}\frac{1}{1-a^2}+\frac{\pi}{1-a^2}\sum_{k=1}^\infty \left(\frac{a}{e^2}\right)^k\\ &= \frac{\pi}{2}\frac{1}{1-a^2}+\frac{\pi}{1-a^2}\frac{a/e^2}{1-a/e^2}\\ &= \frac{\pi}{2}\frac{1}{1-a^2}\left(1+\frac{2a}{e^2-a}\right)=\frac{\pi}{2}\frac{1}{1-a^2}\frac{e^2+a}{e^2-a}\end{aligned}$$

Now rewrite the integral as

$$\frac{1}{1+a^2}\int_0^\infty \frac{1}{(1-\frac{2a}{1+a^2}\cos 2x)(1+x^2)}\,\mathrm{d}x$$

and let $a=3-2\sqrt{2}$, then

$$\begin{aligned}\frac{1}{1+a^2}\int_0^\infty \frac{1}{(1-\frac{1}{3}\cos 2x)(1+x^2)}\,\mathrm{d}x &= \frac{3}{1+a^2}\int_0^\infty \frac{1}{(3-\cos 2x)(1+x^2)}\,\mathrm{d}x\\ &= \frac{\pi}{2}\frac{1}{1-a^2}\frac{e^2+3-2\sqrt{2}}{e^2-3+2\sqrt{2}}\end{aligned}$$

which implies

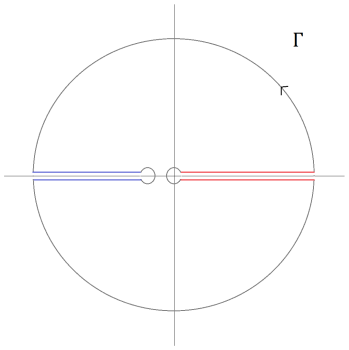
$$\begin{aligned}\int_0^{\pi/2} \frac{1}{1+\sin^2(\tan x)}\,\mathrm{d}x &= \frac{\pi}{3}\frac{1+a^2}{1-a^2}\frac{e^2+3-2\sqrt{2}}{e^2-3+2\sqrt{2}}\\ &= \frac{\pi}{3}\frac{3}{2\sqrt{2}}\frac{e^2+3-2\sqrt{2}}{e^2-3+2\sqrt{2}}\\ &= \frac{\pi}{2\sqrt{2}}\frac{e^2+3-2\sqrt{2}}{e^2-3+2\sqrt{2}}\end{aligned}$$

□

Exercise 1.1.18: Find a closed form for the next integral:

$$\int_0^\infty \frac{\ln^2(x)\ln(1+x)}{(1-x)(x^2+1)}\mathrm{d}x$$

Solution: Integrate $f(z)=\dfrac{\ln^3 z\ln(1+z)}{(1-z)(1+z^2)}$ along this contour.



The integral along the grey portions of the contour vanishes. We also need not worry about the removable singularity at $z = 1$.

The imaginary part of the contour integral is

$$\begin{aligned}\operatorname{Im}\oint_\Gamma f(z)\,\mathrm{d}z =& \operatorname{Im}\int_0^\infty \frac{\left(\ln^3 x-(\ln x+2\pi i)^3\right)\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x+\operatorname{Im}\,2\pi i\int_1^\infty \frac{\ln^3(-x)}{(1+x)(1+x^2)}\mathrm{d}x\\ =&-6\pi\int_0^\infty \frac{\ln^2 x\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x+8\pi^3\mathrm{PV}\int_0^\infty \frac{\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x\\ &+2\pi\int_1^\infty \frac{\ln^3 x}{(1+x)(1+x^2)}\mathrm{d}x-6\pi^3\int_1^\infty \frac{\ln x}{(1+x)(1+x^2)}\mathrm{d}x\end{aligned}$$

I will work out these integrals one by one. The first one is

$$\begin{aligned}\mathrm{PV}\int_0^\infty \frac{\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x=&\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x+\int_0^1 \frac{x(1+x)\ln x}{1-x^4}\mathrm{d}x\\ =&\frac{\pi}{8}\ln 2+\sum_{n=0}^\infty\int_0^1\left(x^{4n+1}+x^{4n+2}\right)\ln x\;\mathrm{d}x\\ =&\frac{\pi}{8}\ln 2-\frac{1}{4}\sum_{n=0}^\infty\frac{1}{(2n+1)^2}-\sum_{n=0}^\infty\frac{1}{(4n+3)^2}\\ =&-\frac{\pi^2}{32}+\frac{\pi}{8}\ln 2-\sum_{n=0}^\infty\frac{1}{(4n+3)^2}\end{aligned}$$

Since

$$\sum_{n=0}^\infty\frac{1}{(4n+1)^2}+\sum_{n=0}^\infty\frac{1}{(4n+3)^2}=\frac{\pi^2}{8}\;,\;\sum_{n=0}^\infty\frac{1}{(4n+1)^2}-\sum_{n=0}^\infty\frac{1}{(4n+3)^2}=\mathbf{G}$$

The sum on the right evaluates to $\dfrac{\pi^2}{16}-\dfrac{\mathbf{G}}{2}$.So

$$\mathrm{PV}\int_0^\infty \frac{\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x=\frac{\mathbf{G}}{2}-\frac{3\pi^2}{32}+\frac{\pi}{8}\ln 2$$

The second one is

$$\begin{aligned}\int_1^\infty \frac{\ln^3 x}{(1+x)(1+x^2)}\mathrm{d}x=&-\int_0^1 \frac{x(1-x)\ln^3 x}{1-x^4}\mathrm{d}x\\ =&-\sum_{n=0}^\infty\int_0^1\left(x^{4n+1}-x^{4n+2}\right)\ln^3 x\;\mathrm{d}x\\ =&\frac{3}{8}\sum_{n=0}^\infty\frac{1}{(2n+1)^4}-6\sum_{n=0}^\infty\frac{1}{(4n+3)^4}\\ =&\frac{\pi^4}{256}-\frac{1}{256}\psi_3\left(\frac{3}{4}\right)\end{aligned}$$

The third one is

$$\int_1^\infty \frac{\ln x}{(1+x)(1+x^2)}\mathrm{d}x=-\int_0^1 \frac{x(1-x)\ln x}{1-x^4}\mathrm{d}x=-\sum_{n=0}^\infty\int_0^1\left(x^{4n+1}-x^{4n+2}\right)\ln x\;\mathrm{d}x$$

$$=\frac{1}{4}\sum_{n=0}^\infty\frac{1}{(2n+1)^2}-\sum_{n=0}^\infty\frac{1}{(4n+3)^2}=\frac{\mathbf{G}}{2}-\frac{\pi^2}{32}$$

Therefore,

$$\operatorname{Im}\oint_\Gamma f(z)\,\mathrm{d}z=-6\pi\int_0^\infty\frac{\ln^2 x\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x+\pi^3\mathbf{G}-\frac{71\pi^5}{128}-\frac{\pi}{128}\psi_3\left(\frac{3}{4}\right)+\pi^4\ln 2$$

By the residue theorem, the contour integral is also equivalent to $2\pi i$ times the sum of residues. Keep in mind that $-\pi<\arg(1+z)\leq\pi$ and $0\leq\arg z<2\pi$. This means that $\ln(-i)=\dfrac{i3\pi}{2}$ but $\ln(1-i)=\dfrac{1}{2}\ln 2-\dfrac{i\pi}{4}$.

$$\begin{aligned}\operatorname{Im}\oint_\Gamma f(z)\,\mathrm{d}z=&\operatorname{Im}\,2\pi i\left(\operatorname{Res}(f,i)+\operatorname{Res}(f,-i)\right)\\=&\frac{\pi^5}{64}-\frac{\pi^4}{32}\ln 2-\frac{27\pi^5}{64}+\frac{27\pi^4}{32}\ln 2\\=&-\frac{13\pi^5}{32}+\frac{13\pi^4}{16}\ln 2\end{aligned}$$

Comparing both equalities, we get

$$\int_0^\infty\frac{\ln^2 x\ln(1+x)}{(1-x)(1+x^2)}\mathrm{d}x=\frac{\pi^2}{6}\mathbf{G}-\frac{1}{768}\psi_3\left(\frac{3}{4}\right)-\frac{19\pi^4}{768}+\frac{\pi^3}{32}\ln 2$$

□

Exercise 1.1.19: Closed form of:

$$I=\int_0^\infty\frac{\tanh(x)\,\tanh(2x)}{x^2}\,\mathrm{d}x$$

Solution: Let $x\mapsto-\ln x$.We get

$$\mathcal{I}=\int_0^1\frac{1-x^2}{1+x^2}\frac{1-x^4}{1+x^4}\frac{\mathrm{d}x}{x\ln^2 x}=\int_0^1\left(\frac{1}{x}-\frac{2x}{1+x^4}\right)\frac{\mathrm{d}x}{\ln^2 x}$$

Define

$$\mathcal{I}(\alpha)=\int_0^1\left(\frac{1}{x}-\frac{2x}{1+x^4}\right)\frac{x^\alpha\mathrm{d}x}{\ln^2 x}$$

Differentiating twice, one obtains

$$\begin{aligned}\mathcal{I}''(\alpha)=&\frac{1}{\alpha}-2\sum_{n=0}^\infty(-1)^n\int_0^1 x^{4n+\alpha+1}\mathrm{d}x\\=&\frac{1}{\alpha}-2\sum_{n=0}^\infty\frac{(-1)^n}{4n+\alpha+2}\\=&\frac{1}{\alpha}-\frac{1}{4}\sum_{n=0}^\infty\frac{1}{n+\frac{\alpha+2}{8}}+\frac{1}{4}\sum_{n=0}^\infty\frac{1}{n+\frac{\alpha+6}{8}}\\=&\frac{1}{\alpha}+\frac{1}{4}\psi_0\left(\frac{\alpha+2}{8}\right)-\frac{1}{4}\psi_0\left(\frac{\alpha+6}{8}\right)\end{aligned}$$

Integrate back once.

$$\mathcal{I}'(\alpha)=\ln\alpha+2\ln\Gamma\left(\frac{\alpha+2}{8}\right)-2\ln\Gamma\left(\frac{\alpha+6}{8}\right)+C_1$$

The constant of integration is, by Striling's asymptotic series, $-3\ln 2$. Now, recall the fact that

$$\int\ln\Gamma(x)\,\mathrm{d}x=\frac{x(1-x)}{2}+\frac{x}{2}\ln(2\pi)+x\ln\Gamma(x)-\ln G(x+1)+C_2$$

Hence, integrating once again, we obtain

$$\begin{aligned}\mathcal{I}(\alpha) =&\alpha\ln\alpha-\alpha-\frac{(\alpha+2)(\alpha-6)}{8}+(\alpha+2)\ln(2\pi)+2(\alpha+2)\ln\Gamma\left(\frac{\alpha+2}{8}\right)\\&-16\ln G\left(\frac{\alpha+10}{8}\right)+\frac{(\alpha-2)(\alpha+6)}{8}-(\alpha+6)\ln(2\pi)\\&-2(\alpha+6)\ln\Gamma\left(\frac{\alpha+6}{8}\right)+16\ln G\left(\frac{\alpha+14}{8}\right)-3\alpha\ln 2+C_2\\=&C_2-4\ln(2\pi)-3\alpha\ln 2+\alpha\ln\alpha+2(\alpha+2)\ln\Gamma\left(\frac{\alpha+2}{8}\right)\\&-2(\alpha+6)\ln\Gamma\left(\frac{\alpha+6}{8}\right)-16\ln G\left(\frac{\alpha+10}{8}\right)+16\ln G\left(\frac{\alpha+14}{8}\right)\end{aligned}$$

One may use the asymptotic series of the Barnes G (derived from the log-gamma integral and Stirling's) to check that $C_2=4\ln(2\pi)$. Letting $\alpha\to 0$,

$$\begin{aligned}\mathcal{I}=&\mathcal{I}(0)\\=&4\ln\Gamma\left(\frac{1}{4}\right)-12\ln\Gamma\left(\frac{3}{4}\right)-16\ln G\left(\frac{5}{4}\right)+16\ln G\left(\frac{7}{4}\right)\end{aligned}$$

It remains to simplify this result. Using the Gamma reflection formula $\Gamma(z)\Gamma(1-z)=\pi\csc(\pi z)$, as well as the Barnes G functional equation $G(z+1)=G(z)\Gamma(z)$, we get

$$\mathcal{I}=-6\ln 2-12\ln\pi-16\ln\left(\frac{G\left(\frac{1}{4}\right)}{G\left(\frac{7}{4}\right)}\right)$$

Now recall that $G(z+1)$ has the infinite product representation

$$G(z+1)=\left(\sqrt{2\pi}\right)^z e^{-\frac{(1+\gamma)z^2+z}{2}}\prod_{k=1}^{\infty}e^{\frac{z^2}{2k}-z}\left(1+\frac{z}{k}\right)^k$$

Taking the logarithm,

$$\begin{aligned}\ln G(z+1)=&-\frac{z}{2}+\frac{z}{2}\ln(2\pi)-\frac{z^2}{2}-\frac{\gamma z^2}{2}+\sum_{k=1}^{\infty}\left\{k\ln\left(1+\frac{z}{k}\right)+\frac{z^2}{2k}-z\right\}\\=&-\frac{z}{2}+\frac{z}{2}\ln(2\pi)-\frac{z^2}{2}-\frac{\gamma z^2}{2}+\sum_{k=1}^{\infty}\left\{\frac{z^2}{2k}-z+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{k^{m-1}}\right\}\\=&-\frac{z}{2}+\frac{z}{2}\ln(2\pi)-\frac{z^2}{2}-\frac{\gamma z^2}{2}+\sum_{k=1}^{\infty}\sum_{m=3}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{k^{m-1}}\\=&-\frac{z}{2}+\frac{z}{2}\ln(2\pi)-\frac{z^2}{2}-\frac{\gamma z^2}{2}+\sum_{m=3}^{\infty}\frac{(-1)^{m-1}\zeta(m-1)}{m}z^m\\=&-\frac{z}{2}+\frac{z}{2}\ln(2\pi)-\frac{z^2}{2}-\frac{\gamma z^2}{2}+\sum_{m=2}^{\infty}\frac{(-1)^{m}\zeta(m)}{m+1}z^{m+1}\end{aligned}$$

Letting $z\mapsto -z$,

$$\ln G(1-z)=\frac{z}{2}-\frac{z}{2}\ln(2\pi)-\frac{z^2}{2}-\frac{\gamma z^2}{2}-\sum_{m=2}^{\infty}\frac{\zeta(m)}{m+1}z^{m+1}$$

It follows that

$$\begin{aligned}\ln\frac{G(1-z)}{G(1+z)} =& z - z\ln(2\pi) - 2\sum_{m=1}^{\infty}\frac{\zeta(2m)}{2m+1}z^{2m+1}\\ =& -z\ln(2\pi) + \int_0^z \pi x\cot(\pi x)\,\mathrm{d}x\end{aligned}$$

Let $z = \frac{3}{4}$.

$$\begin{aligned}\ln\left(\frac{G\left(\frac{1}{4}\right)}{G\left(\frac{7}{4}\right)}\right) =& -\frac{3}{4}\ln(2\pi) + 2\sum_{n=1}^{\infty}\int_0^{\frac{3}{4}}\pi x\sin(2n\pi x)\,\mathrm{d}x\\ =& -\frac{3}{4}\ln(2\pi) + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(3n\pi/2)}{n^2} - \frac{3}{4}\sum_{n=1}^{\infty}\frac{\cos(3n\pi/2)}{n}\\ =& -\frac{3}{4}\ln(2\pi) - \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} + \frac{3}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\\ =& -\frac{3}{8}\ln 2 - \frac{3}{4}\ln\pi - \frac{\mathbf{G}}{2\pi}\end{aligned}$$

Finally,

$$\mathcal{I} = -6\ln 2 - 12\ln\pi - 16\left(-\frac{3}{8}\ln 2 - \frac{3}{4}\ln\pi - \frac{\mathbf{G}}{2\pi}\right) = \frac{8\mathbf{G}}{\pi}$$

□

Exercise 1.1.20: Prove:

$$\int_0^{\infty}\frac{x}{(x^2+1)(e^{2\pi x}+1)}\mathrm{d}x = 1 - \frac{\gamma}{2} - \ln 2$$

where γ is The Euler–Mascheroni constant.

Proof: Method One:

A Generalisation:

$$\begin{aligned}\int_0^{\infty}\frac{x}{(x^2+w^2)(1+e^{2\pi x})}\mathrm{d}x =& \int_0^{\infty}\frac{xe^{-x}}{(x^2+4\pi^2w^2)(1+e^{-x})}\mathrm{d}x &(1)\\ =& \int_0^{\infty}\frac{x}{x^2+4\pi^2w^2}\left(\sum_{n=1}^{\infty}(-1)^{n-1}e^{-nx}\right)\mathrm{d}x &(2)\\ =& \int_0^{\infty}xe^{-x}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{x^2+4n^2\pi^2w^2}\right)\mathrm{d}x &(3)\\ =& \int_0^{\infty}\frac{e^{-x}}{2x} - \frac{e^{-x}}{4w}\operatorname{csch}\left(\frac{x}{2w}\right)\mathrm{d}x &(4)\\ =& -\frac{1}{2}\int_0^1\frac{1}{\ln x} + \frac{x^{1/2w}}{w(1-x^{1/w})}\,\mathrm{d}x &(5)\\ =& -\frac{1}{2}\int_0^1\frac{x^{w-1}}{\ln x} + \frac{x^{w-1/2}}{1-x}\mathrm{d}x &(6)\\ =& -\frac{1}{2}\int_{\infty}^0\int_0^1 x^{t+w-1} + \frac{x^{t+w-1/2}\ln x}{1-x}\,\mathrm{d}x\,\mathrm{d}t &(7)\\ \frac{1}{2}\psi_0\left(w+\frac{1}{2}\right) - \frac{1}{2}\ln w =& -\frac{1}{2}\int_{\infty}^0\frac{1}{t+w} - \psi_1\left(t+w+\frac{1}{2}\right)\,\mathrm{d}t &(8)\end{aligned}$$

Explanation:

(1):Substituted $x \mapsto \dfrac{x}{2\pi}$.

(2):Expanded $\dfrac{1}{1+e^{-x}}$ as a geometric series.

(3):Substituted $x \mapsto \dfrac{x}{n}$.

(4):One can show that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x^2+4n^2\pi^2w^2} = \frac{1}{2wx}\operatorname{csch}\left(\frac{x}{2w}\right)$ using the residue theorem.

(5):Substituted $x \mapsto -\ln x$.

(6):Substituted $x \mapsto x^w$.

(7):Used the fact that $\dfrac{1}{\ln x} = \displaystyle\int_{\infty}^{0} x^t \, \mathrm{d}t$

(8):Used the integral representation of the polygamma function.

The Integral:

Let $w = 1$ to get

$$\int_0^{\infty} \frac{x}{(x^2+1)(e^{2\pi x}+1)}\mathrm{d}x = \frac{1}{2}\psi_0\left(\frac{3}{2}\right) = 1 - \frac{\gamma}{2} - \ln 2$$

Method Two:

Here is an elementary way to evaluate the integral without involving any special functions or advance formulas. Notice that

$$\int_0^{\infty} e^{-y}\sin(xy)\,\mathrm{d}y = \frac{x}{1+x^2}$$

Hence we have

$$\begin{aligned}\int_0^{\infty} \frac{x}{(1+x^2)\,(e^{2\pi x}+1)}\,\mathrm{d}x &= \int_0^{\infty}\int_0^{\infty} \frac{e^{-y}\sin(xy)}{e^{2\pi x}+1}\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{\infty} e^{-y}\int_0^{\infty}\frac{\sin(xy)}{e^{2\pi x}+1}\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

One may use the following technique to evaluate the inner integral

$$\begin{aligned}\int_0^{\infty}\frac{\sin(xy)}{e^{2\pi x}+1}\mathrm{d}x &= \int_0^{\infty}\frac{e^{-2\pi x}\sin(xy)}{1+e^{-2\pi x}}\mathrm{d}x \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty} e^{-2\pi kx}\sin(xy)\,\mathrm{d}x \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{y}{4\pi^2k^2+y^2} \\ &= \frac{1}{4}\left[\frac{2}{y} - \operatorname{csch}\left(\frac{y}{2}\right)\right]\end{aligned}$$

then we obtain

$$\begin{aligned}
\int_0^\infty \frac{x}{(1+x^2)\left(e^{2\pi x}+1\right)}\,\mathrm{d}x &= \frac{1}{4}\int_0^\infty e^{-y}\left[\frac{2}{y}-\operatorname{csch}\left(\frac{y}{2}\right)\right]\mathrm{d}y \\
&= \frac{1}{2}\int_0^\infty e^{-y}\left[\frac{1}{y}-\frac{e^{-y/2}}{1-e^{-y}}\right]\mathrm{d}y \\
&= \frac{1}{2}\int_0^\infty e^{-y}\left[\frac{1}{y}-\frac{1}{1-e^{-y}}+\frac{e^{y/2}}{1+e^{y/2}}\right]\mathrm{d}y \\
&= \underbrace{\frac{1}{2}\int_0^\infty \left[\frac{e^{-y}}{y}-\frac{1}{e^y-1}\right]\mathrm{d}y}_{(*)}+\frac{1}{2}\int_0^\infty \frac{e^{-y/2}}{1+e^{y/2}}\mathrm{d}y \\
&= -\frac{\gamma}{2}+\frac{1}{2}\int_0^\infty \left[e^{-y/2}-\frac{e^{-y/2}}{1+e^{-y/2}}\right]\mathrm{d}y \\
&= 1-\frac{\gamma}{2}-\ln 2
\end{aligned}$$

where $(*)$ is the integral representation of the Euler–Mascheroni constant. □

Exercise 1.1.21: Prove:

$$\int_0^\infty \left(\sqrt{1+x^4}-x^2\right)\,\mathrm{d}x = \frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{\pi}}$$

Proof: Method one: Set

$$x^4=\frac{1}{t}-1 \quad\Rightarrow\quad x=\left(\frac{1-t}{t}\right)^{\frac{1}{4}} \quad\Rightarrow\quad \mathrm{d}x=-\frac{1}{4}t^{-\frac{5}{4}}(1-t)^{-\frac{3}{4}}\,\mathrm{d}t,$$

then the integral turns out to be

$$\begin{aligned}
\int_0^\infty \left(\sqrt{1+x^4}-x^2\right)\,\mathrm{d}x &= \frac{1}{4}\int_0^1 \left(t^{-\frac{7}{4}}(1-t)^{-\frac{3}{4}}-t^{-\frac{7}{4}}(1-t)^{-\frac{1}{4}}\right)\,\mathrm{d}t \\
&= \frac{1}{4}\left[\mathrm{B}\left(-\frac{3}{4},\frac{1}{4}\right)-\mathrm{B}\left(-\frac{3}{4},\frac{3}{4}\right)\right] \\
&\overset{[1]}{=} \frac{1}{4}\left[\mathrm{B}\left(-\frac{3}{4},\frac{1}{4}\right)-0\right] \\
&\overset{[2]}{=} \frac{1}{4}\cdot\frac{\Gamma\left(-\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{2}\right)} \\
&\overset{[3]}{=} \frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{\pi}}.
\end{aligned}$$

Notes :

[1] Use property of Beta function and Euler's reflection formula for the gamma function

$$\mathrm{B}(x,y)=\mathrm{B}(x+1,y)+\mathrm{B}(x,y+1)$$

and

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

[2] $\mathrm{B}(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

[3] $\Gamma(n)=\dfrac{\Gamma(n+1)}{n}$ and $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$

Method Two:

Let's apply the substitution $\sqrt{x^4+1}-x^2=\sqrt{t}$:

$$x^2=\frac{1-t}{2\sqrt{t}},\quad \mathrm{d}x=-\frac{\sqrt{2}}{8}t^{-5/4}(t+1)(1-t)^{-1/2}\mathrm{d}t,$$

$$\begin{aligned}\int_0^\infty\left(\sqrt{x^4+1}-x^2\right)\mathrm{d}x&=\frac{\sqrt{2}}{8}\left(\int_0^1 t^{1/4}(1-t)^{-1/2}\,\mathrm{d}t+\int_0^1 t^{-3/4}(1-t)^{-1/2}\,\mathrm{d}t\right)\\&=\frac{\sqrt{2}}{8}\left(\mathrm{B}\left(\frac{5}{4},\frac{1}{2}\right)+\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right)\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{\pi}}\end{aligned}$$

□

Exercise 1.1.22: Prove this identity:

$$\int_0^\infty\frac{\ln\left(\dfrac{1+x^{4+\sqrt{15}}}{1+x^{2+\sqrt{3}}}\right)}{(1+x^2)\ln x}\mathrm{d}x=\frac{\pi}{4}\left(2+\sqrt{6}\sqrt{3-\sqrt{5}}\right)$$

Proof: **Method One:**

This integral can be evaluated in a closed form for arbitrary real exponents, and does not seem to be related to Herglotz-like integrals.

Assume $a,b\in\mathbb{R}$. Note that

$$\int_0^\infty\frac{\ln\left(\dfrac{1+x^a}{1+x^b}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}=\int_0^\infty\frac{\ln\left(\dfrac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}-\int_0^\infty\frac{\ln\left(\dfrac{1+x^b}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}.$$

Both integrals on the right-hand side have the same shape, so we only need to evaluate one of them:

$$\begin{aligned}&\underbrace{\int_0^\infty\frac{\ln\left(\dfrac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}}_{\text{split the region}}\\&=\int_0^1\frac{\ln\left(\dfrac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}+\underbrace{\int_1^\infty\frac{\ln\left(\dfrac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}}_{\text{change variable } y=1/x}\\&=\int_0^1\frac{\ln\left(\dfrac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}+\underbrace{\int_1^0\frac{\ln\left(\dfrac{1+y^{-a}}{2}\right)}{\ln\left(y^{-1}\right)}\frac{1}{1+y^{-2}}\left(-\frac{1}{y^2}\right)\mathrm{d}y}_{\text{flip the bounds and simplify}}\end{aligned}$$

$$= \int_0^1 \frac{\ln\left(\frac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2} - \underbrace{\int_0^1 \frac{\ln\left(\frac{1+y^{-a}}{2}\right)}{\ln y}\frac{\mathrm{d}y}{1+y^2}}_{\text{rename } y \text{ to } x}$$

$$= \underbrace{\int_0^1 \frac{\ln\left(\frac{1+x^a}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2} - \int_0^1 \frac{\ln\left(\frac{1+x^{-a}}{2}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}}_{\text{combine logarithms}}$$

$$= \int_0^1 \frac{\ln\left(\frac{1+x^a}{1+x^{-a}}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2} = \underbrace{\int_0^1 \frac{\ln\left(\frac{x^a\,(x^{-a}+1)}{1+x^{-a}}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2}}_{\text{cancel } \ 1+x^{-a}}$$

$$= \int_0^1 \frac{\ln\left(x^a\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2} = a\int_0^1 \frac{\mathrm{d}x}{1+x^2} = a\Big(\arctan 1 - \arctan 0\Big)$$

$$= \frac{\pi\, a}{4}.$$

So, finally,

$$\int_0^\infty \frac{\ln\left(\frac{1+x^a}{1+x^b}\right)}{\ln x}\frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{4}(a-b).$$

Therefore,

$$I = \frac{\pi}{4}(2+\sqrt{15}-\sqrt{3})$$

Method Two: Let the considered integral be I. Just to make it easier to write, let $4+\sqrt{15} = a$ and $2+\sqrt{3} = b$. Use the substitution $x = \tan\theta$ to get:

$$I = \int_0^{\pi/2} \frac{\ln\left(\frac{1+(\tan\theta)^a}{1+(\tan\theta)^b}\right)}{\ln\tan\theta}\,\mathrm{d}\theta$$

Next, use the substitution $\theta = \pi/2 - t$ to obtain:

$$I = \int_0^{\pi/2} \frac{\ln\left(\frac{1+(\tan t)^a}{(1+(\tan t)^b)(\tan t)^{a-b}}\right)}{\ln\cot t}\,\mathrm{d}t = \int_0^{\pi/2} \frac{-\ln\left(\frac{1+(\tan\theta)^a}{1+(\tan\theta)^b}\right) + (a-b)\ln\tan\theta}{\ln\tan\theta}\,\mathrm{d}\theta$$

where I used $\ln(\tan(\pi/2-\theta)) = \ln(\cot\theta) = -\ln(\tan\theta)$.

Add the two expressions for I and notice that you are left with:

$$2I = \int_0^{\pi/2} \frac{(a-b)\ln\tan\theta}{\ln\tan\theta}\,d\theta = \frac{\pi}{2}(a-b)$$

$$I = \frac{\pi}{4}(a-b)$$

Therefore,

$$I = \frac{\pi}{4}(2+\sqrt{15}-\sqrt{3})$$

□

Exercise 1.1.23: Prove:

$$I=\int_0^1 \frac{1-x^2+(1+x^2)\ln x}{(x+x^2)\ln^3 x}\mathrm{d}x=6\ln\mathbf{A}-\frac{\ln 4}{3}-\frac{1}{4}$$

where $\mathbf{A}$ is the Glaisher-Kinkelin constant:

$$\mathbf{A}=\exp\left(\frac{1}{12}-\zeta'(-1)\right)$$

Proof: We start from the first Binet's formula:

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{\ln(2\pi)}{2}+\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-t\,z}}{t}\mathrm{d}t$$

Change variable $t=-2\ln x$:

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{\ln(2\pi)}{2}+\frac{1}{2}\int_0^1 x^{2z-1}\frac{1-x^2+(1+x^2)\ln x}{(x^2-1)\ln^2 x}\mathrm{d}x$$

Integrate on interval $0<z<\frac{1}{2}$:

$$\psi^{(-2)}\left(\frac{1}{2}\right)=\frac{1}{16}+\frac{\ln 8}{8}+\frac{\ln\pi}{4}+\frac{1}{4}\int_0^1\frac{1-x^2+(1+x^2)\ln x}{(x+x^2)\ln^3 x}\mathrm{d}x$$

Use formula connecting Barnes G-Function and negapolygamma, and let $z=\frac{1}{2}$:

$$\ln G\left(\frac{1}{2}\right)=\frac{\ln(2\pi)}{4}+\frac{1}{8}-\frac{\ln\pi}{4}-\psi^{(-2)}\left(\frac{1}{2}\right)$$

We have a closed form for $\ln G\left(\frac{1}{2}\right)$ in terms of the Glaisher-Kinkelin constant $\mathbf{A}$:

$$\ln G\left(\frac{1}{2}\right)=-\frac{3\ln\mathbf{A}}{2}-\frac{\ln\pi}{4}+\frac{1}{8}+\frac{\ln 2}{24}$$

Comparing the last two formula, we can get:

$$\psi^{(-2)}\left(\frac{1}{2}\right)=\frac{3\ln\mathbf{A}}{2}+\frac{\ln\pi}{4}+\frac{5\ln 2}{24}$$

Finally,

$$\int_0^1\frac{1-x^2+(1+x^2)\ln x}{(x+x^2)\ln^3 x}\mathrm{d}x=6\ln\mathbf{A}-\frac{\ln 4}{3}-\frac{1}{4}$$

□

Exercise 1.1.24: Evaluate this integral:

$$I=\int_{-1}^1\frac{1}{x}\sqrt{\frac{1+x}{1-x}}\ln\left(\frac{2\,x^2+2\,x+1}{2\,x^2-2\,x+1}\right)\,\mathrm{d}x$$

Solution: I will transform the integral via a substitution, break it up into two pieces and recombine, perform an integration by parts, and perform another substitution to get an integral to which I know a closed form exists. From there, I use a method I know to attack

the integral, but in an unusual way because of the 8th degree polynomial in the denominator of the integrand.

First sub $t = (1-x)/(1+x)$, $\mathrm{d}t = -2/(1+x)^2\mathrm{d}x$ to get

$$2\int_0^\infty \mathrm{d}t \frac{t^{-1/2}}{1-t^2} \ln\left(\frac{5-2t+t^2}{1-2t+5t^2}\right)$$

Now use the symmetry from the map $t \mapsto 1/t$. Break the integral up into two as follows:

$$\begin{aligned}
&2\int_0^1 \mathrm{d}t \frac{t^{-1/2}}{1-t^2} \ln\left(\frac{5-2t+t^2}{1-2t+5t^2}\right) + 2\int_1^\infty \mathrm{d}t \frac{t^{-1/2}}{1-t^2} \ln\left(\frac{5-2t+t^2}{1-2t+5t^2}\right) \\
&= 2\int_0^1 \mathrm{d}t \frac{t^{-1/2}}{1-t^2} \ln\left(\frac{5-2t+t^2}{1-2t+5t^2}\right) + 2\int_0^1 \mathrm{d}t \frac{t^{1/2}}{1-t^2} \ln\left(\frac{5-2t+t^2}{1-2t+5t^2}\right) \\
&= 2\int_0^1 \mathrm{d}t \frac{t^{-1/2}}{1-t} \ln\left(\frac{5-2t+t^2}{1-2t+5t^2}\right)
\end{aligned}$$

Sub $t = u^2$ to get

$$4\int_0^1 \frac{\mathrm{d}u}{1-u^2} \ln\left(\frac{5-2u^2+u^4}{1-2u^2+5u^4}\right)$$

Integrate by parts:

$$\left[2\ln\left(\frac{1+u}{1-u}\right) \ln\left(\frac{5-2u^2+u^4}{1-2u^2+5u^4}\right)\right]_0^1 - 32\int_0^1 \mathrm{d}u \frac{(u^5-6u^3+u)}{(u^4-2u^2+5)\,(5u^4-2u^2+1)} \ln\left(\frac{1+u}{1-u}\right)$$

One last sub: $u = (v-1)/(v+1)$,$\mathrm{d}u = 2/(v+1)^2\mathrm{d}v$, and finally get

$$8\int_0^\infty \mathrm{d}v \frac{(v^2-1)(v^4-6v^2+1)}{v^8+4v^6+70v^4+4v^2+1} \ln v$$

With this form, we may finally conclude that a closed form exists and apply the residue theorem to obtain it. To wit, consider the following contour integral:

$$\oint_C \mathrm{d}z \frac{8(z^2-1)(z^4-6z^2+1)}{z^8+4z^6+70z^4+4z^2+1} \ln^2 z$$

where C is a keyhole contour about the positive real axis. This contour integral is equal to (I omit the steps where I show the integral vanishes about the circular arcs)

$$-i4\pi\int_0^\infty \mathrm{d}v \frac{8(v^2-1)(v^4-6v^2+1)}{v^8+4v^6+70v^4+4v^2+1} \ln v + 4\pi^2\int_0^\infty \mathrm{d}v \frac{8(v^2-1)(v^4-6v^2+1)}{v^8+4v^6+70v^4+4v^2+1}$$

It should be noted that the second integral vanishes; this may be easily seen by exploiting the symmetry about $v \mapsto 1/v$.

On the other hand, the contour integral is $i2\pi$ times the sum of the residues about the poles of the integrand. In general, this requires us to find the zeroes of the eight degree polynomial, which may not be possible analytically. Here, on the other hand, we have many symmetries to exploit, e.g., if a is a root, then $1/a$ is a root, $-a$ is a root, and $\bar{a}$ is a root. For example, we may deduce that

$$z^8+4z^6+70z^4+4z^2+1 = (z^4+4z^3+10z^2+4z+1)(z^4-4z^3+10z^2-4z+1)$$

which exploits the $a \mapsto -a$ symmetry. Now write

$$z^4 + 4z^3 + 10z^2 + 4z + 1 = (z-a)(z-\bar{a})\left(z - \frac{1}{a}\right)\left(z - \frac{1}{\bar{a}}\right)$$

Write $a = re^{i\theta}$ and get the following equations:

$$\left(r + \frac{1}{r}\right)\cos\theta = -2$$

$$\left(r^2 + \frac{1}{r^2}\right) + 4\cos^2\theta = 10$$

From these equations, one may deduce that a solution is $r = \phi + \sqrt{\phi}$ and $\cos\theta = 1/\phi$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Thus the poles take the form

$$z_k = \pm\left(\phi \pm \sqrt{\phi}\right) e^{\pm i \arctan\sqrt{\phi}}$$

Now we have to find the residues of the integrand at these 8 poles. We can break this task up by computing:

$$\sum_{k=1}^{8} \operatorname*{Res}_{z=z_k}\left[\frac{8(z^2-1)(z^4-6z^2+1)\ln^2 z}{z^8+4z^6+70z^4+4z^2+1}\right] = \sum_{k=1}^{8} \operatorname*{Res}_{z=z_k}\left[\frac{8(z^2-1)(z^4-6z^2+1)}{z^8+4z^6+70z^4+4z^2+1}\right]\ln^2 z_k$$

Here things got very messy, but the result is rather unbelievably simple:

$$\operatorname*{Res}_{z=z_k}\left[\frac{8(z^2-1)(z^4-6z^2+1)}{z^8+4z^6+70z^4+4z^2+1}\right] = \operatorname{sgn}[\cos(\arg z_k)]$$

EDIT

Actually, this is a very simple computation. Inspired by @sos440, one may express the rational function of z in a very simple form:

$$\frac{8(z^2-1)(z^4-6z^2+1)}{z^8+4z^6+70z^4+4z^2+1} = -\left[\frac{p'(z)}{p(z)} + \frac{p'(-z)}{p(-z)}\right]$$

where

$$p(z) = z^4 + 4z^3 + 10z^2 + 4z + 1$$

The residue of this function at the poles are then easily seen to be ± 1 according to whether the pole is a zero of $p(z)$ or $p(z)$.

END EDIT

That is, if the pole has a positive real part, the residue of the fraction is $+1$; if it has a negative real part, the residue is 1.

Now consider the log piece. Expanding the square, we get 3 terms:

$$\ln^2|z_k| - (\arg z_k)^2 + i2\ln|z_k|\arg z_k$$

Summing over the residues, we find that because of the ± 1 contributions above, that the first and third terms sum to zero. This leaves the second term. For this, it is crucial that we get

the arguments right, as $\arg z_k \in [0, 2\pi)$. Thus, we have

$$\begin{aligned} I = \int_0^\infty \mathrm{d}v \frac{8(v^2-1)(v^4-6v^2+1)}{v^8+4v^6+70v^4+4v^2+1} \ln v &= \frac{1}{2}\sum_{k=1}^{8} \operatorname{sgn}[\cos(\arg z_k)](\arg z_k)^2 \\ &= \frac{1}{2}[2(\arctan\sqrt{\phi})^2 + 2(2\pi - \arctan\sqrt{\phi})^2 \\ &\quad - 2(\pi - \arctan\sqrt{\phi})^2 - 2(\pi + \arctan\sqrt{\phi})^2] \\ &= 2\pi^2 - 4\pi \arctan\sqrt{\phi} \\ &= 4\pi \operatorname{arccot}\sqrt{\phi} \end{aligned}$$

□

Exercise 1.1.25: Find a closed-form for the following integral:

$$\int_0^\infty \frac{\ln\left(1+x+\sqrt{x^2+2\,x}\right)\,\ln\left(1+\sqrt{x^2+2\,x+2}\right)}{x^2+2x+1}\mathrm{d}x$$

Solution: This integral can be solved by brutal force.

$$I = \int_1^\infty \frac{\ln(x+\sqrt{x^2-1})\ln(1+\sqrt{x^2+1})}{x^2}\,\mathrm{d}x$$

By the knowledge in trigonometry, we obtain

$$\begin{aligned} I &= \int_1^\infty \frac{\operatorname{arcosh} x \cdot (\operatorname{arsinh}(1/x) + \ln x)}{x^2}\,\mathrm{d}x \\ &= \int_0^1 \operatorname{arcosh}(1/x)\cdot(\operatorname{arsinh} x - \ln x)\,\mathrm{d}x. \end{aligned}$$

Note that

$$\int (\operatorname{arsinh} x - \ln x)\,\mathrm{d}x = x\operatorname{arsinh} x - x\ln x + x + 1 - \sqrt{x^2+1}.$$

Here, the constant of integration is chosen so that the integrand becomes $O(x\log x)$ as $x \searrow 0$. Since $\operatorname{arcosh}(1/x) = O(\log x)$ as $x \searrow 0$, we can perform integration by parts to obtain

$$I = \int_0^1 \left(\operatorname{arsinh} x - \ln x + 1 + \frac{1-\sqrt{x^2+1}}{x}\right)\frac{\mathrm{d}x}{\sqrt{1-x^2}}.$$

Plug $x = \sin\theta$. Then

$$I = \int_0^{\frac{\pi}{2}} \left(\operatorname{arsinh}(\sin\theta) - \ln\sin\theta + 1 + \frac{1-\sqrt{1+\sin^2\theta}}{\sin\theta}\right)\mathrm{d}\theta.$$

We divide the integral into 4 parts and consider them separately.

Part 1. By the Taylor expansion of $\operatorname{arsinh} x$,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \operatorname{arsinh}(\sin\theta)\,\mathrm{d}\theta &= \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{1}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\,\mathrm{d}\theta \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \mathbf{G}. \end{aligned}$$

Part 2. We know so many ways to prove that

$$-\int_0^{\frac{\pi}{2}} \ln\sin\theta\,\mathrm{d}\theta = \frac{\pi}{2}\ln 2.$$

Part 3. Do you need an explanation for this?

$$\int_0^{\frac{\pi}{2}} \mathrm{d}\theta = \frac{\pi}{2}.$$

Part 4. Again, by the Taylor expansion,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1-\sqrt{1+\sin^2\theta}}{\sin\theta}\,\mathrm{d}\theta &= -\sum_{n=1}^{\infty}\binom{1/2}{n}\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\,\mathrm{d}\theta \\ &= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}\frac{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+n\right)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)(2n-1)} = \frac{\ln 2}{2}-\frac{\pi}{4}.\end{aligned}$$

Putting all these together, we obtain

$$\int_0^{\infty}\frac{\ln\left(1+x+\sqrt{x^2+2\,x}\right)\ln\left(1+\sqrt{x^2+2\,x+2}\right)}{x^2+2x+1}\mathrm{d}x = \mathbf{G}+\frac{\pi}{4}+\frac{\pi+1}{2}\ln 2$$

where $\mathbf{G}$ is the Catalan constant. □

Exercise 1.1.26: Find a closed-form for the following integral:

$$\int_0^{\infty}\frac{1}{\sqrt[3]{x}}\left(1+\ln\frac{1+e^{x-1}}{1+e^x}\right)\mathrm{d}x$$

Solution: We first remark the following identity:

$$\begin{aligned}\int_0^{\infty}\frac{x^{s-1}}{ze^x-1}\,\mathrm{d}x &= \int_0^{\infty}\frac{z^{-1}x^{s-1}e^{-x}}{1-z^{-1}e^{-x}}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty}z^{-n}\int_0^{\infty}x^{s-1}e^{-nx}\,\mathrm{d}x \\ &= \Gamma(s)\sum_{n=1}^{\infty}\frac{z^{-n}}{n^s} = \Gamma(s)\mathrm{Li}_s(z^{-1})\end{aligned}$$

which initially holds for $|z|>1$, and then extends holomorphically for $z^{-1}\notin(1,\infty]$ since both sides define holomorphic functions on this region. Now, by integrating by parts,

$$\begin{aligned}\int_0^{\infty}\frac{1}{\sqrt[3]{x}}\left(1+\ln\frac{1+e^{x-1}}{1+e^x}\right)\mathrm{d}x &= \frac{3}{2}\int_0^{\infty}x^{2/3}\left(\frac{1}{(-1)e^x-1}-\frac{1}{(-e^{-1})e^x-1}\right)\mathrm{d}x \\ &= \frac{3}{2}\Gamma\left(\frac{5}{3}\right)\left\{\mathrm{Li}_{5/3}(-1)-\mathrm{Li}_{5/3}(-e)\right\} \\ &= -\frac{3}{2}\Gamma\left(\frac{5}{3}\right)\left\{(1-2^{-2/3})\zeta\left(\frac{5}{3}\right)+\mathrm{Li}_{5/3}(-e)\right\}\end{aligned}$$

□

Exercise 1.1.27: Evaluate this integral in a closed form:

$$I=\int_0^1 \frac{\ln(x)}{x-1}\ln\left(1+\frac{1}{\ln^2(x)}\right)\mathrm{d}x$$

Solution: Substitute $x=e^{-y}$:

$$I=\int_0^\infty \frac{y}{e^y-1}\ln(1+y^{-2})\mathrm{d}y.$$

We Start from Binet's Second Formula:

$$\begin{aligned}\ln\Gamma(z)&=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+2\int_0^\infty \frac{\tan^{-1}\left(\frac{t}{z}\right)}{e^{2\pi t}-1}\mathrm{d}t\\&=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+\frac{1}{\pi}\int_0^\infty \frac{\tan^{-1}\left(\frac{y}{2\pi z}\right)}{e^{y}-1}\mathrm{d}y\end{aligned}$$

Integrate from $z=0$ to $\frac{1}{2\pi}$

$$\begin{aligned}\psi^{(-2)}\left(\frac{1}{2\pi}\right)&=-\frac{3}{16\pi^2}+\frac{1}{4\pi}-\frac{\ln(2\pi)}{8\pi^2}+\frac{\ln(2\pi)}{2\pi}\\&\quad+\frac{1}{\pi}\int_0^\infty \frac{1}{e^y-1}\left(\frac{\tan^{-1}y}{2\pi}+\frac{y\ln(1+y^{-2})}{4\pi}\right)\mathrm{d}y\\&=\frac{I}{4\pi^2}-\frac{3}{16\pi^2}+\frac{1}{4\pi}-\frac{\ln(2\pi)}{8\pi^2}+\frac{\ln(2\pi)}{2\pi}+\frac{1}{2\pi^2}\int_0^\infty\left(\frac{\tan^{-1}y}{e^y-1}\right)\mathrm{d}y\\&=\frac{I}{4\pi^2}-\frac{3}{16\pi^2}+\frac{1}{4\pi}-\frac{\ln(2\pi)}{8\pi^2}+\frac{\ln(2\pi)}{2\pi}+\frac{1}{\pi}\int_0^\infty\left(\frac{\tan^{-1}(2\pi t)}{e^{2\pi t}-1}\right)\mathrm{d}t.\end{aligned}$$

Therefore

$$I=4\pi^2\psi^{(-2)}\left(\frac{1}{2\pi}\right)+\frac{3}{4}-\pi+\frac{\ln(2\pi)}{2}-2\pi\ln(2\pi)-4\pi\int_0^\infty\left(\frac{\tan^{-1}(2\pi t)}{e^{2\pi t}-1}\right)\mathrm{d}t.$$

We use Binet's second formula again:

$$\ln\Gamma\left(\frac{1}{2\pi}\right)=\left(\frac{1}{2\pi}-\frac{1}{2}\right)\ln\frac{1}{2\pi}-\frac{1}{2\pi}+\frac{1}{2}\ln(2\pi)+2\int_0^\infty\frac{\tan^{-1}(2\pi t)}{e^{2\pi t}-1}\mathrm{d}t$$

Therefore $4\pi\int_0^\infty\frac{\tan^{-1}(2\pi t)}{e^{2\pi t}-1}\mathrm{d}t=2\pi\ln\Gamma\left(\frac{1}{2\pi}\right)+1-(2\pi-1)\ln(2\pi)$, and

$$I=4\pi^2\psi^{(-2)}\left(\frac{1}{2\pi}\right)-\frac{1}{4}-\pi-\frac{\ln(2\pi)}{2}-2\pi\ln\Gamma\left(\frac{1}{2\pi}\right).$$

Also, we have

$$\psi^{(-2)}(z)=\frac{z}{2}\ln(2\pi)+(z-1)\ln\Gamma(z)-z(z-1)/2-\ln G(z),$$

Therefore

$$4\pi^2\psi^{(-2)}\left(\frac{1}{2\pi}\right)=\pi\ln(2\pi)-(4\pi^2-2\pi)\ln\Gamma\left(\frac{1}{2\pi}\right)-\frac{1}{2}+\pi-4\pi^2\ln G\left(\frac{1}{2\pi}\right)$$

And we have the final answer

$$I=\left(\pi-\frac{1}{2}\right)\ln(2\pi)-\frac{3}{4}-4\pi^2\left(\ln\Gamma\left(\frac{1}{2\pi}\right)+\ln G\left(\frac{1}{2\pi}\right)\right),$$

where $G(z)$ denotes the Barnes G-function. □

Exercise 1.1.28: Evaluate the following integral:

$$\int_0^1\frac{\left[\mathrm{Li}_2\left(\frac{1}{2}\right)-\mathrm{Li}_2\left(\frac{1+x}{2}\right)\right]\ln(1-x)}{1+x}\,\mathrm{d}x$$

Solution: Let $t=\dfrac{1+x}{2}$, then we have

$$\int_0^1\frac{\left[\mathrm{Li}_2\left(\frac{1}{2}\right)-\mathrm{Li}_2\left(\frac{1+x}{2}\right)\right]\ln(1-x)}{1+x}\,\mathrm{d}x=\int_{\frac{1}{2}}^1\frac{\left[\mathrm{Li}_2\left(\frac{1}{2}\right)-\mathrm{Li}_2(t)\right]\ln(2-2t)}{t}\,\mathrm{d}t$$

$$I=\mathrm{Li}_2\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\ln(2-2t)}{t}\,\mathrm{d}t-\int_{\frac{1}{2}}^1\frac{\mathrm{Li}_2(t)\ln(2-2t)}{t}\,\mathrm{d}t$$

$$I=I_1+I_2-I_3-I_4$$

where

$$I_1=\mathrm{Li}_2\left(\frac{1}{2}\right)\ln 2\int_{\frac{1}{2}}^1\frac{\mathrm{d}t}{t}=\mathrm{Li}_2\left(\frac{1}{2}\right)\ln 2\ln(t)\Big|_{\frac{1}{2}}^1=\mathrm{Li}_2\left(\frac{1}{2}\right)\ln^2 2=\frac{\pi^2}{12}\ln^2 2-\frac{\ln^4 2}{2}$$

$$\begin{aligned}I_2&=\mathrm{Li}_2\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\ln(1-t)}{t}\,\mathrm{d}t=-\mathrm{Li}_2\left(\frac{1}{2}\right)\mathrm{Li}_2(t)\Big|_{\frac{1}{2}}^1\\&=\mathrm{Li}_2^2\left(\frac{1}{2}\right)-\mathrm{Li}_2\left(\frac{1}{2}\right)\mathrm{Li}_2(1)=\frac{\ln^4 2}{4}-\frac{\pi^4}{144}\end{aligned}$$

$$\begin{aligned}I_3&=\ln 2\int_{\frac{1}{2}}^1\frac{\mathrm{Li}_2(t)}{t}\,\mathrm{d}t=\ln 2\,\mathrm{Li}_3(t)\Big|_{\frac{1}{2}}^1=\left[\mathrm{Li}_3(1)-\mathrm{Li}_3\left(\frac{1}{2}\right)\right]\ln 2\\&=\frac{\zeta(3)}{8}\ln 2+\frac{\pi^2}{12}\ln^2 2-\frac{1}{6}\ln^4 2\end{aligned}$$

$$\begin{aligned}I_4&=\int_{\frac{1}{2}}^1\frac{\mathrm{Li}_2(t)\ln(1-t)}{t}\,\mathrm{d}t=\left[\mathrm{Li}_2(t)\int\frac{\ln(1-t)}{t}\,\mathrm{d}t\right]_{\frac{1}{2}}^1-\int_{\frac{1}{2}}^1\left[\mathrm{Li}_2'(t)\int\frac{\ln(1-t)}{t}\,\mathrm{d}t\right]\mathrm{d}t\\&=-\mathrm{Li}_2^2(t)\Big|_{\frac{1}{2}}^1-\int_{\frac{1}{2}}^1\frac{\mathrm{Li}_2(t)\ln(1-t)}{t}\,\mathrm{d}t=-\frac{\mathrm{Li}_2^2(t)}{2}\Big|_{\frac{1}{2}}^1=\frac{\ln^4 2}{8}-\frac{\pi^4}{96}-\frac{\pi^2}{24}\ln^2 2\end{aligned}$$

Therefore

$$\int_0^1\frac{\left[\mathrm{Li}_2\left(\frac{1}{2}\right)-\mathrm{Li}_2\left(\frac{1+x}{2}\right)\right]\ln(1-x)}{1+x}\,\mathrm{d}x=\frac{\pi^4}{288}-\frac{5}{24}\ln^4 2+\frac{\pi^2}{24}\ln^2 2-\frac{\zeta(3)}{8}\ln 2$$

□

Exercise 1.1.29: Prove:

$$\int_0^1 \frac{\mathrm{d}x}{1+{}_2F_1\left(\frac{1}{n},x;\frac{1}{n};\frac{1}{n}\right)}=\frac{\ln\left(\frac{2n}{2n-1}\right)}{\ln\left(\frac{n}{n-1}\right)}$$

Proof: First note that

$$\begin{aligned}{}_2F_1\left(\frac{1}{n},x;\frac{1}{n};\frac{1}{n}\right)&=1+\frac{x}{n}+\frac{x(x+1)}{2!n^2}+\frac{x(x+1)(x+2)}{3!n^3}+\frac{x(x+1)(x+2)(x+3)}{4!n^4}+\cdots\\&=1+(-x)\frac{-1}{n}+\frac{(-x)(-x-1)}{2!}\left(\frac{-1}{n}\right)^2+\frac{(-x)(-x-1)(-x-2)}{3!}\left(\frac{-1}{n}\right)^3+\cdots\\&=\left(1-\frac{1}{n}\right)^{-x}\end{aligned}$$

Hence,

$${}_2F_1\left(\frac{1}{n},x;\frac{1}{n};\frac{1}{n}\right)=\left(\frac{n}{n-1}\right)^x$$

Hence,

$$\int_0^1 \frac{\mathrm{d}x}{1+{}_2F_1\left(\frac{1}{n},x;\frac{1}{n};\frac{1}{n}\right)}=\int_0^1 \frac{\mathrm{d}x}{1+\left(\frac{n}{n-1}\right)^x}=\int_0^1 \frac{\mathrm{d}x}{1+e^{kx}}$$

where $e^k=\dfrac{n}{n-1}$

Hence, we are interested in evaluating an integral of the form

$$\int_0^1 \frac{1}{1+e^{kx}}\mathrm{d}x$$

Let $t=1+e^{kx}$. We get $\mathrm{d}t=ke^{kx}\mathrm{d}x=k(t-1)\mathrm{d}x$

$$\int_0^1 \frac{1}{1+e^{kx}}\mathrm{d}x=\int_2^{1+e^k}\frac{1}{k}\frac{\mathrm{d}t}{t(t-1)}=\frac{1}{k}\int_2^{1+e^k}\left(\frac{\mathrm{d}t}{t-1}-\frac{\mathrm{d}t}{t}\right)=\frac{1}{k}\left(k-\ln(e^k+1)+\ln(2)\right)$$

In our case, $e^k=\dfrac{n}{n-1}$ and hence the integral becomes

$$\int_0^1 \frac{\mathrm{d}x}{1+{}_2F_1\left(\frac{1}{n},x;\frac{1}{n};\frac{1}{n}\right)}=1-\frac{\ln(2n-1)-\ln(n-1)}{\ln(n)-\ln(n-1)}+\frac{\ln(2)}{\ln(n)-\ln(n-1)}=\frac{\ln\left(\frac{2n}{2n-1}\right)}{\ln\left(\frac{n}{n-1}\right)}$$

□

Exercise 1.1.30: Prove:

$$I=\int_0^{\pi/2}\theta^2\ln^4(2\cos\theta)\mathrm{d}\theta=\frac{33\pi^7}{4480}+\frac{3\pi}{2}\zeta^2(3)$$

Proof: Method One:

From Table of Integrals, Series, and Products Seventh Edition by I.S. Gradshteyn and I.M. Ryzhik equation 3.631(9) we have

$$\int_0^{\frac{\pi}{2}}\cos^{n-1}x\cos ax\ \mathrm{d}x=\frac{\pi}{2^n n\ \mathrm{B}\left(\frac{n+a+1}{2},\frac{n-a+1}{2}\right)}$$

Proof:

Integrating $(1+z)^p z^q$, for $p, q \geq 0$, in the $z = u + iv$ plane around the contour bounded by the u-axis from 1 to 1 and the upper semicircle of unit radius yields

$$\int_{-1}^{1} (1+z)^p z^q \, \mathrm{d}z = -i \int_0^{\pi} \left(1 + e^{it}\right)^p e^{i(q+1)t} \, \mathrm{d}t,$$

since $(1+z)^p z^q$ is holomorphic within and continuous on and within the given contour. The imaginary part of the RHS is

$$\Im \left[-i \int_0^{\pi} \left(1 + e^{it}\right)^p e^{i(q+1)t} \, \mathrm{d}t \right] = - \int_0^{\pi} \left(2 \cos \frac{t}{2}\right)^p \cos bt \, \mathrm{d}t,$$

where $b = q + \dfrac{1}{2}p + 1$. The LHS integral is equal to

$$\int_0^1 (1+u)^p u^q \, \mathrm{d}u + e^{i\pi q} \int_0^1 (1-u)^p u^q \, \mathrm{d}u$$

of which the imaginary part is

$$\begin{aligned}
\mathrm{B}\left(p+1, q+1\right) \sin \pi q &= -\frac{\Gamma\left(p+1\right)\Gamma\left(q+1\right)}{\Gamma\left(p+q+2\right)} \sin \pi q \\
&= -\frac{\Gamma\left(p+1\right)\Gamma\left(b - \frac{1}{2}p\right)}{\Gamma\left(b + \frac{1}{2}p + 1\right)} \sin \pi \left(b - \frac{1}{2}p\right) \\
&= -\frac{\pi \Gamma\left(p+1\right)}{\Gamma\left(1 + \frac{1}{2}p + b\right)\Gamma\left(1 + \frac{1}{2}p - b\right)}.
\end{aligned}$$

Final step is setting $t = 2x$, $p = n1$, and $a = 2b$.

Thus

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \theta^2 \ln^4(2\cos\theta) \, \mathrm{d}\theta &= \lim_{n \to 5} \lim_{a \to 0} \frac{\partial^6}{\partial n^4 \partial a^2} \left[\int_0^{\frac{\pi}{2}} (2\cos\theta)^{n-1} \cos a\theta \, \mathrm{d}\theta \right] \\
&= \lim_{n \to 5} \lim_{a \to 0} \frac{\partial^6}{\partial n^4 \partial a^2} \left[\frac{\pi}{2n \ \mathrm{B}\left(\frac{n+a+1}{2}, \frac{n-a+1}{2}\right)} \right] \\
&= \frac{33\pi^7}{4480} + \frac{3\pi}{2}\zeta^2(3).
\end{aligned}$$

Method Two:

Note that

$$\ln(1 + e^{i2x}) = \ln(2\cos x) + ix$$

Raise both sides to the fourth power and extract the real part. This yields

$$\Re \ln^4(1 + e^{i2x}) = \ln^4(2\cos x) - 6x^2 \ln^2(2\cos x) + x^4$$

We first have to calculate $\displaystyle\int_0^{\pi/2} x^4 \ln^2(2\cos x) \, \mathrm{d}x$. Squaring the above identity and extracting the real part gives

$$\Re \ln^2(1 + e^{i2x}) = \ln^2(2\cos x) - x^2$$

Then

$$\begin{aligned}\int_0^{\pi/2} x^4 \ln^2(2\cos x)\,\mathrm{d}x &= \Re\int_0^{\pi/2} x^4\ln^2(1+e^{i2x})\,\mathrm{d}x + \frac{\pi^7}{896}\\ &= \frac{\pi^7}{896}+\frac{1}{32}\Im\int_0^1 \frac{\ln^4(-z)\ln^2(1-z)}{z}\mathrm{d}z\\ &= \frac{\pi^7}{896}+\frac{\pi}{8}\int_0^1\frac{\ln^3 z\ln^2(1-z)}{z}\mathrm{d}z-\frac{\pi^3}{8}\int_0^1\frac{\ln z\ln^2(1-z)}{z}\mathrm{d}z\\ &= \frac{\pi^7}{896}-\frac{\pi}{16}\sum_{n=1}^{\infty}H_n\int_0^1 z^n\ln^4 z\,\mathrm{d}z+\frac{\pi^3}{8}\sum_{n=1}^{\infty}H_n\int_0^1 z^n\ln^2 z\,\mathrm{d}z\\ &= \frac{\pi^7}{896}-\frac{3\pi}{2}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^5}+\frac{\pi^3}{4}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\\ &= \frac{5\pi^7}{8064}+\frac{3\pi}{4}\zeta^2(3)\end{aligned}$$

where I have used the result

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^q}=(q+2)\zeta(q+1)-\sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j)$$

which can be easily derived by integrating $f(z)=\dfrac{(\gamma+\psi(-z))^2}{z^q}$ along an infinitely large square.

Moving on to the actual integral,

$$\begin{aligned}\int_0^{\pi/2} x^2\ln^4(2\cos x)\,\mathrm{d}x &= \frac{\pi^7}{384}+\frac{9\pi}{2}\zeta^2(3)+\Re\int_0^{\pi/2}x^2\ln^4(1+e^{i2x})\,\mathrm{d}x\\ &= \frac{\pi^7}{384}+\frac{9\pi}{2}\zeta^2(3)-\frac{1}{8}\Im\int_0^1\frac{\ln^2(-z)\ln^4(1-z)}{z}\,\mathrm{d}z\\ &= \frac{\pi^7}{384}+\frac{9\pi}{2}\zeta^2(3)-\frac{\pi}{4}\int_0^1\frac{\ln^4 z\ln(1-z)}{1-z}\,\mathrm{d}z\\ &= \frac{\pi^7}{384}+\frac{9\pi}{2}\zeta^2(3)+\frac{\pi}{4}\sum_{n=1}^{\infty}H_n\int_0^1 z^n\ln^4 z\,\mathrm{d}z\\ &= \frac{\pi^7}{384}+\frac{9\pi}{2}\zeta^2(3)+6\pi\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^5}\\ &= \frac{\pi^7}{384}+\frac{9\pi}{2}\zeta^2(3)+\frac{\pi^7}{210}-3\pi\zeta^2(3)\\ &= \frac{33\pi^7}{4480}+\frac{3\pi}{2}\zeta^2(3)\end{aligned}$$

as was to be shown.

Just for fun, here is a derivation of

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^q}=(q+2)\zeta(q+1)-\sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j)$$

Consider $f(z) = \dfrac{(\gamma + \psi(-z))^2}{z^q}$. At the positive integers,

$$\begin{aligned}\sum_{n=1}^{\infty} \operatorname{Res}(f, n) &= \sum_{n=1}^{\infty} \operatorname*{Res}_{z=n} \left[\frac{1}{z^q (z-n)^2} + \frac{2H_n}{z^q (z-n)} \right] \\ &= 2 \sum_{n=1}^{\infty} \frac{H_n}{n^q} - q \sum_{n=1}^{\infty} \frac{1}{n^{q+1}} \\ &= 2 \sum_{n=1}^{\infty} \frac{H_n}{n^q} - q\zeta(q+1)\end{aligned}$$

At zero,

$$\begin{aligned}\operatorname{Res}(f, 0) &= [z^{q-1}] \left(\frac{1}{z} - \sum_{k=1}^{\infty} \zeta(k+1) z^k \right)^2 \\ &= [z^{q-1}] \left(-2 \sum_{k=1}^{\infty} \zeta(k+1) z^{k-1} + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \zeta(j+1)\zeta(k-j+2) z^{k+1} \right) \\ &= -2\zeta(q+1) + \sum_{j=1}^{q-2} \zeta(j+1)\zeta(q-j)\end{aligned}$$

where I have removed the first (irrelevant) term and used the Cauchy product in the second equality. Since the sum of the residues is zero, we conclude

$$2 \sum_{n=1}^{\infty} \frac{H_n}{n^q} = (q+2)\zeta(q+1) - \sum_{j=1}^{q-2} \zeta(j+1)\zeta(q-j).$$

□

Exercise 1.1.31: Prove this:

$$I = \int_0^{\infty} \frac{1}{x} \ln \left(\frac{1+x}{1-x} \right)^2 \mathrm{d}x = \pi^2$$

Proof: Allow me to present an approach that uses dilogarithms. Split the integral up into 2 and substitute $x \mapsto \dfrac{1}{x}$ for the second integral. This yields

$$\begin{aligned}\int_0^{\infty} \frac{1}{x} \ln \left(\frac{1+x}{1-x} \right)^2 \mathrm{d}x &= \int_0^1 \frac{1}{x} \ln \left(\frac{1+x}{1-x} \right)^2 \mathrm{d}x + \int_1^{\infty} \frac{1}{x} \ln \left(\frac{1+x}{1-x} \right)^2 \mathrm{d}x \\ &= 4 \int_0^1 \frac{\ln(1+x) - \ln(1-x)}{x} \mathrm{d}x \\ &= 4 \left(\operatorname{Li}_2(1) - \operatorname{Li}_2(-1) \right) \\ &= 4 \left(\frac{\pi^2}{6} + \frac{\pi^2}{12} \right) \\ &= \pi^2\end{aligned}$$

□

Exercise 1.1.32: Solve the following integral:

$$\int_0^1 \frac{\operatorname{Li}_3(x)}{1-x} \ln(x) \, \mathrm{d}x$$

✎ Solution:

$$\begin{aligned}\int_0^1 \frac{\ln x\ \mathrm{Li}_3(x)}{1-x}\mathrm{d}x &= \sum_{n=1}^{\infty} H_n^{(3)} \int_0^1 x^n \ln x\ \mathrm{d}x \\ &= -\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(n+1)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n+1)^5} - \sum_{n=1}^{\infty} \frac{H_{n+1}^{(3)}}{(n+1)^2} \\ &= \zeta(5) - \underbrace{\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}}_{S}\end{aligned}$$

Consider $f(z) = \dfrac{\pi \cot \pi z\ \Psi^{(2)}(-z)}{z^2}$. We know that

$$\begin{aligned}\pi \cot \pi z &= \frac{1}{z-n} - 2\sum_{k=1}^{\infty} \zeta(2k)(z-n)^{2k-1} \\ &\approx \frac{1}{z-n} - 2\zeta(2)(z-n)\end{aligned}$$

and

$$\begin{aligned}\Psi^{(2)}(-z) &= \frac{2}{(z-n)^3} + \sum_{k=2}^{\infty} (-1)^k k(k-1)\left(H_n^{(k+1)} + (-1)^{k+1}\zeta(k+1)\right)(z-n)^{k-2} \\ &\approx \frac{2}{(z-n)^3} + 2\left(H_n^{(3)} - \zeta(3)\right)\end{aligned}$$

At the positive integers,

$$\begin{aligned}\mathrm{Res}(f,n) &= \operatorname*{Res}_{z=n}\left[\frac{2}{z^2(z-n)^4} - \frac{4\zeta(2)}{z^2(z-n)^2} + \frac{2\left(H_n^{(3)} - \zeta(3)\right)}{z^2(z-n)}\right] \\ &= -\frac{8}{n^5} + \frac{8\zeta(2)}{n^3} + \frac{2H_n^{(3)}}{n^2} - \frac{2\zeta(3)}{n^2}\end{aligned}$$

At the negative integers,

$$\begin{aligned}\mathrm{Res}(f,-n) &= \frac{\Psi^{(2)}(n)}{n^2} \\ &= \frac{2H_n^{(3)}}{n^2} - \frac{2\zeta(3)}{n^2} - \frac{2}{n^5}\end{aligned}$$

At $z=0$,

$$\begin{aligned}\mathrm{Res}(f,0) &= [z^1]\left(\frac{1}{z} - 2\zeta(2)z\right)\left(-2\zeta(3) - 12\zeta(5)z^2\right) \\ &= -12\zeta(5) + 4\zeta(2)\zeta(3)\end{aligned}$$

Hence

$$\begin{aligned}4S &= 8\zeta(5) - 8\zeta(2)\zeta(3) + 2\zeta(2)\zeta(3) + 2\zeta(2)\zeta(3) + 2\zeta(5) + 12\zeta(5) - 4\zeta(2)\zeta(3) \\ &= 22\zeta(5) - 8\zeta(2)\zeta(3)\end{aligned}$$

which implies

$$\int_0^1 \frac{\ln x\ \mathrm{Li}_3(x)}{1-x}\mathrm{d}x = \zeta(5) - \frac{22\zeta(5) - 8\zeta(2)\zeta(3)}{4} = 2\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5)$$

□

Exercise 1.1.33: Prove:

$$\int_0^\infty \frac{\ln(x)}{e^x+1}\mathrm{d}x = -\frac{\ln^2(2)}{2}$$

Proof:

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{e^x+1}\mathrm{d}x &= \lim_{a\to 1}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty \frac{x^{a-1}e^{-x}}{1+e^{-x}}\mathrm{d}x && (1)\\
&= \lim_{a\to 1}\frac{\mathrm{d}}{\mathrm{d}a}\sum_{n\geq 0}(-1)^n\int_0^\infty x^{a-1}e^{-(n+1)x}\mathrm{d}x && (2)\\
&= \lim_{a\to 1}\sum_{n\geq 0}(-1)^n\frac{\mathrm{d}}{\mathrm{d}a}\frac{\Gamma(a)}{(n+1)^a} && (3)\\
&= \lim_{a\to 1}\sum_{n\geq 0}(-1)^n\frac{\Gamma(a)\psi(a)-\Gamma(a)\ln(n+1)}{(n+1)^a} && (4)\\
&= \psi(1)\sum_{n\geq 0}\frac{(-1)^n}{n+1} - \sum_{n\geq 0}\frac{(-1)^n\ln(n+1)}{n+1} && (5)\\
&= -\gamma\ln 2 - \left(\frac{1}{2}\ln^2 2 - \gamma\ln 2\right) && (6)\\
&= -\frac{1}{2}\ln^2 2
\end{aligned}$$

Explanation:

(1):Divide numerator and denominator by e^x.

(2):Expand the integrand as a geometric series, swap the order of integration and summation.

(3):Recognise the gamma function.

(4):Quotient rule.

(5):Apply the limit and split the sum into two.

(6):For the first sum, $\psi(1) = -\gamma$, and $\ln(1+1) = \sum_{n\geq 0}\frac{(-1)^n}{n+1}1^{n+1}$. I will now show how to evaluate the second sum.

$$\begin{aligned}
\sum_{n\geq 0}\frac{(-1)^n\ln(n+1)}{n+1} &= \sum_{n\geq 1}\frac{(-1)^{n-1}\ln n}{n} && (7)\\
&= \lim_{s\to 1}\frac{\mathrm{d}}{\mathrm{d}s}\sum_{n\geq 1}\frac{(-1)^n}{n^s}\\
&= -\eta'(1) && (8)\\
&= \frac{1}{2}\ln^2 2 - \gamma\ln 2 && (9)
\end{aligned}$$

(7):Index shift.

(8),(9):See Dirichlet eta function, Stieltjes Constants. □

Exercise 1.1.34: Evaluate the following integrals:

$$\int_0^{\pi/3} \ln^2(\sin x)\,\mathrm{d}x$$

Solution: Disclaimer: I stole this method from Random Variable so I dare not take credit for it.

Note that

$$\ln(1-e^{i2x}) = \ln(\sin x) + \ln 2 + i\left(x - \frac{\pi}{2}\right)$$

Square both sides and extract the real part.

$$\mathrm{Re}\ln^2(1-e^{i2x}) = \ln^2(\sin x) + 2\ln 2\ln(\sin x) + \ln^2 2 - \left(x-\frac{\pi}{2}\right)^2$$

Let us first compute $\mathcal{J} = \int_0^{\frac{\pi}{3}} \ln(\sin x)\mathrm{d}x$.

$$\begin{aligned}
\mathcal{J} &= \mathrm{Re}\int_0^{\frac{\pi}{3}} \ln(1-e^{i2x})\mathrm{d}x - \frac{\pi}{3}\ln 2 \\
&= \frac{1}{2}\mathrm{Im}\int_1^{e^{i\frac{2\pi}{3}}} \frac{\ln(1-z)}{z}\mathrm{d}z - \frac{\pi}{3}\ln 2 \\
&= -\frac{1}{2}\mathrm{Im}\ \mathrm{Li}_2(e^{i2\pi/3}) - \frac{\pi}{3}\ln 2
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{I} &= \int_0^{\frac{\pi}{3}} \ln^2(\sin x)\mathrm{d}x \\
&= \mathrm{Re}\int_0^{\frac{\pi}{3}} \ln^2(1-e^{i2x})\mathrm{d}x - 2\mathcal{J}\ln 2 - \frac{\pi}{3}\ln^2 2 + \frac{13\pi^3}{324} \\
&= \frac{1}{2}\mathrm{Im}\int_1^{e^{i\frac{2\pi}{3}}} \frac{\ln^2(1-z)}{z}\mathrm{d}z - 2\mathcal{J}\ln 2 - \frac{\pi}{3}\ln^2 2 + \frac{13\pi^3}{324} \\
&= \frac{1}{2}\mathrm{Im}\left[\frac{i2\pi}{3}\ln^2(1-e^{i2\pi/3}) + \int_1^{e^{i\frac{2\pi}{3}}} \frac{2\ln z\ln(1-z)}{1-z}\mathrm{d}z\right] - 2\mathcal{J}\ln 2 \\
&\quad - \frac{\pi}{3}\ln^2 2 + \frac{13\pi^3}{324} \\
&= \frac{\pi}{12}\ln^2 3 - \frac{\pi^3}{108} + \frac{1}{2}\mathrm{Im}\left[2\mathrm{Li}_2(1-e^{i2\pi/3})\ln(1-e^{i2\pi/3}) - 2\mathrm{Li}_3(1-e^{i2\pi/3})\right] \\
&\quad + \ln 2\ \mathrm{Im}\ \mathrm{Li}_2(e^{i2\pi/3}) + \frac{2\pi}{3}\ln^2 2 - \frac{\pi}{3}\ln^2 2 + \frac{13\pi^3}{324} \\
&= \frac{5\pi^3}{162} + \frac{\pi}{12}\ln^2 3 + \frac{\pi}{3}\ln^2 2 \\
&\quad - \mathrm{Im}\left[\mathrm{Li}_3(1-e^{i2\pi/3}) - \mathrm{Li}_2(1-e^{i2\pi/3})\ln(1-e^{i2\pi/3}) - \ln 2\mathrm{Li}_2(e^{i2\pi/3})\right]
\end{aligned}$$

The real part of $\mathrm{Li}_2(e^{i2\pi/3})$ is

$$\begin{aligned}\operatorname{Re}\mathrm{Li}_2(e^{i2\pi/3}) &= \sum_{n=1}^{\infty}\frac{\cos(2n\pi/3)}{n^2}\\ &= -\frac{1}{4}\sum_{n=-\infty}^{\infty}\left(\frac{1}{(3n+1)^2}+\frac{1}{(3n+2)^2}\right)+\frac{1}{9}\sum_{n=1}^{\infty}\frac{1}{n^2}\\ &= \frac{1}{4}\left[\operatorname*{Res}_{z=-1/3}\frac{\pi\cot(\pi z)}{(3z+1)^2}+\operatorname*{Res}_{z=-2/3}\frac{\pi\cot(\pi z)}{(3z+2)^2}\right]+\frac{\pi^2}{54}\\ &= -\frac{\pi^2}{18}\end{aligned}$$

and its imaginary part is

$$\begin{aligned}\operatorname{Im}\mathrm{Li}_2(e^{i2\pi/3}) &= \sum_{n=1}^{\infty}\frac{\sin(2n\pi/3)}{n^2}\\ &= \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(3n+1)^2}-\frac{1}{(3n+2)^2}\right)\\ &= \frac{1}{6\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{1}{6\sqrt{3}}\psi_1\left(\frac{2}{3}\right)\end{aligned}$$

By Euler's reflection formula for the dilogarithm,

$$\begin{aligned}\mathrm{Li}_2(1-e^{i2\pi/3}) &= \frac{\pi^2}{6}-\ln(e^{i2\pi/3})\ln(1-e^{i2\pi/3})-\mathrm{Li}_2(e^{i2\pi/3})\\ &= \frac{\pi^2}{9}+i\left(\frac{1}{6\sqrt{3}}\psi_1\left(\frac{2}{3}\right)-\frac{1}{6\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{\pi}{3}\ln 3\right)\end{aligned}$$

Therefore, the imaginary part of $\mathrm{Li}_2(1-e^{i2\pi/3})\ln(1-e^{i2\pi/3})$ is

$$\begin{aligned}&\operatorname{Im}\ln(1-e^{i2\pi/3})\mathrm{Li}_2(1-e^{i2\pi/3})\\ &= \operatorname{Re}\ln(1-e^{i2\pi/3})\operatorname{Im}\mathrm{Li}_2(1-e^{i2\pi/3})+\operatorname{Im}\ln(1-e^{i2\pi/3})\operatorname{Re}\mathrm{Li}_2(1-e^{i2\pi/3})\\ &= -\frac{\pi^3}{54}-\frac{\pi}{6}\ln^2 3+\frac{\ln 3}{12\sqrt{3}}\psi_1\left(\frac{2}{3}\right)-\frac{\ln 3}{12\sqrt{3}}\psi_1\left(\frac{1}{3}\right)\end{aligned}$$

Plucking these values into our previously obtained result yields

$$\begin{aligned}\mathcal{I} = &\frac{\ln 2}{6\sqrt{3}}\psi_1\left(\frac{1}{3}\right)+\frac{\ln 3}{12\sqrt{3}}\psi_1\left(\frac{2}{3}\right)-\frac{\ln 3}{12\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{\ln 2}{6\sqrt{3}}\psi_1\left(\frac{2}{3}\right)\\ &+\frac{\pi}{3}\ln^2 2-\frac{\pi}{12}\ln^2 3+\frac{\pi^3}{81}-\operatorname{Im}\mathrm{Li}_3(1-e^{i2\pi/3})\end{aligned}$$

□

Exercise 1.1.35: Prove that:

$$\int_0^{\pi/2}\ln^2(\cos x)\,\mathrm{d}x = \frac{\pi}{2}\ln^2 2+\frac{\pi^3}{24}$$

Proof: We have that $\ln(2\cos x)$ has a nice Fourier series:

$$\ln(2\cos x) = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\cos(2nx)$$

and since:

$$\int_0^{\pi/2}\cos(2nx)\cos(2mx)\,\mathrm{d}x=\frac{\pi}{4}\delta_{m,n}$$

it follows that:

$$\int_0^{\pi/2}\ln^2(2\cos x)\,\mathrm{d}x=\frac{\pi}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi}{4}\zeta(2)=\frac{\pi^3}{24},$$

while

$$\int_0^{\pi/2}\ln(\cos x)\,\mathrm{d}x=-\frac{\pi}{2}\ln 2$$

is a well-known result. So we have:

$$\int_0^{\pi/2}\ln^2(\cos x)\,\mathrm{d}x=\frac{\pi}{2}\ln^2 2+\frac{\pi^3}{24}.$$

Since

$$\ln(2\sin x)=-\sum_{n=1}^{+\infty}\frac{\cos(2nx)}{n}$$

and

$$\int_0^{\pi/2}\cos(2n_1x)\cos(2n_2x)\cos(2n_3x)\,\mathrm{d}x=\frac{\pi}{8}\delta_{2\cdot\max n_i=(n_1+n_2+n_3)}$$

it follows that

$$\int_0^{\pi/2}\ln^3(2\sin x)\,\mathrm{d}x=-\frac{3\pi}{4}\sum_{n=1}^{+\infty}\frac{H_{n-1}}{n^2}=-\frac{3\pi}{4}\zeta(3)$$

□

Exercise 1.1.36: Prove that:

$$\int_0^1\frac{1-x}{1-x^6}\ln^4 x\,\mathrm{d}x=\frac{16\sqrt{3}}{729}\pi^5+\frac{605}{54}\zeta(5)$$

Proof: Note that

$$\frac{1-x}{1-x^6}=\prod_{n=1}^{5}\frac{1}{x-x_n}=\sum_{n=1}^{5}\frac{b_n}{x-x_n},\quad\begin{cases}x_n=e^{n\pi i/3}\\ b_n=\dfrac{1}{6}x_n\left(x_n-1\right)\end{cases}$$

$$\begin{aligned}&\int_0^1\frac{1-x}{1-x^6}\ln^4 x\,\mathrm{d}x=\sum_{n=1}^{5}b_n\int_0^1\frac{\ln^4 x}{x-x_n}\mathrm{d}x=-24\sum_{n=1}^{5}b_n\,\mathrm{Li}_5\left(\frac{1}{x_n}\right)\\&=4\mathrm{Li}_5\left(e^{-\pi i/3}\right)+4i\sqrt{3}\mathrm{Li}_5\left(e^{-2\pi i/3}\right)-8\mathrm{Li}_5\left(-1\right)-4i\sqrt{3}\mathrm{Li}_5\left(e^{2\pi i/3}\right)+4\mathrm{Li}_5\left(e^{\pi i/3}\right)\\&=8\left[\Re\mathrm{Li}_5\left(e^{\pi i/3}\right)+\sqrt{3}\Im\mathrm{Li}_5\left(e^{2\pi i/3}\right)-\mathrm{Li}_5\left(-1\right)\right]\end{aligned}$$

we used a result of the following picture:

$$\int_0^1 \frac{\ln^4(x)}{x-a}\,dx = -\int_0^1 \frac{\ln^4(a\,[x/a])}{1-x/a}\,\frac{dx}{a} = -\int_0^{1/a} \frac{\ln^4(ax)}{1-x}\,dx$$

$$= -\int_0^{1/a} \ln(1-x)\,4\ln^3(ax)\,\frac{1}{x}\,dx = 4\int_0^{1/a} \mathrm{Li}_2'(x)\ln^3(ax)\,dx$$

$$= -4\int_0^{1/a} \mathrm{Li}_2(x)\;3\ln^2(ax)\,\frac{1}{x}\,dx$$

$$= -12\int_0^{1/a} \mathrm{Li}_3'(x)\ln^2(ax)\,dx = 12\int_0^{1/a} \mathrm{Li}_3(x)\,2\ln(ax)\,\frac{1}{x}\,dx$$

$$= 24\int_0^{1/a} \mathrm{Li}_4'(x)\ln(ax)\,dx = -24\int_0^{1/a} \mathrm{Li}_4(x)\;\frac{1}{x}\,dx = -24\int_0^{1/a} \mathrm{Li}_5'(x)\,dx$$

$$= -24\,\mathrm{Li}_5\left(\frac{1}{a}\right)$$

The following results, but the last one, are found with Jonquiere Inversion Formula:

$$\Re\mathrm{Li}_5\left(e^{\pi i/3}\right) = \frac{25}{54}\zeta(5), \quad \Im\mathrm{Li}_5\left(e^{2\pi i/3}\right) = \frac{2}{729}\pi^5, \quad \mathrm{Li}_5(-1) = -\frac{15}{16}\zeta(5)$$

The last one is found by manipulating the PolyLog serie definition.

$$\int_0^1 \frac{1-x}{1-x^6}\ln^4 x\,dx = 8\left\{\frac{25}{54}\zeta(5) + \sqrt{3}\frac{2}{729}\pi^5 + \frac{15}{16}\zeta(5)\right\}$$

now we get:

$$\int_0^1 \frac{1-x}{1-x^6}\ln^4 x\,dx = \frac{16\sqrt{3}}{729}\pi^5 + \frac{605}{54}\zeta(5)$$

□

Exercise 1.1.37: Prove that:

$$\int_0^1 \frac{\ln(1-x)\ln^2 x}{x-1}dx = \frac{\pi^4}{180}$$

Proof:

$$-\int_0^1 \frac{\ln^2 x\ln(1-x)}{1-x}dx = -\int_0^1 \frac{\ln x\ln(1-x)\ln(1-x)}{x}dx \tag{1}$$

$$= \int_0^1 \frac{\ln x\;\mathrm{Li}_2(x)}{1-x}dx - \int_0^1 \underbrace{\frac{\ln(1-x)\;\mathrm{Li}_2(x)}{x}}_{-\mathrm{Li}_2(x)\mathrm{Li}_2'(x)}dx \tag{2}$$

$$= -\frac{\pi^4}{120} + \frac{\pi^4}{72} \tag{3}$$

$$= \frac{\pi^4}{180}$$

Explanation:

(1):Substitute $x \mapsto 1-x$.

(2):Integrate by parts, with $u = \ln x \ln(1-x)$ and $v = \mathrm{Li}_2(x)$.

(3):For the second, observe that

$$\begin{aligned}\int_0^1 \mathrm{Li}_2(x)\mathrm{Li}_2'(x)\mathrm{d}x &= \left[\frac{1}{2}\mathrm{Li}_2(x)^2\right]_0^1 \\ &= \frac{1}{2}\left(\frac{\pi^2}{6}\right)^2 \\ &= \frac{\pi^4}{72}\end{aligned}$$

□

Exercise 1.1.38: Find a closed-form of this integral:

$$\frac{1}{2}\int_0^\infty \left[\frac{x^2\cos x}{\cosh 2x - \cos x} - \frac{2x^2}{e^{4x} - 2e^{2x}\cos x + 1}\right]\mathrm{d}x$$

Solution: Rewrite the integral into

$$\int_0^\infty \frac{x^2(e^{2x}\cos x - 1)}{e^{4x} - 2e^{2x}\cos x + 1}\,\mathrm{d}x$$

The trigonometric series is the following:

$$\frac{1}{(p^2-1)} + \frac{p^2+1}{(p^2-1)}\sum_{k=1}^\infty \frac{\cos kx}{p^k} = \frac{p\cos x}{p^2 - 2p\cos x + 1};\ \textit{where } p > 1,$$

and

$$\frac{1}{p^2-1} + \frac{2}{p^2-1}\sum_{k=1}^\infty \frac{\cos kx}{p^k} = \frac{1}{p^2 - 2p\cos x + 1};\ \textit{where } p > 1.$$

Subtracting the two gives,

$$\sum_{k=1}^\infty \frac{\cos kx}{p^k} = \frac{p\cos x - 1}{p^2 - 2p\cos x + 1};\ \textit{where } p > 1.$$

With $p = e^{2x}$,

$$\begin{aligned}\int_0^\infty \mathrm{d}x\,\frac{x^2(e^{2x}\cos x - 1)}{e^{4x} - 2e^{2x}\cos x + 1} &= \int_0^\infty \mathrm{d}x \sum_{k=1}^\infty x^2 e^{-2kx}\cos kx \\ &= \sum_{k=1}^\infty \int_0^\infty \mathrm{d}x\, x^2 e^{-2kx}\cos kx \\ &= \sum_{k=1}^\infty \left[\frac{\partial^2}{\partial k^2}\int_0^\infty \mathrm{d}x\,\frac{1}{4}e^{-2kx}\cos nx\right]_{n=k} \\ &= \sum_{k=1}^\infty \left[\frac{\partial^2}{\partial k^2}\frac{1}{4}\cdot\frac{2k}{4k^2+n^2}\right]_{n=k} \\ &= \sum_{k=1}^\infty \left[\frac{4k(4k^2-3n^2)}{(4k^2+n^2)^3}\right]_{n=k} \\ &= \sum_{k=1}^\infty \frac{4}{125k^3} \\ &= \frac{4}{125}\zeta(3)\end{aligned}$$

□

Exercise 1.1.39: Find the value:

$$I=\int_0^{\frac{\pi}{2}}\left(\ln\left(1+\tan^4 x\right)\right)^2\frac{2\cos^2 x}{2-\left(\sin\left(2x\right)\right)^2}\mathrm{d}x$$

Solution: After performing the initial arctangent substitution, the resultant integral may be rewritten as the second derivative of a beta function and evaluated accordingly:

$$\begin{aligned}
I&=\int_0^{\frac{\pi}{2}}\frac{2\cos^2\theta}{2-\sin^2\left(2\theta\right)}\ln^2\left(1+\tan^4\theta\right)\mathrm{d}\theta\\
&=\int_0^{\infty}\frac{\ln^2\left(1+x^4\right)}{1+x^4}\mathrm{d}x\\
&=\left.\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\right|_{\alpha=1}\int_0^{\infty}\frac{\mathrm{d}x}{(1+x^4)^{\alpha}}\\
&=\left.\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\right|_{\alpha=1}\int_0^{\infty}\frac{\frac{1}{4}u^{-\frac{3}{4}}}{(1+u)^{\alpha}}\mathrm{d}u\\
&=\left.\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\right|_{\alpha=1}\mathrm{B}\left(\frac{1}{4},\alpha-\frac{1}{4}\right)\\
&=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\left[-\zeta(2)+\left(\gamma+\psi\left(\frac{3}{4}\right)\right)^2+\psi'\left(\frac{3}{4}\right)\right]\\
&=\frac{\sqrt{2}\,\pi}{4}\left[-\frac{\pi^2}{6}+\left(\frac{\pi}{2}-\ln 8\right)^2+\pi^2-8\mathbf{G}\right]\\
&=-2\sqrt{2}\,\pi\,\mathbf{G}+\frac{13\pi^3}{24\sqrt{2}}+\frac{\pi\ln^2 8}{2\sqrt{2}}-\frac{\pi^2\ln 8}{2\sqrt{2}}
\end{aligned}$$

□

Exercise 1.1.40: Find a closed form for this integral:

$$\mathcal{I}=\int_0^1\frac{\ln^2 x}{\sqrt{x^2-x+1}}\mathrm{d}x$$

Solution: We begin by transforming the integral via an Euler substitution of the first kind:

$$\sqrt{x^2-x+1}=x+t\implies x=\frac{1-t^2}{1+2t},$$

$$\implies \mathrm{d}x=\frac{(-2)\left(1+t+t^2\right)}{(1+2t)^2}\,\mathrm{d}t,$$

$$\implies \sqrt{x^2-x+1}=\frac{1-t^2}{1+2t}+t=\frac{1+t+t^2}{1+2t}.$$

Under the transformation, the integral $\mathcal{I}$ becomes

$$\begin{aligned}\mathcal{I} &= \int_0^1 \frac{\ln^2(x)}{\sqrt{x^2-x+1}}\,\mathrm{d}x \\ &= 2\int_0^1 \frac{\ln^2\left(\dfrac{1-t^2}{1+2t}\right)}{1+2t}\,\mathrm{d}t; \quad \left[\sqrt{x^2-x+1}=x+t\right] \\ &= 2\int_0^1 \frac{\left[\ln(1-t)+\ln(1+t)-\ln(1+2t)\right]^2}{1+2t}\,\mathrm{d}t.\end{aligned}$$

Next, using the algebraic identity

$$(a+b-c)^2 = a^2+2b^2-(a-b)^2+(a-c)^2-2bc,$$

with $a=\ln(1-t)\wedge b=\ln(1+t)\wedge c=\ln(1+2t)$, we may expand the integral $\mathcal{I}$ as a sum of five simpler logarithmic integrals:

$$\begin{aligned}\mathcal{I} = 2\int_0^1 \frac{\left[\ln(1-t)+\ln(1+t)-\ln(1+2t)\right]^2}{1+2t}\,\mathrm{d}t &= 2\int_0^1 \frac{\ln^2(1-t)}{1+2t}\,\mathrm{d}t + 4\int_0^1 \frac{\ln^2(1+t)}{1+2t}\,\mathrm{d}t \\ -2\int_0^1 \frac{\left[\ln(1-t)-\ln(1+t)\right]^2}{1+2t}\,\mathrm{d}t + 2\int_0^1 \frac{\left[\ln(1-t)-\ln(1+2t)\right]^2}{1+2t}\,\mathrm{d}t & \\ -4\int_0^1 \frac{\ln(1+t)\ln(1+2t)}{1+2t}\,\mathrm{d}t = 2\int_0^1 \frac{\ln^2(1-t)}{1+2t}\,\mathrm{d}t + 4\int_0^1 \frac{\ln^2(1+t)}{1+2t}\,\mathrm{d}t & \\ -2\int_0^1 \frac{\ln^2\left(\dfrac{1-t}{1+t}\right)}{1+2t}\,\mathrm{d}t + 2\int_0^1 \frac{\ln^2\left(\dfrac{1-t}{1+2t}\right)}{1+2t}\,\mathrm{d}t - 4\int_0^1 \frac{\ln(1+t)\ln(1+2t)}{1+2t}\,\mathrm{d}t &\end{aligned}$$

Now, we'll find it convenient to introduce the following two auxiliary functions:

$$J(z) := \int_0^1 \frac{\ln^2(y)}{1+zy}\,\mathrm{d}y; \quad z \geq -1,$$

and

$$H(a,c) := \int_0^1 \frac{\ln^2(1+ay)}{1+cy}\,\mathrm{d}y; \quad a \geq -1 \wedge c > -1.$$

As we shall see, the integral $\mathcal{I}$ can be expressed entirely in terms of the auxiliary functions J

and H, and hence, entirely in terms of elementary functions and the standard polylogarithms:

$$\begin{aligned}
\mathcal{I} &= 2\int_0^1 \frac{\ln^2(1-t)}{1+2t}\,\mathrm{d}t + 4\int_0^1 \frac{\ln^2(1+t)}{1+2t}\,\mathrm{d}t - 2\int_0^1 \frac{\ln^2\left(\frac{1-t}{1+t}\right)}{1+2t}\,\mathrm{d}t + 2\int_0^1 \frac{\ln^2\left(\frac{1-t}{1+2t}\right)}{1+2t}\,\mathrm{d}t \\
&\quad - \int_0^1 \frac{4\ln(1+t)\ln(1+2t)}{1+2t}\,\mathrm{d}t = 2\int_0^1 \frac{\ln^2(u)}{3-2u}\,\mathrm{d}u; \quad [1-t=u] + 4\,H(1,2) \\
&\quad - \int_0^1 \frac{4\ln^2(v)}{(3-v)(1+v)}\,\mathrm{d}v; \quad \left[\frac{1-t}{1+t}=v\right] + 2\int_0^1 \frac{\ln^2(w)}{1+2w}\,\mathrm{d}w; \quad \left[\frac{1-t}{1+2t}=w\right] \\
&\quad - \left(\Big[\ln(1+t)\ln^2(1+2t)\Big]_0^1 - \int_0^1 \frac{\ln^2(1+2t)}{1+t}\,\mathrm{d}t\right); \quad I.B.P.s \\
&= \frac{2}{3}\int_0^1 \frac{\ln^2(u)}{1-\frac{2}{3}u}\,\mathrm{d}u + 4\,H(1,2) - \frac{1}{3}\int_0^1 \frac{\ln^2(v)}{1-\frac{1}{3}v}\,\mathrm{d}v - \int_0^1 \frac{\ln^2(v)}{1+v}\,\mathrm{d}v + 2\,J(2) \\
&\quad - \ln(2)\ln^2(3) + H(2,1) \\
&= \frac{2}{3}\,J\left(-\frac{2}{3}\right) + 4\,H(1,2) - \frac{1}{3}\,J\left(-\frac{1}{3}\right) - J(1) + 2\,J(2) - \ln(2)\ln^2(3) + H(2,1)
\end{aligned}$$

The function H can be expressed in terms of J, dilogarithms, and elementary functions. Define $\gamma := \dfrac{a-c}{c}$. For $0 < a \wedge 0 < c$, we have $-1 < \gamma$ and

$$\begin{aligned}
H(a,c) &= \int_0^1 \frac{\ln^2(1+ay)}{1+cy}\,\mathrm{d}y = \int_{\frac{1}{1+a}}^1 \frac{\ln^2(x)}{x\left[c+(a-c)\,x\right]}\,\mathrm{d}x; \quad \left[\frac{1}{1+ay}=x\right] \\
&= \frac{1}{c}\int_{\frac{1}{1+a}}^1 \frac{\ln^2(x)}{x}\,\mathrm{d}x - \frac{a-c}{c}\int_{\frac{1}{1+a}}^1 \frac{\ln^2(x)}{c+(a-c)\,x}\,\mathrm{d}x = \frac{1}{3c}\ln^3(1+a) \\
&\quad - \frac{a-c}{c^2}\int_{\frac{1}{1+a}}^1 \frac{\ln^2(x)}{1+\left(\frac{a-c}{c}\right)x}\,\mathrm{d}x = \frac{1}{3c}\ln^3(1+a) - \frac{\gamma}{c}\int_{\frac{1}{1+a}}^1 \frac{\ln^2(x)}{1+\gamma x}\,\mathrm{d}x \\
&= \frac{1}{3c}\ln^3(1+a) - \frac{\gamma}{c}\int_0^1 \frac{\ln^2(x)}{1+\gamma x}\,\mathrm{d}x + \frac{\gamma}{c}\int_0^{\frac{1}{1+a}} \frac{\ln^2(x)}{1+\gamma x}\,\mathrm{d}x \\
&= \frac{1}{3c}\ln^3(1+a) - \frac{\gamma}{c}J(\gamma) + \frac{\gamma}{c\,(1+a)}\int_0^1 \frac{\ln^2\left(\frac{w}{1+a}\right)}{1+\left(\frac{\gamma}{1+a}\right)w}\,\mathrm{d}w; \quad [(1+a)\,x=w] \\
&= \frac{1}{3c}\ln^3(1+a) - \left(\frac{a-c}{c^2}\right)J\left(\frac{a-c}{c}\right) \\
&\quad + \frac{\gamma}{c\,(1+a)}\int_0^1 \frac{\ln^2(w) - 2\ln(w)\ln(1+a) + \ln^2(1+a)}{1+\left(\frac{\gamma}{1+a}\right)w}\,\mathrm{d}w \\
&= \frac{1}{3c}\ln^3(1+a) - \left(\frac{a-c}{c^2}\right)J\left(\frac{a-c}{c}\right) + \frac{\gamma}{c\,(1+a)}\int_0^1 \frac{\ln^2(w)}{1+\left(\frac{\gamma}{1+a}\right)w}\,\mathrm{d}w \\
&\quad - \frac{2}{c}\ln(1+a)\int_0^1 \frac{\left(\frac{\gamma}{1+a}\right)\ln(w)}{1+\left(\frac{\gamma}{1+a}\right)w}\,\mathrm{d}w + \frac{1}{c}\ln^2(1+a)\int_0^1 \frac{\left(\frac{\gamma}{1+a}\right)}{1+\left(\frac{\gamma}{1+a}\right)w}\,\mathrm{d}w
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3c}\ln^3(1+a) - \left(\frac{a-c}{c^2}\right) J\left(\frac{a-c}{c}\right) + \frac{\gamma}{c(1+a)} J\left(\frac{\gamma}{1+a}\right) \\
&\quad + \frac{2}{c}\ln(1+a)\int_0^1 \frac{\ln\left(1+\left(\frac{\gamma}{1+a}\right)w\right)}{w}\,\mathrm{d}w + \frac{1}{c}\ln^2(1+a)\ln\left(1+\left(\frac{\gamma}{1+a}\right)\right) \\
&= \frac{1}{3c}\ln^3(1+a) - \left(\frac{a-c}{c^2}\right) J\left(\frac{a-c}{c}\right) + \frac{a-c}{c^2(1+a)} J\left(\frac{a-c}{c(1+a)}\right) \\
&\quad - \frac{2}{c}\ln(1+a)\operatorname{Li}_2\left(-\left(\frac{\gamma}{1+a}\right)\right) + \frac{1}{c}\ln^2(1+a)\ln\left(\frac{a(1+c)}{(1+a)c}\right) \\
&= \frac{a-c}{c^2(1+a)} J\left(\frac{a-c}{c(1+a)}\right) - \frac{a-c}{c^2} J\left(\frac{a-c}{c}\right) - \frac{2}{c}\ln(1+a)\operatorname{Li}_2\left(\frac{c-a}{c(1+a)}\right) \\
&\quad + \frac{1}{3c}\ln^3(1+a) + \frac{1}{c}\ln^2(1+a)\ln\left(\frac{a(1+c)}{(1+a)c}\right)
\end{aligned}
$$

The function J reduces to the trilogarithm. For $-1 \le z \wedge z \neq 0$,

$$
\begin{aligned}
J(z) &= \int_0^1 \frac{\ln^2(y)}{1+zy}\,\mathrm{d}y \\
&= -\frac{2}{z}\operatorname{Li}_3(-z).
\end{aligned}
$$

Thus, continuing from where we left off

$$
\begin{aligned}
\mathcal{I} &= \frac{2}{3} J\left(-\frac{2}{3}\right) - \frac{1}{3} J\left(-\frac{1}{3}\right) - J(1) + 2J(2) \\
&\quad + 4H(1,2) + H(2,1) - \ln(2)\ln^2(3) \\
&= 2\operatorname{Li}_3\left(\frac{2}{3}\right) - 2\operatorname{Li}_3\left(\frac{1}{3}\right) + 2\operatorname{Li}_3(-1) - 2\operatorname{Li}_3(-2) \\
&\quad + 4\left[-\operatorname{Li}_3\left(\frac{1}{4}\right) + \operatorname{Li}_3\left(\frac{1}{2}\right) - \ln(2)\operatorname{Li}_2\left(\frac{1}{4}\right) - \frac{5}{6}\ln^3(2) + \frac{1}{2}\ln^2(2)\ln(3)\right] \\
&\quad + \left[2\operatorname{Li}_3(-1) - 2\operatorname{Li}_3\left(-\frac{1}{3}\right) - \ln(9)\operatorname{Li}_2\left(-\frac{1}{3}\right) - \frac{2}{3}\ln^3(3) + \ln(4)\ln^2(3)\right] \\
&\quad - \ln(2)\ln^2(3)
\end{aligned}
$$

So we get

$$
\begin{aligned}
\mathcal{I} &= 2\operatorname{Li}_3\left(\frac{2}{3}\right) - 2\operatorname{Li}_3\left(\frac{1}{3}\right) - 2\operatorname{Li}_3\left(-\frac{1}{3}\right) - 2\operatorname{Li}_3(-2) \\
&\quad - 4\operatorname{Li}_3\left(\frac{1}{4}\right) + 4\operatorname{Li}_3\left(\frac{1}{2}\right) + 4\operatorname{Li}_3(-1) \\
&\quad - 4\ln(2)\operatorname{Li}_2\left(\frac{1}{4}\right) - 2\ln(3)\operatorname{Li}_2\left(-\frac{1}{3}\right) \\
&\quad - \frac{10}{3}\ln^3(2) + 2\ln^2(2)\ln(3) + \ln(2)\ln^2(3) - \frac{2}{3}\ln^3(3).
\end{aligned}
$$

□

Exercise 1.1.41: Find a closed form for this integral:

$$
I = \int_0^\infty {}_3F_2\left(\left.\begin{matrix}\frac{5}{8},\frac{5}{8},\frac{9}{8}\\ \frac{1}{2},\frac{13}{8}\end{matrix}\right| -x\right)^2 \frac{\mathrm{d}x}{\sqrt{x}}
$$

✎ **Solution:** Looks like we have a more general result:

$$
\begin{aligned}
I(a) &= \int_0^\infty {}_3F_2\left(\begin{matrix} a, a, a+\frac{1}{2} \\ \frac{1}{2}, a+1 \end{matrix}\,\middle|\, -x\right)^2 \frac{\mathrm{d}x}{\sqrt{x}} \\
&= \frac{2^{4a-2}a^2}{\Gamma(2a)^2} G_{4\,4}^{3\,3}\left(1\,\middle|\, \begin{matrix} \frac{1}{2}, 1, 1; 2a+\frac{1}{2} \\ 2a-\frac{1}{2}, 2a-\frac{1}{2}, 2a; 0 \end{matrix}\right) \\
&= \frac{2^{4a-2}a^2}{\Gamma(2a)^2} G_{6\,6}^{4\,4}\left(1\,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2}, 1, 1; 2a, 2a+\frac{1}{2} \\ 2a-\frac{1}{2}, 2a-\frac{1}{2}, 2a, 2a; 0, \frac{1}{2} \end{matrix}\right) \\
&= \frac{4\pi a^2}{\Gamma(2a)^2} G_{3\,3}^{2\,2}\left(1\,\middle|\, \begin{matrix} 1, 1; 4a \\ 4a-1, 4a-1; 0 \end{matrix}\right).
\end{aligned}
$$

Here Mathematica gives when $a = \frac{5}{8}$,

$$
\begin{aligned}
&G_{3\,3}^{2\,2}\left(z\,\middle|\, \begin{matrix} 1, 1; \frac{5}{2} \\ \frac{3}{2}, \frac{3}{2}; 0 \end{matrix}\right) \\
&= \frac{2\sqrt{\pi}}{3}\left(\ln(\sqrt{z}+\sqrt{z+1}) + z^{3/2}\ln(1+\sqrt{z+1}) - z^{3/2}\ln(\sqrt{z}) - \sqrt{z}\sqrt{z+1}\right).
\end{aligned}
$$

So

$$
G_{3\,3}^{2\,2}\left(1\,\middle|\, \begin{matrix} 1, 1; \frac{5}{2} \\ \frac{3}{2}, \frac{3}{2}; 0 \end{matrix}\right) = \frac{2\sqrt{\pi}}{3}(2\ln(1+\sqrt{2}) - \sqrt{2}).
$$

Now we have:

$$
\int_0^\infty {}_3F_2\left(\begin{matrix} \frac{5}{8}, \frac{5}{8}, \frac{9}{8} \\ \frac{1}{2}, \frac{13}{8} \end{matrix}\,\middle|\, -x\right)^2 \frac{\mathrm{d}x}{\sqrt{x}} = I\left(\frac{5}{8}\right) = \frac{50\,\pi^{3/2}}{3\,\Gamma^2\left(\frac{1}{4}\right)}\left(\ln\left(3+\sqrt{8}\right) - \sqrt{2}\right)
$$

□

Exercise 1.1.42: Find a closed form evaluation of the following improper definite integral involving logarithms:

$$
I = \int_0^1 \frac{1}{x}\ln(x)\ln^3(1-x)\,\mathrm{d}x
$$

✎ **Solution:** We could use the series expansion

$$
\frac{\ln(1-x)}{1-x} = -\sum_{n=1}^\infty H_n x^n,
$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is the harmonic number. Then we get

$$
I = \int_0^1 \frac{\ln(1-x)\ln^3 x}{1-x}\mathrm{d}x = -\sum_{n=1}^\infty H_n \int_0^1 x^n \ln^3 x\,\mathrm{d}x = 6\sum_{n=1}^\infty \frac{H_n}{(n+1)^4}.
$$

The last series could be evaluated using Euler's formula

$$\sum_{n=1}^{\infty} \frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3)$$

(see formula (20) here: http://mathworld.wolfram.com/HarmonicNumber.html):

$$I = 6\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4} = 6\sum_{n=1}^{\infty} \left(\frac{H_{n+1}}{(n+1)^4} - \frac{1}{(n+1)^5} \right) = 6(2\zeta(5) - \zeta(2)\zeta(3))$$

□

Exercise 1.1.43: Find a possible methods of computing the following integral:

$$\int_0^{\infty} \frac{\tanh(x)}{x\left[1 - 2\cosh(2x)\right]^2} \mathrm{d}x$$

Solution: A possible way I see of doing this is to apply the substitution $x \mapsto -\ln x$, which yields

$$-\int_0^1 \frac{x^3(1-x^2)}{(1+x^2)(1-x^2+x^4)^2} \frac{\mathrm{d}x}{\ln(x)}$$

So

$$\begin{aligned}
\mathcal{I} &= -\int_0^1 \frac{x^3(1-x^2)}{(1+x^2)(1-x^2+x^4)^2} \frac{\mathrm{d}x}{\ln x} = -\int_0^1 \frac{x^3(1-x^2)}{(1+x^6)(1-x^2+x^4)} \frac{\mathrm{d}x}{\ln x} \\
&= -\int_0^1 \frac{x^3(1-x^4)}{(1+x^6)^2} \frac{\mathrm{d}x}{\ln x} = -\int_0^1 \frac{z(1-z^{4/3})}{(1+z^2)^2} \frac{\mathrm{d}z}{3z^{2/3}\ln(z^{1/3})} = \int_0^1 \mathrm{d}z \frac{1}{(1+z^2)^2} \frac{z^{5/3}-z^{1/3}}{\ln z} \\
&= \int_0^1 \mathrm{d}z \frac{1}{(1+z^2)^2} \int_{1/3}^{5/3} \mathrm{d}\mu\, z^{\mu} = \int_{1/3}^{5/3} \mathrm{d}\mu \int_0^1 \mathrm{d}z \frac{z^{\mu}}{(1+z^2)^2} = \int_{1/3}^{5/3} \mathrm{d}\mu \left[-\frac{1}{4} + \frac{\mu-1}{4}\beta\left(\frac{\mu-1}{2}\right) \right] \\
&= -\frac{1}{3} + \int_{1/3}^{5/3} \mathrm{d}\mu \left[\frac{\mu-1}{4}\beta\left(\frac{\mu-1}{2}\right) \right] = -\frac{1}{3} + \int_{-1/3}^{1/3} \mathrm{d}t\, t\beta(t) \\
&= -\frac{1}{3} + \int_{-1/3}^{1/3} \mathrm{d}t\, \frac{t}{2} \left[\psi\left(\frac{t+1}{2}\right) - \psi\left(\frac{t}{2}\right) \right] = -\frac{1}{3} + \int_{-1/3}^{1/3} \mathrm{d}t\, \frac{t}{2}\psi\left(\frac{t+1}{2}\right) - \int_{-1/3}^{1/3} \mathrm{d}t\, \frac{t}{2}\psi\left(\frac{t}{2}\right) \\
&= -\frac{1}{3} + \int_{1/3}^{2/3} \mathrm{d}u\, (2u-1)\psi(u) - 2\int_{-1/6}^{1/6} \mathrm{d}u\, u\psi(u) \\
&= -\frac{1}{3} - \int_{1/3}^{2/3} \mathrm{d}u\, \psi(u) + 2\int_{1/3}^{2/3} \mathrm{d}u\, u\psi(u) - 2\int_{-1/6}^{1/6} \mathrm{d}u\, u\psi(u) \\
&= -\frac{1}{3} + \ln\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \right) + 2\int_{1/3}^{2/3} \mathrm{d}u\, u\psi(u) - 2\int_{-1/6}^{1/6} \mathrm{d}u\, u\psi(u) \\
&= -\frac{1}{3} + \ln\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \right) + 2\int_{1/3}^{2/3} \mathrm{d}u\, u\psi(u) - 2\int_{5/6}^{7/6} \mathrm{d}v\, (1-v)\psi(1-v)
\end{aligned}$$

$$= -\frac{1}{3} + \ln\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right) + 2\left[u\ln\Gamma(u) - \psi^{(-2)}(u)\right]_{1/3}^{2/3}$$
$$+ 2\left[(1-v)\ln\Gamma(1-v) - \psi^{(-2)}(1-v)\right]_{5/6}^{7/6}$$
$$= \ln\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right) - \frac{5\pi}{9\sqrt{3}} - \frac{\ln(2\pi)}{3} + \frac{2}{3}\ln\left(\frac{\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{1}{3}\right)}\right) + \frac{5\psi^{(1)}\left(\frac{1}{3}\right)}{6\sqrt{3}\,\pi}$$
$$= -\frac{5\pi}{9\sqrt{3}} - \frac{\ln 3}{6} + \frac{5\psi^{(1)}\left(\frac{1}{3}\right)}{6\sqrt{3}\,\pi}$$

□

Exercise 1.1.44: Prove:

$$I = \int_0^{\pi} \ln^2\left(\tan\frac{x}{4}\right)\mathrm{d}x = \frac{\pi^3}{4}.$$

Proof: Let $u = \tan\frac{x}{4}$, then $\mathrm{d}u = \frac{1}{4}\sec^2\frac{x}{4}\mathrm{d}x = \frac{1}{4}(1+u^2)\mathrm{d}x$. The integral becomes

$$\int_0^{\pi} \ln^2\left(\tan\frac{x}{4}\right)\mathrm{d}x = 4\int_0^1 \frac{\ln^2 u}{1+u^2}\mathrm{d}u = 4\sum_{n\geq 0}(-1)^n\int_0^1 x^{2n}\ln^2 x\mathrm{d}x$$
$$= 4\sum_{n\geq 0}(-1)^n \lim_{a\to 2n}\frac{\mathrm{d}^2}{\mathrm{d}a^2}\int_0^1 x^a\mathrm{d}x = 4\sum_{n\geq 0}(-1)^n\lim_{a\to 2n}\frac{\mathrm{d}}{\mathrm{d}a}\left(-\frac{1}{(a+1)^2}\right) = 8\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^3}$$
$$= 8\beta(3) = \frac{\pi^3}{4}$$

□

Exercise 1.1.45: Prove:

$$I = \int_0^{\pi/4}\frac{\ln\tan x}{\cos 2x}\mathrm{d}x = -\frac{\pi^2}{8}$$

Proof: **Method One:**

Introduce variables t and y such that $t = \tan x = e^{-y}$, we have

$$\int_0^{\pi/4}\frac{\ln\tan x}{\cos 2x}\mathrm{d}x = \int_0^1 \frac{\ln t}{\frac{1-t^2}{1+t^2}}\frac{\mathrm{d}t}{1+t^2}$$
$$= -\int_0^{\infty}\frac{y}{1-e^{-2y}}e^{-y}\mathrm{d}y = -\int_0^{\infty}\sum_{k=0}^{\infty} y e^{-(2k+1)y}\mathrm{d}y$$
$$= -\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = -\left[\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right]$$
$$= -\left(1-\frac{1}{4}\right)\zeta(2) = -\frac{\pi^2}{8}$$

Method Two:

Let $\tan x = e^{-t}$, then

$$\cos 2x = \cos^2 x - \sin^2 x = \frac{1-e^{-2t}}{1+e^{-2t}}$$

and

$$\mathrm{d}x = -\frac{e^{-t}\,\mathrm{d}t}{1+e^{-2t}}.$$

Therefore

$$\int_0^{\pi/4} \frac{\ln\tan x}{\cos 2x}\mathrm{d}x = -\int_\infty^0 \frac{\ln e^{-t}}{\frac{1-e^{-2t}}{1+e^{-2t}}}\cdot\frac{e^{-t}\,\mathrm{d}t}{1+e^{-2t}} = -\int_0^\infty \frac{te^{-t}}{1-e^{-2t}}\mathrm{d}t$$

It can be solved by using IBP. Let $u = t \;\to \mathrm{d}u = \mathrm{d}t$ and $\mathrm{d}v = \dfrac{e^{-t}}{1-e^{-2t}}\mathrm{d}t$

$$\begin{aligned} v &= \int \frac{e^{-t}}{1-e^{-2t}}\mathrm{d}t = -\int \frac{\mathrm{d}\left(e^{-t}\right)}{1-e^{-2t}} \quad \Rightarrow \quad y = e^{-t} \\ &= -\int \frac{\mathrm{d}y}{1-y^2} = -\frac{1}{2}\left(\int \frac{\mathrm{d}y}{1-y} + \int \frac{\mathrm{d}y}{1+y}\right) \\ &= \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \frac{1}{2}\ln\left(1-e^{-t}\right) - \frac{1}{2}\ln\left(1+e^{-t}\right) \end{aligned}$$

Hence

$$\begin{aligned} -\int_0^\infty \frac{te^{-t}}{1-e^{-2t}}\mathrm{d}t &= \left[\frac{t}{2}\ln\left(1-e^{-t}\right) + \frac{t}{2}\ln\left(1+e^{-t}\right)\right]_0^\infty \\ &\quad \frac{1}{2}\int_0^\infty \ln\left(1-e^{-t}\right)\,\mathrm{d}t - \frac{1}{2}\int_0^\infty \ln\left(1+e^{-t}\right)\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty \ln\left(1-e^{-t}\right)\,\mathrm{d}t - \frac{1}{2}\int_0^\infty \ln\left(1+e^{-t}\right)\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^1 \ln(1-y)\,\mathrm{d}y - \frac{1}{2}\int_0^1 \ln(1+y)\,\mathrm{d}y \end{aligned}$$

Since $|y| < 1$, we can use Maclaurin series for natural logarithm.

$$\begin{aligned} -\int_0^\infty \frac{te^{-t}}{1-e^{-2t}}\mathrm{d}t &= \frac{1}{2}\int_0^1 \ln(1-y)\,\mathrm{d}y - \frac{1}{2}\int_0^1 \ln(1+y)\,\mathrm{d}y \\ &= -\frac{1}{2}\int_0^1 \sum_{n=1}^\infty \frac{y^n}{n}\,\mathrm{d}y - \frac{1}{2}\int_0^1 \sum_{n=1}^\infty (-1)^{n+1}\frac{y^n}{n}\,\mathrm{d}y \\ &= -\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n^2} \\ &= -\frac{1}{2}\zeta(2) - \frac{1}{2}\eta(2) \\ &= -\frac{1}{2}\zeta(2) - \frac{1}{4}\zeta(2) \\ &= -\frac{\pi^2}{8} \end{aligned}$$

where $\zeta(s)$ is Riemann zeta function, $\eta(s)$ is Dirichlet eta function, and $\zeta(2) = \dfrac{\pi^2}{6}$. □

Exercise 1.1.46: Prove:

$$\int_0^1 \ln\frac{\left(x+\sqrt{1-x^2}\right)^2}{\left(x-\sqrt{1-x^2}\right)^2}\frac{x\,\mathrm{d}x}{1-x^2} = \frac{\pi^2}{2}$$

☞ **Proof:** With $x=\sin\theta$:

$$\int_0^1 \ln\frac{\left(x+\sqrt{1-x^2}\right)^2}{\left(x-\sqrt{1-x^2}\right)^2}\frac{x\,\mathrm{d}x}{1-x^2} = -\int_0^{\pi/2}\ln\left(\left[\frac{\tan\theta-1}{\tan\theta+1}\right]^2\right)\tan\theta\mathrm{d}\theta$$
$$= -\int_0^{\pi/2}\ln\left(\tan^2\left(\theta-\frac{\pi}{4}\right)\right)\tan\theta\mathrm{d}\theta = -\int_{-\pi/4}^{\pi/4}\ln\left(\tan^2\theta\right)\tan\left(\theta+\frac{\pi}{4}\right)\mathrm{d}\theta$$
$$= -\int_0^{\pi/4}\ln\left(\tan^2\theta\right)\left[\frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}\right]\mathrm{d}\theta$$
$$= -2\int_0^{\pi/4}\ln\left(\tan\theta\right)\frac{2\sec^2\theta}{1-\tan^2\theta}\mathrm{d}\theta = -4\overbrace{\int_0^1\frac{\ln t}{1-t^2}\mathrm{d}t}^{-\frac{\pi^2}{8}} = \frac{\pi^2}{2}$$

□

Exercise 1.1.47: Prove that:

$$\int_0^1\frac{\ln(1-x^2)}{x}\mathrm{d}x = -\frac{\pi^2}{12}$$

without using series expansion.

An easy way to calculate the above integral is using series expansion. Here is an example:

$$\int_0^1\frac{\ln(1-x^2)}{x}\mathrm{d}x = -\int_0^1\frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2n}}{n}\mathrm{d}x = -\sum_{n=0}^{\infty}\frac{1}{n}\int_0^1 x^{2n-1}\mathrm{d}x$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{12}$$

☞ **Proof:** Substitute $u=x^2$. Then,

$$\int_0^1\frac{\ln\left(1-x^2\right)}{x}\mathrm{d}x = \int_0^1\frac{\ln\left(1-u\right)}{\sqrt{u}}\cdot\frac{\mathrm{d}u}{2\sqrt{u}} = \frac{1}{2}\int_0^1\frac{\ln\left(1-u\right)}{u}\mathrm{d}u$$
$$= -\frac{1}{2}\operatorname{Li}_2\left(u\right)\Big|_0^1 = -\frac{1}{2}\operatorname{Li}_2\left(1\right) = -\frac{\pi^2}{12}$$

I fully anticipate there will be a not small number of people who call this cheating, but it's certainly cheating with style!

Edit: Here's a slightly more satisfying result. Substitute $u=x^2$ first like before, and next substitute $u=1-e^{-w}$. Then,

$$\int_0^1\frac{\ln\left(1-x^2\right)}{x}\mathrm{d}x = \int_0^1\frac{\ln\left(1-u\right)}{\sqrt{u}}\cdot\frac{\mathrm{d}u}{2\sqrt{u}} = \frac{1}{2}\int_0^1\frac{\ln\left(1-u\right)}{u}\mathrm{d}u = \frac{1}{2}\int_0^\infty\frac{-w}{1-e^{-w}}(e^{-w})\mathrm{d}w$$
$$= -\frac{1}{2}\int_0^\infty\frac{w}{e^w-1}\mathrm{d}w = -\frac{1}{2}\Gamma(2)\zeta(2) = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}$$

□

Exercise 1.1.48: Evaluate the integral:

$$\int_0^1\frac{\ln x\ \operatorname{Li}_2(x)}{1-x}\mathrm{d}x$$

✎ **Solution:** Applying the reflection substitution about the interval $[0,1]$,$x \mapsto 1-x$, and using Euler's dilogarithm identity,

$$\operatorname{Li}_2(1-x) = \zeta(2) - \operatorname{Li}_2(x) - \ln(1-x)\ln(x), \quad \text{(Euler)},$$

we can split up the integral into a sum of three integrals, the first two of which have simple anti-derivatives in terms of the dilogarithm function:

$$\begin{aligned}
&\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{1-x}\mathrm{d}x = \int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(1-x)}{x}\mathrm{d}x \\
&= \zeta(2)\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x - \int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x - \int_0^1 \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x \\
&= \zeta(2)\left[-\operatorname{Li}_2(1)\right] - \left[-\frac{1}{2}\operatorname{Li}_2(1)^2\right] - \int_0^1 \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x \\
&= -\frac{1}{2}\zeta(2)^2 - \int_0^1 \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x = -\frac{\pi^4}{72} - \int_0^1 \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x
\end{aligned}$$

The last integral can evaluated as derivatives of a beta function.

$$\begin{aligned}
&\int_0^1 \frac{\ln^2(1-x)\ln(x)}{x}\mathrm{d}x = \lim_{\mu\to 0}\lim_{\nu\to 1}\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\ln^2(1-x)\ln(x)\mathrm{d}x \\
&= \lim_{\mu\to 0}\lim_{\nu\to 1}\frac{\partial}{\partial\mu}\frac{\partial^2}{\partial\nu^2}\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\mathrm{d}x = \lim_{\mu\to 0}\lim_{\nu\to 1}\frac{\partial}{\partial\mu}\frac{\partial^2}{\partial\nu^2}\mathrm{B}(\mu,\nu) \\
&= \lim_{\mu\to 0}\lim_{\nu\to 1}\frac{\partial}{\partial\mu}\mathrm{B}(\mu,\nu)\left[(\psi(\nu)-\psi(\mu+\nu))^2 + \psi^{(1)}(\nu) - \psi^{(1)}(\mu+\nu)\right] \\
&= \lim_{\mu\to 0}\lim_{\nu\to 1}\mathrm{B}(\mu,\nu)[(\psi(\mu)-\psi(\mu+\nu))\left((\psi(\nu)-\psi(\mu+\nu))^2 - \psi^{(1)}(\mu+\nu) + \psi^{(1)}(\nu)\right) \\
&- 2(\psi(\nu)-\psi(\mu+\nu))\psi^{(1)}(\mu+\nu) - \psi^{(2)}(\mu+\nu)] \\
&= \lim_{\mu\to 0}\mathrm{B}(\mu,1)[(\psi(\mu)-\psi(\mu+1))\left((\psi(1)-\psi(\mu+1))^2 - \psi^{(1)}(\mu+1) + \psi^{(1)}(1)\right) \\
&- 2(\psi(1)-\psi(\mu+1))\psi^{(1)}(\mu+1) - \psi^{(2)}(\mu+1)] \\
&= \lim_{\mu\to 0}\frac{1}{\mu}\left[-\frac{1}{\mu}\left(H_\mu^2 - \psi^{(1)}(\mu+1) + \psi^{(1)}(1)\right) + 2H_\mu\psi^{(1)}(\mu+1) - \psi^{(2)}(\mu+1)\right] \\
&= \lim_{\mu\to 0}\left[\frac{2H_\mu\psi^{(1)}(\mu+1)}{\mu} - \frac{H_\mu^2 - \psi^{(1)}(\mu+1) + \zeta(2)}{\mu^2} - \frac{\psi^{(2)}(\mu+1)}{\mu}\right] \\
&= \lim_{\mu\to 0}\left[\frac{2H_\mu\psi^{(1)}(\mu+1)}{\mu} - \frac{H_\mu^2 - \psi^{(1)}(\mu+1) + \zeta(2) + \mu\,\psi^{(2)}(\mu+1)}{\mu^2}\right] \\
&= 2\zeta^2(2) - \frac{11\pi^4}{180} = -\frac{\pi^4}{180}
\end{aligned}$$

Hence, the integral come to a value of:

$$\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{1-x}\mathrm{d}x = \frac{\pi^4}{180} - \frac{\pi^4}{72} = -\frac{\pi^4}{120}$$

Appendix

The most common integral representation for the dilogarithm function is,

$$\operatorname{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\mathrm{d}t.$$

Hence, the value of the first integral is:

$$\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x = -\operatorname{Li}_2(1).$$

Note that the integral representation implies that $\operatorname{Li}_2(0) = 0$. The value of the dilogarithm function at $z = 1$ is given by the zeta function: $\operatorname{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$.
The second integral may be found readily via integration by parts:

$$\begin{aligned}\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x &= -\operatorname{Li}_2(x)^2\Big|_0^1 - \int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x \\ &= 2\int_0^1 \frac{\ln(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x = -\operatorname{Li}_2(1)^2.\end{aligned}$$

□

Exercise 1.1.49: Find a closed form solution to this integral:

$$\int_{-\infty}^{\infty} \frac{2ie^{-ibx}(bx-i)\left[a(-ix)^a\,\Gamma(-a,-ix)\right]^n}{x^2+\epsilon^2}\mathrm{d}x$$

Given $n \in \mathbb{N}^+ \wedge a, b, \epsilon \in \mathbb{R} \wedge 1 < a \wedge 1 \leq b \leq n \wedge 0 < \epsilon$.

Solution: We may use the following integral representation for the incomplete gamma function:

$$\Gamma(\alpha, z) = \frac{z^\alpha e^{-z}}{\Gamma(1-\alpha)}\int_0^\infty \frac{t^{-\alpha}e^{-t}}{z+t}\,\mathrm{d}t; \quad |\arg(z)| < \pi,\ \Re(\alpha) < 1.$$

Define $I_n(a, b; \epsilon)$ via the integral

$$I_n(a,b;\epsilon) = \int_{-\infty}^{\infty} \mathrm{d}x\,\frac{2ie^{-ibx}(bx-i)\left[a(-ix)^a\,\Gamma(-a,-ix)\right]^n}{x^2+\epsilon^2}.$$

Then,

$$\begin{aligned}I_n(a,b;\epsilon) &= \int_{-\infty}^{\infty} \mathrm{d}x\,\frac{2ie^{-ibx}(bx-i)\left[a(-ix)^a\,\Gamma(-a,-ix)\right]^n}{x^2+\epsilon^2} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\,\frac{2ie^{-ibx}(bx-i)}{x^2+\epsilon^2}\left[\frac{e^{ix}}{\Gamma(a)}\int_0^\infty \frac{t^a e^{-t}}{t-ix}\,\mathrm{d}t\right]^n \\ &= \frac{1}{[\Gamma(a)]^n}\int_{-\infty}^{\infty} \mathrm{d}x\,\frac{2ie^{-ibx}(bx-i)}{x^2+\epsilon^2}e^{inx}\left[\int_0^\infty \frac{t^a e^{-t}}{t-ix}\,\mathrm{d}t\right]^n\end{aligned}$$

$$
\begin{aligned}
&= \frac{2i}{[\Gamma(a)]^n}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(bx-i)\,e^{i(n-b)x}}{x^2+\epsilon^2}\left[\int_0^{\infty}\frac{t^a e^{-t}}{t-ix}\,\mathrm{d}t\right]^n \\
&= \frac{2i}{[\Gamma(a)]^n}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(bx-i)\,e^{i(n-b)x}}{x^2+\epsilon^2}\prod_{k=1}^{n}\int_0^{\infty}\mathrm{d}t_k\,\frac{t_k^a e^{-t_k}}{t_k-ix} \\
&= \frac{2i}{[\Gamma(a)]^n}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(bx-i)\,e^{i(n-b)x}}{x^2+\epsilon^2}\int_{[0,\infty)^n}\mathrm{d}^n\mathbf{t}\,e^{-\sum_{k=1}^{n}t_k}\prod_{k=1}^{n}\left(\frac{t_k^a}{t_k-ix}\right) \\
&= \frac{2i}{[\Gamma(a)]^n}\int_{[0,\infty)^n}\mathrm{d}^n\mathbf{t}\left(\prod_{k=1}^{n}t_k^a\right)e^{-\sum_{k=1}^{n}t_k}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(bx-i)\,e^{i(n-b)x}}{(x^2+\epsilon^2)\prod_{k=1}^{n}(t_k-ix)} \\
&= \frac{2i}{[\Gamma(a)]^n}\int_{[0,\infty)^n}\mathrm{d}^n\mathbf{t}\,\frac{\left(\prod_{k=1}^{n}t_k^a\right)}{\exp\left(\sum_{k=1}^{n}t_k\right)}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(-bx-i)\,e^{-i(n-b)x}}{(x^2+\epsilon^2)\prod_{k=1}^{n}(t_k+ix)} \\
&=: \frac{2i}{[\Gamma(a)]^n}\int_{[0,\infty)^n}\mathrm{d}^n\mathbf{t}\,\frac{\left(\prod_{k=1}^{n}t_k^a\right)}{\exp\left(\sum_{k=1}^{n}t_k\right)}f_n(b;\epsilon),
\end{aligned}
$$

where for $n\in\mathbb{N}^+\wedge b,\epsilon\in\mathbb{R}\wedge 1\le b\le n\wedge 0<\epsilon$ we've defined the auxiliary function denoting the innermost integration,

$$f_n(b;\epsilon):=\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(-bx-i)\,e^{-i(n-b)x}}{(x^2+\epsilon^2)\prod_{k=1}^{n}(t_k+ix)}.$$

For the integration over x, I will appeal to a well-suited proposition from Gradshteyn's Tables. Now, I usually try to make my posts as self-contained as possible by proving any non-trivial lemmas I plan on using, rather than taking the lazy way out and simply citing the result from an outside source. But I can't resist doing so, partly out of laziness, but mostly just to show off how Gradshteyn can on occasion be eerily clairvoyant.

Gradshteyn 3.386: Given the conditions

$$-1<\Re(\nu_0)\wedge 0<\Re(\beta_k)\wedge\sum_{k=0}^{n}\Re(\nu_k)<1\wedge 0<p,$$

we have the following two results:

$$\int_{-\infty}^{\infty}\frac{(ix)^{\nu_0}\,e^{-ipx}\prod_{k=1}^{n}(\beta_k+ix)^{\nu_k}}{\beta_0-ix}\mathrm{d}x=2\pi e^{-\beta_0 p}\beta_0^{\nu_0}\prod_{k=1}^{n}(\beta_0+\beta_k)^{\nu_k},$$

and

$$\int_{-\infty}^{\infty}\frac{(ix)^{\nu_0}\,e^{-ipx}\prod_{k=1}^{n}(\beta_k+ix)^{\nu_k}}{\beta_0+ix}\mathrm{d}x=0.$$

The following partial fraction decomposition is easily verified:

$$\frac{-bx-i}{x^2+\epsilon^2}=\frac{1}{2i\epsilon}\left[\frac{(1-b\epsilon)}{\epsilon-ix}+\frac{(1+b\epsilon)}{\epsilon+ix}\right].$$

Then, assuming $0 < \Re(t_k) \wedge b < n$,

$$\begin{aligned} f_n(b;\epsilon) &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{(-bx - i)\, e^{-i(n-b)x}}{(x^2+\epsilon^2)\prod_{k=1}^{n}(t_k + ix)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{2i\epsilon}\left[\frac{(1-b\epsilon)}{\epsilon - ix} + \frac{(1+b\epsilon)}{\epsilon + ix}\right] \frac{e^{-i(n-b)x}}{\prod_{k=1}^{n}(t_k + ix)} \\ &= \frac{1-b\epsilon}{2i\epsilon}\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{-i(n-b)x}}{(\epsilon - ix)\prod_{k=1}^{n}(t_k + ix)} \\ &+ \frac{1+b\epsilon}{2i\epsilon}\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{-i(n-b)x}}{(\epsilon + ix)\prod_{k=1}^{n}(t_k + ix)} \\ &= \frac{1-b\epsilon}{2i\epsilon} \cdot \frac{2\pi e^{-(n-b)\epsilon}}{\prod_{k=1}^{n}(\epsilon + t_k)}. \end{aligned}$$

Thus,

$$\begin{aligned} I_n(a,b;\epsilon) &= \frac{2i}{[\Gamma(a)]^n}\int_{[0,\infty)^n} \mathrm{d}^n\mathbf{t} \, \frac{(\prod_{k=1}^{n} t_k^a)}{\exp(\sum_{k=1}^{n} t_k)}\, f_n(b;\epsilon) \\ &= \frac{2\pi\,(1-b\epsilon)\, e^{-(n-b)\epsilon}}{[\Gamma(a)]^n\,\epsilon}\int_{[0,\infty)^n} \mathrm{d}^n\mathbf{t} \, \frac{(\prod_{k=1}^{n} t_k^a)}{\exp(\sum_{k=1}^{n} t_k)} \cdot \frac{1}{\prod_{k=1}^{n}(\epsilon + t_k)} \\ &= \frac{2\pi\,(1-b\epsilon)\, e^{-(n-b)\epsilon}}{[\Gamma(a)]^n\,\epsilon}\int_{[0,\infty)^n} \mathrm{d}^n\mathbf{t} \, \prod_{k=1}^{n}\left(\frac{t_k^a e^{-t_k}}{\epsilon + t_k}\right) \\ &= \frac{2\pi\,(1-b\epsilon)\, e^{-(n-b)\epsilon}}{[\Gamma(a)]^n\,\epsilon}\left[\int_0^{\infty} \mathrm{d}t \, \left(\frac{t^a e^{-t}}{\epsilon + t}\right)\right]^n \\ &= \frac{2\pi\,(1-b\epsilon)\, e^{-(n-b)\epsilon}}{[\Gamma(a)]^n\,\epsilon}\left[\epsilon^a e^{\epsilon}\,\Gamma(1+a)\,\Gamma(-a,\epsilon)\right]^n \\ &= 2\pi a^n\,(1-b\epsilon)\,\epsilon^{na-1}e^{b\epsilon}\left[\Gamma(-a,\epsilon)\right]^n \end{aligned}$$

□

Exercise 1.1.50: Evaluate the following definite integral:

$$\int_0^1 \frac{\ln x \, \mathrm{d}x}{\sqrt[3]{x(1-x^2)^2}}$$

Solution: We have:

$$I = \frac{1}{4}\int_0^1 z^{-2/3}(1-z)^{-2/3}\ln z \, \mathrm{d}z$$

hence:

$$I = \frac{1}{4}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\int_0^1 z^{-2/3+\alpha}(1-z)^{-2/3}\,\mathrm{d}z\right)\Big|_{\alpha=0} = \frac{\Gamma\left(\frac{1}{3}\right)}{4}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\Gamma\left(\frac{1}{3}+\alpha\right)}{\Gamma\left(\frac{2}{3}+\alpha\right)}\right)\Big|_{\alpha=0}$$

and by using the identity $\Gamma' = \Gamma \cdot \frac{\mathrm{d}}{\mathrm{d}z}(\ln\Gamma) = \Gamma\cdot\psi$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\Gamma\left(\frac{1}{3}+\alpha\right)}{\Gamma\left(\frac{2}{3}+\alpha\right)}\right)\Big|_{\alpha=0} = \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\left(\psi\left(\frac{1}{3}\right) - \psi\left(\frac{2}{3}\right)\right)$$

so:

$$I=\frac{\Gamma\left(\frac{1}{3}\right)^3}{8}\cdot\frac{\sqrt{3}}{\pi}\left(\psi\left(\frac{1}{3}\right)-\psi\left(\frac{2}{3}\right)\right)$$

now we have:

$$\int_0^1\frac{\ln x\,\mathrm{d}x}{\sqrt[3]{x(1-x^2)^2}}=-\frac{1}{8}\left[\Gamma\left(\frac{1}{3}\right)\right]^3$$

□

Exercise 1.1.51: Sophomore's dream:

$$\int_0^1 x^{-x}\mathrm{d}x=\sum_{n=1}^{\infty}n^{-n}$$

Proof: Method One:

Here's a bird's-eye view of the chain of arguments for how you get from the integral to the series:

$$\int_0^1 x^{-x}\mathrm{d}x=\int_0^1\exp\left(-x\ln x\right)\mathrm{d}x=\int_0^1\sum_{n=0}^{\infty}\frac{(-x\ln x)^n}{n!}\mathrm{d}x$$
$$=\sum_{n=0}^{\infty}\int_0^1\frac{(-x\ln x)^n}{n!}\mathrm{d}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^n\left(\ln x\right)^n\mathrm{d}x$$
$$\cdots$$
$$=\sum_{n=0}^{\infty}(n+1)^{-(n+1)}=\sum_{n=1}^{\infty}n^{-n}.$$

Thus, to fill in the missing blanks in the middle of the proof, it will suffice to prove:

$$\int_0^1 x^n\left(\ln x\right)^n\mathrm{d}x=(-1)^n(n+1)^{-(n+1)}n!.$$

You can still figure out this integral by integrating by parts iteratively, as follows:

$$\int_0^1 x^n\left(\ln x\right)^n\mathrm{d}x=\frac{1}{n+1}x^{n+1}\left(\ln x\right)^n\Big|_0^1-\int_0^1\frac{1}{n+1}x^{n+1}\frac{n\left(\ln x\right)^{n-1}}{x}\mathrm{d}x$$
$$=-\frac{n}{n+1}\int_0^1 x^n\left(\ln x\right)^{n-1}\mathrm{d}x$$
$$=(-1)^2\frac{n(n-1)}{(n+1)^2}\int_0^1 x^n\left(\ln x\right)^{n-2}\mathrm{d}x$$
$$=(-1)^3\frac{n(n-1)(n-2)}{(n+1)^3}\int_0^1 x^n\left(\ln x\right)^{n-3}\mathrm{d}x$$
$$\cdots$$
$$=(-1)^n\frac{n!}{(n+1)^n}\int_0^1 x^n\left(\ln x\right)^{n-n}\mathrm{d}x$$
$$=(-1)^n\frac{n!}{(n+1)^{n+1}}.$$

However, if you know that $n! = \Gamma(n+1) = \int_0^\infty u^n e^{-u} \mathrm{d}u$, you can see that the identity we are trying to prove is:

$$\begin{aligned}\int_0^1 x^n \left(\ln x\right)^n \mathrm{d}x &= (-1)^n (n+1)^{-(n+1)} \int_0^\infty u^n e^{-u} \mathrm{d}u \\ &= (n+1)^{-1} \int_0^\infty \frac{(-1)^n u^n}{(n+1)^n} e^{-u} \mathrm{d}u \\ &= (n+1)^{-1} \int_0^\infty \left(\frac{-u}{(n+1)}\right)^n e^{-u} \mathrm{d}u\end{aligned}$$

This last line strongly suggests the substitution,

$$\ln x = \frac{-u}{(n+1)},$$

or equivalently,

$$x = \exp\left(\frac{-u}{n+1}\right).$$

Method Two:

$$\begin{aligned}\int_0^1 x^{-x} dx &= \int_0^\infty \left(e^{-t}\right)^{-e^{-t}} e^{-t}\, dt = \int_0^\infty e^{te^{-t}} e^{-t}\, dt \\ &= \int_0^\infty \left[\sum_{n=0}^\infty \frac{1}{n!} t^n e^{-nt}\right] e^{-t}\, dt \\ &= \sum_{n=0}^\infty \frac{1}{n!} \int_0^\infty t^n e^{-(n+1)t} dt \\ &= \sum_{n=1}^\infty \frac{1}{(n-1)!} \int_0^\infty e^{-nt} t^{n-1}\, dt \\ &= \sum_{n=1}^\infty \frac{1}{(n-1)!} n^{-n} \int_0^\infty e^{-s} s^{n-1}\, ds \\ &= \sum_{n=1}^\infty \frac{1}{(n-1)!} n^{-n} \Gamma(n) = \sum_{n=1}^\infty n^{-n}. \square\end{aligned}$$

□

Exercise 1.1.52: How to find:

$$\int_0^\infty \frac{x}{1+e^x} \mathrm{d}x$$

Solution: Method One:

One way to work this one out is to rewrite the integral as

$$\int_0^\infty \frac{x\, e^{-x}}{1+e^{-x}} \mathrm{d}x$$

then expand the denominator in a Taylor series. The representation of this integral becomes

$$\int_0^\infty x \sum_{k=0}^\infty (-1)^k e^{-(k+1)x} \mathrm{d}x$$

Because the sum and integral converge, we can change the order of summation and integration:

$$\sum_{k=0}^\infty (-1)^k \int_0^\infty x\, e^{-(k+1)x} \mathrm{d}x = \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2}$$

So,

$$\int_0^\infty \frac{x}{1+e^x}\mathrm{d}x = \frac{\pi^2}{12}$$

Method Two:

Another way to work this, which you may or may not be aware of, is to use Cauchy's theorem. Consider the contour integral

$$\oint_C \frac{z^2}{1+e^z}\mathrm{d}z$$

where C is the rectangle with vertices 0, R, $R+i2\pi$, and $i2\pi$, with a semicircular segment of radius ϵ centered at $i\pi$ into the rectangle. The contour integral then becomes

$$\int_0^R \frac{x^2}{1+e^x}\mathrm{d}x + i\int_0^{2\pi} \frac{(R+iy)^2}{1+e^R e^{iy}}\mathrm{d}y + \int_R^0 \frac{(x+i2\pi)^2}{1+e^x}\mathrm{d}x$$
$$+i\int_{2\pi}^{\pi+\epsilon} \frac{(iy)^2}{1+e^{iy}}\mathrm{d}x + i\epsilon\int_{\pi/2}^{-\pi/2} e^{i\phi}\frac{(i\pi+\epsilon e^{i\phi})^2}{1-e^{\epsilon e^{i\phi}}}\mathrm{d}\phi + i\int_{\pi-\epsilon}^0 \frac{(iy)^2}{1+e^{iy}}\mathrm{d}y$$

As $R\to\infty$, the second integral vanishes. As $\epsilon\to 0$, the fifth integral becomes

$$i\epsilon\int_{\pi/2}^{-\pi/2} e^{i\phi}\frac{-\pi^2}{-\epsilon e^{i\phi}}\mathrm{d}\phi = -i\pi^3$$

The contour integral is then

$$-i4\pi\int_0^\infty \frac{x}{1+e^x}\mathrm{d}x + 4\pi^2\int_0^\infty \frac{\mathrm{d}x}{1+e^x} + iPV\int_0^{2\pi}\frac{y^2}{1+e^{iy}}\mathrm{d}y - i\pi^3$$

Note that PV denotes the Cauchy principal value. Also note that

$$iPV\int_0^{2\pi}\frac{y^2}{1+e^{iy}}\mathrm{d}y = i\frac{1}{2}\int_0^{2\pi} y^2\mathrm{d}y + \frac{1}{2}PV\int_0^{2\pi} y^2\tan\left(\frac{y}{2}\right)\mathrm{d}y$$
$$= i\frac{4\pi^3}{3} + \frac{1}{2}PV\int_0^{2\pi} y^2\tan\left(\frac{y}{2}\right)\mathrm{d}y$$

By Cauchy's theorem, the contour integral is zero, which means that both the real and imaginary parts are zero. From the imaginary part of the contour integral being zero, we have

$$\int_0^\infty \frac{x}{1+e^x}\mathrm{d}x = \frac{\pi^2}{12}$$

Method Three:

$$\int_0^\infty \frac{x}{1+e^x}\mathrm{d}x = \int_0^\infty x\left(\frac{1}{1+e^x} - \frac{1}{e^x-1}\right)\mathrm{d}x + \int_0^\infty \frac{x}{e^x-1}\mathrm{d}x$$
$$= -2\int_0^\infty \frac{x}{e^{2x}-1}\mathrm{d}x + \int_0^\infty \frac{x}{e^x-1}\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\mathrm{d}x$$
$$= -\frac{1}{2}\int_0^\infty \ln\left(1-e^{-x}\right)\mathrm{d}x = \frac{1}{2}\int_0^\infty \sum_{n=1}^\infty \frac{e^{-nx}}{n}\mathrm{d}x = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty e^{-nx}\mathrm{d}x$$
$$= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{12}$$

Method Four:

Knowing that

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{1+e^x}\mathrm{d}x$$

where $\eta(s)$ is Dirichlet η function,

$$\Rightarrow \int_0^\infty \frac{x^{s-1}}{1+e^x}\mathrm{d}x = \left(1-2^{1-s}\right)\zeta(s)\,\Gamma(s)$$

So,

$$\int_0^\infty \frac{x}{e^x+1}\mathrm{d}x = \frac{1}{2}\zeta(2)\Gamma(2) = \frac{\pi^2}{12}$$

□

Exercise 1.1.53: Evaluate the integral:

$$\int \frac{1}{x^2\left(x^4+1\right)^{3/4}}\,\mathrm{d}x$$

Solution: The following method feels more systematic to me:

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2\left(1+x^4\right)^{3/4}} &= \frac{1}{4}\int \frac{4x^3\,\mathrm{d}x}{x^5\left(1+x^4\right)^{3/4}} \\
&= \frac{1}{4}\int \frac{\mathrm{d}t}{t^{5/4}\left(1+t\right)^{3/4}}; \quad x^4 = t \\
&= \frac{1}{4}\int \left(\frac{1+t}{t}\right)^{-3/4}\cdot\frac{1}{t^2}\,\mathrm{d}t \\
&= -\frac{1}{4}\int u^{-3/4}\,\mathrm{d}u; \quad \frac{1+t}{t} = u \\
&= -\sqrt[4]{u}+C = -\sqrt[4]{\frac{1+t}{t}}+C \\
&= -\sqrt[4]{\frac{1+x^4}{x^4}}+C = -\frac{\sqrt[4]{1+x^4}}{x}+C
\end{aligned}$$

□

Exercise 1.1.54: Evaluate:

$$\int e^{x\sin x+\cos x}\left(\frac{x^4\cos^3 x - x\sin x+\cos x}{x^2\cos^2 x}\right)\mathrm{d}x$$

Solution: Let

$$\begin{aligned}
I &= \int e^{x\sin x+\cos x}\left[\frac{x^4\cos^3 x - x\sin x+\cos x}{x^2\cos^2 x}\right]\mathrm{d}x \\
&= \int e^{x\sin x+\cos x}\left[\frac{\cos x - x\sin x+x^4\cos^3 x}{x^2\cos^2 x}\right]\mathrm{d}x \\
&= \int e^{x\sin x+\cos x}\left[\frac{1}{x^2\cos x}-\frac{\sin x}{x\cos^2 x}+x^2\cos x\right]\mathrm{d}x \\
&= \int e^{x\sin x+\cos x}\left[\frac{1}{x^2\cos x}-\frac{\sin x}{x\cos^2 x}+1+x^2\cos x-1\right]\mathrm{d}x \\
&= \int e^{x\sin x+\cos x}\left[\left(\frac{1}{x^2\cos x}-\frac{\sin x}{x\cos^2 x}+1\right)+(x\cos x)\left(x-\frac{1}{x\cos x}\right)\right]\mathrm{d}x
\end{aligned}$$

Notice that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x-\frac{1}{x\cos x}\right] = \frac{1}{x^2\cos x}-\frac{\sin x}{x\cos^2 x}+1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x+\cos x)=x\cos x$$

Now use

$$\int e^t[f(t)+f'(t)]\mathrm{d}t=e^t\cdot f(t)+C$$

to get

$$\int e^{x\sin x+\cos x}\left(\frac{x^4\cos^3 x-x\sin x+\cos x}{x^2\cos^2 x}\right)\mathrm{d}x=e^{x\sin x+\cos x}\left(x-\frac{1}{x\cos x}\right)+C$$

□

Exercise 1.1.55: Evaluate:

$$\int_0^1 \ln^2(\sqrt{1+x}-\sqrt{1-x})\mathrm{d}x$$

Solution: Let $\sqrt{1+x}-\sqrt{1-x}\mapsto 2\sin x$. We get, for the first integral,

$$\begin{aligned}
&\int_0^1 \ln^2\left(\sqrt{1+x}-\sqrt{1-x}\right)\,\mathrm{d}x=\int_0^{\frac{\pi}{4}} 2\cos 2x\ln^2(2\sin x)\,\mathrm{d}x\\
=&\sin 2x\ln^2(2\sin x)\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}2\left(1+\cos 2x\right)\ln(2\sin x)\,\mathrm{d}x\\
=&\frac{1}{4}\ln^2 2+2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,\mathrm{d}x-\sin 2x\ln(2\sin x)\Big|_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}2\cos^2 x\,\mathrm{d}x\\
=&\frac{1}{4}\ln^2 2+\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}-\frac{1}{2}\ln 2+\left[x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}\\
=&\mathbf{G}+\frac{1}{4}\ln^2 2-\frac{1}{2}\ln 2+\frac{\pi}{4}+\frac{1}{2}
\end{aligned}$$

For the third integral,

$$\begin{aligned}
&\int_0^1 \ln^3\left(\sqrt{1+x}-\sqrt{1-x}\right)\,\mathrm{d}x=\int_0^{\frac{\pi}{4}} 2\cos 2x\ln^3(2\sin x)\,\mathrm{d}x\\
=&\sin 2x\ln^3(2\sin x)\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}3(1+\cos 2x)\ln^2(2\sin x)\,\mathrm{d}x\\
=&\frac{1}{8}\ln^3 2-\Re\int_0^{\frac{\pi}{4}}3\ln^2(1-e^{i2x})\,\mathrm{d}x-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}3x^2\,\mathrm{d}x\\
&-\frac{3}{2}\sin 2x\ln^2(2\sin x)\Big|_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}6\cos^2 x\ln(2\sin x)\,\mathrm{d}x\\
=&-\frac{3\mathbf{G}}{2}+\frac{1}{8}\ln^3 2-\frac{7\pi^3}{64}-\frac{3}{8}\ln^2 2+\frac{3}{4}\ln 2-\frac{3\pi}{8}-\frac{3}{4}-\Re\int_0^{\frac{\pi}{4}}3\ln^2(1-e^{i2x})\,\mathrm{d}x
\end{aligned}$$

The remaining integral can be evaluated like so

$$\begin{aligned}
&-\Re\int_0^{\frac{\pi}{4}} 3\ln^2(1-e^{i2x})\,\mathrm{d}x\\
=&-\frac{3}{2}\Im\int_1^i \frac{\ln^2(1-z)}{z}\mathrm{d}z\\
=&\Im\left\{-\frac{3}{2}\ln i\ln^2(1-i)-3\int_1^i\frac{\ln z\ln(1-z)}{1-z}\mathrm{d}z\right\}\\
=&\frac{3\pi^3}{64}-\frac{3\pi}{16}\ln^2 2-\Im\left\{3\mathrm{Li}_2(1-i)\ln(1-i)-3\mathrm{Li}_3(1-i)\right\}
\end{aligned}$$

Plucking $z=i$ in the dilogarithm reflection formula,

$$\underbrace{\mathrm{Li}_2(i)}_{-\frac{\pi^2}{48}+i\mathbf{G}}+\mathrm{Li}_2(1-i)=\frac{\pi^2}{6}\underbrace{-\ln i\ln(1-i)}_{-\frac{\pi^2}{8}-i\frac{\pi}{4}\ln 2}$$

which implies

$$\mathrm{Li}_2(1-i)=\frac{\pi^2}{16}-i\left(\mathbf{G}+\frac{\pi}{4}\ln 2\right)$$

Thus we have

$$-\Re\int_0^{\frac{\pi}{4}} 3\ln^2(1-e^{i2x})\,\mathrm{d}x=3\Im\mathrm{Li}_3(1-i)+\frac{3\mathbf{G}}{2}\ln 2+\frac{3\pi^3}{32}+\frac{3\pi}{16}\ln^2 2$$

Finally,

$$\begin{aligned}
\int_0^1\ln^3\left(\sqrt{1+x}-\sqrt{1-x}\right)\,\mathrm{d}x=&3\Im\mathrm{Li}_3(1-i)+\frac{3\mathbf{G}}{2}\ln 2-\frac{3\mathbf{G}}{2}-\frac{\pi^3}{64}+\frac{3\pi}{16}\ln^2 2\\
&+\frac{1}{8}\ln^3 2-\frac{3}{8}\ln^2 2+\frac{3}{4}\ln 2-\frac{3\pi}{8}-\frac{3}{4}
\end{aligned}$$

□

Exercise 1.1.56: Prove this result below:

$$\int_0^1\left(\ln\frac{1}{x}\right)^{-\frac{1}{2}}\mathrm{d}x=\sqrt{\pi}$$

and

$$\int_0^1\left(\ln\frac{1}{x}\right)^{\frac{1}{2}}\mathrm{d}x=\frac{\sqrt{\pi}}{2}$$

Proof: Rewrite $\ln(1/x)=-\ln(x)$ and use the substitution $-\ln x=t^2\Rightarrow \mathrm{d}x=-2te^{-t^2}\mathrm{d}t$ to obtain:

$$\int_0^1\frac{\mathrm{d}x}{\sqrt{\ln\frac{1}{x}}}=\int_0^\infty 2e^{-t^2}\,\mathrm{d}t=\sqrt{\pi}$$

For the second one,

$$\int_0^1\sqrt{\ln\frac{1}{x}}\mathrm{d}x=x\sqrt{\ln\frac{1}{x}}\Big|_0^1-\int_0^1 x\frac{\mathrm{d}x}{(-2x)\sqrt{\ln\frac{1}{x}}}=\frac{1}{2}\cdot\int_0^1\frac{\mathrm{d}x}{\sqrt{\ln\frac{1}{x}}}=\frac{\sqrt{\pi}}{2}$$

□

Exercise 1.1.57: Evaluate this integral:

$$\int_0^\infty \frac{x}{x^2+1}\frac{\mathrm{d}x}{\sinh\left(\frac{\pi}{2}x\right)}$$

Solution: I will solve the general form

$$\begin{aligned}
&\int_0^\infty \frac{x}{x^2+1}\frac{\mathrm{d}x}{\sinh(ax)} = \int_0^\infty \frac{x}{x^2+a^2}\frac{\mathrm{d}x}{\sinh(x)} = \int_0^\infty\int_0^\infty e^{-at}\frac{\sin(xt)}{\sinh(x)}\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_0^\infty e^{-at}\int_0^\infty \frac{\sin(xt)}{\sinh(x)}\,\mathrm{d}x\,\mathrm{d}t = \frac{\pi}{2}\int_0^\infty e^{-at}\tanh\left(\frac{\pi}{2}t\right)\,\mathrm{d}t\\
&= \int_0^\infty e^{-zx}\tanh(x)\,\mathrm{d}t \quad ;\left(z=\frac{2}{\pi}a\right) = \int_0^\infty \frac{e^{-zx}(1-e^{-2x})}{e^{-2x}+1}\,\mathrm{d}x
\end{aligned}$$

By splitting the integral we have

$$\int_0^\infty \frac{e^{-zx}}{e^{-2x}+1}\,\mathrm{d}x = \sum_{n\geq 0}\int_0^\infty e^{-x(2n+z)}\,\mathrm{d}x = \sum_{n\geq 0}\frac{(-1)^n}{2n+z} = \frac{1}{4}\left(\psi\left(\frac{1}{2}+\frac{z}{4}\right)-\psi\left(\frac{z}{4}\right)\right)$$

$$\int_0^\infty \frac{-e^{-x(z+2)}}{e^{-2x}+1}\,\mathrm{d}x = -\sum_{n\geq 0}\frac{(-1)^n}{z+2+2n} = -\frac{1}{4}\left(-\psi\left(\frac{1}{2}+\frac{z}{4}\right)+\psi\left(1+\frac{z}{4}\right)\right)$$

Hence we have

$$\begin{aligned}
\int_0^\infty \frac{x}{x^2+1}\frac{\mathrm{d}x}{\sinh(ax)} &= \frac{1}{4}\left(2\psi\left(\frac{1}{2}+\frac{z}{4}\right)-\psi\left(1+\frac{z}{4}\right)-\psi\left(\frac{z}{4}\right)\right)\\
&= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{z}{4}\right)-\frac{1}{2}\psi\left(\frac{z}{4}\right)-z\\
&= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{a}{2\pi}\right)-\frac{1}{2}\psi\left(\frac{a}{2\pi}\right)-\frac{2}{\pi}a
\end{aligned}$$

Let $a=\pi/2$

$$\begin{aligned}
\int_0^\infty \frac{x}{x^2+1}\frac{\mathrm{d}x}{\sinh(\frac{\pi}{2}x)} &= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{2}\psi\left(\frac{1}{4}\right)-1\\
&= \frac{\pi}{2}\cot\left(\frac{\pi}{4}\right)-1\\
&= \frac{\pi}{2}-1
\end{aligned}$$

□

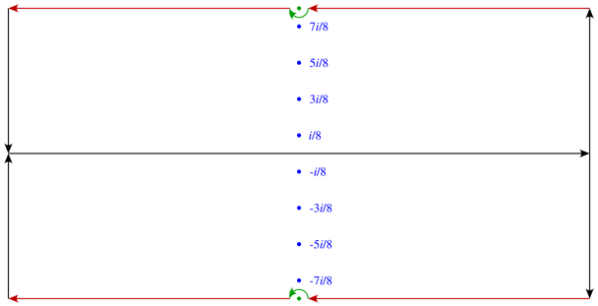
Exercise 1.1.58: Evaluate the integral:

$$I=\int_0^\infty \frac{1}{(e^{\pi x}+e^{-\pi x})(16+x^2)}\mathrm{d}x$$

Solution: Let's compute $(-1)^n\int_{-\infty}^\infty \frac{\operatorname{sech}(n\pi x)}{x^2+1}\mathrm{d}x$

$\operatorname{sech}(n\pi x)$ has simple poles at $x=\frac{2k+1}{2n}i$, with residue $\frac{(-1)^k}{n\pi i}$.

$\frac{1}{x^2+1}$ has simple poles at $x=\pm i$, with residue $\frac{1}{\pm 2i}$.



Note that the integrals along the vertical segments vanish at ∞. The integral along the upper rectangular contour is the integral in question: $\int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2+1}\mathrm{d}x$, minus the Principal Value integral along the red path: $\int_{-\infty+i}^{\infty+i} \frac{\operatorname{sech}(n\pi x)}{x^2+1}\mathrm{d}x$, minus πi times the residue at $x=i$: $\pi i \times (-1)^n \frac{1}{2i} = (-1)^n \frac{\pi}{2}$. This is also $2\pi i$ times the sum of the residues inside the contour: $\sum_{k=0}^{n-1} \frac{(-1)^k}{n} \frac{2}{1-\left(\frac{2k+1}{2n}\right)^2}$.
Putting these together gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2+1}\mathrm{d}x \\
&= \frac{1}{2}\left((-1)^n \int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2+2ix}\mathrm{d}x + (-1)^n\frac{\pi}{2} + \sum_{k=0}^{n-1}\frac{(-1)^k}{n}\frac{2}{1-\left(\frac{2k+1}{2n}\right)^2}\right) \\
&+ \frac{1}{2}\left((-1)^n \int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2-2ix}\mathrm{d}x + (-1)^n\frac{\pi}{2} + \sum_{k=0}^{n-1}\frac{(-1)^k}{n}\frac{2}{1-\left(\frac{2k+1}{2n}\right)^2}\right) \\
&= (-1)^n \int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2+4}\mathrm{d}x + (-1)^n\frac{\pi}{2} + 8n\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n)^2-(2k+1)^2} \\
&= (-1)^n\left(\frac{1}{2}\int_{-\infty}^{\infty} \frac{\operatorname{sech}(2n\pi x)}{x^2+1}\mathrm{d}x + \frac{\pi}{2}\right) + 2\sum_{k=0}^{n-1}(-1)^k\left(\frac{1}{2n+2k+1}+\frac{1}{2n-2k-1}\right) \\
&= (-1)^n\left(\frac{1}{2}\int_{-\infty}^{\infty} \frac{\operatorname{sech}(2n\pi x)}{x^2+1}\mathrm{d}x + \frac{\pi}{2} - 4\sum_{k=0}^{n-1}(-1)^k\frac{1}{2k+1} + 2\sum_{k=0}^{2n-1}(-1)^k\frac{1}{2k+1}\right)
\end{aligned}
$$

Rearranging the last line

$$
\begin{aligned}
&(-1)^n \int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2+1}\mathrm{d}x + 4\sum_{k=0}^{n-1}(-1)^k\frac{1}{2k+1} - \pi \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{\operatorname{sech}(2n\pi x)}{x^2+1}\mathrm{d}x + 4\sum_{k=0}^{2n-1}(-1)^k\frac{1}{2k+1} - \pi\right) \\
&\to 0
\end{aligned}
$$

Therefore,

$$
(-1)^n \int_{-\infty}^{\infty} \frac{\operatorname{sech}(n\pi x)}{x^2+1}\mathrm{d}x = \pi - 4\sum_{k=0}^{n-1}(-1)^k\frac{1}{2k+1} \tag{1}
$$

Now,applying (1), with $n=4$, to the question:

$$\int_0^\infty \frac{\mathrm{d}x}{\left(e^{\pi x}+e^{-\pi x}\right)\left(16+x^2\right)} = \frac{1}{4}\int_{-\infty}^{\infty}\frac{\operatorname{sech}(\pi x)\,\mathrm{d}x}{16+x^2} = \frac{1}{16}\int_{-\infty}^{\infty}\frac{\operatorname{sech}(4\pi x)\,\mathrm{d}x}{1+x^2}$$
$$= \frac{\pi}{16}-\frac{1}{4}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\right) = \frac{\pi}{16}-\frac{19}{105}$$

□

Exercise 1.1.59: Find a general expression for:

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\mathrm{d}x$$

Solution: Let $x^2 \mapsto x$, then

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\mathrm{d}x = \frac{1}{2}\int_0^1 x^n(1-x)^{-1/2}\,\mathrm{d}x = \frac{1}{2}\beta\left(n+1,\frac{1}{2}\right) = \frac{1}{2}\frac{n!\sqrt{\pi}}{\left(n+\frac{1}{2}\right)\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}}$$
$$= \frac{2^{2n}}{(2n+1)\binom{2n}{n}}$$

if $n\in\mathbb{N}$. Similarly,

$$\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\mathrm{d}x = \frac{1}{2}\beta\left(n+\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}\frac{\frac{(2n)!}{2^{2n}n!}\pi}{n!} = \frac{\pi}{2^{2n+1}}\binom{2n}{n}$$

□

Exercise 1.1.60: Find a general expression for:

$$\int_0^1 \left(\frac{\ln(1+x)}{1+x}\right)^n \mathrm{d}x \quad n=1,2,3,\ldots$$

Solution: **Derivation:**

$$\int_0^1 \frac{\ln^n(1+x)}{(1+x)^n}\mathrm{d}x = \int_1^2 \frac{\ln^n x}{x^n}\mathrm{d}x$$
$$= (-1)^n\int_{\frac{1}{2}}^1 x^{n-2}\ln^n x\,\mathrm{d}x$$
$$= (-1)^n\frac{\partial^n}{\partial n^n}\int_{\frac{1}{2}}^1 x^{n-2}\mathrm{d}x$$
$$= (-1)^n\frac{\partial^n}{\partial n^n}\left[\frac{1}{n-1}-\frac{1}{2^{n-1}(n-1)}\right]$$
$$= (-1)^n\left[\frac{(-1)^n n!}{(n-1)^{n+1}}-\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{n-j}(n-j)!}{(n-1)^{n-j+1}}\frac{(-1)^j\ln^j(2)}{2^{n-1}}\right]$$
$$= \frac{n!}{(n-1)^{n+1}}-\frac{n!}{2^{n-1}}\sum_{j=0}^{n}\frac{\ln^j 2}{j!(n-1)^{n-j+1}}$$

□

Exercise 1.1.61: Calculate this integral:

$$\int_0^\infty \left(\frac{x}{1-x}\right)^{\frac{1}{3}} \frac{1}{1+x^2}\mathrm{d}x$$

Solution: Let

$$f(z) = \frac{|z|^{1/3}e^{i\varphi_1/3}|z-1|^{-1/3}e^{-i\varphi_2/3}}{1+z^2}$$

where $\varphi_1 = \arg z$, $\varphi_2 = \arg(z-1)$, $0 \le \varphi_1, \varphi_2 \le 2\pi$. Integrating $f(z)$ over a dumbbell contour, we get

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_0^1 \frac{x^{1/3}e^{i(0)/3}(1-x)^{-1/3}e^{-i(\pi)/3}}{1+x^2}\mathrm{d}x + \int_1^0 \frac{x^{1/3}e^{i(2\pi)/3}(1-x)^{-1/3}e^{-i(\pi)/3}}{1+x^2}\mathrm{d}x \\
&= -i\sqrt{3}\int_0^1 \frac{x^{1/3}(1-x)^{-1/3}}{1+x^2}\mathrm{d}x \\
&= 2\pi i\left(\sum_{\pm}\operatorname*{Res}_{z=\pm i} f(z) - \operatorname*{Res}_{z=0}\frac{(1-z)^{-1/3}}{1+z^2}\right) \\
&= 2\pi i\left(\frac{e^{\pi i/6}\times 2^{-1/6}\times e^{-\pi i/4} - e^{\pi i/2}\times 2^{-1/6}\times e^{-5\pi i/12}}{2i}\right) \\
&= -i\frac{\pi}{\sqrt[3]{4}}\left(\sqrt{3}-1\right)
\end{aligned}$$

which implies

$$\int_0^1 \frac{x^{1/3}(1-x)^{-1/3}}{1+x^2}\mathrm{d}x = \frac{\pi}{\sqrt[3]{4}}\left(1-\frac{1}{\sqrt{3}}\right)$$

For the remaining integral, one can apply the substitution $x \mapsto 1-x^3$ to get

$$\begin{aligned}
\int_0^1 \frac{(x-1)^{-1/3}}{1+x^2}\mathrm{d}x &= -\int_0^1 \frac{3x}{(x^3-1)^2+1}\mathrm{d}x \\
&= -\sum_{\omega\in S}\frac{1}{2(\omega^4-\omega)}\int_0^1 \frac{x-\sigma+it}{(x-\sigma)^2+t^2}\mathrm{d}x \\
&= -\sum_{\omega\in S}\frac{1}{2(\omega^4-\omega)}\left[\frac{1}{2}\ln\left((x-\sigma)^2+t^2\right) + i\arctan\left(\frac{x-\sigma}{t}\right)\right]_0^1 \\
&= -\sum_{\omega\in S}\frac{\ln\left(\frac{(1-\sigma)^2+t^2}{\sigma^2+t^2}\right) + 2i\left(\arctan\left(\frac{1-\sigma}{t}\right)+\arctan\left(\frac{\sigma}{t}\right)\right)}{4(\omega^4-\omega)}
\end{aligned}$$

where $\omega = \sigma + it, \sigma \in \mathbb{R}$ and

$$S = \left\{\sqrt[6]{2}e^{\pi i/12}, \sqrt[6]{2}e^{7\pi i/12}, \sqrt[6]{2}e^{3\pi i/4}, \sqrt[6]{2}e^{5\pi i/4}, \sqrt[6]{2}e^{17\pi i/12}, \sqrt[6]{2}e^{23\pi i/12}\right\}$$

Writing out the 6 terms manually then using Mathematica to simplify, we find that the sum (which is equivalent to a hypergeometric function as pointed out by Lucian) evaluates to

$$\begin{aligned}
\frac{3}{2}{}_3\mathrm{F}_2\left(\left.\begin{matrix}1,1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{6}\end{matrix}\right| -1\right) &= \frac{\pi\sqrt[3]{2}}{12}(3+\sqrt{3}) + \frac{\ln 2}{3\sqrt[3]{4}} + \frac{\ln(\sqrt[3]{2}-1)}{4\sqrt[3]{4}} - \frac{\ln(2+\sqrt[3]{4}+\sqrt[3]{16})}{2\sqrt[3]{4}} \\
&\quad + \frac{\sqrt{3}}{2\sqrt[3]{4}}\operatorname{artanh}\left(\sqrt{\frac{3}{5-\sqrt[3]{4}}}\right) + \frac{\sqrt{3}-1}{2\sqrt[3]{4}}\arctan\left(\frac{\sqrt{3}+\sqrt[3]{16}-1}{\sqrt{3}+1}\right) \\
&\quad - \frac{\sqrt{3}+1}{2\sqrt[3]{4}}\arctan\left(\frac{\sqrt{3}-\sqrt[3]{16}+1}{\sqrt{3}-1}\right) - \frac{\arctan(1+\sqrt[3]{2})}{\sqrt[3]{4}}
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{x^{1/3}(1-x)^{-1/3}}{1+x^2}\mathrm{d}x = &-\frac{\pi\sqrt[3]{2}}{12}(3+\sqrt{3})-\frac{\ln 2}{3\sqrt[3]{4}}-\frac{\ln(\sqrt[3]{2}-1)}{4\sqrt[3]{4}}+\frac{\ln(2+\sqrt[3]{4}+\sqrt[3]{16})}{2\sqrt[3]{4}}\\&-\frac{\sqrt{3}}{2\sqrt[3]{4}}\operatorname{artanh}\left(\sqrt{\frac{3}{5-\sqrt[3]{4}}}\right)-\frac{\sqrt{3}-1}{2\sqrt[3]{4}}\arctan\left(\frac{\sqrt{3}+\sqrt[3]{16}-1}{\sqrt{3}+1}\right)\\&+\frac{\sqrt{3}+1}{2\sqrt[3]{4}}\arctan\left(\frac{\sqrt{3}-\sqrt[3]{16}+1}{\sqrt{3}-1}\right)+\frac{\arctan(1+\sqrt[3]{2})}{\sqrt[3]{4}}\\&+\frac{\pi}{\sqrt[3]{4}}\left(1-\frac{1}{\sqrt{3}}\right)\end{aligned}$$

□

Exercise 1.1.62: Calculate this integral:

$$\int_0^1 x\tan(\pi x)\ln(\sin(\pi x))\mathrm{d}x$$

Solution: Method One:

$$\begin{aligned}\mathcal{I}=&\int_0^1 x\tan(\pi x)\ln(\sin(\pi x))\ \mathrm{d}x\\=&\left(\int_0^{1/2}+\int_{1/2}^1\right)x\tan(\pi x)\ln(\sin(\pi x))\ \mathrm{d}x &(1)\\=&\int_0^{1/2}(2x-1)\tan(\pi x)\ln(\sin(\pi x))\ \mathrm{d}x &(2)\\=&-\frac{2}{\pi^2}\int_0^{\pi/2}x\cot x\ln(\cos x)\ \mathrm{d}x &(3)\\=&-\frac{1}{\pi}\int^{\pi/2}\tan x\ln(\sin x)\ \mathrm{d}x+\frac{2}{\pi^2}\int_0^{\pi/2}x\tan x\ln(\sin x)\ \mathrm{d}x &(4)\\=&\frac{2}{\pi^2}\int_0^{\pi/2}\ln(\sin x)\ln(\cos x)\ \mathrm{d}x-\frac{2}{\pi^2}\int_0^{\pi/2}x\tan x\ln(\sin x)\ \mathrm{d}x &(5)\\=&\frac{1}{\pi^2}\int_0^{\pi/2}\ln(\sin x)\ln(\cos x)\ \mathrm{d}x-\frac{1}{2\pi}\int_0^{\pi/2}\tan x\ln(\sin x)\ \mathrm{d}x &(6)\\=&\frac{1}{8\pi^2}\frac{\partial^2\mathrm{B}}{\partial a\partial b}\left(\frac{1}{2},\frac{1}{2}\right)-\frac{1}{8\pi}\frac{\partial\mathrm{B}}{\partial b}(0^+,1) &(7)\\=&\frac{1}{8\pi^2}\left(4\pi\ln^2 2-\frac{\pi^3}{6}\right)-\frac{1}{8\pi}\left(-\frac{\pi^2}{6}\right)=\frac{\ln^2 2}{2\pi}\end{aligned}$$

Explanation:

(1):Split the integral at $\frac{1}{2}$.

(2):Substituted $x\mapsto 1-x$ in the second integral.

(3):Substituted $x\mapsto\frac{1}{2}-\frac{x}{\pi}$.

(4):Substituted $x\mapsto\frac{\pi}{2}-x$.

(5):Integrated by parts.

(6):Took the average of (2) and (3).

(7):$\mathrm{B}(a,b)=2\int_0^{\pi/2}\sin^{2a-1}x\cos^{2b-1}x\ \mathrm{d}x$

Method Two: Integrating by parts, we get

$$\begin{aligned}
&\int_0^1 x\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x\\
&=\int_0^{1/2}(2x-1)\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x &(1)\\
&=-\frac{1}{\pi}\int_0^{1/2}(2x-1)\ln(\sin(\pi x))\,\mathrm{d}\ln(\cos(\pi x)) &(2)\\
&=\frac{2}{\pi}\int_0^{1/2}\ln(\sin(\pi x))\ln(\cos(\pi x))\,\mathrm{d}x+\int_0^{1/2}(2x-1)\cot(\pi x)\ln(\cos(\pi x))\,\mathrm{d}x &(3)\\
&=\frac{2}{\pi}\int_0^{1/2}\ln(\sin(\pi x))\ln(\cos(\pi x))\,\mathrm{d}x+\int_0^{1/2}(-2x)\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x &(4)\\
&=\frac{1}{\pi}\int_0^{1/2}\ln(\sin(\pi x))\ln(\cos(\pi x))\,\mathrm{d}x-\frac{1}{2}\int_0^{1/2}\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x &(5)
\end{aligned}$$

Explanation:
(1):Subtract $\frac{1}{2}\int_0^1\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x=0$ and use the symmetry of $x\mapsto 1-x$.
(2):Apply $\mathrm{d}\ln(\cos(\pi x))=\pi\tan(\pi x)\,\mathrm{d}x$.
(3):Integrate by Parts.
(4):Substitute $x\mapsto\frac{1}{2}-x$.
(5):Average (1) and (2).
Using

$$\ln(\sin(x))=-\ln(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}$$

$$\ln(\cos(x))=-\ln(2)-\sum_{k=1}^{\infty}(-1)^k\frac{\cos(2kx)}{k}$$

we get

$$\begin{aligned}
&\int_0^{1/2}\ln(\sin(\pi x))\ln(\cos(\pi x))\,\mathrm{d}x\\
&=\int_0^{1/2}(-\ln(2))^2\,\mathrm{d}x-\ln(2)\int_0^{1/2}\sum_{k=1}^{\infty}\left[-1-(-1)^k\right]\frac{\cos(2\pi kx)}{k}\,\mathrm{d}x\\
&+\int_0^{1/2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^k\frac{\cos(2\pi jx)}{j}\frac{\cos(2\pi kx)}{k}\,\mathrm{d}x\\
&=\frac{\ln(2)^2}{2}-0+\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=\frac{\ln(2)^2}{2}-\frac{\pi^2}{48}
\end{aligned}$$

since $\int_0^{1/2}\cos(2\pi jx)\cos(2\pi kx)\,\mathrm{d}x=0$ when $j\neq k$.
Substituting $t=\sin(\pi x)$ and $e^{-u/2}=t$, we get

$$\begin{aligned}
&\int_0^{1/2}\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x=\frac{1}{\pi}\int_0^1\frac{t}{1-t^2}\ln(t)\,\mathrm{d}t=-\frac{1}{4\pi}\int_0^{\infty}\frac{e^{-u}}{1-e^{-u}}u\,\mathrm{d}u\\
&=-\frac{1}{4\pi}\int_0^{\infty}\sum_{k=1}^{\infty}ue^{-ku}\,\mathrm{d}u=-\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}=-\frac{\pi}{24}
\end{aligned}$$

So,we have

$$\int_0^1 x\tan(\pi x)\ln(\sin(\pi x))\,\mathrm{d}x=\frac{\ln(2)^2}{2\pi}$$

□

Exercise 1.1.63: Compute the integral:

$$\int_{-\infty}^{\infty} \frac{\sin ax - a\sin x}{x^3(x^2+1)}\,\mathrm{d}x$$

Solution:

$$\begin{aligned}\sin(ax) - a\sin(x) &= \frac{1}{2i}\left(\left[e^{iax} - e^{-iax}\right] - \left[ae^{ix} - ae^{-ix}\right]\right)\\ &= \frac{1}{2i}\left(\left[e^{iax} - ae^{ix}\right] - \left[e^{-iax} - ae^{-ix}\right]\right)\\ &= \frac{1}{2i}\left(\left[-1+a+e^{iax} - ae^{ix}\right] - \left[-1+a+e^{-iax} - ae^{-ix}\right]\right)\end{aligned}$$

In fact, we cannot perform the integral using f and the half-doughnut contour without showing that the $\frac{1}{z^2}$ term of $f(z)$ is 0. Expanding, we get

$$\frac{e^{iaz} - ae^{iz}}{z^3(z^2+1)} \sim \frac{1-a}{z^3} - \frac{a^2-3z+2}{2z} - i\frac{a^3-a}{6} + O(z)$$

Furthermore, we need to show that for the small semi-circle $\gamma = re^{i[0,\pi]}$,

$$\int_\gamma \frac{\mathrm{d}z}{z^3} = \left[-\frac{1}{2z^2}\right]_{+r}^{-r} = 0$$

Thus, as RandomVariable has noted previously, we can do the integral either with or without the addition of $1+a$, but with the $1+a$, we only have to deal with simple poles.

Regarding the change in the residues at i and i, those changes are cancelled by the change in the residue at 0. In fact,

$$\operatorname*{Res}_{z=0}\frac{-1+a}{z^3(z^2+1)} = (1-a)\ ,\ \operatorname*{Res}_{z=i}\frac{-1+a}{z^3(z^2+1)} = -\frac{1-a}{2}\ ,\ \operatorname*{Res}_{z=-i}\frac{-1+a}{z^3(z^2+1)} = -\frac{1-a}{2}$$

However, by moving the contour away from the singularities things become simpler.

Since $\dfrac{\sin(az) - a\sin(z)}{z^3(z^2+1)}$ is bounded in the rectangle

$$[-R,R] \cup R + [0,i/2] \cup [R,-R] + i/2 \cup -R + [i/2,0]$$

we get that all singularities are removable. Therefore,

$$\int_{-\infty}^{\infty} \frac{\sin(ax) - a\sin(x)}{x^3(x^2+1)}\,\mathrm{d}x = \int_{-\infty+i/2}^{\infty+i/2} \frac{\sin(az) - a\sin(z)}{z^3(z^2+1)}\,\mathrm{d}z$$

Using the contours

$$\gamma_+ = [-R,R] + i/2 \cup \Re^{i[0,\pi]} + i/2\ ,\ \gamma_- = [-R,R] + i/2 \cup \Re^{-i[0,\pi]} + i/2$$

where γ_+ circles the singularity at i counterclockwise and γ_- circles the singularities at 0 and i clockwise, we get

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\sin(ax) - a\sin(x)}{x^3(x^2+1)}\,\mathrm{d}x = \frac{1}{2i}\int_{\gamma_+} \frac{e^{iaz} - ae^{iz}}{z^3(z^2+1)}\,\mathrm{d}z - \frac{1}{2i}\int_{\gamma_-} \frac{e^{-iaz} - ae^{-iz}}{z^3(z^2+1)}\,\mathrm{d}z\\ &= \pi \operatorname*{Res}_{z=i}\left[\frac{e^{iaz} - ae^{iz}}{z^3(z^2+1)}\right] + \pi \operatorname*{Res}_{z=0}\left[\frac{e^{-iaz} - ae^{-iz}}{z^3(z^2+1)}\right] + \pi \operatorname*{Res}_{z=-i}\left[\frac{e^{-iaz} - ae^{-iz}}{z^3(z^2+1)}\right]\\ &= \pi\left[\frac{e^{-a} - ae^{-1}}{2}\right] + \pi\left[-\frac{a^2-3a+2}{2}\right] + \pi\left[\frac{e^{-a} - ae^{-1}}{2}\right] = \pi\left[e^{-a} - ae^{-1} - \frac{a^2-3a+2}{2}\right]\end{aligned}$$

□

Exercise 1.1.64: Compute the following integrals:

$$I_n = \int_0^1 \frac{\ln^n(1+x)}{x}\mathrm{d}x \ , \ J_n = \int_0^1 \frac{\ln^n(1+x)}{1+x}\mathrm{d}x$$

Solution: **Derivation:**

$$\begin{aligned}
\mathcal{I}_n &= \int_1^2 \frac{\ln^n x}{x-1}\mathrm{d}x && (1)\\
&= (-1)^n \int_{\frac{1}{2}}^1 \frac{\ln^n x}{x-x^2}\mathrm{d}x && (2)\\
&= \frac{\ln^{n+1}(2)}{n+1} + (-1)^n \sum_{k=1}^{\infty} \int_{\frac{1}{2}}^1 x^{k-1}\ln^n x \ \mathrm{d}x && (3)\\
&= \frac{\ln^{n+1}(2)}{n+1} + (-1)^n \sum_{k=1}^{\infty} \frac{\partial^n}{\partial k^n}\left[\frac{1}{k} - \frac{1}{k2^k}\right]\\
&= \frac{\ln^{n+1}(2)}{n+1} + (-1)^n \sum_{k=1}^{\infty} \left[\frac{(-1)^n n!}{k^{n+1}} - \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j j!}{k^{j+1}}\frac{(-1)^{n-j}\ln^{n-j}(2)}{2^k}\right] && (4)\\
&= \frac{\ln^{n+1}(2)}{n+1} + n! \sum_{k=1}^{\infty}\left[\frac{1}{k^{n+1}} - \sum_{j=0}^{n}\frac{\ln^{n-j}(2)}{k^{j+1}2^k(n-j)!}\right]\\
&= \frac{\ln^{n+1}(2)}{n+1} + n!\left[\zeta(n+1) - \sum_{j=0}^{n}\frac{\ln^{n-j}(2)}{(n-j)!}\mathrm{Li}_{j+1}\left(\frac{1}{2}\right)\right] && (5)\\
&= n!\zeta(n+1) - \frac{n\ln^{n+1}(2)}{n+1} - \sum_{j=0}^{n-1}\frac{n!\ln^j(2)}{j!}\mathrm{Li}_{n-j+1}\left(\frac{1}{2}\right) && (6)
\end{aligned}$$

Explanation:

(1):Let $x \mapsto x-1$.

(2):Let $x \mapsto x^{-1}$.

(3):Partial fractions, expand $(1-x)^{-1}$.

(4):Apply Leibniz's Generalised Product Rule.

(5):Recognise the zeta and polylog terms.

(6):Let $j \mapsto n-j$, rearrange terms.

The second integral is

$$J_n = \int_0^1 \frac{\ln^n(1+x)}{1+x}\mathrm{d}x = \left[\frac{\ln^{n+1}(1+x)}{n+1}\right]_0^1 = \frac{\ln^{n+1}(2)}{n+1}$$

□

Exercise 1.1.65: Evaluate following integral:

$$\int_0^{\infty} \frac{\ln(1+16x^2)}{1+25x^2}\mathrm{d}x$$

Solution: Consider Following parametric Integral

$$I(\alpha) = \int_0^{\infty} \frac{\ln(1+\alpha^2x^2)}{1+25x^2}\mathrm{d}x$$

We have $I(0) = 0$ and $I(4)$ is required integral
By differentiating under integral sign we get

$$I'(\alpha) = \int_0^\infty \frac{2\alpha x^2}{(1+25x^2)(1+\alpha^2x^2)}\mathrm{d}x = \frac{\pi}{5(\pi+\alpha)}$$

Integrating back we get

$$I(\alpha) = \frac{1}{5}\pi\ln(\alpha+5) + c$$

Since we have $I(0) = 0 \implies c = -\frac{1}{5}\pi\ln(5)$

$$I(\alpha) = \frac{1}{5}\pi\ln(\alpha+5) - \frac{1}{5}\pi\ln(5)$$

By putting $\alpha = 4$ we get required integral,

$$I(4) = \frac{1}{5}\pi\ln(9) - \frac{1}{5}\pi\ln(5) = \frac{1}{5}\pi\ln\left(\frac{9}{5}\right)$$

So,

$$\int_0^\infty \frac{\ln(1+16x^2)}{1+25x^2}\mathrm{d}x = \frac{1}{5}\pi\ln\left(\frac{9}{5}\right)$$

□

Exercise 1.1.66: Prove:

$$\int_0^1 \ln\Gamma(x)\sin(2n\pi x)\ \mathrm{d}x = \frac{\gamma+\ln(2n\pi)}{2\pi n}$$

Proof: Begin with the infinite product representation of the gamma function.

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{k=1}^{\infty} e^{\frac{x}{k}}\left(1+\frac{x}{k}\right)$$

Take the logarithm and multiply throughout by $\sin(2n\pi x)$.

$$\ln\Gamma(x)\sin(2n\pi x) = -(\gamma x+\ln x)\sin(2n\pi x) + \sum_{k=1}^{\infty}\left\{\frac{x}{k} - \ln\left(1+\frac{x}{k}\right)\right\}\sin(2n\pi x)$$

Integrating the non-sum terms from 0 to 1,

$$\begin{aligned}
\int_0^1 (-\gamma x - \ln x)\sin(2n\pi x)\ \mathrm{d}x =& \frac{\gamma}{2n\pi} + \frac{\ln x\cos(2n\pi x)}{2n\pi}\bigg|_0^1 - \int_0^1 \frac{\cos(2n\pi x)}{2n\pi x}\mathrm{d}x \\
=& \frac{\gamma}{2n\pi} + \left[\frac{\ln x\cos(2n\pi x) - \mathrm{Ci}(2n\pi x)}{2n\pi}\right]_0^1 \\
=& \frac{\gamma - \mathrm{Ci}(2n\pi)}{2n\pi} - \lim_{\epsilon\to 0}\frac{\ln\epsilon - \mathrm{Ci}(2n\pi\epsilon)}{2n\pi} \\
=& \frac{\gamma - \mathrm{Ci}(2n\pi)}{2n\pi} - \lim_{\epsilon\to 0}\frac{\ln\epsilon - \ln(2n\pi) - \ln\epsilon - \gamma + \mathcal{O}(\epsilon)}{2n\pi} \\
=& \frac{2\gamma + \ln(2n\pi) - \mathrm{Ci}(2n\pi)}{2n\pi}
\end{aligned}$$

Integrate the remaining terms from 0 to 1.

$$\begin{aligned}
&\int_0^1 \sum_{k=1}^{\infty} \left\{ \frac{x}{k} - \ln\left(1+\frac{x}{k}\right) \right\} \sin(2n\pi x)\ \mathrm{d}x \\
=& -\frac{1}{2n\pi} \sum_{k=1}^{\infty} \left\{ \frac{1}{k} + \mathrm{Ci}(2n\pi k + 2n\pi) - \mathrm{Ci}(2n\pi k) - \ln\left(1+\frac{1}{k}\right) \right\} \\
=& -\frac{1}{2n\pi} \left[\sum_{k=1}^{\infty} \left\{ \frac{1}{k} - \ln\left(1+\frac{1}{k}\right) \right\} + \lim_{N\to\infty} \left(\sum_{k=2}^{N+1} \mathrm{Ci}(2n\pi k) - \sum_{k=1}^{N} \mathrm{Ci}(2n\pi k) \right) \right] \\
=& -\frac{1}{2n\pi} \left[\gamma + \lim_{N\to\infty} \left(\mathrm{Ci}(2n\pi + 2n\pi N) - \mathrm{Ci}(2n\pi) \right) \right] = \frac{\mathrm{Ci}(2n\pi) - \gamma}{2n\pi}
\end{aligned}$$

Adding them together,

$$\int_0^1 \ln\Gamma(x) \sin(2n\pi x)\ \mathrm{d}x = \frac{2\gamma + \ln(2n\pi) - \mathrm{Ci}(2n\pi) + \mathrm{Ci}(2n\pi) - \gamma}{2n\pi} = \frac{\gamma + \ln(2n\pi)}{2n\pi}$$

□

Exercise 1.1.67: Evaluate the integral below

$$\int_0^{\infty} \left(\frac{e^{-x}}{\sinh(x)} - \frac{e^{-3x}}{x} \right) \mathrm{d}x$$

Solution: Let $u = e^{-x}$, then

$$\int_0^{\infty} \left(\frac{2e^{-x}}{e^x - e^{-x}} - \frac{e^{-3x}}{x} \right) \mathrm{d}x = \int_0^1 \left(\frac{2u}{1-u^2} + \frac{u^2}{\ln u} \right) \mathrm{d}u$$

Denote

$$\mathcal{I}(a) = \int_0^1 \left(\frac{2u}{1-u^2} + \frac{u^2}{\ln u} \right) u^a\ \mathrm{d}u$$

Differentiating under the integral sign, we get

$$\begin{aligned}
\mathcal{I}'(a) =& \int_0^1 \left(\frac{2u^{a+1}\ln u}{1-u^2} + u^{a+2} \right) \mathrm{d}u = \frac{1}{a+3} + 2\sum_{n=0}^{\infty} \int_0^1 u^{2n+a+1} \ln u\ \mathrm{d}u \\
=& \frac{1}{a+3} - 2\sum_{n=0}^{\infty} \frac{1}{(2n+a+2)^2} = \frac{1}{a+3} - \frac{1}{2}\psi_1\left(\frac{a+2}{2}\right)
\end{aligned}$$

Integrating back,

$$\mathcal{I}(a) = \ln(a+3) - \psi_0\left(\frac{a+2}{2}\right) + C$$

where

$$C = \lim_{a\to\infty} \left[\psi_0\left(\frac{a+2}{2}\right) - \ln(a+3) \right]$$

and by Stirling's formula, we know that $\psi_0(z) \sim \ln z$ for large z. So

$$C = \lim_{a\to\infty} \left[\ln\left(\frac{a+2}{a+3}\right) - \ln 2 \right] = -\ln 2$$

and therefore,

$$\int_0^{\infty} \left(\frac{2e^{-x}}{e^x - e^{-x}} - \frac{e^{-3x}}{x} \right) \mathrm{d}x = \mathcal{I}(0) = \gamma + \ln\left(\frac{3}{2}\right)$$

□

Exercise 1.1.68: Evaluate:

$$\int_0^{\pi/12} \ln(\tan x)\mathrm{d}x$$

Solution: Using the Fourier series of $\ln(\tan x)$,

$$\begin{aligned}
&\int_0^{\frac{\pi}{12}} \ln(\tan x)\ \mathrm{d}x = -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\frac{\pi}{12}}\cos\Big[(4n+2)x\Big]\ \mathrm{d}x = -\sum_{n=0}^{\infty}\frac{\sin\left[(2n+1)\frac{\pi}{6}\right]}{(2n+1)^2}\\
=&-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(12n+3)^2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+5)^2}\\
&+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+7)^2}+\sum_{n=0}^{\infty}\frac{1}{(12n+9)^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(12n+11)^2}\\
=&-\frac{1}{9}\underbrace{\sum_{n=0}^{\infty}\left[\frac{1}{(4n+1)^2}-\frac{1}{(4n+3)^2}\right]}_{\mathbf{G}}-\frac{1}{2}\mathbf{G}-\frac{1}{2}\underbrace{\sum_{n=0}^{\infty}\left[\frac{1}{(12n+3)^2}-\frac{1}{(12n+9)^2}\right]}_{\frac{1}{9}\mathbf{G}}\\
=&\left(-\frac{1}{9}-\frac{1}{2}-\frac{1}{18}\right)\mathbf{G}=-\frac{2}{3}\mathbf{G}
\end{aligned}$$

Things could be made clearer if we explicitly write out the terms of the sums. For the **pink** sums,

$$\begin{aligned}
&-\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{1}{11^2}+\cdots\right)\\
=&-\frac{1}{2}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots\right)-\frac{1}{2}\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}-\cdots\right)\\
=&-\frac{1}{2}G-\frac{1}{2}\cdot\frac{1}{9}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right)=-\frac{5}{9}\mathbf{G}
\end{aligned}$$

□

Exercise 1.1.69: Closed form for:

$$\int_0^{\infty}\frac{(\arctan x)^2\ln^2(1+x^2)}{x^2}\mathrm{d}x$$

Solution: Let $\gamma = \lim_{R\to\infty}[-R,R]\cup Re^{i[0,\pi]}$. Observe that

$$\oint_\gamma \frac{\ln^4(1-iz)}{z^2}\mathrm{d}z = \frac{1}{8}\int_0^{\infty}\frac{\ln^4(1+x^2)}{x^2}\mathrm{d}x - 3\int_0^{\infty}\frac{\ln^2(1+x^2)\arctan^2 x}{x^2}\mathrm{d}x + 2\int_0^{\infty}\frac{\arctan^4 x}{x^2}\mathrm{d}x = 0$$

since

The integral over the arc vanishes as $\mathcal{O}\left(\frac{\ln^4 R}{R}\right)$.

The imaginary part is odd and vanishes over a symmetric interval.

The singularity at 0 is removable.

Therefore,

$$\begin{aligned}
&\int_0^\infty \frac{\ln^2(1+x^2)\arctan^2 x}{x^2}\mathrm{d}x = \frac{1}{24}\int_0^\infty \frac{\ln^4(1+x^2)}{x^2}\mathrm{d}x + \frac{2}{3}\int_0^\infty \frac{\arctan^4 x}{x^2}\mathrm{d}x \\
&= \frac{1}{3}\int_0^\infty \frac{\ln^3(1+x^2)}{1+x^2}\mathrm{d}x + \frac{2}{3}\int_0^{\frac{\pi}{2}} x^4\csc^2 x\ \mathrm{d}x = -\frac{8}{3}\int_0^{\frac{\pi}{2}} \ln^3(\cos x)\ \mathrm{d}x + \frac{8}{3}\int_0^{\frac{\pi}{2}} x^3\cot x\ \mathrm{d}x \\
&= -\frac{1}{6}\frac{\partial^3 \mathrm{B}}{\partial b^3}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{16}{3}\sum_{n=1}^\infty \int_0^{\frac{\pi}{2}} x^3\sin(2nx)\mathrm{d}x \\
&= 2\pi\zeta(3) + \frac{\pi^3}{3}\ln 2 + \frac{4\pi}{3}\ln^3 2 + 2\pi\sum_{n=1}^\infty \frac{(-1)^n}{n^3} + \frac{\pi^3}{3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \\
&= 2\pi\zeta(3) + \frac{\pi^3}{3}\ln 2 + \frac{4\pi}{3}\ln^3 2 + 2\pi\left(-\frac{3}{4}\zeta(3)\right) + \frac{\pi^3}{3}\ln 2 \\
&= \frac{\pi}{2}\zeta(3) + \frac{2\pi^3}{3}\ln 2 + \frac{4\pi}{3}\ln^3 2
\end{aligned}$$

□

Exercise 1.1.70: Calculate:

$$\int_0^1 \frac{\ln(x)\ln\left(\frac{1}{2}\left(1+\sqrt{1-x^2}\right)\right)}{x}\mathrm{d}x$$

Solution: Here is an alternative solution.

$$\int_0^1 \frac{\ln x \ln\left(\frac{1+\sqrt{1-x^2}}{2}\right)}{x}\ \mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\ln x \ln\left(\frac{1+\sqrt{1-x}}{2}\right)}{x}\ \mathrm{d}x \tag{1}$$

$$= \frac{1}{4}\int_0^1 \frac{\ln(1-x)}{1-x}\ln\left(\frac{1+\sqrt{x}}{2}\right)\ \mathrm{d}x = \frac{1}{16}\int_0^1 \frac{\ln^2(1-x)}{\sqrt{x}(1+\sqrt{x})}\frac{1-\sqrt{x}}{1-\sqrt{x}}\ \mathrm{d}x \tag{2}$$

$$= -\frac{1}{96}\int_0^1 x^{-3/2}\ln^3(1-x)\ \mathrm{d}x = \frac{1}{48}\lim_{q\to 1}\frac{\partial^3}{\partial q^3}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(q)}{\Gamma\left(q-\frac{1}{2}\right)} \tag{3}$$

Explanation:

(1): Substituted $x \mapsto \sqrt{x}$.

(2):Substituted $x \mapsto 1-x$ and integrated by parts.

(3):Integrated by parts and used the integral representation of the Beta function.

Using Wolfram Alpha (or differentiating by hand),

$$\frac{d^3}{dq^3}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(q)}{48\,\Gamma\left(q-\frac{1}{2}\right)}\right)=\frac{1}{48}\sqrt{\pi}\left(3\,\Gamma(q)\,\psi^{(0)}(q)\left(\frac{\psi^{(0)}\left(q-\frac{1}{2}\right)^2}{\Gamma\left(q-\frac{1}{2}\right)}-\frac{\psi^{(1)}\left(q-\frac{1}{2}\right)}{\Gamma\left(q-\frac{1}{2}\right)}\right)-\frac{3\,\psi^{(0)}\left(q-\frac{1}{2}\right)\left(\Gamma(q)\,\psi^{(0)}(q)^2+\Gamma(q)\,\psi^{(1)}(q)\right)}{\Gamma\left(q-\frac{1}{2}\right)}+\Gamma(q)\left(-\frac{\psi^{(0)}\left(q-\frac{1}{2}\right)^3}{\Gamma\left(q-\frac{1}{2}\right)}+\frac{3\,\psi^{(1)}\left(q-\frac{1}{2}\right)\psi^{(0)}\left(q-\frac{1}{2}\right)}{\Gamma\left(q-\frac{1}{2}\right)}-\frac{\psi^{(2)}\left(q-\frac{1}{2}\right)}{\Gamma\left(q-\frac{1}{2}\right)}\right)+\frac{\Gamma(q)\,\psi^{(0)}(q)^3+3\,\Gamma(q)\,\psi^{(1)}(q)\,\psi^{(0)}(q)+\Gamma(q)\,\psi^{(2)}(q)}{\Gamma\left(q-\frac{1}{2}\right)}\right)$$

$\Gamma(x)$ is the gamma function

$\psi^{(n)}(x)$ is the n^{th} derivative of the digamma function

Setting $q=1$ gives us the required result.

$$\int_0^1 \frac{\ln x \ln\left(\frac{1+\sqrt{1-x^2}}{2}\right)}{x}\,\mathrm{d}x=\frac{\zeta(3)}{4}-\frac{\pi^2}{24}\ln 2+\frac{\ln^3 2}{6}$$

□

Exercise 1.1.71: Calculate the value of the integrals:

$$\int_0^1\left(\frac{\arctan x}{x}\right)^2\mathrm{d}x\ ,\quad \int_0^1\left(\frac{\arctan x}{x}\right)^3\mathrm{d}x\ ,\quad \int_0^1\frac{\arctan^2 x\ln x}{x}\mathrm{d}x$$

Solution: For the first one,

$$\int_0^1\frac{\arctan^2 x}{x^2}\mathrm{d}x=\int_0^{\frac{\pi}{4}}x^2\csc^2 x\,\mathrm{d}x=-x^2\cot x\Big|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}x\cot x\,\mathrm{d}x$$

$$=-\frac{\pi^2}{16}+4\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{4}}x\sin(2nx)\,\mathrm{d}x=-\frac{\pi^2}{16}+\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\cos(n\pi/2)}{n}$$

$$=-\frac{\pi^2}{16}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}=\mathbf{G}+\frac{\pi}{4}\ln 2-\frac{\pi^2}{16}$$

For the second one,

$$\int_0^1\frac{\arctan^3 x}{x^3}\mathrm{d}x=\int_0^{\frac{\pi}{4}}x^3\cot x\csc^2 x\,\mathrm{d}x=-\frac{1}{2}x^3\cot^2 x\Big|_0^{\frac{\pi}{4}}+\frac{3}{2}\int_0^{\frac{\pi}{4}}x^2\cot^2 x\,\mathrm{d}x$$

$$=-\frac{\pi^3}{128}-\frac{3}{2}\int_0^{\frac{\pi}{4}}x^2\,\mathrm{d}x+\frac{3}{2}\int_0^{\frac{\pi}{4}}x^2\csc^2 x\,\mathrm{d}x=-\frac{\pi^3}{64}+\frac{3}{2}\left(\mathbf{G}+\frac{\pi}{4}\ln 2-\frac{\pi^2}{16}\right)$$

$$=\frac{3}{2}\mathbf{G}-\frac{\pi^3}{64}+\frac{3\pi}{8}\ln 2-\frac{3\pi^2}{32}$$

For the third one,

$$\int_0^1\frac{\arctan^2 x\ln x}{x}\mathrm{d}x=-\int_0^1\frac{\arctan x\ln^2 x}{1+x^2}\mathrm{d}x=-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{2k+1}\int_0^1 x^{2n+1}\ln^2 x\,\mathrm{d}x$$

$$=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\left(H_{2n+1}-\frac{1}{2}H_n\right)}{(n+1)^3}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{n^3}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}$$

Let us compute the generating function of $\frac{H_n}{n^3}$.

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_n}{n^3}z^n=\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1\frac{(xz)^n\ln^2 x}{x}\,\mathrm{d}x\\
=&-\frac{1}{2}\int_0^1\frac{\ln^2 x\ln(1-xz)}{x(1-xz)}\mathrm{d}x=-\frac{1}{2}\int_0^z\frac{\ln^2\left(\frac{x}{z}\right)\ln(1-x)}{x(1-x)}\mathrm{d}x\\
=&-\frac{1}{2}\int_0^z\frac{\ln^2 x\ln(1-x)}{x(1-x)}\mathrm{d}x+\ln z\int_0^z\frac{\ln x\ln(1-x)}{x(1-x)}\mathrm{d}x-\frac{\ln^2 z}{2}\int_0^z\frac{\ln(1-x)}{x(1-x)}\mathrm{d}x\\
=&-\frac{1}{2}\int_0^z\frac{\ln^2 x\ln(1-x)}{1-x}\mathrm{d}x-\frac{1}{2}\int_0^z\frac{\ln^2 x\ln(1-x)}{x}\mathrm{d}x\\
&+\ln z\int_0^z\frac{\ln x\ln(1-x)}{x}\mathrm{d}x+\ln z\int_0^z\frac{\ln x\ln(1-x)}{1-x}\mathrm{d}x\\
&-\frac{\ln^2 z}{2}\int_0^z\frac{\ln(1-x)}{x}\mathrm{d}x-\frac{\ln^2 z}{2}\int_0^z\frac{\ln(1-x)}{1-x}\mathrm{d}x
\end{aligned}$$

The second integral is

$$\begin{aligned}
-\frac{1}{2}\int_0^z\frac{\ln^2 x\ln(1-x)}{x}\mathrm{d}x=&\frac{1}{2}\mathrm{Li}_2(x)\ln^2 x\Big|_0^z-\int_0^z\frac{\mathrm{Li}_2(x)\ln x}{x}\mathrm{d}x\\
=&\frac{1}{2}\mathrm{Li}_2(z)\ln^2 z-\mathrm{Li}_3(x)\ln x\Big|_0^z+\int_0^z\frac{\mathrm{Li}_3(x)}{x}\mathrm{d}x\\
=&\mathrm{Li}_4(z)-\mathrm{Li}_3(z)\ln z+\frac{1}{2}\mathrm{Li}_2(z)\ln^2 z
\end{aligned}$$

The third integral is

$$\begin{aligned}
\ln z\int_0^z\frac{\ln x\ln(1-x)}{x}\mathrm{d}x=&-\mathrm{Li}_2(x)\ln x\ln z\Big|_0^z+\ln z\int_0^z\frac{\mathrm{Li}_2(x)}{x}\mathrm{d}x\\
=&\mathrm{Li}_3(z)\ln z-\mathrm{Li}_2(z)\ln^2 z
\end{aligned}$$

The fourth integral is

$$\begin{aligned}
\ln z\int_0^z\frac{\ln x\ln(1-x)}{1-x}\mathrm{d}x=&\mathrm{Li}_2(1-x)\ln(1-x)\ln z\Big|_0^z+\ln z\int_0^z\frac{\mathrm{Li}_2(1-x)}{1-x}\mathrm{d}x\\
=&\zeta(3)\ln z-\mathrm{Li}_3(1-z)\ln z+\mathrm{Li}_2(1-z)\ln(1-z)\ln z
\end{aligned}$$

The fifth integral is

$$-\frac{\ln^2 z}{2}\int_0^z\frac{\ln(1-x)}{x}\mathrm{d}x=\frac{1}{2}\mathrm{Li}_2(z)\ln^2 z$$

The sixth integral is

$$-\frac{\ln^2 z}{2}\int_0^z\frac{\ln(1-x)}{1-x}\mathrm{d}x=\frac{1}{4}\ln^2 z\ln^2(1-z)$$

Putting all of this together,

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}z^n=-\frac{1}{2}\int_0^z\frac{\ln^2 x\ln(1-x)}{1-x}\mathrm{d}x+\mathrm{Li}_4(z)-\mathrm{Li}_3(1-z)\ln z+\zeta(3)\ln z$$
$$+\mathrm{Li}_2(1-z)\ln z\ln(1-z)+\frac{1}{4}\ln^2 z\ln^2(1-z)$$

By Landen's trilogarithm identity, the remaining integral is

$$-\frac{1}{2}\int_0^z\frac{\ln^2 x\ln(1-x)}{1-x}\mathrm{d}x=\frac{1}{4}\ln^2 z\ln^2(1-z)-\frac{1}{2}\int_0^z\frac{\ln x\ln^2(1-x)}{x}\mathrm{d}x$$
$$=\frac{1}{4}\ln^2 z\ln^2(1-z)+\mathrm{Li}_4(z)+\int_0^z\frac{\mathrm{Li}_3(1-x)}{x}\mathrm{d}x+\int_0^{\frac{z}{z-1}}\frac{\mathrm{Li}_3(x)}{x(1-x)}\mathrm{d}x-\frac{1}{6}\int_0^z\frac{\ln^3(1-x)}{x}\mathrm{d}x$$
$$+\frac{\pi^2}{6}\mathrm{Li}_2(z)-\zeta(3)\ln z+\zeta(3)\ln\epsilon$$

The first integral is

$$\int_0^z\frac{\mathrm{Li}_3(1-x)}{x}\mathrm{d}x=\mathrm{Li}_3(1-x)\ln x\Big|_0^z+\int_0^z\frac{\mathrm{Li}_2(1-x)\ln x}{1-x}\mathrm{d}x=\mathrm{Li}_3(1-z)\ln z$$
$$-\zeta(3)\ln\epsilon+\frac{1}{2}\mathrm{Li}_2^2(1-x)\Big|_0^z=\mathrm{Li}_3(1-z)\ln z+\frac{1}{2}\mathrm{Li}_2^2(1-z)-\frac{\pi^4}{72}-\zeta(3)\ln\epsilon$$

The second integral is

$$\int_0^{\frac{z}{z-1}}\frac{\mathrm{Li}_3(x)}{x(1-x)}\mathrm{d}x=\mathrm{Li}_4\left(\frac{z}{z-1}\right)-\mathrm{Li}_3(x)\ln(1-x)\Big|_0^{\frac{z}{z-1}}+\int_0^{\frac{z}{z-1}}\frac{\mathrm{Li}_2(x)\ln(1-x)}{x}\mathrm{d}x$$
$$=\mathrm{Li}_4\left(\frac{z}{z-1}\right)+\mathrm{Li}_3\left(\frac{z}{z-1}\right)\ln(1-z)-\frac{1}{2}\mathrm{Li}_2^2\left(\frac{z}{z-1}\right)$$

The third integral is

$$-\frac{1}{6}\int_0^z\frac{\ln^3(1-x)}{x}\mathrm{d}x=-\frac{1}{6}\ln x\ln^3(1-x)\Big|_0^z-\frac{1}{2}\int_0^z\frac{\ln x\ln^2(1-x)}{1-x}\mathrm{d}x$$
$$=-\frac{1}{6}\ln z\ln^3(1-z)-\frac{1}{2}\mathrm{Li}_2(1-x)\ln^2(1-x)\Big|_0^z-\int_0^z\frac{\mathrm{Li}_2(1-x)\ln(1-x)}{1-x}\mathrm{d}x$$
$$=-\frac{1}{6}\ln z\ln^3(1-z)-\frac{1}{2}\mathrm{Li}_2(1-z)\ln^2(1-z)+\mathrm{Li}_3(1-x)\ln(1-x)\Big|_0^z-\mathrm{Li}_4(1-x)\Big|_0^z$$
$$=-\mathrm{Li}_4(1-z)+\mathrm{Li}_3(1-z)\ln(1-z)-\frac{1}{2}\mathrm{Li}_2(1-z)\ln^2(1-z)-\frac{1}{6}\ln z\ln^3(1-z)+\frac{\pi^4}{90}$$

After consolidating all the terms and simplifying using the reflection and Landen formulae, we get

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}z^n=2\mathrm{Li}_4(z)+\mathrm{Li}_4\left(\frac{z}{z-1}\right)-\mathrm{Li}_4(1-z)-\mathrm{Li}_3(z)\ln(1-z)-\frac{1}{2}\mathrm{Li}_2^2\left(\frac{z}{z-1}\right)$$
$$+\frac{1}{2}\mathrm{Li}_2(z)\ln^2(1-z)+\frac{1}{2}\mathrm{Li}_2^2(z)+\frac{1}{6}\ln^4(1-z)-\frac{1}{6}\ln z\ln^3(1-z)$$
$$+\frac{\pi^2}{12}\ln^2(1-z)+\zeta(3)\ln(1-z)+\frac{\pi^4}{90}$$

By letting $z=1$, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}=2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{11\pi^4}{360}+\frac{7}{4}\zeta(3)\ln 2-\frac{\pi^2}{12}\ln^2 2+\frac{1}{12}\ln^4 2$$

You may sub $z=1$ and verify this yourself using the inversion formulae. Also observe that in general,

$$\begin{aligned}\mathrm{Li}_s(i)&=\sum_{n=1}^{\infty}\frac{\cos(n\pi/2)}{n^s}+i\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^s}=2^{-s}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}+i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}\\&=\left(2^{1-2s}-2^{-s}\right)\zeta(s)+i\beta(s)\end{aligned}$$

So we have

$$\mathrm{Li}_2(i)=-\frac{\pi^2}{48}+i\mathbf{G}\ ,\ \mathrm{Li}_3(i)=-\frac{3}{32}\zeta(3)+i\frac{\pi^3}{32}\ ,\ \mathrm{Li}_4(i)=-\frac{7\pi^4}{11520}+i\beta(4)$$

Using the inversion (and reflection) formulae, we also get

$$\begin{aligned}\mathrm{Li}_2\left(\frac{1-i}{2}\right)&=-\frac{\pi^2}{6}-\frac{1}{2}\ln^2(-1-i)-\mathrm{Li}_2(1+i)\\&=\frac{5\pi^2}{96}-\frac{1}{8}\ln^2 2+i\left(\frac{\pi}{8}\ln 2-\mathbf{G}\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Li}_4\left(\frac{1-i}{2}\right)&=-\mathrm{Li}_4(1+i)-\frac{1}{24}\ln^4(-1-i)-\frac{\pi^2}{12}\ln^2(-1-i)-\frac{7\pi^4}{360}\\&=-\mathrm{Li}_4(1+i)+\frac{1313\pi^4}{92160}+\frac{11\pi^2}{768}\ln^2 2-\frac{1}{384}\ln^4 2+i\left(\frac{7\pi^3}{256}\ln 2+\frac{\pi}{64}\ln^3 2\right)\end{aligned}$$

Therefore,

$$\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{n^3}=\Re\sum_{n=1}^{\infty}\frac{H_n}{n^3}i^n=-2\Re\mathrm{Li}_4(1+i)+\frac{29\pi^4}{2304}+\frac{35}{64}\zeta(3)\ln 2+\frac{\pi^2}{64}\ln^2 2$$

Finally,

$$\begin{aligned}&\int_0^1\frac{\arctan^2 x\ln x}{x}\mathrm{d}x\\=&-4\Re\mathrm{Li}_4(1+i)-\frac{1}{4}\mathrm{Li}_4\left(\tfrac{1}{2}\right)+\frac{167\pi^4}{5760}+\frac{7}{8}\zeta(3)\ln 2+\frac{\pi^2}{24}\ln^2 2-\frac{1}{96}\ln^4 2\end{aligned}$$

□

Exercise 1.1.72: Closed form for:

$$I(\nu^{-1},1)=\int_0^{\infty}\frac{\arctan^2 x\arctan(\nu x)}{x^2}\mathrm{d}x,\ \text{for } \nu>0$$

Solution: First, start off by re-expressing the integral

$$\begin{aligned}&\int_0^{\pi}x^2\cos(nx)\cot\left(\frac{x}{2}\right)\,\mathrm{d}x=-\Re\int_C\frac{z+1}{z-1}z^{n-1}\ln^2 z\,\mathrm{d}z\\&=(-1)^n\int_0^1\frac{1-x}{1+x}x^{n-1}(\ln^2 x-\pi^2)\,\mathrm{d}x-\int_0^1\frac{1+x}{1-x}x^{n-1}\ln^2 x\,\mathrm{d}x\end{aligned}$$

where C is the arc joining $z=1$ to $z=1$. (The first equality follows from $z=e^{ix}$, whereas the second follows from the fact that the integral along $[-1,\epsilon]\cup\epsilon\exp(i[\pi,0])\cup[\epsilon,1]\cup C$ is 0 since $z=1$ is a removable singularity and the indent around $z=0$ vanishes.)
Next, note that

$$I_\nu(\nu^{-1},1)=\int_0^{\frac{\pi}{2}}\frac{x^2}{\tan x(\cos^2 x+\nu^2\sin^2 x)}\mathrm{d}x=\frac{1}{4}\int_0^{\pi}\frac{x^2}{\tan\left(\frac{x}{2}\right)(1+(1-\nu^2)\cos x+\nu^2)}\mathrm{d}x$$
$$=\frac{1}{8\nu}\int_0^{\pi}\left(x^2\cot\left(\frac{x}{2}\right)+2\sum_{n=1}^{\infty}\left(\frac{\nu-1}{\nu+1}\right)^n x^2\cos(nx)\cot\left(\frac{x}{2}\right)\right)\mathrm{d}x$$
$$=\frac{\pi^2}{4\nu}\ln 2-\frac{7\zeta(3)}{8\nu}-\frac{\xi}{4\nu}\left(\int_0^1\frac{1-x}{(1+x)(1+\xi x)}(\ln^2 x-\pi^2)+\frac{1+x}{(1-x)(1-\xi x)}\ln^2 x\ \mathrm{d}x\right)$$

Here $\xi=\dfrac{\nu-1}{\nu+1}$. Utilising the partial fraction decompositions

$$\frac{1-x}{(1+x)(1+\xi x)}=\frac{\nu+1}{1+x}-\frac{\nu}{1+\xi x}\ ,\ \frac{1+x}{(1-x)(1-\xi x)}=\frac{\nu+1}{1-x}-\frac{\nu}{1-\xi x}$$

in tandem with the easily verifiable fact

$$\int_0^1\frac{\ln^2 x}{1+\lambda x}\mathrm{d}x=-\frac{2\mathrm{Li}_3(-\lambda)}{\lambda}$$

yields

$$I_\nu(\nu^{-1},1)$$
$$=\frac{\pi^2}{4\nu}\ln 2-\frac{7\zeta(3)}{8\nu}-\frac{\xi}{4\nu}\left(\frac{\pi^2\nu}{\xi}\ln(1+\xi)-\pi^2(\nu+1)\ln 2+\frac{7\zeta(3)}{2}(\nu+1)-\frac{4\nu}{\xi}\chi_3(\xi)\right)$$
$$=\chi_3\left(\frac{\nu-1}{\nu+1}\right)-\frac{7\zeta(3)}{8}+\frac{\pi^2}{4}\ln\left(1+\frac{1}{\nu}\right)$$

where $\chi_s(z)=\sum\limits_{n\ \text{odd}}\dfrac{z^n}{n^s}=\dfrac{1}{2}\left(\mathrm{Li}_s(z)-\mathrm{Li}_s(-z)\right)$ is the Legendre-chi function.
Integrating back,

$$I_\nu(\nu^{-1},1)$$
$$=\underbrace{\frac{\pi^2}{4}\ln\left(\frac{(1+\nu)^{1+\nu}}{\nu^\nu}\right)-\frac{7\zeta(3)}{8}\nu}_{\text{Let this be C}}+2\int_1^{\frac{1-\nu}{1+\nu}}\frac{\chi_3(v)}{(1+v)^2}\mathrm{d}v$$
$$=C-\frac{2\chi_3(v)}{1+v}\bigg|_1^{\frac{1-\nu}{1+\nu}}+2\int_1^{\frac{1-\nu}{1+\nu}}\frac{\chi_2(v)}{v(1+v)}\mathrm{d}v$$
$$=C+(1-\nu)\chi_3\left(\frac{1-\nu}{1+\nu}\right)-\frac{7\zeta(3)}{8}-2\chi_2(v)\ln(1+v)\big|_1^{\frac{1-\nu}{1+\nu}}+\int_1^{\frac{1-\nu}{1+\nu}}\frac{\ln(1+v)\ln\left(\frac{1+v}{1-v}\right)}{v}\mathrm{d}v$$
$$=C+(1-\nu)\chi_3\left(\frac{1-\nu}{1+\nu}\right)-\frac{7\zeta(3)}{8}+2\chi_2\left(\frac{1-\nu}{1+\nu}\right)\ln\left(\frac{1+\nu}{2}\right)+\frac{\pi^2}{4}\ln 2$$
$$+\frac{1}{2}\int_1^{\frac{1-\nu}{1+\nu}}\frac{\ln^2(1+v)-\ln^2(1-v)+\ln^2\left(\frac{1-v}{1+v}\right)}{v}\mathrm{d}v$$

Repeatedly integrating by parts, it is not hard to derive that

$$\begin{aligned}
\frac{1}{2}\int_1^{\frac{1-\nu}{1+\nu}} \frac{\ln^2(1+v)}{v}\mathrm{d}v =& -\frac{1}{6}\ln^3\left(\frac{1+\nu}{2}\right)+\frac{7\zeta(3)}{8}-\mathrm{Li}_3\left(\frac{1+\nu}{2}\right)+\mathrm{Li}_2\left(\frac{1+\nu}{2}\right)\ln\left(\frac{1+\nu}{2}\right)\\
&+\frac{1}{2}\ln\left(\frac{1-\nu}{2}\right)\ln^2\left(\frac{1+\nu}{2}\right)\\
-\frac{1}{2}\int_1^{\frac{1-\nu}{1+\nu}} \frac{\ln^2(1-v)}{v}\mathrm{d}v =& \mathrm{Li}_3\left(\frac{2\nu}{1+\nu}\right)-\mathrm{Li}_2\left(\frac{2\nu}{1+\nu}\right)\ln\left(\frac{2\nu}{1+\nu}\right)-\frac{1}{2}\ln\left(\frac{1-\nu}{1+\nu}\right)\ln^2\left(\frac{2\nu}{1+\nu}\right)\\
\frac{1}{2}\int_1^{\frac{1-\nu}{1+\nu}} \frac{\ln^2\left(\frac{1-v}{1+v}\right)}{v}\mathrm{d}v =& -2\chi_3(\nu)+2\chi_2(\nu)\ln\nu+\frac{1}{2}\ln^2\nu\ln\left(\frac{1-\nu}{1+\nu}\right)
\end{aligned}$$

Therefore, we get, for $I(\nu^{-1},1)$,

$$\begin{aligned}
I(\nu^{-1},1) =& (1-\nu)\chi_3\left(\frac{1-\nu}{1+\nu}\right)-2\chi_3(\nu)+\mathrm{Li}_3\left(\frac{2\nu}{1+\nu}\right)-\mathrm{Li}_3\left(\frac{1+\nu}{2}\right)-\frac{7\zeta(3)}{8}\nu\\
&+2\chi_2(\nu)\ln\nu+2\chi_2\left(\frac{1-\nu}{1+\nu}\right)\ln\left(\frac{1+\nu}{2}\right)-\mathrm{Li}_2\left(\frac{2\nu}{1+\nu}\right)\ln\left(\frac{2\nu}{1+\nu}\right)\\
&+\mathrm{Li}_2\left(\frac{1+\nu}{2}\right)\ln\left(\frac{1+\nu}{2}\right)+\frac{\pi^2}{4}\ln\left(\frac{(1+\nu)^{1+\nu}}{\nu^\nu}\right)+\frac{\pi^2}{4}\ln 2\\
&+\frac{1}{2}\ln^2\nu\ln\left(\frac{1-\nu}{1+\nu}\right)+\frac{1}{2}\ln\left(\frac{1-\nu}{2}\right)\ln^2\left(\frac{1+\nu}{2}\right)-\frac{1}{6}\ln^3\left(\frac{1+\nu}{2}\right)\\
&-\frac{1}{2}\ln\left(\frac{1-\nu}{1+\nu}\right)\ln^2\left(\frac{2\nu}{1+\nu}\right)
\end{aligned}$$

If no mistakes were made, this formula should hold for $0<\nu\leq 1$. Further simplifications to the formula may be possible through some polylogarithm identities. □

Exercise 1.1.73: Closed form for the following integral:

$$\int_0^{\frac{\pi}{2}}\left[\tan x\arctan\Big(\beta\tan(\mu\tan x)\Big)-\tan x\arctan\Big(\alpha\tan(\mu\tan x)\Big)\right]\mathrm{d}x$$

Solution: First, one may note that

$$\int_0^\infty \frac{x\sin(ax)}{x^2+b^2}\mathrm{d}x=\operatorname*{Res}_{z=ib}\frac{\pi z e^{iaz}}{z^2+b^2}=\frac{\pi}{2}e^{-ab}$$

Applying the substitution $\mu\tan x\mapsto x$, we see that $\mathcal{I}$ is equal to

$$\begin{aligned}
\mathscr{F}(\alpha,\beta,\mu) &= \int_\alpha^\beta\int_0^\infty \frac{x}{x^2+\mu^2}\frac{\tan x}{1+\lambda^2\tan^2 x}\mathrm{d}x\,\mathrm{d}\lambda\\
&= \int_\alpha^\beta\int_0^\infty \frac{x}{x^2+4\mu^2}\frac{\sin x}{1+\lambda^2+(1-\lambda^2)\cos x}\mathrm{d}x\,\mathrm{d}\lambda
\end{aligned}$$

Observe that

$$\sum_{n=1}^\infty a^n\sin(nx)=\Im\sum_{n=1}^\infty(ae^{ix})^n=\Im\frac{ae^{ix}}{1-ae^{ix}}=\frac{a\sin x}{1-2a\cos x+a^2}\quad(\text{for }|a|<1)$$

Letting $a\mapsto\dfrac{b}{a}$,

$$\frac{ab\sin x}{a^2-2ab\cos x+b^2}=\sum_{n=1}^\infty\left(\frac{b}{a}\right)^n\sin(nx)\quad(\text{for }|b|<a)$$

Setting $a=\dfrac{\lambda+1}{\sqrt{2}}$ and $b=\dfrac{\lambda-1}{\sqrt{2}}$, then denoting $\xi=\dfrac{\lambda-1}{\lambda+1}$, we get

$$\frac{\sin x}{1+\lambda^2+(1-\lambda^2)\cos x}=\frac{2}{\lambda^2-1}\sum_{n=1}^{\infty}\xi^n\sin(nx)$$

Integrating termwise,

$$\begin{aligned}\mathscr{F}(\alpha,\beta,\mu)&=\int_\alpha^\beta\frac{2}{\lambda^2-1}\sum_{n=1}^{\infty}\xi^n\int_0^\infty\frac{x\sin(nx)}{x^2+4\mu^2}\mathrm{d}x\ \mathrm{d}\lambda\\&=\int_\alpha^\beta\frac{\pi}{\lambda^2-1}\frac{e^{-2\mu}\xi}{1-e^{-2\mu}\xi}\mathrm{d}\lambda=\frac{\pi}{2}\int_{\frac{\alpha-1}{\alpha+1}}^{\frac{\beta-1}{\beta+1}}\frac{\mathrm{d}\xi}{e^{2\mu}-\xi}\end{aligned}$$

It follows that

$$\mathscr{F}(\alpha,\beta,\mu)=\frac{\pi}{2}\ln\left(\frac{e^{2\mu}-\dfrac{\alpha-1}{\alpha+1}}{e^{2\mu}-\dfrac{\beta-1}{\beta+1}}\right)$$

□

Exercise 1.1.74: Closed form for:

$$\int_0^{\pi/3}\ln(1+\sin x)\ln(1-\sin x)\,\mathrm{d}x$$

Solution: Integral expressed in terms of $F_\pm(x,n)$

For $2x\in(-\pi,\pi)$, one may write the integrand as

$$\prod_{\pm}\ln(1\pm\sin x)=2f_-(2\bar{x},2)-2f_-(\bar{x},2)-2f_+(\bar{x},2)-2\ln 2f_-(2\bar{x},1)+2\bar{x}(2\bar{x}-\pi)+\ln^2 2$$

where $4\bar{x}=\pi-2x$ and $f_\pm(x,n)=\operatorname{Re}\ln^n(1\pm e^{2ix})$. Now note that for $n=1,2$, $f_\pm(x,n)$ has antiderivatives $F_\pm(x,n)$ which can be obtained through integration by parts. To be specific,

$$\begin{aligned}F_-(x,1)&=\operatorname{Re}\frac{i}{2}\mathrm{Li}_2(e^{2ix})\\F_-(x,2)&=\operatorname{Re}\frac{i}{2}\left(2\mathrm{Li}_3(1-e^{2ix})-2\mathrm{Li}_2(1-e^{2ix})\ln(1-e^{2ix})-\ln(e^{2ix})\ln^2(1-e^{2ix})\right)\\F_+(x,2)&=\operatorname{Re}\frac{i}{2}\left(2\mathrm{Li}_3(z)-2\mathrm{Li}_2(z)\ln(z)-\ln^2 z\ln(1-z)+\frac{\ln^3 z}{3}\right)\end{aligned}$$

where $z=(1+e^{2ix})^{-1}$.

As the integrand has no poles in the first quadrant, we are allowed to simply plug in the limits into these antiderivatives. This gives

$$\begin{aligned}&\int_0^{\frac{\pi}{3}}\prod_{\pm}\ln(1\pm\sin x)\ dx\\&=2F_-\left(\frac{\pi}{2},2\right)-2F_-\left(\frac{\pi}{6},2\right)-4F_-\left(\frac{\pi}{4},2\right)+4F_-\left(\frac{\pi}{12},2\right)-4F_+\left(\frac{\pi}{4},2\right)\\&\quad+4F_+\left(\frac{\pi}{12},2\right)-2\ln 2F_+\left(\frac{\pi}{2},1\right)+2\ln 2F_+\left(\frac{\pi}{6},1\right)+\frac{\pi}{3}\ln^2 2-\frac{23\pi^3}{324}\end{aligned}$$

It remains to simplify these polylogarithmic expressions.

Simplification of $F_-\left(\frac{\pi}{2},1\right)$ and $F_-\left(\frac{\pi}{6},1\right)$

Evidently, $F_+\left(\frac{\pi}{2},1\right)=\text{Re}\left(\frac{i}{2}\text{Li}_2(-1)\right)=0$, while the value of

$$F_+\left(\frac{\pi}{6},1\right)=\text{Re}\left(\frac{i}{2}\text{Li}_2(e^{\pi i/3})\right)=\frac{-\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)+\psi_1\left(\frac{5}{6}\right)}{48\sqrt{3}}=\frac{\pi^2}{6\sqrt{3}}-\frac{\psi_1\left(\frac{1}{3}\right)}{4\sqrt{3}}$$

can be deduced by writing it as a sum and applying the duplication formula followed by the reflection formula twice.

Simplification of $F_-\left(\frac{\pi}{2},2\right)$ and $F_-\left(\frac{\pi}{6},2\right)$

Use the polylogarithm inversion formulae to deduce that $F_-\left(\frac{\pi}{2},2\right)=0$. Since $1-e^{\pi i/3}=e^{-\pi i/3}$ lies on the unit circle it is easy to verify that

$$F_+\left(\frac{\pi}{6},2\right)=\frac{\pi^3}{324}$$

using the known Fourier series identities for $\sum\cos(n\theta)n^{-2}$ and $\sum\sin(n\theta)n^{-3}$.

Simplification of $F_-\left(\frac{\pi}{4},2\right)$ and $F_-\left(\frac{\pi}{4},2\right)$

The 3 facts

$$\begin{aligned}\text{Li}_2(1-i)&=\frac{\pi^2}{16}-i\left(\frac{\pi}{4}\ln 2+\mathbf{G}\right)\\ \text{Li}_2\left(\frac{1-i}{2}\right)&=\frac{5\pi^2}{96}-\frac{\ln^2 2}{8}+i\left(\frac{\pi}{8}\ln 2-\mathbf{G}\right)\\ -\text{Im Li}_3\left(\frac{1-i}{2}\right)&=\text{Im Li}_3(1-i)+\frac{7\pi^3}{128}+\frac{3\pi}{32}\ln^2 2\end{aligned}$$

(which respectively follow from the dilogarithm reflection formula and Landen's di/trilogarithm identities) allow us to conclude, after some algebra,

$$F_-\left(\frac{\pi}{4},2\right)=-F_+\left(\frac{\pi}{4},2\right)=-\text{Im Li}_3(1-i)-\frac{\mathbf{G}}{2}\ln 2-\frac{\pi^3}{32}-\frac{\pi}{16}\ln^2 2$$

So $F_-\left(\frac{\pi}{4},2\right)+F_+\left(\frac{\pi}{4},2\right)=0$ surprisingly convenient equality indeed.

Simplification of $F_-\left(\frac{\pi}{12},2\right)+F_-\left(\frac{\pi}{12},2\right)$

This is the most tedious part of the evaluation. We have the identity

$$\begin{aligned}\text{Li}_3\left(\frac{1-z}{1+z}\right)-\text{Li}_3\left(-\frac{1-z}{1+z}\right)=2\text{Li}_3\left(1-z\right)+2\text{Li}_3\left(\frac{1}{1+z}\right)-\frac{1}{2}\text{Li}_3\left(1-z^2\right)\\ -\frac{\ln^3(1+z)}{3}+\frac{\pi^2}{6}\ln(1+z)-\frac{7\zeta(3)}{4}\end{aligned}$$

and it so happens that when $z=e^{\pi i/6}$, $(1-z)(1+z)^{-1}=-(2-\sqrt{3})i$ is purely imaginary and $1-z^2$ lies on the unit circle. Therefore

$$\begin{aligned}4\text{Im}\left(\text{Li}_3\left(1-e^{\pi i/6}\right)+\text{Li}_3\left(\frac{1}{1+e^{\pi i/6}}\right)\right)&=-4\text{Ti}_3\left(2-\sqrt{3}\right)-\frac{17\pi^3}{288}+\frac{\pi}{24}\ln^2(2-\sqrt{3})\\ &=4\text{Ti}_3\left(2+\sqrt{3}\right)-\frac{89\pi^3}{288}-\frac{23\pi}{24}\ln^2(2+\sqrt{3})\end{aligned}$$

since $16\text{Ti}_3(z)+16\text{Ti}_3(z^{-1})=\pi^3+4\pi\ln^2 z$. Furthermore, it is not hard to get

$$\text{Li}_2\left(e^{\pi i/6}\right)=\frac{13\pi^2}{144}+i\left(\frac{\psi_1\left(\frac{1}{3}\right)}{8\sqrt{3}}+\frac{2\mathbf{G}}{3}-\frac{\pi^2}{12\sqrt{3}}\right)$$

by applying its definition, so by the dilogarithm reflection formula,

$$\text{Li}_2\left(1-e^{\pi i/6}\right)=\frac{\pi^2}{144}+i\left(\frac{\pi^2}{12\sqrt{3}}-\frac{\psi_1\left(\frac{1}{3}\right)}{8\sqrt{3}}-\frac{2\mathbf{G}}{3}+\frac{\pi}{12}\ln(2+\sqrt{3})\right)$$

By a similar process we obtain

$$\text{Li}_2\left(\frac{1}{1+e^{\pi i/6}}\right)=\frac{23\pi^2}{288}-\frac{\ln^2(2+\sqrt{3})}{8}+i\left(\frac{\psi_1\left(\frac{1}{3}\right)}{8\sqrt{3}}-\frac{\pi^2}{12\sqrt{3}}-\frac{2\mathbf{G}}{3}+\frac{\pi}{24}\ln(2+\sqrt{3})\right)$$

after an application of the inversion formula to $z=1+e^{\pi i/6}$. After some further manipulations using these values, we eventually arrive at

$$4F_-\left(\frac{\pi}{12},2\right)+4F_+\left(\frac{\pi}{12},2\right)=-4\text{Ti}_3\left(2+\sqrt{3}\right)+\frac{8\mathbf{G}}{3}\ln(2+\sqrt{3})+\frac{5\pi}{6}\ln^2(2+\sqrt{3})+\frac{137\pi^3}{648}$$

The Closed Form

Assimilating all our results, we indeed get

$$\int_0^{\pi/3}\ln(1+\sin x)\ln(1-\sin x)\,dx=-4\text{Ti}_3\left(2+\sqrt{3}\right)-\frac{\psi_1\left(\frac{1}{3}\right)}{2\sqrt{3}}\ln 2+\frac{8\mathbf{G}}{3}\ln\left(2+\sqrt{3}\right)+\frac{29\pi^3}{216}+\frac{5\pi}{6}\ln^2\left(2+\sqrt{3}\right)+\frac{\pi}{3}\ln^2 2+\frac{\pi^2}{3\sqrt{3}}\ln 2$$

□

Exercise 1.1.75: Prove:

$$\int_0^\infty\frac{\sin\left(\pi x^2\right)}{x+2}\text{d}x=\frac{1}{4}\left(\pi-2\pi C\left(2\sqrt{2}\right)-2\pi S\left(2\sqrt{2}\right)+2\text{Si}(4\pi)\right)$$

Proof: Integrating $f(z)=\dfrac{e^{iaz^2}}{z+b}$ along $[0,R]\cup Re^{i[0,\pi/2]}\cup i[R,0]$ gives

$$\int_0^\infty\frac{\sin(ax^2)}{x+b}\,dx=\int_0^\infty\frac{b\cos(ay^2)-y\sin(ay^2)}{y^2+b^2}\,dy$$

To compute the first integral, we consider the function $I(a)=\displaystyle\int_0^\infty\frac{e^{iay^2}}{y^2+b^2}\,dy$ such that $-iI'(a)+b^2I(a)=\dfrac{\sqrt{\pi}}{2\sqrt{2a}}(1+i)$. Solving this ode while noting the initial value $I(0)=\dfrac{\pi}{2b}$,

$$\begin{aligned}I(a)&=-e^{-iab^2}\left(\frac{\sqrt{\pi}}{2\sqrt{2}}(1-i)\right)\int\frac{\cos(ab^2)+i\sin(ab^2)}{\sqrt{a}}\,da\\&=-e^{-iab^2}\left(\frac{\sqrt{\pi}}{2\sqrt{2}}(1-i)\right)\left[\frac{\sqrt{2\pi}}{b}C\left(b\sqrt{\frac{2a}{\pi}}\right)+i\frac{\sqrt{2\pi}}{b}S\left(b\sqrt{\frac{2a}{\pi}}\right)-\frac{\sqrt{\pi}}{b\sqrt{2}}(1+i)\right]\end{aligned}$$

Taking the real part of $bI(a)$,

$$\int_0^\infty \frac{b\cos(ay^2)}{y^2+b^2}\,\mathrm{d}y$$
$$= \frac{\pi}{2}\left[C\left(b\sqrt{\frac{2a}{\pi}}\right)\Big(\sin(ab^2)-\cos(ab^2)\Big) - S\left(b\sqrt{\frac{2a}{\pi}}\right)\Big(\sin(ab^2)+\cos(ab^2)\Big)+\cos(ab^2)\right]$$

The second integral readily reduces to sine and cosine integrals.

$$\int_0^\infty \frac{y\sin(ay^2)}{y^2+b^2}\,\mathrm{d}y = \frac{1}{2}\int_0^\infty \frac{\sin(ay)}{y+b^2}\,\mathrm{d}y = \frac{1}{2}\int_{b^2}^\infty \frac{\sin(ay)\cos(ab^2)-\cos(ay)\sin(ab^2)}{y}\,\mathrm{d}y$$
$$= \frac{1}{2}\Big[\mathrm{Si}(ay)\cos(ab^2)-\mathrm{Ci}(ay)\sin(ab^2)\Big]_{b^2}^\infty$$
$$= \frac{\pi}{4}\cos(ab^2) - \frac{1}{2}\mathrm{Si}(ab^2)\cos(ab^2) + \frac{1}{2}\mathrm{Ci}(ab^2)\sin(ab^2)$$

Therefore, we have a generalised result that holds for positive, real a, b.

$$\int_0^\infty \frac{\sin(ax^2)}{x+b}\,\mathrm{d}x = \frac{\pi}{2}\left[C\left(b\sqrt{\frac{2a}{\pi}}\right)\Big(\sin(ab^2)-\cos(ab^2)\Big) - S\left(b\sqrt{\frac{2a}{\pi}}\right)\Big(\sin(ab^2)+\cos(ab^2)\Big)\right.$$
$$\left.+\frac{1}{2}\cos(ab^2)\right] + \frac{1}{2}\Big(\mathrm{Si}(ab^2)\cos(ab^2) - \mathrm{Ci}(ab^2)\sin(ab^2)\Big)$$

Setting $a=\pi, b=2$ reproduces the identity stated in the question. □

Exercise 1.1.76: Find a closed form for the following integral:

$$\int_0^{1/2} (2x-1)^6 \ln^2(2\sin\pi x)\,\mathrm{d}x$$

Solution: It doesn't take much to show that the integral is equivalent to

$$\mathscr{J} = \frac{64}{\pi^7}\int_0^{\frac{\pi}{2}} x^6\ln^2(2\cos x)\,\mathrm{d}x$$

Use the identity $\ln^2(2\cos x) = \mathrm{Re}\ln^2(1+e^{i2x}) + x^2$ to get

$$\mathscr{J} = \frac{\pi^2}{72} + \frac{64}{\pi^7}\Re\int_0^{\frac{\pi}{2}} x^6\ln^2(1+e^{i2x})\,\mathrm{d}x = \frac{\pi^2}{72} + \frac{1}{2\pi^7}\Im\int_1^{-1}\frac{\ln^6 z\ln^2(1+z)}{z}\mathrm{d}z$$
$$= \frac{\pi^2}{72} - \frac{3}{\pi^6}\int_0^1\frac{\ln^5 z\ln^2(1-z)}{z}\mathrm{d}z + \frac{10}{\pi^4}\int_0^1\frac{\ln^3 z\ln^2(1-z)}{z}\mathrm{d}z - \frac{3}{\pi^2}\int_0^1\frac{\ln z\ln^2(1-z)}{z}\mathrm{d}z$$
$$= \frac{\pi^2}{72} + \frac{720}{\pi^6}\sum_{n=1}^\infty\frac{H_n}{(n+1)^7} - \frac{120}{\pi^4}\sum_{n=1}^\infty\frac{H_n}{(n+1)^5} + \frac{6}{\pi^2}\sum_{n=1}^\infty\frac{H_n}{(n+1)^3}$$
$$= \frac{\pi^2}{72} + \frac{720}{\pi^6}\left(\frac{\pi^8}{7560} - \zeta(3)\zeta(5)\right) - \frac{120}{\pi^4}\left(\frac{\pi^6}{1260} - \frac{1}{2}\zeta^2(3)\right) + \frac{6}{\pi^2}\left(\frac{\pi^4}{360}\right)$$
$$= \frac{11\pi^2}{360} + \frac{60}{\pi^4}\zeta^2(3) - \frac{720}{\pi^6}\zeta(3)\zeta(5)$$

□

Exercise 1.1.77: Find a closed-form for:

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\psi_3(n+1) = -\int_0^1\frac{\ln(1+x)\ln^3 x}{1-x}\mathrm{d}x$$

where $\psi_3(x)$ is the polygamma function of order 3.

✎ **Solution:** We can break up the integral into

$$\begin{aligned}
&-\int_0^1 \frac{\ln^3 x \ln(1+x)}{1-x}\mathrm{d}x = \int_0^1 \frac{\ln^3 x \ln(1-x)}{1-x}\mathrm{d}x - \int_0^1 \frac{(1+x)\ln^3 x \ln(1-x^2)}{(1+x)(1-x)}\mathrm{d}x\\
=&\int_0^1 \frac{\ln^3 x \ln(1-x)}{1-x}\mathrm{d}x - \int_0^1 \frac{\ln^3 x \ln(1-x^2)}{1-x^2}\mathrm{d}x - \int_0^1 \frac{x\ln^3 x \ln(1-x^2)}{1-x^2}\mathrm{d}x\\
=&\frac{15}{16}\int_0^1 \frac{\ln^3 x \ln(1-x)}{1-x}\mathrm{d}x - \frac{1}{16}\int_0^1 \frac{x^{-1/2}\ln^3 x \ln(1-x)}{1-x}\mathrm{d}x\\
=&\frac{15}{16}\frac{\partial^4\beta}{\partial a^3\partial b}(1,0^+) - \frac{1}{16}\frac{\partial^4\beta}{\partial a^3\partial b}(0.5,0^+)
\end{aligned}$$

After differentiating and expanding at $b=0$ (with the help of Mathematica),

$$\begin{aligned}
&\frac{\partial^4\beta}{\partial a^3\partial b}(a,0^+)\\
=&\left[\frac{\Gamma(a)}{\Gamma(a+b)}\left(\frac{1}{b}+\mathcal{O}(1)\right)\left(\left(-\frac{\psi_4(a)}{2}+(\gamma+\psi_0(a))\psi_3(a)+3\psi_1(a)\psi_2(a)\right)b+\mathcal{O}(b^2)\right)\right]_{b=0}\\
=&-\frac{1}{2}\psi_4(a)+(\gamma+\psi_0(a))\psi_3(a)+3\psi_1(a)\psi_2(a)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-\int_0^1 \frac{\ln^3 x \ln(1+x)}{1-x}\mathrm{d}x\\
=&-\frac{15}{32}\psi_4(1)+\frac{45}{16}\psi_1(1)\psi_2(1)+\frac{1}{32}\psi_4(0.5)+\frac{1}{8}\psi_3(0.5)\ln 2-\frac{3}{16}\psi_1(0.5)\psi_2(0.5)\\
=&-12\zeta(5)+\frac{3\pi^2}{8}\zeta(3)+\frac{\pi^4}{8}\ln 2
\end{aligned}$$

□

Exercise 1.1.78: Prove that:

$$\int_{-1}^1 \frac{1}{\pi^2+(2\operatorname{arctanh}(x))^2}\mathrm{d}x = \frac{1}{6}$$

Here is a supplementary question

$$\int_{-1}^1 \frac{\ln(1-x)}{\pi^2+(2\operatorname{arctanh}(x))^2}\mathrm{d}x = \frac{1}{4}+\frac{\gamma}{6}+\frac{\ln(2)}{6}-2\ln(\mathbf{A})$$

where $\mathbf{A}$ is Glaisher–Kinkelin constant.

✎ **Solution:** **Approach 1:**

For the first integral

$$
\begin{aligned}
2\int_0^1 \frac{\mathrm{d}x}{\pi^2+\ln^2\left(\frac{1-x}{1+x}\right)} &= -\frac{4}{\pi}\mathrm{Im}\int_0^1 \frac{\mathrm{d}x}{(\ln x+\pi i)(1+x)^2} && (1)\\
&= -\frac{4}{\pi}\mathrm{Im}\int_{-1}^1 \frac{\mathrm{d}x}{\ln x(1-x)^2} && (2)\\
&= \frac{4}{\pi}\mathrm{Im}\left[\int_0^\pi \frac{ie^{i\phi}}{i\phi(1-e^{i\phi})^2}\,\mathrm{d}\phi - \frac{\pi i}{2}\operatorname*{Res}_{z=1}\frac{1}{\ln z(z-1)^2}\right] && (3)\\
&= \frac{4}{\pi}\mathrm{Im}\left[\int_0^\pi \frac{\mathrm{d}\phi}{-4\phi\sin^2\left(\frac{\phi}{2}\right)} - \frac{\pi i}{2}\operatorname*{Res}_{z=1}\frac{1}{\ln z(z-1)^2}\right]\\
&= \frac{4}{\pi}\left(-\frac{\pi}{2}\right)\operatorname*{Res}_{z=1}\left[\frac{1}{(z-1)^3}+\frac{1}{2(z-1)^2}-\frac{1}{12(z-1)}+\mathcal{O}(1)\right] && (4)\\
&= \frac{1}{6}
\end{aligned}
$$

Explanation:

(1):Substitute $x\mapsto\dfrac{1-x}{1+x}$.

(2):Substitute $x\mapsto -x$ and extend the integration interval to $[1,1]$. The integral over $[0,1]$ contributes nothing since it is purely real.

(3):Change the path of integration to the unit semicircular arc in the upper half plane. We pick up the residue at $z=1$ in the process.

(4):The first integral in the previous line is purely real; Expand $\dfrac{1}{\ln z(z-1)^2}$ as a Laurent series.

Approach 2:

Here is another approach that avoids complex analysis.

$$
\begin{aligned}
&2\int_0^1 \frac{\mathrm{d}x}{\pi^2+\ln^2\left(\frac{1-x}{1+x}\right)} = -\frac{4}{\pi}\Im\int_0^1 \frac{\mathrm{d}x}{\ln(-x)(1+x)^2} = \frac{4}{\pi}\int_0^\infty\int_0^1 \frac{x^s\sin(\pi s)}{(1+x)^2}\,\mathrm{d}x\,\mathrm{d}s\\
&= \frac{4}{\pi}\sum_{k=0}^\infty(-1)^k\int_0^1\int_0^1 \frac{x^{s+k}\sin(\pi s)}{(1+x)^2}\,\mathrm{d}x\,\mathrm{d}s = \frac{4}{\pi}\int_0^1\sin(\pi s)\int_0^1\frac{x^s}{(1+x)^3}\,\mathrm{d}x\,\mathrm{d}s\\
&= \frac{4}{\pi}\int_0^1\sin(\pi s)\left[-\frac{1}{8}-\frac{s}{4}+\frac{s(s-1)}{4}\left(\psi_0\left(\frac{s}{2}\right)-\psi_0\left(\frac{s-1}{2}\right)\right)\right]\mathrm{d}s\\
&= \frac{1}{\pi}\int_0^1\sin(\pi s)s(s-1)\left[\psi_0\left(\frac{s}{2}\right)-\psi_0\left(\frac{s+1}{2}\right)\right]\mathrm{d}s\\
&= \frac{2}{\pi}\int_0^1\ln\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\sin(\pi s)s(1-s)\right)\mathrm{d}s
\end{aligned}
$$

Using the fact

$$
\ln\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} = \sum_{n\in 2\mathbb{N}_0+1}\left[\frac{1}{n}\cos(n\pi s)+2\frac{\gamma+\ln(2n\pi)}{n\pi}\sin(n\pi s)\right]\quad,\quad 0<s\le 1
$$

in tandem with the results

$$\int_0^1 \sin((2n+1)\pi s)\left(\frac{\mathrm{d}}{\mathrm{d}s}\sin(\pi s)s(1-s)\right)\,\mathrm{d}s = 0$$

$$\int_0^1 \cos((2n+1)\pi s)\left(\frac{\mathrm{d}}{\mathrm{d}s}\sin(\pi s)s(1-s)\right)\,\mathrm{d}s = \begin{cases} \dfrac{\pi}{12}+\dfrac{1}{4\pi}, & \text{if } n=0 \\ \dfrac{2n+1}{4\pi}\left(\dfrac{1}{(n+1)^2}-\dfrac{1}{n^2}\right), & \text{if } n\in\mathbb{N} \end{cases}$$

yields

$$2\int_0^1 \frac{\mathrm{d}x}{\pi^2+\ln^2\left(\frac{1-x}{1+x}\right)} = \frac{2}{\pi}\left[\frac{\pi}{12}+\frac{1}{4\pi}+\frac{1}{4\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)\right] = \frac{1}{6}$$

Supplementary Problem:

Substituting $x\mapsto\tanh x$ then $2x\mapsto\ln x$,

$$\begin{aligned}\int_{-1}^1 \frac{\ln(1-x)}{\pi^2+\ln^2\left(\frac{1-x}{1+x}\right)}\,\mathrm{d}x &= \int_{-1}^1 \frac{\ln 2}{\pi^2+\ln^2\left(\frac{1-x}{1+x}\right)}\,\mathrm{d}x + \int_{-\infty}^{\infty}\frac{\ln\left(\frac{1-\tanh x}{2}\right)}{(\pi^2+4x^2)\cosh^2 x}\,\mathrm{d}x \\ &= \frac{\ln 2}{6} - \int_0^{\infty}\frac{2\ln(1+x)}{(\pi^2+\ln^2 x)(1+x)^2}\,\mathrm{d}x \\ &= \frac{\ln 2}{6} - \int_0^1 \frac{1}{2\pi i}\int_C f(z,x)\,\mathrm{d}z\,\mathrm{d}x\end{aligned}$$

where $f(z,x)=\dfrac{2z}{(1+xz)(\ln z-\pi i)(1+z)^2}$ and C is the keyhole contour deformed along the branch cut $[0,\infty]$. We note that

$$\frac{1}{(\ln z-\pi i)(1+z)^2} \sim_{-1} -\frac{1}{(z+1)^3}+\frac{1}{2(z+1)^2}+\frac{1}{12(z+1)}+\mathcal{O}(1)$$

and by the residue theorem, the contour integral is

$$\begin{aligned}&\frac{1}{2\pi i}\int_C f(z,x)\,\mathrm{d}z \\ =\ &\operatorname*{Res}_{z=-1/x} f(z,x)+\operatorname*{Res}_{z=-1} f(z,x) \\ =\ &\frac{2\left(-\frac{1}{x}\right)}{x\left(\ln\left(-\frac{1}{x}\right)-\pi i\right)\left(1-\frac{1}{x}\right)^2}-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{\mathrm{d}}{\mathrm{d}z}\right)\frac{2z}{1+xz}\bigg|_{z=-1}-\frac{1}{6(1-x)} \\ =\ &\frac{2x}{(1-x)^3}+\frac{1}{(1-x)^2}-\frac{1}{6(1-x)}+\frac{2}{(1-x)^2\ln x}\end{aligned}$$

Next, let

$$I(s)=\int_0^1\left(\frac{2x}{(1-x)^3}+\frac{1}{(1-x)^2}-\frac{1}{6(1-x)}+\frac{2}{(1-x)^2\ln x}\right)x^s\,\mathrm{d}x$$

such that the integral we seek is $\dfrac{\ln 2}{6}-I(0)$. Differentiating twice and integrating by parts,

$$\begin{aligned}I''(s) &= 1+\int_0^1\left(\frac{s^2x^{s-1}\ln^2 x}{1-x}+\frac{2sx^{s-1}\ln x}{1-x}-\frac{x^s\ln^2 x}{6(1-x)}\right)\,\mathrm{d}x \\ &= 1-s^2\psi_2(s)-2s\psi_1(s)+\frac{1}{6}\psi_2(s+1)\end{aligned}$$

and we may integrate back to obtain

$$I'(s) = \frac{1}{2} + s - s^2\psi_1(s) + \frac{1}{6}\psi_1(s+1)$$
$$I(s) = L - s\ln(2\pi) - \frac{s}{2} + \frac{3s^2}{2} + \frac{1}{6}\psi_0(s+1) - s^2\psi_0(s) + 2\ln G(s+1)$$

where

$$\begin{aligned} L &= \lim_{s\to\infty}\left(s\ln(2\pi) + \frac{s}{2} - \frac{3s^2}{2} - \frac{1}{6}\psi_0(s+1) + s^2\psi_0(s) - 2\ln G(s+1)\right) \\ &= \lim_{s\to\infty}\left(s\ln(2\pi) + \frac{s}{2} - \frac{3s^2}{2} - \frac{1}{6}\ln s + s^2\ln s - \frac{s}{2}\right. \\ &\left. -\frac{1}{12} - s^2\ln s + \frac{3s^2}{2} - s\ln(2\pi) + \frac{1}{6}\ln s - 2\zeta'(-1) + \mathcal{O}\left(\frac{1}{s}\right)\right) \\ &= -2\zeta'(-1) - \frac{1}{12} \end{aligned}$$

Letting $s = 0$ and noting that $\psi_0(1) = -\gamma$, we arrive at

$$\int_{-1}^{1} \frac{\ln(1-x)}{\pi^2 + \ln^2\left(\frac{1-x}{1+x}\right)}\,\mathrm{d}x = 2\zeta'(-1) + \frac{\gamma}{6} + \frac{\ln 2}{6} + \frac{1}{12}$$

□

Exercise 1.1.79: Evaluate the following integral:

$$\int_0^1 \frac{\tanh^{-1}(x)\ln(x)}{(1-x)x(x+1)}\,\mathrm{d}x$$

Solution: The integral can be rewritten as

$$\int_0^1 \frac{\operatorname{artanh} x\ln x}{x(1-x)(1+x)}\mathrm{d}x = \int_0^1 \frac{\operatorname{artanh} x\ln x}{x}\mathrm{d}x + \int_0^1 \frac{x\operatorname{artanh} x\ln x}{1-x^2}\mathrm{d}x$$

The first integral is

$$\begin{aligned} \int_0^1 \frac{\operatorname{artanh} x\ln x}{x}\mathrm{d}x &= \chi_2(x)\ln x\Big|_0^1 - \int_0^1 \frac{\chi_2(x)}{x}\mathrm{d}x \\ &= -\chi_3(1) = -\frac{7}{8}\zeta(3) \end{aligned}$$

The second integral is

$$\begin{aligned} \int_0^1 \frac{x\operatorname{artanh} x\ln x}{1-x^2}\mathrm{d}x &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{2k+1}\int_0^1 x^{2n+2}\ln x\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty}\frac{\frac{1}{2}H_n - H_{2n+1}}{(2n+3)^2} \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^2} \end{aligned}$$

For the first sum, consider $f(z)=\dfrac{(\gamma+\psi_0(-z))^2}{(2z+1)^2}$. At the positive integers,

$$\begin{aligned}\sum_{n=1}^{\infty}\operatorname{Res}(f,n)&=\sum_{n=1}^{\infty}\operatorname*{Res}_{z=n}\left[\frac{1}{(2z+1)^2(z-n)^2}+\frac{2H_n}{(2z+1)^2(z-n)}\right]\\&=2\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}-\frac{7}{2}\zeta(3)+4\end{aligned}$$

At $z=0$,

$$\operatorname{Res}(f,0)=\operatorname*{Res}_{z=0}\frac{1}{z^2(2z+1)^2}=-4$$

At $z=-\dfrac{1}{2}$,

$$\operatorname{Res}\left(f,-\frac{1}{2}\right)=\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}z}(\gamma+\psi_0(-z))^2\bigg|_{z=-\frac{1}{2}}=\frac{\pi^2}{2}\ln 2$$

Since the sum of residues is zero,

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}=\frac{7}{4}\zeta(3)-\frac{\pi^2}{4}\ln 2$$

For the second sum, consider $f(z)=\dfrac{\pi\cot(\pi z)(\gamma+\psi_0(-2z))}{(2z+1)^2}$. Note that the poles at the half integers are cancelled by the zeroes of $\pi\cot(\pi z)$. At the positive integers,

$$\begin{aligned}\sum_{n=1}^{\infty}\operatorname{Res}(f,n)&=\sum_{n=1}^{\infty}\operatorname*{Res}_{z=n}\left[\frac{1}{2(2z+1)^2(z-n)^2}+\frac{H_{2n}}{(2z+1)^2(z-n)}\right]\\&=\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^2}-\frac{7}{4}\zeta(3)+2\end{aligned}$$

At the negative integers,

$$\begin{aligned}\sum_{n=1}^{\infty}\operatorname{Res}(f,-n)&=\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}+1\\&=\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^2}+\frac{7}{8}\zeta(3)\end{aligned}$$

At $z=0$,

$$\operatorname{Res}(f,0)=-2$$

Since the sum of residues is zero,

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^2}=\frac{7}{16}\zeta(3)$$

Hence, the original integral is simply

$$\int_0^1\frac{\operatorname{artanh}x\ln x}{x(1-x)(1+x)}\mathrm{d}x=-\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln 2$$

□

Exercise 1.1.80: Evaluate the following integral:

$$\int_0^{\pi/4} \sqrt{\tan x}\sqrt{1-\tan x}\,\mathrm{d}x$$

Solution: **Metnod One:**

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sqrt{1-\tan x}\,\mathrm{d}x &= \int_0^1 \frac{\sqrt{y(1-y)}}{1+y^2}\,\mathrm{d}y \quad \Rightarrow \quad y=\tan x \\
&= \int_0^\infty \frac{\sqrt{t}}{(1+t)(1+2t+2t^2)}\,\mathrm{d}t \quad \Rightarrow \quad t=\frac{y}{1-y} \\
&= \int_0^\infty \frac{2z^2}{(1+z^2)(1+2z^2+2z^4)}\,\mathrm{d}z \quad \Rightarrow \quad z^2=t \\
&= 2\int_0^\infty \left[\frac{2z^2}{1+2z^2+2z^4}+\frac{1}{1+2z^2+2z^4}-\frac{1}{1+z^2}\right]\mathrm{d}z \\
&= \int_{-\infty}^\infty \left[\frac{2z^2}{1+2z^2+2z^4}+\frac{1}{1+2z^2+2z^4}-\frac{1}{1+z^2}\right]\mathrm{d}z \\
&= I_1+I_2-\pi
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-\infty}^\infty \frac{2z^2}{1+2z^2+2z^4}\,\mathrm{d}z = \int_{-\infty}^\infty \frac{1}{z^2+\frac{1}{2z^2}+1}\,\mathrm{d}z = \int_{-\infty}^\infty \frac{1}{\left(z-\frac{1}{\sqrt{2}z}\right)^2+1+\sqrt{2}}\,\mathrm{d}z \\
&= \int_{-\infty}^\infty \frac{1}{z^2+1+\sqrt{2}}\,\mathrm{d}z = \frac{\pi}{\sqrt{1+\sqrt{2}}}
\end{aligned}$$

where the 4th line we use identity

$$\int_{-\infty}^{\infty} f\left(x\right)\,dx = \int_{-\infty}^{\infty} f\left(x-\frac{a}{x}\right)\,dx \qquad , \qquad \text{for } a>0.$$

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{x^4+ax^2+b^2} &= \int_0^\infty \frac{\mathrm{d}x}{x^2}\frac{1}{(x-b/x)^2+2b+a} \\
&= \frac{1}{b}\int_0^\infty \frac{\mathrm{d}x}{(x-b/x)^2+2b+a} \\
&= \frac{1}{b}\int_0^\infty \frac{\mathrm{d}x}{x^2+2b+a} \\
&= \frac{\pi}{2b\sqrt{2b+a}}
\end{aligned}$$

Hence,

$$I_2 = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{z^4+z^2+\frac{1}{2}}\,\mathrm{d}z = \pi\sqrt{\frac{\sqrt{2}-1}{2}}$$

Combine all the results together, we finally get

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sqrt{1-\tan x}\;dx = \frac{\pi}{\sqrt[4]{2}}\sqrt{\frac{2+\sqrt{2}}{2}}-\pi$$

Method Two:

Here is a complex analysis approach. Integrate

$$f(z)=\frac{\sqrt{z}\sqrt{z-1}}{z^2+1}=\frac{|z|^{\frac{1}{2}}|z-1|^{\frac{1}{2}}e^{i\varphi}}{z^2+1}$$

where $\varphi=\frac{1}{2}\left(\arg z+\arg(z-1)\right)$, $0\le\arg z,\arg(z-1)\le 2\pi$, along a dumbbell contour. Just above $[0,1]$, $\varphi=\frac{\pi}{2}$, and just below $[0,1]$, $\varphi=\frac{3\pi}{2}$. So the contour integral is

$$\begin{aligned}2i\int_0^1\frac{\sqrt{x}\sqrt{1-x}}{1+x^2}\mathrm{d}x=&2\pi i\left[\operatorname*{Res}_{z=i}f(z)+\operatorname*{Res}_{z=-i}f(z)-\operatorname*{Res}_{z=0}\frac{f(z^{-1})}{z^2}\right]\\=&2\pi i\left[\frac{\sqrt[4]{2}}{2i}e^{i5\pi/8}-\frac{\sqrt[4]{2}}{2i}e^{i11\pi/8}-1\right]\\=&2\pi i\left[\frac{\sqrt[4]{2}}{2i}\left(e^{i3\pi/8}-e^{-i3\pi/8}\right)-1\right]\\=&2\pi i\left[\sqrt[4]{2}\sin\left(\frac{3\pi}{8}\right)-1\right]\end{aligned}$$

Therefore

$$\int_0^{\frac{\pi}{4}}\sqrt{\tan x-\tan^2 x}\,\mathrm{d}x=\int_0^1\frac{\sqrt{x}\sqrt{1-x}}{1+x^2}\mathrm{d}x=\pi\left[\sqrt[4]{2}\sin\left(\frac{3\pi}{8}\right)-1\right]$$

□

Exercise 1.1.81: Prove the following integral:

$$\int_0^1\frac{\ln 2-\ln\left(1+x^2\right)}{1-x}\,\mathrm{d}x=\frac{5\pi^2}{48}-\frac{\ln^2 2}{4}$$

Proof: 1.Differentiation under the Integral Sign:

Denote

$$\mathcal{I}(a)=\int_0^1\frac{-2x\ln(1-ax)}{1+x^2}\mathrm{d}x$$

Differentiating yields

$$\begin{aligned}\mathcal{I}'(a)=&\int_0^1\frac{2x^2}{(1+x^2)(1-ax)}\mathrm{d}x\\=&-\frac{2}{1+a^2}\int_0^1\left(\frac{ax}{1+x^2}+\frac{1}{1+x^2}-\frac{1}{1-ax}\right)\mathrm{d}x\\=&-\frac{a\ln 2}{1+a^2}-\frac{\pi}{2}\frac{1}{1+a^2}-\frac{2\ln(1-a)}{a(1+a^2)}\end{aligned}$$

Integrating back,

$$\begin{aligned}\mathcal{I}(1)=&\int_0^1\left(-\frac{a\ln 2}{1+a^2}-\frac{\pi}{2}\frac{1}{1+a^2}-\frac{2\ln(1-a)}{a(1+a^2)}\right)\mathrm{d}a\\=&-\frac{1}{2}\ln^2 2-\frac{\pi}{2}\arctan 1+2\mathrm{Li}_2(1)-\int_0^1\frac{-2a\ln(1-a)}{1+a^2}\mathrm{d}a\\=&\frac{5\pi^2}{24}-\frac{1}{2}\ln^2 2-\mathcal{I}(1)\end{aligned}$$

and a simple rearrangement yields

$$\mathcal{I}(1) = \frac{5\pi^2}{48} - \frac{1}{4}\ln^2 2$$

2.Contour Integration:

Contour integration is, admittedly, an extremely inefficient method to tackle this integral. Nevertheless, I will post this approach purely for the fun of it.

Integrate $f(z) = \dfrac{\ln 2 - \ln(1+z^2)}{1-z}$ along a unit quarter circle in the first quadrant with indents at the pole $z = 1$ and the branch point $z = i$. Both of the contributions from the indents tend to zero. Since the contour encloses no poles,

$$\int_0^1 \frac{\ln 2 - \ln(1+x^2)}{1-x}\mathrm{d}x + \left(\int_{\text{arc}} + \int_{\text{line}}\right)\frac{\ln 2 - \ln(1+z^2)}{1-z}\mathrm{d}z = 0$$

Around the arc,

$$\begin{aligned}
\Re\int_{\text{arc}} \frac{\ln 2 - \ln(1+z^2)}{1-z}\mathrm{d}z =& \Re\int_0^{\frac{\pi}{2}} \frac{\ln 2 - \ln(1+e^{i2\theta})}{1-e^{i\theta}} ie^{i\theta}\mathrm{d}\theta \\
=& \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\cos\theta)\cot\frac{\theta}{2}\,\mathrm{d}\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta\,\mathrm{d}\theta \\
=& -\int_0^1 \frac{2\operatorname{artanh} t^2}{t(1+t^2)}\mathrm{d}t - \frac{\pi^2}{16} \\
=& -\chi_2(1) + \frac{1}{4}\ln^2(1+t^2)\Big|_0^1 - \int_0^1 \frac{t\ln(1-t^2)}{1+t^2} - \frac{\pi^2}{16} \\
=& -\frac{3\pi^2}{16} + \frac{1}{4}\ln^2 2 - \frac{1}{2}\int_0^1 \frac{\ln(1-t)}{1+t}\mathrm{d}t
\end{aligned}$$

Along the line segment on the imaginary axis,

$$\begin{aligned}
\Re\int_{\text{line}} \frac{\ln 2 - \ln(1+z^2)}{1-z}\mathrm{d}z =& \Re\int_1^0 i\frac{\ln 2 - \ln(1-y^2)}{1-iy}\mathrm{d}y \\
=& \int_0^1 \frac{y\ln 2}{1+y^2}\mathrm{d}y - \int_0^1 \frac{y\ln(1-y^2)}{1+y^2}\mathrm{d}y \\
=& \frac{1}{2}\ln^2 2 - \frac{1}{2}\int_0^1 \frac{\ln(1-y)}{1+y}\mathrm{d}y
\end{aligned}$$

Hence we have

$$\int_0^1 \frac{\ln 2 - \ln(1+x^2)}{1-x}\mathrm{d}x = \frac{3\pi^2}{16} - \frac{3}{4}\ln^2 2 + \int_0^1 \frac{\ln(1-x)}{1+x}\mathrm{d}x$$

For the sake of consistency only, I will use contour integration once again to evaluate the remaining integral though it can be computed easily with real methods. This time, integrate $g(z) = \dfrac{\ln(1-z)}{1+z}$ along a unit quarter circle with an indent at $z = 1$. The contribution from the indent vanishes, hence leaving us with

$$\int_0^1 \frac{\ln(1-x)}{1+x}\mathrm{d}x + \left(\int_{\text{arc}} + \int_{\text{line}}\right)\frac{\ln(1-z)}{1+z}\mathrm{d}z = 0$$

Around the arc,

$$\begin{aligned}\Re\int_{\text{arc}}\frac{\ln(1-z)}{1+z}\mathrm{d}z=&\Re\int_0^{\frac{\pi}{2}}\frac{\ln(1-e^{i\theta})}{1+e^{i\theta}}ie^{i\theta}\ \mathrm{d}\theta\\=&-\int_0^{\frac{\pi}{4}}\ln(2\sin x)\tan x\ \mathrm{d}x-\frac{1}{4}\int_0^{\frac{\pi}{2}}(x-\pi)\ \mathrm{d}x\\=&\frac{3\pi^2}{32}-\frac{1}{2}\ln^2 2-\frac{1}{4}\underbrace{\int_0^1\frac{\ln t}{1+t}\mathrm{d}t}_{\mathrm{Li}_2(-1)}+\frac{1}{4}\int_0^1\frac{\ln(1+t)}{1+t}\mathrm{d}t\\=&\frac{11\pi^2}{96}-\frac{3}{8}\ln^2 2\end{aligned}$$

Along the line segment on the imaginary axis,

$$\begin{aligned}\Re\int_{\text{line}}\frac{\ln(1-z)}{1+z}\mathrm{d}z=&\Re\int_1^0\frac{i\ln(1-iy)}{1+iy}\mathrm{d}y\\=&-\int_0^1\frac{\arctan y}{1+y^2}\mathrm{d}y-\frac{1}{4}\int_0^1\frac{\ln(1+y)}{1+y}\mathrm{d}y\\=&\left[-\frac{1}{2}\arctan^2 y-\frac{1}{8}\ln^2(1+y)\right]_0^1\\=&-\frac{\pi^2}{32}-\frac{1}{8}\ln^2 2\end{aligned}$$

Thus we have

$$\int_0^1\frac{\ln(1-x)}{1+x}\mathrm{d}x=-\frac{\pi^2}{12}+\frac{1}{2}\ln^2 2$$

and consequently,

$$\int_0^1\frac{\ln 2-\ln(1+x^2)}{1-x}\mathrm{d}x=\frac{5\pi^2}{48}-\frac{1}{4}\ln^2 2$$

□

Exercise 1.1.82: Evaluate:

$$\int_0^1\ln^2(1-x)\ln^2(x)\,\mathrm{d}x$$

Solution: Using the series of $\ln^2(1-x)$,

$$\begin{aligned}\int_0^1\ln^2 x\ln^2(1-x)\ \mathrm{d}x=&\sum_{n=1}^{\infty}\frac{2H_n}{n+1}\int_0^1 x^{n+1}\ln^2 x\ \mathrm{d}x\\=&\sum_{n=1}^{\infty}\frac{4H_n}{(n+1)(n+2)^3}\end{aligned}$$

Then integrate $f(z)=\dfrac{(\gamma+\psi_0(-z))^2}{(z+1)(z+2)^3}$ along an infinitely large square. The integral vanishes which implies the sum of its residues is zero. At the positive integers,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathrm{Res}(f,n)=&\sum_{n=1}^{\infty}\operatorname*{Res}_{z=n}\left[\frac{1}{(z+1)(z+2)^3(z-n)^2}+\frac{2H_n}{(z+1)(z+2)^3(z-n)}\right]\\=&\sum_{n=1}^{\infty}\frac{2H_n}{(n+1)(n+2)^3}-\sum_{n=1}^{\infty}\frac{4n+5}{(n+1)^2(n+2)^4}\\=&\sum_{n=1}^{\infty}\frac{2H_n}{(n+1)(n+2)^3}+\frac{\pi^4}{30}+2\zeta(3)-\frac{91}{16}\end{aligned}$$

At $z=0$,

$$\begin{aligned}\operatorname{Res}(f,0) &= \operatorname*{Res}_{z=0} \frac{1}{z^2(z+1)(z+2)^3} \\ &= -\frac{5}{16}\end{aligned}$$

At $z=1$,

$$\operatorname{Res}(f,-1) = 0$$

At $z=2$,

$$\begin{aligned}\operatorname{Res}(f,-2) &= \frac{1}{2}\lim_{z\to -2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{(\gamma+\psi_0(-z))^2}{z+1} \\ &= -\frac{\pi^4}{36}+2\zeta(3)+\frac{2\pi^2}{3}-6\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 \ln^2 x \ln^2(1-x)\,\mathrm{d}x &= 2\left[-\frac{\pi^4}{30}-2\zeta(3)+\frac{91}{16}+\frac{5}{16}+\frac{\pi^4}{36}-2\zeta(3)-\frac{2\pi^2}{3}+6\right] \\ &= -\frac{\pi^4}{90}-8\zeta(3)-\frac{4\pi^2}{3}+24\end{aligned}$$

□

Exercise 1.1.83: Prove:

$$\int_0^{\frac{\pi}{2}} \ln\left(\frac{\ln^2\sin\theta}{\pi^2+\ln^2\sin\theta}\right)\frac{\ln\cos\theta}{\tan\theta}\,\mathrm{d}\theta = \frac{\pi^2}{4}$$

Proof: I will use the following result:

$$\lim_{N\to\infty}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k(2k)!}N^{2k}-\ln N\right] = \gamma$$

Letting $\ln(\sin\theta) = -x$ and using $\mathcal{I}$ to denote the integral in question,

$$\begin{aligned}
\mathcal{I} =& \frac{1}{2}\int_0^\infty \ln(1-e^{-2x})\ln\left(\frac{x^2}{\pi^2+x^2}\right)\,\mathrm{d}x \\
=& -\frac{\partial}{\partial a}\bigg|_{a=0} \sum_{n=1}^\infty \frac{1}{n}\int_0^\infty x^a e^{-2nx}\,\mathrm{d}x + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty e^{-2nx}\ln(\pi^2+x^2)\,\mathrm{d}x \\
=& -\frac{\partial}{\partial a}\bigg|_{a=0} \frac{\Gamma(a+1)\zeta(a+2)}{2^{a+1}} + \frac{1}{2}\sum_{n=1}^\infty \frac{\ln\pi}{n^2} + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{xe^{-2nx}}{\pi^2+x^2}\mathrm{d}x \\
=& -\frac{1}{2}\Gamma'(1)\zeta(2) - \frac{1}{2}\Gamma(1)\zeta'(2) + \frac{1}{2}\Gamma(1)\zeta(2)\ln 2 + \frac{1}{2}\zeta(2)\ln\pi \\
&+ \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty \int_0^\infty e^{-2nx}e^{-xy}\cos\pi y\,\mathrm{d}x\,\mathrm{d}y \\
=& \frac{\pi^2}{12}(\gamma+\ln 2\pi) - \frac{1}{2}\zeta'(2) + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{\cos\pi y}{y+2n}\mathrm{d}y \\
=& \frac{\pi^2}{12}(\gamma+\ln 2\pi) - \frac{1}{2}\zeta'(2) + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{\cos(y+2n\pi)}{y+2n\pi}\mathrm{d}y \\
=& \frac{\pi^2}{12}(\gamma+\ln 2\pi) - \frac{1}{2}\zeta'(2) + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\int_{2n\pi}^\infty \frac{\cos y}{y}\mathrm{d}y \\
=& \frac{\pi^2}{12}(\gamma+\ln 2\pi) - \frac{1}{2}\zeta'(2) + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\left[\left(\int_0^\infty - \int_0^{2n\pi}\right)\frac{\cos y-1}{y}\mathrm{d}y + \ln y\bigg|_{2n\pi}^\infty\right] \\
=& \frac{\pi^2}{12}(\gamma+\ln 2\pi) - \frac{1}{2}\zeta'(2) + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^{2n\pi} \frac{1-\cos y}{y}\mathrm{d}y - \frac{1}{2}\sum_{n=1}^\infty \frac{\ln(2n\pi)}{n^2} \\
&+ \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^\infty \frac{1}{n^2}\left[\ln N - \sum_{k=1}^\infty \frac{(-1)^{k+1}}{2k(2k)!}N^{2k}\right] \\
=& \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^1 \frac{1-\cos 2n\pi y}{y}\mathrm{d}y \\
=& \frac{1}{2}\int_0^1 \left(\frac{\pi^2}{6y} - \pi^2 y + \pi^2 - \frac{\pi^2}{6y}\right)\,\mathrm{d}y = \frac{\pi^2}{2}\int_0^1 (1-y)\,\mathrm{d}y = \frac{\pi^2}{4}
\end{aligned}$$

□

Exercise 1.1.84: Find a closed-form for:

$$\int_0^1 \frac{\mathrm{Li}_3^2(-x)}{x^2}\,\mathrm{d}x$$

Solution: Repeatedly integrate by parts.

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{Li}_3^2(-x)}{x^2}\mathrm{d}x = -\frac{\mathrm{Li}_3^2(-x)}{x}\bigg|_0^1 + \int_0^1 \frac{2\mathrm{Li}_2(-x)\mathrm{Li}_3(-x)}{x^2}\mathrm{d}x \\
=& -\frac{9}{16}\zeta^2(3) - \frac{2\mathrm{Li}_2(-x)\mathrm{Li}_3(-x)}{x}\bigg|_0^1 + \int_0^1 \frac{2\mathrm{Li}_2^2(-x)}{x^2}\mathrm{d}x - \int_0^1 \frac{2\mathrm{Li}_3(-x)\ln(1+x)}{x^2}\mathrm{d}x \\
=& -\frac{9}{16}\zeta^2(3) - \frac{3}{4}\zeta(2)\zeta(3) - \frac{2\mathrm{Li}_2^2(-x)}{x}\bigg|_0^1 - \int_0^1 \frac{4\mathrm{Li}_2(-x)\ln(1+x)}{x^2}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\mathrm{Li}_3(-x)\ln(1+x)}{x}\bigg|_0^1-\int_0^1\frac{2\mathrm{Li}_3(-x)}{x(1+x)}\mathrm{d}x-\int_0^1\frac{2\mathrm{Li}_2(-x)\ln(1+x)}{x^2}\mathrm{d}x\\
=&-\frac{9}{16}\zeta^2(3)-\frac{3}{4}\zeta(2)\zeta(3)-\frac{1}{2}\zeta^2(2)+\frac{7}{4}\zeta(4)-\frac{3}{2}\zeta(3)\ln 2+\int_0^1\frac{2\mathrm{Li}_3(-x)}{1+x}\mathrm{d}x\\
&+\frac{6\mathrm{Li}_2(-x)\ln(1+x)}{x}\bigg|_0^1-\int_0^1\frac{6\mathrm{Li}_2(-x)}{x(1+x)}\mathrm{d}x+\int_0^1\frac{6\ln^2(1+x)}{x^2}\mathrm{d}x\\
=&-\frac{9}{16}\zeta^2(3)-\frac{3}{4}\zeta(2)\zeta(3)-\frac{1}{2}\zeta^2(2)+\frac{7}{4}\zeta(4)-\frac{3}{2}\zeta(3)\ln 2+\frac{9}{2}\zeta(3)-3\zeta(2)\ln 2\\
&+\int_0^1\frac{6\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x+\int_0^1\frac{2\mathrm{Li}_3(-x)}{1+x}\mathrm{d}x-\frac{6\ln^2(1+x)}{x}\bigg|_0^1+\int_0^1\frac{12\ln(1+x)}{x(1+x)}\mathrm{d}x\\
=&-\frac{9}{16}\zeta^2(3)-\frac{3}{4}\zeta(2)\zeta(3)-\frac{1}{2}\zeta^2(2)+\frac{7}{4}\zeta(4)-\frac{3}{2}\zeta(3)\ln 2+\frac{9}{2}\zeta(3)-3\zeta(2)\ln 2\\
&+6\zeta(2)-12\ln^2 2+\int_0^1\frac{6\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x+\int_0^1\frac{2\mathrm{Li}_3(-x)}{1+x}\mathrm{d}x
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_0^1\frac{6\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x=&6\mathrm{Li}_2(-x)\ln(1+x)\bigg|_0^1+\int_0^1\frac{6\ln^2(1+x)}{x}\mathrm{d}x\\
=&\frac{3}{2}\zeta(3)-3\zeta(2)\ln 2
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1\frac{2\mathrm{Li}_3(-x)}{1+x}\mathrm{d}x=&2\mathrm{Li}_3(-x)\ln(1+x)\bigg|_0^1-\int_0^1\underbrace{\frac{2\mathrm{Li}_2(-x)\ln(1+x)}{x}\mathrm{d}x}_{2\mathrm{Li}_2(-x)\ \mathrm{d}\mathrm{Li}_2(-x)}\\
=&\frac{1}{4}\zeta^2(2)-\frac{3}{2}\zeta(3)\ln 2
\end{aligned}
$$

The closed form is

$$
\begin{aligned}
\int_0^1\frac{\mathrm{Li}_3^2(-x)}{x^2}\mathrm{d}x=&-\frac{9}{16}\zeta^2(3)-\frac{3}{4}\zeta(2)\zeta(3)-\frac{1}{4}\zeta^2(2)+\frac{7}{4}\zeta(4)-3\zeta(3)\ln 2\\
&+6\zeta(3)-6\zeta(2)\ln 2+6\zeta(2)-12\ln^2 2
\end{aligned}
$$

□

Exercise 1.1.85: Find a closed-form for:

$$I=\int_0^{\pi/2}\tan^{-1}\left(\frac{\cos(x)}{\sin(x)-1-\sqrt{2}}\right)\tan(x)\mathrm{d}x$$

Solution: **Step 1: Introducing an extra parameter**

Define

$$\phi(\alpha)=\int_0^{\frac{\pi}{2}}\arctan\left(\frac{\sin x}{\cos x-\frac{1}{\alpha}}\right)\cot x\ \mathrm{d}x$$

Differentiating yields

$$\begin{aligned}\phi'(\alpha) =& -\int_0^{\frac{\pi}{2}} \frac{\cos x}{1-2\alpha\cos x+\alpha^2}\mathrm{d}x \\ =& \frac{\pi}{4\alpha} - \frac{1+\alpha^2}{2\alpha(1-\alpha^2)}\int_0^{\frac{\pi}{2}} \frac{1-\alpha^2}{1-2\alpha\cos x+\alpha^2}\mathrm{d}x\end{aligned}$$

Step 2: Evaluation of $\phi'(\alpha)$

For $|\alpha|<1$, the following identity holds.

$$\frac{1-\alpha^2}{1-2\alpha\cos x+\alpha^2} = 1+2\sum_{n=1}^{\infty}\alpha^n\cos(nx)$$

Therefore,

$$\begin{aligned}\phi'(\alpha) =& \frac{\pi}{4\alpha} - \frac{1+\alpha^2}{2\alpha(1-\alpha^2)}\left(\frac{\pi}{2}+2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\alpha^{2n+1}\right) \\ =& -\frac{\pi\alpha}{2(1-\alpha^2)} - \frac{\arctan\alpha}{\alpha} - \frac{2\alpha\arctan\alpha}{1-\alpha^2} \qquad (1)\end{aligned}$$

Step 3: The Closed Form

Integrating back, we get

$$\phi(\sqrt{2}-1) = \left(\frac{\pi}{4}+\arctan\alpha\right)\ln(1-\alpha^2)\Big|_0^{\sqrt{2}-1} - \int_0^{\sqrt{2}-1}\left[\frac{\arctan\alpha}{\alpha}+\frac{\ln(1-\alpha^2)}{1+\alpha^2}\right]\mathrm{d}\alpha \qquad (2)$$

$$= \frac{3\pi}{8}\ln(2\sqrt{2}-2) \underbrace{- \int_0^{\frac{\pi}{8}} 2x\csc 2x\ \mathrm{d}x + \int_0^{\frac{\pi}{8}} 2\ln(\cos x)\ \mathrm{d}x} - \int_0^{\frac{\pi}{8}}\ln(\cos 2x)\ \mathrm{d}x \qquad (3)$$

$$= \frac{3\pi}{8}\ln(2\sqrt{2}-2) - x\ln(\tan x)\Big|_0^{\frac{\pi}{8}} + \int_0^{\frac{\pi}{8}}\ln\left(\frac{1}{2}\sin 2x\right)\ \mathrm{d}x - \int_0^{\frac{\pi}{8}}\ln(\cos 2x)\ \mathrm{d}x \qquad (4)$$

$$= \frac{\pi}{4}\ln(2\sqrt{2}-2) + \int_0^{\frac{\pi}{8}}\ln(\tan 2x)\ \mathrm{d}x$$

$$= \frac{\pi}{4}\ln(2\sqrt{2}-2) - 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\frac{\pi}{8}}\cos\left((8n+4)x\right)\ \mathrm{d}x \qquad (5)$$

$$= \frac{\pi}{4}\ln(2\sqrt{2}-2) - 2\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{(2n+1)(8n+4)}$$

$$= \frac{\pi}{4}\ln(2\sqrt{2}-2) - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \qquad (6)$$

$$= \frac{\pi}{4}\ln(2\sqrt{2}-2) - \frac{\mathbf{G}}{2} \qquad (7)$$

Explanation:

(2): Integrated (1) from 0 to $\sqrt{2}-1$. Integrated $\dfrac{2\alpha\arctan\alpha}{1-\alpha^2}$ by parts.

(3): Applied the substitution $\alpha=\tan x$.

(4): Integrated $-2x\csc 2x$ by parts.

(5): Used the Fourier series of $\ln(\tan 2x)$.

(6): $\cos(n\pi)=(-1)^n$ for $n\in\mathbb{N}$.

(7): Used the definition of Catalan's constant. □

Exercise 1.1.86: Prove the following integral:

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)\ln^2(x)}{x}\mathrm{d}x = -\frac{\zeta(6)}{3}$$

Proof: Method One:

Hint.

Observe that

$$\frac{d}{dx}\mathrm{Li}_{p+1}(x) = \frac{\mathrm{Li}_p(x)}{x} \tag{1}$$

and

$$\int_0^1 x^\alpha \,\mathrm{Li}_p(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^p}\int_0^1 x^{\alpha+n}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^p(n+\alpha+1)}, \quad \alpha > -2. \tag{2}$$

Differentiating (2) twice w. r. t. α gives

$$\int_0^1 x^\alpha \,\mathrm{Li}_p(x)\ln x\,\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{1}{n^p(n+\alpha+1)^2} \tag{3}$$

$$\int_0^1 x^\alpha \,\mathrm{Li}_p(x)\ln^2 x\,\mathrm{d}x = 2\sum_{n=1}^{\infty}\frac{1}{n^p(n+\alpha+1)^3}. \tag{4}$$

Now set $I := \displaystyle\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)\ln^2(x)}{x}\mathrm{d}x.$
We may write

$$\begin{aligned}
I &= \int_0^1 \mathrm{Li}'_3(x)\ln(1-x)\ln^2 x\,\mathrm{d}x \\
&= \mathrm{Li}_3(x)\ln(1-x)\ln^2 x\Big|_0^1 - \int_0^1 \mathrm{Li}_3(x)\left(\ln(1-x)\ln^2 x\,\mathrm{d}x\right)'\,\mathrm{d}x \\
&= \int_0^1 \mathrm{Li}_3(x)\frac{\ln^2 x}{1-x}\mathrm{d}x - 2\int_0^1 \frac{\mathrm{Li}_3(x)}{x}\ln(1-x)\ln x\,\mathrm{d}x \\
&= \int_0^1 \mathrm{Li}_3(x)\frac{\ln^2 x}{1-x}\mathrm{d}x - 2\left(\mathrm{Li}_4(x)\ln(1-x)\ln x\Big|_0^1 - \int_0^1 \mathrm{Li}_4(x)\left(\ln(1-x)\ln x\,\mathrm{d}x\right)'\,\mathrm{d}x\right) \\
&= \int_0^1 \mathrm{Li}_3(x)\frac{\ln^2 x}{1-x}\mathrm{d}x - 2\int_0^1 \mathrm{Li}_4(x)\frac{\ln x}{1-x}\mathrm{d}x + 2\int_0^1 \mathrm{Li}_4(x)\frac{\ln(1-x)}{x}\mathrm{d}x \\
&= \int_0^1 \mathrm{Li}_3(x)\frac{\ln^2 x}{1-x}\mathrm{d}x - 2\int_0^1 \mathrm{Li}_4(x)\frac{\ln x}{1-x}\mathrm{d}x + 2\int_0^1 \frac{\mathrm{Li}_5(x)-\zeta(5)}{1-x}\mathrm{d}x.
\end{aligned}$$

Then, using (4),

$$\int_0^1 \mathrm{Li}_3(x)\frac{\ln^2 x}{1-x}\mathrm{d}x = \sum_{n=0}^{\infty}\int_0^1 x^n\mathrm{Li}_3(x)\ln^2 x\mathrm{d}x = 2\sum_{n,k\geq 1}^{\infty}\frac{1}{k^3(k+n)^3} = \zeta^2(3)-\zeta(6)$$

similarly, using (3),

$$-2\int_0^1 \mathrm{Li}_4(x)\frac{\ln x}{1-x}\mathrm{d}x = -2\sum_{n=0}^{\infty}\int_0^1 x^n\mathrm{Li}_4(x)\ln x\mathrm{d}x = 2\sum_{n,k\geq 1}^{\infty}\frac{1}{k^4(k+n)^2} = -2\zeta^2(3)+\frac{25\zeta(6)}{6}^*$$

and using (2),

$$2\int_0^1 \frac{\mathrm{Li}_5(x)-\zeta(5)}{1-x}\mathrm{d}x = 2\sum_{n=0}^{\infty}\int_0^1 x^n\left(\mathrm{Li}_5(x)-\zeta(5)\right)\mathrm{d}x = -2\sum_{n,k\geq 1}\frac{1}{k^4 n(k+n)} = -2\sum_{k=1}^{\infty}\frac{H_k}{k^5}$$

From Euler's standard formula we get

$$-2\sum_{k=1}^{\infty}\frac{H_k}{k^5} = \zeta^2(3) - \frac{7\zeta(6)}{2}$$

Putting all this together, we end up with

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)\ln^2(x)}{x}\mathrm{d}x = -\zeta(6) + \frac{25\zeta(6)}{6} - \frac{7\zeta(6)}{2} = -\frac{\zeta(6)}{3}.$$

*We readily have

$$\begin{aligned}
2\sum_{n,k\geq 1}\frac{1}{k^4(k+n)^2} &= 2\sum_{k\geq 1}\frac{1}{k^4}\sum_{n=1}^{\infty}\frac{1}{(n+k)^2} = 2\sum_{k\geq 1}\frac{1}{k^4}\left(\zeta(2)-\sum_{n=1}^{k}\frac{1}{n^2}\right)\\
&= 2\sum_{k\geq 1}^{\infty}\frac{1}{k^4}\left(\zeta(2)-\frac{1}{k^2}-\sum_{n=1}^{k-1}\frac{1}{n^2}\right) = 2\zeta(2)\zeta(4)-2\zeta(6)-2\zeta(4,2)\\
&= \frac{3}{2}\zeta(6)-2\zeta(4,2)
\end{aligned}$$

where $\zeta(4,2)$ denotes a Multi Zeta Values (MZVs) namely

$$\zeta(4,2) := \sum_{k\geq 1}^{\infty}\frac{1}{k^4}\sum_{n=1}^{k-1}\frac{1}{n^2}.$$

Obtaining a reduction formula for $\zeta(4,2)$ is a tough part in this evaluation. We have

$$\zeta(4,2) = \zeta^2(3) - \frac{4}{3}\zeta(6),$$

and a proof, using MZVs algebra, may be found here

http://arxiv.org/pdf/1012.5192.pdf [p.8(3.10)].

Method Two:

By integrating by parts and expanding using $\sum_{n=1}^{\infty} H_n^{(p)}x^n = \frac{\mathrm{Li}_p(x)}{1-x}$, we obtain

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2 x\ln(1-x)}{x} &= \frac{1}{3}\int_0^1 \frac{\ln^3 x\ln^2(1-x)}{x}\mathrm{d}x + \frac{1}{3}\int_0^1 \frac{\mathrm{Li}_2(x)\ln^3 x}{1-x}\mathrm{d}x\\
&= \frac{1}{6}\int_0^1 \frac{\ln^4 x\ln(1-x)}{1-x}\mathrm{d}x + \frac{1}{3}\int_0^1 \frac{\mathrm{Li}_2(x)\ln^3 x}{1-x}\mathrm{d}x\\
&= -\frac{1}{6}\sum_{n=1}^{\infty}H_n\int_0^1 x^n\ln^4 x\,\mathrm{d}x + \frac{1}{3}\sum_{n=1}^{\infty}H_n^{(2)}\int_0^1 x^n\ln^3 x\,\mathrm{d}x\\
&= -4\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^5} - 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^4} = -4\sum_{n=1}^{\infty}\frac{H_n}{n^5} - 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} + 6\zeta(6)
\end{aligned}$$

Using the following formulae,

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^q} = (q+2)\zeta(q+1) - \sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j) \tag{1}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q} = \zeta(p)\zeta(q) + \zeta(p+q) - \sum_{n=1}^{\infty}\frac{H_n^{(q)}}{n^p} \tag{2}$$

we can compute the two sums as such:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\zeta(6)+\sum_{n=1}^{\infty}\frac{H_{n-1}^{(2)}}{n^4}=\zeta(6)+\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{k=1}^{n-1}\frac{1}{(n-k)^2} \tag{3}$$

$$=\zeta(6)+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^2(n+k)^4} \tag{4}$$

$$\begin{aligned}=\zeta(6)&-\sum_{k=1}^{\infty}\frac{4}{k^5}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+k}\right)+\sum_{k=1}^{\infty}\frac{1}{k^4}\sum_{n=1}^{\infty}\frac{1}{n^2}\\&+\sum_{k=1}^{\infty}\frac{3}{k^4}\sum_{n=1}^{\infty}\frac{1}{(n+k)^2}+\sum_{k=1}^{\infty}\frac{2}{k^3}\sum_{n=1}^{\infty}\frac{1}{(n+k)^3}\\&+\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{n=1}^{\infty}\frac{1}{(n+k)^4}\end{aligned} \tag{5}$$

$$\begin{aligned}=\zeta(6)&-4\sum_{k=1}^{\infty}\frac{H_k}{k^5}+\zeta(2)\zeta(4)+3\zeta(2)\zeta(4)-3\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}\\&+2\zeta^2(3)-2\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^3}+\zeta(2)\zeta(4)-\sum_{k=1}^{\infty}\frac{H_k^{(4)}}{k^2}\end{aligned}$$

$$=-8\zeta(6)+3\zeta^2(3)+4\zeta(2)\zeta(4)-2\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4} \tag{6}$$

$$=-\frac{8}{3}\zeta(6)+\zeta^2(3)+\frac{4}{3}\zeta(2)\zeta(4) \tag{7}$$

Explanation:

(1):See here

http://math.stackexchange.com/questions/469023/.

(3):Expand $H_{n-1}^{(2)}$.

(4):Reverse the order of summation.

(5):Partial Fractions.

(6):Simplify the sums with (1) and (2).

(7):Move the unknown sum over to the other side and divide by 3.

Proof of (2):

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}&=\sum_{n=1}^{\infty}\frac{1}{n^q}\sum_{k=1}^{n}\frac{1}{n^p}\\&=\sum_{n=1}^{\infty}\frac{1}{n^q}\left(\sum_{k=1}^{\infty}\frac{1}{k^p}-\sum_{k=n}^{\infty}\frac{1}{k^p}+\frac{1}{n^p}\right)\\&=\sum_{n=1}^{\infty}\frac{1}{n^q}\sum_{k=1}^{\infty}\frac{1}{k^p}+\sum_{n=1}^{\infty}\frac{1}{n^{p+q}}-\sum_{k=1}^{\infty}\frac{1}{k^p}\sum_{n=1}^{k}\frac{1}{n^q}\\&=\zeta(p)\zeta(q)+\zeta(p+q)-\sum_{n=1}^{\infty}\frac{H_n^{(q)}}{n^p}\end{aligned}$$

Hence your integral is

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2 x\ln(1-x)}{x}\mathrm{d}x = \frac{13}{3}\zeta(6) - \frac{8}{3}\zeta(2)\zeta(4) = -\frac{\pi^6}{2835}$$

Method Three:

We have,

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)\ln^2 x}{x}\,\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\,\mathrm{Li}_2(x)\ln^2 x\,\mathrm{d}x$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}\frac{1}{m^2}\int_0^1 x^{m+n-1}\ln^2 x\,\mathrm{d}x = -2\sum_{n,m=1}^{\infty}\frac{1}{nm^2(m+n)^3}$$
$$= -\sum_{n,m=1}^{\infty}\frac{1}{nm^2(m+n)^3} - \sum_{n,m=1}^{\infty}\frac{1}{mn^2(m+n)^3} = -\sum_{n,m=1}^{\infty}\frac{1}{n^2m^2(m+n)^2} = -\frac{1}{3}\zeta(6)$$

The last double summation is proved here

http://math.stackexchange.com/a/1008037/129017. □

Exercise 1.1.87: Closed form for:

$$\int_0^{\pi}\arctan\left(\frac{\ln(\sin x)}{x}\right)\mathrm{d}x$$

Solution: Let

$$I(s) = \int_0^{\pi}\arctan\left(\frac{\ln(s\sin x)}{x}\right)\mathrm{d}x$$

Differentiate under the integral sign to get

$$I'(s) = \frac{1}{s}\int_0^{\pi}\frac{x}{x^2+\ln^2(s\sin x)}\mathrm{d}x = -\frac{1}{s}\Im\int_0^{\pi}\frac{1}{\ln\left(\dfrac{se^{i2x}-s}{2i}\right)}\mathrm{d}x$$
$$= -\frac{1}{s}\Im\int_{|z|=1}\frac{1}{\ln\left(\dfrac{sz-s}{2i}\right)}\frac{\mathrm{d}z}{2iz} = -\frac{1}{s}\Im\frac{\pi}{\ln\left(-\dfrac{s}{2i}\right)}$$
$$= -\frac{1}{s}\Im\frac{\pi}{\ln\left(\dfrac{s}{2}\right)+\dfrac{\pi i}{2}}\frac{\ln\left(\dfrac{s}{2}\right)-\dfrac{\pi i}{2}}{\ln\left(\dfrac{s}{2}\right)-\dfrac{\pi i}{2}} = \frac{1}{2s}\frac{\pi^2}{\ln^2\left(\dfrac{s}{2}\right)+\dfrac{\pi^2}{4}}$$

where the fourth equality follows from the residue theorem and the fact that the indent around the branch point $z=1$ produces no contribution as $\epsilon\to 0$.

Integrating back,

$$I(1) = I(\infty) + \frac{\pi^2}{2}\int_{s=\infty}^{s=1}\frac{1}{\ln^2\left(\dfrac{s}{2}\right)+\dfrac{\pi^2}{4}}\mathrm{d}\ln\left(\frac{s}{2}\right) = \frac{\pi^2}{2} + \frac{\pi^2}{2}\frac{2}{\pi}\arctan\left(\frac{2\ln\left(\dfrac{s}{2}\right)}{\pi}\right)\Bigg|_{\infty}^{1}$$
$$= -\pi\arctan\left(\frac{2\ln 2}{\pi}\right)$$

□

Exercise 1.1.88: Find an elegant approach for the integral:

$$I_1=\int_0^1 \frac{\ln(x)}{\sqrt{x(x+1)}}\mathrm{d}x,\quad I_2=\int_0^1 \frac{\ln^2(x)}{\sqrt{x(x+1)}}\mathrm{d}x$$

Solution: As an alternative, we may apply the substitution $x\mapsto \sinh^2 x$.

$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x(x+1)}}\mathrm{d}x &= 4\int_0^{-\ln(\sqrt{2}-1)} \ln(\sinh x)\,\mathrm{d}x\\ &= 4\int_0^{-\ln(\sqrt{2}-1)} \left(\ln\left(1-e^{-2x}\right)-\ln 2+x\right)\,\mathrm{d}x\\ &= -4\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{-\ln(\sqrt{2}-1)} e^{-2nx}\,\mathrm{d}x+4\ln 2\ln\left(\sqrt{2}-1\right)+2\ln^2\left(\sqrt{2}-1\right)\\ &= 2\sum_{n=1}^{\infty}\frac{\left(3-2\sqrt{2}\right)^n-1}{n^2}+4\ln 2\ln\left(\sqrt{2}-1\right)+2\ln^2\left(\sqrt{2}-1\right)\\ &= 2\mathrm{Li}_2\left(3-2\sqrt{2}\right)-\frac{\pi^2}{3}+4\ln 2\ln\left(\sqrt{2}-1\right)+2\ln^2\left(\sqrt{2}-1\right)\end{aligned}$$

Similarly, for the second integral,

$$\int_0^1 \frac{\ln^2 x}{\sqrt{x(x+1)}}\,\mathrm{d}x=8\int_0^{-\ln(\sqrt{2}-1)}\ln^2(\sinh x)\,\mathrm{d}x=8(A+B+C)$$

where

$$\begin{aligned}A&=\int_0^{-\ln(\sqrt{2}-1)}\left(x^2-2x\ln 2-2\ln 2\ln\left(1-e^{-2x}\right)+\ln^2 2\right)\,\mathrm{d}x\\ &=-\mathrm{Li}_2\left(3-2\sqrt{2}\right)\ln 2+\frac{\pi^2}{6}\ln 2-\ln^2 2\ln\left(\sqrt{2}-1\right)-\ln 2\ln^2\left(\sqrt{2}-1\right)-\frac{1}{3}\ln^3\left(\sqrt{2}-1\right)\end{aligned}$$

and

$$\begin{aligned}B&=\int_0^{-\ln(\sqrt{2}-1)}2x\ln\left(1-e^{-2x}\right)\,\mathrm{d}x\\ &=-2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{-\ln(\sqrt{2}-1)}xe^{-2nx}\,\mathrm{d}x\\ &=2\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{xe^{-2nx}}{2n}+\frac{e^{-2nx}}{4n^2}\right]_0^{-\ln(\sqrt{2}-1)}\\ &=-\ln\left(\sqrt{2}-1\right)\sum_{n=1}^{\infty}\frac{\left(3-2\sqrt{2}\right)^n}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(3-2\sqrt{2}\right)^n}{n^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\\ &=\frac{1}{2}\mathrm{Li}_3\left(3-2\sqrt{2}\right)-\frac{\zeta(3)}{2}-\mathrm{Li}_2\left(3-2\sqrt{2}\right)\ln\left(\sqrt{2}-1\right)\end{aligned}$$

and

$$\begin{aligned}C&=\int_0^{-\ln(\sqrt{2}-1)}\ln^2\left(1-e^{-2x}\right)\,\mathrm{d}x=2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\int_0^{-\ln(\sqrt{2}-1)}e^{-2nx}\,\mathrm{d}x\\ &=-\sum_{n=1}^{\infty}\frac{\left(H_n-\frac{1}{n}\right)\left(\left(3-2\sqrt{2}\right)^n-1\right)}{n^2}\end{aligned}$$

Using the well-known generating function

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}z^n=\operatorname{Li}_3(z)-\operatorname{Li}_3(1-z)+\operatorname{Li}_2(1-z)\ln(1-z)+\frac{1}{2}\ln z\ln^2(1-z)+\zeta(3)$$

we obtain

$$C=\operatorname{Li}_3\left(2\sqrt{2}-2\right)-\operatorname{Li}_2\left(2\sqrt{2}-2\right)\ln\left(2\sqrt{2}-2\right)-\ln\left(\sqrt{2}-1\right)\ln^2 2-2\ln 2\ln^2\left(\sqrt{2}-1\right)$$
$$-\ln^3\left(\sqrt{2}-1\right)$$

Therefore,

$$\int_0^1\frac{\ln^2 x}{\sqrt{x(x+1)}}\,\mathrm{d}x=4\operatorname{Li}_3\left(3-2\sqrt{2}\right)+8\operatorname{Li}_3\left(2\sqrt{2}-2\right)-4\zeta(3)-8\operatorname{Li}_2\left(3-2\sqrt{2}\right)\ln\left(2\sqrt{2}-2\right)$$
$$-8\operatorname{Li}_2\left(2\sqrt{2}-2\right)\ln\left(2\sqrt{2}-2\right)+\frac{4\pi^2}{3}\ln 2-24\ln 2\ln^2\left(\sqrt{2}-1\right)$$
$$-16\ln^2 2\ln\left(\sqrt{2}-1\right)-\frac{32}{3}\ln^3\left(\sqrt{2}-1\right)$$

□

Exercise 1.1.89: Find a closed form for this integral:

$$I=\int_0^{\infty}\arctan\left(\frac{2\pi}{x-\ln x+\ln\left(\frac{\pi}{2}\right)}\right)\frac{\mathrm{d}x}{x+1}$$

Solution: Computing a Related Contour Integral:

Define

$$f(z)=\frac{i}{2}\frac{z-1}{1+az}\left(\frac{1}{z-\ln z+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{z-\ln z+2\pi i+\ln\left(\frac{\pi}{2}\right)}\right)$$

and let γ denote a keyhole contour deformed around $[0,\infty]$. Restricting the argument between 0 and 2π, it is not hard to see that $f(z)$ has poles at $z=-\frac{1}{a}$, $z=-W_{-1}\left(-\frac{\pi}{2}\right)=\frac{\pi i}{2}$, and $z=-W_0\left(-\frac{\pi}{2}\right)=-\frac{\pi i}{2}$. The residues at these poles are

$$\operatorname*{Res}_{z=\frac{\pi i}{2}}f(z)=\frac{i}{2}\frac{\frac{\pi i}{2}-1}{\frac{\pi i}{2}a+1}\frac{1}{1-\frac{2}{\pi i}},\quad \operatorname*{Res}_{z=-\frac{\pi i}{2}}f(z)=\frac{i}{2}\frac{\frac{\pi i}{2}+1}{\frac{\pi i}{2}a-1}\frac{1}{1+\frac{2}{\pi i}}$$

$$\operatorname*{Res}_{z=-\frac{1}{a}}f(z)=-\frac{i}{2}u'(a)\left(\frac{1}{u(a)+\ln\left(\frac{\pi}{2}\right)-\pi i}+\frac{1}{u(a)+\ln\left(\frac{\pi}{2}\right)+\pi i}\right)$$

where $u(a)=\ln a-\frac{1}{a}$. By the residue theorem,

$$\oint_\gamma f(z)\,\mathrm{d}z=2\pi i\sum_{z_k\in\{-a^{-1},\pm\pi i/2\}}\operatorname*{Res}_{z=z_k}f(z)=\pi\left(\frac{2\left(u(a)+\ln\left(\frac{\pi}{2}\right)\right)}{\left(u(a)+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}u'(a)-\frac{2\pi^2 a}{\pi^2a^2+4}\right)$$

Parameterisation of the Contour Integral:

We take the argument of z to be 0 above the branch cut, and 2π below the branch cut. Also, the contribution from the big arc is clearly $2\pi i \times \frac{i}{2} \times \frac{1}{a} \times (1+1) = -\frac{2\pi}{a}$. Taking all of these points into consideration, we eventually arrive at

$$\begin{aligned}
&\oint_\gamma f(z)\,\mathrm{d}z + \frac{2\pi}{a} \\
&= \frac{i}{2}\int_0^\infty \frac{x-1}{1+ax}\left(-\frac{1}{x-\ln|x|-2\pi i+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{x-\ln|x|+2\pi i+\ln\left(\frac{\pi}{2}\right)+\pi^2}\right)\mathrm{d}x \\
&= 2\pi\int_0^\infty \frac{x-1}{\left(x-\ln x+\ln\left(\frac{\pi}{2}\right)\right)^2+4\pi^2}\frac{\mathrm{d}x}{1+ax}
\end{aligned}$$

Obtaining the Closed Form:

Integrating with respect to a, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{2\pi\left(1-\frac{1}{x}\right)\ln(1+ax)}{\left(x-\ln x+\ln\left(\frac{\pi}{2}\right)\right)^2+4\pi^2}\,\mathrm{d}x = \pi\int\left(\frac{2\left(u(a)+\ln\left(\frac{\pi}{2}\right)\right)u'(a)}{\left(u(a)+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}-\frac{2\pi^2 a}{\pi^2a^2+4}+\frac{2}{a}\right)\mathrm{d}a \\
&= \pi\left(\ln\left(\left(u(a)+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2\right)-\ln\left(\pi^2a^2+4\right)+\ln a^2\right)+\text{const.} \\
&= \pi\ln\left(\frac{\left(\ln a-\frac{1}{a}+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}{\pi^2+\frac{4}{a^2}}\right)+\text{const.}
\end{aligned}$$

Letting $a \to 0$, we find that the constant term is $\pi\ln 4$. Plugging in $a = 1a$ and integrating by parts, we finally arrive at the closed form.

$$\begin{aligned}
&\int_0^\infty \frac{2\pi\left(1-\frac{1}{x}\right)\ln(1+x)}{\left(x-\ln x+\ln\left(\frac{\pi}{2}\right)\right)^2+4\pi^2}\,\mathrm{d}x = \int_0^\infty \arctan\left(\frac{2\pi}{x-\ln x+\ln\left(\frac{\pi}{2}\right)}\right)\frac{\mathrm{d}x}{1+x} \\
&= \pi\ln\left(\frac{\left(\ln a-\frac{1}{a}+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}{\frac{\pi^2}{4}+\frac{1}{a^2}}\right)\Bigg|_{a=1} = \pi\ln\left(\frac{\ln^2\left(\frac{\pi}{2}\right)-2\ln\left(\frac{\pi}{2}\right)+1+\pi^2}{\frac{\pi^2}{4}+1}\right)
\end{aligned}$$

□

Exercise 1.1.90: Consider the Clausen function $\mathrm{Cl}_2(x)$ that can be defined for $0 < x < 2\pi$ in several equivalent ways:

$$\begin{aligned}
\mathrm{Cl}_2(x) &= -\int_0^x \ln\left(2\sin\left(\frac{t}{2}\right)\right)\mathrm{d}t \\
&= \sum_{n=1}^\infty \frac{\sin(nx)}{n^2} = \Im\,\mathrm{Li}_2\left(e^{ix}\right) \\
&= i\left(\frac{\pi^2}{6}+\frac{x^2}{4}-\frac{\pi x}{2}-\mathrm{Li}_2\left(e^{ix}\right)\right)
\end{aligned}$$

Define

$$I(p) = \int_0^{2\pi} \mathrm{Cl}_2(x)^2\, x^p \,\mathrm{d}x,$$

prove:

$$I(0) \stackrel{?}{=} \frac{\pi^5}{90},\quad I(1) \stackrel{?}{=} \frac{\pi^6}{90},\quad I(2) \stackrel{?}{=} \frac{44\,\pi^7}{2835},\quad I(3) \stackrel{?}{=} \frac{23\,\pi^8}{945},$$

and find a closed form for $I(4)$.

✎ **Solution: The General Case:**

Consider instead the integral

$$J(\eta) = \Re \int_0^{2\pi} x^\eta\, \mathrm{Li}_2(e^{ix})^2 \,\mathrm{d}x$$

for $\eta \in \mathbb{N}_0$. Deforming the contour around $z = 0$, we get

$$\begin{aligned} J(\eta) &= \Re \oint_{|z|=1} \frac{\mathrm{Li}_2(z)^2 \ln^\eta(z)}{i^{\eta+1} z}\mathrm{d}z = \Re \int_0^1 \frac{((\ln(x)+2\pi i)^\eta - \ln^\eta(x))\,\mathrm{Li}_2(x)^2}{i^{\eta+1}x}\mathrm{d}x \\ &= \Re \sum_{k=0}^{\eta-1} \binom{\eta}{k} \frac{(2\pi)^{\eta-k}}{i^{k+1}} \int_0^1 \frac{\mathrm{Li}_2(x)^2 \ln^k(x)}{x}\mathrm{d}x \\ &= \sum_{k=0}^{\lfloor \eta/2\ -1\rfloor} \binom{\eta}{2k+1} (-1)^{k+1} (2\pi)^{\eta-2k-1} \int_0^1 \frac{\mathrm{Li}_2(x)^2 \ln^{2k+1}(x)}{x}\mathrm{d}x \end{aligned}$$

Relating this to $I(\eta)$,

$$I(\eta) = \pi^{\eta+5}\left(\frac{2^{\eta+1}}{36(\eta+1)} - \frac{2^{\eta+2}}{6(\eta+2)} + \frac{2^{\eta+3}}{3(\eta+3)} - \frac{2^{\eta+4}}{4(\eta+4)} + \frac{2^{\eta+5}}{16(\eta+5)}\right) - J(\eta)$$

The Case $I(0)$:

It is evident that $J(0) = 0$ since it is a vacuous summation. Thus

$$I(0) = \pi^{\eta+5}\left(\frac{2^{\eta+1}}{36(\eta+1)} - \frac{2^{\eta+2}}{6(\eta+2)} + \frac{2^{\eta+3}}{3(\eta+3)} - \frac{2^{\eta+4}}{4(\eta+4)} + \frac{2^{\eta+5}}{16(\eta+5)}\right)\Bigg|_{\eta=0} = \frac{\pi^5}{90}$$

The Case $I(1)$:

In this case, $J(1) = 0$ as well. Hence

$$I(1) = \pi^{\eta+5}\left(\frac{2^{\eta+1}}{36(\eta+1)} - \frac{2^{\eta+2}}{6(\eta+2)} + \frac{2^{\eta+3}}{3(\eta+3)} - \frac{2^{\eta+4}}{4(\eta+4)} + \frac{2^{\eta+5}}{16(\eta+5)}\right)\Bigg|_{\eta=1} = \frac{\pi^6}{90}$$

The Case $I(2)$:

For $\eta = 2$,

$$\pi^{\eta+5}\left(\frac{2^{\eta+1}}{36(\eta+1)} - \frac{2^{\eta+2}}{6(\eta+2)} + \frac{2^{\eta+3}}{3(\eta+3)} - \frac{2^{\eta+4}}{4(\eta+4)} + \frac{2^{\eta+5}}{16(\eta+5)}\right)\Bigg|_{\eta=2} = \frac{16\pi^7}{945}$$

and

$$J(2) = \binom{2}{1}(-1)(2\pi)\int_0^1 \frac{\mathrm{Li}_2(x)^2 \ln x}{x}\mathrm{d}x = 4\pi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^2 n^2 (m+n)^2} = \frac{4\pi^7}{2835}$$

The double sum $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2n^2(m+n)^2}=\frac{\pi^6}{2835}$ has been evaluated here http://math.stackexchange.com/questions/1007926/.

Thus

$$I(2)=\frac{16\pi^7}{945}-\frac{4\pi^7}{2835}=\frac{44\pi^7}{2835}$$

The Case $I(3)$:

When $\eta=3$,

$$\pi^{\eta+5}\left(\frac{2^{\eta+1}}{36(\eta+1)}-\frac{2^{\eta+2}}{6(\eta+2)}+\frac{2^{\eta+3}}{3(\eta+3)}-\frac{2^{\eta+4}}{4(\eta+4)}+\frac{2^{\eta+5}}{16(\eta+5)}\right)\Bigg|_{\eta=3}=\frac{\pi^8}{35}$$

and

$$J(3)=\binom{3}{1}(-1)(2\pi)^2\int_0^1\frac{\mathrm{Li}_2(x)^2\ln x}{x}\mathrm{d}x=12\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2n^2(m+n)^2}=\frac{4\pi^8}{945}$$

Therefore

$$I(3)=\frac{\pi^8}{35}-\frac{4\pi^8}{945}=\frac{23\pi^8}{945}$$

The Case $I(4)$:

For $\eta=4$,

$$\pi^{\eta+5}\left(\frac{2^{\eta+1}}{36(\eta+1)}-\frac{2^{\eta+2}}{6(\eta+2)}+\frac{2^{\eta+3}}{3(\eta+3)}-\frac{2^{\eta+4}}{4(\eta+4)}+\frac{2^{\eta+5}}{16(\eta+5)}\right)\Bigg|_{\eta=4}=\frac{16\pi^9}{315}$$

and

$$\begin{aligned}
J(4)&=\binom{4}{1}(-1)(2\pi)^3\left(-\frac{\pi^6}{2835}\right)+\binom{4}{3}(2\pi)\int_0^1\frac{\mathrm{Li}_2(x)^2\ln^3 x}{x}\mathrm{d}x\\
&=\frac{32\pi^9}{2835}+4\pi\int_0^1\frac{\mathrm{Li}_2(x)\ln^4 x\ln(1-x)}{x}\mathrm{d}x\\
&=\frac{32\pi^9}{2835}+\frac{4\pi}{5}\int_0^1\frac{\mathrm{Li}_2(x)\ln^5 x}{1-x}\mathrm{d}x+\frac{4\pi}{15}\int_0^1\frac{\ln^6 x\ln(1-x)}{1-x}\mathrm{d}x\\
&=\frac{32\pi^9}{2835}-96\pi\sum_{m=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^6}-192\pi\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^7}\\
&=-\frac{8\pi^9}{567}+192\pi\zeta(3)\zeta(5)-96\pi\zeta(6,2)
\end{aligned}$$

giving us

$$\begin{aligned}
I(4)&=\frac{184\pi^9}{2835}-192\pi\zeta(3)\zeta(5)+96\pi\zeta(6,2)\\
&=\frac{184\pi^9}{2835}-192\pi\zeta(3)\zeta(5)+96\pi\left(\frac{\pi^6}{945}-1\right)\\
&=\frac{184\pi^9}{2835}+\frac{32\pi^7}{315}-192\pi\zeta(3)\zeta(5)-96\pi
\end{aligned}$$

□

Exercise 1.1.91: Find a closed form for:

$$I=\int_0^{\infty}\frac{x-\sin x}{(e^x-1)\,x^2}\,\mathrm{d}x$$

✎ **Solution:** The laplace transform of $\dfrac{x-\sin x}{x^2}$ is given by

$$\begin{aligned}\mathcal{L}_s\left(\frac{x-\sin x}{x^2}\right) &= \int_\infty^s \int_\infty^t \frac{1}{u^2} - \frac{1}{1+u^2}\,\mathrm{d}u\,\mathrm{d}t \qquad (1)\\ &= \int_\infty^s -\frac{1}{t} - \arctan t + \frac{\pi}{2}\,\mathrm{d}t\\ &= -\ln s + \frac{1}{2}\ln(1+s^2) - s\arctan s + \frac{\pi}{2}s - 1\\ &= \frac{1}{2}\ln\left(1+\frac{1}{s^2}\right) + s\arctan\left(\frac{1}{s}\right) - 1\end{aligned}$$

Thus

$$\begin{aligned}\int_0^\infty \frac{x-\sin x}{x^2(e^x-1)}\mathrm{d}x &= \sum_{n=1}^\infty \mathcal{L}_n\left(\frac{x-\sin x}{x^2}\right) \qquad (2)\\ &= \frac{1}{2}\sum_{n=1}^\infty \ln\left(1+\frac{1}{n^2}\right) + \sum_{n=1}^\infty \left(n\arctan\left(\frac{1}{n}\right) - 1\right)\\ &= \frac{1}{2}\ln\frac{\pi z}{\pi}\prod_{n=1}^\infty \left(1+\frac{z^2}{n^2}\right)\Bigg|_{z=1} + \sum_{n=1}^\infty\sum_{k=1}^\infty \frac{(-1)^k}{(2k+1)n^{2k}} \qquad (3)\\ &= \frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) + \sum_{k=1}^\infty \frac{(-1)^k\zeta(2k)}{(2k+1)} \qquad (4)\\ &= \frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) + \frac{1}{2i}\int_0^i (1-\pi z\cot(\pi z))\,\mathrm{d}z \qquad (5)\\ &= \frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) + \frac{1}{2} - \frac{1}{8\pi}\int_1^{\exp(-2\pi)} \frac{\ln u(1+u)}{u(1-u)}\mathrm{d}u \qquad (6)\\ &= \frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) + \frac{1}{2} - \frac{1}{8\pi}\left[2\mathrm{Li}_2(1-u) + \frac{\ln^2 u}{2}\right]_1^{\exp(-2\pi)} \qquad (7)\\ &= \frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) - \frac{1}{4\pi}\mathrm{Li}_2\left(1-e^{-2\pi}\right)\end{aligned}$$

Explanation:

(1): Differentiated under the integral twice.

(2): Expanded $(e^x-1)^{-1}$.

(3): Expanded $\arctan\left(n^{-1}\right)$.

(4): Recognised the Weierstrass product for sinh, summed in n.

(5): Used the fact that $\pi z\cot(\pi z) = 1 - 2\sum_{k=1}^\infty \zeta(2k)z^{2k}$.

(6): Substituted $u = e^{2\pi i z}$.

(7): $\dfrac{\ln u(1+u)}{u(1-u)} = \dfrac{2\ln u}{1-u} + \dfrac{\ln u}{u}$. □

Exercise 1.1.92: Find a closed form for the following integral:

$$\int_0^1 \frac{\ln^3(1+x)}{x}\mathrm{d}x$$

✎ **Solution:** Letting $u = \ln(1+x)$,

$$\int_0^1 \frac{\ln^3(1+x)}{x}\,\mathrm{d}x = \int_0^{\ln 2} \frac{u^3}{e^u-1}e^u\,\mathrm{d}u = \int_0^{\ln 2} \frac{u^3}{1-e^{-u}}\,\mathrm{d}u = \int_0^{\ln 2} u^3 \sum_{n=0}^{\infty} e^{-nu}\,\mathrm{d}u$$
$$= \sum_{n=0}^{\infty}\int_0^{\ln 2} u^3 e^{-nu}\,\mathrm{d}u = \int_0^{\ln 2} u^3\,\mathrm{d}u + \sum_{n=1}^{\infty}\int_0^{\ln 2} u^3 e^{-nu}\,\mathrm{d}u = \frac{\ln^4(2)}{4} + \sum_{n=1}^{\infty}\int_0^{\ln 2} u^3 e^{-nu}\,\mathrm{d}u$$

Then integrating by parts 3 times,

$$\begin{aligned}
&\int_0^1 \frac{\ln^3(1+x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^4(2)}{4} - \sum_{n=1}^{\infty} e^{-nu}\left(\frac{6}{n^4}+\frac{6u}{n^3}+\frac{3u^2}{n^2}+\frac{u^3}{n}\right)\Bigg|_0^{\ln 2} \\
&= \frac{\ln^4(2)}{4} - \sum_{n=1}^{\infty}\left[\frac{1}{2^n}\left(\frac{6}{n^4}+\frac{6\ln 2}{n^3}+\frac{3\ln^2(2)}{n^2}+\frac{\ln^3(2)}{n}\right) - 6\zeta(4)\right] \\
&= -\frac{3\ln^4(2)}{4} - 6\mathrm{Li}_4\left(\frac{1}{2}\right) - 6\ln(2)\mathrm{Li}_3\left(\frac{1}{2}\right) - 3\ln^2(2)\mathrm{Li}_2\left(\frac{1}{2}\right) + 6\zeta(4)
\end{aligned}$$

The answer could of course be simplified using the known values of $\mathrm{Li}_2\left(\frac{1}{2}\right)$, $\mathrm{Li}_3\left(\frac{1}{2}\right)$, and $\zeta(4)$.

http://mathworld.wolfram.com/Dilogarithm.html

http://mathworld.wolfram.com/Trilogarithm.html

https://en.wikipedia.org/wiki/Riemann_zeta_function#Specific_values □

Exercise 1.1.93: Evaluate:

$$\int_0^{\frac{\pi}{2}} \arctan(\sin x)\,\mathrm{d}x$$

✎ **Solution:** Integrating by parts, we get

$$\int_0^{\pi/2} \sin^{2k+1}(x)\,\mathrm{d}x = \frac{2k}{2k+1}\int_0^{\pi/2}\sin^{2k-1}(x)\,\mathrm{d}x \tag{1}$$

Therefore, by induction, we have

$$\int_0^{\pi/2} \sin^{2k+1}(x)\,\mathrm{d}x = \frac{2^k\,k!}{(2k+1)!!} \tag{2}$$

Thus,

$$\int_0^{\pi/2} \arctan(\sin(x))\,\mathrm{d}x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\int_0^{\pi/2}\sin^{2k+1}(x)\,\mathrm{d}x \tag{3a}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\frac{2^k\,k!}{(2k+1)!!} \tag{3b}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\frac{4^k}{\binom{2k}{k}} \tag{3c}$$

Explanation:

(3a): use the series for $\arctan(x)$.

(3b): use (2).

(3c): rewrite (3b) using central binomial coefficients.

In this answer http://math.stackexchange.com/a/1052675,it is shown that

$$\sum_{n=0}^{\infty}\frac{(-4)^n}{(2n+1)^2\binom{2n}{n}}=\frac{\pi^2}{8}-\frac{1}{2}\text{arcsinh}^2(1) \tag{4}$$

Therefore, combining (3) and (4), we have

$$\int_0^{\pi/2}\arctan(\sin(x))\,\mathrm{d}x=\frac{\pi^2}{8}-\frac{1}{2}\text{arcsinh}^2(1)$$

□

Exercise 1.1.94: Evaluate:

$$\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\mathrm{d}\theta$$

Solution: Substituting $t=\cos\theta$ is a good first step. After that, note that the integrand is now an even function of t, so the integral from $t=1$ to $t=1$ will be equal to twice the integral $t=0$ to $t=1$. Next, I would recommend rewriting the integrand using the partial fraction decomposition,

$$\frac{2}{1-t^2}=\frac{1}{1+t}+\frac{1}{1-t},$$

and the trigonometric power-reduction identity,

$$\cos^2\varphi=\frac{1+\cos(2\varphi)}{2}.$$

Thus,

$$\begin{aligned}\mathcal{I}&=\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\mathrm{d}\theta=\int_{-1}^{1}\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1-t^2}\mathrm{d}t=2\int_0^1\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1-t^2}\,\mathrm{d}t\\&=\int_0^1\left(\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1+t}+\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1-t}\right)\mathrm{d}t=\int_0^1\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1+t}\,\mathrm{d}t+\int_0^1\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1-t}\,\mathrm{d}t\\&=\frac{1}{2}\int_0^1\frac{1+\cos(\pi t)}{1+t}\,\mathrm{d}t+\frac{1}{2}\int_0^1\frac{1+\cos(\pi t)}{1-t}\,\mathrm{d}t\end{aligned}$$

Transform the first integral in the last line above using the substitution $t=2u1$ and transform the second integral using the substitution $t=1v$ using the respective trigonometric identities:

$$\cos(\pi(2u-1))=-\cos(2\pi u),$$

and

$$\cos(\pi(1-v))=\cos(\pi)\cos(\pi v)+\sin(\pi)\sin(\pi v)=-\cos(\pi v),$$

both of which can be derived from the single angle-addition identity,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

This yields,

$$\begin{aligned}\mathcal{I} &= \frac{1}{2}\int_0^1 \frac{1+\cos(\pi t)}{1+t}\,\mathrm{d}t + \frac{1}{2}\int_0^1 \frac{1+\cos(\pi t)}{1-t}\,\mathrm{d}t \\ &= \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{1+\cos(\pi(2u-1))}{u}\,\mathrm{d}u + \frac{1}{2}\int_0^1 \frac{1+\cos(\pi(1-v))}{v}\,\mathrm{d}v \\ &= \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{1-\cos(2\pi u)}{u}\,\mathrm{d}u + \frac{1}{2}\int_0^1 \frac{1-\cos(\pi v)}{v}\,\mathrm{d}v \\ &= \frac{1}{2}\int_{\pi}^{2\pi} \frac{1-\cos(x)}{x}\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} \frac{1-\cos(x)}{x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1-\cos(x)}{x}\,\mathrm{d}x = \frac{1}{2}\operatorname{Cin}(2\pi) = \frac{\gamma+\ln(2\pi)-\operatorname{Ci}(2\pi)}{2}\end{aligned}$$

□

Exercise 1.1.95: Evaluate:

$$\int_0^{\pi} e^{-ik\cos\theta}\sin^2\theta\,\mathrm{d}\theta$$

Solution: As a first step, you might substitute $x = -\cos\theta$. Next, a little algebra puts the integral in a form recognizable as a common integral representation for the Bessel function of the first kind of order one:

$$\begin{aligned}\mathcal{I} &= \int_0^{\pi} e^{-ik\cos\theta}\sin^2\theta\,\mathrm{d}\theta = \int_0^{\pi} e^{-ik\cos\theta}\sqrt{1-\cos^2\theta}\,\sin\theta\,\mathrm{d}\theta \\ &= \int_{-1}^1 e^{ikx}\sqrt{1-x^2}\,\mathrm{d}x = \int_0^1 e^{ikx}\sqrt{1-x^2}\,\mathrm{d}x + \int_{-1}^0 e^{ikx}\sqrt{1-x^2}\,\mathrm{d}x \\ &= \int_0^1 e^{ikx}\sqrt{1-x^2}\,\mathrm{d}x + \int_0^1 e^{-ikx}\sqrt{1-x^2}\,\mathrm{d}x = 2\int_0^1 \frac{e^{ikx}+e^{-ikx}}{2}\sqrt{1-x^2}\,\mathrm{d}x \\ &= 2\int_0^1 \cos(kx)\sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi\,J_1(k)}{k}\end{aligned}$$

Anyway, if you are not confident with Bessel functions, you can just notice that:

$$\begin{aligned}I = \int_0^{\pi} e^{-ik\cos\theta}\sin^2\theta\,\mathrm{d}\theta &= 2\int_0^{\pi/2}\cos(k\cos\theta)\sin^2\theta\,\mathrm{d}\theta \\ &= \frac{2}{k}\int_0^{\pi/2}\sin(k\cos\theta)\cos\theta\,\mathrm{d}\theta\end{aligned}$$

where the last equality follows from integration by parts. Since:

$$\int_0^{\pi/2}\cos^{2m}\theta\,\mathrm{d}\theta = \frac{\pi}{4^m}\binom{2m}{m},$$

exploiting the Taylor series of the sine function we get:

$$I = \frac{2}{k}\sum_{n=0}^{+\infty}(-1)^n\frac{k^{2n+1}}{(2n+1)!}\int_0^{\pi/2}\cos^{2n+2}\theta\,\mathrm{d}\theta = \frac{\pi}{4}\sum_{n=0}^{+\infty}\frac{(-1)^n k^{2n}}{4^n(2n+1)!}\binom{2n+2}{n+1}$$

or just:

$$I = \frac{\pi}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{k^2}{4}\right)^n$$

that is a pretty fast converging series. □

Exercise 1.1.96: Find a closed form expression of the integral:

$$\int_1^\infty \frac{\ln^3 x}{x(x-1)}\mathrm{d}x$$

Solution: **Method One:**

The change of variables $x \leftarrow 1/x$ shows that

$$\begin{aligned} I &= \int_1^\infty \frac{\ln^3 x}{x(x-1)}\mathrm{d}x = \int_0^1 \frac{-\ln^3 x}{1-x}\mathrm{d}x = -\sum_{n=0}^\infty \int_0^1 x^n \ln^3 x\,\mathrm{d}x \\ &= \sum_{n=0}^\infty \frac{6}{(n+1)^4} = 6\zeta(4) = \frac{\pi^4}{15} \end{aligned}$$

Indeed, generally, the change of variables $x = e^{-t}$ shows that

$$\int_0^1 x^n \ln^p(1/x)\,\mathrm{d}x = \int_0^\infty e^{-(n+1)t}t^p\mathrm{d}t = \frac{1}{(n+1)^{p+1}}\int_0^\infty e^{-u}u^p\mathrm{d}u = \frac{p!}{(n+1)^{p+1}}$$

Method Two:

First, make the substitutions $x = \dfrac{1}{u}$ followed by $u = 1-z$, and then integrate by parts twice. Substitute $z = 1w$, then integrate by parts again:

$$\begin{aligned} &\int_1^\infty \frac{\ln^3 x}{x(1-x)}\mathrm{d}x = \int_1^0 \frac{\ln^3 \frac{1}{u}}{\frac{1}{u}(1-\frac{1}{u})}\cdot\frac{(-\mathrm{d}u)}{u^2} = -\int_0^1 \frac{\ln^3 u}{1-u}\mathrm{d}u = -\int_0^1 \frac{\ln^3(1-z)}{z}\mathrm{d}z \\ &= -\left[\ln(z)\ln^3(1-z)\right]_0^1 + \int_0^1 \frac{3\ln(z)\ln^2(1-z)}{z-1}\mathrm{d}z = -3\int_0^1 \frac{\ln(z)\ln^2(1-z)}{1-z}\mathrm{d}z \\ &= -3\left[\ln^2(1-z)\,\mathrm{Li}_2(1-z)\right]_0^1 + 3\int_0^1 \frac{2\ln(1-z)\,\mathrm{Li}_2(1-z)}{z-1}\mathrm{d}z \\ &= -6\int_0^1 \frac{\ln(1-z)\,\mathrm{Li}_2(1-z)}{1-z}\mathrm{d}z = -6\int_0^1 \frac{\ln(w)\,\mathrm{Li}_2(w)}{w}\mathrm{d}w \\ &= -6\left[\ln(w)\,\mathrm{Li}_3(w)\right]_0^1 + 6\int_0^1 \frac{\mathrm{Li}_3(w)}{w}\mathrm{d}w = 6\int_0^1 \frac{\mathrm{Li}_3(w)}{w}\mathrm{d}w = 6\,\mathrm{Li}_4(1) = \frac{\pi^4}{15} \end{aligned}$$

□

Exercise 1.1.97: Evaluate:

$$\frac{2}{\pi}\int_0^{\pi/2} \sqrt[8]{x^2+\ln^2\cos x}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\ln^2\cos x}{x^2+\ln^2\cos x}}}}\,\mathrm{d}x$$

and

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{1}{\sqrt[8]{x^2+\ln^2\cos x}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\ln^2\cos x}{x^2+\ln^2\cos x}}}}\,\mathrm{d}x$$

Solution: Let s be a real number such that $1 < s < 1$. Then

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(s\arctan\left(-\dfrac{x}{\ln\cos x}\right)\right)}{(x^2+\ln^2\cos x)^{s/2}}\,\mathrm{d}x = \frac{\pi}{2}\frac{1}{\ln^s 2} \tag{1}$$

Proof. First assume that $0 < s < 1$. Then we may write

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(s\arctan\left(-\dfrac{x}{\ln\cos x}\right)\right)}{(x^2+\ln^2\!\cos x)^{s/2}}\mathrm{d}x = \frac{1}{\Gamma(s)}\int_0^{\frac{\pi}{2}}\int_0^{+\infty} u^{s-1}\cos(ux)\,e^{u\ln\cos x}\,\mathrm{d}u\,\mathrm{d}x \tag{2}$$

$$= \frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}\int_0^{\frac{\pi}{2}}\cos^u x\cos(ux)\,\mathrm{d}x\,\mathrm{d}u \tag{3}$$

$$= \frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}\frac{\pi}{2^{u+1}}\,\mathrm{d}u \tag{4}$$

$$= \frac{\pi}{2}\frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}e^{-u\ln 2}\,\mathrm{d}u \tag{5}$$

$$= \frac{\pi}{2}\frac{1}{\Gamma(s)}\frac{\Gamma(s)}{\ln^s 2} \tag{6}$$

$$= \frac{\pi}{2}\frac{1}{\ln^s 2} \tag{7}$$

where we have used Fubini's theorem and the classic results.

$$\int_0^{+\infty} u^{s-1}\cos(au)\,e^{-bu}\,\mathrm{d}u = \Gamma(s)\frac{\cos\left(s\arctan\left(\dfrac{a}{b}\right)\right)}{(a^2+b^2)^{s/2}},\ (\Re(s)>0, b>0, a>0) \tag{8}$$

$$\int_0^{\frac{\pi}{2}}\cos^u x\cos(ux)\,\mathrm{d}x = \frac{\pi}{2^{u+1}},\quad u>-1. \tag{9}$$

We may extend identity (7) by analytic continuation to obtain (1).

Example 1. We have

$$\int_0^{\pi/2}\sqrt{\sqrt{x^2+\ln^2\!\cos x}-\ln\cos x}\,\mathrm{d}x = \frac{\pi}{2}\sqrt{\ln 2} \tag{10}$$

and

$$\int_0^{\pi/2}\frac{1}{\sqrt{x^2+\ln^2\!\cos x}}\sqrt{\sqrt{x^2+\ln^2\!\cos x}-\ln\cos x}\,\mathrm{d}x = \frac{\pi}{\sqrt{\ln 2}} \tag{11}$$

Proof. Let $0 < x < \dfrac{\pi}{2}$ and set $t := \arctan\left(-\dfrac{x}{\ln\cos x}\right)$. Observe that $0 < t < \frac{\pi}{2}$ and

$$\cos t = \cos\left(\arctan\left(-\frac{x}{\ln\cos x}\right)\right) = -\frac{\ln\cos x}{\sqrt{x^2+\ln^2\!\cos x}},$$

$$\cos\left(\frac{t}{2}\right) = \sqrt{\frac{1}{2}+\frac{1}{2}\cos t}$$

then put successively $s = -\dfrac{1}{2}$, $s = \dfrac{1}{2}$ in (1) to obtain (10) and (11).

Example 2.

$$\frac{2}{\pi}\int_0^{\pi/2}\sqrt[8]{x^2+\ln^2\!\cos x}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\ln^2\!\cos x}{x^2+\ln^2\!\cos x}}}}\,\mathrm{d}x = \sqrt[4]{\ln 2} \tag{12}$$

and

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt[8]{x^2+\ln^2\!\cos x}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\ln^2\!\cos x}{x^2+\ln^2\!\cos x}}}}\,\mathrm{d}x = \frac{1}{\sqrt[4]{\ln 2}} \tag{13}$$

Proof. Let $0<x<\dfrac{\pi}{2}$ and set $t:=\arctan\left(-\dfrac{x}{\ln\cos x}\right)$. Observe that $0<t<\frac{\pi}{2}$ and

$$\cos t=\cos\left(\arctan\left(-\frac{x}{\ln\cos x}\right)\right)=-\frac{\ln\cos x}{\sqrt{x^2+\ln^2\cos x}},$$

$$\cos\left(\frac{t}{4}\right)=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos t}},$$

then put successively $s=-\dfrac{1}{4}$, $s=\dfrac{1}{4}$ in (1) to obtain (12) and (13).

Example n.

$$R_n^+:=\frac{2}{\pi}\int_0^{\pi/2}\sqrt[2^n]{x^2+\ln^2\cos x}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\ln^2\cos x}{x^2+\ln^2\cos x}}}}}\,\mathrm{d}x$$

and

$$R_n^-:=\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt[2^n]{x^2+\ln^2\cos x}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\ln^2\cos x}{x^2+\ln^2\cos x}}}}}\,\mathrm{d}x.$$

Then

$$R_n^+=\sqrt[2^n]{\ln 2} \tag{14}$$

and

$$R_n^-=\frac{1}{\sqrt[2^n]{\ln 2}}. \tag{15}$$

Proof. Let $0<x<\dfrac{\pi}{2}$ and set $t:=\arctan\left(-\dfrac{x}{\ln\cos x}\right)$. Observe that $0<t<\frac{\pi}{2}$ and

$$\cos t=\cos\left(\arctan\left(-\frac{x}{\ln\cos x}\right)\right)=-\frac{\ln\cos x}{\sqrt{x^2+\ln^2\cos x}},$$

$$\cos\left(\frac{t}{2^n}\right)=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos t}}}},$$

then put successively $s=-\dfrac{1}{2^n}$, $s=\dfrac{1}{2^n}$, $n\geq 1$,, in (1) to obtain (14) and (15). □

Exercise 1.1.98: Prove:

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+\left[\,x+\tan\left(\,x\,\right)\,\right]^2}=\pi$$

Proof: We may use the following result

http://mathworld.wolfram.com/GlassersMasterTheorem.html,

which goes back to G. Boole (1857) :

$$\int_{-\infty}^{+\infty}f\left(x-\frac{a_1}{x-\lambda_1}-\cdots-\frac{a_n}{x-\lambda_n}\right)\mathrm{d}x=\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x \tag{1}$$

with $a_i > 0, \lambda_i \in \mathbb{R}$ and f sufficiently 'regular'.

Observe that, for $x \neq n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, we have

$$\cot x = \lim_{N \to +\infty} \left(\frac{1}{x} + \frac{1}{x+\pi} + \frac{1}{x-\pi} + \cdots + \frac{1}{x+N\pi} + \frac{1}{x-N\pi} \right)$$

leading to (see Theorem 10.3 p. 14 here

http://arxiv.org/pdf/1004.2445.pdf

and see achille's answer

http://math.stackexchange.com/questions/1015462/

giving a route to prove it)

$$\int_{-\infty}^{+\infty} f\left(x - \cot x\right) \mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x \tag{2}$$

with $f(x) = \dfrac{1}{1 + \left(\dfrac{\pi}{2} - x\right)^2}$.

On the one hand, from (2),

$$\begin{aligned} \int_{-\infty}^{+\infty} f\left(x - \cot x\right) \mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x &= \int_{-\infty}^{+\infty} \frac{1}{1 + \left(\dfrac{\pi}{2} - x\right)^2}\, \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, \mathrm{d}x = \pi \end{aligned} \tag{3}$$

On the other hand, with the change of variable $x \to \dfrac{\pi}{2} - x$,

$$\int_{-\infty}^{+\infty} f\left(x - \cot x\right) \mathrm{d}x = \int_{-\infty}^{+\infty} f\left(\frac{\pi}{2} - x - \tan x\right) \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{1}{1 + (x + \tan x)^2} \mathrm{d}x \tag{4}$$

Combining (3) and (4) gives

$$\int_{-\infty}^{+\infty} \frac{1}{1 + (x + \tan x)^2} \mathrm{d}x = \pi$$

□

Exercise 1.1.99: Evaluate:

$$\int \frac{x^n}{1 + x + \dfrac{x^2}{2} + \cdots + \dfrac{x^n}{n!}} \mathrm{d}x$$

Solution: You may observe that

$$\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right)' = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

giving

$$\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right)' = \frac{x^n}{n!}$$

and

$$\int \frac{x^n}{1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}}\mathrm{d}x = n!\int \frac{\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}\right)-\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}\right)'}{1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}}\,\mathrm{d}x$$

$$= n!\int \mathrm{d}x - n!\int \frac{\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}\right)'}{\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}\right)}\,\mathrm{d}x$$

Thus

$$\int \frac{x^n}{1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}}\mathrm{d}x = n!\,x - n!\ln\left|1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}\right| + C$$

□

Exercise 1.1.100: Evaluate:

$$\int_{-3}^{3}\frac{x^8}{1+e^{2x}}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_{-3}^{3}\frac{x^8}{1+e^{2x}}\mathrm{d}x &= \int_{-3}^{0}\frac{x^8}{1+e^{2x}}\mathrm{d}x + \int_{0}^{3}\frac{x^8}{1+e^{2x}}\mathrm{d}x\\ &= \int_{0}^{3}\frac{x^8}{1+e^{-2x}}\mathrm{d}x + \int_{0}^{3}\frac{x^8}{1+e^{2x}}\mathrm{d}x = \int_{0}^{3}\frac{e^{2x}x^8}{1+e^{2x}}\mathrm{d}x + \int_{0}^{3}\frac{x^8}{1+e^{2x}}\mathrm{d}x\\ &= \int_{0}^{3}x^8\mathrm{d}x = 2187\end{aligned}$$

□

Exercise 1.1.101: Evaluate:

$$\int_0^{\pi/3}\ln^2\left(\frac{\sin x}{\sin(x+\pi/3)}\right)\mathrm{d}x$$

Solution: Set $t = \dfrac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)}$, then

$$\int_0^{\frac{\pi}{3}}\ln^2\left[\frac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)}\right]\mathrm{d}x = \frac{\sqrt{3}}{2}\int_0^1\frac{\ln^2 t}{1-t+t^2}\,\mathrm{d}t. \tag{1}$$

Method One:

$$\begin{aligned}\int_0^1\frac{\ln^2 t}{1-t+t^2}\,\mathrm{d}t &= \int_0^1\frac{1+t}{1+t^3}\ln^2 t\,\mathrm{d}t\\ &= \int_0^1\sum_{k=0}^{\infty}(-1)^k t^{3k}\,(1+t)\ln^2 t\,\mathrm{d}t\\ &= \sum_{k=0}^{\infty}(-1)^k\left(\int_0^1 t^{3k}\ln^2 t\,\mathrm{d}t + \int_0^1 t^{3k+1}\ln^2 t\,\mathrm{d}t\right).\end{aligned} \tag{2}$$

Using

$$\int_0^1 x^\alpha \ln^n x \, \mathrm{d}x = \frac{(-1)^n n!}{(\alpha+1)^{n+1}}, \qquad \text{for } \; n = 0, 1, 2, \ldots \tag{3}$$

http://math.stackexchange.com/questions/877829/

then (2) turns out to be

$$\begin{aligned}\int_0^1 \frac{\ln^2 t}{1-t+t^2} \, \mathrm{d}t &= \sum_{k=0}^{\infty} (-1)^k \left[\frac{2}{(3k+1)^3} + \frac{2}{(3k+2)^3} \right] \\ &= \frac{2}{3^3} \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{\left(k+\frac{1}{3}\right)^3} + \frac{1}{\left(k+\frac{2}{3}\right)^3} \right]. \end{aligned} \tag{4}$$

Consider the alternating series in term of polygamma function

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)^{n+1}} = \frac{1}{(-2)^{n+1} n!} \left[\psi_n\left(\frac{z}{2}\right) - \psi_n\left(\frac{z+1}{2}\right) \right] \quad ; \quad \text{for } n \in \mathbb{Z}_+ \tag{5}$$

and the reflection formula of polygamma function

$$\psi_n(1-z) + (-1)^{n+1} \psi_n(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot(\pi z). \tag{6}$$

Using (5) and (6), then (4) becomes

$$\begin{aligned}\int_0^1 \frac{\ln^2 x}{1+x+x^2} \, \mathrm{d}x &= \frac{2}{3^3} \cdot \frac{1}{(-2)^3 \cdot 2!} \left[\psi_2\left(\frac{1}{6}\right) - \psi_2\left(\frac{2}{3}\right) + \psi_2\left(\frac{1}{3}\right) - \psi_2\left(\frac{5}{6}\right) \right] \\ &= -\frac{1}{6^3} \left[\psi_2\left(\frac{1}{6}\right) - \psi_2\left(\frac{5}{6}\right) + \psi_2\left(\frac{1}{3}\right) - \psi_2\left(\frac{2}{3}\right) \right] \\ &= -\frac{1}{216} \left[\pi \left. \frac{d^2}{dz^2} \cot(\pi z) \right|_{z=\frac{5}{6}} + \pi \left. \frac{d^2}{dz^2} \cot(\pi z) \right|_{z=\frac{2}{3}} \right] \\ &= -\frac{1}{216} \left[-8\sqrt{3}\pi^3 - \frac{8\sqrt{3}}{9} \pi^3 \right] \\ &= \frac{10\sqrt{3}}{243} \pi^3. \end{aligned}$$

Thus

$$\int_0^{\frac{\pi}{3}} \ln^2 \left[\frac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)} \right] \mathrm{d}x = \frac{\sqrt{3}}{2} \int_0^1 \frac{\ln^2 t}{1-t+t^2} \, dt = \frac{5}{81} \pi^3$$

Method Two:

Set $t = \dfrac{1}{x}$, from (1) we obtain

$$\int_0^{\frac{\pi}{3}} \ln^2 \left[\frac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)} \right] \mathrm{d}x = \frac{\sqrt{3}}{2} \int_0^1 \frac{\ln^2 t}{1-t+t^2} \, \mathrm{d}t = \frac{\sqrt{3}}{4} \int_0^{\infty} \frac{\ln^2 x}{1-x+x^2} \, \mathrm{d}x.$$

Consider

$$\int_0^{\infty} \frac{x^\alpha \, \mathrm{d}x}{1+2x\cos\beta+x^2} = \frac{\pi \sin(\alpha\beta)}{\sin(\alpha\pi)\sin\beta}. \tag{7}$$

http://math.stackexchange.com/questions/890210/

Differentiating (7) with respect to α twice, setting $\alpha=0$ and $\beta=\dfrac{2}{3}\pi$, we will obtain

$$\int_0^\infty \frac{\partial^2}{\partial\alpha^2}\left[\frac{x^\alpha\,\mathrm{d}x}{1+2x\cos\beta+x^2}\right]_{\alpha=0\ ;\ \beta=\frac{2}{3}\pi} = \frac{\partial^2}{\partial\alpha^2}\left[\frac{\pi\sin(\alpha\beta)}{\sin(\alpha\pi)\sin\beta}\right]_{\alpha=0\ ;\ \beta=\frac{2}{3}\pi}$$

$$\int_0^\infty \frac{\ln^2 x}{1-x+x^2}\,\mathrm{d}x = \frac{20}{81\sqrt{3}}\pi^3$$

Thus

$$\int_0^{\frac{\pi}{3}} \ln^2\left[\frac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)}\right]\,\mathrm{d}x = \frac{\sqrt{3}}{4}\int_0^\infty \frac{\ln^2 x}{1-x+x^2}\,\mathrm{d}x = \frac{5}{81}\pi^3$$

□

Exercise 1.1.102: Find a closed form for this integral:

$$I=\int_1^\infty \frac{1-x+\ln x}{x\,(1+x^2)\ln^2 x}\mathrm{d}x$$

Solution: Set $x=\dfrac{1}{t}$, then

$$\int_1^\infty \frac{1-x+\ln x}{x\,(1+x^2)\ln^2 x}\,\mathrm{d}x = \int_0^1 \frac{t-1-t\ln t}{(1+t^2)\ln^2 t}\,\mathrm{d}t.$$

Now consider

$$\mathcal{I}(\alpha)=\int_0^1 t^\alpha\cdot\frac{t-1-t\ln t}{(1+t^2)\ln^2 t}\,\mathrm{d}t.$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}\alpha^2} &= \int_0^1 \frac{t^\alpha}{1+t^2}(t-1-t\ln t)\,\mathrm{d}t \\
&= \int_0^1 \sum_{n=0}^\infty (-1)^n t^{2n+\alpha}(t-1-t\ln t)\,\mathrm{d}t \\
&= \sum_{n=0}^\infty (-1)^n \int_0^1 \left(t^{2n+\alpha+1}-t^{2n+\alpha}-t^{2n+\alpha+1}\ln t\right)\,\mathrm{d}t \qquad (1)\\
&= \sum_{n=0}^\infty (-1)^n \left[\frac{1}{2n+\alpha+2}-\frac{1}{2n+\alpha+1}+\frac{1}{(2n+\alpha+2)^2}\right] \qquad (2)\\
&= -\frac{1}{4}\left[\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha+4}{4}\right)\right]+\frac{1}{4}\left[\psi\left(\frac{\alpha+1}{4}\right)-\psi\left(\frac{\alpha+3}{4}\right)\right] \\
&\quad +\frac{1}{16}\left[\psi_1\left(\frac{\alpha+2}{4}\right)-\psi_1\left(\frac{\alpha+4}{4}\right)\right] \qquad (3)\\
\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\alpha} &= -\ln\Gamma\left(\frac{\alpha+2}{4}\right)+\ln\Gamma\left(\frac{\alpha+4}{4}\right)+\ln\Gamma\left(\frac{\alpha+1}{4}\right)-\ln\Gamma\left(\frac{\alpha+3}{4}\right) \\
&\quad +\frac{1}{4}\left[\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha+4}{4}\right)\right] \qquad (4)\\
&= \ln\Gamma\left[\frac{(\alpha+1)(\alpha+4)}{(\alpha+2)(\alpha+3)}\right]+\frac{1}{4}\left[\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha+4}{4}\right)\right] \qquad (5)\\
\mathcal{I}(\alpha) &= \int \ln\Gamma\left[\frac{(\alpha+1)(\alpha+4)}{(\alpha+2)(\alpha+3)}\right]\,\mathrm{d}\alpha+\ln\Gamma\left(\frac{\alpha+2}{4}\right)-\ln\Gamma\left(\frac{\alpha+4}{4}\right).
\end{aligned}$$

Since $0 < t < 1$, then as $\alpha \to \infty$, implying $\mathcal{I}(\alpha) \to 0$ and $\mathcal{I}'(\alpha) \to 0$. Thus

$$\mathcal{I}(0) = \int_1^\infty \frac{1 - x + \ln x}{x\,(1+x^2)\ln^2 x}\,\mathrm{d}x = \frac{\ln 2}{6} + \frac{\ln \pi}{2} - 6\ln \mathbf{A} + \frac{2}{\pi}\mathbf{G},$$

where **A** is the Glaisher–Kinkelin constant and **G** is the Catalan constant.

Notes:

(1) $\displaystyle\int_0^1 x^\alpha \ln^n x\,\mathrm{d}x = \frac{(-1)^n n!}{(\alpha+1)^{n+1}}, \qquad \text{for } \; n = 0, 1, 2, \ldots$

http://math.stackexchange.com/questions/877829/

(2) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)^{m+1}} = \frac{1}{(-2)^{m+1} m!}\left[\psi_m\left(\frac{z}{2}\right) - \psi_m\left(\frac{z+1}{2}\right)\right]$

http://math.stackexchange.com/questions/305124/

(3) $\displaystyle\psi_m(z) := \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\ln\Gamma(z)$

https://en.wikipedia.org/wiki/Polygamma_function

(4) $\displaystyle\int \ln\Gamma(z)\,\mathrm{d}z = \frac{z(1-z)}{2} + \frac{z}{2}\ln 2\pi + z\ln\Gamma(z) - \ln \mathrm{G}(1+z) + C$

https://en.wikipedia.org/wiki/Barnes_G-function

where $\mathrm{G}(\cdot)$ is the Barnes G-function and C is a constant of integration. Also for $a, b > 0$

$$\int \ln\Gamma\left(\frac{x+a}{b}\right)\,\mathrm{d}x = b\;\psi^{(-2)}\left(\frac{x+a}{b}\right) + C.$$

(5) As $\alpha \to \infty$, both of sides tend to 0, hence the constant of integration is 0. □

Exercise 1.1.103: Find a closed form for this integral:

$$\int_0^1 \left(\left\{\frac{1}{x}\right\} - \frac{1}{2}\right)\frac{\ln(x)}{x}\mathrm{d}x$$

Solution: We may obtain the following closed form.

$$\int_0^1 \left(\left\{\frac{1}{x}\right\} - \frac{1}{2}\right)\frac{\ln(x)}{x}\mathrm{d}x = \frac{\ln^2(2\pi)}{4} - \frac{\gamma^2}{4} + \frac{\pi^2}{48} - \frac{\gamma_1}{2} - 1 \tag{1}$$

where x denotes the **fractional part** of x, γ denotes the Euler–Mascheroni constant and where γ_1 denotes the Stieltjes constant defined by

$$\gamma_1 = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{\ln k}{k} - \frac{\ln^2 N}{2}\right).$$

Consequently, we have the numerical evaluation:

$$\int_0^1 \left(\left\{\frac{1}{x}\right\} - \frac{1}{2}\right)\frac{\ln(x)}{x}\mathrm{d}x = 0.00317822795429242560505 00\ldots. \tag{2}$$

Here is an approach.

Step 1. Let ss be a complex number such that $0 < \Re s < 1$. Then

$$\int_0^1 x^{s-1}\left\{\frac{1}{x}\right\}\mathrm{d}x = -\frac{1}{1-s} - \frac{\zeta(s)}{s} \tag{3}$$

where x denotes the fractional part of x and where ζ denotes the Riemann zeta function.

Proof. Let us assume that $0 < \Re s < 1$. We may write

$$\begin{aligned}
&\int_0^1 x^{s-1}\left\{\frac{1}{x}\right\}\mathrm{d}x = \sum_{k=1}^{\infty}\int_{1/(k+1)}^{1/k} x^{s-1}\left\{\frac{1}{x}\right\}\mathrm{d}x\\
&= \sum_{k=1}^{\infty}\int_k^{k+1}\{x\}\frac{\mathrm{d}x}{x^{s+1}} = \sum_{k=1}^{\infty}\int_k^{k+1}(x-k)\frac{\mathrm{d}x}{x^{s+1}}\\
&= \sum_{k=1}^{\infty}\int_0^1 \frac{v}{(v+k)^{s+1}}\mathrm{d}v = \sum_{k=1}^{\infty}\int_0^1\left(\frac{1}{(v+k)^s}-\frac{k}{(v+k)^{s+1}}\right)\mathrm{d}v\\
&= \sum_{k=1}^{\infty}\left(\frac{1}{(-s+1)(v+k)^{s-1}}+\frac{k}{s(v+k)^s}\right)\Bigg|_0^1 = -\frac{1}{1-s}-\frac{\zeta(s)}{s}.
\end{aligned}$$

Step 2. We have

$$\int_0^1 x^{s-1}\left(\left\{\frac{1}{x}\right\}-\frac{1}{2}\right)\ln(x)\mathrm{d}x = -\frac{1}{(1-s)^2}+\frac{1}{2s^2}+\frac{\zeta(s)}{s^2}-\frac{\zeta'(s)}{s}. \tag{4}$$

Using (3), we readily get

$$\int_0^1 x^{s-1}\left(\left\{\frac{1}{x}\right\}-\frac{1}{2}\right)\mathrm{d}x = -\frac{1}{1-s}-\frac{1}{2s}-\frac{\zeta(s)}{s}$$

which we differentiate with respect to s to obtain (4).

Step 3. For s near 0, we take into account the Taylor series expansion of the Riemann ζ function:

$$\begin{aligned}
\zeta(s) &= -\frac{1}{2}-\frac{\ln(2\pi)}{2}s+\left(\frac{\gamma^2}{4}-\frac{\pi^2}{48}+\ln(2\pi)-\frac{\ln^2(2\pi)}{4}+\frac{\gamma_1}{2}\right)s^2+\mathcal{O}(s^3)\\
\zeta'(s) &= -\frac{\ln(2\pi)}{2}+\left(\frac{\gamma^2}{2}-\frac{\pi^2}{24}+2\ln(2\pi)-\frac{\ln^2(2\pi)}{2}+\gamma_1\right)s+\mathcal{O}(s^2)
\end{aligned}$$

and upon letting s tend to 0^+ in (4) we obtain (1).

Remark: A related result to (3).

http://math.stackexchange.com/questions/875076/ □

Exercise 1.1.104: Prove:

$$\int_0^1 \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots+\cfrac{1}{1+\psi(\{1/x\}+1)}}}}\mathrm{d}x = \frac{F_{n-1}}{F_n}-\frac{(-1)^n}{F_n^2}\ln\left(\frac{F_{n+2}-F_n\gamma}{F_{n+1}-F_n\gamma}\right)$$

where $\{x\} = x - \lfloor x\rfloor$ denotes the **fractional part** of x, γ is the **Euler-Mascheroni constant**, F_n are the **Fibonacci numbers**, $\psi := \Gamma'/\Gamma$ is the **digamma function** and where the **continued fraction** has a total of n horizontal bars.

Proof: Recall that the digamma function $\psi := \Gamma'/\Gamma$ admits the following expansion, coming from the Weierstrass infinite product representation of the Γ function,

$$\psi(x+1) = -\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right), \quad x > -1,$$

where γ is the Euler-Mascheroni constant. By differentiation, one obtains

$$\psi'(x+1) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}, \quad x > -1.$$

Let f be integrable on $(0,1)$. Then

$$\int_0^1 f\left(\{1/x\}\right) \mathrm{d}x = \int_0^1 f(x)\, \psi'(x+1) \mathrm{d}x \tag{1}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x, ψ' being the derivative of the digamma function $\psi := \Gamma'/\Gamma$.

Proof. One may write

$$\begin{aligned}
&\int_0^1 f\left(\{1/x\}\right) \mathrm{d}x = \sum_{k=1}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} f\left(\{1/x\}\right) \mathrm{d}x = \sum_{k=1}^{\infty} \int_k^{k+1} f\left(\{u\}\right) \frac{\mathrm{d}u}{u^2} \\
&= \sum_{k=1}^{\infty} \int_k^{k+1} f\left(u-k\right) \frac{\mathrm{d}u}{u^2} = \sum_{k=1}^{\infty} \int_0^1 f\left(v\right) \frac{\mathrm{d}v}{(v+k)^2} \\
&= \int_0^1 f\left(v\right) \sum_{k=1}^{\infty} \frac{1}{(v+k)^2} \mathrm{d}v = \int_0^1 f(v)\, \psi'(v+1)\, \mathrm{d}v
\end{aligned}$$

where the interchange between the infinite sum and the integration is allowed by the uniform bound:

$$\left| \sum_{k=1}^{N} \frac{1}{(v+k)^2} \right| < \frac{\pi^2}{6}, \quad N \geq 1,\, 0 \leq v \leq 1.$$

One of the consequences of **Theorem 1.1.1** is that the transformation $x \to \gamma + \psi(\{1/x\}+1)$ leaves the Lebesgue measure on $(0,1)$ invariant.

Theorem 1.1.1

Let f be integrable on $(0,1)$. Then

$$\int_0^1 f\left(\gamma + \psi(\{1/x\}+1)\right) \mathrm{d}x = \int_0^1 f(x)\, \mathrm{d}x \tag{2}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x, γ being the Euler-Mascheroni constant and $\psi := \Gamma'/\Gamma$..

Proof. From (1), one gets

$$\begin{aligned}
\int_0^1 f\left(\gamma + \psi(\{1/x\}+1)\right) \mathrm{d}x &= \int_0^1 f\left(\gamma + \psi(x+1)\right) \psi'(x+1)\, \mathrm{d}x \\
&= \int_0^1 f\left(\gamma + \psi(x+1)\right) \left(\gamma + \psi(x+1)\right)' \mathrm{d}x \\
&= \int_0^1 f(u)\, \mathrm{d}u,
\end{aligned}$$

using the change of variables $u = \gamma + \psi(x+1)$ which gives $u(0) = \gamma + \psi(1) = 0$ and $u(1) = \gamma + \psi(2) = 1$.

Proposition 1.1.1

Let $n = 0, 1, 2, \cdots$. Then

$$\int_0^1 (\psi(\{1/x\}+1))^n \, \mathrm{d}x = \sum_{k=0}^{n} \frac{(-1)^k}{n-k+1}\binom{n}{k}\gamma^k \tag{3}$$

and

$$\gamma^n = (-1)^n \sum_{k=0}^{n}\binom{n}{k} B_k \int_0^1 (\psi(\{1/x\}+1))^{n-k}\, \mathrm{d}x. \tag{4}$$

where γ is the Euler constant, $\psi := \Gamma'/\Gamma$ and B_k are the Bernoulli numbers.

Theorem 1.1.2 enables one to evaluate a great variety of integrals involving the digamma function in the integrand.

Proof. Using (1), we may prove (3) by writing

$$\begin{aligned}
\int_0^1 (\psi(\{1/x\}+1))^n \, \mathrm{d}x &= \int_0^1 (\gamma + \psi(\{1/x\}+1) - \gamma)^n \, \mathrm{d}x \\
&= \sum_{k=0}^{n} (-1)^{n-k}\gamma^{n-k}\binom{n}{k}\int_0^1 (\gamma + \psi(\{1/x\}+1))^k \, \mathrm{d}x \\
&= \sum_{k=0}^{n} (-1)^{n-k}\gamma^{n-k}\binom{n}{k}\int_0^1 x^k \, \mathrm{d}x \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{n-k+1}\binom{n}{k}\gamma^k.
\end{aligned}$$

Identity (4) is then deduced from (4) by appealing to an inversion combinatorial sum (**J. Riordan**, Inverse Relations and Combinatorial Identities. The American Mathematical Monthly, Vol. 71, No. 5, May 1964, p. 495).

$$\int_0^1 \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots+\cfrac{1}{1+\psi(\{1/x\}+1)}}}}\mathrm{d}x = \frac{F_{n-1}}{F_n} - \frac{(-1)^n}{F_n^2}\ln\left(\frac{F_{n+2}-F_n\gamma}{F_{n+1}-F_n\gamma}\right) \tag{5}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x, γ is the Euler-Mascheroni constant, F_n are the Fibonacci numbers, $\psi := \Gamma'/\Gamma$ is the digamma function and where the continued fraction has a total of n horizontal bars.

Proof. Using (2), we just have to evaluate the corresponding elementary rational func-

tion. One may prove by a simple induction that

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots+\cfrac{1}{1+x}}}}=\frac{F_nx+F_{n+1}}{F_{n+1}x+F_n}$$

where the continued fraction has n horizontal bars, leading easily to (5). □

Exercise 1.1.105: Find the value of the integral:

$$\int_0^\infty \frac{x^{\frac{2}{5}}}{1+x^2}\mathrm{d}x$$

Solution: Substitution $x=t^5$ to obtain

$$\int_0^\infty \frac{5t^6}{1+t^{10}}\mathrm{d}t$$

One may recall the Euler beta function

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

with a remarkable case

$$\int_0^1 x^{a-1}(1-x)^{-a}\mathrm{d}x=\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin(\pi a)}.$$

Then, by the change of variable $x=\dfrac{1}{1+t^{10}}$, giving $t=x^{-1/10}(1-x)^{1/10}$, we get

$$\int_0^\infty \frac{5t^6}{1+t^{10}}\mathrm{d}t=\frac{1}{2}\int_0^1 x^{-7/10}(1-x)^{7/10-1}\mathrm{d}x=\frac{\pi}{2\sin(7\pi/10)}=\frac{\sqrt{5}-1}{2}\pi$$

□

Exercise 1.1.106: Evaluate:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{e^x}{(e^x+1)(x^2+1)}\mathrm{d}x$$

Solution:

$$\begin{aligned}
\int_{-\sqrt{3}}^{\sqrt{3}} \frac{e^x}{(e^x+1)(x^2+1)}\mathrm{d}x &= \int_{-\sqrt{3}}^{0} \frac{e^x}{(e^x+1)(x^2+1)}\mathrm{d}x+\int_0^{\sqrt{3}} \frac{e^x}{(e^x+1)(x^2+1)}\mathrm{d}x\\
&=\int_0^{\sqrt{3}} \frac{e^{-x}}{(e^{-x}+1)(x^2+1)}\mathrm{d}x+\int_0^{\sqrt{3}} \frac{e^x}{(e^x+1)(x^2+1)}\mathrm{d}x\\
&=\int_0^{\sqrt{3}} \frac{1}{(e^x+1)(x^2+1)}\mathrm{d}x+\int_0^{\sqrt{3}} \frac{e^x}{(e^x+1)(x^2+1)}\mathrm{d}x\\
&=\int_0^{\sqrt{3}} \frac{(1+e^x)}{(e^x+1)(x^2+1)}\mathrm{d}x=\int_0^{\sqrt{3}} \frac{1}{x^2+1}\mathrm{d}x=[\arctan x]_0^{\sqrt{3}}=\frac{\pi}{3}
\end{aligned}$$

□

Exercise 1.1.107: Let $P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ and evaluate

$$\int \frac{2n! \sin x + x^n}{e^x + \sin x + \cos x + P_n(x)} \, \mathrm{d}x$$

Solution: Observe that

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

is such that

$$P_n'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} = P_n(x) - \frac{x^n}{n!}.$$

Setting

$$f(x) := e^x + \cos x + \sin x + P_n(x)$$

we then have

$$f'(x) = e^x - \sin x + \cos x + P_n(x) - \frac{x^n}{n!}$$

and

$$f(x) - f'(x) = 2 \sin x + \frac{x^n}{n!}$$

Hence the integral may be rewritten as

$$\begin{aligned}
\int \frac{2n! \sin x + x^n}{e^x + \sin x + \cos x + P_n(x)} \mathrm{d}x &= n! \int \frac{f(x) - f'(x)}{f(x)} \mathrm{d}x \\
&= n! \int \mathrm{d}x - n! \int \frac{f'(x)}{f(x)} \mathrm{d}x \\
&= n!\, x - n!\, \ln |f(x)| + C \\
&= n!\, x - n!\, \ln |e^x + \cos x + \sin x + P_n(x)| + C
\end{aligned}$$

□

Exercise 1.1.108: Evaluate this integral:

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{\ln(\tan(x))} + \frac{1}{1 - \tan(x)} \right) \mathrm{d}x$$

Solution: By the change of variable $u = \tan x$, we readily have

$$I = \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \frac{1}{1+u^2} \mathrm{d}u.$$

We set

$$I(s) := \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \frac{u^s}{1+u^2} \mathrm{d}u, \quad s > -1.$$

The expected integral is thus $I(0)$.

Let's differentiate $I(s)$.

We get

$$\begin{aligned} I'(s) &= \int_0^1 \left(\frac{1}{\ln u}+\frac{1}{1-u}\right)\frac{u^s \ln u}{1+u^2}\mathrm{d}u \\ &= \int_0^1 \left(1+\frac{\ln u}{1-u}\right)\frac{u^s}{1+u^2}\mathrm{d}u \\ &= \int_0^1 \frac{u^s}{1+u^2}\mathrm{d}u + \int_0^1 \frac{(1+u)u^s \ln u}{(1-u^2)(1+u^2)}\mathrm{d}u \\ &= \int_0^1 \frac{u^s(1-u^2)}{1-u^4}\mathrm{d}u + \int_0^1 \frac{u^s \ln u}{1-u^4}\mathrm{d}u + \int_0^1 \frac{u^{s+1}\ln u}{1-u^4}\mathrm{d}u. \end{aligned}$$

By the change of variable $v = u^4$ in each of the preceding integrals and recalling the well known integral representations for the digamma function $\psi := \Gamma'/\Gamma$ and for its derivative,

$$\psi(s) = -\gamma + \int_0^1 \frac{1-v^{s-1}}{1-v}\mathrm{d}v, \quad s>0,$$

$$\psi'(s) = -\int_0^1 \frac{s^{s-1}\ln v}{1-v}\mathrm{d}v, \quad s>0,$$

we obtain

$$I'(s) = \frac{1}{4}\psi\left(\frac{s+3}{4}\right) - \frac{1}{4}\psi\left(\frac{s+1}{4}\right) - \frac{1}{16}\psi'\left(\frac{s+2}{4}\right) - \frac{1}{16}\psi'\left(\frac{s+1}{4}\right). \tag{2}$$

Since

$$\left|\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)\frac{u^s}{1+u^2}\right| < u^s, \quad 0<u<1,\, s>-1,$$

giving

$$|I(s)| \le \int_0^1 \left|\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)\frac{u^s}{1+u^2}\right|\mathrm{d}u < \int_0^1 u^s \mathrm{d}u = \frac{1}{s+1},$$

then, as $s\to+\infty$, we have $I(s)\to 0$.

We deduce that

$$I(s) = \log\Gamma\left(\frac{s+3}{4}\right) - \ln\Gamma\left(\frac{s+1}{4}\right) - \frac{1}{4}\psi\left(\frac{s+2}{4}\right) - \frac{1}{4}\psi\left(\frac{s+1}{4}\right),\ s>-1, \tag{3}$$

and, for $s=0$,

$$I = \frac{\pi}{8} + \frac{7}{4}\ln 2 + \frac{\gamma}{2} + \ln\pi - 2\ln\Gamma\left(\frac{1}{4}\right)$$

where have used special values of the digamma function,

$$\begin{aligned} \psi\left(\frac{1}{2}\right) &= -\gamma - 2\ln 2, \\ \psi\left(\frac{1}{4}\right) &= -\gamma + \frac{\pi}{2} - 3\ln 2, \end{aligned}$$

and the complement/reflection formula

$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sin(\frac{\pi}{4})} = \pi\sqrt{2}.$$

□

Exercise 1.1.109: Evaluate:

$$\int_0^{\frac{\pi}{2}}\left(\frac{1}{\ln(\tan x)}+\frac{1}{1-\tan x}\right)^2\mathrm{d}x$$

and

$$\int_0^{\frac{\pi}{2}}\left(\frac{1}{\ln(\tan x)}+\frac{1}{1-\tan x}\right)^3\mathrm{d}x$$

Solution: Using the method shown in **Exercise 1.1.108**, I have found the following closed form.

$$\int_0^{\frac{\pi}{2}}\left(\frac{1}{\ln(\tan x)}+\frac{1}{1-\tan x}\right)^2\mathrm{d}x=3\ln 2-\frac{4}{\pi}\mathrm{G}-\frac{1}{2},$$

So We have

$$\int_0^{\frac{\pi}{2}}\left(\frac{1}{\ln(\tan x)}+\frac{1}{1-\tan x}\right)^3\mathrm{d}x=\frac{9}{2}\ln 2-\frac{6}{\pi}\mathrm{G}-\frac{3}{4}-\frac{\pi}{8}\tag{1}$$

Proof. Set

$$I_n:=\int_0^{\frac{\pi}{2}}\left(\frac{1}{\ln(\tan x)}+\frac{1}{1-\tan x}\right)^n\mathrm{d}x,\ n=0,1,2....\tag{2}$$

Clearly

$$I_0=\frac{\pi}{2}.$$

Recall that

$$\tan\left(\frac{\pi}{2}-x\right)=\frac{1}{\tan x}$$

giving

$$\begin{aligned}\frac{1}{\ln\tan\left(\frac{\pi}{2}-x\right)}+\frac{1}{1-\tan\left(\frac{\pi}{2}-x\right)}&=\frac{1}{\ln\left(\frac{1}{\tan x}\right)}+\frac{1}{1-\frac{1}{\tan x}}\\&=-\frac{1}{\ln\left(\tan x\right)}-\frac{1}{1-\tan x}+1\end{aligned}\tag{3}$$

Hence, by the change of variable $x\to\frac{\pi}{2}-x$, we readily have

$$I_1=\int_0^{\frac{\pi}{2}}\left(-\frac{1}{\ln(\tan x)}-\frac{1}{1-\tan x}+1\right)\mathrm{d}x=-I_1+\frac{\pi}{2}\tag{4}$$

$$I_1=\frac{\pi}{4}.$$

Similarly,

$$\int_0^{\frac{\pi}{2}}\left(\frac{1}{\ln(\tan x)}+\frac{1}{1-\tan x}\right)^3\mathrm{d}x=\int_0^{\frac{\pi}{2}}\left(-\frac{1}{\ln\left(\tan x\right)}-\frac{1}{1-\tan x}+1\right)^3\mathrm{d}x\tag{5}$$

and, by the binomial expansion,

$$I_3=-I_3+3I_2-3I_1+I_0$$

$$I_3=-I_3+3I_2-3\frac{\pi}{4}+\frac{\pi}{2}$$

$$I_3 = \frac{3}{2} I_2 - \frac{\pi}{8} \tag{6}$$

As the result shown in (1). □

Exercise 1.1.110: Prove:

$$\int_0^1 \sqrt{\frac{\{1/x\}}{1-\{1/x\}}} \frac{\mathrm{d}x}{1-x} = \pi$$

where x denotes the fractional part of x.

Proof: With the substitution $1/x = t$, the integral is:

$$I = \int_1^\infty \sqrt{\frac{\{t\}}{1-\{t\}}} \frac{\mathrm{d}t}{t(t-1)} = \sum_{n=1}^\infty \int_n^{n+1} \sqrt{\frac{t-n}{n+1-t}} \frac{\mathrm{d}t}{t(t-1)}$$

Next, use the substitution $\dfrac{t-n}{n+1-t} = x^2 \Rightarrow t = n+1-\dfrac{1}{1+x^2} \Rightarrow \mathrm{d}t = \dfrac{2x}{(1+x^2)^2}\,\mathrm{d}x$

$$I = 2\sum_{n=1}^\infty \int_0^\infty \frac{x^2}{(x^2(n+1)+n)(nx^2+n-1)}\,\mathrm{d}x$$

$$\Rightarrow I = 2\sum_{n=1}^\infty \frac{1}{n(n+1)} \int_0^\infty \frac{x^2}{\left(x^2+\dfrac{n}{n+1}\right)\left(x^2+\dfrac{n-1}{n}\right)}\,\mathrm{d}x$$

The above integral can be evaluated by decomposing into partial fractions

$$\begin{aligned} I &= \sum_{n=1}^\infty \frac{1}{n(n+1)} \frac{\pi}{\left(\sqrt{\dfrac{n}{n+1}} + \sqrt{\dfrac{n-1}{n}}\right)} \\ &= \pi \sum_{n=1}^\infty \frac{1}{\sqrt{n(n+1)}(n+\sqrt{n^2-1})} \\ &= \pi \sum_{n=1}^\infty \frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}} \\ &= \pi \sum_{n=1}^\infty \left(\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n-1}{n}}\right) \end{aligned}$$

The final sum telescopes and its value is 1. Hence,

$$I = \pi$$

Let me suggest a general approach.

We start with a theorem.

Theorem 1.1.2

Let ff be a piecewise continuous function on $(0,1)$ verifying $\int_0^1 \frac{|f(x)|}{x}\,\mathrm{d}x < \infty$. Then

$$\int_0^1 f\left(\{1/x\}\right)\frac{\mathrm{d}x}{1-x} = \int_0^1 f(x)\frac{\mathrm{d}x}{x} \tag{1}$$

where x denotes the fractional part of x.

Proof. One has

$$\begin{aligned}
\int_0^1 f\left(\{1/x\}\right)\frac{\mathrm{d}x}{1-x} &= \sum_{k=1}^{\infty}\int_{\frac{1}{k+1}}^{\frac{1}{k}} f\left(\{1/x\}\right)\frac{\mathrm{d}x}{1-x}\\
&= \sum_{k=1}^{\infty}\int_k^{k+1} f\left(\{u\}\right)\frac{\mathrm{d}u}{u(u-1)}\\
&= \sum_{k=1}^{\infty}\int_k^{k+1} f\left(u-k\right)\frac{\mathrm{d}u}{u(u-1)}\\
&= \sum_{k=1}^{\infty}\int_0^{1} f\left(v\right)\frac{\mathrm{d}v}{(v+k)(v+k-1)}\\
&= \int_0^1 f\left(v\right)\sum_{k=1}^{\infty}\frac{1}{(v+k)(v+k-1)}\mathrm{d}v\\
&= \int_0^1 f(v)\frac{\mathrm{d}v}{v},
\end{aligned}$$

where the permutation between the infinite sum and the integration is allowed by the dominated convergence theorem: let $v\in(0,1)$, as N tends to ∞,

$$\begin{aligned}
&f\left(v\right)\sum_{k=1}^{N}\frac{1}{(v+k)(v+k-1)}\\
&= f(v)\sum_{k=1}^{N}\left(\frac{1}{v+k-1}-\frac{1}{v+k}\right) = \frac{f(v)}{v}-\frac{f(v)}{v+N}\longrightarrow\frac{f(v)}{v}
\end{aligned}$$

and one has

$$\left|f\left(v\right)\sum_{k=1}^{N}\frac{1}{(v+k)(v+k-1)}\right| = \left|\frac{f(v)}{v}-\frac{f(v)}{v+N}\right| = \frac{|f(v)|}{v}-\frac{|f(v)|}{v+N}\le\frac{|f(v)|}{v}$$

the latter function being such that $\int_0^1\frac{|f(v)|}{v}\,\mathrm{d}v<\infty$ by hypothesis.

Now, to get our result one may apply (1) with $f(x)=\sqrt{\frac{x}{1-x}}$. Clearly f satisfies the hypotheses of our Theorem and

$$\int_0^1 |f(x)|\,\frac{\mathrm{d}x}{x} = \int_0^1\left(\frac{x}{1-x}\right)^{1/2}\frac{\mathrm{d}x}{x} = 2\int_0^1\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = \pi.$$

Similarly, inserting successively

$$f(x) := \sqrt{\frac{x}{1-a^2x}}, \qquad 0 < a \leq 1$$

and

$$f(x) := \sqrt[n]{\frac{x}{1-x}}, \qquad n = 2, 3, \cdots$$

into (1) gives some generalizations.

Proposition 1.1.2

Let $0 < a \leq 1$. Then

$$\int_0^1 \sqrt{\frac{\{1/x\}}{1-a^2\{1/x\}}} \frac{\mathrm{d}x}{1-x} = \frac{2\arcsin a}{a}$$

where x denotes the fractional part of x.

The initial integral is obtained with $a = 1$.

Proposition 1.1.3

Let $n = 2, 3, \cdots$. Then

$$\int_0^1 \sqrt[n]{\frac{\{1/x\}}{1-\{1/x\}}} \frac{\mathrm{d}x}{1-x} = \frac{\pi}{\sin(\pi/n)}$$

where x denotes the fractional part of x.

The initial integral is obtained with $n = 2$.

The (new) identity (1) may be seen as a dual of the Gauss famous invariance result in ergodic theory:

$$\int_0^1 f\left(\{1/x\}\right) \frac{\mathrm{d}x}{x+1} = \int_0^1 f(x) \frac{\mathrm{d}x}{x+1}, \quad f \in L^1(0,1)$$

where x denotes the fractional part of x. □

Exercise 1.1.111: Prove:

$$\int_0^\infty \frac{x^4}{(x^4+x^2+1)^3} \mathrm{d}x = \frac{\pi}{48\sqrt{3}}$$

☞ **Proof:**

$$\begin{aligned}
&\int_0^\infty \frac{x^4}{\left(x^4+x^2+1\right)^3}\mathrm{d}x = \int_0^\infty \frac{x^4}{\left(x^2+\frac{1}{x^2}+1\right)^3 x^6}\mathrm{d}x \\
&= \int_0^\infty \frac{1}{\left(x^2+\frac{1}{x^2}+1\right)^3}\frac{\mathrm{d}x}{x^2} = \int_0^\infty \frac{1}{\left(x^2+\frac{1}{x^2}+1\right)^3}\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \frac{1}{\left(x^2+\frac{1}{x^2}+1\right)^3}\left(1+\frac{1}{x^2}\right)\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{1}{\left(\left(x-\frac{1}{x}\right)^2+3\right)^3}\mathrm{d}\left(x-\frac{1}{x}\right) \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{\left(u^2+3\right)^3}\mathrm{d}u = \frac{1}{4}\,\partial_a^2\left(\int_{-\infty}^{+\infty}\frac{1}{\left(u^2+a\right)}\mathrm{d}u\right)\Big|_{a=3} = \frac{1}{4}\,\partial_a^2\left(\frac{\pi}{\sqrt{a}}\right)\Big|_{a=3} \\
&= \frac{\pi}{48\sqrt{3}}
\end{aligned}$$

□

Exercise 1.1.112: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{x^2\cdot\ln\sin x}{\sin^2 x}\mathrm{d}x$$

✎ **Solution:** Since $\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$, we have

$$\begin{aligned}
\int \frac{\ln(\sin(x))}{\sin^2(x)}\,\mathrm{d}x &= -\int \ln(\sin(x))\,\mathrm{d}\cot(x) \\
&= -\cot(x)\ln(\sin(x)) + \int(\csc^2(x)-1)\,\mathrm{d}x \\
&= -\cot(x)\ln(\sin(x)) - \cot(x) - x
\end{aligned}$$

Then since $\int \cot(x)\,\mathrm{d}x = \ln(\sin(x))$ we have

$$\begin{aligned}
&-\int(\cot(x)\ln(\sin(x)) + \cot(x) + x)\,\mathrm{d}x \\
&= -\frac{1}{2}\ln(\sin(x))^2 - \ln(\sin(x)) - \frac{1}{2}x^2
\end{aligned}$$

Therefore, since $\int x^2 f''(x)\,\mathrm{d}x = x^2 f'(x) - 2xf(x) + 2\int f(x)\,\mathrm{d}x$

$$\begin{aligned}
&\int_0^{\pi/2} x^2\frac{\ln(\sin(x))}{\sin^2(x)}\,\mathrm{d}x \\
&= -\frac{\pi^3}{8} + \frac{\pi^3}{8} - \int_0^{\pi/2}\left(\ln(\sin(x))^2 + 2\ln(\sin(x)) + x^2\right)\mathrm{d}x
\end{aligned}$$

Using (2) and (9) from this answer

http://math.stackexchange.com/a/300319,

we get

$$\begin{aligned}
\int_0^{\pi/2} x^2\frac{\ln(\sin(x))}{\sin^2(x)}\,\mathrm{d}x &= -\left(\frac{\pi^3}{24} + \frac{\pi}{2}\ln(2)^2\right) + 2\left(\frac{\pi}{2}\ln(2)\right) - \frac{\pi^3}{24} \\
&= \pi\ln(2) - \frac{\pi}{2}\ln(2)^2 - \frac{\pi^3}{12}
\end{aligned}$$

□

Exercise 1.1.113: Find the value of:

$$\int_0^{\pi/2} \ln(2015+\sin^2 x)\,\mathrm{d}x$$

Solution: Set

$$I(a) := \int_0^{\pi/2} \ln(a+\sin^2 x)\mathrm{d}x, \qquad a \geq 0.$$

Then differentiating under the integral sign with respect to a you get

$$\begin{aligned} I'(a) &= \int_0^{\pi/2} \frac{1}{a+\sin^2 x}\mathrm{d}x = \int_0^{\infty} \frac{1}{a+\dfrac{t^2}{t^2+1}}\frac{\mathrm{d}t}{t^2+1} \quad (t=\tan x) \\ &= \int_0^{\infty} \frac{1}{(a+1)t^2+a}\mathrm{d}t = \frac{\pi}{2}\frac{1}{\sqrt{a(a+1)}} = \pi\left(\ln\left(\sqrt{a}+\sqrt{a+1}\right)\right)\Big|_a' \end{aligned}$$

Thus

$$\int_0^{\pi/2} \ln(a+\sin^2 x)\mathrm{d}x = \pi\ln\left(\sqrt{a}+\sqrt{a+1}\right)+C$$

with $C = I(0) = -\pi\ln 2$ giving

$$\int_0^{\pi/2} \ln(a+\sin^2 x)\mathrm{d}x = \pi\ln\left(\frac{\sqrt{a}+\sqrt{a+1}}{2}\right), \quad a \geq 0.$$

Hence the initial integral is obtained with $a = 2015$,

$$\int_0^{\pi/2} \ln(2015+\sin^2 x)\mathrm{d}x = \pi\ln\left(\frac{\sqrt{2015}+\sqrt{2016}}{2}\right)$$

□

Exercise 1.1.114: Closed form for:

$$\int_0^1 \frac{\ln^2\ln\left(\dfrac{1}{x}\right)}{1+x+x^2}\mathrm{d}x$$

Solution:

$$\begin{aligned} \int_0^1 \frac{\ln^2\ln\left(\frac{1}{x}\right)}{1+x+x^2}\mathrm{d}x &= \int_0^{\infty}\left(1-e^{-t}\right)\frac{\ln^2 t}{1-e^{-3t}}e^{-t}\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\int_0^{\infty}\left(e^{-t}-e^{-2t}\right)e^{-3nt}\ln^2 t\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\left(\int_0^{\infty}e^{-(3n+1)t}\ln^2 t\,\mathrm{d}t - \int_0^{\infty}e^{-(3n+2)t}\ln^2 t\,\mathrm{d}t\right) \\ &= \sum_{n=0}^{\infty}\partial_s^2\left(\int_0^{\infty}t^s e^{-(3n+1)t}\,\mathrm{d}t - \int_0^{\infty}t^s e^{-(3n+2)t}\,\mathrm{d}t\right)\Big|_{s=0} \\ &= \partial_s^2\left(\Gamma(s+1)\left(\sum_{n=0}^{\infty}\frac{1}{(3n+1)^{s+1}} - \sum_{n=0}^{\infty}\frac{1}{(3n+2)^{s+1}}\right)\right)\Big|_{s=0} \\ &= \partial_s^2\left(\frac{\Gamma(s+1)}{3^{s+1}}\left(\zeta\left(s+1,\frac{1}{3}\right)-\zeta\left(s+1,\frac{2}{3}\right)\right)\right)\Big|_{s=0} \end{aligned}$$

Then, using the Laurent series expansion of the Hurwitz zeta function near 1,

$$\zeta(s+1,a)=\frac{1}{s}+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(a)s^k,\quad a>0,\ s\to 0,$$

with $\gamma_0(a)=-\psi(a)=-\Gamma'(a)/\Gamma(a)$, we get

$$\frac{\pi^3\sqrt{3}}{54}+\frac{\sqrt{3}\pi}{9}(\gamma+\ln 3)^2+\frac{2}{3}(\gamma+\ln 3)\left(\gamma_1\left(\frac{1}{3}\right)-\gamma_1\left(\frac{2}{3}\right)\right)+\frac{1}{3}\left(\gamma_2\left(\frac{1}{3}\right)-\gamma_2\left(\frac{2}{3}\right)\right),$$

where

$$\gamma_1\left(\frac{1}{3}\right)-\gamma_1\left(\frac{2}{3}\right)=\frac{\pi}{\sqrt{3}}\left(6\ln\Gamma\left(\frac{1}{3}\right)-\gamma+\frac{\ln 3}{2}-4\ln(2\pi)\right).$$

□

Exercise 1.1.115: Calculate this integral:

$$\int_0^1\frac{1-x}{\ln x}\left(x+x^2+x^{2^2}+x^{2^3}+\cdots\right)\mathrm{d}x$$

Solution: We have

$$\int_0^1\frac{1-x}{\ln x}\left(x+x^2+x^{2^2}+x^{2^3}+\cdots\right)\mathrm{d}x=-\ln 3. \tag{1}$$

Proof. One may recall that, using **Frullani's integral**, we have

$$\int_0^1\frac{x^{a-1}-x^{b-1}}{\ln x}\,\mathrm{d}x=\ln\frac{a}{b}\quad(a,b>0). \tag{2}$$

Considering a finite sum in the integrand, we get

$$\begin{aligned}
&\int_0^1\frac{1-x}{\ln(x)}(x+x^2+x^{2^2}+\cdots+x^{2^N})\,\mathrm{d}x\qquad(N=0,1,2,\cdots)\\
&=\sum_{n=0}^{N}\int_0^1\frac{x^{2^n}-x^{2^n+1}}{\ln x}\,\mathrm{d}x=\sum_{n=0}^{N}\ln\frac{2^n+1}{2^n+2}\qquad(\text{using }(2))\\
&=\sum_{n=0}^{N}\left(\ln(2^n+1)-\ln(2^{n-1}+1)-\ln 2\right)\quad(\text{a telescoping sum})\\
&=\ln(2^N+1)-\ln(2^{-1}+1)-(N+1)\ln 2=-\ln 3+\ln\left(1+1/2^N\right),
\end{aligned}$$

then, letting $N\to+\infty$, leads to (1). □

Exercise 1.1.116: Prove:

$$\int_0^\infty\frac{\ln x}{\cosh^2 x}\,\mathrm{d}x=\ln\frac{\pi}{4}-\gamma$$

Proof: Method One:

Let $I(a)=\displaystyle\int_0^\infty\frac{x^a}{\cosh^2 x}\,\mathrm{d}x$.
Then if $\Re(a)>0$,

$$\begin{aligned}
I(a)&=4\int_0^\infty\frac{x^a}{(e^x+e^{-x})^2}\,\mathrm{d}x=4\int_0^\infty x^a\frac{e^{-2x}}{(1+e^{-2x})^2}\,\mathrm{d}x\\
&=4\int_0^\infty x^a\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-2nx}\,\mathrm{d}x=4\sum_{n=1}^{\infty}(-1)^{n-1}n\int_0^\infty x^ae^{-2nx}\,\mathrm{d}x\\
&=4\sum_{n=1}^{\infty}(-1)^{n-1}n\frac{\Gamma(a+1)}{(2n)^{a+1}}=\frac{2\Gamma(a+1)}{2^a}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^a}=\frac{2\Gamma(a+1)\eta(a)}{2^a}
\end{aligned}$$

where $\eta(a)$ is the Dirichlet eta function.

But by analytic continuation, the above formula is valid if $\mathrm{Re}(a) > -1$.

Then differentiating under the integral sign we get

$$I'(a) = \int_0^\infty \frac{x^a \ln(x)}{\cosh^2 x}\,\mathrm{d}x = 2\frac{[\Gamma'(a+1)\eta(a) + \Gamma(a+1)\eta'(a)]2^a - \Gamma(a+1)\eta(a)\ln(2)2^a}{2^{2a}}$$

which means

$$\int_0^\infty \frac{\ln x}{\cosh^2 x}\,\mathrm{d}x = I'(0) = 2\Big(\Gamma'(1)\eta(0) + \eta'(0) - \eta(0)\ln(2)\Big).$$

The eta values can be determined using the relation

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

and the known Riemann zeta values

$$\zeta(0) = -\frac{1}{2}$$

and

$$\zeta'(0) = -\frac{1}{2}\ln(2\pi).$$

Therefore,

$$\int_0^\infty \frac{\ln x}{\cosh^2 x}\,\mathrm{d}x = 2\left[-\gamma\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \frac{1}{2}\ln(2)\right] = \ln\left(\frac{\pi}{4}\right) - \gamma$$

Method Two:

Let the considered integral be denoted by I. We have

$$\begin{aligned}
I &= \int_0^1 \frac{\ln x}{\cosh^2 x}\mathrm{d}x + \int_1^\infty \frac{\ln x}{\cosh^2 x}\mathrm{d}x \\
&= \Big[(\ln x)\tanh x\Big]_0^1 - \int_0^1 \frac{\tanh x}{x}\mathrm{d}x + \Big[(\ln x)(\tanh x - 1)\Big]_1^\infty - \int_1^\infty \frac{\tanh x - 1}{x}\mathrm{d}x \\
&= 2\int_1^\infty \frac{\mathrm{d}x}{x(1+e^{2x})} - \int_0^1 \frac{1-e^{-2x}}{x(1+e^{-2x})}\mathrm{d}x \\
&= 2\int_1^\infty \frac{\mathrm{d}x}{x(1+e^{2x})} - \int_0^1 \frac{1-e^{-2x}}{x}\left(\frac{1}{1+e^{-2x}} - 1\right)\mathrm{d}x - \int_0^1 \frac{1-e^{-2x}}{x}\mathrm{d}x \\
&= 2\int_1^\infty \frac{\mathrm{d}x}{x(1+e^{2x})} + \int_0^1 \frac{e^{-2x}-e^{-4x}}{x(1+e^{-2x})}\mathrm{d}x - \int_0^1 \frac{1-e^{-2x}}{x}\mathrm{d}x \\
&= 2\int_1^\infty \frac{\mathrm{d}x}{x(1+e^{2x})} - \int_1^\infty \frac{e^{-2x}-e^{-4x}}{x(1+e^{-2x})}\mathrm{d}x + \int_0^\infty \frac{e^{-2x}-e^{-4x}}{x(1+e^{-2x})}\mathrm{d}x - \int_0^1 \frac{1-e^{-2x}}{x}\mathrm{d}x \\
&= \int_1^\infty \frac{1+e^{-2x}}{x(1+e^{2x})}\mathrm{d}x + \int_0^\infty \frac{e^{-2x}-e^{-4x}}{x(1+e^{-2x})}\mathrm{d}x - \int_0^1 \frac{1-e^{-2x}}{x}\mathrm{d}x \\
&= \int_1^\infty \frac{e^{-2x}}{x}\mathrm{d}x - \int_0^1 \frac{1-e^{-2x}}{x}\mathrm{d}x + \int_0^\infty \frac{e^{-2x}-e^{-4x}}{x(1+e^{-2x})}\mathrm{d}x \\
&= \underbrace{\int_2^\infty \frac{e^{-t}}{t}\mathrm{d}t - \int_0^2 \frac{1-e^{-t}}{t}\mathrm{d}t}_{A} + \underbrace{\int_0^\infty \frac{e^{-t}-e^{-2t}}{t(1+e^{-t})}\mathrm{d}t}_{B}
\end{aligned}$$

Now, note that

$$\begin{aligned} A &= \int_2^\infty \frac{e^{-t}}{t}\mathrm{d}t - \int_0^1 \frac{1-e^{-t}}{t}\mathrm{d}t - \int_1^2 \frac{1-e^{-t}}{t}\mathrm{d}t \\ &= \int_2^\infty \frac{e^{-t}}{t}\mathrm{d}t - \int_0^1 \frac{1-e^{-t}}{t}\mathrm{d}t - \ln 2 + \int_1^2 \frac{e^{-t}}{t}\mathrm{d}t \\ &= \int_1^\infty \frac{e^{-t}}{t}\mathrm{d}t - \int_0^1 \frac{1-e^{-t}}{t}\mathrm{d}t - \ln 2 = -\gamma - \ln 2 \end{aligned}$$

To calculate B we can do the following

$$\begin{aligned} B &= \int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\left(\sum_{n=0}^\infty (-1)^n e^{-nt}\right)\mathrm{d}t = \sum_{n=0}^\infty (-1)^n \int_0^\infty \frac{e^{-(n+1)t}-e^{-(n+2)t}}{t}\mathrm{d}t \\ &= \sum_{n=0}^\infty (-1)^n \ln\left(\frac{n+2}{n+1}\right) = \lim_{m\to\infty}\sum_{n=0}^{2m-2}(-1)^n \ln\left(\frac{n+2}{n+1}\right) \\ &= \lim_{m\to\infty}\ln\left(\prod_{k=1}^m \frac{2k}{2k-1}\prod_{k=1}^{m-1}\frac{2k}{2k+1}\right) = \lim_{m\to\infty}\ln\left(\frac{(2m)!!(2m-2)!!}{((2m-1)!!)^2}\right) = \ln\left(\frac{\pi}{2}\right) \end{aligned}$$

By Stirling's formula. This yields $I = \ln\pi - 2\ln 2 - \gamma$. □

Exercise 1.1.117: Evaluate:

$$\int_0^\infty \frac{\arctan x}{x(x^2+1)}\,\mathrm{d}x$$

Solution: Consider

$$I(a) := \int_0^\infty \frac{\arctan(ax)}{x(x^2+1)}\,\mathrm{d}x, \quad 0 < a < 1, \tag{1}$$

and obtain

$$I'(a) = \int_0^\infty \frac{1}{(x^2+1)(a^2x^2+1)}\,\mathrm{d}x.$$

By using partial fraction decomposition, we have

$$\frac{1}{(x^2+1)(a^2x^2+1)} = \frac{1}{(1-a^2)(x^2+1)} - \frac{a^2}{(1-a^2)(a^2x^2+1)}$$

giving

$$\begin{aligned} I'(a) &= \frac{1}{(1-a^2)}\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x - \frac{a^2}{(1-a^2)}\int_0^\infty \frac{1}{a^2x^2+1}\,\mathrm{d}x \\ &= \frac{1}{(1-a^2)}[\arctan x]_0^\infty - \frac{a^2}{(1-a^2)}\left[\frac{\arctan(ax)}{a}\right]_0^\infty \\ &= \frac{1}{(1-a^2)}\frac{\pi}{2} - \frac{a}{(1-a^2)}\frac{\pi}{2} = \frac{\pi}{2}\frac{1}{1+a} \end{aligned} \tag{2}$$

Since $I(0) = 0$, by integrating (2), you easily get

$$\int_0^\infty \frac{\arctan(ax)}{x(x^2+1)}\,\mathrm{d}x = \frac{\pi}{2}\ln(a+1), \qquad 0 \le a < 1,$$

from which, by letting $a \to 1^-$, you deduce

$$\int_0^\infty \frac{\arctan x}{x(x^2+1)}\,\mathrm{d}x = \frac{\pi}{2}\ln 2$$

□

Exercise 1.1.118: Prove:

$$\int_0^\infty e^{-x}\prod_{n=1}^\infty\left(1-e^{-24nx}\right)\mathrm{d}x=\frac{\pi^2}{6\sqrt{3}}$$

Proof: To be somewhat explicit. One may perform the change of variable, $q=e^{-x}$, $\mathrm{d}q=-e^{-x}\mathrm{d}x$, giving

$$\int_0^\infty e^{-x}\prod_{n=1}^\infty\left(1-e^{-24nx}\right)\mathrm{d}x=\int_0^1\prod_{n=1}^\infty\left(1-q^{24n}\right)\mathrm{d}q$$

then use the identity (the **Euler pentagonal number theorem**)

https://en.wikipedia.org/wiki/Jacobi_triple_product#Properties

https://en.wikipedia.org/wiki/Pentagonal_number_theorem

$$\prod_{n=1}^\infty(1-q^n)=\sum_{-\infty}^\infty(-1)^n q^{\frac{3n^2-n}{2}}$$

to get

$$\int_0^\infty e^{-x}\prod_{n=1}^\infty\left(1-e^{-24nx}\right)\mathrm{d}x=\sum_{-\infty}^\infty\int_0^1(-1)^n q^{12(3n^2-n)}\mathrm{d}q=\sum_{-\infty}^\infty\frac{(-1)^n}{(6n-1)^2}=\frac{\pi^2}{6\sqrt{3}}$$

The last equality may be obtained by converting the series to the Hurwitz zeta function and by using the **multiplication theorem**.

http://dlmf.nist.gov/25.11#v □

Exercise 1.1.119: Evaluate:

$$\int_0^1\left(\frac{1}{\ln x}+\frac{1}{1-x}\right)^2\mathrm{d}x$$

Solution: Using the fact that $\dfrac{1}{\ln(x)}=-\lim\limits_{n\to\infty}\dfrac{1/n}{1-x^{1/n}}$ yields

$$\begin{aligned}\int_0^1\left(\frac{1}{1-x}+\frac{1}{\ln(x)}\right)^2\mathrm{d}x&=\lim_{n\to\infty}\int_0^1\left(\frac{1}{1-x}-\frac{1/n}{1-x^{1/n}}\right)^2\mathrm{d}x\\&=\lim_{n\to\infty}\int_0^1\left(\frac{1}{(1-x)^2}-\frac{2/n}{(1-x)(1-x^{1/n})}+\frac{1/n^2}{(1-x^{1/n})^2}\right)\mathrm{d}x\end{aligned}\tag{1}$$

For each term in the last integral, we can use the power series for the integrand to get

$$\int_0^a\frac{1}{(1-x)^2}\mathrm{d}x=\frac{a}{1-a}\tag{2}$$

$$\int_0^a\frac{1}{(1-x)(1-x^{1/n})}\mathrm{d}x=\int_0^a\left(\sum_{k=0}^\infty x^{k/n}\left\lfloor\frac{k+n}{n}\right\rfloor\right)\mathrm{d}x=\sum_{k=n}^\infty\frac{n}{k}a^{k/n}\left\lfloor\frac{k}{n}\right\rfloor\tag{3}$$

$$\int_0^a\frac{1}{(1-x^{1/n})^2}\mathrm{d}x=\int_0^a\left(\sum_{k=0}^\infty(k+1)x^{k/n}\right)\mathrm{d}x=\sum_{k=n}^\infty(k-n+1)\frac{n}{k}a^{k/n}\tag{4}$$

Combining (2), (3), and (4) as in (1) yields

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{1-x}+\frac{1}{\ln(x)}\right)^2 \mathrm{d}x \\
&= \lim_{a\to 1}\lim_{n\to\infty}\int_0^a \left(\frac{1}{(1-x)^2}-\frac{2/n}{(1-x)(1-x^{1/n})}+\frac{1/n^2}{(1-x^{1/n})^2}\right)\mathrm{d}x && (5)\\
&= \lim_{a\to 1}\frac{a}{1-a}+\lim_{n\to\infty}\sum_{k=n}^{\infty} a^{k/n}\left(-\frac{2}{k}\left\lfloor\frac{k}{n}\right\rfloor+\frac{k-n+1}{kn}\right) && (6)\\
&= \lim_{a\to 1}\frac{a}{1-a}+\frac{a}{\ln(a)}+\lim_{n\to\infty}\sum_{k=n}^{\infty} a^{k/n}\left(-\frac{2}{k}\left\lfloor\frac{k}{n}\right\rfloor+\frac{2k-n+1}{kn}\right) && (7)\\
&= \frac{1}{2}+\int_1^{\infty}\left(-\frac{2}{x}\lfloor x\rfloor+2-\frac{1}{x}\right)\mathrm{d}x && (8)\\
&= \frac{1}{2}+\sum_{k=1}^{\infty}\left(2-(2k+1)\ln\left(\frac{k+1}{k}\right)\right) && (9)\\
&= \frac{1}{2}+\lim_{n\to\infty}2n+2\sum_{k=2}^{n}\ln(k)-(2n+1)\ln(n+1) && (10)\\
&= \frac{1}{2}+\lim_{n\to\infty}2n+\ln(2\pi)+(2n+1)\ln(n)-2n-(2n+1)\ln(n+1) && (11)\\
&= \frac{1}{2}+\ln(2\pi)-\lim_{n\to\infty}(2n+1)\ln(1+1/n) && (12)\\
&= \ln(2\pi)-\frac{3}{2} && (13)
\end{aligned}$$

Explanation:

(5):Apply (1).

(6):Apply (2), (3), and (4).

(7): $\lim_{n\to\infty}\sum_{k=n}^{\infty}\frac{1}{n}a^{k/n}=\lim_{n\to\infty}\frac{1}{n}\frac{a}{1-a^{1/n}}=-\frac{a}{\ln(a)}$

(8): $\lim_{a\to 1}\frac{a}{1-a}+\frac{a}{\ln(a)}=\frac{1}{2}$ and a Riemann Sum

(9):Break the integral into integer intervals

(10):Collapse the telescoping sum

(11):Stirling's Approximation

(12):Arithmetic

(13): $\lim_{n\to\infty}(2n+1)\ln(1+1/n)=2$ □

Exercise 1.1.120: Prove:

$$I:=\int_0^{\infty}\frac{x}{\sinh(ax)}\mathrm{d}x=\frac{\pi^2}{4a^2}$$

Proof:

$$\begin{aligned}
&\int_0^{+\infty}\frac{x}{\sinh(ax)}\,\mathrm{d}x=2\int_0^{+\infty}\frac{x}{e^{ax}-e^{-ax}}\,\mathrm{d}x=2\int_0^{+\infty}\frac{x}{1-e^{-2ax}}e^{-ax}\,\mathrm{d}x\\
&=2\int_0^{+\infty}x\sum_{n=0}^{\infty}e^{-a(2n+1)x}\mathrm{d}x=2\sum_{n=0}^{\infty}\int_0^{+\infty}x\,e^{-a(2n+1)x}\mathrm{d}x=\frac{2}{a^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{2}{a^2}\frac{\pi^2}{8}=\frac{\pi^2}{4a^2}
\end{aligned}$$

□

Exercise 1.1.121: Prove:

$$\int_0^1 \ln\ln\left(\frac{1+x}{1-x}\right)\cdot\frac{\ln x}{1-x^2}\,\mathrm{d}x=\frac{\pi^2}{24}\,\ln\left(\frac{\mathbf{A}^{36}}{16\,\pi^3}\right)$$

where **A** is the **Glaisher—Kinkelin constant**.

Proof: Change of variable $x\to\ln\frac{1+x}{1-x}$ we get $\frac{1}{2}\int_0^\infty \ln x\cdot\ln\tanh\left(\frac{x}{2}\right)\mathrm{d}x$,and use the integral $\int\frac{1}{\sinh x}\mathrm{d}x=\ln\tanh\left(\frac{x}{2}\right)+C$, we have

$$\begin{aligned}
&-2\int_0^1 \ln\ln\left(\frac{1+x}{1-x}\right)\frac{\ln x}{1-x^2}\mathrm{d}x=-\int_0^\infty \ln x\ \ln\tanh\left(\frac{x}{2}\right)\mathrm{d}x\\
&=\int_0^\infty \ln x\int_x^\infty\frac{1}{\sinh t}\mathrm{d}t\mathrm{d}x=\int_0^\infty\int_0^t \ln x\frac{1}{\sinh t}\mathrm{d}x\mathrm{d}t\\
&=\int_0^\infty\frac{x\ln x-x}{\sinh x}\mathrm{d}x=\int_0^\infty(x\ln x-x)\frac{2e^{-x}}{1-e^{-2x}}\,\mathrm{d}x\\
&=\int_0^\infty(x\ln x-x)2\sum_{n=0}^\infty e^{-x(2n+1)}\mathrm{d}x=2\sum_{n=0}^\infty\left(\frac{\mathrm{d}}{\mathrm{d}s}\frac{\Gamma(s)}{(2n+1)^s}\bigg|_{s=2}-\frac{1}{(2n+1)^2}\right)\\
&=2\sum_{n=0}^\infty\left(\frac{\Gamma'(2)-1}{(2n+1)^2}-\frac{\ln(2n+1)}{(2n+1)^2}\right)=2(\Gamma'(2)-1)\frac{\pi^2}{8}-2\left(\sum_{n=1}^\infty\frac{\ln(n)}{n^2}-\sum_{n=1}^\infty\frac{\ln(2n)}{(2n)^2}\right)\\
&=2(\Gamma'(2)-1)\frac{\pi^2}{8}+2\zeta'(2)+\frac{1}{2}\left(\ln 2\frac{\pi^2}{6}-\zeta'(2)\right)=-\frac{\pi^2}{12}\ln\left(\frac{A^{36}}{16\pi^3}\right)
\end{aligned}$$

Hence we get

$$\int_0^1 \ln\ln\left(\frac{1+x}{1-x}\right)\cdot\frac{\ln x}{1-x^2}\,\mathrm{d}x=\frac{\pi^2}{24}\,\ln\left(\frac{\mathbf{A}^{36}}{16\,\pi^3}\right)$$

□

Exercise 1.1.122: Evaluate:

$$\int_0^1\left(\frac{1}{\ln x}+\frac{1}{1-x}-\frac{1}{2}\right)\frac{\mathrm{d}x}{1-x}$$

and find a closed form for:

$$I(s):=\int_0^1\left(\frac{1}{\ln x}+\frac{1}{1-x}-\frac{1}{2}\right)\frac{x^s}{1-x}\mathrm{d}x,\quad \Re(s)>-1$$

Solution: Recall Binet's formula

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+\int_0^\infty\left(\frac{1}{2}-\frac{1}{x}+\frac{1}{e^x-1}\right)\frac{e^{-zx}}{x}\mathrm{d}x,\quad \Re z>0$$

which, upon making $x=-\ln v$, can be written as

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)-\int_0^1\left(\frac{1}{\ln v}+\frac{1}{1-v}-\frac{1}{2}\right)\frac{v^{z-1}}{\ln v}\mathrm{d}v,\ \Re z>0$$

and Gauss' formula

$$-\psi(z)+\ln z=\int_0^1\left(\frac{1}{\ln v}+\frac{1}{1-v}\right)v^{z-1}\mathrm{d}v.$$

One may check that

$$\left(\frac{1}{\ln u}+\frac{1}{1-u}-\frac{1}{2}\right)\frac{u^s}{1-u}=u^s\frac{\mathrm{d}}{\mathrm{d}u}\left\{u\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)\right\}$$
$$+\left(\frac{1}{\ln u}+\frac{1}{1-u}-\frac{1}{2}\right)\frac{u^s}{\ln u}-\frac{1}{2}\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)u^s.$$

Hence our integral $I(s)$ is the sum of three integrals.

The first,

$$I_1(s)=\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)u^{s+1}\Big|_0^1-s\int_0^1\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)u^s\mathrm{d}u,$$

using Gauss' formula,

$$I_1(s)=\frac{1}{2}+s\left(\psi(s+1)-\ln(s+1)\right).$$

The second, applying Binet's formula,

$$I_2(s)=\int_0^1\left(\frac{1}{\ln u}+\frac{1}{1-u}-\frac{1}{2}\right)\frac{u^s}{\ln u}\mathrm{d}u$$
$$=-\ln\Gamma(s+1)+(s+1/2)\ln(s+1)-s-1+\frac{1}{2}\ln(2\pi).$$

The third, using Gauss' formula once more

$$I_3(s)=-\frac{1}{2}\int_0^1\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)u^s\mathrm{d}u=\frac{1}{2}\left(\psi(s+1)-\ln(s+1)\right).$$

Consequently, $I(s)=I_1(s)+I_2(s)+I_3(s)$ is given by

$$I(s):=\int_0^1\left(\frac{1}{\ln u}+\frac{1}{1-u}-\frac{1}{2}\right)\frac{u^s}{1-u}\mathrm{d}u$$
$$=-\ln\Gamma(s+1)+\left(s+\frac{1}{2}\right)\psi(s+1)-s-\frac{1}{2}+\frac{1}{2}\ln(2\pi).$$

We have, with $s=0$,

$$\int_0^1\left(\frac{1}{\log x}+\frac{1}{1-x}-\frac{1}{2}\right)\frac{\mathrm{d}x}{1-x}=-\frac{1}{2}+\frac{1}{2}\ln(2\pi)-\frac{1}{2}\gamma$$

□

Exercise 1.1.123: Evaluate the following integral:

$$\int_0^{\pi/2}\frac{x\sin x\cos x}{(a^2\cos^2x+b^2\sin^2x)^2}\mathrm{d}x$$

Solution: The case $a^2=b^2$ being simple, let's just consider, by symmetry, the case $a>b>0$.

Observe that

$$\partial_x\left(\frac{1}{a^2\cos^2x+b^2\sin^2x}\right)=2(a^2-b^2)\frac{\cos x\sin x}{(a^2\cos^2x+b^2\sin^2x)^2}$$

then, integrating by parts, you may write

$$\begin{aligned}
I(a,b) &= \int_0^{\pi/2} \frac{x\cos x\sin x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\mathrm{d}x \\
&= \frac{x}{2(a^2-b^2)}\,\frac{1}{a^2\cos^2 x + b^2\sin^2 x}\bigg|_0^{\pi/2} - \frac{1}{2(a^2-b^2)}\int_0^{\pi/2}\frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} \\
&= \frac{\pi}{4(a^2-b^2)b^2} - \frac{1}{2(a^2-b^2)}\int_0^{\pi/2}\frac{1}{\left(a^2+b^2\dfrac{\sin^2 x}{\cos^2 x}\right)}\frac{1}{\cos^2 x}\mathrm{d}x \\
&= \frac{\pi}{4(a^2-b^2)b^2} - \frac{1}{2(a^2-b^2)}\int_0^{\pi/2}\frac{1}{(a^2+b^2\tan^2 x)}(\tan x)'\mathrm{d}x \\
&= \frac{\pi}{4(a^2-b^2)b^2} - \frac{1}{2(a^2-b^2)}\frac{1}{ab}\arctan\left(\frac{b}{a}\tan x\right)\bigg|_0^{\pi/2} \\
&= \frac{\pi}{4(a^2-b^2)b^2} - \frac{\pi}{4(a^2-b^2)}\frac{1}{ab} = \frac{\pi}{4(a+b)ab^2}
\end{aligned}$$

□

Exercise 1.1.124: Evaluate:

$$\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}\,\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}\,\mathrm{d}x = 2\int_0^{\infty}\frac{x^2e^x}{(e^x+1)^2}\,\mathrm{d}x = -2\int_0^{\infty}x^2\,\mathrm{d}\frac{1}{e^x+1} \\
&= 4\int_0^{\infty}\frac{x}{e^x+1}\,\mathrm{d}x = 4\Gamma(2)\,\eta(2) = 2\zeta(2) = \frac{\pi^2}{3}
\end{aligned}$$

□

Exercise 1.1.125: Evaluate:

$$\int_0^{\infty}\frac{\ln^2(1-e^{-x})x^5}{e^x-1}\,\mathrm{d}x$$

Solution: By the change of variable, $u = e^{-x}$, you get

$$\begin{aligned}
\int_0^{\infty}\frac{\ln^2(1-e^{-x})x^5}{e^x-1}\,\mathrm{d}x &= -\int_0^1\frac{\ln^2(1-u)\ln^5 u}{1-u}\,\mathrm{d}u \\
&= -\partial_b^2\partial_a^5\left(\int_0^1 u^{a-1}(1-u)^{b-1}\mathrm{d}u\right)\bigg|_{a=1,b=0} \\
&= -\partial_b^2\partial_a^5\left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)\bigg|_{a=1,b=0} \\
&= \frac{4}{3}\pi^4\zeta(3) + 20\pi^2\zeta(5) - 360\zeta(7)
\end{aligned}$$

□

Exercise 1.1.126: Evaluate:

$$\int_0^1\ln\left(\frac{1+x}{1-x}\right)\frac{\mathrm{d}x}{x}$$

✎ **Solution:** By the change of variable

$$t = \frac{1-x}{1+x}$$

we get

$$\int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{\mathrm{d}x}{x} = -2\int_0^1 \frac{\ln t}{1-t^2}\mathrm{d}t = -2\sum_{n=0}^{\infty}\int_0^1 t^{2n}\ln t\,\mathrm{d}t = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{4}$$

□

Exercise 1.1.127: Evaluate:

$$\int_0^1 \frac{x^3}{2(2-x^2)(1+x^2)+3\sqrt{(2-x^2)(1+x^2)}}\,\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
I &= \int_0^1 \frac{x^3}{2(2-x^2)(1+x^2)+3\sqrt{(2-x^2)(1+x^2)}}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \frac{t}{2(2-t)(1+t)+3\sqrt{(2-t)(1+t)}}\,\mathrm{d}t \qquad (1)\\
&= \frac{1}{3}\int_{\sqrt{2}}^{1/\sqrt{2}} \frac{u^2-2}{(1+u)^2(u^2+1)}\,\mathrm{d}u \qquad (2)\\
&= \frac{1}{3}\left[\int_{\sqrt{2}}^{1/\sqrt{2}} \frac{3u}{2\left(u^2+1\right)}\,\mathrm{d}u - \int_{\sqrt{2}}^{1/\sqrt{2}} \frac{3}{2(u+1)}\,\mathrm{d}u - \int_{\sqrt{2}}^{1/\sqrt{2}} \frac{1}{2(u+1)^2}\,\mathrm{d}u\right] \qquad (3)\\
&= \frac{1}{3}\left[\frac{3}{4}\ln\left(u^2+1\right) - \frac{3}{2}\ln(u+1) + \frac{1}{2(u+1)}\right]_{\sqrt{2}}^{1/\sqrt{2}} = \frac{1}{3}\left[\frac{3-2\sqrt{2}}{2}\right]
\end{aligned}$$

Hence

$$\int_0^1 \frac{x^3}{2(2-x^2)(1+x^2)+3\sqrt{(2-x^2)(1+x^2)}}\,\mathrm{d}x = \frac{1}{6}\left(3-2\sqrt{2}\right)$$

Explanation:

(1):Substitute $x^2 = t \iff 2x\,\mathrm{d}x = \mathrm{d}t$.

(2):Using Type III Euler Substitution

http://en.wikipedia.org/wiki/Euler_substitution

$$\begin{aligned}
\sqrt{(2-t)(t+1)} = (t+1)u &\iff t+1 = \frac{3}{u^2+1}\\
&\iff t = \frac{2-u^2}{u^2+1}\\
&\iff \mathrm{d}t = -\frac{6u}{\left(u^2+1\right)^2}\,\mathrm{d}u
\end{aligned}$$

(3):Using Partial Fraction Decomposition

$$\frac{2-u^2}{(1+u)^2(u^2+1)} = -\frac{3u}{2\left(u^2+1\right)} + \frac{3}{2(u+1)} + \frac{1}{2(u+1)^2}$$

□

Exercise 1.1.128: Evaluate:

$$\int_0^\infty \frac{1}{x^n+1}\,\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int_0^\infty \frac{1}{x^n+1}\,\mathrm{d}x = \int_0^\infty \int_0^\infty e^{-(x^n+1)t}\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^\infty e^{-t}\int_0^\infty e^{-x^n t}\,\mathrm{d}t\,\mathrm{d}x = \int_0^\infty e^{-t}\left(\int_0^\infty e^{-x^n t}\,\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{1}{n}\int_0^\infty t^{-\frac{1}{n}}e^{-t}\left(\int_0^\infty u^{\frac{1}{n}-1}e^{-u}\mathrm{d}u\right)\mathrm{d}t = \frac{1}{n}\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right) \\
&= \frac{\pi}{n\sin\frac{\pi}{n}}
\end{aligned}$$

□

Exercise 1.1.129: Evaluate:

$$\int_0^\infty \frac{\sqrt[3]{x+1}-\sqrt[3]{x}}{\sqrt{x}}\,\mathrm{d}x$$

Solution: You may perform a change of variable $u=\dfrac{1}{1+x}$, $x=\dfrac{1}{u}-1$, in your initial integral to obtain

$$\begin{aligned}
\int_0^\infty \frac{\sqrt[3]{x+1}-\sqrt[3]{x}}{\sqrt{x}}\,\mathrm{d}x &= \int_0^1 \frac{(1/u)^{\frac{1}{3}}-(1/u-1)^{\frac{1}{3}}}{(1/u-1)^{\frac{1}{2}}}\frac{1}{u^2}\mathrm{d}u \\
&= \int_0^1 \frac{1-(1-u)^{\frac{1}{3}}}{(1-u)^{\frac{1}{2}}}u^{-11/6}\mathrm{d}u.
\end{aligned}$$

Now, observe that using **Beta Function** we get, for $s>0$,

$$\int_0^1 \frac{1-(1-u)^{\frac{1}{3}}}{(1-u)^{\frac{1}{2}}}u^s\mathrm{d}u = \frac{\Gamma(1/2)\,\Gamma(1+s)}{\Gamma(3/2+s)} - \frac{\Gamma(5/6)\,\Gamma(1+s)}{\Gamma(11/6+s)} \tag{1}$$

and since both sides of (1) are analytic for $\Re s>-2$ then the identity can be extended to this case allowing one to write

$$\begin{aligned}
\int_0^1 \frac{1-(1-u)^{\frac{1}{3}}}{(1-u)^{\frac{1}{2}}}u^{-11/6}\mathrm{d}u &= \lim_{s\to -11/6}\left(\frac{\Gamma(1/2)\,\Gamma(1+s)}{\Gamma(3/2+s)} - \frac{\Gamma(5/6)\,\Gamma(1+s)}{\Gamma(11/6+s)}\right) \\
&= \lim_{s\to -11/6}\Gamma(1+s)\left(\frac{\sqrt{\pi}}{\Gamma(3/2+s)} - \frac{\Gamma(5/6)\,(11/6+s)}{\Gamma(17/6+s)}\right) \\
&= \frac{2\sqrt{\pi}\,\Gamma(1/6)}{5\Gamma(2/3)},
\end{aligned}$$

with some algebra, consequently

$$\int_0^\infty \frac{\sqrt[3]{x+1}-\sqrt[3]{x}}{\sqrt{x}}\,\mathrm{d}x = \frac{2\sqrt{\pi}\,\Gamma(1/6)}{5\Gamma(2/3)}$$

□

Exercise 1.1.130: Find a closed form for:

$$\int_0^1 \ln(-\ln x)\ln\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$$

Solution: We have the following closed form.

$$\int_0^1 \ln(-\ln x)\ln\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x = \gamma_1 - 2\ln^2 2 - 2\gamma\ln 2 - \gamma_1\left(1,\frac{1}{2}\right) \tag{⋆}$$

where γ_1 is the Stieltjes constant,

$$\gamma_1 = \lim_{N\to+\infty}\left(\sum_{n=1}^{N}\frac{\ln n}{n} - \int_1^N \frac{\ln t}{t}\,\mathrm{d}t\right)$$

and where $\gamma_1(a,b)$ is the poly-Stieltjes constant,

$$\gamma_1(a,b) = \lim_{N\to+\infty}\left(\sum_{n=1}^{N}\frac{\ln(n+a)}{n+b} - \int_1^N \frac{\ln t}{t}\,\mathrm{d}t\right).$$

Proof. One may recall the classic integral representation of the Euler gamma function

$$\frac{\Gamma(s)}{(a+1)^s} = \int_0^\infty t^{s-1}e^{-(a+1)t}\,\mathrm{d}t, \qquad s>0,\ a>-1. \tag{1}$$

By differentiating (1) with respect to s, putting $s=1$ and making the change of variable $x=e^{-t}$, we get

$$\int_0^1 x^a\ln\left(-\ln x\right)\,\mathrm{d}x = -\frac{\gamma+\ln(a+1)}{a+1}, \qquad a>-1, \tag{2}$$

where $\gamma = \lim_{N\to+\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \int_1^N\frac{\mathrm{d}t}{t}\right)$ is the Euler-Mascheroni constant.
From the standard Taylor series expansion,

$$-\ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad |x|<1, \tag{3}$$

one gets

$$\ln(1+x)-\ln(1-x) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \qquad |x|<1. \tag{4}$$

One may write the given integral as

$$\int_0^1 \ln(-\ln x)\ln\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x = \int_0^1 \ln(-\ln x)\left(\ln(1+x)-\ln(1-x)\right)\mathrm{d}x$$

then, inserting (4) into the latter integrand and using (2), we obtain

$$\begin{aligned}
\int_0^1 \ln(-\ln x)\ln\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x &= 2\int_0^1 \ln(-\ln x)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\,\mathrm{d}x\\
&= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1 x^{2n+1}\ln(-\ln x)\,\mathrm{d}x\\
&= -2\sum_{n=0}^{\infty}\frac{\gamma+\ln(2n+2)}{(2n+1)(2n+2)}\\
&= -\sum_{n=0}^{\infty}\frac{2\left(\gamma+\ln 2\right)}{(2n+1)(2n+2)} - \sum_{n=1}^{\infty}\frac{2\ln(n+1)}{(2n+1)(2n+2)}.
\end{aligned} \tag{5}$$

On the one hand, using **Abel's theorem** and using (3), one has

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2}{(2n+1)(2n+2)} &= \lim_{x\to 1^-}\sum_{n=0}^{\infty}\frac{2x^{2n+2}}{(2n+1)(2n+2)}\\ &= \lim_{x\to 1^-}\left((1+x)\ln(1+x)+(1-x)\ln(1-x)\right)\\ &= 2\ln 2. \end{aligned}\tag{6}$$

On the other hand, using Theorem 2 here

http://math.stackexchange.com/questions/364452/

one has

$$\sum_{n=1}^{\infty}\frac{2\ln(n+1)}{(2n+1)(2n+2)} = -\gamma_1+\gamma_1\left(1,\frac{1}{2}\right),\tag{7}$$

since $\gamma_1(1,1)=\gamma_1$.

Finally, bringing all the steps together gives $(\star)$. □

Exercise 1.1.131: Evaluate:

$$\int_0^{\pi}\frac{x\cos x}{1+\sin^2 x}\mathrm{d}x$$

Solution: Start with a Fourier series expansion of $\dfrac{1}{1+\sin^2 x}$. For $0\le x\le\pi$, we easily obtain

$$\begin{aligned}\frac{1}{1+\sin^2 x} &= \Re\frac{1}{1-i\sin x}\\ &= \Re\frac{2e^{ix}}{2-(e^{ix}-1)^2}\\ &= \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\sum_{n=2}^{\infty}\left(\sqrt{2}-1\right)^n\left(1+(-1)^n\right)\cos(nx)\end{aligned}\tag{1}$$

The series in (1) is normally convergent on $[0,\pi]$, we are then allowed to integrate termwise:

$$\int_0^{\pi}\frac{x\cos x}{1+\sin^2 x}\mathrm{d}x = \frac{\sqrt{2}}{2}\sum_{n=2}^{\infty}(\sqrt{2}-1)^n(1+(-1)^n)\int_0^{\pi}x\cos x\cos(nx)\,\mathrm{d}x.$$

The latter integral is classic, we write

$$2\cos x\cos(nx)=\cos((n+1)x)+\cos((n-1)x)$$

then integrate twice by parts to get

$$-2\int_0^{\pi}x\cos x\cos(nx)\,\mathrm{d}x=\frac{1}{(n+1)^2}+\frac{1}{(n-1)^2},\quad n=2,3,\ldots.$$

It is easy to see that

$$\sum_{n=2}^{\infty}(1+(-1)^n)\left(\frac{x^n}{(n+1)^2}+\frac{x^n}{(n-1)^2}\right)=\left(x+\frac{1}{x}\right)\left(\mathrm{Li}_2(x)-\mathrm{Li}_2(-x)\right),\quad |x|<1,$$

where $\mathrm{Li}_2(\cdot)$ is the **dilogarithm function**. Now, putting $x := \sqrt{2}-1$ in the preceding identity and using the following closed form

$$\mathrm{Li}_2(\sqrt{2}-1)-\mathrm{Li}_2(-(\sqrt{2}-1))=\frac{\text{ß}^2}{8}-\frac{1}{2}\ln^2(\sqrt{2}+1),$$

Hence we get

$$\int_0^{\pi}\frac{x\cos x}{1+\sin^2 x}\mathrm{d}x=\ln^2(\sqrt{2}+1)-\frac{\pi^2}{4}$$

□

Exercise 1.1.132: Find a closed form for:

$$\int_0^{\infty}\frac{\sqrt[\phi]{x}\,\arctan x}{(x^{\phi}+1)^2}\mathrm{d}x$$

where ϕ is the **golden ratio**:

$$\phi=\frac{1+\sqrt{5}}{2}$$

Solution: Since $\frac{1}{\phi}=\phi-1$, we get

$$\int_0^{\infty}\frac{\sqrt[\phi]{x}\,\arctan(x)}{(x^{\phi}+1)^2}\mathrm{d}x=\int_0^{\infty}\frac{x^{\phi}\arctan(x)}{(x^{\phi}+1)^2}\frac{\mathrm{d}x}{x} \tag{1}$$

$$=\int_0^{\infty}\frac{x^{\phi}(\frac{\pi}{2}-\arctan(x))}{(x^{\phi}+1)^2}\frac{\mathrm{d}x}{x} \tag{2}$$

Average (1) and (2) to get

$$\int_0^{\infty}\frac{\sqrt[\phi]{x}\,\arctan(x)}{(x^{\phi}+1)^2}\mathrm{d}x=\frac{\pi}{4}\int_0^{\infty}\frac{x^{\phi}}{(x^{\phi}+1)^2}\frac{\mathrm{d}x}{x} \tag{3}$$

$$=\frac{\pi}{4\phi}\int_0^{\infty}\frac{x}{(x+1)^2}\frac{\mathrm{d}x}{x} \tag{4}$$

$$=\frac{\pi}{4\phi} \tag{5}$$

Explanation:
(1): $\frac{1}{\phi}=\phi-1$.
(2): Substitute $x\mapsto\frac{1}{x}$.
(3): Average (1) and (2).
(4): Substitute $x\mapsto x^{1/\phi}$.
(5): $\int_0^{\infty}\frac{\mathrm{d}x}{(x+1)^2}=\left[-\frac{1}{x+1}\right]_0^{\infty}=1$.

□

Exercise 1.1.133: Evaluate:

$$\int_0^{\infty}\frac{\mathrm{d}x}{(1+x^2)(1+x^a)}$$

Solution: With a change of variable

$$\int_0^{\infty}\frac{\mathrm{d}x}{(1+x^2)(1+x^a)}\overset{x\to 1/x}{=}\int_0^{\infty}\frac{x^a\mathrm{d}x}{(1+x^2)(1+x^a)}$$

Adding and dividing by two yields

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(1+x^a)} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)} = \frac{\pi}{4}$$

□

Exercise 1.1.134: Evaluate the **Fresnel integral**:

$$\int_0^\infty \sin x^2 \,\mathrm{d}x = \sqrt{\frac{\pi}{8}}$$

Solution: Method One:

Let $u = x^2$, then

$$\int_0^\infty \sin(u)\frac{\mathrm{d}u}{2\sqrt{u}}$$

The real analysis way of evaluating this integral is to consider a parametric family:

$$I(\epsilon) = \int_0^\infty \frac{\sin(u)}{2\sqrt{u}}\mathrm{e}^{-\epsilon u}\mathrm{d}u = \frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\int_0^\infty u^{2n+\frac{1}{2}}\mathrm{e}^{-\epsilon u}\mathrm{d}u$$

$$= \frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\Gamma\left(2n+\frac{3}{2}\right)\epsilon^{-\frac{3}{2}-2n} = \frac{1}{2\epsilon^{3/2}}\sum_{n=0}^\infty \left(-\frac{1}{\epsilon^2}\right)^n \frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma\left(2n+2\right)}$$

$$\overset{\Gamma-\text{duplication}}{=} \frac{1}{2\epsilon^{3/2}}\sum_{n=0}^\infty \left(-\frac{1}{\epsilon^2}\right)^n \frac{\Gamma\left(n+\frac{3}{4}\right)\Gamma\left(n+\frac{5}{4}\right)}{\sqrt{2}n!\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{1}{(2\epsilon)^{3/2}}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}{}_2F_1\left(\frac{3}{4},\frac{5}{4};\frac{3}{2};-\frac{1}{\epsilon^2}\right)$$

$$\overset{\text{Euler integral}}{=} \frac{1}{(2\epsilon)^{3/2}}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}\frac{1}{\mathrm{B}\left(\frac{5}{4},\frac{3}{2}-\frac{5}{4}\right)}\int_0^1 x^{\frac{5}{4}-1}(1-x)^{\frac{3}{2}-\frac{5}{4}-1}\left(1+\frac{x}{\epsilon^2}\right)^{-3/4}\mathrm{d}x$$

$$= \frac{1}{2^{3/2}}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)}\int_0^1 x^{\frac{5}{4}-1}(1-x)^{\frac{1}{4}-1}\left(\epsilon^2+x\right)^{-3/4}\mathrm{d}x$$

Now we are ready to compute $\lim_{\epsilon\to 0} I(\epsilon)$:

$$\lim_{\epsilon\to 0} I(\epsilon) = \frac{1}{2^{3/2}}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{4}-1}\,\mathrm{d}x = \frac{1}{2^{3/2}}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2^{3/2}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Method Two:

Aside from some trigonometric substitutions and identities, we will need the following identity, which can be shown using integration by parts twice:

$$\int_0^\infty \cos(\alpha t)e^{-\lambda t}\,\mathrm{d}t = \frac{\lambda}{\alpha^2+\lambda^2} \tag{1}$$

We will also use the standard arctangent integral:

$$\int_0^\infty \frac{\mathrm{d}t}{a^2+t^2} = \frac{\pi}{2a} \tag{2}$$

Now

$$\begin{aligned}
&\left(\int_0^\infty \sin(x^2)e^{-\lambda x^2}\,\mathrm{d}x\right)^2 \\
&= \int_0^\infty\int_0^\infty \sin(x^2)\sin(y^2)e^{-\lambda(x^2+y^2)}\,\mathrm{d}y\,\mathrm{d}x && (3.1)\\
&= \frac{1}{2}\int_0^\infty\int_0^\infty \left(\cos(x^2-y^2)-\cos(x^2+y^2)\right)e^{-\lambda(x^2+y^2)}\,\mathrm{d}y\,\mathrm{d}x && (3.2)\\
&= \frac{1}{2}\int_0^{\pi/2}\int_0^\infty \left(\cos(r^2\cos(2\phi))-\cos(r^2)\right)e^{-\lambda r^2}\,r\,\mathrm{d}r\,\mathrm{d}\phi && (3.3)\\
&= \frac{1}{4}\int_0^{\pi/2}\int_0^\infty \left(\cos(s\cos(2\phi))-\cos(s)\right)e^{-\lambda s}\,\mathrm{d}s\,\mathrm{d}\phi && (3.4)\\
&= \frac{1}{4}\int_0^{\pi/2}\left(\frac{\lambda}{\cos^2(2\phi)+\lambda^2}-\frac{\lambda}{1+\lambda^2}\right)\mathrm{d}\phi && (3.5)\\
&= \frac{1}{2}\int_0^{\pi/4}\frac{\lambda}{\cos^2(2\phi)+\lambda^2}\,\mathrm{d}\phi-\frac{\lambda\pi/8}{1+\lambda^2} && (3.6)\\
&= \frac{1}{4}\int_0^{\pi/4}\frac{\lambda\,\mathrm{d}\tan(2\phi)}{1+\lambda^2+\lambda^2\tan^2(2\phi)}-\frac{\lambda\pi/8}{1+\lambda^2} && (3.7)\\
&= \frac{1}{4}\int_0^\infty\frac{\mathrm{d}t}{1+\lambda^2+t^2}-\frac{\lambda\pi/8}{1+\lambda^2} && (3.8)\\
&= \frac{\pi/8}{\sqrt{1+\lambda^2}}-\frac{\lambda\pi/8}{1+\lambda^2} && (3.9)
\end{aligned}$$

(3.1) change the square of the integral into a double integral.

(3.2) use $2\sin(A)\sin(B) = \cos(A-B)-\cos(A+B)$.

(3.3) convert to polar coordinates.

(3.4) substitute $s = r^2$.

(3.5) apply (1).

(3.6) pull out the constant and apply symmetry to reduce the domain of integration.

(3.7) multiply numerator and denominator by $\sec^2(2\phi)$.

(3.8) substitute $t = \lambda\tan(2\phi)$.

(3.9) apply (2).

Finally, take the square root of both sides of (3) and let $\lambda \to 0^+$ to get

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}$$

also in the same way,we get

$$\int_0^\infty \cos(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}$$

□

Exercise 1.1.135: Evaluate:

$$\int_0^\pi \ln(\sin(x))\,\mathrm{d}x$$

✎ **Solution:** Start with

$$\int_0^{\pi/2} \ln(\sin(x))\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln(\sin(x))\,\mathrm{d}x = \int_0^{\pi/2} \ln(\sin(2x))\,\mathrm{d}x$$
$$= \int_0^{\pi/2} \Big(\ln(2) + \ln(\sin(x)) + \ln(\cos(x))\Big)\,\mathrm{d}x = \frac{\pi}{2}\ln(2) + 2\int_0^{\pi/2} \ln(\sin(x))\,\mathrm{d}x$$

Therefore,

$$\int_0^{\pi/2} \ln(\sin(x))\,\mathrm{d}x = -\frac{\pi}{2}\ln(2)$$

Thus

$$\int_0^{\pi} \ln(\sin(x))\,\mathrm{d}x = -\pi\ln(2)$$

□

Exercise 1.1.136: Calculate:

$$\int_0^{\infty} \frac{x^3}{e^x - 1}\,\mathrm{d}x$$

✎ **Solution:** In general,

$$\int_0^{\infty} \frac{x^n}{e^x-1}\mathrm{d}x = \sum_{k=1}^{\infty}\int_0^{\infty} x^n e^{-kx}\mathrm{d}x = \sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\int_0^{\infty} x^n e^{-x}\mathrm{d}x = \zeta(n+1)\Gamma(n+1)$$

In the particular case of $n = 3$, we get

$$\int_0^{\infty} \frac{x^3}{e^x-1}\mathrm{d}x = \zeta(4)\Gamma(4) = \frac{\pi^4}{90}\cdot 6 = \frac{\pi^4}{15}$$

□

Exercise 1.1.137: Prove:

$$\int_0^1 \left(\frac{\ln t}{1-t}\right)^2 \mathrm{d}t = \frac{\pi^2}{3}$$

☞ **Proof:** Consider

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}. \tag{1}$$

Differentiating (1) with respect to t yields

$$\sum_{k=1}^{\infty} k t^{k-1} = \frac{1}{(1-t)^2}. \tag{2}$$

Multiplying both sides of (2) by $\ln^2 t$ and integrating from $t = 0$ to $t = 1$ yields

$$\int_0^1 \left(\frac{\ln t}{1-t}\right)^2 \mathrm{d}t = \int_0^1 \sum_{k=1}^{\infty} k t^{k-1}\ln^2 t\;\mathrm{d}t = \sum_{k=1}^{\infty} k\int_0^1 t^{k-1}\ln^2 t\;\mathrm{d}t. \tag{3}$$

Using formula

$$\int_0^1 x^{\alpha}\ln^n x\;\mathrm{d}x = \frac{(-1)^n n!}{(\alpha+1)^{n+1}}, \qquad \text{for } \; n = 0, 1, 2, \ldots \tag{4}$$

then (3) becomes

$$\int_0^1 \left(\frac{\ln t}{1-t}\right)^2 \mathrm{d}t = 2\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{3}$$

□

Exercise 1.1.138: Evaluate:

$$\int_0^\infty \ln\left(\frac{x^2+2kx\cos b+k^2}{x^2+2kx\cos a+k^2}\right)\frac{\mathrm{d}x}{x}$$

Solution: Method One:

First note that by substituting $x \mapsto kx$, we get

$$\int_0^\infty \ln\left(\frac{x^2+2kx\cos(a)+k^2}{x^2+2kx\cos(b)+k^2}\right)\frac{\mathrm{d}x}{x} = \int_0^\infty \ln\left(\frac{x^2+2x\cos(a)+1}{x^2+2x\cos(b)+1}\right)\frac{\mathrm{d}x}{x}$$

Let $u = \dfrac{x+\cos(a)}{\sin(a)}$. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty \ln\left(\frac{x^2+2x\cos(a)+1}{x^2+2x+1}\right)\frac{\mathrm{d}x}{x} = -2\int_0^\infty \frac{\sin(a)}{x^2+2x\cos(a)+1}\,\mathrm{d}x\\
&= -2\int_0^\infty \frac{\sin(a)}{(x+\cos(a))^2+\sin^2(a)}\,\mathrm{d}x = -\frac{2}{\sin(a)}\int_0^\infty \frac{1}{\frac{(x+\cos(a))^2}{\sin^2(a)}+1}\,\mathrm{d}x\\
&= -2\int_{\cot(a)}^\infty \frac{1}{u^2+1}\,\mathrm{d}u = -2a
\end{aligned}$$

Integrating in a gives

$$\int_0^\infty \ln\left(\frac{x^2+2x\cos(a)+1}{x^2+2x+1}\right)\frac{\mathrm{d}x}{x} = -a^2$$

Therefore, by subtraction,

$$\int_0^\infty \ln\left(\frac{x^2+2kx\cos(a)+k^2}{x^2+2kx\cos(b)+k^2}\right)\frac{\mathrm{d}x}{x} = b^2-a^2$$

Method Two:

$$\begin{aligned}
&\int_0^\infty \ln\left(\frac{x^2+2kx\cos b+k^2}{x^2+2kx\cos a+k^2}\right)\frac{\mathrm{d}x}{x} = \int_0^\infty \frac{1}{x}\int_a^b \frac{\mathrm{d}}{\mathrm{d}y}\ln\left(x^2+2kx\cos y+k^2\right)\mathrm{d}y\,\mathrm{d}x\\
&= -\int_0^\infty \frac{1}{x}\int_a^b \frac{2kx\sin y}{x^2+2kx\cos y+k^2}\,\mathrm{d}y\,\mathrm{d}x = 2k\int_b^a\int_0^\infty \frac{\sin y\,\mathrm{d}x}{x^2+2kx\cos y+k^2}\,\mathrm{d}y\\
&= 2k\int_b^a\int_0^\infty \frac{\sin y\,\mathrm{d}x}{(x+k\cos y)^2+k^2-k^2\cos^2 y}\,\mathrm{d}y = 2k\int_b^a \sin y\underbrace{\int_0^\infty \frac{\mathrm{d}x}{(x+k\cos y)^2+k^2\sin^2 y}}_{(k\sin y)\,u=x+k\cos y}\mathrm{d}y\\
&= 2\int_b^a\int_{\cot y}^\infty \frac{\mathrm{d}u}{u^2+1}\,\mathrm{d}y = 2\int_b^a\left(\frac{\pi}{2}-\arctan\left(\cot y\right)\right)\mathrm{d}y = 2\int_b^a y\,\mathrm{d}y = a^2-b^2
\end{aligned}$$

Method Three:

Another approach is to use the Fourier series

$$\sum_{k=1}^{\infty}\frac{x^k\cos(ka)}{k} = -\frac{1}{2}\ln\left(x^2-2x\cos(a)+1\right)\ ,\ |x|<1$$

which can be derived from the Maclaurin series of $\ln(1-z)$ by replacing z with xe^{ia} and equating the real parts on both sides.

$$\int_0^\infty \ln\left(\frac{x^2+2kx\cos(b)+k^2}{x^2+2kx\cos(a)+k^2}\right)\frac{\mathrm{d}x}{x} = \int_0^\infty \ln\left(\frac{u^2+2u\cos(b)+1}{u^2+2u\cos(a)+1}\right)\frac{\mathrm{d}u}{u} \tag{1}$$

$$= 2\int_0^1 \ln\left(\frac{u^2+2u\cos(b)+1}{u^2+2u\cos(a)+1}\right)\frac{\mathrm{d}u}{u} \tag{2}$$

$$= 4\int_0^1 \frac{1}{u}\sum_{k=1}^{\infty}(-u)^k\frac{\cos(ka)-\cos(kb)}{k}\mathrm{d}u = 4\sum_{k=1}^{\infty}(-1)^k\frac{\cos(ka)-\cos(kb)}{k}\int_0^1 u^{k-1}\,\mathrm{d}u$$

$$= 4\sum_{k=1}^{\infty}(-1)^k\frac{\cos(ka)-\cos(kb)}{k^2} = 4\left(\frac{a^2}{4}-\frac{b^2}{4}\right) \tag{3}$$

$$= a^2-b^2$$

(1): Let $u=\dfrac{x}{k}$.

(2): Separate the integral into two integrals, namely one over the interval $(0,1)$ and one over the interval $(1,\infty)$, and replace u with $\dfrac{1}{u}$ in the second integral.

(3): By integrating the Fourier series $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\sin(ka)}{k} = -\frac{a}{2}$, $|a|<\pi$, we get

$$\sum_{k=1}^{\infty}(-1)^k\frac{\cos(ka)}{k^2} = \frac{a^2}{4}-\frac{\pi^2}{12}\ ,\ |a|<\pi.$$

□

Exercise 1.1.139: Evaluate:

$$\int_0^\infty \frac{x^{1/n}}{1+x^2}\,\mathrm{d}x$$

Solution: This is easily converted to a Beta integral. Let $x^2=\dfrac{t}{1-t}$, then

$$\int_0^\infty \frac{x^{1/n}}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 t^{\frac{1}{2n}-\frac{1}{2}}(1-t)^{-\frac{1}{2n}-\frac{1}{2}}\,\mathrm{d}t \tag{1}$$

$$= \frac{1}{2}\mathrm{B}\left(\frac{1}{2n}+\frac{1}{2},-\frac{1}{2n}+\frac{1}{2}\right) \tag{2}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2n}+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2n}+\frac{1}{2}\right) \tag{3}$$

$$= \frac{\pi}{2}\sec\left(\frac{\pi}{2n}\right) \tag{4}$$

Explanation: (1):Substitute $x^2=\dfrac{t}{1-t}$.

(2):$\mathrm{B}(\alpha,\beta)=\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$.

(3):$\mathrm{B}(\alpha,\beta)=\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(4):$\Gamma(\alpha)\Gamma(1-\alpha)=\dfrac{\pi}{\sin(\pi\alpha)}$

□

Exercise 1.1.140: Evaluate:

$$\int_0^{\pi/3}\Big((\sqrt{3}\cos\vartheta-\sin\vartheta)\sin\vartheta\Big)^{1/2}\cos\vartheta\,\mathrm{d}\vartheta$$

✎ **Solution:** First, an identity:

$$(\sqrt{3}\cos(\theta)-\sin(\theta))\sin(\theta)=2\sin\left(\frac{\pi}{3}-\theta\right)\sin(\theta)=\cos\left(\frac{\pi}{3}-2\theta\right)-\cos\left(\frac{\pi}{3}\right)$$
$$=\frac{1}{2}-2\sin^2\left(\frac{\pi}{6}-\theta\right)=\frac{1}{2}-2\sin^2\left(\theta-\frac{\pi}{6}\right)$$

Now, the integral

$$\int_0^{\pi/3}\left((\sqrt{3}\cos(\theta)-\sin(\theta))\sin(\theta)\right)^{1/2}\cos(\theta)\,\mathrm{d}\theta=\int_0^{\pi/3}\left(\frac{1}{2}-2\sin^2(\theta-\frac{\pi}{6})\right)^{1/2}\cos(\theta)\,\mathrm{d}\theta$$
$$=\frac{1}{\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2(\theta)\right)^{1/2}\cos(\theta+\frac{\pi}{6})\,\mathrm{d}\theta$$
$$=\frac{1}{\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2(\theta)\right)^{1/2}\left(\frac{\sqrt{3}}{2}\cos(\theta)-\frac{1}{2}\sin(\theta)\right)\,\mathrm{d}\theta$$
$$=\frac{\sqrt{3}}{2\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2(\theta)\right)^{1/2}\cos(\theta)\,\mathrm{d}\theta=\frac{\sqrt{3}}{2\sqrt{2}}\int_{-\pi/6}^{\pi/6}\left(1-4\sin^2(\theta)\right)^{1/2}\,\mathrm{d}\sin(\theta)$$
$$=\frac{\sqrt{3}}{4\sqrt{2}}\int_{-\pi/2}^{\pi/2}\left(1-\sin^2(\theta)\right)^{1/2}\,\mathrm{d}\sin(\theta)=\frac{\sqrt{3}}{4\sqrt{2}}\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,\mathrm{d}\theta=\frac{\pi\sqrt{3}}{8\sqrt{2}}$$

□

Exercise 1.1.141: Evaluate:

$$\int_0^\infty\frac{x^{\mu-1}}{1+x^\nu}\,\mathrm{d}x$$

where $0<\mu<\nu$.

✎ **Solution:** By the substitution $x=\tan^{2/\nu}\theta$, we have

$$\frac{\mathrm{d}x}{1+x^\nu}=\frac{2}{\nu}\tan^{(2/\nu)-1}\theta\,\mathrm{d}\theta.$$

Thus

$$\int_0^\infty\frac{x^{\mu-1}}{1+x^\nu}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{2}{\nu}\tan^{\frac{2\mu}{\nu}-1}\theta\,\mathrm{d}\theta=\frac{1}{\nu}B\left(\frac{\mu}{\nu},1-\frac{\mu}{\nu}\right)$$
$$=\frac{1}{\nu}\Gamma\left(\frac{\mu}{\nu}\right)\Gamma\left(1-\frac{\mu}{\nu}\right)=\frac{\pi}{\nu}\csc\left(\frac{\pi\mu}{\nu}\right)$$

where the last equality follows from Euler reflexion formula. □

Exercise 1.1.142: Prove:

$$\int_0^\infty\frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\mathrm{d}x=\frac{2-\sqrt{2}}{4}$$

☞ **Proof: Contours**

Since the integrand is even,

$$\int_0^\infty\frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x=\frac{1}{2}\int_{-\infty}^\infty\frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x$$

Define

$$f(z)=\frac{\cos(\pi z^2)}{\sinh(2\pi z)\sinh^2(\pi z)}$$

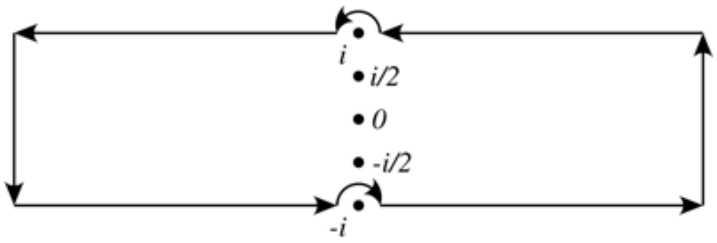
Note that because

$$f(x \pm i) = \frac{-\cos(\pi x^2)\cosh(2\pi x) \pm i \sin(\pi x^2)\sinh(2\pi x)}{\sinh(2\pi x)\sinh^2(\pi x)}$$

we have

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_{-\infty}^{\infty} \Big[f(x-i) - f(x+i)\Big]\,\mathrm{d}x \\ &= -2i\int_{-\infty}^{\infty} \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x \\ &= 2\pi i \times \text{the sum of the residuesinside the contour}\end{aligned}$$

where γ is the contour



Therefore,

$$\int_0^{\infty} \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x = -\frac{\pi}{2} \times \text{the sum of the residuesinside the contour}$$

Residues

near 0 :

$$f(z) = \frac{\cos(\pi z^2)}{\sinh(2\pi z)\sinh^2(\pi z)} = \frac{1 - \frac{1}{2}\pi^2 z^4 + O(z^8)}{2\pi z\left(1 + \frac{2}{3}\pi^2 z^2 + O(z^4)\right)\pi^2 z^2\left(1 + \frac{1}{3}\pi^2 z^2 + O(z^4)\right)}$$

$$= \frac{1 - \pi^2 z^2}{2\pi^3 z^3} + O(z) \implies \text{residue} = -\frac{1}{2\pi}$$

at $\pm i/2$, use L'Hosptal :

$$\begin{aligned}\text{residue} &= \lim_{z\to \pm i/2} \frac{(z \mp i/2)\cos(\pi z^2)}{\sinh(2\pi z)\sinh^2(\pi z)} = \frac{1}{2\pi\cosh(\pm\pi i)}\frac{\cos(-\pi/4)}{\sinh^2(\pm\pi i/2)} \\ &= \frac{1}{2\pi\cos(\pm\pi)}\frac{\sqrt{2}/2}{-\sin^2(\pm\pi/2)} = \frac{\sqrt{2}}{4\pi}\end{aligned}$$

near $\pm i$:

$$\begin{aligned}f(z \pm i) &= \frac{-\cos(\pi z^2)\cosh(2\pi z) \pm i\sin(\pi z^2)\sinh(2\pi z)}{\sinh(2\pi z)\sinh^2(\pi z)} \\ &= \frac{-\left(1 - \frac{1}{2}\pi^2 z^4 + O(z^8)\right)\left(1 + 2\pi^2 z^2 + O(z^4)\right) + O(z^3)}{2\pi z\left(1 + \frac{2}{3}\pi^2 z^2 + O(z^4)\right)\pi^2 z^2\left(1 + \frac{1}{3}\pi^2 z^2 + O(z^4)\right)} \\ &= -\frac{1 + \pi^2 z^2}{2\pi^3 z^3} + O(1) \implies \text{residue} = -\frac{1}{2\pi}\end{aligned}$$

Result

Thus,

$$\int_0^\infty \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x = -\frac{\pi}{2}\left(-\frac{1}{2\pi}-\frac{1}{2\pi}+\frac{\sqrt{2}}{4\pi}+\frac{\sqrt{2}}{4\pi}\right) = \frac{2-\sqrt{2}}{4}$$

□

Exercise 1.1.143: Compute the integral:

$$\int_0^\infty \frac{1}{(1+x^\varphi)^\varphi}\,\mathrm{d}x$$

where, $\varphi = \dfrac{\sqrt{5}+1}{2}$ is the **Golden Ratio**.

Solution: Since $\dfrac{1}{\varphi} = \varphi - 1$,

$$\begin{aligned}\int_0^\infty \frac{1}{(1+x^\varphi)^\varphi}\,\mathrm{d}x &= (\varphi-1)\int_0^\infty \frac{x^{\varphi-2}}{(1+x)^\varphi}\,\mathrm{d}x \qquad (1)\\ &= (\varphi-1)\mathrm{B}(\varphi-1,1) = (\varphi-1)\frac{\Gamma(\varphi-1)}{\Gamma(\varphi)} = \frac{\Gamma(\varphi)}{\Gamma(\varphi)} = 1\end{aligned}$$

Explanation: (1): substitute $x \mapsto x^{\varphi-1}$ noting that $\varphi(\varphi-1)=1$. □

Exercise 1.1.144: Prove:

$$\int_0^1 \frac{t^2-1}{(t^2+1)\ln t}\mathrm{d}t = 2\ln\left(\frac{2\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)$$

Proof:

$$\begin{aligned}&\int_0^1 \frac{t+1}{t^2+1}\frac{t-1}{\ln(t)}\,\mathrm{d}t = \int_0^1 \frac{t+1}{t^2+1}\int_0^1 t^x\,\mathrm{d}x\,\mathrm{d}t = \int_0^1\int_0^1 \frac{t^{x+1}+t^x}{t^2+1}\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_0^1\left(\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}-\frac{1}{x+4}+\ldots\right)\mathrm{d}x = \left(\ln\left(\frac{2}{1}\right)+\ln\left(\frac{3}{2}\right)\right)\\ &\quad -\left(\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5}{4}\right)\right)+\cdots = \ln\left(\frac{3}{1}\right)-\ln\left(\frac{5}{3}\right)+\ln\left(\frac{7}{5}\right)-\ln\left(\frac{9}{7}\right)+\ln\left(\frac{11}{9}\right)-\ldots\\ &= \ln\left(\frac{3}{1}\cdot\frac{3}{5}\cdot\frac{7}{5}\cdot\frac{7}{9}\cdot\frac{11}{9}\cdots\right) = \lim_{n\to\infty}\ln\left(\frac{\Gamma\left(\frac{5}{4}\right)^2}{\Gamma\left(\frac{3}{4}\right)^2}\frac{\Gamma\left(\frac{4n+3}{4}\right)^2}{\Gamma\left(\frac{4n+5}{4}\right)^2}(4n+3)\right) = 2\ln\left(\frac{2\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)\end{aligned}$$

□

Exercise 1.1.145: Evaluate:

$$\int_0^1 \frac{\ln(1+x)-\ln(1-x)}{\left(1+\ln^2 x\right)x}\,\mathrm{d}x$$

Solution: Recall the **Frullani Integral**:

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x = \ln\left(\frac{b}{a}\right) \qquad (1)$$

and scale equation (4) from this answer

http://math.stackexchange.com/a/176180

to get

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2+u^2}=\frac{\pi}{4u}\tanh\left(\frac{\pi u}{2}\right) \tag{2}$$

Then

$$\int_0^1\frac{\ln(1+x)-\ln(1-x)}{\left(1+\ln^2(x)\right)x}\,\mathrm{d}x=\int_0^{\infty}\frac{\ln\left(1+e^{-u}\right)-\ln\left(1-e^{-u}\right)}{1+u^2}\,\mathrm{d}u \tag{3a}$$

$$=2\sum_{k=0}^{\infty}\int_0^{\infty}\frac{e^{-(2k+1)u}}{2k+1}\frac{\mathrm{d}u}{1+u^2} \tag{3b}$$

$$=2\sum_{k=0}^{\infty}\int_0^{\infty}\frac{e^{-u}\,\mathrm{d}u}{(2k+1)^2+u^2} \tag{3c}$$

$$=2\int_0^{\infty}\frac{\pi}{4u}\tanh\left(\frac{\pi u}{2}\right)e^{-u}\,\mathrm{d}u \tag{3d}$$

$$=\frac{\pi}{2}\int_0^{\infty}\frac{1-e^{-\pi u}}{u}\frac{e^{-u}}{1+e^{-\pi u}}\,\mathrm{d}u \tag{3e}$$

$$=\frac{\pi}{2}\int_0^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{e^{-(k\pi+1)u}-e^{-((k+1)\pi+1)u}}{u}\,\mathrm{d}u \tag{3f}$$

$$=\frac{\pi}{2}\sum_{k=0}^{\infty}(-1)^k\ln\left(\frac{(k+1)\pi+1}{k\pi+1}\right) \tag{3g}$$

$$=\frac{\pi}{2}\sum_{k=0}^{\infty}\ln\left(\frac{(2k+1)\pi+1}{2k\pi+1}\frac{(2k+1)\pi+1}{(2k+2)\pi+1}\right) \tag{3h}$$

$$=\lim_{n\to\infty}\frac{\pi}{2}\ln\left[\prod_{k=0}^{n}\frac{k+\frac{1}{2}+\frac{1}{2\pi}}{k+\frac{1}{2\pi}}\frac{k+\frac{1}{2}+\frac{1}{2\pi}}{k+1+\frac{1}{2\pi}}\right] \tag{3i}$$

$$=\lim_{n\to\infty}\frac{\pi}{2}\ln\left[\frac{\Gamma\left(n+\frac{3}{2}+\frac{1}{2\pi}\right)^2}{\Gamma\left(\frac{1}{2}+\frac{1}{2\pi}\right)^2}\frac{\Gamma\left(\frac{1}{2\pi}\right)\Gamma\left(1+\frac{1}{2\pi}\right)}{\Gamma\left(n+1+\frac{1}{2\pi}\right)\Gamma\left(n+2+\frac{1}{2\pi}\right)}\right] \tag{3j}$$

$$=\frac{\pi}{2}\ln\left[\frac{\Gamma\left(\frac{1}{2\pi}\right)\Gamma\left(1+\frac{1}{2\pi}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2\pi}\right)^2}\right] \tag{3k}$$

$$=\pi\ln\left[\frac{\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2\pi}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2\pi}\right)}\right] \tag{3l}$$

Explanation:

(3a): Substitute $x=e^{-u}$.

(3b): $\ln\left(\frac{1+x}{1-x}\right)=2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$.

(3c): Substitute $u \mapsto \dfrac{u}{2k+1}$.
(3d): apply (2).
(3e): $\tanh\left(\dfrac{\pi u}{2}\right) = \dfrac{1-e^{-\pi u}}{1+e^{-\pi u}}$.
(3f): $\dfrac{1}{1+x} = \displaystyle\sum_{k=0}^{\infty}(-1)^k x^k$.
(3g): apply (1).
(3h): combine $2k$ and $2k+1$ terms.
(3i): change a sum of logs to a log of a product.
(3j): write products as ratios of Gamma functions.
(3k): apply **Gautschi's Inequality**.
(3l): $\Gamma(1+x) = x\Gamma(x)$. □

Exercise 1.1.146: Evaluate:

$$\int_0^{\infty} \frac{\sin^{2n+1}(x)}{x}\,\mathrm{d}x$$

Solution: There is a theorem that states if $f(x)$ is continuous and π-peridodic on $\mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} f(x)\,\mathrm{d}x = \int_0^{\pi} f(x)\,\mathrm{d}x.$$

See Graham Hesketh's comment for a way to prove this.

Using this theorem,

$$\begin{aligned}\int_0^{\infty} \frac{\sin^{2n+1}(x)}{x}\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^{2n+1}(x)}{x}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\sin^{2n}(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{\pi} \sin^{2n}(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^{2n}(x)\,\mathrm{d}x \\ &= \frac{\pi}{2^{2n+1}}\binom{2n}{n}\end{aligned}$$

□

Exercise 1.1.147: Evaluate:

$$\int_0^{\infty} \ln(t)^n\, e^{-t}\,\mathrm{d}t$$

Solution: Using the definition

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\, e^{-t}\,\mathrm{d}t$$

and taking the derivative n times, we get

$$\Gamma^{(n)}(1) = \int_0^{\infty} \ln(t)^n\, e^{-t}\,\mathrm{d}t$$

we also have

$$\Gamma'(x+1) = \Gamma(x+1)\left(-\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right)\right) \qquad (*)$$

Taking the derivative of $(*)$ at $x=0$ yields

$$\begin{aligned}\Gamma''(1) &= \Gamma'(1)\,(-\gamma) + \Gamma(1)\zeta(2)\\ &= \gamma^2 + \zeta(2)\end{aligned}$$

Taking the second derivative of $(*)$ at $x=0$ yields

$$\begin{aligned}\Gamma'''(1) &= \Gamma''(1)(-\gamma) + 2\Gamma'(1)\zeta(2) + \Gamma(1)(-2\zeta(3))\\ &= -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3)\end{aligned}$$

We can use **Leibniz rule** and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(-\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right)\right) = (-1)^{n+1} n!\,\zeta(n+1) \qquad \text{for } n \geq 1$$

applied to $(*)$, to get the recursion

$$\Gamma^{(n)}(1) = -\gamma\,\Gamma^{(n-1)}(1) + (n-1)! \sum_{k=2}^{n} (-1)^k \zeta(k) \frac{\Gamma^{(n-k)}(1)}{(n-k)!}$$

□

Exercise 1.1.148: Prove:

$$\int_0^1 \ln\left(1 + \frac{\ln^2 x}{4\,\pi^2}\right) \frac{\ln(1-x)}{x} \mathrm{d}x = -\pi^2\left(4\,\zeta'(-1) + \frac{2}{3}\right)$$

Proof: Let I denote the integral. Then

$$\begin{aligned}
I &= -\int_0^1 \ln\left(1 + \frac{\ln^2 x}{4\pi^2}\right) \mathrm{Li}_2'(x)\,\mathrm{d}x\\
&= \left[-\ln\left(1 + \frac{\ln^2 x}{4\pi^2}\right)\mathrm{Li}_2(x)\right]_0^1 + 2\int_0^1 \frac{\ln x}{4\pi^2 + \ln^2 x}\frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x\\
&= -2\int_0^\infty \frac{t}{4\pi^2 + t^2}\mathrm{Li}_2(e^{-t})\,\mathrm{d}t \qquad (x = e^{-t})\\
&= -2\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty \frac{t}{4\pi^2+t^2}\,e^{-nt}\,\mathrm{d}t = -2\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty\left(\int_0^\infty \cos(2\pi u)e^{-tu}\,\mathrm{d}u\right)e^{-nt}\,\mathrm{d}t\\
&= -2\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty \frac{\cos(2\pi u)}{u+n}\,\mathrm{d}u\\
&= -2\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty \frac{\cos(2\pi n u)}{u+1}\,\mathrm{d}u \qquad (u \mapsto nu)\\
&= -2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2}\int_0^1 \frac{\cos(2\pi n u)}{u+k}\,\mathrm{d}u = -2\sum_{k=1}^{\infty}\int_0^1 \frac{1}{u+k}\left(\sum_{n=1}^{\infty}\frac{\cos(2\pi n u)}{n^2}\right)\mathrm{d}u.
\end{aligned}$$

Now we invoke the Fourier series of the Bernoulli polynomial $B_2(x)$:

http://en.wikipedia.org/wiki/Bernoulli_polynomials

$$\sum_{n=1}^{\infty}\frac{\cos 2\pi n x}{n^2} = \pi^2 B_2(x) = \pi^2\left(x^2 - x + \frac{1}{6}\right), \quad 0 \leq x \leq 1.$$

Then it follows that

$$I=-2\pi^2\sum_{k=1}^{\infty}\int_0^1\frac{u^2-u+\frac{1}{6}}{u+k}\mathrm{d}u=\pi^2\sum_{k=1}^{\infty}\left\{2k+1+2\left(k^2+k+\frac{1}{6}\right)\ln\left(\frac{k}{k+1}\right)\right\}$$

Now we consider the exponential of the partial sum:

$$\exp\left[\sum_{k=1}^{N}\left\{2k+1+2\left(k^2+k+\frac{1}{6}\right)\ln\left(\frac{k}{k+1}\right)\right\}\right]=e^{N^2+2N}\prod_{k=1}^{N}\left(\frac{k}{k+1}\right)^{2k^2+2k+\frac{1}{3}}$$
$$=\frac{e^{N^2+2N}}{(N+1)^{2N^2+2N+\frac{1}{3}}}\prod_{k=1}^{N}k^{4k}=\frac{e^{2N}}{\left(1+\frac{1}{N}\right)^{2N^2+2N+\frac{1}{3}}}\left\{\frac{e^{N^2/4}}{N^{N^2/2+N/2+1/12}}\prod_{k=1}^{N}k^k\right\}^4$$

In view of the definition of **Glaisher-Kinkelin constant A**, we have

$$\lim_{N\to\infty}\exp\left[\sum_{k=1}^{N}\left\{2k+1+2\left(k^2+k+\frac{1}{6}\right)\ln\left(\frac{k}{k+1}\right)\right\}\right]=\frac{A^4}{e}.$$

This, together with the identity $\ln\mathbf{A}=\frac{1}{12}-\zeta'(-1)$, yields

$$I=\pi^2(4\ln\mathbf{A}-1)=-\pi^2\left(4\zeta'(-1)+\frac{2}{3}\right)$$

□

Exercise 1.1.149: Evaluate:

$$\int_0^1\ln(1-x)\ln(1+x)\ln^2x\,\mathrm{d}x$$

Solution: This answer is split into 3 main steps.

Step 1: Expressing the integral as a sum

$$\int_0^1\ln(1+x)\ln(1-x)\ln^2x\,\mathrm{d}x$$
$$=\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1x^{j+k}\ln^2x\,\mathrm{d}x=2\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\sum_{k=1}^{\infty}\frac{1}{k(k+j+1)^3}$$
$$=2\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\sum_{k=1}^{\infty}\frac{1}{(j+1)^3k}-\frac{1}{(j+1)^3(k+j+1)}-\frac{1}{(j+1)^2(k+j+1)^2}-\frac{1}{(j+1)(k+j+1)^3}$$
$$=2\sum_{j=1}^{\infty}\frac{(-1)^jH_{j+1}}{j(j+1)^3}-2\sum_{j=1}^{\infty}\frac{(-1)^j\left[\zeta(2)-H_{j+1}^{(2)}\right]}{j(j+1)^2}-2\sum_{j=1}^{\infty}\frac{(-1)^j\left[\zeta(3)-H_{j+1}^{(3)}\right]}{j(j+1)}$$

Step 2a: Value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}=\frac{1}{2}\ln^22-\frac{\pi^2}{12}$$

See here for the details.

(http://math.stackexchange.com/questions/275643/)

Step 2b: Value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}=-\frac{5}{8}\zeta(3)$$

See here for the details.

(http://math.stackexchange.com/questions/275643/)

Step 2c: Value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}=\int_0^{-1}\frac{1}{y}\left[\int_0^{y}\frac{1}{x}\left[\int_0^{x}\frac{\ln(1-t)}{t(t-1)}\mathrm{d}t\right]\mathrm{d}x\right]\mathrm{d}y$$
$$=2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{11\pi^4}{360}+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{\pi^2}{12}\ln^2 2$$

Tunk-Fey did a calculation of this type here.

(http://math.stackexchange.com/questions/909228/)

Step 2d: Value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n}$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n}x^n=\int_0^{x}\frac{\mathrm{Li}_2(t)}{t(1-t)}\mathrm{d}t=\mathrm{Li}_3(x)+\int_0^{x}\frac{\mathrm{Li}_2(t)}{1-t}\mathrm{d}t$$
$$=\mathrm{Li}_3(x)-\mathrm{Li}_2(x)\ln(1-x)-\int_0^{x}\frac{\ln^2(1-t)}{t}\mathrm{d}t$$
$$=\mathrm{Li}_3(x)-\mathrm{Li}_2(x)\ln(1-x)+\int_1^{1-x}\frac{\ln^2 t}{1-t}\mathrm{d}t$$
$$=\mathrm{Li}_3(x)-\mathrm{Li}_2(x)\ln(1-x)-\ln^2(1-x)\ln x+\int_1^{1-x}\frac{2\ln(1-t)\ln t}{t}\mathrm{d}t$$
$$=\mathrm{Li}_3(x)-\mathrm{Li}_2(x)\ln(1-x)-\ln^2(1-x)\ln x-2\mathrm{Li}_2(1-x)\ln(1-x)+\int_1^{1-x}\frac{2\mathrm{Li}_2(t)}{t}\mathrm{d}t$$
$$=\mathrm{Li}_3(x)-\mathrm{Li}_2(x)\ln(1-x)-\ln^2(1-x)\ln x-2\mathrm{Li}_2(1-x)\ln(1-x)+2\mathrm{Li}_3(1-x)-2\zeta(3)$$

Therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n}=\mathrm{Li}_3(-1)-\mathrm{Li}_2(-1)\ln 2-\ln^2 2\ln(-1)-2\mathrm{Li}_2(2)\ln 2+2\mathrm{Li}_3(2)-2\zeta(3)$$
$$=-\zeta(3)+\frac{\pi^2}{12}\ln 2$$

You can use polylogarithm identities to simplify the last equation. I took the easy way out and used Wolfram Alpha. Note that contour integration is a slightly more efficient method to solve this sum, however this method is required if I want to solve $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2}$ as well.

Step 2e: Value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{n}$

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}x^n=\int_0^{x}\frac{\mathrm{Li}_3(t)}{t(1-t)}\mathrm{d}t=\mathrm{Li}_4(x)+\int_0^{x}\frac{\mathrm{Li}_3(t)}{1-t}\mathrm{d}t=\mathrm{Li}_4(x)-\mathrm{Li}_3(x)\ln(1-x)$$
$$-\int_0^{x}\frac{-\ln(1-t)\mathrm{Li}_2(t)}{t}\mathrm{d}t=\mathrm{Li}_4(x)-\mathrm{Li}_3(x)\ln(1-x)-\frac{1}{2}\mathrm{Li}_2^2(x)$$

Therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(3)}}{n}=\mathrm{Li}_4(-1)-\mathrm{Li}_3(-1)\ln 2-\frac{1}{2}\mathrm{Li}_2^2(-1)=-\frac{19\pi^4}{1440}+\frac{3}{4}\zeta(3)\ln 2$$

Step 2f: Value of $\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2}$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n=\mathrm{Li}_4(x)-2\zeta(3)\ln x+\frac{1}{2}\mathrm{Li}_2^2(x)$$
$$+\int\frac{-\ln^2(1-x)\ln x}{x}\mathrm{d}x+2\int\frac{\mathrm{Li}_3(1-x)-\mathrm{Li}_2(1-x)\ln(1-x)}{x}\mathrm{d}x$$

The green integral is

$$\int\frac{-\ln^2(1-x)\ln x}{x}\mathrm{d}x=-\frac{1}{2}\ln^2 x\ln^2(1-x)-\int\frac{\ln^2 x\ln(1-x)}{1-x}\mathrm{d}x$$
$$=-\frac{1}{2}\ln^2 x\ln^2(1-x)+\sum_{n=1}^{\infty}H_n\int x^n\ln^2 x\,\mathrm{d}x=-\frac{1}{2}\ln^2 x\ln^2(1-x)+\sum_{n=1}^{\infty}H_n\partial_n^2\frac{x^{n+1}}{n+1}$$
$$=-\frac{1}{2}\ln^2 x\ln^2(1-x)+\ln^2 x\sum_{n=1}^{\infty}\frac{H_n x^{n+1}}{n+1}-2\ln x\sum_{n=1}^{\infty}\frac{H_n x^{n+1}}{(n+1)^2}+2\sum_{n=1}^{\infty}\frac{H_n x^{n+1}}{(n+1)^3}$$
$$=2\ln x\mathrm{Li}_3(x)-2\mathrm{Li}_4(x)-2\ln x\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n+2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n$$

The orange integral is

$$2\int\frac{\mathrm{Li}_3(1-x)-\mathrm{Li}_2(1-x)\ln(1-x)}{x}\mathrm{d}x$$
$$=2\mathrm{Li}_3(1-x)\ln x-2\mathrm{Li}_2(1-x)\ln x\ln(1-x)+2\int\frac{\ln(1-x)\ln^2 x}{1-x}\mathrm{d}x$$
$$=2\mathrm{Li}_3(1-x)\ln x-2\mathrm{Li}_2(1-x)\ln x\ln(1-x)-\ln^2 x\ln^2(1-x)$$
$$-4\ln x\mathrm{Li}_3(x)+4\mathrm{Li}_4(x)+4\ln x\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n-4\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n$$

So

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}x^n$$
$$=3\mathrm{Li}_4(x)+2\mathrm{Li}_3(1-x)\ln x-2\mathrm{Li}_3(x)\ln x-2\zeta(3)\ln x+\frac{1}{2}\mathrm{Li}_2^2(x)$$
$$-2\mathrm{Li}_2(1-x)\ln x\ln(1-x)-\ln^2 x\ln^2(1-x)+2\ln x\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}x^n+C$$

Therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2}=3\mathrm{Li}_4(-1)+2\mathrm{Li}_3(2)\ln(-1)-2\mathrm{Li}_3(-1)\ln(-1)-2\zeta(3)\ln(-1)$$
$$+\frac{1}{2}\mathrm{Li}_2^2(-1)-2\mathrm{Li}_2(2)\ln(-1)\ln(2)-\ln^2(-1)\ln^2 2$$
$$+2\ln(-1)\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}$$
$$=\frac{17\pi^4}{480}-4\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{1}{6}\ln^4 2-\frac{7}{2}\zeta(3)\ln 2+\frac{\pi^2}{6}\ln^2 2$$

Step 3a: Evaluating $2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}}{j(j+1)^3}$

$$2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}}{j(j+1)^3} = 2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}}{j} - \frac{(-1)^j H_{j+1}}{(j+1)^3} - \frac{(-1)^j H_{j+1}}{(j+1)^2} - \frac{(-1)^j H_{j+1}}{j+1}$$
$$= 2\sum_{j=1}^{\infty}\frac{(-1)^j H_j}{j} + 2\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)} + 2\sum_{j=1}^{\infty}\frac{(-1)^j H_j}{j^3} + 2 + 2\sum_{j=1}^{\infty}\frac{(-1)^j H_j}{j^2} + 2 + 2\sum_{j=1}^{\infty}\frac{(-1)^j H_j}{j^3} + 2$$
$$= 4\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{11\pi^4}{180} + \frac{1}{6}\ln^4 2 + \frac{7}{2}\zeta(3)\ln 2 - \frac{5}{4}\zeta(3) - \frac{\pi^2}{6}\ln^2 2 - \frac{\pi^2}{3} + 2\ln^2 2 - 4\ln 2 + 8$$

Step 3b: Evaluating $-\frac{\pi^2}{3}\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)^2}$

$$-\frac{\pi^2}{3}\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)^2} = -\frac{\pi^2}{3}\sum_{j=1}^{\infty}\frac{(-1)^j}{j} - \frac{(-1)^j}{(j+1)^2} - \frac{(-1)^j}{j+1}$$
$$= \frac{\pi^2}{3}\ln 2 + \frac{\pi^4}{36} - \frac{\pi^2}{3} + \frac{\pi^2}{3}\ln 2 - \frac{\pi^2}{3} = \frac{\pi^4}{36} + \frac{2\pi^2}{3}\ln 2 - \frac{2\pi^2}{3}$$

Step 3c: Evaluating $2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(2)}}{j(j+1)^2}$

$$2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(2)}}{j(j+1)^2} = 2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(2)}}{j} - \frac{(-1)^j H_{j+1}^{(2)}}{(j+1)^2} - \frac{(-1)^j H_{j+1}^{(2)}}{j+1}$$
$$= 4\sum_{j=1}^{\infty}\frac{(-1)^j H_j^{(2)}}{j} + 2\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)^2} + 2\sum_{j=1}^{\infty}\frac{(-1)^j H_j^{(2)}}{j^2} + 2 + 2$$
$$= -8\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{17\pi^4}{240} - \frac{1}{3}\ln^4 2 - 7\zeta(3)\ln 2 - 4\zeta(3) + \frac{\pi^2}{3}\ln^2 2 + \frac{\pi^2}{3}\ln 2 - \frac{\pi^2}{6} - 4\ln 2 + 8$$

Step 3d: Evaluating $-2\zeta(3)\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)}$

$$-2\zeta(3)\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)} = -2\zeta(3)\sum_{j=1}^{\infty}\frac{(-1)^j}{j} + 2\zeta(3)\sum_{j=1}^{\infty}\frac{(-1)^j}{j+1} = 4\zeta(3)\ln 2 - 2\zeta(3)$$

Step 3e: Evaluating $2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(3)}}{j(j+1)}$

$$2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(3)}}{j(j+1)} = 2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(3)}}{j} - 2\sum_{j=1}^{\infty}\frac{(-1)^j H_{j+1}^{(3)}}{j+1}$$
$$= 4\sum_{j=1}^{\infty}\frac{(-1)^j H_j^{(3)}}{j} + 2\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+1)^3} + 2$$
$$= -\frac{19\pi^4}{360} + 3\zeta(3)\ln 2 - \frac{3}{2}\zeta(3) - \frac{\pi^2}{6} - 4\ln 2 + 8$$

Step 4: Obtaining the final result

Summing the results from steps 3a, 3b, 3c, 3d and 3e gives

$$\int_0^1 \ln(1+x)\ln(1-x)\ln^2 x\,\mathrm{d}x = 24 - \frac{4\pi^2}{3} - \frac{11\pi^4}{720} - 12\ln 2 + 2\ln^2 2 - \frac{1}{6}\ln^4 2 + \pi^2\ln 2 + \frac{\pi^2}{6}\ln^2 2 - 4\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{35}{4}\zeta(3) + \frac{7}{2}\zeta(3)\ln 2$$

□

Exercise 1.1.150: Compute the following integral:

$$\int_0^{\frac{\pi}{4}} \left(\left[\frac{(1-x^2)\ln(1+x^2) + (1+x^2) - (1-x^2)\ln(1-x^2)}{(1-x^4)(1+x^2)} \right] \cdot x \cdot \exp\left[\frac{x^2-1}{x^2+1} \right] \right) \mathrm{d}x$$

Solution: Rewrite

$$\begin{aligned}
&\int \left[\frac{(1-x^2)\ln(1+x^2) + (1+x^2) - (1-x^2)\ln(1-x^2)}{(1-x^4)(1+x^2)} \right] x\ \exp\left[\frac{x^2-1}{x^2+1} \right]\ \mathrm{d}x \\
&= -\int \left[\frac{(1-x^2)\ln\left(\frac{1-x^2}{1+x^2}\right) - (1+x^2)}{(1-x^2)(1+x^2)(1+x^2)} \right] x\ \exp\left[-\frac{1-x^2}{1+x^2} \right]\ \mathrm{d}x \\
&= -\frac{1}{4}\int \left[\frac{(1-x^2)\ln\left(\frac{1-x^2}{1+x^2}\right) - (1+x^2)}{(1-x^2)} \right] \frac{2x}{1+x^2}\ \exp\left[-\frac{1-x^2}{1+x^2} \right]\ \frac{2\,\mathrm{d}x}{1+x^2} \\
&= -\frac{1}{4}\int \left[\ln\left(\frac{1-x^2}{1+x^2}\right) - \frac{1+x^2}{1-x^2} \right] \frac{2x}{1+x^2}\ \exp\left[-\frac{1-x^2}{1+x^2} \right]\ \frac{2\,\mathrm{d}x}{1+x^2}. \qquad (1)
\end{aligned}$$

Now, consider **Weierstrass substitution**:

$$x = \tan\frac{t}{2}\ ,\ \sin t = \frac{2x}{1+x^2}\ ,\ \cos t = \frac{1-x^2}{1+x^2}\ ,\ \text{ and }\ \mathrm{d}t = \frac{2\,\mathrm{d}x}{1+x^2}.$$

The integral in (1) turns out to be

$$-\frac{1}{4}\int \left[\ln(\cos t) - \frac{1}{\cos t} \right] \sin t\ \exp[-\cos t]\ \mathrm{d}t. \qquad (2)$$

Let $y = \cos t \ \Rightarrow\ \mathrm{d}y = -\sin t\ \mathrm{d}t$, then (2) becomes

$$\frac{1}{4}\int \left[\ln y - \frac{1}{y} \right] e^{-y}\ \mathrm{d}y = \frac{1}{4}\left[\int e^{-y}\ln y\ \mathrm{d}y - \int \frac{e^{-y}}{y}\ \mathrm{d}y \right]. \qquad (3)$$

The second integral in the RHS (3) can be evaluated by using IBP. Taking $u = e^{-y} \ \Rightarrow\ \mathrm{d}u = -e^{-y}\ \mathrm{d}y$ and $\mathrm{d}v = \frac{1}{y}\ \mathrm{d}y \ \Rightarrow\ v = \ln y$, then

$$\int \frac{e^{-y}}{y}\ \mathrm{d}y = e^{-y}\ln y + \int e^{-y}\ln y\ \mathrm{d}y. \qquad (4)$$

Substituting (4) to (3), we obtain

$$\frac{1}{4}\left[\int e^{-y}\ln y\,\mathrm{d}y - e^{-y}\ln y - \int e^{-y}\ln y\,\mathrm{d}y\right] = -\frac{1}{4}e^{-y}\ln y + C.$$

Thus

$$\int\left[\frac{(1-x^2)\ln(1+x^2)+(1+x^2)-(1-x^2)\ln(1-x^2)}{(1-x^4)(1+x^2)}\right]x\ \exp\left[\frac{x^2-1}{x^2+1}\right]\ \mathrm{d}x$$
$$= -\frac{1}{4}\exp\left[-\frac{1-x^2}{1+x^2}\right]\ln\left|\frac{1-x^2}{1+x^2}\right| + C$$

and

$$\int_0^{\frac{\pi}{4}}\left[\frac{(1-x^2)\ln(1+x^2)+(1+x^2)-(1-x^2)\ln(1-x^2)}{(1-x^4)(1+x^2)}\right]x\ \exp\left[\frac{x^2-1}{x^2+1}\right]\ dx$$
$$= -\frac{1}{4}\exp\left[-\frac{16-\pi^2}{16+\pi^2}\right]\ln\left|\frac{16-\pi^2}{16+\pi^2}\right|$$

□

Exercise 1.1.151: Evaluate:

$$\int_0^\infty \frac{\cos\ x}{1+x^2}\mathrm{d}x$$

Solution: Method One:

Consider the function $f(t)=e^{-a|t|}$, then the Fourier transform of $f(t)$ is given by

$$F(\omega)=\mathcal{F}[f(t)]=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,\mathrm{d}t=\int_{-\infty}^{\infty}e^{-a|t|}e^{-i\omega t}\,\mathrm{d}t$$
$$=\int_{-\infty}^{0}e^{at}e^{-i\omega t}\,\mathrm{d}t+\int_0^{\infty}e^{-at}e^{-i\omega t}\,\mathrm{d}t=\lim_{u\to-\infty}\frac{e^{(a-i\omega)t}}{a-i\omega}\Big|_{t=u}^{0}-\lim_{v\to\infty}\frac{e^{-(a+i\omega)t}}{a+i\omega}\Big|_{0}^{t=v}$$
$$=\frac{1}{a-i\omega}+\frac{1}{a+i\omega}=\frac{2a}{\omega^2+a^2}$$

Next, the inverse Fourier transform of $F(\omega)$ is

$$f(t)=\mathcal{F}^{-1}[F(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}\,\mathrm{d}\omega$$
$$e^{-a|t|}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{\omega^2+a^2}e^{i\omega t}\,\mathrm{d}\omega$$
$$\frac{\pi e^{-a|t|}}{a}=\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega^2+a^2}\,\mathrm{d}\omega. \tag{1}$$

Now, rewrite

$$\int_0^\infty\frac{\cos 2x}{x^2+4}\,\mathrm{d}x=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\Re\left(e^{2ix}\right)}{x^2+2^2}\,\mathrm{d}x. \tag{2}$$

Comparing (2) to (1) yield $t=2$ and $a=2$. Thus,

$$\int_0^\infty\frac{\cos 2x}{x^2+4}\,\mathrm{d}x=\frac{1}{2}\frac{\pi e^{-2\cdot|2|}}{2}=\frac{\pi}{4e^4}$$

and

$$\int_0^\infty \frac{\cos x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2e}$$

Method Two:

Note that:

$$\int_{y=0}^\infty e^{-(x^2+4)y}\,\mathrm{d}y = \frac{1}{x^2+4},$$

therefore

$$\int_{x=0}^\infty \int_{y=0}^\infty e^{-(x^2+4)y}\cos 2x\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \frac{\cos 2x}{x^2+4}\,\mathrm{d}x$$

Rewrite $\cos 2x = \Re\left(e^{-2ix}\right)$, then

$$\begin{aligned}\int_0^\infty \frac{\cos 2x}{x^2+4}\,dx &= \int_{x=0}^\infty \int_{y=0}^\infty e^{-(x^2+4)y}\cos 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{y=0}^\infty \int_{x=0}^\infty e^{-(yx^2+2ix+4y)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{y=0}^\infty e^{-4y}\int_{x=0}^\infty e^{-(yx^2+2ix)}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

In general

$$\begin{aligned}\int_{x=0}^\infty e^{-(ax^2+bx)}\,\mathrm{d}x &= \int_{x=0}^\infty \exp\left(-a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)\right)\mathrm{d}x \\ &= \exp\left(\frac{b^2}{4a}\right)\int_{x=0}^\infty \exp\left(-a\left(x+\frac{b}{2a}\right)^2\right)\mathrm{d}x\end{aligned}$$

Let $u = x+\dfrac{b}{2a} \;\to\; \mathrm{d}u = \mathrm{d}x$, then

$$\int_{x=0}^\infty e^{-(ax^2+bx)}\,\mathrm{d}x = \exp\left(\frac{b^2}{4a}\right)\int_{x=0}^\infty \exp\left(-a\left(x+\frac{b}{2a}\right)^2\right)\mathrm{d}x = \exp\left(\frac{b^2}{4a}\right)\int_{u=0}^\infty e^{-au^2}\,\mathrm{d}u$$

The last form integral is **Gaussian integral** that equals to $\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}$. Hence

$$\int_{x=0}^\infty e^{-(ax^2+bx)}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right).$$

Thus

$$\int_{x=0}^\infty e^{-(yx^2+2ix)}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{y}}\exp\left(\frac{(2i)^2}{4y}\right) = \frac{1}{2}\sqrt{\frac{\pi}{y}}\exp\left(-\frac{1}{y}\right).$$

Next

$$\int_0^\infty \frac{\cos 2x}{x^2+4}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\int_{y=0}^\infty \frac{\exp\left(-4y-\dfrac{1}{y}\right)}{\sqrt{y}}\,\mathrm{d}y.$$

In general

$$\begin{aligned}\int_{y=0}^{\infty}\frac{\exp\left(-ay-\frac{b}{y}\right)}{\sqrt{y}}\,\mathrm{d}y &= 2\int_{v=0}^{\infty}\exp\left(-av^2-\frac{b}{v^2}\right)\,\mathrm{d}v\\ &= 2\int_{v=0}^{\infty}\exp\left(-a\left(v^2+\frac{b}{av^2}\right)\right)\,\mathrm{d}v\\ &= 2\int_{v=0}^{\infty}\exp\left(-a\left(v^2-2\sqrt{\frac{b}{a}}+\frac{b}{av^2}+2\sqrt{\frac{b}{a}}\right)\right)\,\mathrm{d}v\\ &= 2\int_{v=0}^{\infty}\exp\left(-a\left(v-\frac{1}{v}\sqrt{\frac{b}{a}}\right)^2-2\sqrt{ab}\right)\,\mathrm{d}v\\ &= 2\exp(-2\sqrt{ab})\int_{v=0}^{\infty}\exp\left(-a\left(v-\frac{1}{v}\sqrt{\frac{b}{a}}\right)^2\right)\,\mathrm{d}v\end{aligned}$$

The trick to solve the last integral is by setting

$$I=\int_{v=0}^{\infty}\exp\left(-a\left(v-\frac{1}{v}\sqrt{\frac{b}{a}}\right)^2\right)\,\mathrm{d}v.$$

Let $t=-\frac{1}{v}\sqrt{\frac{b}{a}}\ \rightarrow\ v=-\frac{1}{t}\sqrt{\frac{b}{a}}\ \rightarrow\ \mathrm{d}v=\frac{1}{t^2}\sqrt{\frac{b}{a}}\,\mathrm{d}t$, then

$$I_t=\sqrt{\frac{b}{a}}\int_{t=0}^{\infty}\frac{\exp\left(-a\left(-\frac{1}{t}\sqrt{\frac{b}{a}}+t\right)^2\right)}{t^2}\,\mathrm{d}t.$$

Let $t=v\ \rightarrow\ \mathrm{d}t=\mathrm{d}v$, then

$$I_t=\int_{t=0}^{\infty}\exp\left(-a\left(t-\frac{1}{t}\sqrt{\frac{b}{a}}\right)^2\right)\,\mathrm{d}t.$$

Adding the two I_ts yields

$$2I=I_t+I_t=\int_{t=0}^{\infty}\left(1+\frac{1}{t^2}\sqrt{\frac{b}{a}}\right)\exp\left(-a\left(t-\frac{1}{t}\sqrt{\frac{b}{a}}\right)^2\right)\,\mathrm{d}t.$$

Let $s=t-\frac{1}{t}\sqrt{\frac{b}{a}}\ \rightarrow\ \mathrm{d}s=\left(1+\frac{1}{t^2}\sqrt{\frac{b}{a}}\right)\mathrm{d}t$ and for $0<t<\infty$ is corresponding to $-\infty<s<\infty$, then

$$I=\frac{1}{2}\int_{s=-\infty}^{\infty}e^{-as^2}\,\mathrm{d}s=\frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Thus

$$\begin{aligned}\int_{y=0}^{\infty}\frac{\exp\left(-ay-\frac{b}{y}\right)}{\sqrt{y}}\,\mathrm{d}y &= 2\exp(-2\sqrt{ab})\int_{v=0}^{\infty}\exp\left(-a\left(v-\frac{1}{v}\sqrt{\frac{b}{a}}\right)^2\right)\,\mathrm{d}v\\ &= \sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}\end{aligned}$$

and

$$\int_0^\infty \frac{\cos 2x}{x^2+4}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\int_{y=0}^\infty \frac{\exp\left(-4y-\frac{1}{y}\right)}{\sqrt{y}}\,\mathrm{d}y = \frac{\sqrt{\pi}}{2}\cdot\sqrt{\frac{\pi}{4}}e^{-2\sqrt{4\cdot 1}} = \frac{\pi}{4e^4}$$

Hence

$$\int_0^\infty \frac{\cos x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2e}$$

□

Exercise 1.1.152: Evaluate:

$$\int_0^1 \ln^3\left(1+x+x^2\right)\mathrm{d}x$$

Solution: A Recurrence Relation

I will use the notation

$$\mathcal{A}_n = \int_0^1 \ln^n(1+x+x^2)\,\mathrm{d}x \quad , \quad \mathcal{B}_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln^n\left(\frac{3}{4\cos^2 x}\right)\,\mathrm{d}x$$

Integrating by parts and applying the substitution $x+\frac{1}{2}\mapsto\frac{\sqrt{3}}{2}\tan x$, it is evident that

$$\mathcal{A}_n = n\sqrt{3}\mathcal{B}_{n-1} - 2n\mathcal{A}_{n-1} + \frac{3}{2}(\ln 3)^n$$

We may use this recurrence to compute $\mathcal{A}_n$ for small positive integer values of n.

Evaluation of $\mathcal{A}_1$

We immediately have

$$\mathcal{A}_1 = 1\times\sqrt{3}\times\frac{\pi}{6} - 2\times 1\times 1 + \frac{3}{2}\ln 3 = \frac{\pi}{2\sqrt{3}} + \frac{3}{2}\ln 3 - 2$$

Evaluation of $\mathcal{A}_2$

We first compute $\mathcal{B}_1$ by exploiting a Fourier series.

$$\begin{aligned}
\mathcal{B}_1 &= \frac{\pi}{6}\ln 3 - 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(2\cos x)\,\mathrm{d}x = \frac{\pi}{6}\ln 3 + 2\sum_{n=1}^\infty \frac{(-1)^n}{n}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos(2nx)\,\mathrm{d}x \\
&= \frac{\pi}{6}\ln 3 + \sum_{n=1}^\infty \frac{(-1)^n}{n^2}\left(\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right) \\
&= \frac{\pi}{6}\ln 3 - \frac{1}{12\sqrt{3}}\sum_{n=0}^\infty\left[\frac{1}{\left(n+\frac{1}{3}\right)^2} - \frac{1}{\left(n+\frac{2}{3}\right)^2}\right] \\
&= -\frac{1}{6\sqrt{3}}\psi_1\left(\frac{1}{3}\right) + \frac{\pi^2}{9\sqrt{3}} + \frac{\pi}{6}\ln 3
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{A}_2 &= 2\sqrt{3}\left(-\frac{1}{6\sqrt{3}}\psi_1\left(\frac{1}{3}\right) + \frac{\pi^2}{9\sqrt{3}} + \frac{\pi}{6}\ln 3\right) - 4\left(\frac{\pi}{2\sqrt{3}} + \frac{3}{2}\ln 3 - 2\right) + \frac{3}{2}\ln^2 3 \\
&= -\frac{1}{3}\psi_1\left(\frac{1}{3}\right) + \frac{2\pi^2}{9} + \frac{\pi}{\sqrt{3}}\ln 3 + \frac{3}{2}\ln^2 3 - \frac{2\pi}{\sqrt{3}} - 6\ln 3 + 8
\end{aligned}$$

Simplification of Some Li_2, Li_3 Terms

I will simplify the terms

$$\mathrm{Li}_2(e^{-\pi i/3}),\ \mathrm{Li}_2(1-e^{2\pi i/3}),\ \Im\mathrm{Li}_3(e^{\pi i/3}),\ \Im\mathrm{Li}_3(e^{-\pi i/3}),\ \Im\mathrm{Li}_3(e^{2\pi i/3})$$

The identities (for $0<\theta<2\pi$),

$$\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}=\frac{\theta^2}{4}-\frac{\pi\theta}{2}+\frac{\pi^2}{6}\ ,\ \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^3}=\frac{\theta^3}{12}-\frac{\pi\theta^2}{4}+\frac{\pi^2\theta}{6}$$

(which can be derived by considering $\Im\ln(1-e^{i\theta})$ and integrating), give us

$$\begin{aligned}
\Im\mathrm{Li}_3(e^{\pm\pi i/3})&=\pm\frac{5\pi^3}{162}\ ,\ \Im\mathrm{Li}_3(e^{2\pi i/3})=\frac{2\pi^3}{81}\\
\mathrm{Li}_2(e^{-\pi i/3})&=\frac{\pi^2}{36}-i\sum_{n=1}^{\infty}\frac{\sin(n\pi/3)}{n^2}\\
&=\frac{\pi^2}{36}-\frac{i\sqrt{3}}{2}\sum_{n=0}^{\infty}\left[\frac{1}{(6n+1)^2}+\frac{1}{(6n+2)^2}-\frac{1}{(6n+4)^2}-\frac{1}{(6n+5)^2}\right]\\
&=\frac{\pi^2}{36}-\frac{i}{24\sqrt{3}}\left(\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{1}{3}\right)-\psi_1\left(\frac{2}{3}\right)-\psi_1\left(\frac{5}{6}\right)\right)\\
&=\frac{\pi^2}{36}-i\left(\frac{1}{2\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{\pi^2}{3\sqrt{3}}\right)
\end{aligned}$$

Furthermore, the dilogarithm reflection formula states

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)=\frac{\pi^2}{6}-\ln z\ln(1-z)$$

Hence

$$\begin{aligned}
\mathrm{Li}_2(1-e^{2\pi i/3})&=\frac{\pi^2}{6}-\frac{2\pi i}{3}\left(\frac{\ln 3}{2}-\frac{\pi i}{6}\right)-\left(-\frac{\pi^2}{18}+i\sum_{n=1}^{\infty}\frac{\sin(2n\pi/3)}{n^2}\right)\\
&=\frac{\pi^2}{6}-\frac{2\pi i}{3}\left(\frac{\ln 3}{2}-\frac{\pi i}{6}\right)-\left(-\frac{\pi^2}{18}+\frac{i\sqrt{3}}{2}\sum_{n=0}^{\infty}\left[\frac{1}{(3n+1)^2}-\frac{1}{(3n+2)^2}\right]\right)\\
&=\frac{\pi^2}{9}-i\left(\frac{1}{3\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{2\pi^2}{9\sqrt{3}}+\frac{\pi}{3}\ln 3\right)
\end{aligned}$$

Evaluation of $\mathcal{A}_3$

Similarly, we start with the evaluation of $\mathcal{B}_2$.

$$\begin{aligned}
\mathcal{B}_2&=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\ln^2 3-4\ln 3\ln(2\cos x)+4x^2+4\Re\ln^2(1+e^{2ix})\ \mathrm{d}x\\
&=-\frac{\ln 3}{3\sqrt{3}}\psi_1\left(\frac{1}{3}\right)+\frac{7\pi^3}{162}+\frac{2\pi^2}{9\sqrt{3}}\ln 3+\frac{\pi}{6}\ln^2 3+8\Re\sum_{n=1}^{\infty}\frac{(-1)^n H_{n-1}}{n}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}e^{2inx}\ \mathrm{d}x\\
&=-\frac{\ln 3}{3\sqrt{3}}\psi_1\left(\frac{1}{3}\right)+\frac{7\pi^3}{162}+\frac{2\pi^2}{9\sqrt{3}}\ln 3+\frac{\pi}{6}\ln^2 3-4\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sin\left(\frac{2\pi n}{3}\right)-\sin\left(\frac{\pi n}{3}\right)\right)\\
&+4\Im\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(e^{2\pi in/3}-e^{\pi in/3}\right)\\
&=-\frac{\ln 3}{3\sqrt{3}}\psi_1\left(\frac{1}{3}\right)+\frac{11\pi^3}{162}+\frac{2\pi^2}{9\sqrt{3}}\ln 3+\frac{\pi}{6}\ln^2 3\\
&+4\Im\left[\mathrm{Li}_3(z)-\mathrm{Li}_3(1-z)+\mathrm{Li}_2(1-z)\ln(1-z)+\frac{1}{2}\ln z\ln^2(1-z)+\zeta(3)\right]_{e^{\pi i/3}}^{e^{2\pi i/3}}
\end{aligned}$$

where I used the generating function of $\dfrac{H_n}{n^2}$. Using results derived in the previous section,

$$\begin{aligned}
&\Im\left[\mathrm{Li}_3(z)-\mathrm{Li}_3(1-z)+\mathrm{Li}_2(1-z)\ln(1-z)+\frac{1}{2}\ln z\ln^2(1-z)+\zeta(3)\right]_{e^{\pi i/3}}^{e^{2\pi i/3}}\\
&=\frac{2\pi^3}{81}-\frac{5\pi^3}{162}+\left(-\frac{5\pi^3}{162}\right)-\Im\mathrm{Li}_3(1-e^{2\pi i/3})\\
&+\Im\left(\frac{\pi^2}{9}-i\left(\frac{1}{3\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{2\pi^2}{9\sqrt{3}}+\frac{\pi}{3}\ln 3\right)\right)\left(\frac{\ln 3}{2}-\frac{\pi i}{6}\right)\\
&-\Im\left(\frac{\pi^2}{36}-i\left(\frac{1}{2\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{\pi^2}{3\sqrt{3}}\right)\right)\left(-\frac{\pi i}{3}\right)+\frac{\pi^3}{108}+\frac{\pi}{12}\ln^2 3\\
&=-\Im\mathrm{Li}_3(1-e^{2\pi i/3})-\frac{\ln 3}{6\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{\pi^3}{27}+\frac{\pi^2}{9\sqrt{3}}\ln 3-\frac{\pi}{12}\ln^2 3
\end{aligned}$$

Therefore

$$\mathcal{B}_2=-4\Im\mathrm{Li}_3(1-e^{2\pi i/3})-\frac{\ln 3}{\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{13\pi^3}{162}+\frac{2\pi^2}{3\sqrt{3}}\ln 3-\frac{\pi}{6}\ln^2 3$$

and finally,

$$\begin{aligned}
\mathcal{A}_3=&3\sqrt{3}\left(-4\Im\mathrm{Li}_3(1-e^{2\pi i/3})-\frac{\ln 3}{\sqrt{3}}\psi_1\left(\frac{1}{3}\right)-\frac{13\pi^3}{162}+\frac{2\pi^2}{3\sqrt{3}}\ln 3-\frac{\pi}{6}\ln^2 3\right)\\
&-6\left(-\frac{1}{3}\psi_1\left(\frac{1}{3}\right)+\frac{2\pi^2}{9}+\frac{\pi}{\sqrt{3}}\ln 3+\frac{3}{2}\ln^2 3-\frac{2\pi}{\sqrt{3}}-6\ln 3+8\right)+\frac{3}{2}\ln^3 3\\
=&-12\sqrt{3}\Im\mathrm{Li}_3(1-e^{2\pi i/3})+(2-3\ln 3)\psi_1\left(\frac{1}{3}\right)+\frac{3}{2}\ln^3 3-\left(\frac{\sqrt{3}\pi}{2}+9\right)\ln^2 3\\
&+(2\pi^2-2\sqrt{3}\pi)\ln 3-\left(\frac{13\sqrt{3}\pi^3}{54}+\frac{4\pi^2}{3}-4\sqrt{3}\pi-36\ln 3+48\right)
\end{aligned}$$

□

Exercise 1.1.153: Evaluate:

$$\int_0^{\pi/6}\ln^2(2\sin x)\,\mathrm{d}x$$

Solution: Using the principal branch of the logarithm and assuming that $0<x<\pi$, we have

$$\ln(1-e^{2ix})=\ln(e^{-ix}-e^{ix})+\ln(e^{ix})=\ln(-2i\sin x)+ix=\ln(2\sin x)-\frac{i\pi}{2}+ix$$

Squaring both sides and integrating,

$$\int_0^{\pi/6}\left(\ln(2\sin x)-\frac{i\pi}{2}+ix\right)^2\,\mathrm{d}x=\int_0^{\pi/6}\ln^2(1-e^{2ix})\,\mathrm{d}x.$$

Then equating the real parts on both sides of the equation, we get

$$\begin{aligned}
\int_0^{\pi/6}\ln^2(2\sin x)\,\mathrm{d}x&=\int_0^{\pi/6}\left(x-\frac{\pi}{2}\right)^2\,dx+\Re\int_0^{\pi/6}\ln^2(1-e^{2ix})\,\mathrm{d}x\\
&=\frac{19\pi^3}{648}+\Re\int_C\ln^2(1-z)\frac{\mathrm{d}z}{2iz}\\
&=\frac{19\pi^3}{648}+\frac{1}{2}\,\Im\int_C\frac{\ln^2(1-z)}{z}\,\mathrm{d}z
\end{aligned}$$

where C is the portion of the unit circle from $z = 1$ to $z = e^{\pi i/3}$.
But since $\dfrac{\ln^2(1-z)}{z}$ is analytic for $\Re(z) < 1$,

$$\int_C \frac{\ln^2(1-z)}{z}\,\mathrm{d}z = \int_1^{e^{\pi i/3}} \frac{\ln^2(1-z)}{z}\,\mathrm{d}z.$$

Then integrating by parts twice, we get

$$\Im \int_1^{e^{\pi i/3}} \frac{\ln^2(1-z)}{z}\,\mathrm{d}z = \Im\ \ln^2(1-z)\ln(z)\Big|_1^{e^{\pi i/3}} + 2\,\Im \int_1^{e^{\pi i/3}} \frac{\ln(1-z)\ln(z)}{1-z}\,\mathrm{d}z$$

$$= \Im\ \ln^2(e^{-\pi i/3})\ln(e^{\pi i/3}) + 2\,\Im\ \ln(1-z)\mathrm{Li}_2(1-z)\Big|_1^{e^{\pi i/3}} + 2\,\Im \int_1^{e^{\pi i/3}} \frac{\mathrm{Li}_2(1-z)}{1-z}\,\mathrm{d}z$$

$$= -\frac{\pi^3}{27} - \frac{2\pi}{3}\Im\ i\ \mathrm{Li}_2(e^{-\pi i/3}) - 2\,\Im\ \mathrm{Li}_3(1-z)\Big|_1^{e^{\pi i/3}} = -\frac{\pi^3}{27} - \frac{2\pi}{3}\Im\ i\ \mathrm{Li}_2(e^{-\pi i/3})$$

$$- 2\,\Im\ \mathrm{Li}_3(e^{-\pi i/3}) = -\frac{\pi^3}{27} - \frac{2\pi}{3}\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n^2} + 2\sum_{n=1}^{\infty}\frac{\sin(n\pi/3)}{n^3}$$

Integrating both sides of the Fourier series

$$\sum_{n=1}^{\infty}\frac{\sin(k\theta)}{k} = \frac{\pi-\theta}{2}\ ,\ 0 < \theta < 2\pi$$

we get

$$\sum_{n=1}^{\infty}\frac{\cos(k\theta)}{k^2} = \frac{\theta^2}{4} - \frac{\pi\theta}{2} + \frac{\pi^2}{6}$$

And integrating a second time,

$$\sum_{n=1}^{\infty}\frac{\sin(k\theta)}{k^3} = \frac{\theta^3}{12} - \frac{\pi\theta^2}{4} + \frac{\pi^2\theta}{6}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n^2} = \frac{\pi^2}{36}\ ,\ \sum_{n=1}^{\infty}\frac{\sin(n\pi/3)}{n^3} = \frac{5\pi^3}{162}$$

So finally we have

$$\int_0^{\pi/6} \ln^2(2\sin x)\,\mathrm{d}x = \frac{19\pi^3}{648} + \frac{1}{2}\left[-\frac{\pi^3}{27} - \frac{2\pi}{3}\left(\frac{\pi^2}{36}\right) + 2\left(\frac{5\pi^3}{162}\right)\right] = \frac{7\pi^3}{216}$$

□

Exercise 1.1.154: Evaluate:

$$\int_0^{\frac{\pi}{2}} \ln^2\left(\cos^2 x\right)\mathrm{d}x$$

Solution: Start with

$$\int_0^{\pi/2}\ln(\sin(x))\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi}\ln(\sin(x))\,\mathrm{d}x = \int_0^{\pi/2}\ln(\sin(2x))\,\mathrm{d}x$$

$$= \int_0^{\pi/2}\Big(\ln(2) + \ln(\sin(x)) + \ln(\cos(x))\Big)\,\mathrm{d}x = \frac{\pi}{2}\ln(2) + 2\int_0^{\pi/2}\ln(\sin(x))\,\mathrm{d}x \qquad (1)$$

Therefore,

$$\int_0^{\pi/2} \ln(\sin(x))\,\mathrm{d}x = -\frac{\pi}{2}\ln(2) \tag{2}$$

Next

$$\begin{aligned}\int_0^{\pi/2} \ln^2(\sin(x))\,\mathrm{d}x &= \int_0^{\pi/2} \Big(\ln(2)+\ln(\sin(x))+\ln(\cos(x))\Big)^2\,\mathrm{d}x\\ &= \frac{\pi}{2}\ln^2(2) + 4\ln(2)\int_0^{\pi/2}\ln(\sin(x))\,\mathrm{d}x + 2\int_0^{\pi/2}\ln(\sin(x))\ln(\cos(x))\,\mathrm{d}x\\ &\quad + 2\int_0^{\pi/2}\ln^2(\sin(x))\,\mathrm{d}x\end{aligned} \tag{3}$$

Using (2) in (3) yields

$$\int_0^{\pi/2} \ln^2(\sin(x))\,\mathrm{d}x = \frac{3}{2}\pi\ln^2(2) - 2\int_0^{\pi/2}\ln(\sin(x))\ln(\cos(x))\,\mathrm{d}x \tag{4}$$

As in this answer,

http://math.stackexchange.com/a/294364

we can use contour integration to get that

$$\begin{aligned}\int_0^\infty \frac{\ln^2(x)}{1-x^2}\mathrm{d}x = \int_0^\infty \frac{\ln^2(ix)}{1+x^2}\mathrm{d}ix &= i\int_0^\infty \frac{\left(\frac{\pi}{2}i+\ln(x)\right)^2}{1+x^2}\mathrm{d}x\\ &= i\int_0^\infty \frac{\ln^2(x)-\frac{\pi^2}{4}}{1+x^2}\mathrm{d}x - \pi\int_0^\infty\frac{\ln(x)}{1+x^2}\mathrm{d}x\end{aligned} \tag{5}$$

Looking at the imaginary part of (5), we see that

$$\int_0^\infty \frac{\ln^2(x)}{1+x^2}\mathrm{d}x = \frac{\pi^3}{8} \tag{6}$$

With a change of variables, (6) becomes

$$\begin{aligned}\frac{\pi^3}{8} &= \int_0^{\pi/2}\ln^2(\tan(x))\,\mathrm{d}x\\ &= \int_0^{\pi/2}\Big(\ln^2(\sin(x))+\ln^2(\cos(x))-2\ln(\sin(x))\ln(\cos(x))\Big)\,\mathrm{d}x\end{aligned} \tag{7}$$

which yields

$$\int_0^{\pi/2}\ln^2(\sin(x))\,\mathrm{d}x = \frac{\pi^3}{16} + \int_0^{\pi/2}\ln(\sin(x))\ln(\cos(x))\,\mathrm{d}x \tag{8}$$

Adding twice (8) to (4) and dividing by 3 gives

$$\int_0^{\pi/2}\ln^2(\sin(x))\,\mathrm{d}x = \frac{\pi^3}{24} + \frac{1}{2}\pi\ln^2(2)$$

Therefore,

$$\int_0^{\pi/2}\ln^2(\cos^2(x))\,\mathrm{d}x = \frac{\pi^3}{6} + 2\pi\ln^2(2)$$

□

Exercise 1.1.155: Evaluate:

$$\int_0^1 \mathrm{Li}_2(x)\,\mathrm{d}x$$

Solution:

$$\int_0^1 \mathrm{Li}_2(x)\,\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty} \frac{x^k}{k^2}\,\mathrm{d}x = \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2}$$
$$= \sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{k} + \frac{1}{k+1}\right) = \zeta(2) - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{\pi^2}{6} - 1$$

where

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=2}^{n+1}\frac{1}{k}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

□

Exercise 1.1.156: Evaluate:

$$I = \int_0^{\frac{\pi}{4}} \ln(\sin x)\ \mathrm{d}x \quad \text{and} \quad J = \int_0^{\frac{\pi}{4}} \ln(\cos x)\ \mathrm{d}x$$

Solution: Method One:

$$I + J = \int_0^{\frac{\pi}{4}} \ln(\sin x \cos x)\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{2}\sin 2x\right)\ \mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\ \mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln 2\ \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin y)\ \mathrm{d}y - \frac{\pi}{4}\ln 2 \qquad \Rightarrow \qquad \text{set } y = 2x$$
$$= -\frac{\pi}{2}\ln 2$$

and

$$I - J = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin x}{\cos x}\right)\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(\tan x)\ \mathrm{d}x = \int_0^1 \frac{\ln t}{1+t^2}\ \mathrm{d}t \qquad \Rightarrow \qquad \text{set } t = \tan x$$
$$= \int_0^1 \sum_{n=1}^{\infty} (-1)^n t^{2n} \ln t\ \mathrm{d}t = \sum_{n=1}^{\infty}(-1)^n \int_0^1 t^{2n}\ln t\ \mathrm{d}t = -\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2} = -\mathbf{G}$$

where $\mathbf{G}$ is Catalan's constant. Therefore

$$I = \int_0^{\frac{\pi}{4}} \ln(\sin x)\ \mathrm{d}x = -\frac{1}{2}\left(\mathbf{G} + \frac{\pi}{2}\ln 2\right)$$

and

$$J = \int_0^{\frac{\pi}{4}} \ln(\cos x)\ \mathrm{d}x = \frac{1}{2}\left(\mathbf{G} - \frac{\pi}{2}\ln 2\right)$$

Method Two:

$$\int_0^{\frac{\pi}{4}} \ln(\sin x)\,\mathrm{d}x = -\int_0^{\frac{\pi}{4}}\left(\ln 2 + \sum_{k=1}^{\infty}\frac{\cos 2kx}{k}\right)\mathrm{d}x = -\frac{\pi}{4}\ln 2 - \sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{4}}\cos 2kx\ \mathrm{d}x$$
$$= -\frac{\pi}{4}\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{2}\right)}{k^2}\ ,\ k \to 2n+1$$
$$= -\frac{\pi}{4}\ln 2 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2} = -\frac{\pi}{4}\ln 2 - \frac{\mathbf{G}}{2}$$

□

Exercise 1.1.157: Evaluate:

$$\int_0^{\pi/4} \ln^2(\sin x)\ \mathrm{d}x$$

Solution: Following the same approach as in **Exercise 1.1.153**

$$\int_0^{\pi/4} \ln^2(2\sin x)\ \mathrm{d}x = \int_0^{\pi/4} \ln^2(2)\ \mathrm{d}x + 2\ln 2 \int_0^{\pi/4} \ln(\sin x)\ \mathrm{d}x + \int_0^{\pi/4} \ln^2(\sin x)\ \mathrm{d}x$$
$$= \frac{\pi}{4}\ln^2(2) - \ln(2)\left(\mathbf{G} + \frac{\pi}{2}\ln(2)\right) + \int_0^{\pi/4} \ln^2(\sin x)\ \mathrm{d}x$$
$$= \int_0^{\pi/4} \left(x - \frac{\pi}{2}\right)^2 \mathrm{d}x + \Re \int_0^{\pi/4} \ln^2(1-e^{2ix})\ \mathrm{d}x = \frac{7\pi^3}{192} + \frac{1}{2}\Im \int_1^i \frac{\ln^2(1-z)}{z}\ \mathrm{d}z$$
$$= \frac{7\pi^3}{192} + \frac{1}{2}\Im\left(\ln^2(1-i)\ln(i) + 2\ln(1-i)\mathrm{Li}_2(1-i) - 2\mathrm{Li}_3(1-i)\right)$$
$$= \frac{7\pi^3}{192} + \frac{1}{2}\left(\frac{\pi}{8}\ln^2(2) - \frac{\pi^3}{32} + \ln(2)\ \Im\ \mathrm{Li}_2(1-i) - \frac{\pi}{2}\Re\ \mathrm{Li}_2(1-i) - 2\ \Im\ \mathrm{Li}_3(1-i)\right)$$

Therefore,

$$\int_0^{\pi/4} \ln^2(\sin x)\ \mathrm{d}x = \frac{\pi^3}{48} + \mathbf{G}\ln(2) + \frac{5\pi}{16}\ln^2(2) + \frac{\ln(2)}{2}\Im\ \mathrm{Li}_2(1-i) - \frac{\pi}{4}\Re\ \mathrm{Li}_2(1-i) - \Im\ \mathrm{Li}_3(1-i)$$

The answer could be further simplified using the **dilogarithm reflection formula**

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln(x)\ln(1-x)$$

and the fact that

$$\mathrm{Li}_2(i) = -\frac{\pi^2}{48} + i\mathbf{G}\ ,\ \mathrm{Li}_2(1-i) = \frac{\pi^2}{16} - i\mathbf{G} - \frac{i\pi}{4}\ln(2)$$

So

$$\int_0^{\pi/4} \ln^2(\sin x)\ \mathrm{d}x = \frac{\pi^3}{192} + \mathbf{G}\frac{\ln(2)}{2} + \frac{5\pi}{16}\ln^2(2) - \Im\ \mathrm{Li}_3(1-i)$$

□

Exercise 1.1.158: Evaluate:

$$\int_0^{\pi/3} x\ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x$$

Solution: Use the fourier expansion of $\ln\sin x$ which is

$$\ln\sin x = -\ln 2 - \sum_{k=1}^{\infty}\frac{\cos 2kx}{k}$$

Then the rest is pretty straightforward.

$$\int_0^{\pi/3} x\ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x = -\int_0^{\pi/3} x\sum_{k=1}^{\infty}\frac{\cos kx}{k}\,\mathrm{d}x = -\sum_{k=1}^{\infty}\int_0^{\pi/3}\frac{x}{k}\cos kx\,\mathrm{d}x$$
$$= -\sum_{k=1}^{\infty}\left(\frac{\pi}{3k^2}\sin\frac{k\pi}{3} + \frac{1}{k^3}\cos\frac{k\pi}{3} - \frac{1}{k^3}\right) = -\frac{\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \mathrm{Cl}_3\left(\frac{\pi}{3}\right) + \zeta(3)$$

where $\mathrm{Cl}_n(x)$ are the **Clausen function**. Let a_n be $(-1)^{n/3}$ if n is a multiple of 3 and otherwise zero. Then we have

$$\begin{aligned}\mathrm{Cl}_3\left(\frac{\pi}{3}\right) &= \sum_{n=1}^{\infty}\frac{1}{n^3}\cos\frac{n\pi}{3} = \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{(-1)^{n-1}}{2}+\frac{3}{2}a_n\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(3n)^3} = \frac{\zeta(3)}{3}\end{aligned}$$

and from the relation of the **clausen function** and the **Barnes G function**, we have

$$\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = 2\pi\ln\left(\frac{G(5/6)}{G(7/6)}\right)+\frac{\pi}{3}\ln(2\pi)$$

Plugging altogether gives

$$\int_0^{\pi/3} x\ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x = \frac{2\zeta(3)}{3}-\frac{\pi^2}{9}\ln(2\pi)-\frac{2\pi^2}{3}\ln\left(\frac{G(5/6)}{G(7/6)}\right)$$

□

Exercise 1.1.159: Evaluate:

$$\int_{\frac{1}{2}}^{1}\frac{\ln(2x-1)}{\sqrt[6]{x(1-x)(1-2x)^4}}\,\mathrm{d}x$$

Solution: Setting $u = 2x-1$ yields

$$\frac{1}{\sqrt[3]{4}}\int_0^1\frac{\ln u}{\sqrt[6]{(1+u)(1-u)u^4}}\,\mathrm{d}u = \frac{1}{\sqrt[3]{4}}\int_0^1\frac{\ln u}{\sqrt[6]{(1-u^2)u^4}}\,\mathrm{d}u. \tag{1}$$

Setting $t = u^2$ yields

$$\frac{1}{\sqrt[3]{128}}\int_0^1\frac{\ln t}{(1-t)^{\frac{1}{6}}t^{\frac{5}{6}}}\,\mathrm{d}t = \frac{1}{\sqrt[3]{128}}\int_0^1(1-t)^{-\frac{1}{6}}t^{-\frac{5}{6}}\ln t\,\mathrm{d}t. \tag{2}$$

Now, consider **Beta function**

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t. \tag{3}$$

Differentiating (4) with respect to x yields

$$\begin{aligned}\frac{\partial}{\partial x}\mathrm{B}(x,y) &= \int_0^1\frac{\partial}{\partial x}\left(t^{x-1}(1-t)^{y-1}\right)\mathrm{d}t\\ (\psi(x)-\psi(x+y))\mathrm{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\ln t\,\mathrm{d}t,\end{aligned} \tag{4}$$

where $\psi(\cdot)$ is the **digamma function**.

Using (4) then (2) turns out to be

$$\frac{1}{\sqrt[3]{128}}\left(\psi\left(\frac{1}{6}\right)-\psi(1)\right)\mathrm{B}\left(\frac{1}{6},\frac{5}{6}\right) = -\frac{\pi}{\sqrt[3]{256}}\left(\pi\sqrt{3}+4\ln 2+3\ln 3\right)$$

□

Exercise 1.1.160: Evaluate:

$$\int_0^1 \frac{\ln\ln\left(\frac{1}{x}\right)}{x^2-x+1}\mathrm{d}x$$

Solution: Let $x=e^{-u}$. Then:

$$\int_0^1 \frac{\ln\ln\left(\frac{1}{x}\right)}{x^2-x+1}\mathrm{d}x=\int_0^\infty \frac{e^{-u}\ln u}{e^{-2u}-e^{-u}+1}\mathrm{d}u$$

Let:

$$\begin{aligned}
I(a)&=\int_0^\infty \frac{e^{-u}u^a}{e^{-2u}-e^{-u}+1}\mathrm{d}u=\int_0^\infty \frac{e^{-u}(1+e^{-u})u^a}{1+e^{-3u}}\mathrm{d}u\\
&=\int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-(3n+1)u}(1+e^{-u})u^a\ \mathrm{d}u=\int_0^\infty \sum_{n=0}^\infty (-1)^n (e^{-(3n+1)u}+e^{-(3n+2)u})u^a\ \mathrm{d}u\\
&=\Gamma(a+1)\sum_{n=0}^\infty (-1)^n\left(\frac{1}{(3n+1)^{a+1}}+\frac{1}{(3n+2)^{a+1}}\right)\\
&=6^{-a-1}\Gamma(a+1)\left[\zeta\left(a+1,\frac{1}{6}\right)+\zeta\left(a+1,\frac{1}{3}\right)-\zeta\left(a+1,\frac{2}{3}\right)-\zeta\left(a+1,\frac{5}{6}\right)\right]
\end{aligned}$$

Hence:

$$\begin{aligned}
I'(0)&=\frac{1}{6}\left[(\gamma-\ln 6)\left(\psi_0\left(\frac{1}{6}\right)-\psi_0\left(\frac{5}{6}\right)+\psi_0\left(\frac{1}{3}\right)-\psi_0\left(\frac{2}{3}\right)\right)\right.\\
&\quad\left.-\gamma_1\left(\frac{1}{6}\right)-\gamma_1\left(\frac{1}{3}\right)+\gamma_1\left(\frac{2}{3}\right)+\gamma_1\left(\frac{5}{6}\right)\right]\\
&=\frac{1}{6}\left[-\frac{4\pi(\gamma-\ln 6)}{\sqrt{3}}-\gamma_1\left(\frac{1}{6}\right)-\gamma_1\left(\frac{1}{3}\right)+\gamma_1\left(\frac{2}{3}\right)+\gamma_1\left(\frac{5}{6}\right)\right]
\end{aligned}$$

where $\gamma_k(x)$ is the k-th **Stieltjes constant**. Using:

$$\psi_0(1-z)-\psi_0(z)=\pi\cot(\pi z)$$

it is easy to get:

$$\psi_0\left(\frac{1}{6}\right)-\psi_0\left(\frac{5}{6}\right)+\psi_0\left(\frac{1}{3}\right)-\psi_0\left(\frac{2}{3}\right)=-\frac{4\pi}{\sqrt{3}}$$

For $\gamma_1\left(\frac{p}{q}\right)$, we have to use the following formula:

$$\gamma_1\left(\frac{p}{q}\right)-\gamma_1\left(1-\frac{p}{q}\right)=-\pi(\ln(2\pi q)+\gamma)\cot\frac{\pi p}{q}-2\pi\sum_{j=1}^{q-1}\ln\Gamma\left(\frac{j}{q}\right)\sin\left(\frac{2\pi jp}{q}\right)$$

from this.Hence we have:

$$\gamma_1\left(\frac{1}{3}\right)-\gamma_1\left(\frac{2}{3}\right)=-\frac{\pi}{2\sqrt{3}}\left[2\gamma-\ln 3+8\ln(2\pi)-12\ln\Gamma\left(\frac{1}{3}\right)\right]$$

$$\gamma_1\left(\frac{1}{6}\right)-\gamma_1\left(\frac{5}{6}\right)=-\pi\sqrt{3}\left[\gamma+\ln\left(\frac{12\pi\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}\right)\right]=-\pi\sqrt{3}\left[\gamma+\ln\left(\frac{2^{14/3}\pi^4}{\sqrt{3}\Gamma^6\left(\frac{1}{3}\right)}\right)\right]$$

Putting all the results together, finally we have:

$$I'(0) = \int_0^1 \frac{\ln\ln\left(\frac{1}{x}\right)}{x^2-x+1}\mathrm{d}x$$
$$= \frac{\pi}{12\sqrt{3}}\left[\ln\frac{268435456}{531441} + 32\ln\pi - 48\ln\Gamma\left(\frac{1}{3}\right)\right]$$

□

Exercise 1.1.161: Prove:

$$\int_0^\infty \ln^2 x \frac{1+x^2}{1+x^4}\mathrm{d}x = \frac{3\pi^3}{16\sqrt{2}}$$

Proof: Let

$$I(\mu) = \int_0^\infty x^\mu \frac{1+x^2}{1+x^4}\,\mathrm{d}x$$

and then take $I''(0)$. By the ubiquitous formula

$$\int_0^\infty \frac{x^a}{1+x^b}\,\mathrm{d}x = \frac{\pi}{b\sin\left(\frac{\pi(a+1)}{b}\right)}$$

we obtain

$$I(\mu) = \frac{\pi}{4}\left[\frac{1}{\sin\left(\frac{\pi(\mu+1)}{4}\right)} + \frac{1}{\left(\frac{\pi(\mu+3)}{4}\right)}\right]$$

This gives (after some simple algebra)

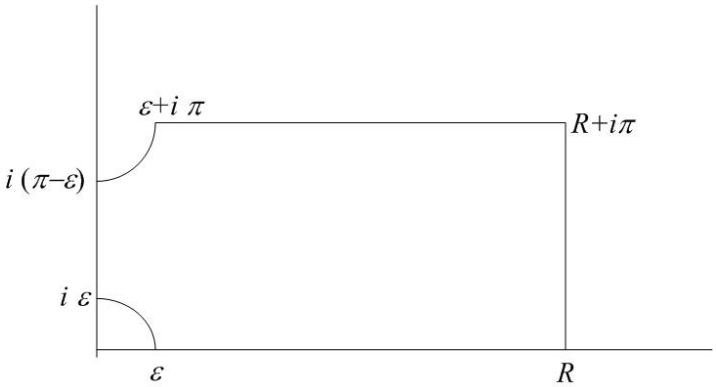
$$I''(0) = \int_0^\infty \frac{1+x^2}{1+x^4}\ln^2 x\,\mathrm{d}x = \frac{\pi}{4}\left[\frac{3\pi^2}{8\sqrt{2}} + \frac{3\pi^2}{8\sqrt{2}}\right] = \frac{3\pi^3}{16\sqrt{2}}$$

□

Exercise 1.1.162: Evaluate:

$$\int_0^\infty e^{-ax}\sin(ax)\left(\cot(x)+\coth(x)\right)\mathrm{d}x$$

Solution: We use a rectangular contour C with notches cut out as follows:



Let us first consider the following integral:

$$\oint_C e^{-bz}\coth z\mathrm{d}x$$

Note that we will be taking $b=(1i)a$. By Cauchy's Integral Theorem, this integral is zero because there are no poles inside (or on) C. On the other hand, the contour integral is evaluated along C and may be broken up into six pieces as follows:

$$\begin{aligned}&\int_\epsilon^R e^{-bx}\coth x\mathrm{d}x+i\int_0^\pi e^{-b(R+iy)}\coth(R+iy)\mathrm{d}x+\\&\int_R^\epsilon e^{b(x+i\pi)}\coth(x+i\pi)\mathrm{d}x+i\epsilon\int_0^{-\pi/2}e^{i\phi}\,e^{-b\left(i\pi+\epsilon e^{i\phi}\right)}\coth\left(\epsilon e^{i\phi}\right)\mathrm{d}x+\\&i\int_{\pi-\epsilon}^\epsilon e^{-iby}\coth iy\mathrm{d}x+i\epsilon\int_0^{\pi/2}e^{i\phi}\,e^{-b\epsilon e^{i\phi}}\coth\left(\epsilon e^{i\phi}\right)\mathrm{d}x\end{aligned}$$

We take the limits as $\epsilon\to0^+$ and $R\to\infty$. The second integral vanishes in this limit because $\Re b>0$. Setting this sum of integrals to zero and rearranging a bit, we get

$$\left(1-e^{-i\pi b}\right)\int_0^\infty e^{-bx}\coth x\mathrm{d}x=\int_0^\pi e^{-iby}\cot y\mathrm{d}y+i\frac{\pi}{2}\left(1+e^{-i\pi b}\right)$$

Note that, in the first integral on the RHS, I used the fact that $\coth iy=-i\cot y$.
Now we are ready to begin discussing the original integral. From the above equation, we take imaginary parts; from the fact that $b=(1i)a$, we may write

$$\begin{aligned}&\int_0^\infty e^{-ax}\sin ax\,(\coth x+\cot x)\mathrm{d}x=\frac{\pi}{2}\frac{1+(-1)^ae^{-\pi a}}{1-(-1)^ae^{-\pi a}}+\\&\int_0^\infty e^{-ax}\sin ax\cot x\mathrm{d}x-\frac{1}{1-(-1)^ae^{-\pi a}}\int_0^\pi e^{-ax}\sin ax\cot x\mathrm{d}x\end{aligned}$$

Remember that a is a positive integer. I will now prove that the last two terms on the RHS cancel each other out.
Let's take the case where $a=2n$, i.e., a is even. Then

$$\begin{aligned}&\int_0^\pi e^{-2nx}\sin 2nx\cot x\mathrm{d}x=\int_0^\pi e^{-2nx}\frac{\sin 2nx}{\sin x}\cos x\mathrm{d}x\\&=\int_0^\pi e^{-2nx}\,2\sum_{k=0}^{n-1}\cos\left[(2k+1)x\right]\cos x\mathrm{d}x=\int_0^\pi e^{-2nx}\left[1+2\sum_{k=1}^{n-1}\cos 2kx+\cos 2nx\right]\\&=\frac{1-e^{-2\pi}}{2n}+2\sum_{k=1}^{n-1}\Re\left[\int_0^\pi e^{-(2n-i2k)x}\mathrm{d}x\right]+\Re\left[\int_0^\pi e^{-(2n-i2n)x}\mathrm{d}x\right]\\&=\frac{1-e^{-2\pi}}{2n}+2\sum_{k=1}^{n-1}\Re\left[\frac{1-e^{-2n\pi}}{2n-i2k}\right]+\Re\left[\frac{1-e^{-2n\pi}}{2n-i2n}\right]\\&=\frac{1-e^{-2\pi}}{2n}+2\left(1-e^{-2n\pi}\right)\sum_{k=1}^{n-1}\Re\left[\frac{2n+i2k}{4n^2+4k^2}\right]+\left(1-e^{-2n\pi}\right)\Re\left[\frac{2n+i2n}{4n^2+4n^2}\right]\\&=\left(1-e^{-2n\pi}\right)\left[\frac{1}{n}+n\sum_{k=1}^{n-1}\frac{1}{n^2+k^2}\right]\end{aligned}$$

Similarly, one may show that

$$\int_0^\infty e^{-2nx} \sin 2nx \cot x \mathrm{d}x = \frac{1}{n} + n \sum_{k=1}^{n-1} \frac{1}{n^2+k^2}$$
$$= \frac{1}{1-e^{-2n\pi}} \int_0^\pi e^{-2nx} \sin 2nx \cot x \mathrm{d}x$$

The cancellation occurs when a is even. One may show that this cancellation also occurs when a is odd, i.e., when $a = 2n1$ in a similar fashion, using the fact that

$$\frac{\sin(2n-1)x}{\sin x} = 1 + 2\sum_{k=1}^{n-1} \cos 2kx$$

We may then write

$$\int_0^\infty e^{-ax} \sin ax \, (\coth x + \cot x) \mathrm{d}x = \frac{\pi}{2} \frac{1+(-1)^a e^{-\pi a}}{1-(-1)^a e^{-\pi a}}$$

or, in terms of hyperbolic functions, for $n \in \mathbb{N}$:

$$\int_0^\infty e^{-ax} \sin ax \, (\coth x + \cot x) \mathrm{d}x = \begin{cases} \frac{\pi}{2} \coth \pi n & a = 2n \\ \frac{\pi}{2} \tanh\left(n+\frac{1}{2}\right)\pi & a = 2n+1 \end{cases}$$

□

Exercise 1.1.163: Evaluate:

$$\int_0^\infty \frac{x^3}{(x^4+1)(e^x-1)} \mathrm{d}x$$

Solution: **Step 1.** Let us introduce the function

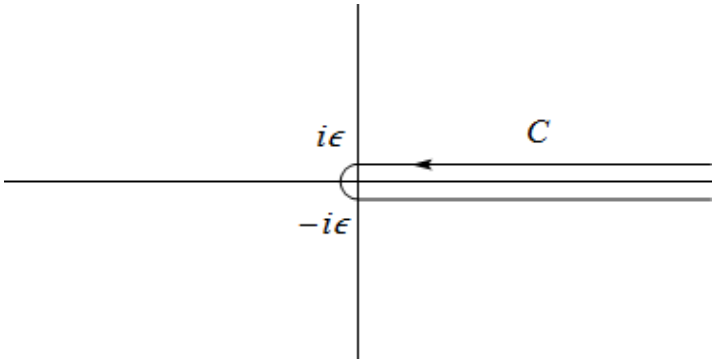
$$I(s) = \int_0^\infty \frac{x^s}{(x^4+1)(e^x-1)} \, \mathrm{d}x$$

It is easy to see that

$$I(s) + I(s+4) = \Gamma(s+1)\zeta(s+1). \tag{1}$$

This allows us to extend $I(s)$ as a meromorphic function on $\mathbb{C}$.

Step 2. Consider a contour C starting from $\infty + \epsilon i$, making a counter-clockwise turn around $z = 0$ and going back to $\infty - \epsilon i$ as follows:



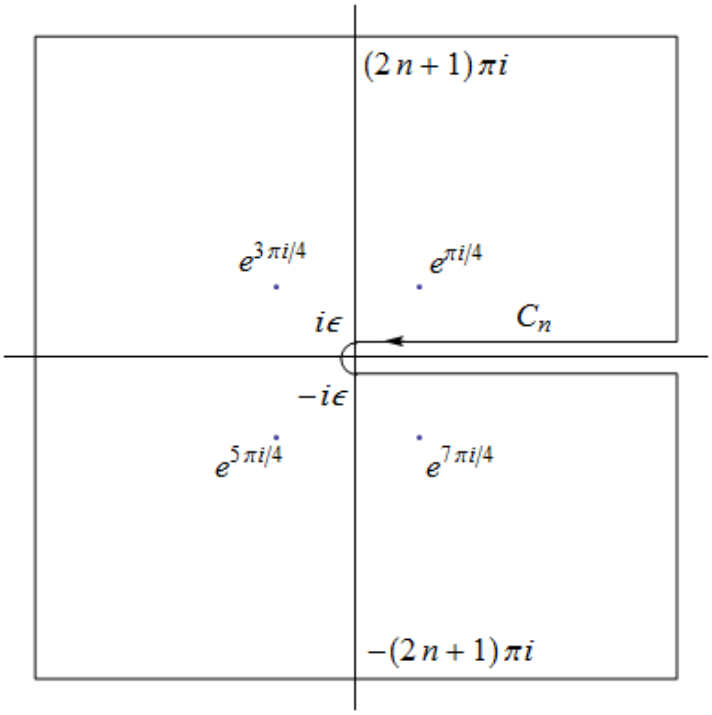
If we use a logarithm function with the branch cut $[0,\infty)$, we see that

$$(e^{2\pi is}-1)I(s)=\int_C \frac{z^s}{(z^4+1)(e^z-1)}\,\mathrm{d}z. \tag{2}$$

Also, for $1<s<2$ we can confirm that (2) is rewritten as

$$(e^{2\pi is}-1)I(s)=\lim_{n\to\infty}\int_{C_n} \frac{z^s}{(z^4+1)(e^z-1)}\,\mathrm{d}z,$$

where C_n is the contour given by



Here, the condition $1<s<2$ is introduced in order to create an appropriate decay speed for the integral along the contour $C-C_n$. By applying the Cauchy integration formula, we have

$$(e^{2\pi is}-1)I(s)=-2\pi i\sum_{\omega^4=-1}\mathrm{Res}_{z=\omega}\frac{z^s}{(z^4+1)(e^z-1)}-2\pi i\sum_{n\neq 0}\mathrm{Res}_{z=2\pi in}\frac{z^s}{(z^4+1)(e^z-1)}.$$

Simplifying,

$$I(s)=\frac{\pi}{\sin\pi s}\frac{e^{-i\pi s}}{4}\sum_{\omega^4=-1}\frac{\omega^{s+1}}{e^\omega-1}-\frac{2^s\pi^{s+1}}{\sin\dfrac{\pi s}{2}}\sum_{n=1}^{\infty}\frac{n^s}{1+(2\pi n)^4} \tag{3}$$

Since both sides define a meromorphic function for $\Re s<3$, they coincide on this range.

Step 3. Combining (1)(1) and (3)(3), we have

$$\begin{aligned}I(s+3)&=\Gamma(s)\zeta(s)-I(s-1)\\&=\Gamma(s)\zeta(s)-\frac{\pi}{\sin\pi s}\frac{e^{-i\pi s}}{4}\sum_{\omega^4=-1}\frac{\omega^s}{e^\omega-1}-\frac{(2\pi)^s}{2\cos\dfrac{\pi s}{2}}\sum_{n=1}^{\infty}\frac{n^s}{n\,(1+(2\pi n)^4)}\end{aligned}$$

Taking $s \to 0$, we obtain

$$\int_0^\infty \frac{x^3}{(x^4+1)(e^x-1)}\,\mathrm{d}x = \frac{\gamma}{2} - \ln\sqrt{2\pi} + \frac{\pi}{4}\frac{\sin\frac{1}{\sqrt{2}}}{\cosh\frac{1}{\sqrt{2}} - \cos\frac{1}{\sqrt{2}}} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n\,(1+(2\pi n)^4)}$$

To calculate the infinite sum and simplify the result we need the following:

$$\frac{1}{n\,((2\,\pi\,n)^4+1)} = \frac{1}{n} - \frac{1}{4}\left(\frac{1}{n+\frac{(-1)^{1/4}}{2\pi}} + \frac{1}{n+\frac{(-1)^{-1/4}}{2\pi}} + \frac{1}{n+\frac{(-1)^{3/4}}{2\pi}} + \frac{1}{n+\frac{(-1)^{-3/4}}{2\pi}}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+x}\right) = \gamma + \psi(1+x)$$

and the formulas (8),(9) from here.

http://mathworld.wolfram.com/PolygammaFunction.html

We get

$$\int_0^\infty \frac{x^3}{(x^4+1)(e^x-1)}\mathrm{d}x = -\frac{\pi}{\sqrt{8}} - \ln\sqrt{2\pi} - \frac{1}{2}\Re\,\psi\left(\frac{\sqrt[4]{-1}}{2\pi}\right)$$

□

Exercise 1.1.164:

Solution:

□

Exercise 1.1.165:

Solution:

□

Exercise 1.1.166:

Solution:

□

Exercise 1.1.167:

Solution:

□

Exercise 1.1.168:

Solution:

□

Exercise 1.1.169:

Solution:

□

Exercise 1.1.170:

Solution:

□

*

1.2 Integrals and Series

✎ *

Exercise 1.2.1: Prove:

$$\int_0^\infty \frac{e^{\cos(ax)}\cos\left(\sin(ax)+bx\right)}{c^2+x^2}\mathrm{d}x = \frac{\pi}{2c}e^{-bc+e^{-ac}}$$

where $a, b, c > 0$.

☞ **Proof:**

$$\begin{aligned}\int_0^\infty \frac{e^{\cos(ax)}\cos\left(\sin(ax)+bx\right)}{c^2+x^2}\mathrm{d}x &= \frac{1}{2}\,\Re\int_{-\infty}^{\infty}\frac{e^{e^{iax}}e^{ibx}}{c^2+x^2}\mathrm{d}x\\ &= \Re\; i\pi\;\mathrm{Res}\left[\frac{e^{e^{iaz}}e^{ibz}}{c^2+z^2}, ic\right]\\ &= \pi\frac{e^{e^{-ac}}e^{-bc}}{2c}\\ &= \frac{\pi}{2c}e^{-bc+e^{-ac}}\end{aligned}$$

□

Exercise 1.2.2: Prove:

$$\int_0^{\frac{\pi}{2}} \frac{\sin\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\mathrm{d}x = \frac{\pi}{2bc}\sin\left(\frac{2ac}{b+c}\right)$$

where $b, c > 0$.

☞ **Proof:** Let I denote the integral. Then:

$$\begin{aligned}I &= \int_0^{\frac{\pi}{2}} \frac{\sin\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\mathrm{d}x = \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\mathrm{d}x\\ &= \frac{1}{4}\int_{-\pi}^{\pi}\frac{\sin\left(2a\cos^2\frac{x}{2}\right)\cosh(a\sin x)}{b^2\cos^2\frac{x}{2}+c^2\sin^2\frac{x}{2}}\mathrm{d}x = \frac{1}{4}\int_{-\pi}^{\pi}\frac{\sin\left(a(1+\cos x)\right)\cosh(a\sin x)}{b^2\left(\frac{1+\cos x}{2}\right)+c^2\left(\frac{1-\cos x}{2}\right)}\mathrm{d}x\\ &= \frac{1}{2}\Re\int_{-\pi}^{\pi}\frac{\sin\left(a+ae^{ix}\right)}{b^2+c^2+(b^2-c^2)\cos(x)}\mathrm{d}x\end{aligned}$$

Now substitute $z = e^{ix}$:

$$\begin{aligned}I &= \frac{1}{2}\Re\int_{|z|=1}\frac{\sin\left(a+az\right)}{b^2+c^2+(b^2-c^2)\frac{z+z^{-1}}{2}}\frac{\mathrm{d}z}{iz} = \Re\int_{|z|=1}\frac{z\sin(a+az)}{2z(b^2+c^2)+(b^2-c^2)(z^2+1)}\frac{\mathrm{d}z}{iz}\\ &= \frac{1}{b^2-c^2}\Im\int_{|z|=1}\frac{\sin(a+az)}{\left(z+\frac{b+c}{b-c}\right)\left(z+\frac{b-c}{b+c}\right)}\mathrm{d}z\end{aligned}$$

Using Residue Theorem,

$$\begin{aligned} I &= \frac{1}{b^2-c^2}\Im\left(2\pi i\ \text{Res}_{z=-\frac{b-c}{b+c}}\frac{\sin(a+az)}{\left(z+\dfrac{b+c}{b-c}\right)\left(z+\dfrac{b-c}{b+c}\right)}\right) \\ &= 2\pi\frac{1}{b^2-c^2}\frac{1}{\dfrac{4bc}{b^2-c^2}}\sin\left(\frac{2ac}{b+c}\right) \\ &= \frac{\pi}{2bc}\sin\left(\frac{2ac}{b+c}\right) \end{aligned}$$

□

Exercise 1.2.3: Prove:

$$\int_0^\infty \frac{\sin\left(2\cos^2 x\right)\cosh(\sin 2x)}{1+x^2}\mathrm{d}x = \frac{\pi}{2}\sin\left(1+\frac{1}{e^2}\right)$$

Proof: Notice that the magnitude of $\sin(1+e^{2iz}) = \sin\left(1+e^{2i(x+iy)}\right) = \sin\left(1+e^{-2y}\cos 2x + ie^{-2y}\sin 2x\right)$ is never large in the upper half-plane since $e^{-2y}\sin 2x$ is never large in the upper half-plane.
So $\displaystyle\int\frac{\sin(1+e^{2iz})}{1+z^2}\mathrm{d}z$ vanishes along the upper half of $|z|=R$ as $R\to\infty$.
And therefore,

$$\begin{aligned} \int_0^\infty \frac{\sin(2\cos^2 x)\cosh(\sin 2x)}{1+x^2}\ \mathrm{d}x &= \frac{1}{2}\ \Re\int_{-\infty}^{\infty}\frac{\sin(1+e^{2ix})}{1+x^2}\ \mathrm{d}x \\ &= \pi\ \Re\ i\ \text{Res}\left[\frac{\sin(1+e^{2iz})}{1+z^2}, i\right] \\ &= \frac{\pi}{2}\sin\left(1+\frac{1}{e^2}\right) \end{aligned}$$

□

Exercise 1.2.4: Prove:

$$\int_0^1\left(\text{erf}^{-1}\sqrt{x}\right)^2\mathrm{d}x = \frac{1}{2}+\frac{1}{\pi}$$

Proof:

$$\begin{aligned} \int_0^1\left(\text{erf}^{-1}(\sqrt{x})\right)^2\ \mathrm{d}x &= \frac{4}{\sqrt{\pi}}\int_0^\infty u^2e^{-u^2}\text{erf}(u)\ \mathrm{d}u = \frac{8}{\pi}\int_0^\infty\int_0^1 u^2e^{-u^2}u\ e^{-u^2t^2}\ \mathrm{d}t\ \mathrm{d}u \\ &= \frac{8}{\pi}\int_0^1\int_0^\infty u^3e^{-(1+t^2)u^2}\ \mathrm{d}u\ \mathrm{d}t = \frac{4}{\pi}\int_0^1\frac{1}{(1+t^2)^2}\ \mathrm{d}t \\ &= \frac{4}{\pi}\left(\frac{\pi}{8}+\frac{1}{4}\right) = \frac{1}{2}+\frac{1}{\pi} \end{aligned}$$

□

Exercise 1.2.5: Prove:

$$\int_0^1\left(E(k)-\frac{\pi}{2}\right)\frac{\mathrm{d}k}{k} = \pi\ln 2 - 2\mathbf{G} + 1 - \frac{\pi}{2}$$

where $E(k)$ is the **complete elliptic integral of the second kind**.

☞ **Proof:**

$$\begin{aligned}\int_0^1 \left(E(k)-\frac{\pi}{2}\right)\frac{\mathrm{d}k}{k} &= \int_0^1 \left(\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta-\frac{\pi}{2}\right)\frac{\mathrm{d}k}{k}\\ &= \int_0^1\int_0^{\pi/2}\frac{\sqrt{1-k^2\sin^2\theta}-1}{k}\,\mathrm{d}\theta\,\mathrm{d}k\\ &= \int_0^{\pi/2}\int_0^1\frac{\sqrt{1-k^2\sin^2\theta}-1}{k}\,\mathrm{d}k\,\mathrm{d}\theta\end{aligned}$$

It is easy to show that:

$$\begin{aligned}\int_0^1\frac{\sqrt{1-k^2\sin^2\theta}-1}{k}\,\mathrm{d}k &= \left(\sqrt{1-k^2\sin^2\theta}-\ln\left(\sqrt{1-k^2\sin^2\theta}+1\right)\right)\Big|_0^1\\ &= \ln 2-1+\cos\theta-\ln\left(1+\cos\theta\right)\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{\pi/2}\int_0^1\frac{\sqrt{1-k^2\sin^2\theta}-1}{k}\,\mathrm{d}k\,\mathrm{d}\theta &= \int_0^{\pi/2}\left(\ln 2-1+\cos\theta-\ln\left(1+\cos\theta\right)\right)\,\mathrm{d}\theta\\ &= \frac{\pi}{2}\left(\ln 2-1\right)+1-\int_0^{\pi/2}\ln(1+\cos\theta)\,\mathrm{d}\theta\end{aligned}$$

The final integral can be evaluated as:

$$\begin{aligned}\int_0^{\pi/2}\ln(1+\cos\theta)\,\mathrm{d}\theta &= \int_0^{\pi/2}\ln\left(2\cos^2\frac{\theta}{2}\right)\,\mathrm{d}\theta\\ &= \int_0^{\pi/2}\ln 2\,\mathrm{d}\theta+2\int_0^{\pi/2}\ln\left(\cos\frac{\theta}{2}\right)\,\mathrm{d}\theta\\ &= \frac{\pi}{2}\ln 2+4\int_0^{\pi/4}\ln(\cos\theta)\,\mathrm{d}\theta\\ &= \frac{\pi}{2}\ln 2+4\left(-\frac{\pi\ln 2}{4}+\frac{\mathbf{G}}{2}\right)=2\mathbf{G}-\frac{\pi}{2}\ln 2\end{aligned}$$

So we have:

$$\int_0^1\left(E(k)-\frac{\pi}{2}\right)\frac{\mathrm{d}k}{k}=\pi\ln 2-2\mathbf{G}+1-\frac{\pi}{2}$$

□

Exercise 1.2.6: Prove:

$$\int_0^\infty\left(\frac{1}{2}-S(x)\right)\ln(x)\,\mathrm{d}x=-\frac{2+\gamma+\ln\left(\frac{\pi}{2}\right)}{2\pi}$$

where $S(x)=\displaystyle\int_0^x\sin\left(\frac{\pi t^2}{2}\right)\mathrm{d}t$ is a Fresnel integral and γ is the Euler's constant.

☞ **Proof:** Let $F(z)=\displaystyle\int_0^\infty\left(\frac{1}{2}-S(x)\right)x^{z-1}\,\mathrm{d}x$, where $0<z<3$.
Then the integral that we seek is simply $F'(1)$.

$F(z)$ can be computed as follows:

$$\begin{aligned}
F(z) &= \int_0^\infty \left(\frac{1}{2} - S(x)\right) x^{z-1}\,\mathrm{d}x \\
&= \int_0^\infty \int_x^\infty \sin\left(\frac{\pi t^2}{2}\right) dt\; x^{z-1}\,\mathrm{d}x \\
&= \int_0^\infty \int_1^\infty x^z \sin\left(\frac{\pi x^2 t^2}{2}\right)\,\mathrm{d}t\,\mathrm{d}x \\
&= \frac{1}{2}\int_1^\infty \int_0^\infty x^{\frac{z-1}{2}} \sin\left(\frac{\pi t^2 x}{2}\right)\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right)^{-(z+1)/2}\Gamma\left(\frac{1+z}{2}\right)\sin\left(\frac{\pi(z+1)}{4}\right)\int_1^\infty t^{-1-z}\,\mathrm{d}t \\
&= \frac{1}{2z}\left(\frac{\pi}{2}\right)^{-(z+1)/2}\Gamma\left(\frac{1+z}{2}\right)\sin\left(\frac{\pi(z+1)}{4}\right)
\end{aligned}$$

Calculating the derivative by hand is a little tedious. I used Mathematica to do it.

$$F'(z) = \frac{\left(\begin{array}{c} 2^{\frac{z-5}{2}}\pi^{\frac{1}{2}(-z-1)}\Gamma\left(\frac{z+1}{2}\right)\left\{\pi z\cos\left(\frac{1}{4}\pi(z+1)\right)\right. \\ \left.+2\sin\left(\frac{1}{4}\pi(z+1)\right)\left(z\ln\left(\frac{2}{\pi}\right)+z\psi_0\left(\frac{z+1}{2}\right)-2\right)\right\}\end{array}\right)}{z^2}$$

$$\Longrightarrow\; F'(1) = -\frac{2+\gamma+\ln\left(\frac{\pi}{2}\right)}{2\pi}$$

We also get

$$\begin{aligned}
F'\left(\frac{1}{2}\right) &= \int_0^\infty \left(\frac{1}{2} - S(x)\right)\frac{\ln x}{\sqrt{x}}\,\mathrm{d}x \\
&= \frac{\sqrt{1+\sqrt{2}}\left(-8-2\gamma+\sqrt{2}\pi-2\ln(4\pi)\right)}{4\pi^{3/4}}\Gamma\left(\frac{3}{4}\right)
\end{aligned}$$

□

Exercise 1.2.7: Prove:

$$\int_0^\infty \ln(x)\ln\left(1+\frac{a^n}{x^n}\right)\mathrm{d}x = \pi a\cdot\csc\left(\frac{\pi}{n}\right)\left(\ln(a) - \frac{\pi}{n}\cot\left(\frac{\pi}{n}\right) - 1\right)$$

Proof:

$$I(a) = \int_0^\infty \ln(x)\ln\left(1+\frac{a^n}{x^n}\right)\mathrm{d}x \Rightarrow I'(a) = na^{n-1}\int_0^\infty \frac{\ln x}{x^n+a^n}\,\mathrm{d}x$$

With the substitution $x^n = a^n t$,

$$\begin{aligned}
I'(a) &= \int_0^\infty \frac{t^{-\left(1-\frac{1}{n}\right)}\ln(at^{1/n})}{1+t}\,\mathrm{d}t \\
&= \ln a\int_0^\infty \frac{t^{-\left(1-\frac{1}{n}\right)}}{1+t}\,\mathrm{d}t + \frac{1}{n}\int_0^\infty \frac{t^{-\left(1-\frac{1}{n}\right)}\ln t}{1+t}\,\mathrm{d}t \\
&= \ln a\, J_1 + \frac{1}{n}J_2
\end{aligned}$$

Since,

$$\int_0^\infty \frac{t^{-b}}{1+t}\,\mathrm{d}t = \frac{\pi}{\sin(b\pi)} \text{ for } 0<b<1 \Rightarrow J_1 = \pi\csc\left(\frac{\pi}{n}\right)$$

and

$$J_2 = -\frac{\mathrm{d}}{\mathrm{d}b}\left.\frac{\pi}{\sin(b\pi)}\right|_{b=1-\frac{1}{n}} = -\pi^2\cot\left(\frac{\pi}{n}\right)\csc\left(\frac{\pi}{n}\right)$$

Hence,

$$I'(a) = \pi\csc\left(\frac{\pi}{n}\right)\ln a - \frac{\pi^2}{n}\cot\left(\frac{\pi}{n}\right)\csc\left(\frac{\pi}{n}\right)$$

$$\Rightarrow I(a) = \pi\csc\left(\frac{\pi}{n}\right)(a\ln a - a) - \frac{1}{n}\pi^2 a\cot\left(\frac{\pi}{n}\right)\csc\left(\frac{\pi}{n}\right) + C$$

With $a = 0$ and $I(0) = 0$, $C = 0$.

$$\Rightarrow I(a) = a\pi\csc\left(\frac{\pi}{n}\right)\left(\ln a - \frac{\pi}{n}\cot\left(\frac{\pi}{n}\right) - 1\right)\ ,\ n>0$$

□

Exercise 1.2.8: Prove:

$$\int_0^\pi \left(\frac{e+\cos(x)}{1+2e\cos(x)+e^2}\right)^2 \mathrm{d}x = \frac{\pi}{2e^2}\left(\frac{2e^2-1}{e^2-1}\right)$$

Proof: Use the following:

$$\frac{e+\cos x}{e^2+2e\cos(x)+1} = \sum_{n=0}^{\infty}\frac{(-1)^n\cos(nx)}{e^{n+1}}$$

The integral can then be written as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{e^{m+n+2}}\int_0^\pi \cos(nx)\cos(mx)\,\mathrm{d}x$$

For $m \neq n$, the integral is zero. We only consider those cases where $m = n$.

$$\sum_{n=0}^{\infty}\frac{1}{e^{2n+2}}\int_0^\pi \cos^2(nx)\,\mathrm{d}x = \frac{\pi}{e^2} + \frac{\pi}{2e^2}\sum_{n=1}^{\infty}\frac{1}{e^{2n}}$$

$$= \frac{\pi}{e^2} + \frac{\pi}{2e^2}\frac{1}{e^2-1} = \frac{\pi}{2e^2}\left(\frac{2e^2-1}{e^2-1}\right)$$

□

Exercise 1.2.9: Prove:

$$\int_0^\pi x\tan^{-1}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\left\{\frac{\pi^2}{4} - \ln^2(\sqrt{2}-1)\right\}$$

Proof: For $|r| \leq 1$,

$$\sum_{k=1}^{\infty}\frac{r^k\sin k\theta}{k} = \Im\sum_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k} = -\Im\ln(1-re^{i\theta})$$

$$= \arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right)$$

which implies

$$\begin{aligned}2\sum_{k=0}^{\infty}\frac{\sin(2k+1)\theta}{2k+1}r^{2k+1} &= \sum_{k=1}^{\infty}\frac{r^k\sin k\theta}{k}-\sum_{k=1}^{\infty}\frac{(-1)^k r^k\sin k\theta}{k}\\ &= \arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right)-\arctan\left(\frac{-r\sin\theta}{1+r\cos\theta}\right)\\ &= \arctan\left(\frac{2r\sin\theta}{1-r^2}\right)\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{\pi} x\arctan\left(\frac{2r\sin x}{1-r^2}\right)\mathrm{d}x &= 2\int_0^{\pi} x\sum_{n=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}r^{2k+1}\,\mathrm{d}x\\ &= 2\sum_{n=0}^{\infty}\frac{r^{2k+1}}{2k+1}\int_0^{\pi} x\sin(2k+1)x\,\mathrm{d}x\\ &= 2\pi\sum_{k=0}^{\infty}\frac{r^{2k+1}}{(2k+1)^2}\\ &= \pi\Big(\mathrm{Li}_2(r)-\mathrm{Li}_2(-r)\Big)\end{aligned}$$

Then

$$\begin{aligned}\int_0^{\pi} x\arctan(\sin x)\,\mathrm{d}x &= \pi\Big(\mathrm{Li}_2(\sqrt{2}-1)-\mathrm{Li}_2(1-\sqrt{2})\Big)\\ &= \frac{\pi}{2}\left\{\frac{\pi^2}{4}-\ln^2(\sqrt{2}-1)\right\}\end{aligned}$$

□

Exercise 1.2.10: Prove:

$$\int_0^{\pi} x\csc(x)\ln(1+a\cdot\sin(x))\mathrm{d}x = \frac{\pi^2}{2}\sin^{-1}(a)-\frac{\pi}{2}\left(\sin^{-1}(a)\right)^2,\quad a<1$$

Proof: Method One:

It can be shown that:

$$I(a) = \pi\int_0^{\pi/2}\csc(x)\ln(1+a\sin x)\,\mathrm{d}x$$

$$\Rightarrow I'(a) = \pi\int_0^{\pi/2}\frac{\mathrm{d}x}{1+a\sin x}$$

The above integral can be solved using Weierstrass Substitution

$$I'(a) = \frac{2\pi}{\sqrt{1-a^2}}\left(\tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)\right)$$

Next step is to simplify the arctangents. With the substitution $a=\cos(2\theta)$ in the first one,

$$\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\right) = \tan^{-1}(\cot\theta) = \frac{\pi}{2}-\frac{1}{2}\cos^{-1}a = \frac{\pi}{4}+\frac{1}{2}\sin^{-1}a$$

Simplify the second one with the substitution $a=\sin\theta$

$$\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right) = \tan^{-1}(\tan\theta) = \sin^{-1}a$$

Hence,

$$I'(a) = \frac{2\pi}{\sqrt{1-a^2}}\left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1} a\right) = \frac{\pi^2}{2}\frac{1}{\sqrt{1-a^2}} - \pi\frac{\sin^{-1} a}{\sqrt{1-a^2}}$$

$$\Rightarrow I(a) = \frac{\pi^2}{2}\sin^{-1} a - \frac{\pi}{2}\left(\sin^{-1} a\right)^2$$

□

Exercise 1.2.11: Prove:

$$\int_0^\infty \left(\frac{1}{2} - S(x)\right)\sin(2px)\,\mathrm{d}x = \frac{1}{4p}\left\{\sin\left(\frac{2p^2}{\pi}\right) - \cos\left(\frac{2p^2}{\pi}\right) + 1\right\}$$

Proof:

$$\begin{aligned}\int_0^\infty \left(\frac{1}{2} - S(x)\right)\sin(2px)\,\mathrm{d}x &= \int_0^\infty \int_x^\infty \sin\left(\frac{\pi t^2}{2}\right)\sin(2px)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\infty \int_0^t \sin\left(\frac{\pi t^2}{2}\right)\sin(2px)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2p}\int_0^\infty \sin\left(\frac{\pi t^2}{2}\right)(1-\cos(2pt))\,\mathrm{d}t \\ &= \frac{1}{2p}\left(\frac{1}{2} + \Im\left(\int_0^\infty e^{-i\frac{\pi t^2}{2}}\cos(2pt)\,\mathrm{d}t\right)\right)\end{aligned}$$

Since,

$$\int_0^\infty e^{-ax^2}\cos(mx)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-m^2/(4a)} \Rightarrow \int_0^\infty e^{-i\frac{\pi t^2}{2}}\cos(2pt)\,\mathrm{d}t = \frac{1}{\sqrt{2i}}e^{i2p^2/\pi}$$

$$\Rightarrow \Im\left(\frac{1}{\sqrt{2i}}e^{i2p^2/\pi}\right) = \frac{1}{2}\left(\sin\frac{2p^2}{\pi} - \cos\frac{2p^2}{\pi}\right)$$

Hence,

$$\int_0^\infty \left(\frac{1}{2} - S(x)\right)\sin(2px)\,\mathrm{d}x = \frac{1}{4p}\left(1 + \sin\frac{2p^2}{\pi} - \cos\frac{2p^2}{\pi}\right)\ ,\ p > 0$$

Method Two:

First of all note that

$$\frac{1}{2} - S(x) = \int_x^\infty \sin\left(\frac{\pi t^2}{2}\right)\mathrm{d}t = x\int_1^\infty \sin\left(\frac{\pi t^2 x^2}{2}\right)\,\mathrm{d}t$$

Then,

$$\begin{aligned}\int_0^\infty \left(\frac{1}{2}-S(x)\right)\sin(2px)\,\mathrm{d}x &= \int_0^\infty\int_1^\infty x\sin\left(\frac{\pi t^2x^2}{2}\right)\sin(2px)\,\mathrm{d}t\,\mathrm{d}x\\ &= -\frac{1}{2}\int_0^\infty\int_1^\infty \frac{\partial}{\partial p}\left\{\sin\left(\frac{\pi t^2x^2}{2}\right)\cos(2px)\right\}\,\mathrm{d}t\,\mathrm{d}x\\ &= -\frac{1}{2}\int_1^\infty \frac{\partial}{\partial p}\int_0^\infty \sin\left(\frac{\pi t^2x^2}{2}\right)\cos(2px)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\frac{1}{4}\int_1^\infty \frac{1}{t}\frac{\partial}{\partial p}\left\{\cos\left(\frac{2p^2}{\pi t^2}\right)-\sin\left(\frac{2p^2}{\pi t^2}\right)\right\}\mathrm{d}t\\ &= \frac{p}{\pi}\int_1^\infty \frac{1}{t^3}\left\{\sin\left(\frac{2p^2}{\pi t^2}\right)+\cos\left(\frac{2p^2}{\pi t^2}\right)\right\}\mathrm{d}t\\ &= \frac{p}{2\pi}\int_0^1 \left\{\sin\left(\frac{2p^2}{\pi}t\right)+\cos\left(\frac{2p^2}{\pi}t\right)\right\}\mathrm{d}t \quad (1/t^2\mapsto t)\\ &= \frac{1}{4p}\left\{\sin\left(\frac{2p^2}{\pi}\right)-\cos\left(\frac{2p^2}{\pi}\right)+1\right\}\end{aligned}$$

□

Exercise 1.2.12: Evaluate:

$$\int_0^1 \frac{\mathrm{Li}_2(x)^3}{x}\,\mathrm{d}x$$

Solution: **Step 1.Reduction to Double Euler Sums**

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(x)^3}{x}\mathrm{d}x &= \sum_{m,n=1}^\infty \frac{1}{m^2n^2}\int_0^1 x^{m+n-1}\mathrm{Li}_2(x)\,\mathrm{d}x\\ &= \sum_{m,n=1}^\infty \frac{1}{m^2n^2}\left(\frac{\zeta(2)}{m+n}-\frac{H_{n+m}}{(n+m)^2}\right)\\ &= \zeta(2)\sum_{m,n=1}^\infty \frac{1}{m^2n^2(m+n)}-\sum_{m,n=1}^\infty \frac{1}{m^2n^2}\frac{H_{n+m}}{(n+m)^2}\end{aligned}$$

Step 2.Evaluation of Double Euler Sums (I)

Notice that with the substitution $r=m+n$, the double Euler sum may be transformed into a single sum as

$$\begin{aligned}\sum_{m,n=1}^\infty \frac{1}{m^2n^2(m+n)} &= \sum_{r=2}^\infty \frac{1}{r}\sum_{k=1}^{r-1}\frac{1}{k^2(r-k)^2}\\ &= \sum_{r=2}^\infty \frac{1}{r}\sum_{k=1}^{r-1}\left(\frac{1}{r^2k^2}+\frac{1}{r^2(r-k)^2}+\frac{2}{kr^3}+\frac{2}{r^3(r-k)}\right)\\ &= \sum_{r=2}^\infty \frac{1}{r}\left(2\frac{H_r^{(2)}}{r^2}+4\frac{H_r}{r^3}-6\frac{1}{r^4}\right)\\ &= -6\zeta(5)+2\sum_{r=1}^\infty \frac{H_r^{(2)}}{r^3}+4\sum_{r=1}^\infty \frac{H_r}{r^4}\\ &= -6\zeta(5)+2\left(3\zeta(2)\zeta(3)-\frac{9\zeta(5)}{2}\right)+4(3\zeta(5)-\zeta(2)\zeta(3))\\ &= -3\zeta(5)+2\zeta(2)\zeta(3)\end{aligned}$$

Step 3.Evaluation of Double Euler Sums (II)

Repeating the first 3 steps as above,

$$\begin{aligned}\sum_{m,n=1}^{\infty}\frac{H_{m+n}}{m^2n^2(m+n)^2}&=\sum_{r=1}^{\infty}\frac{H_r}{r^2}\left(2\frac{H_r^{(2)}}{r^2}+4\frac{H_r}{r^3}-6\frac{1}{r^4}\right)\\&=-6\sum_{r=1}^{\infty}\frac{H_r}{r^6}+4\sum_{r=1}^{\infty}\frac{(H_r)^2}{r^5}+2\sum_{r=1}^{\infty}\frac{H_rH_r^{(2)}}{r^4}\\&=-6(-\zeta(4)\zeta(3)-\zeta(2)\zeta(5)+4\zeta(7))\\&+4\left(-\frac{5}{2}\zeta(4)\zeta(3)-\zeta(2)\zeta(5)+6\zeta(7)\right)+2\sum_{r=1}^{\infty}\frac{H_rH_r^{(2)}}{r^4}\\&=-4\zeta(4)\zeta(3)+2\zeta(2)\zeta(5)+2\sum_{r=1}^{\infty}\frac{H_rH_r^{(2)}}{r^4}\end{aligned}$$

Substituting every thing in equation (1), we get

$$\int_0^1\frac{\mathrm{Li}_2(x)^3}{x}\mathrm{d}x=\boxed{9\zeta(4)\zeta(3)-5\zeta(2)\zeta(5)-2\sum_{r=1}^{\infty}\frac{H_rH_r^{(2)}}{r^4}}$$

Step 4.Evaluation of Non-Linear Euler Sum

Using the methods shown in this paper,

http://www.combinatorics.org/ojs/index.php/eljc/article/view/v3i1r23/pdf

which obtained

$$\sum_{r=1}^{\infty}\frac{H_rH_r^{(2)}}{r^4}=\frac{3}{4}\zeta(4)\zeta(3)+2\zeta(2)\zeta(5)-\frac{51\zeta(7)}{16}$$

Step 5.Final Answer

$$\begin{aligned}\int_0^1\frac{\mathrm{Li}_2(x)^3}{x}\mathrm{d}x&=9\zeta(4)\zeta(3)-5\zeta(2)\zeta(5)-2\sum_{r=1}^{\infty}\frac{H_rH_r^{(2)}}{r^4}\\&=9\zeta(4)\zeta(3)-5\zeta(2)\zeta(5)-2\left(\frac{3}{4}\zeta(4)\zeta(3)+2\zeta(2)\zeta(5)-\frac{51\zeta(7)}{16}\right)\\&=\frac{15}{2}\zeta(3)\zeta(4)-9\zeta(2)\zeta(5)+\frac{51\zeta(7)}{8}\end{aligned}$$

With the following generalization

$$\begin{aligned}2\int_0^1\frac{\mathrm{Li}_q(x)\mathrm{Li}_{q-1}(x)\ln(1-x)}{x}\,\mathrm{d}x&=\sum_{m=2}^{q-1}(-1)^{m-1}\zeta(q-m+1)S_{q,m}\\&+(-1)^{q-1}\sum_{k\geq1}\frac{H_k^{(q)}H_k}{k^q}-\mathfrak{H}(q,q)\\&+\zeta(q)\zeta(q+1)-\zeta(q)\mathfrak{H}(q-1,1)\end{aligned}$$

where $\mathfrak{H}(p,q)$ is reducible to zeta terms.

The solution for the question could be done by putting $q=3$ and applying integration by parts.

Source :http://mathhelpboards.com/calculus-10/polylog-integrals-8044.html □

Exercise 1.2.13: Prove:

$$\int_{-\infty}^{\infty} \frac{x \ln|x| \ln|x-1|}{(x^2+1)(x^2+3)}\,\mathrm{d}x = \frac{\pi^2}{24}\ln\left(\frac{8}{9}\right)$$

Proof: let $f(z) = \dfrac{z \ln z \ln(z-1)}{(z^2+1)(z^2+3)}$.

Place the branch cut on the real axis and define $f(z)$ to be real-valued just above the positive real axis and to the right of $z = 1$.

Then integrating around a contour that consists of the negative real axis, the half-line just above the positive real axis, and the upper half of $|z| = R$,

$$\begin{aligned}\int_{-\infty}^{0} \frac{x\,(\ln|x| + i\pi)\,(\ln|x-1| + i\pi)}{(x^2+1)(x^2+3)}\,\mathrm{d}x + \int_{0}^{1} \frac{x \ln|x|\,(\ln|x-1| + i\pi)}{(x^2+1)(x^2+3)}\,\mathrm{d}x \\ + \int_{1}^{\infty} \frac{x \ln|x| \ln|x-1|}{(x^2+1)(x^2+3)}\,\mathrm{d}x \\ = 2\pi i\Big(\mathrm{Res}[f(z), i] + \mathrm{Res}[f(z), i\sqrt{3}]\Big).\end{aligned}$$

And equating the real parts on both sides of the equation,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x \ln|x| \ln|x-1|}{(x^2+1)(x^2+3)}\,\mathrm{d}x - \pi^2 \int_{-\infty}^{0} \frac{x}{(x^2+1)(x^2+3)}\,\mathrm{d}x \\ = \Re\; 2\pi i\Big(\mathrm{Res}[f(z), i] + \mathrm{Res}[f(z), i\sqrt{3}]\Big)\end{aligned}$$

where

$$\begin{aligned}\int_{-\infty}^{0} \frac{x}{(x^2+1)(x^2+3)}\,\mathrm{d}x &= -\int_{0}^{\infty} \frac{x}{(x^2+1)(x^2+3)}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{0}^{\infty} \left(\frac{x}{x^2+3} - \frac{x}{x^2+1}\right)\mathrm{d}x \\ &= \frac{1}{4}\ln\left(\frac{x^2+3}{x^2+1}\right)\Bigg|_{0}^{\infty} \\ &= -\frac{\ln 3}{4},\end{aligned}$$

$$\mathrm{Res}[f(z), i] = \frac{i}{(2i)(2)}\left(\frac{i\pi}{2}\right)\left(\frac{\ln 2}{2} + \frac{3\pi i}{4}\right),$$

and

$$\mathrm{Res}[f(z), i\sqrt{3}] = \frac{i\sqrt{3}}{(-2)(2i\sqrt{3})}\left(\frac{\ln 3}{2} + \frac{i\pi}{2}\right)\left(\ln 2 + \frac{2\pi i}{3}\right).$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x \ln|x| \ln|x-1|}{(x^2+1)(x^2+3)}\,\mathrm{d}x &= -\frac{\pi^2}{4}\ln 3 - \frac{\pi^2}{8}\ln 2 + \frac{\pi^2}{6}\ln 3 + \frac{\pi^2}{4}\ln 2 \\ &= \frac{\pi^2}{24}\left(-\ln 3^6 - \ln 2^3 + \ln 3^4 + \ln 2^6\right) \\ &= \frac{\pi^2}{24}\ln\left(\frac{8}{9}\right)\end{aligned}$$

□

Exercise 1.2.14: Prove:

$$\int_0^\infty \frac{\sinh(ax)\sin(bx)}{\big(\cosh ax+\cos bx\big)^2}\,\mathrm{d}x=\frac{b}{a^2+b^2}\quad(a,b>0)$$

Proof: Namely,

$$2\sum_{n=1}^{\infty}(-1)^{n-1}\sin(bnx)e^{-anx}=\frac{\sin bx}{\cosh ax+\cos bx}$$

and

$$2\sum_{n=1}^{\infty}(-1)^{n-1}\cos(bnx)e^{-anx}=\frac{e^{-ax}+\cos bx}{\cosh ax+\cos bx}.$$

Differentiating the first identity with respect to a,

$$2\sum_{n=1}^{\infty}(-1)^{n-1}n\sin(bnx)e^{-anx}=\frac{\sinh(ax)\sin(bx)}{(\cosh ax+\cos bx)^2}.$$

Therefore,

$$\int_0^\infty \frac{\sinh(ax)\sin(bx)}{(\cosh ax+\cos bx)^2}\,\mathrm{d}x=2\int_0^\infty\sum_{n=1}^{\infty}(-1)^{n-1}n\sin(bnx)e^{-anx}\,\mathrm{d}x.$$

Changing the order of integration and summation at this point will lead to a nonsensical result, namely that the integral is not finite.

But this is not entirely unexpected since Fubini's theorem is not satisfied.

That is

$$2\int_0^\infty\sum_{n=1}^{\infty}\left|(-1)^{n-1}n\sin(bnx)e^{-anx}\right|\,\mathrm{d}x\not<\infty.$$

So that the theorem is satisfied, write the right hand side as

$$\begin{aligned}
&2\lim_{\epsilon\to0}\int_\epsilon^\infty\sum_{n=1}^{\infty}(-1)^{n-1}n\sin(bnx)e^{-anx}\mathrm{d}x\\
&=2\lim_{\epsilon\to0}\sum_{n=1}^{\infty}(-1)^{n-1}n\int_\epsilon^\infty\sin(bnx)e^{-anx}\,\mathrm{d}x\\
&=\frac{2}{a^2+b^2}\lim_{\epsilon\to0}\sum_{n=1}^{\infty}(-1)^{n-1}\left(a\sin(bn\epsilon)e^{-an\epsilon}+b\cos(bn\epsilon)e^{-an\epsilon}\right)\\
&=\frac{2}{a^2+b^2}\lim_{\epsilon\to0}\left(\frac{a}{2}\frac{\sin b\epsilon}{\cosh a\epsilon+\cos b\epsilon}+\frac{b}{2}\frac{e^{-a\epsilon}+\cos b\epsilon}{\cosh a\epsilon+\cos b\epsilon}\right)\\
&=\frac{b}{a^2+b^2}
\end{aligned}$$

□

Exercise 1.2.15: Prove:

$$\int_0^{\frac{\pi}{2}}\frac{\ln(\sin(x))}{a-b\sin^2(x)}\mathrm{d}x=-\frac{\pi}{4\sqrt{a(a-b)}}\ln\left(\frac{2a-b+2\sqrt{a(a-b)}}{a}\right)$$

Proof: The integral is clearly equal to

$$\frac{1}{2a}\int_0^{\pi/2}\frac{\ln\cos^2x}{1-(b/a)\cos^2x}\mathrm{d}x$$

Hence we just need to show that

$$\int_0^{\pi/2} \frac{\ln\cos^2 x}{1-c\cos^2 x}\mathrm{d}x = -\frac{\pi}{2\sqrt{1-c}}\ln\left(2-c+2\sqrt{1-c}\right) = -\frac{\pi}{\sqrt{1-c}}\ln\left(1+\sqrt{1-c}\right)$$

which is equivalent to

$$\int_0^{\pi/2} \frac{\ln\cos^2 x}{1-(1-d^2)\cos^2 x}\mathrm{d}x = -\frac{\pi}{d}\ln(1+d)$$

I will show the above using real methods, silently assuming throughout that $0 < d < 1$.
We have

$$\int \frac{\mathrm{d}x}{1-(1-d^2)\cos^2 x} = \frac{1}{d}\arctan\left(\frac{\tan x}{d}\right),$$

as can be verified by differentiating back.
Hence integrating by parts yields

$$\int \frac{\ln\cos^2 x}{1-(1-d^2)\cos^2 x}\mathrm{d}x = \frac{1}{d}\arctan\left(\frac{\tan x}{d}\right)\ln\cos^2 x + \frac{2}{d}\int \tan x \arctan\left(\frac{\tan x}{d}\right)\mathrm{d}x.$$

Substituting $u = \tan x$ in the second integral and integrating by parts again yields

$$\int \frac{\ln\cos^2 x}{1-(1-d^2)\cos^2 x}\mathrm{d}x = \frac{1}{d}\arctan\left(\frac{\tan x}{d}\right)\ln\cos^2 x + \frac{1}{d}\arctan\left(\frac{\tan x}{d}\right)\ln(1+\tan^2 x) - \int \frac{\ln(1+u^2)}{d^2+u^2}\,\mathrm{d}u.$$

Observe that the first two terms cancel each other. Now that we have gotten rid of the divergent part, we can plug in the limits $x = 0, x = \pi/2$ and $u = 0, u = \infty$.
We calculate the last integral as follows: It can evidently be written as

$$\frac{1}{d}\int_0^\infty \frac{\ln(1+d^2u^2)}{1+u^2}\mathrm{d}u$$

Now consider the integral

$$I(f) := \int_0^\infty \frac{\ln(1+fx^2)}{1+x^2}\mathrm{d}x$$

such that

$$I'(f) = \int_0^\infty \frac{x^2}{(1+x^2)(1+fx^2)}\mathrm{d}x = \frac{1}{f-1}\int_0^\infty \left[\frac{1}{1+x^2} - \frac{1}{1+fx^2}\right]\mathrm{d}x = \frac{\pi}{2}\frac{1-1/\sqrt{f}}{f-1},$$

so

$$I(f) = \int_0^f \frac{\pi}{2}\frac{1}{\sqrt{f'}+1}\frac{1}{\sqrt{f'}}\mathrm{d}f' = \pi\ln(\sqrt{f}+1).$$

This yields the desired result:

$$\int_0^{\pi/2} \frac{\ln\cos^2 x}{1-(1-d^2)\cos^2 x}\mathrm{d}x = \frac{1}{d}\int_0^\infty \frac{\ln(1+d^2u^2)}{1+u^2}\mathrm{d}u = -\frac{\pi}{d}\ln(1+d)\,.$$

Hence we have:

$$\int_0^{\frac{\pi}{2}} \frac{\ln(\sin(x))}{a-b\sin^2(x)}\mathrm{d}x = -\frac{\pi}{4\sqrt{a(a-b)}}\ln\left(\frac{2a-b+2\sqrt{a(a-b)}}{a}\right)$$

□

Exercise 1.2.16: Prove:

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\tan^{-1}\left(\frac{11-6x}{4\sqrt{21}}\right)\mathrm{d}x = \frac{\pi^2}{6}$$

Proof: **1.**Denote $x=\sin\varphi$ and recall that $\arctan x = \frac{1}{2i}\ln\frac{1+ix}{1-ix}$. One then obtains the integral

$$\frac{1}{2i}\int_{-\pi/2}^{\pi/2}\ln\frac{4\sqrt{21}+i(11-6\sin\varphi)}{4\sqrt{21}-i(11-6\sin\varphi)}\mathrm{d}\varphi = \frac{1}{4i}\int_0^{2\pi}\ln\frac{4\sqrt{21}+i(11-6\cos\varphi)}{4\sqrt{21}-i(11-6\cos\varphi)}\mathrm{d}\varphi, \tag{1}$$

where at the last step we first used the symmetry of sine function to extend the integration to interval $[-\pi,\pi]$, and then made use of periodicity to shift the integration interval and to replace sin by cos.

2.There is a well-known integral (see, for example, here)

http://math.stackexchange.com/q/397337/73025

$$\int_0^{2\pi}\ln\left(1+r^2-2r\cos\varphi\right)\mathrm{d}\varphi = \begin{cases}0, & \text{for } |r|<1,\\ 2\pi\ln r^2, & \text{for } |r|>1.\end{cases} \tag{2}$$

Obviously, our integral above is a difference of two integrals of this type. Some care should be taken over multivaluedness of logarithms. This can be handled by saying that the arguments of $4\sqrt{21}\pm i(11-6\cos\varphi)$ belong to $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

3.Now we have

$$4\sqrt{21}\pm i(11-6\cos\varphi) = A_\pm\left(1+r_\pm^2-2r_\pm\cos\varphi\right)$$

with

$$r_\pm = \frac{11-4\sqrt{7}}{3}e^{\pm i\pi/3}, \qquad A_\pm = (11+4\sqrt{7})e^{\pm i\pi/6}.$$

Since $|r_\pm|<1$, the integral (1) reduces to

$$\frac{1}{4i}\cdot 2\pi\cdot\ln\frac{A_+}{A_-} = \frac{\pi^2}{6}$$

□

Exercise 1.2.17: For $a,b>0$,prove:

$$\int_0^\infty\left(\frac{e^{-ax}-e^{-bx}}{x^3}-(b-a)\frac{e^{-bx}}{x^2}-\frac{(b-a)^2}{2}\frac{e^{-bx}}{x}\right)\mathrm{d}x$$
$$= \frac{3}{4}a^2+\frac{1}{4}b^2-ab+\frac{a^2}{2}\ln\left(\frac{b}{a}\right)$$

Proof: Denote the integral by $I(a,b)$. By Taylor expanding the integral near zero, it is clear that the integral converges absolutely. In the following, f,g,h denote arbitrary functions

of one variable.

Differentiating with respect to b gives:

$$\partial_b I(a,b) = \int_0^\infty \frac{(b-a)^2}{2} e^{-bx} \mathrm{d}x = \frac{(b-a)^2}{2b},$$

whence, upon integrating back,

$$I(a,b) = \frac{1}{4}b^2 - ab + \frac{1}{2}a^2 \ln b + f(a).$$

Differentiating twice with respect to aa gives a well-known (**Frullani**) integral:

$$\partial_a^2 I(a,b) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \mathrm{d}x = \ln b - \ln a.$$

Effecting two elementary integrations with respect to a yields

$$I(a,b) = \frac{3}{4}a^2 + \frac{1}{2}a^2(\ln b - \ln a) + a\, g(b) + h(b).$$

It follows that

$$\partial_a I(a,b) = \int_0^\infty \frac{e^{-bx} - e^{-ax}}{x^2} - (a-b)\frac{e^{-bx}}{x} \mathrm{d}x = g(b) + a \ln \frac{b}{a} + a.$$

Setting $a = b$ here yields zero on the left-hand side and $g(b) + b$ on the right-hand side. Therefore $g(b) = b$.

Comparing the two expressions for $I(a,b)$, we deduce that

$$-ab + f(a) + \frac{a^2}{2} \ln b = a\, g(b) + \frac{3}{4}a^2 + \frac{a^2}{2} \ln \frac{b}{a} = -ab + \frac{3}{4}a^2 + \frac{a^2}{2} \ln \frac{b}{a}$$

and hence that

$$I(a,b) = \frac{a^2}{2} \ln \frac{b}{a} + \frac{3}{4}a^2 - ab + \frac{1}{4}b^2$$

□

Exercise 1.2.18: Prove:

$$\int_0^{\pi/2} \frac{x}{1+\sin x} \mathrm{d}x = \ln 2$$

Proof: Method One:

$$\begin{aligned}
\int_0^{\pi/2} \frac{x}{1+\sin x} \mathrm{d}x &= \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin^2\left(\frac{x}{2} + \frac{\pi}{4}\right)} \mathrm{d}x \\
&= \int_{\pi/4}^{\pi/2} \frac{2u - \frac{\pi}{2}}{\sin^2 u} \mathrm{d}u \\
&= -2u \cot u \Big|_{\pi/4}^{\pi/2} + 2 \int_{\pi/4}^{\pi/2} \cot u \, \mathrm{d}u - \frac{\pi}{2} \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 u} \mathrm{d}u \\
&= -2u \cot u + 2 \ln(\sin u) + \frac{\pi}{2} \cot u \Big|_{\pi/4}^{\pi/2} \\
&= 2\left(\frac{\pi}{4}\right)(1) - 2 \ln\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{2}(1) \\
&= \ln 2
\end{aligned}$$

□

Exercise 1.2.19: Prove:

$$\int_0^{\pi} \frac{x}{\left(\sin(x)+\sqrt{2}\right)^2}\,\mathrm{d}x = \frac{(\pi-2)\pi}{2\sqrt{2}}$$

Proof: Making the substitution $t = \tan\left(\frac{x}{2}\right)$,

$$\begin{aligned}\int_0^{\pi} \frac{x}{(\sin x+\sqrt{2})^2}\,\mathrm{d}x &= \frac{\pi}{2}\int_0^{\pi} \frac{1}{(\sin x+\sqrt{2})^2}\,\mathrm{d}x \\ &= \frac{\pi}{2}\int_0^{\infty} \frac{1+t^2}{(t^2+\sqrt{2}t+1)^2}\,\mathrm{d}t.\end{aligned}$$

Now let $f(z) = \dfrac{(1+z^2)\ln(-z)}{(z^2+\sqrt{2}z+1)^2}$ and integrate around a keyhole contour where the branch cut for $\ln(-z)$ is placed along the positive real axis.

Then

$$\begin{aligned}\int_0^{\infty} \frac{1+t^2}{(t^2+\sqrt{2}t+1)^2}\,\mathrm{d}t &= -\left(\operatorname{Res}\left[f(z),\frac{-1+i}{\sqrt{2}}\right]+\operatorname{Res}\left[f(z),\frac{-1-i}{\sqrt{2}}\right]\right) \\ &= \frac{(\pi-2)}{2\sqrt{2}}+\frac{(\pi-2)}{2\sqrt{2}} = \frac{(\pi-2)}{\sqrt{2}}.\end{aligned}$$

Therefore,

$$\int_0^{\pi} \frac{x}{\left(\sin(x)+\sqrt{2}\right)^2}\,\mathrm{d}x = \frac{\pi}{2}\left(\frac{(\pi-2)}{\sqrt{2}}\right) = \frac{(\pi-2)\pi}{2\sqrt{2}}$$

Method Two:

The following integral is easier to evaluate:

$$\int_0^{\pi} \frac{\mathrm{d}x}{\csc a+\sin x} = \pi\tan a - 2a\tan a$$

Differentiate both the sides by a to get:

$$\int_0^{\pi} \frac{\cot a\csc a}{(\csc a+\sin x)^2}\,\mathrm{d}x = \sec^2 a(-2a-\sin(2a)+\pi)$$

With $a = \pi/4$,

$$\int_0^{\pi} \frac{dx}{(\sin x+\sqrt{2})^2}\,\mathrm{d}x = \frac{\pi-2}{\sqrt{2}}$$

Hence,

$$\int_0^{\pi} \frac{x}{\left(\sin(x)+\sqrt{2}\right)^2}\,\mathrm{d}x = \frac{\pi}{2}\left(\frac{(\pi-2)}{\sqrt{2}}\right) = \frac{(\pi-2)\pi}{2\sqrt{2}}$$

□

Exercise 1.2.20: Prove:

$$\int_0^{\frac{\pi}{2}} e^{\cos(x)}\cos(\sin(x))\mathrm{d}x = \frac{\pi}{2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!(2k-1)} = \frac{\pi}{2}+\mathrm{Si}(1)$$

☞ **Proof:**

$$\int_0^{\pi/2} e^{\cos x}\cos(\sin x)\,\mathrm{d}x = \Re\int_0^{\pi/2} e^{e^{ix}}\,\mathrm{d}x = \Re\frac{1}{i}\int_C \frac{e^z}{z}\,\mathrm{d}z$$

where C is the portion of the unit circle in the first quadrant traversed counterclockwise. But $\dfrac{e^z}{z}$ is analytic in a simply connected domain that contains $z=1$ and $z=i$. Therefore,

$$\begin{aligned}\int_C \frac{e^z}{iz}\,\mathrm{d}z &= \Big(\mathrm{Ei}(i) - \mathrm{Ei}(1)\Big)\\ &= \Big(\mathrm{Ci}(1) + i\mathrm{Si}(1) + \frac{i\pi}{2} - \mathrm{Ei}(1)\Big).\end{aligned}$$

And

$$\begin{aligned}\int_0^{\pi/2} e^{\cos x}\cos(\sin x)\,\mathrm{d}x &= \Re\frac{1}{i}\Big(\mathrm{Ci}(1) + i\mathrm{Si}(1) + \frac{i\pi}{2} - \mathrm{Ei}(1)\Big)\\ &= \mathrm{Si}(1) + \frac{\pi}{2}\end{aligned}$$

□

Exercise 1.2.21: Evaluate:

$$\int_0^\infty \frac{x\ln(x)}{(1+x)^2(1+x^3)}\mathrm{d}x$$

✎ **Solution:** Using the expansion

$$\frac{x}{(1+x)^2(1+x^3)} = \frac{1}{3}\left[\frac{1}{1+x^3} - \frac{1}{(1+x)^3}\right],$$

it reduces to the sum of two integrals.

$$\begin{aligned}I_1 &= \int_0^\infty \frac{\ln x}{1+x^3}\mathrm{d}x = \frac{1}{9}\int_0^\infty \frac{x^{-2/3}\ln x}{1+x}\mathrm{d}x = \frac{1}{9}\partial_a|_{1/3}\int_0^\infty \frac{x^{a-1}}{1+x}\mathrm{d}x\\ &= \frac{1}{9}\partial_a|_{1/3}\frac{\pi}{\sin(\pi a)} = -\frac{2\pi^2}{27}.\end{aligned}$$

$$\begin{aligned}I_2 &= \int_0^\infty \frac{\ln x}{(1+x)^3}\mathrm{d}x = \partial_a|_1\int_0^\infty \frac{x^{a-1}}{(1+x)^3}\mathrm{d}x = \partial_a|_1\frac{\Gamma(a)\Gamma(3-a)}{\Gamma(3)}\\ &= \frac{1}{2}\left[\psi(1) - \psi(2)\right] = -\frac{1}{2},\end{aligned}$$

where I used $\psi(1) = -\gamma$ and $\psi(2) = 1-\gamma$.

Taking the linear combination yields the answer

$$\int_0^\infty \frac{x\ln(x)}{(1+x)^2(1+x^3)}\mathrm{d}x = \frac{1}{6} - \frac{2\pi^2}{81}$$

□

Exercise 1.2.22: Evaluate:

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)\mathrm{d}x$$

✎ **Solution:**

$$I = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right) \mathrm{d}x = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{-2x}{1+x^2}\right) \mathrm{d}x$$
$$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \left(\pi - \cos^{-1}\left(\frac{2x}{1+x^2}\right)\right) \mathrm{d}x$$
$$\Rightarrow I = \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \mathrm{d}x = \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(-1 + \frac{1}{2(1+x^2)} + \frac{1}{2(1-x^2)}\right) \mathrm{d}x$$
$$\Rightarrow I = -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} + \frac{\pi}{4} \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

□

Exercise 1.2.23: Evaluate:

$$\int_0^\pi \frac{\sin\theta \left[(1+2a\cos\theta+a^2)^{s/2} \sin\left(s \arctan\dfrac{\sin\theta}{a+\cos\theta}\right)\right]}{1-2b\cos\theta+b^2} \mathrm{d}\theta$$

✎ **Solution:** Assume that $f(a+z)$ has a Taylor expansion at $z=0$ and that the expansion converges absolutely for points on the unit circle.

Then

$$f(a+e^{i\theta}) - f(a+e^{-i\theta})$$
$$= f'(a)\left(e^{i\theta} - e^{-i\theta}\right) + \frac{f''(a)}{2!}\left(e^{2i\theta} - e^{-2i\theta}\right) + \frac{f'''(a)}{3!}\left(e^{3i\theta} - e^{-3i\theta}\right) + \dots$$
$$= 2i\left(f'(a)\sin\theta + \frac{f''(a)}{2!}\sin 2\theta + \frac{f'''(a)}{3!}\sin 3\theta + \dots\right).$$

And using the Fourier series $\displaystyle\sum_{n=1}^{\infty} b^n \sin n\theta = \frac{b\sin\theta}{1-2b\cos\theta+b^2}$ $(|b|<1)$,

$$\int_0^\pi \frac{f(a+e^{i\theta}) - f(a+e^{-i\theta})}{1-2b\cos\theta+b^2} \sin\theta \, \mathrm{d}\theta$$
$$= \frac{2i}{b}\left[\int_0^\pi \left(f'(a)\sin\theta + \frac{f''(a)}{2!}\sin 2\theta + \frac{f'''(a)}{3!}\sin 3\theta + \dots\right)\right.$$
$$\left. \cdot \left(b\sin\theta + b^2\sin 2\theta + b^3\sin 3\theta + \dots\right) \mathrm{d}\theta\right]$$
$$= \frac{2i}{b}\left(f'(a)b\frac{\pi}{2} + \frac{f''(a)}{2!}b^2\frac{\pi}{2} + \frac{f'''(a)}{3!}b^3\frac{\pi}{2} + \dots\right)$$
$$= \frac{i\pi}{b}\Big(f(a+b) - f(a)\Big).$$

Now let $f(z) = z^s$.

Then if $a > 1$,

$$\int_0^\pi \frac{(a+e^{i\theta})^s - (a+e^{-i\theta})^s}{1-2b\cos\theta+b^2} \sin\theta \, \mathrm{d}\theta = \frac{i\pi}{b}\Big((a+b)^s - a^s\Big).$$

But

$$(a+e^{i\theta})^s-(a+e^{-i\theta})^s=\left(\left((a+\cos\theta)^2+\sin^2\theta\right)^{s/2}\right.$$
$$\left.\cdot\left[\exp\left(is\arctan\frac{\sin\theta}{a+\cos\theta}\right)-\exp\left(-is\arctan\frac{\sin\theta}{a+\cos\theta}\right)\right]\right)$$
$$=2i(1+2a\cos\theta+a^2)^{s/2}\sin\left(s\arctan\frac{\sin\theta}{a+\cos\theta}\right).$$

And the result is

$$\int_0^\pi\frac{\sin\theta\left[(1+2a\cos\theta+a^2)^{s/2}\sin\left(s\arctan\frac{\sin\theta}{a+\cos\theta}\right)\right]}{1-2b\cos\theta+b^2}\,\mathrm{d}\theta=\frac{\pi}{2b}\Big((a+b)^s-a^s\Big)$$

If you let $f(z)=\sin z$, you get

$$\int_0^\pi\frac{\sinh(\sin\theta)\cos(a+\cos\theta)\sin\theta}{1-2b\cos\theta+b^2}\mathrm{d}\theta=\frac{\pi}{2b}\Big(\sin(a+b)-\sin(a)\Big)$$

where there is still the restriction on b but no restriction on a.

And if you let $f(z)=e^z$, you get

$$\int_0^\pi\frac{e^{\cos\theta}\sin(\sin\theta)\sin\theta}{1-2b\cos\theta+b^2}\mathrm{d}\theta=\frac{\pi}{2b}\left(e^b-1\right)$$

In a similar manner, and under the same restrictions, one can derive

$$\int_0^\pi\frac{\sin\theta\left[(1+2a\cos\theta+a^2)^{s/2}\sin\left(s\arctan\frac{\sin\theta}{a+\cos\theta}\right)\right]}{(1-2b\cos\theta+b^2)^2}\,\mathrm{d}\theta=\frac{\pi s}{2(1-b^2)}(a+b)^{s-1}$$

Differentiating the identity

$$1+2\sum_{n=1}^\infty b^n\cos n\theta=\frac{1-b^2}{1-2b\cos\theta+b^2}\ ,\ (|b|<1)$$

you get

$$\sum_{n=1}^\infty b^{n-1}n\sin n\theta=\frac{(1-b^2)\sin\theta}{(1-2b\cos\theta+b^2)^2}.$$

Then under the same assumptions as the other integral,

$$\int_0^\pi\frac{f(a+e^{i\theta})-f(a+e^{-i\theta})}{(1-2b\cos\theta+b^2)^2}\sin\theta\ \mathrm{d}\theta$$
$$=\frac{2i}{1-b^2}\left[\int_0^\pi\left(f'(a)\sin\theta+\frac{f''(a)}{2!}\sin2\theta+\frac{f'''(a)}{3!}\sin3\theta+\ldots\right)\right.$$
$$\left.\cdot\left(\sin\theta+2b\sin2\theta+3b^2\sin3\theta+\ldots\right)\ \mathrm{d}\theta\right]$$
$$=\frac{2i}{1-b^2}\left(f'(a)\frac{\pi}{2}+\frac{f''(a)}{2!}2b\frac{\pi}{2}+\frac{f'''(a)}{3!}3b^2\frac{\pi}{2}+\ldots\right)$$
$$=\frac{i\pi}{1-b^2}f'(a+b).$$

And again letting $f(z)=z^s$ where $a>1$ and $s>0$,

$$2i\int_0^\pi \frac{\sin\theta\left[(1+2a\cos\theta+a^2)^{s/2}\sin\left(s\arctan\dfrac{\sin\theta}{a+\cos\theta}\right)\right]}{(1-2b\cos\theta+b^2)^2}\,\mathrm{d}\theta=\frac{i\pi}{(1-b^2)}s(a+b)^{s-1}.$$

And the result follows.

Letting $f(z)=\sin z$, you get

$$\int_0^\pi \frac{\sinh(\sin\theta)\cos(a+\cos\theta)\sin\theta}{(1-2b\cos\theta+b^2)^2}\mathrm{d}\theta=\frac{\pi}{2(1-b^2)}\cos(a+b)$$

While letting $f(z)=e^z$, you get

$$\int_0^\pi \frac{e^{\cos\theta}\sin(\sin\theta)\sin\theta}{(1-2b\cos\theta+b^2)^2}\,\mathrm{d}\theta=\frac{\pi}{2(1-b^2)}e^b$$

In addition,two other formulas which I didn't mention are

$$\boxed{\begin{aligned}&\int_0^\pi \frac{f(a+e^{i\theta})+f(a+e^{-i\theta})}{1-2b\cos\theta+b^2}\,\mathrm{d}\theta=\frac{2\pi}{1-b^2}f(a+b)\\[2ex]&\int_0^\pi \frac{\Big(f(a+e^{i\theta})+f(a+e^{-i\theta})\Big)(1-b\cos\theta)}{1-2b\cos\theta+b^2}\,\mathrm{d}\theta=\pi\Big(f(a)+f(a+b)\Big)\end{aligned}}$$

□

Exercise 1.2.24: Prove:

$$\int_0^\infty \frac{e^{-zx}}{1-e^{-x}}\frac{\cos ax-\cos bx}{x}\,\mathrm{d}x=\ln\left|\frac{\Gamma(z+ia)}{\Gamma(z+ib)}\right|,\ z>0$$

Proof:

$$\begin{aligned}&\int_0^\infty \frac{e^{-zx}}{1-e^{-x}}\frac{\cos ax-\cos bx}{x}\,\mathrm{d}x=\int_0^\infty \frac{\cos ax-\cos bx}{x}\sum_{k=0}^\infty e^{-(z+k)x}\,\mathrm{d}x\\
&=\sum_{k=0}^\infty\int_0^\infty \frac{\cos ax-\cos bx}{x}e^{-(z+k)x}\,\mathrm{d}x=\sum_{k=0}^\infty\int_0^\infty\int_0^\infty(\cos ax-\cos bx)\,e^{-(z+k)x}e^{-xt}\mathrm{d}t\,\mathrm{d}x\\
&=\sum_{k=0}^\infty\int_0^\infty\int_0^\infty(\cos ax-\cos bx)\,e^{-(z+k+t)x}\,\mathrm{d}x\,\mathrm{d}t\\
&=\sum_{k=0}^\infty\int_0^\infty\left(\frac{z+k+t}{(z+k+t)^2+a^2}-\frac{z+k+t}{(z+k+t)^2+b^2}\right)\mathrm{d}t\\
&=\frac{1}{2}\sum_{k=0}^\infty\ln\left(\frac{(z+k)^2+b^2}{(z+k)^2+a^2}\right)=\ln\prod_{k=0}^\infty\frac{\left(1+\dfrac{b^2}{(z+k)^2}\right)^{1/2}}{\left(1+\dfrac{a^2}{(z+k)^2}\right)^{1/2}}\end{aligned}$$

Euler's infinite product representation of the gamma function is

$$\Gamma(x)=\lim_{n\to\infty}n!n^x\prod_{k=0}^n\frac{1}{x+k}.$$

So

$$\frac{\Gamma(x)}{\Gamma(x+y)} = \lim_{n\to\infty} \frac{1}{n^y} \prod_{k=0}^{n} \left(\frac{x+y+k}{x+k}\right) = \lim_{n\to\infty} \frac{1}{n^y} \prod_{k=0}^{n} \left(1+\frac{y}{x+k}\right),$$

which implies

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right| = \lim_{n\to\infty} \prod_{k=0}^{n} \left|1+\frac{iy}{x+k}\right| = \prod_{k=0}^{\infty} \left(1+\frac{y^2}{(x+k)^2}\right)^{1/2}.$$

Therefore,

$$\int_0^\infty \frac{e^{-zx}}{1-e^{-x}} \frac{\cos ax - \cos bx}{x}\, \mathrm{d}x = \ln\left|\frac{\Gamma(z)}{\Gamma(z+ib)}\frac{\Gamma(z+ia)}{\Gamma(z)}\right| = \ln\left|\frac{\Gamma(z+ia)}{\Gamma(z+ib)}\right|$$

For the particular case $z = 1$, this simplifies to

$$\int_0^\infty \frac{e^{-x}}{1-e^{-x}} \frac{\cos ax - \cos bx}{x}\, \mathrm{d}x = \frac{1}{2} \ln\left(\frac{a \sinh \pi b}{b \sinh \pi a}\right)$$

□

Exercise 1.2.25: For $|a| < 1$,prove:

$$\int_0^\pi \frac{\sin 2nx}{\sin x} \frac{1}{1-2a\cos x + a^2}\, \mathrm{d}x = \frac{2\pi a}{1-a^2}\frac{1-a^{2n}}{1-a^2}$$
$$\int_0^\pi \frac{\sin 2nx}{\sin x} \frac{\cos x}{1-2a\cos x + a^2}\, \mathrm{d}x = \pi\,\frac{1+a^2}{1-a^2}\frac{1-a^{2n}}{1-a^2}$$
$$\int_0^\pi \frac{\sin 2nx}{\sin x} \ln\left(1-2a\cos x + a^2\right)\, \mathrm{d}x = -2\pi \sum_{k=1}^{n} \frac{a^{2k-1}}{2k-1}$$

Proof: (a) Consider I_n as the given integral, then

$$\begin{aligned} I_n - I_{n-1} &= 2\int_0^\pi \frac{\cos((2n-1)x)}{a^2-2a\cos x+1}\,\mathrm{d}x \\ &= \frac{2}{1-a^2}\int_0^\pi \cos((2n-1)x)\left(1+2\sum_{k=1}^{\infty} a^k\cos(kx)\right)\mathrm{d}x \\ \Rightarrow I_n - I_{n-1} &= \frac{4}{1-a^2}\sum_{k=1}^{\infty} a^k \int_0^\pi \cos((2n-1)x)\cos(kx)\,\mathrm{d}x \\ &= \frac{4}{1-a^2}a^{2n-1}\int_0^\pi \cos^2((2n-1)x)\,\mathrm{d}x = \frac{2\pi}{1-a^2}a^{2n-1} \end{aligned}$$

Hence,

$$I_n = \frac{2\pi}{1-a^2}\sum_{r=1}^{n} a^{2r-1} = \frac{2\pi a}{1-a^2}\frac{1-a^{2n}}{1-a^2}$$

(b) Proceeding in a similar manner as before,

$$\begin{aligned} I_n - I_{n-1} &= \frac{4}{1-a^2}\sum_{k=1}^{\infty} a^k \int_0^\pi \cos((2n-1)x)\cos x\cos(kx)\,\mathrm{d}x \\ &= \frac{2}{1-a^2}\sum_{k=1}^{\infty} a^k \int_0^\pi \cos(kx)\left(\cos(2nx)+\cos((2n-2)x)\right)\mathrm{d}x \\ \Rightarrow I_n - I_{n-1} &= \frac{\pi}{1-a^2}\left(a^{2n}+a^{2n-2}\right) = \pi\frac{1+a^2}{1-a^2}a^{2n-2} \end{aligned}$$

Hence,

$$I_n = \pi \frac{1+a^2}{1-a^2} \sum_{r=1}^{n} a^{2r-2} = \pi \frac{1+a^2}{1-a^2} \frac{1-a^{2n}}{1-a^2}$$

(c) Consider

$$J(a) = \int_0^\pi \frac{\sin(2nx)}{\sin x} \ln(a^2 - 2a\cos x + 1)\,\mathrm{d}x$$
$$\Rightarrow J'(a) = 2\int_0^\pi \frac{\sin(2nx)}{\sin x} \frac{a - \cos x}{a^2 - 2a\cos x + 1}\,\mathrm{d}x$$

Set

$$I_n = 2\int_0^\pi \frac{\sin(2nx)}{\sin x} \frac{a - \cos x}{a^2 - 2a\cos x + 1}\,\mathrm{d}x$$

$$\Rightarrow I_n - I_{n-1} = 4\int_0^\pi \cos((2n-1)x) \frac{a - \cos x}{a^2 - 2a\cos x + 1}\,\mathrm{d}x$$
$$= -4\sum_{k=0}^{\infty} a^k \int_0^\pi \cos((2n-1)x)\cos((k+1)x)\,\mathrm{d}x$$
$$\Rightarrow I_n - I_{n-1} = -2\pi a^{2n-2} \Rightarrow J'(a) = I_n = -2\pi \sum_{k=1}^{n} a^{2k-2}$$

Hence,

$$J(a) = -2\pi \sum_{k=1}^{n} \frac{a^{2k-1}}{2k-1}$$

The constant of integration is zero since $J(0) = 0$. □

Exercise 1.2.26: For $|a| < 1$,prove:

$$\int_0^\pi \frac{1}{(1 - 2a\cos x + a^2)^3}\,\mathrm{d}x = \frac{\pi}{(1-a^2)^5}\left(1 + 4a^2 + a^4\right)$$

and

$$\int_0^\pi \frac{1}{(1 - 2a\cos x + a^2)^4}\,\mathrm{d}x = \frac{\pi}{(1-a^2)^7}\left(1 + 9a^2 + 9a^4 + a^6\right)$$

In general,

$$\int_0^\pi \frac{1}{(1 - 2a\cos x + a^2)^n}\,\mathrm{d}x = \frac{\pi}{(1-a^2)^{2n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 a^{2k}$$

Proof: Let $I(n) = \int_0^\pi \frac{1}{(1 - 2a\cos x + a^2)^n}\,\mathrm{d}x$.

Then

$$I(1) = \int_0^\pi \frac{1}{1 - 2a\cos x + a^2}\,\mathrm{d}x = \frac{1}{1-a^2}\int_0^\pi \left(1 + 2\sum_{n=1}^{\infty} a^n \cos(nx)\right)\,\mathrm{d}x$$
$$= \frac{1}{1-a^2}\int_0^\pi \mathrm{d}x + \frac{2a^n}{1-a^2}\sum_{n=1}^{\infty}\int_0^\pi \cos(nx)\,\mathrm{d}x$$
$$= \frac{\pi}{1-a^2} + 0$$
$$= \frac{\pi}{1-a^2}.$$

And differentiating $I(n)$ with respect to a,

$$\begin{aligned}\frac{\mathrm{d}I(n)}{\mathrm{d}a} &= n\int_0^\pi \frac{2\cos x - 2a}{(1-2a\cos x+a^2)^{n+1}}\,\mathrm{d}x \\ &= \frac{n}{a}\int_0^\pi \frac{1-a^2-(1-2a\cos x+a^2)}{(1-2a\cos x+a^2)^{n+1}}\,\mathrm{d}x \\ &= \frac{n(1-a^2)}{a}I(n+1) - \frac{n}{a}I(n),\end{aligned}$$

which implies

$$\begin{aligned}I(n+1) &= \frac{1}{1-a^2}\left(I(n)+\frac{a}{n}\frac{\mathrm{d}I(n)}{\mathrm{d}a}\right) \\ &= \frac{1}{1-a^2}\left(I(n)+a^n\frac{\mathrm{d}I(n)}{\mathrm{d}a^n}\right) \\ &= \frac{1}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a^n}a^nI(n).\end{aligned}$$

So

$$\begin{aligned}I(2) &= \frac{1}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a}aI(1) = \frac{\pi}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a}\frac{a}{1-a^2} \\ &= \frac{\pi}{1-a^2}\frac{(1-a^2)-a(-2a)}{(1-a^2)^2} \\ &= \frac{\pi}{(1-a^2)^3}(1+a^2)\end{aligned}$$

$$\begin{aligned}I(3) &= \frac{1}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a^2}a^2I(2) = \frac{\pi}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a^2}\frac{a^2+a^4}{(1-a^2)^3} \\ &= \frac{\pi}{1-a^2}\frac{(1+4a^3\frac{1}{2a})(1-a^2)^3-(a^2+a^4)3(1-a^2)^2(-1)}{(1-a^2)^6} \\ &= \frac{\pi}{(1-a^2)^5}\left((1+2a^2)(1-a^2)+3(a^2+a^4)\right) \\ &= \frac{\pi}{(1-a^2)^5}\left(1+4a^2+a^4\right)\end{aligned}$$

$$\begin{aligned}I(4) &= \frac{1}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a^3}a^3I(3) = \frac{\pi}{1-a^2}\frac{\mathrm{d}}{\mathrm{d}a^3}\frac{a^3+4a^5+a^7}{(1-a^2)^5} \\ &= \frac{\pi}{1-a^2}\frac{(3a^2+20a^4+7a^6)\frac{1}{3a^2}(1-a^2)^5-(a^3+4a^5+a^7)5(1-a^2)^4(-2a)\frac{1}{3a^2}}{(1-a^2)^{10}} \\ &= \frac{\pi}{(1-a^2)^7}\frac{1}{3}\left((3+20a^2+7a^4)(1-a^2)+10(a^2+4a^4+a^6)\right) \\ &= \frac{\pi}{(1-a^2)^7}\left(1+9a^2+9a^4+a^6\right)\end{aligned}$$

And so on. □

Exercise 1.2.27: Prove:

$$\int_0^\pi e^{a\cos x}\sin(a\sin x)\csc x\,\mathrm{d}x = \pi\sinh(a)$$

and

$$\int_0^{\pi} e^{a\cos x}\sin(a\sin x)\cot x\ \mathrm{d}x=\pi\left(\cosh(a)-1\right)$$

☞ **Proof: Method One:**Using contour integration, one may obtain

$$\begin{aligned}\int_0^{\pi} e^{a\cos x}\sin(a\sin x)\csc x\ \mathrm{d}x=&\Im\int_{-\pi}^{\pi}\frac{ie^{ae^{ix}}}{e^{ix}-e^{-ix}}\mathrm{d}x\\=&\Im\oint_{|z|=1}\frac{e^{az}}{z^2-1}\mathrm{d}z\\=&\Im\ i\pi\Big[\mathrm{Res}(f,1)+\mathrm{Res}(f,-1)\Big]\\=&\Im\ i\pi\left(\frac{e^a-e^{-a}}{2}\right)\\=&\pi\sinh a\end{aligned}$$

and

$$\begin{aligned}\int_0^{\pi} e^{a\cos x}\sin(a\sin x)\cot x\ \mathrm{d}x=&\frac{1}{2}\Im\int_{-\pi}^{\pi}\frac{ie^{ae^{ix}}(e^{ix}+e^{-ix})}{e^{ix}-e^{-ix}}\mathrm{d}x\\=&\frac{1}{2}\Im\oint_{|z|=1}\frac{e^{az}(z^2+1)}{z^3-z}\mathrm{d}z\\=&\Im\ i\pi\left[\frac{1}{2}\mathrm{Res}(f,1)+\mathrm{Res}(f,0)+\frac{1}{2}\mathrm{Res}(f,-1)\right]\\=&\Im\ i\pi\left(\frac{e^a+e^{-a}}{2}-1\right)\\=&\pi\left(\cosh a-1\right)\end{aligned}$$

Method Two: Using a viable real method $\displaystyle\sum_{k=0}^{\infty}\frac{\sin(xk)}{k!}=e^{\cos x}\sin(\sin x)$. Expanding $\Im e^{ae^{ix}}$,

$$\int_0^{\pi} e^{a\cos x}\sin(a\sin x)\csc x\ \mathrm{d}x=\sum_{n=1}^{\infty}\frac{a^n}{n!}\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x$$

If we denote

$$\mathcal{I}_n=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x$$

then

$$\mathcal{I}_{2n+2}-\mathcal{I}_{2n}=\int_{-\pi}^{\pi}\frac{\cos\left((2n+1)x\right)\cos x}{\sin x}\mathrm{d}x=0$$

and since

$$\mathcal{I}_1=\pi,\qquad \mathcal{I}_2=0$$

we have

$$\mathcal{I}_n=\begin{cases}\pi, & \text{n odd}\\ 0, & \text{n even}\end{cases}$$

So the integral is

$$\int_0^{\pi} e^{a\cos x}\sin(a\sin x)\csc x\ \mathrm{d}x=\sum_{n=0}^{\infty}\frac{\pi a^{2n+1}}{(2n+1)!}=\pi\sinh a$$

Similarly,

$$\int_0^\pi e^{a\cos x}\sin(a\sin x)\cot x\ \mathrm{d}x = \sum_{n=1}^\infty \frac{a^n}{n!}\int_0^\pi \frac{\sin(nx)\cos x}{\sin x}\mathrm{d}x$$

Denote

$$\mathcal{J}_n = \frac{1}{2}\int_{-\pi}^{\pi}\frac{\sin(nx)\cos x}{\sin x}\mathrm{d}x$$

and the relation $\mathcal{J}_{2n+2} - \mathcal{J}_{2n} = 0$ also applies here. Since

$$\mathcal{J}_1 = 0, \qquad \mathcal{J}_2 = \pi$$

then

$$\mathcal{J}_n = \begin{cases}\pi, & \text{n even}\\ 0, & \text{n odd}\end{cases}$$

So

$$\int_0^\pi e^{a\cos x}\sin(a\sin x)\cot x\ \mathrm{d}x = \sum_{n=1}^\infty \frac{\pi a^{2n}}{(2n)!} = \pi\left(\cosh a - 1\right)$$

□

Exercise 1.2.28: Evaluate:

$$\int_0^{\frac{\pi}{2}}\Big[\tan x\arctan\left(5\tan(2\tan x)\right) - \tan x\arctan\left(3\tan(2\tan x)\right)\Big]\mathrm{d}x$$

Solution: Let $\mathcal{I}$ denote the integral in question.

First, one may note that

$$\int_0^\infty \frac{x\sin(ax)}{x^2+b^2}\mathrm{d}x = \operatorname*{Res}_{z=ib}\frac{\pi z e^{iaz}}{z^2+b^2} = \frac{\pi}{2}e^{-ab}$$

Applying the substitution $2\tan x \mapsto x$, we see that $\mathcal{I}$ is equal to

$$\begin{aligned}\mathcal{I} &= \int_3^5\int_0^\infty \frac{x}{x^2+4}\frac{\tan x}{1+\mu^2\tan^2 x}\mathrm{d}x\ \mathrm{d}\mu\\ &= \int_3^5\int_0^\infty \frac{x}{x^2+16}\frac{\sin x}{1+\mu^2+(1-\mu^2)\cos x}\mathrm{d}x\ \mathrm{d}\mu\end{aligned}$$

Observe that

$$\sum_{n=1}^\infty a^n\sin(nx) = \Im\sum_{n=1}^\infty (ae^{ix})^n = \Im\frac{ae^{ix}}{1-ae^{ix}} = \frac{a\sin x}{1-2a\cos x+a^2} \quad (\text{for } |a|<1)$$

Letting $a \mapsto \dfrac{b}{a}$,

$$\frac{ab\sin x}{a^2-2ab\cos x+b^2} = \sum_{n=1}^\infty\left(\frac{b}{a}\right)^n\sin(nx) \quad (\text{for } |b|<a)$$

Setting $a = \dfrac{\mu+1}{\sqrt{2}}$ and $b = \dfrac{\mu-1}{\sqrt{2}}$, then denoting $\xi = \dfrac{\mu-1}{\mu+1}$, we get

$$\frac{\sin x}{1+\mu^2+(1-\mu^2)\cos x} = \frac{2}{\mu^2-1}\sum_{n=1}^\infty \xi^n\sin(nx)$$

Integrating termwise,

$$\begin{aligned}\mathcal{I} &= \int_3^5 \frac{2}{\mu^2-1}\sum_{n=1}^{\infty}\xi^n\int_0^{\infty}\frac{x\sin(nx)}{x^2+16}\mathrm{d}x\,\mathrm{d}\mu\\ &= \int_3^5 \frac{\pi}{\mu^2-1}\frac{e^{-4}\xi}{1-e^{-4}\xi}\mathrm{d}\mu = \frac{\pi}{2}\int_{\frac{1}{2}}^{\frac{2}{3}}\frac{\mathrm{d}\xi}{e^4-\xi}\\ &= \left[-\frac{\pi}{2}\ln|e^4-\xi|\right]_{\frac{1}{2}}^{\frac{2}{3}} = \frac{\pi}{2}\ln\left(1+\frac{1}{6e^4-4}\right)\end{aligned}$$

□

Exercise 1.2.29: Prove:

$$\int_0^{\pi}\frac{x\sin x\ln\left(\tan\frac{x}{2}\right)}{3+\cos(2x)}\mathrm{d}x = -\frac{\pi\mathbf{G}}{4}+\frac{\pi^2}{16}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Proof: Method One:

The Weierstrass substitution reduces it to

$$\int_0^{\infty}\frac{2x\ln x\arctan x}{1+x^4}\mathrm{d}x$$

Then,we have the formula below

$$\begin{aligned}I(b) &= \int_0^{\infty}\frac{x^b\ln x\arctan x}{1+x^4}\mathrm{d}x\\ &= \frac{\pi}{4}\left[\frac{\beta'\left(\frac{3-b}{4}\right)}{8\sin\left(\frac{\pi b}{2}\right)}+\frac{\pi\beta\left(\frac{3-b}{4}\right)\cos\left(\frac{\pi b}{2}\right)}{4\sin^2\left(\frac{\pi b}{2}\right)}\right.\\ &\quad\left.+\frac{1}{16}\pi\left(\frac{\beta\left(\frac{1}{4}\right)\sin\left(\frac{\pi b}{4}\right)}{\cos^2\left(\frac{\pi b}{4}\right)}-\frac{\beta\left(\frac{3}{4}\right)\cos\left(\frac{\pi b}{4}\right)}{\sin^2\left(\frac{\pi b}{4}\right)}\right)\right]\end{aligned}$$

Here

$$\beta(x) := \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right]$$

is the **incomplete beta function**.

I will now prove it. Consider the integral

$$K(a) := \int_0^{\infty}\frac{x^b\ln(1+ax)}{1+x^4}\mathrm{d}x$$

Here we temporarily consider b to be fixed.

We have $K(0)=0$, and

$$\begin{aligned}K' &= \int_0^{\infty}\frac{x^{b+1}}{(1+ax)(1+x^4)}\mathrm{d}x\\ &= \frac{1}{1+a^4}\int_0^{\infty}x^{b+1}\left[\frac{a^4}{1+ax}+\frac{1-ax+a^2x^2-a^3x^3}{1+x^4}\right]\mathrm{d}x\\ &= \frac{1}{1+a^4}\left[a^{2-b}\frac{\pi}{\sin(\pi(b+2))}+\frac{\pi}{4}\left(\frac{1}{\sin(\pi(b+2)/4)}-\frac{a}{\sin(\pi(b+3)/4)}\right.\right.\\ &\quad\left.\left.+\frac{a^2}{\sin(\pi(b+4)/4)}-\frac{a^3}{\sin(\pi(b+5)/4)}\right)\right]\end{aligned}$$

Now let us calculate

$$K(i) = \int_0^i K'(a)\mathrm{d}a.$$

In calculating this integral, we encounter five integrals of the form

$$\int_0^i \frac{a^{\xi-1}}{1+a^4}\mathrm{d}a$$

where ξ is some exponent. Making a change of variable turns this into

$$\xi^i \int_0^1 \frac{t^{\xi-1}}{1+t^4}\mathrm{d}t$$

Two of the integrals can be straightforwardly evaluated. The other three can be expressed in terms of the β function using the property

$$\int_0^1 \frac{t^{\xi-1}}{1+t^\eta}\mathrm{d}t = \frac{1}{\eta}\beta\left(\frac{\xi}{\eta}\right).$$

The result is

$$K(i) = \frac{\pi}{\sin(\pi b)} i^{3-b}\frac{1}{4}\beta\left(\frac{3-b}{4}\right) + \frac{\pi}{4}\left[\frac{i\beta(1/4)}{4\cos(\pi b/4)} + \frac{\pi/2}{4\cos(\pi(b+1)/4)} + \frac{i\beta(3/4)}{4\sin(\pi b/4)} + \frac{\ln 2}{4\sin(\pi(b+1)/4)}\right].$$

The imaginary part gives, after simplifying the first term a bit,

$$\begin{aligned}\int_0^\infty \frac{x^b \arctan x}{1+x^4}\mathrm{d}x &= \Im\int_0^\infty \frac{x^b \ln(1+ix)}{1+x^4}\mathrm{d}x \\ &= \frac{\pi}{4}\left[-\frac{\beta\left(\frac{3-b}{4}\right)}{2\sin\left(\frac{\pi b}{2}\right)} + \frac{\beta\left(\frac{3}{4}\right)}{4\sin\left(\frac{\pi b}{4}\right)} + \frac{\beta\left(\frac{1}{4}\right)}{4\cos\left(\frac{\pi b}{4}\right)}\right].\end{aligned}$$

Taking the derivative with respect to b yields the desired result.

Now in order to get the value for integral, it remains to multiply by 2 and set $b = 1$. This gives

$$I = \frac{\pi}{16}\beta'\left(\frac{1}{2}\right) + \frac{\pi^2\sqrt{2}}{32}\left[\beta\left(\frac{1}{4}\right) - \beta\left(\frac{3}{4}\right)\right].$$

Here

$$\beta'\left(\frac{1}{2}\right) = \frac{1}{4}\left[\psi'\left(\frac{3}{4}\right) - \psi'\left(\frac{1}{4}\right)\right] = -4\mathbf{G}$$

and

$$\begin{aligned}\beta\left(\frac{1}{4}\right) - \beta\left(\frac{3}{4}\right) &= \frac{1}{2}\left[\psi\left(\frac{5}{8}\right) - \psi\left(\frac{1}{8}\right) - \psi\left(\frac{7}{8}\right) + \psi\left(\frac{3}{8}\right)\right] \\ &= \sqrt{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right),\end{aligned}$$

which follows from the properties

$$\psi(1-x) = \psi(x) + \pi\cot\pi x,$$

$$\psi\left(\frac{5}{8}\right) - \psi\left(\frac{1}{8}\right) = \sqrt{2}\left[\pi + 2\coth^{-1}\sqrt{2}\right] = \sqrt{2}\left[\pi + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right].$$

Hence we have

$$\int_0^{\pi} \frac{x\sin x \ln\left(\tan\frac{x}{2}\right)}{3+\cos(2x)}\mathrm{d}x = -\frac{\pi\mathbf{G}}{4}+\frac{\pi^2}{16}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Method Two:

Integrated $f(z)=\dfrac{z\ln z\arctan z}{1+z^4}$ along a semicircle in the UHP deformed around the branch point $z=i$, and with an indent around the branch point $z=0$. The integral over all the arcs disappear.Then letting $\mathscr{I}$ denote the integral,

$$\begin{aligned}
&\Re\oint_\Gamma f(z)\,\mathrm{d}z\\
=&\int_0^\infty \frac{x\ln x\arctan x}{1+x^4}\,\mathrm{d}x+\Re\int_{-\infty}^0 \frac{x(\ln|x|+i\pi)\arctan x}{1+x^4}\mathrm{d}x+\Re\, 2\pi i\int_1^\infty \frac{iy\ln(iy)}{1+y^4}\frac{i}{2}i\,\mathrm{d}y\\
=&\mathscr{I}+\underbrace{\pi\int_1^\infty \frac{y\ln y}{1+y^4}\mathrm{d}y}_{\text{sub } y\mapsto\sqrt{y}\text{ then } y\mapsto y^{-1}}=\mathscr{I}+\frac{\pi\mathbf{G}}{4}
\end{aligned}$$

The contour integral is also equal to $2\pi i$ times the sum of residues.

$$\begin{aligned}
\Re\oint_\Gamma f(z)\,\mathrm{d}z=&\Re\, 2\pi i\left(\mathrm{Res}(f,e^{i\pi/4})+\mathrm{Res}(f,e^{i3\pi/4})\right)\\
=&\Re\, 2\pi i\left(\frac{e^{i\pi/4}}{4e^{i3\pi/4}}\frac{i\pi}{4}\arctan e^{i\pi/4}+\frac{e^{i3\pi/4}}{4e^{i9\pi/4}}\frac{i3\pi}{4}\arctan e^{i3\pi/4}\right)\\
=&\Re\, 2\pi i\left(\frac{\pi}{16}\arctan e^{i\pi/4}-\frac{3\pi}{16}\arctan e^{i3\pi/4}\right)\\
=&-\frac{\pi^2}{16}\ln(3-2\sqrt{2})
\end{aligned}$$

It follows that

$$\mathscr{I}=-\frac{\pi\mathbf{G}}{4}-\frac{\pi^2}{16}\ln(3-2\sqrt{2})$$

□

Exercise 1.2.30: Evaluate:

$$\int_0^\infty \frac{x\ln(1+x^2)}{1+x^4}\,\mathrm{d}x$$

Solution:

$$\begin{aligned}
\int_0^\infty \frac{x\ln(1+x^2)}{1+x^4}\,\mathrm{d}x&=\int_0^1 \frac{x\ln(1+x^2)}{1+x^4}\,\mathrm{d}x+\int_1^\infty \frac{x\ln(1+x^2)}{1+x^4}\,\mathrm{d}x\\
&=\int_0^1 \frac{x\ln(1+x^2)}{1+x^4}\,\mathrm{d}x+\int_0^1 \frac{u(\ln(1+u^2)-2\ln u)}{1+u^4}\,\mathrm{d}u\\
&=2\int_0^1 \frac{x\ln(1+x^2)}{1+x^4}\,\mathrm{d}x-2\int_0^1 \frac{u\ln u}{1+u^4}\,\mathrm{d}u\\
&=\int_0^1 \frac{\ln(1+w)}{1+w^2}\,\mathrm{d}w-2\int_0^1 \frac{u\ln u}{1+u^4}\,\mathrm{d}u\\
&=\frac{\pi}{8}\ln 2-2\left(-\frac{\mathbf{G}}{4}\right)\\
&=\frac{\pi}{8}\ln 2+\frac{\mathbf{G}}{2}
\end{aligned}$$

□

Exercise 1.2.31: Evaluate:

$$\int_0^\infty \frac{\sqrt{t}}{1+t}\frac{\ln^2(5t+1)}{5t+1}\mathrm{d}t$$

Solution: Starting from

$$\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x = \mathrm{B}(a,b),$$

substitute

$$x = \left[1+\frac{p}{p+1}\frac{1}{t}\right]^{-1}$$

to get

$$\int_0^\infty t^{a-1}\left((p+1)\,t+p\right)^{-a-b}\mathrm{d}t = \frac{\mathrm{B}(a,b)}{p^b(1+p)^a}.$$

Integrating once with respect to p gives:

$$\int_0^\infty \frac{t^{a-1}}{t+1}\left((p+1)\,t+p\right)^{1-a-b}\mathrm{d}t = (1-a-b)\mathrm{B}(a,b)J + \frac{\pi}{\sin\pi(1-b)},$$

where

$$J = \int_0^p \frac{1}{q^b(1+q)^a}\mathrm{d}q.$$

Differentiating with respect to b gives

$$\begin{aligned}&\int_0^\infty \frac{t^{a-1}}{t+1}\frac{\ln\left[(p+1)\,t+p\right]}{\left[(p+1)\,t+p\right]^{a+b-1}}\mathrm{d}t\\ =&\mathrm{B}(a,b)\left[J+(a+b-1)\left(\psi(b)-\psi(a+b)\right)J+(a+b-1)\partial_b J\right]-\frac{\pi^2\cos\pi(1-b)}{\sin^2\pi(1-b)}.\end{aligned}$$

From now on we take $a = 3/2$. Also we will set b equal to $1/2$ after differentiation. This gives

$$\int_0^\infty \frac{t^{1/2}}{t+1}\frac{\ln\left[(p+1)\,t+p\right]}{(p+1)\,t+p}\mathrm{d}t = \frac{1}{2}\pi\left[\partial_b J - J\ln 4\right].$$

and (after another differentiation)

$$\begin{aligned}&\int_0^\infty \frac{t^{1/2}}{t+1}\frac{\ln^2\left[(p+1)\,t+p\right]}{(p+1)\,t+p}\mathrm{d}t\\ =&\frac{1}{6}\pi\left[6\ln(4)\partial_b J-\left(\pi^2+3\ln^2(4)\right)J+6\pi^2\right]-\frac{1}{2}\pi\partial_b^2 J.\end{aligned}$$

We now need to find the integrals J, $\partial_b J$ and $\partial_b^2 J$. The first two are elementary.

$$\begin{aligned}J &= \int_0^p \frac{1}{q^{1/2}(1+q)^{3/2}}\mathrm{d}q = 2\sqrt{\frac{p}{1+p}}.\\ \partial_b J &= \int_0^p \frac{-\ln q}{q^{1/2}(1+q)^{3/2}}\mathrm{d}q = -2\sqrt{\frac{p}{1+p}}\ln p + 4\sqrt{p}.\\ \partial_b^2 J &= \int_0^p \frac{\ln^2 q}{q^{1/2}(1+q)^{3/2}}\mathrm{d}q = 2\sqrt{\frac{p}{1+p}}\ln^2 p - 16\int_0^{\sqrt{p}}\frac{\ln q}{\sqrt{1+q^2}}\mathrm{d}q.\end{aligned}$$

To find the last integral, substitute $q = \sinh x$:

$$\int \frac{\ln q}{\sqrt{1+q^2}}\mathrm{d}q = \int \ln\sinh x\mathrm{d}x = \int \left[-\ln 2 + \ln e^x + \ln\left(1-e^{-2x}\right)\right]\mathrm{d}x$$
$$= -x\ln 2 + \frac{1}{2}x^2 + \int \sum_{k\geq 1}\frac{-e^{-2kx}}{k}\mathrm{d}x = -x\ln 2 + \frac{1}{2}x^2 + \frac{1}{2}\mathrm{Li}(e^{-2x}).$$

Now we will put $p = 1/4$. Observe that $\operatorname{arcsinh}\sqrt{p} = \ln(p+\sqrt{1+p^2}) = \ln\phi$.
This gives

$$\begin{aligned} J &= \frac{2}{\sqrt{5}} \\ \partial_b J &= \frac{2}{\sqrt{5}}\ln 4 + 4\ln\phi \\ \partial_b^2 J &= \frac{2}{\sqrt{5}}\ln^2 4 - 16\left[-\ln\phi\ln 2 + \frac{1}{2}\ln^2\phi + \frac{1}{2}\mathrm{Li}(\phi^{-2}) - \frac{1}{2}\mathrm{Li}(1)\right] \\ &= \frac{2}{\sqrt{5}}\ln^2 4 - 16\left[-\ln\phi\ln 2 + \frac{\pi^2}{30} - \frac{\pi^2}{12}\right] \\ &= \frac{2}{\sqrt{5}}\ln^2 4 + 16\ln\phi\ln 2 + \frac{4}{5}\pi^2. \end{aligned}$$

To find the value of the original integral, we write

$$\int_0^\infty \frac{\sqrt{t}}{1+t}\frac{\ln^2(5t+1)}{5t+1}\mathrm{d}t = \frac{1}{4}\int_0^\infty \frac{\sqrt{t}}{1+t}\frac{\ln^2\left(\frac{5}{4}t+\frac{1}{4}\right) + 2\ln 4\ln\left(\frac{5}{4}t+\frac{1}{4}\right) + \ln^2 4}{\frac{5}{4}t+\frac{1}{4}}\mathrm{d}t$$

Each of the integrals on the right-hand side can be expressed in terms of $J, \partial_b J$ and $\partial_b^2 J$ by the formulas above. Doing this gives the result

$$\int_0^\infty \frac{\sqrt{t}}{1+t}\frac{\ln^2(5t+1)}{5t+1}\mathrm{d}t = \frac{\pi^3}{15}\left(9-\sqrt{5}\right) + 4\pi\left(1-\frac{1}{\sqrt{5}}\right)\ln^2 2 + 8\pi\ln 2\ln\phi$$

□

Exercise 1.2.32: Prove:

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{\sqrt{1+x^4}}{1-x^4}\mathrm{d}x = \frac{1}{2\sqrt{2}}\left[\frac{\pi}{2} + \ln(2+\sqrt{5}) - \arctan\frac{\sqrt{5}}{2}\right]$$

Proof: Letting $x = \sqrt{\tan u}$,

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{\sqrt{1+x^4}}{1-x^4}\,\mathrm{d}x &= \int_0^{\arctan(1/2)} \frac{\sec u}{\sec^2(u)\cos(2u)}\frac{\sec^2 u}{2\sqrt{\tan u}}\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^{\arctan(1/2)} \frac{1}{\cos(2u)\sqrt{\sin(2u)/2}}\,\mathrm{d}u \\ &= \frac{\sqrt{2}}{2}\int_0^{\arctan(1/2)} \frac{\cos(2u)}{(1-\sin^2(2u))\sqrt{\sin(2u)}}\,\mathrm{d}u \\ &= \frac{\sqrt{2}}{2}\int_0^{2/\sqrt{5}} \frac{w}{(1-w^4)w}\,\mathrm{d}w \\ &= \frac{\sqrt{2}}{2}\int_0^{2/\sqrt{5}} \frac{1}{1-w^4}\,\mathrm{d}w. \end{aligned}$$

From there we could use partial fractions to get the answer. □

Exercise 1.2.33: Evaluate:

$$\int_0^1 \frac{\ln^2(1+x^2)}{1+x}\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int_0^1 \frac{\ln^2(1+x^2)}{1+x}\mathrm{d}x\\
=&\int_0^1 \frac{\ln^2(1-y^2)}{1+iy} i\ \mathrm{d}y - \int_0^{\frac{\pi}{2}} \frac{\ln^2(1+e^{i2\theta})}{1+e^{i\theta}} ie^{i\theta}\ \mathrm{d}\theta\\
=&\underbrace{\int_0^1 \frac{y\ln^2(1-y^2)}{1+y^2}\mathrm{d}y}_{I_1} + \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \tan\left(\frac{\theta}{2}\right)\ln^2(2\cos\theta)\ \mathrm{d}\theta}_{I_2} + \underbrace{\int_0^{\frac{\pi}{2}} \theta\ln(2\cos\theta)\ \mathrm{d}\theta}_{I_3} \underbrace{-\frac{1}{2}\int_0^{\frac{\pi}{2}} \theta^2\tan\left(\frac{\theta}{2}\right)\ \mathrm{d}\theta}_{I_4}
\end{aligned}$$

Evaluation of I_1:

$$\begin{aligned}
I_1 =&\frac{1}{2}\int_0^1 \frac{\ln^2(1-y)}{1+y}\mathrm{d}y = \frac{1}{4}\int_0^1 \frac{\ln^2 y}{1-y/2}\mathrm{d}y\\
=&\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 y^n\ln^2 y\ \mathrm{d}y = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}(n+1)^3}\\
=&\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\ln 2 + \frac{1}{6}\ln^3 2
\end{aligned}$$

Evaluation of I_2:

$$\begin{aligned}
I_2 =&\int_0^1 \frac{x}{1+x^2}\ln^2\left(2\frac{1-x^2}{1+x^2}\right)\ \mathrm{d}x\\
=&\frac{1}{2}\int_0^1 \frac{1}{1+x}\ln^2\left(2\frac{1-x}{1+x}\right)\ \mathrm{d}x\\
=&\frac{1}{2}\int_0^1 \frac{\ln^2 x}{1+x}\mathrm{d}x + \ln 2\int_0^1 \frac{\ln x}{1+x}\mathrm{d}x + \frac{1}{2}\ln^2 2\int_0^1 \frac{1}{1+x}\mathrm{d}x\\
=&\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln^2 x\ \mathrm{d}x + \ln 2\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln x\ \mathrm{d}x + \frac{1}{2}\ln^3 2\\
=&\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} - \ln 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} + \frac{1}{2}\ln^3 2\\
=&\frac{3}{4}\zeta(3) - \frac{\pi^2}{12}\ln 2 + \frac{1}{2}\ln^3 2
\end{aligned}$$

Evaluation of I_3:

$$\begin{aligned}
I_3 =&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}}\theta\cos(2n\theta)\ \mathrm{d}\theta\\
=&-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\\
=&-\frac{7}{16}\zeta(3)
\end{aligned}$$

Evaluation of I_4:

$$\begin{aligned}
I_4 &= \left[\theta^2 \ln\left(\cos\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \theta \ln\left(\cos\frac{\theta}{2}\right)\,\mathrm{d}\theta \\
&= -\frac{\pi^2}{8}\ln 2 + 2\ln 2\int_0^{\frac{\pi}{2}} \theta\,\mathrm{d}\theta - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}}\theta\cos(n\theta)\,\mathrm{d}\theta \\
&= \frac{\pi^2}{8}\ln 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - \pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(n\pi/2)}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(n\pi/2)}{n^3} \\
&= \frac{21}{16}\zeta(3) - \pi\mathbf{G} + \frac{\pi^2}{8}\ln 2
\end{aligned}$$

Result:

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1+x^2)}{1+x}\mathrm{d}x &= \left(\frac{7}{8}+\frac{3}{4}-\frac{7}{16}+\frac{21}{16}\right)\zeta(3) - \pi\mathbf{G} + \left(-\frac{\pi^2}{12}-\frac{\pi^2}{12}+\frac{\pi^2}{8}\right)\ln 2 + \left(\frac{1}{6}+\frac{1}{2}\right)\ln^3 2 \\
&= \frac{5}{2}\zeta(3) - \pi\mathbf{G} - \frac{\pi^2}{24}\ln 2 + \frac{2}{3}\ln^3 2
\end{aligned}$$

□

Exercise 1.2.34: For $0 \le a < \pi/2$,prove:

$$\int_0^a \sqrt{\tan^2(a) - \tan^2(x)}\,\mathrm{d}x = \frac{\pi}{2}\Big(\sec(a) - 1\Big)$$

Proof: Let $\tan x = \tan a \sin u \implies \mathrm{d}x = \dfrac{\tan a \cos u}{1 + \tan^2 a \sin^2 u}\mathrm{d}u.$

$$\begin{aligned}
\int_0^a \sqrt{\tan^2 a - \tan^2 x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\tan^2 a - \tan^2 a \sin^2 u}{1 + \tan^2 a \sin^2 u}\mathrm{d}u \\
&= \int_0^{\frac{\pi}{2}} \frac{\sec^2 a}{1 + \tan^2 a \sin^2 u}\mathrm{d}u - \int_0^{\frac{\pi}{2}}\mathrm{d}u \\
&= \int_0^{\frac{\pi}{2}} \frac{\sec^2 a \csc^2 u}{\sec^2 a + \cot^2 u}\mathrm{d}u - \frac{\pi}{2} \\
&= \int_0^{\infty} \frac{\sec^2 a}{\sec^2 a + t^2}\mathrm{d}t - \frac{\pi}{2} \\
&= \frac{\sec^2 a}{\sec a}\arctan(t\cos a)\Big|_0^{\infty} - \frac{\pi}{2} \\
&= \frac{\pi}{2}(\sec a - 1)
\end{aligned}$$

□

Exercise 1.2.35: Evaluate:

$$\int_0^{\infty} \frac{e^{-x}\ln x}{1 - 2e^{-x}\cos(2\pi a) + e^{-2x}}\mathrm{d}x$$

✎ **Solution:** Actually, the sum can be evaluated using the Fourier expansion of $\ln\Gamma(x)$.

$$\begin{aligned}
&\int_0^\infty \frac{e^{-x}\ln x}{1-2e^{-x}\cos(2\pi a)+e^{-2x}}\,\mathrm{d}x\\
&=\frac{1}{\sin(2\pi a)}\int_0^\infty \ln(x)\sum_{k=1}^\infty e^{-kx}\sin(2\pi ak)\,\mathrm{d}x\\
&=\frac{1}{\sin(2\pi a)}\sum_{k=1}^\infty \sin(2\pi ak)\int_0^\infty \ln(x)e^{-kx}\,\mathrm{d}x\\
&=-\frac{1}{\sin(2\pi a)}\sum_{k=1}^\infty \frac{\ln k+\gamma}{k}\sin(2\pi ak)\\
&=-\frac{1}{\sin(2\pi a)}\left(\pi\ln\Gamma(a)-\frac{\pi\ln(2\pi)}{2}-\ln(2\pi)\sum_{k=1}^\infty\frac{\sin(2\pi na)}{n}-\frac{\pi}{2}\sum_{k=1}^\infty\frac{\cos(2\pi na)}{n}\right)\\
&=-\frac{1}{\sin(2\pi a)}\left(\pi\ln\Gamma(a)-\frac{\pi\ln(2\pi)}{2}-\ln(2\pi)\frac{\pi-2\pi a}{2}+\frac{\pi}{2}\ln(2\sin\pi a)\right)\\
&=-\frac{\pi}{2\sin(2\pi a)}\left(\ln\Gamma^2(a)-\ln(\pi)+\ln(2\pi)^{2a-1}+\sin(\pi a)\right)\\
&=\frac{\pi}{2\sin(2\pi a)}\ln\left(\frac{\pi}{(2\pi)^{2a-1}\sin(\pi a)\Gamma^2(a)}\right)
\end{aligned}$$

□

Exercise 1.2.36: Evaluate:

$$\int_0^1 \sin(2\pi y)\ln\Gamma\left(\frac{5}{\pi}+y\right)\,\mathrm{d}y$$

✎ **Solution:** Consider

$$\int_{C_R} e^{-2z}\ln\Gamma\left(\frac{z+5i}{\pi i}\right)\mathrm{d}z=0$$

where C_R is the rectangular contour with vertices 0, R, $R+\pi i$, πi. Parameterising along C_R, one obtains

$$\int_0^R e^{-2x}\ln\left(\frac{\pi}{x+5i}\right)\mathrm{d}x=i\int_0^\pi e^{-2iy}\ln\Gamma\left(\frac{y+5}{\pi}\right)\mathrm{d}y-\int_{R+i[0,\pi]} e^{-2z}\ln\Gamma\left(\frac{z+5i}{\pi i}\right)\mathrm{d}z$$

As $R\to\infty$, the last integral $\to 0$. Taking the limit and equating the real parts of both sides of the equation,

$$\begin{aligned}
\int_0^\pi \sin(2y)\ln\Gamma\left(\frac{y+5}{\pi}\right)\mathrm{d}y=&\frac{1}{2}\ln\pi-\frac{1}{2}\int_0^\infty e^{-2x}\ln(x^2+25)\mathrm{d}x &(1)\\
=&\frac{1}{2}\ln\left(\frac{\pi}{5}\right)-\frac{1}{2}\int_0^\infty\frac{xe^{-2x}}{x^2+25}\mathrm{d}x &(2)\\
=&\frac{1}{2}\ln\left(\frac{\pi}{5}\right)+\frac{1}{2}\frac{\partial}{\partial s}\left[\int_0^\infty\frac{e^{-sx}}{x^2+25}\mathrm{d}x\right]_{s=2} &(3)
\end{aligned}$$

Denote the laplace transform of $\dfrac{1}{x^2+25}$ as $I(s)$. Then $I(s)$ satisfies the differential equation

$$25I(s)+I''(s)=\frac{1}{s} \tag{4}$$

By inspection, two solutions to the corresponding homogeneous ODE are $I_1(s) = \cos(5s)$ and $I_2(s) = \sin(5s)$. Their Wronskian is given by $\mathbf{W}(I_1(s), I_2(s)) = 5\cos^2 5s + 5\sin^2(5s) = 5$ and the general solution is

$$I(s) = -\frac{1}{5}\cos(5s)\int \frac{\sin(5s)}{s}\mathrm{d}s + \frac{1}{5}\sin(5s)\int \frac{\cos(5s)}{s}\mathrm{d}s \tag{5}$$

$$= -\frac{1}{5}\cos(5s)\mathrm{Si}(5s) + \frac{1}{5}\sin(5s)\mathrm{Ci}(5s) + C_1\cos(5s) + C_2\sin(5s) \tag{6}$$

Setting $s = 0$, it is easy to see that $C_1 = I(0) = \dfrac{\pi}{10}$. Differentiating,

$$I'(s) = \cos(5s)\mathrm{Ci}(5s) + \sin(5s)\mathrm{Si}(5s) - \frac{\pi}{2}\sin(5s) + 5C_2\cos(5s) \tag{7}$$

As $s \to \infty$, $I'(s) \to 0$. Since the first three terms $\to 0$, this implies that $C_2 = 0$. Plugging in $s = 2$,

$$I'(2) = \cos(10)\mathrm{Ci}(10) + \sin(10)\mathrm{Si}(10) - \frac{\pi}{2}\sin(10) \tag{8}$$

Finally, make the substitution $y \mapsto \pi y$ in the LHS of (1). Equating (1), (3) and (8), we arrive at

$$\int_0^1 \sin(2\pi y)\ln\Gamma\left(\frac{5}{\pi} + y\right)\mathrm{d}y = \frac{1}{2\pi}\left(\ln\left(\frac{\pi}{5}\right) + \cos(10)\mathrm{Ci}(10) + \sin(10)\mathrm{Si}(10)\right) - \frac{1}{4}\sin(10)$$

□

Exercise 1.2.37: Prove:

$$\int_0^\infty \frac{e^x \ln(x^2+a^2)}{1 - 2e^x\cos(2\pi b) + e^{2x}}\mathrm{d}x = \frac{\pi}{\sin(2\pi b)}\ln\left((2\pi)^{(1-2b)}\frac{\Gamma\left(1 - b + \dfrac{a}{2\pi}\right)}{\Gamma\left(b + \dfrac{a}{2\pi}\right)}\right)$$

Proof: Let

$$I = \int_0^\infty \frac{e^x \ln(x^2+a^2)}{1 - 2e^x\cos(2\pi b) + e^{2x}}\mathrm{d}x$$

Consider the contour integral

$$\int_{C_R} f(z)\mathrm{d}z = \int_{C_R} \frac{e^z}{1 - 2e^z\cos(2\pi b) + e^{2z}}\ln\Gamma\left(\frac{z + ia}{2\pi i}\right)\mathrm{d}z$$

where C_R is a rectangle with vertices R, R, $R + 2\pi i$, $-R + 2\pi i$. As $\mathrm{Re}(z) \to \infty$, the integrand $\to 0$ as $\mathcal{O}(Re^{-R}\ln R)$. Taking the limit as $R \to \infty$, we get

$$\begin{aligned}
\Re \lim_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z &= \Re\int_{-\infty}^{\infty} \frac{e^x}{1 - 2e^x\cos(2\pi b) + e^{2x}}\ln\left(\frac{2\pi i}{x + ia}\right)\mathrm{d}x \\
&= -I + 2\ln(2\pi)\int_0^\infty \frac{e^x}{1 - 2e^x\cos(2\pi b) + e^{2x}}\mathrm{d}x \\
&= -I + \frac{2\ln(2\pi)}{\sin(2\pi b)}\sum_{n=1}^{\infty}\sin(2n\pi b)\int_0^\infty e^{-nx}\mathrm{d}x \\
&= -I + \frac{2\ln(2\pi)}{\sin(2\pi b)}\sum_{n=1}^{\infty}\frac{\sin(2n\pi b)}{n} \\
&= -I + \frac{\pi}{\sin(2\pi b)}\ln(2\pi)^{(1-2b)}
\end{aligned}$$

On the other hand, the contour integral is, by the residue theorem, equivalent to $2\pi i$ multiplied by the residues of $f(z)$ at $2\pi ib$ and $2\pi i(1b)$. They are

$$\begin{aligned}
2\pi i \operatorname*{Res}_{z=2\pi ib} f(z) =& 2\pi i \frac{e^{2\pi ib}}{e^{2\pi ib}(e^{2\pi ib}-e^{-2\pi ib})} \ln\Gamma\left(\frac{2\pi ib+ia}{2\pi i}\right) \\
=& \frac{\pi}{\sin(2\pi b)} \ln\Gamma\left(b+\frac{a}{2\pi}\right) \\
2\pi i \operatorname*{Res}_{z=2\pi i(1-b)} f(z) =& 2\pi i \frac{e^{-2\pi ib}}{e^{-2\pi ib}(e^{-2\pi ib}-e^{2\pi ib})} \ln\Gamma\left(\frac{2\pi i(1-b)+ia}{2\pi i}\right) \\
=& -\frac{\pi}{\sin(2\pi b)} \ln\Gamma\left(1-b+\frac{a}{2\pi}\right)
\end{aligned}$$

Therefore the closed form for I is

$$I = \int_0^\infty \frac{e^x \ln(x^2+a^2)}{1-2e^x\cos(2\pi b)+e^{2x}}\mathrm{d}x = \frac{\pi}{\sin(2\pi b)} \ln\left((2\pi)^{(1-2b)} \frac{\Gamma\left(1-b+\frac{a}{2\pi}\right)}{\Gamma\left(b+\frac{a}{2\pi}\right)}\right)$$

□

Exercise 1.2.38: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{\ln(1+\sin^2(x))-\ln(1+\cos^2(x))}{\sin^2(x)-\cos^2(x)}\mathrm{d}x$$

Solution: Method One:

$$I = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\sin^2 x)-\ln(1+\cos^2 x)}{\sin^2 x-\cos^2 x}\mathrm{d}x = \frac{1}{4}\int_0^{2\pi} \ln\left(\frac{3+\cos x}{3-\cos x}\right)\frac{\mathrm{d}x}{\cos x}$$

Let $r = 3-2\sqrt{2}$ in

$$\ln\left(\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}\right) = 4\sum_{n=0}^\infty \frac{r^{2n+1}}{2n+1}\cos\left((2n+1)\theta\right)$$

to get

$$\begin{aligned}
I =& \sum_{n=0}^\infty \frac{(3-2\sqrt{2})^{2n+1}}{2n+1}\int_0^{2\pi}\frac{\cos((2n+1)x)}{\cos x}\mathrm{d}x = 2\pi\sum_{n=0}^\infty \frac{(-1)^n(3-2\sqrt{2})^{2n+1}}{2n+1} \\
=& 2\pi\arctan\left(3-2\sqrt{2}\right)
\end{aligned}$$

Method Two:

Substitute $t = \tan x$ to get

$$I = J(\sqrt{2}),$$

where

$$J(a) = \int_0^\infty \frac{\ln(a^2t^2+1)-\ln(t^2+a^2)}{t^2-1}\mathrm{d}t,$$

We have $J(1) = 0$ and

$$\begin{aligned} J'(a) &= 2a \int_0^\infty \frac{1}{t^2-1}\left[\frac{t^2}{a^2t^2+1} - \frac{1}{t^2+a^2}\right] \mathrm{d}t \\ &= a \int_{-\infty}^\infty \frac{1}{t^2-1}\left[\frac{t^2}{a^2t^2+1} - \frac{1}{t^2+a^2}\right] \mathrm{d}t \\ &= a(2\pi i)\left[\frac{1}{a^2(i/a)^2-1}\frac{(i/a)^2}{2a^2(i/a)} - \frac{1}{(ia)^2-1}\frac{1}{2ia}\right] \\ &= \frac{2\pi}{1+a^2}, \end{aligned}$$

whence

$$I = J(\sqrt{2}) = 2\pi\left[\arctan\sqrt{2} - \arctan 1\right] = 2\pi\left[\arctan\sqrt{2} - \frac{\pi}{4}\right]$$

□

Exercise 1.2.39: Evaluate:

$$\int_0^\pi \frac{\cos(nx)}{a-\cos(x)}\mathrm{d}x\ , \quad a > 1$$

Solution:

$$\begin{aligned} \int_0^\pi \frac{\cos(nx)}{a-\cos x}\mathrm{d}x &= \Re \int_{|z|=1} \frac{z^n}{a - \frac{z^2+1}{2z}} \frac{\mathrm{d}z}{2iz} \\ &= -\Im \int_{|z|=1} \frac{z^n}{z^2-2az+1}\mathrm{d}z \\ &= -\Im\ 2\pi i \left(\frac{\left(a-\sqrt{a^2-1}\right)^n}{2\left(a-\sqrt{a^2-1}\right)-2a}\right) \\ &= \frac{\pi\left(a-\sqrt{a^2-1}\right)^n}{\sqrt{a^2-1}} \end{aligned}$$

□

Exercise 1.2.40: Prove:

$$\int_0^\infty \frac{\tan^{-1}(x/a)\tan^{-1}(x/b)}{x^2}\mathrm{d}x = \frac{\pi}{2}\ln\left(\frac{(a+b)^{1/a+1/b}}{a^{1/b}b^{1/a}}\right)$$

Proof: Let $f(z) = \dfrac{\ln\left(1-i\frac{z}{a}\right)\arctan\left(\frac{z}{b}\right)}{z^2}$ and integrate along γ, which is the classic semicircle deformed around $[ib, i\infty]$.

$$\int_\gamma f(z)\ \mathrm{d}z = \int_{-\infty}^\infty f(x)\ \mathrm{d}x + 2\pi i \int_b^\infty \frac{\ln\left(1-i\frac{iy}{a}\right)\frac{i}{2}}{-y^2} i\ \mathrm{d}y = 0$$

Equating imaginary parts (and dividing by 2),

$$\begin{aligned}\int_0^\infty \frac{\arctan\left(\frac{x}{a}\right)\arctan\left(\frac{x}{b}\right)}{x^2}\mathrm{d}x &= \frac{\pi}{2}\int_b^\infty \frac{\ln\left(1+\frac{y}{a}\right)}{y^2}\mathrm{d}y \\ &= \frac{\pi}{2}\left[\frac{1}{a}\ln\left(1+\frac{a}{y}\right)+\frac{1}{y}\ln\left(1+\frac{y}{a}\right)\right]_\infty^b \\ &= \frac{\pi}{2}\ln\left(\frac{(a+b)^{\frac{1}{a}+\frac{1}{b}}}{a^{\frac{1}{b}}b^{\frac{1}{a}}}\right)\end{aligned}$$

□

Exercise 1.2.41: Prove:

$$\int_{-\infty}^\infty \frac{\cos(ax)+2x\sin(ax)}{\cosh(\pi x)\left(x^2+\frac{1}{4}\right)}\,\mathrm{d}x = 4e^{-a/2}\ln(1+e^a)$$

Proof: Method One:

Note that

$$\int_{-\infty}^\infty \frac{\cos(ax)+2x\sin(ax)}{\cosh(\pi x)\left(x^2+\frac{1}{4}\right)}\mathrm{d}x = 4\int_{-\infty}^\infty \frac{e^{iax}}{\cosh(\pi x)(1+2ix)}\mathrm{d}x = I(a)$$

Then

$$I'(a)+\frac{1}{2}I(a) = 2\int_{-\infty}^\infty \frac{e^{iax}}{\cosh(\pi x)}\mathrm{d}x$$

Integrate over a rectangle with vertices $-R, R, R+i, -R+i$ to get

$$(1+e^{-a})\int_{-\infty}^\infty \frac{e^{iax}}{\cosh(\pi x)}\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=i/2}\frac{e^{iaz}}{\cosh(\pi z)} = 2e^{-a/2}$$

Therefore

$$I(a) = e^{-a/2}\int \frac{4}{1+e^{-a}}\mathrm{d}a = 4e^{-a/2}\ln(1+e^a)$$

Method Two:

For $-1 < \Re(s) < 0$,

$$\mathcal{M}\{\ln(1+x)\}(s) = \frac{\pi}{s\sin(\pi s)}.$$

To prove this notice that

$$\begin{aligned}\int_0^\infty x^{s-1}\ln(1+x)\,\mathrm{d}x &= \int_0^\infty\int_0^x \frac{x^{s-1}}{1+t}\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\infty\int_t^\infty \frac{x^{s-1}}{1+t}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{s}\int_0^\infty \frac{t^s}{1+t}\,\mathrm{d}t \\ &= -\frac{1}{s}\frac{\pi}{\sin\left(\pi(s+1)\right)} \\ &= \frac{\pi}{s\sin(\pi s)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{M}^{-1}\left\{\frac{\pi}{s\sin(\pi s)}\right\}(x) &= \ln(1+x)\\ &= \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} x^{-s}\,\frac{\pi}{s\sin(\pi s)}\,\mathrm{d}s\\ &= \frac{x^{1/2}}{2}\int_{-\infty}^{\infty} x^{-it}\,\frac{\frac{1}{2}+it}{\left(t^2+\frac{1}{4}\right)\cosh(\pi t)}\,\mathrm{d}t.\end{aligned}$$

Equating the real parts and rearranging,

$$\int_{-\infty}^{\infty}\frac{\cos(t\ln x)+2t\sin(t\ln x)}{\cosh(\pi t)\left(t^2+\frac{1}{4}\right)}\,\mathrm{d}t = 4x^{-1/2}\ln(1+x)$$

Now replace $\ln x$ with x.

$$\int_{-\infty}^{\infty}\frac{\cos(xt)+2t\sin(xt)}{\cosh(\pi t)\left(t^2+\frac{1}{4}\right)}\,\mathrm{d}t = 4e^{-x/2}\ln(1+e^{x})$$

□

Exercise 1.2.42: Prove:

$$\int_0^{\pi/2}\frac{e^{-\cos^2(x)}\cos\left(\frac{\sin 2x}{2}+\tan x-x\right)}{\cos x}\,\mathrm{d}x = \frac{\pi J_0(2)}{e}$$

$J_0(x)$ is the **Bessel function of the first kind** of order zero.

Proof:

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\frac{e^{-\cos^2 x}\cos\left(\frac{\sin 2x}{2}+\tan x-x\right)}{\cos x}\mathrm{d}x\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{1}{1+x^2}+i\left(\frac{x}{1+x^2}+x-\arctan x\right)\right)}{\sqrt{1+x^2}}\mathrm{d}x\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ix}\exp\left(-\frac{1}{1+ix}\right)}{1+ix}\mathrm{d}x\\ &= \frac{1}{2}\lim_{\epsilon\to 0}\int_0^{2\pi}\frac{\exp\left(i\left(i+\epsilon e^{i\phi}\right)-\frac{1}{1+i(i+\epsilon e^{i\phi})}\right)}{1+i(i+\epsilon e^{i\phi})}i\epsilon e^{i\phi}\,\mathrm{d}\phi\\ &= \frac{1}{2e}\lim_{\epsilon\to 0}\int_0^{2\pi}\exp\left(i\epsilon e^{i\phi}-\frac{1}{i\epsilon e^{i\phi}}\right)\,\mathrm{d}\phi\\ &= \frac{1}{2e}\lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k i^{n-2k}\int_0^{2\pi}\epsilon^{n-2k}e^{i(n-2k)\phi}\,\mathrm{d}\phi\end{aligned}$$

The integral inside evaluates to 0 when $n \neq 2k$, and to 2π when $n = 2k$. Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{e^{-\cos^2 x}\cos\left(\frac{\sin 2x}{2}+\tan x - x\right)}{\cos x}\mathrm{d}x &= \frac{\pi}{e}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\binom{2n}{n}\\ &= \frac{\pi}{e}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma\left(n+0+1\right)}\left(\frac{2}{2}\right)^{2n+0}\\ &= \frac{\pi}{e}J_0(2)\end{aligned}$$

A generalization of the problem is

$$\int_0^{\pi/2} \frac{e^{-\cos^2(t)}\cos\left(\frac{\sin 2t}{2}+x\tan t - t\right)}{\cos t}\,\mathrm{d}t = \frac{\pi J_0(2\sqrt{x})}{e^x}\ ,\ x>0$$

For the case $x = 0$.

$$\begin{aligned}\int_0^{\pi/2} \frac{e^{-\cos^2(t)}\cos\left(\frac{\sin 2t}{2}-t\right)}{\cos t}\,\mathrm{d}t &= \frac{1}{2}\,\Re\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{1}{1+it}\right)}{1+it}\mathrm{d}t\\ &= -\frac{1}{2}\,\Re\ i\pi\,\mathrm{Res}\left[\frac{\exp\left(-\frac{1}{1+iz}\right)}{1+iz},\infty\right]\\ &= \frac{1}{2}\,\Re\ i\pi\lim_{|z|\to\infty} z\,\frac{\exp\left(-\frac{1}{1+iz}\right)}{1+iz}\\ &= \frac{1}{2}\,\Re\ i\pi\left(\frac{1}{i}\right)\\ &= \frac{\pi}{2}\end{aligned}$$

□

Exercise 1.2.43: Prove:

$$\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{2}+it\right)}{\cosh\pi t}\,\mathrm{d}t = -2e\,\mathrm{Ei}(-1)$$

Where $\mathrm{Ei}(x)$ is the **exponential integral**.

Proof: Method One:

Previously it was derived that

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{\cosh(\pi x)}\mathrm{d}x = \frac{2e^{-\frac{a}{2}}}{1+e^{-a}}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{2}+it\right)}{\cosh(\pi t)}\mathrm{d}t &= \int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t)}\int_0^{\infty}x^{-\frac{1}{2}+it}e^{-x}\mathrm{d}x\ \mathrm{d}t\\ &= \int_0^{\infty}x^{-\frac{1}{2}}e^{-x}\int_{-\infty}^{\infty}\frac{e^{it\ln x}}{\cosh(\pi t)}\mathrm{d}t\ \mathrm{d}x\\ &= \int_0^{\infty}x^{-\frac{1}{2}}e^{-x}\frac{2x^{-\frac{1}{2}}}{1+\frac{1}{x}}\mathrm{d}x\\ &= \int_0^{\infty}\frac{2e^{-x}}{1+x}\mathrm{d}x\\ &= 2e\int_1^{\infty}\frac{e^{-x}}{x}\mathrm{d}x\\ &= -2e\mathrm{Ei}(-1)\end{aligned}$$

Method Two:

Using Parseval's formula for the Mellin transform.

$$\begin{aligned}\int_0^{\infty}\frac{e^{-t}}{1+t}\,\mathrm{d}t = -e\mathrm{Ei}(-1) &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\mathcal{M}\left\{e^{-t}\right\}(s)\,\mathcal{M}\left\{\frac{1}{1+t}\right\}(1-s)\,\mathrm{d}s\\ &= \frac{1}{2i}\int_{1/2-i\infty}^{1/2+i\infty}\Gamma(s)\csc(\pi s)\ \mathrm{d}s\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{2}+it\right)}{\cosh\pi t}\,\mathrm{d}t\\ &= \int_0^{\infty}\frac{\Re\,\Gamma\left(\frac{1}{2}+it\right)}{\cosh\pi t}\,\mathrm{d}t\end{aligned}$$

More generally, for all real values of x,

$$\begin{aligned}\int_0^{\infty}\frac{e^{-e^xt}}{1+t}\,\mathrm{d}t = -e^{e^x}\mathrm{Ei}(-e^x) &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}e^{-xs}\mathcal{M}\left\{e^{-t}\right\}(s)\,\mathcal{M}\left\{\frac{1}{1+t}\right\}(1-s)\ \mathrm{d}s\\ &= \frac{e^{-x/2}}{2}\int_{-\infty}^{\infty}\frac{e^{-ixt}\,\Gamma\left(\frac{1}{2}+it\right)}{\cosh\pi t}\,\mathrm{d}t\\ &= e^{-x/2}\int_0^{\infty}\frac{\cos(xt)\,\Re\,\Gamma\left(\frac{1}{2}+it\right)+\sin(xt)\,\Im\,\Gamma\left(\frac{1}{2}+it\right)}{\cosh\pi t}\,\mathrm{d}t\end{aligned}$$

□

Exercise 1.2.44: Evaluate:

$$\int_0^{\infty}\frac{(7t^4+12t^2+6)\ln(1+t^2)-3t^2(t^4+3t^2+2)}{t^3(1+t^2)^2\ln^2(1+t^2)}\mathrm{d}t$$

Solution: Start with the integral representation

$$\ln G(z+1) = \frac{z}{2}\ln(2\pi)+\int_0^{\infty}\left(\frac{1-e^{-zx}}{4\sinh^2\left(\frac{x}{2}\right)}-\frac{z}{x}+\frac{z^2}{2}e^{-x}\right)\frac{\mathrm{d}x}{x}$$

Integrating both sides with respect to z from 0 to 1 gives

$$\int_0^1 \ln G(z+1)\ \mathrm{d}z = \frac{1}{4}\ln(2\pi) + \int_0^\infty \left(\frac{x + e^{-x} - 1}{4\sinh^2\left(\frac{x}{2}\right)} - \frac{1}{2x} + \frac{1}{6}e^{-x} \right) \frac{\mathrm{d}x}{x}$$

The **LHS** is

$$\begin{aligned}
\int_0^1 \ln G(z+1)\ \mathrm{d}z &= -\int_0^1 z\left(\frac{1}{2}(1+\ln(2\pi)) - z + z\psi_0(z)\right)\mathrm{d}z \\
&= \frac{1}{12} - \frac{1}{4}\ln(2\pi) + 2\int_0^1 z\ln\Gamma(z)\ \mathrm{d}z \\
&= \frac{1}{12} + \frac{1}{4}\ln(2\pi) + \sum_{n=1}^\infty \frac{1}{n}\int_0^1 z\cos(2n\pi z)\ \mathrm{d}z \\
&\quad + \frac{2}{\pi}\sum_{n=1}^\infty \frac{(\gamma+\ln(2\pi)) + \ln n}{n}\int_0^1 z\sin(2n\pi z)\ \mathrm{d}z \\
&= \frac{1}{12} + \frac{1}{4}\ln(2\pi) - \frac{1}{\pi^2}\sum_{n=1}^\infty \frac{(\gamma+\ln(2\pi)) + \ln n}{n^2} \\
&= \frac{1}{12}(1+\ln(2\pi)) - \frac{\gamma}{6} + \frac{\zeta'(2)}{\pi^2} \\
&= \frac{1}{12} + \frac{1}{4}\ln(2\pi) - 2\ln\mathbf{A}
\end{aligned}$$

Substituting $x = \ln(1+t^2)$ in the integral on the **RHS** yields

$$\frac{1}{12} + \frac{1}{4}\ln(2\pi) - 2\ln\mathbf{A} = \frac{1}{4}\ln(2\pi) + \int_0^\infty \frac{(7t^4+12t^2+6)\ln(1+t^2) - 3t^2(t^4+3t^2+2)}{3t^3(1+t^2)^2\ln^2(1+t^2)}\mathrm{d}t$$

It follows that

$$\int_0^\infty \frac{(7t^4+12t^2+6)\ln(1+t^2) - 3t^2(t^4+3t^2+2)}{t^3(1+t^2)^2\ln^2(1+t^2)}\mathrm{d}t = \frac{1}{4} - 6\ln\mathbf{A}$$

□

Exercise 1.2.45: Evaluate:

$$\int_0^1 \frac{\ln\ln\left(\frac{1}{x}\right)}{1+x^2}\mathrm{d}x$$

Solution: Let

$$I(a) = \int_0^1 \frac{\ln\left(\frac{1}{x}\right)^{a-1}}{1+x^2}\mathrm{d}x$$

so that we are after $I'(1)$.

Substituting $t = \ln(1/x)$ the integral becomes

$$\begin{aligned}
I(a) &= \int_0^\infty t^{a-1}\frac{e^{-t}}{1+e^{-2t}}\mathrm{d}t = \int_0^\infty t^{a-1}\sum_{n=0}^\infty (-1)^n e^{-t(2n+1)}\mathrm{d}t \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^a}\int_0^\infty t^{a-1}e^{-t}\mathrm{d}t = \Gamma(a)\beta(a)
\end{aligned}$$

where $\beta(x)$ is **Dirichlet's beta function**.

So

$$\int_0^1 \frac{\ln\ln\left(\frac{1}{x}\right)}{1+x^2}\mathrm{d}x = \lim_{a\to 1} I'(a) = \lim_{a\to 1} \Gamma'(a)\beta(a) + \Gamma(a)\beta'(a) = \beta'(1) - \frac{\pi\gamma}{4}$$
$$= \frac{\pi}{4}\left(2\ln(2) + 3\ln(\pi) - 4\ln\Gamma\left(\frac{1}{4}\right)\right)$$

where

$$\beta'(1) = \frac{\pi}{4}\left(\gamma + 2\ln(2) + 3\ln(\pi) - 4\ln\Gamma\left(\frac{1}{4}\right)\right)$$

□

Exercise 1.2.46: Prove:

$$\int_0^1 \frac{\cos(1+x^2)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4} - \pi\,\mathrm{S}\left(\sqrt{\frac{2}{\pi}}\right)\mathrm{C}\left(\sqrt{\frac{2}{\pi}}\right)$$

where $\mathrm{S}(z)$ and $\mathrm{C}(z)$ are the Fresnel integrals defined as

$$\mathrm{S}(z) = \int_0^z \sin\left(\frac{\pi x^2}{2}\right)\mathrm{d}x\ ,\ \mathrm{C}(z) = \int_0^z \cos\left(\frac{\pi x^2}{2}\right)\mathrm{d}x$$

Proof:

$$\int_0^1 \frac{\cos(1+x^2)}{1+x^2}\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{1+x^2} - \int_0^1\int_0^1 \sin(t+tx^2)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{\pi}{4} - \int_0^1\int_0^1 \sin t\cos(tx^2) - \cos t\sin(tx^2)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{\pi}{4} - \sqrt{\frac{\pi}{2}}\int_0^1 \frac{\sin t\ \mathrm{C}\left(\sqrt{\frac{2t}{\pi}}\right) - \cos t\ \mathrm{S}\left(\sqrt{\frac{2t}{\pi}}\right)}{\sqrt{t}}\,\mathrm{d}t$$
$$= \frac{\pi}{4} - \sqrt{2\pi}\int_0^1 \sin(t^2)\ \mathrm{C}\left(\sqrt{\frac{2}{\pi}}t\right) - \cos(t^2)\ \mathrm{S}\left(\sqrt{\frac{2}{\pi}}t\right)\,\mathrm{d}t$$
$$= \frac{\pi}{4} - \sqrt{2\pi}\left(\sqrt{\frac{\pi}{2}}\mathrm{C}\left(\sqrt{\frac{2}{\pi}}\right)\mathrm{S}\left(\sqrt{\frac{2}{\pi}}\right)\right.$$
$$\left. - \int_0^1 \left[\cos(t^2)\mathrm{S}\left(\sqrt{\frac{2}{\pi}}\right) - \cos(t^2)\mathrm{S}\left(\sqrt{\frac{2}{\pi}}\right)\right]\,\mathrm{d}t\right)$$
$$= \frac{\pi}{4} - \pi\ \mathrm{C}\left(\sqrt{\frac{2}{\pi}}\right)\mathrm{S}\left(\sqrt{\frac{2}{\pi}}\right)$$

Similarly,

$$\int_0^1 \frac{\sin(1+x^2)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}\left(\mathrm{C}^2\left(\sqrt{\frac{2}{\pi}}\right) - \mathrm{S}^2\left(\sqrt{\frac{2}{\pi}}\right)\right)$$

□

Exercise 1.2.47: Prove:

$$\int_0^\infty \frac{\tanh^3(x)}{x^3}\,\mathrm{d}x = \frac{1}{\pi^4}\left(186\,\zeta(5) - 7\pi^2\zeta(3)\right)$$

☞ **Proof:** **Method One:**

Denoting $\alpha_n = \dfrac{(2n+1)\pi i}{2}$, we have the Laurent series expansion

$$\tanh^3 z = \frac{1}{(z-\alpha_n)^3} + \frac{1}{z-\alpha_n} + \mathcal{O}\left((z-\alpha_n)\right)$$

Thus

$$\begin{aligned}\operatorname*{Res}_{z=\alpha_n} \frac{\tanh^3 z}{z^3} &= \operatorname*{Res}_{z=\alpha_n}\left[\frac{1}{z^3(z-\alpha_n)^3} + \frac{1}{z^3(z-\alpha_n)}\right] \\ &= \frac{6}{\alpha_n^5} + \frac{1}{\alpha_n^3} = \frac{192}{(2n+1)^5\pi^5 i} - \frac{8}{(2n+1)^3\pi^3 i}\end{aligned}$$

Integrating $\dfrac{\tanh^3 z}{z^3}$ along a semicircle in the UHP and noting that 0 is just a removable singularity, as well as the fact that the integral over the arc vanishes, we get

$$\begin{aligned}\int_0^\infty \frac{\tanh^3 x}{x^3}\,\mathrm{d}x &= \pi\sum_{n=0}^{\infty}\frac{192}{(2n+1)^5\pi^5} - \frac{8}{(2n+1)^3\pi^3} \\ &= \frac{1}{\pi^4}\times 192\times\left(1-\frac{1}{32}\right)\times\zeta(5) - \frac{1}{\pi^2}\times 8\times\left(1-\frac{1}{8}\right)\times\zeta(3) \\ &= \frac{186\zeta(5)}{\pi^4} - \frac{7\zeta(3)}{\pi^2}\end{aligned}$$

Method Two:

The idea from **sos440's post**

http://math.stackexchange.com/questions/1309701/

(which was about $\displaystyle\int_0^\infty \frac{\tanh^{2n}(x)}{x^2}\,\mathrm{d}x$) is that summing the residues in the upper half-plane, we find

$$\begin{aligned}\sum_{n=0}^{\infty}\operatorname{Res}\left[\frac{\tanh^3(z)}{z^3}, i\pi\left(n+\frac{1}{2}\right)\right] &= \sum_{n=0}^{\infty}\operatorname{Res}\left[\frac{\tanh^3(z+i\pi n)}{(z+i\pi n)^3}, \frac{i\pi}{2}\right] \\ &= \sum_{n=0}^{\infty}\operatorname{Res}\left[\frac{\tanh^3(z)}{(z+i\pi n)^3}, \frac{i\pi}{2}\right] \\ &= \operatorname{Res}\left[\tanh^3(z)\sum_{n=0}^{\infty}\frac{1}{(z+i\pi n)^3}, \frac{i\pi}{2}\right] \\ &= \operatorname{Res}\left[\tanh^3(z)\frac{1}{2i\pi^3}\,\psi_2\left(\frac{z}{i\pi}\right), \frac{i\pi}{2}\right] \\ &= \frac{1}{2i\pi^3}\operatorname{Res}\left[\coth^3(z)\,\psi_2\left(\frac{1}{2}+\frac{z}{i\pi}\right), 0\right].\end{aligned}$$

And since

$$\coth^3(z)\,\psi^2\left(\frac{1}{2}+\frac{z}{i\pi}\right) = \left(\frac{1}{z^3}+\frac{1}{z}+\ldots\right)\left(\psi_2\left(\frac{1}{2}\right)+\frac{1}{i\pi}\psi_3\left(\frac{1}{2}\right)z - \frac{1}{\pi^2}\psi_4\left(\frac{1}{2}\right)\frac{z^2}{2!}+\ldots\right),$$

we get

$$\begin{aligned}\int_0^\infty \frac{\tanh^3 x}{x^3}\,\mathrm{d}x &= i\pi\,\frac{1}{2i\pi^3}\left[-\frac{1}{2\pi^2}\psi_4\left(\frac{1}{2}\right)+\psi_2\left(\frac{1}{2}\right)\right] \\ &= \frac{1}{4\pi^4}\left[2\pi^2\psi_2\left(\frac{1}{2}\right)-\psi_4\left(\frac{1}{2}\right)\right] \\ &= \frac{186\zeta(5)}{\pi^4} - \frac{7\zeta(3)}{\pi^2}\end{aligned}$$

where

$$\psi_2\left(\frac{1}{2}\right) = -14\zeta(3) \ , \ \psi_4\left(\frac{1}{2}\right) = -744\zeta(5)$$

□

Exercise 1.2.48: For $b \geq a \geq 0$ and $p > 0$,prove:

$$\Re \int_0^\infty \cos(ax)\, \frac{f(e^{ibx})}{x^2+p^2}\, \mathrm{d}x = \frac{\pi}{2p}\Big[f(e^{-bp})\cosh(ap) - f(0)\sinh(ap)\Big]$$

and

$$\Im \int_0^\infty \sin(ax)\, \frac{f(e^{ibx})}{x^2+p^2}\, \mathrm{d}x = \frac{\pi}{2p}\sinh(ap)\Big[f(e^{-bp}) - f(0)\Big]$$

Proof:

$$\begin{aligned}
&\Re \int_0^\infty \cos(ax)\, \frac{f(e^{ibx})}{x^2+p^2}\, \mathrm{d}x \\
&= \int_0^\infty \frac{\cos(ax)}{x^2+p^2} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cos(nbx)\, \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \int_0^\infty \frac{\cos(ax)\cos(nbx)}{x^2+p^2}\, \mathrm{d}x \\
&= f(0)\int_0^\infty \frac{\cos(ax)}{x^2+p^2}\, \mathrm{d}x + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \int_0^\infty \frac{\cos(ax)\cos(nbx)}{x^2+p^2}\, \mathrm{d}x \\
&= f(0)\frac{\pi}{2p}e^{-ap} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\int_0^\infty \frac{\cos\Big((a+nb)x\Big)}{x^2+p^2}\, \mathrm{d}x + \int_0^\infty \frac{\cos\Big((a-nb)x\Big)}{x^2+p^2}\, \mathrm{d}x \right) \\
&= \frac{\pi}{2p} f(0)e^{-ap} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{\pi}{2p} \left(e^{-(a+nb)p} + e^{-(nb-a)p} \right) \\
&= \frac{\pi}{2p} f(0)e^{-ap} + \frac{\pi}{2p}\cosh(ap) \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} e^{-nbp} \\
&= \frac{\pi}{2p} f(0)e^{-ap} + \frac{\pi}{2p}\cosh(ap)\Big(f(e^{-bp}) - f(0) \Big) \\
&= \frac{\pi}{2p}\Big(f(e^{-bp})\cosh(ap) - f(0)\sinh(ap) \Big)
\end{aligned}$$

So, for example,

$$\int_0^\infty \frac{\cos(ax)e^{\cos(bx)}\cos(\sin bx)}{x^2+p^2}\, \mathrm{d}x = \frac{\pi}{2p}\Big(e^{e^{-bp}}\cosh(ap) - \sinh(ap) \Big)$$

The other one is similar. □

Exercise 1.2.49: Prove:

$$\int_0^1 \frac{1-x^{n-1}}{(1-x)(1-x^n)} (-\ln(x))^{m-1} \mathrm{d}x = \left(1 - \frac{1}{n^m}\right)\Gamma(m)\zeta(m)$$

☞ **Proof:**

$$\begin{aligned}
\int_0^1 \frac{1-x^{n-1}}{(1-x)(1-x^n)}(-\ln x)^{m-1}\,\mathrm{d}x &= \int_0^1 \frac{1+\cdots+x^{n-2}+x^{n-1}-x^{n-1}}{(1-x)(1+\cdots+x^{n-1})}(-\ln x)^{m-1} \\
&= \int_0^1 \frac{(-\ln x)^{m-1}}{1-x} - \frac{x^{n-1}(-\ln x)^{m-1}}{1-x^n}\,\mathrm{d}x \\
&= \left(1-\frac{1}{n^m}\right)\int_0^1 \frac{(-\ln x)^{m-1}}{1-x}\,\mathrm{d}x \\
&= \left(1-\frac{1}{n^m}\right)\sum_{n=0}^{\infty}\int_0^1 x^n(-\ln x)^{m-1}\,\mathrm{d}x \\
&= \left(1-\frac{1}{n^m}\right)\sum_{n=0}^{\infty}\int_0^{\infty} x^{m-1}e^{-(n+1)x}\,\mathrm{d}x \\
&= \left(1-\frac{1}{n^m}\right)\sum_{n=0}^{\infty}\frac{1}{(n+1)^m}\Gamma(m) \\
&= \left(1-\frac{1}{n^m}\right)\Gamma(m)\zeta(m)
\end{aligned}$$

□

Exercise 1.2.50: Evaluate:

$$\int_0^{\infty} \sin(a\tan x)\frac{\cot x}{x^2+p^2}\,\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
\int_0^{\infty} \sin(a\tan x)\frac{\cot x}{x^2+p^2}\,\mathrm{d}x &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{2}+n\pi}^{\frac{\pi}{2}+n\pi} \sin(a\tan x)\frac{\cot x}{x^2+p^2}\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(a\tan x)\frac{\cot x}{(x+n\pi)^2+p^2}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(a\tan x)\cot x\left[-\sum_{\pm}\operatorname*{Res}_{z=\frac{-x\pm ip}{\pi}}\frac{\pi\cot(\pi z)}{(x+z\pi)^2+p^2}\right]\mathrm{d}x \\
&= \frac{1}{4ip}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(a\tan x)\cot x\left(\cot(x-ip)-\cot(x+ip)\right)\,\mathrm{d}x \\
&= \frac{\tanh p}{2p}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(a\tan x)\cot x\left(\frac{\tan^2 x+1}{\tan^2 x+\tanh^2 p}\right)\,\mathrm{d}x \\
&= \frac{\tanh p}{2p}\int_{-\infty}^{\infty}\frac{\sin(ax)}{x(x^2+\tanh^2 p)}\,\mathrm{d}x \\
&= \frac{\coth p}{2p}\int_{-\infty}^{\infty}\frac{\sin(ax)}{x}-\frac{x\sin(ax)}{x^2+\tanh^2 p}\,\mathrm{d}x \\
&= \frac{\pi\coth p}{2p}\left(1-e^{-a\tanh p}\right)
\end{aligned}$$

□

Exercise 1.2.51: Evaluate:

$$\int_0^{\infty}\frac{x\sin(ax)}{\cosh\left(\frac{\pi x}{4}\right)(x^2+1)}\mathrm{d}x$$

✎ **Solution:** The function $f(z)=\dfrac{ze^{iaz}}{\cosh\left(\dfrac{\pi z}{4}\right)(1+z^2)}$ has simple poles at $z=i$ and $z=(4n+2)i$ in the upper half plane. Noting Jordan's lemma and summing the residues,

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin(ax)}{\cosh\left(\dfrac{\pi x}{4}\right)(1+x^2)}\,\mathrm{d}x\\
&=\Re\pi\left(\frac{ie^{ia(i)}}{\cosh\left(\dfrac{\pi i}{4}\right)(2i)}+\sum_{n=0}^{\infty}\frac{(4n+2)ie^{ia(4n+2)i}}{\dfrac{\pi}{4}\sinh\left(\dfrac{\pi(4n+2)i}{4}\right)(1+((4n+2)i)^2)}\right)\\
&=\frac{\pi}{\sqrt{2}}e^{-a}-2e^{-2a}\sum_{n=0}^{\infty}(-e^{-4a})^n\left(\frac{1}{4n+1}+\frac{1}{4n+3}\right)\\
&=\frac{\pi}{\sqrt{2}}e^{-a}-2e^{-2a}\sum_{n=0}^{\infty}(-e^{-4a})^n\int_0^1 x^{4n}(1+x^2)\,\mathrm{d}x\\
&=\frac{\pi}{\sqrt{2}}e^{-a}-2e^{-2a}\int_0^1\frac{1+x^2}{1+e^{-4a}x^4}\,\mathrm{d}x
\end{aligned}$$

Partial fraction decomposition (or WolframAlpha) yields

$$\begin{aligned}
&\int_0^1\frac{1+x^2}{1+e^{-4a}x^4}\,\mathrm{d}x\\
&=\frac{1}{4\sqrt{2}}e^{a}\left[(e^{2a}-1)\ln\left(\frac{1-\sqrt{2}e^{-a}x+e^{-2a}x^2}{1+\sqrt{2}e^{-a}x+e^{-2a}x^2}\right)+2(e^{2a}+1)\arctan\left(\frac{2\sqrt{2}e^{-a}x}{1+1-2e^{-2a}x^2}\right)\right]_0^1\\
&=\frac{1}{4\sqrt{2}}e^{a}\left[(e^{2a}-1)\ln\left(\frac{e^{a}-\sqrt{2}+e^{-a}}{e^{a}+\sqrt{2}+e^{-a}}\right)+2(e^{2a}+1)\arctan\left(\frac{\sqrt{2}}{e^{a}-e^{-a}}\right)\right]\\
&=\frac{1}{4\sqrt{2}}e^{a}\left[(e^{2a}-1)\ln\left(\frac{2\cosh a-\sqrt{2}}{2\cosh a+\sqrt{2}}\right)+2(e^{2a}+1)\arctan\left(\frac{1}{\sqrt{2}\sinh a}\right)\right]
\end{aligned}$$

Therefore, we indeed have

$$\begin{aligned}
&\int_0^{\infty}\frac{x\sin(ax)}{\cosh\left(\dfrac{\pi x}{4}\right)(1+x^2)}\,\mathrm{d}x\\
&=\frac{\pi}{\sqrt{2}}e^{-a}-\frac{1}{2\sqrt{2}}e^{-a}\left[(e^{2a}-1)\ln\left(\frac{2\cosh a-\sqrt{2}}{2\cosh a+\sqrt{2}}\right)+2(e^{2a}+1)\arctan\left(\frac{1}{\sqrt{2}\sinh a}\right)\right]\\
&=\frac{\pi}{\sqrt{2}}e^{-a}-\frac{\sinh a}{\sqrt{2}}\ln\left(\frac{2\cosh a-\sqrt{2}}{2\cosh a+\sqrt{2}}\right)-\sqrt{2}\cosh a\arctan\left(\frac{1}{\sqrt{2}\sinh a}\right)
\end{aligned}$$

□

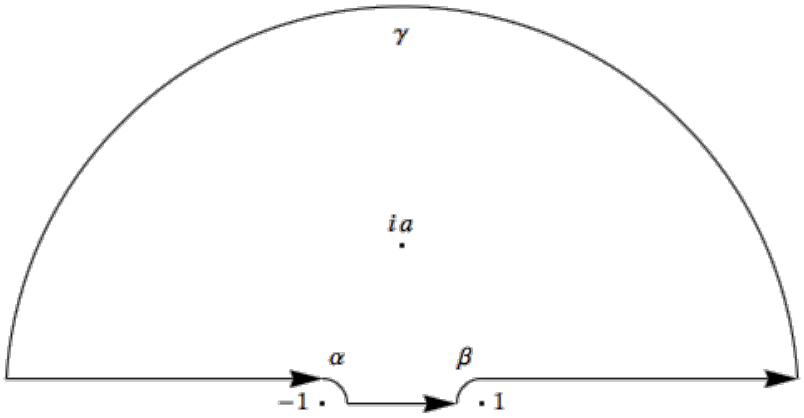
Exercise 1.2.52: Evaluate:

$$\int_0^{\infty}\frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left|\frac{x+1}{x-1}\right|\,\mathrm{d}x$$

✎ **Solution:** Let $f(z)=\dfrac{z(1-z^2)}{(z^2+a^2)^2}\ln\left(\dfrac{1+z}{1-z}\right)$ and consider the contour integral

$$\oint_C\frac{z(1-z^2)}{(z^2+a^2)^2}\ln\left(\frac{1+z}{1-z}\right)\,\mathrm{d}z$$

where C is the following contour



1. γ is a large semicircle of radius R centered at the origin.

2. α and β are quarter circle indents around the branch points $1, 1$ respectively of radius ϵ.

3. There are 3 line segments joining

- $-\sqrt{R^2-\epsilon^2}+i\epsilon$ and $-1+i\epsilon$
- $-1+\epsilon$ and $1-\epsilon$
- $1+i\epsilon$ and $\sqrt{R^2-\epsilon^2}+i\epsilon$

For the line segments above the branch cuts, we have

$$\begin{aligned}
&\int_{-\sqrt{R^2-\epsilon^2}+i\epsilon}^{-1+i\epsilon} f(z)\,\mathrm{d}z + \int_{1+i\epsilon}^{\sqrt{R^2-\epsilon^2}+i\epsilon} f(z)\,\mathrm{d}z \\
&= \int_{-\sqrt{R^2-\epsilon^2}+i\epsilon}^{-1+i\epsilon} \frac{z(1-z^2)}{(z^2+a^2)^2}\left\{\ln\left(\frac{|1+z|}{1-z}\right)+i\pi\right\}\mathrm{d}z \\
&+ \int_{1+i\epsilon}^{\sqrt{R^2-\epsilon^2}+i\epsilon} \frac{z(1-z^2)}{(z^2+a^2)^2}\left\{\ln\left(\frac{1+z}{|1-z|}\right)-i\pi\right\}\mathrm{d}z \\
&\to 2\int_1^\infty \frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left(\frac{x+1}{x-1}\right)\mathrm{d}x \quad \text{as } R\to\infty \text{ and } \epsilon\to 0^+
\end{aligned}$$

Similarly for the other segment,

$$\int_{-1+\epsilon}^{1+\epsilon} f(z)\,\mathrm{d}z \to 2\int_0^1 \frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left(\frac{1+x}{1-x}\right)\mathrm{d}x \quad \text{as } \epsilon\to 0^+$$

And also,

$$\int_\gamma f(z)\,\mathrm{d}z \to \pi^2 \text{ as } R\to\infty$$

$$\int_\alpha f(z)\,\mathrm{d}z \to 0 \quad , \int_\beta f(z)\,\mathrm{d}z \to 0 \text{ as } \epsilon\to 0^+$$

So by the residue theorem, we get

$$\begin{aligned}
&\lim_{\substack{R\to\infty\\ \epsilon\to 0^+}} \oint_C f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}_{z=ia} \frac{z(1-z^2)}{(z^2+a^2)^2}\ln\left(\frac{1+z}{1-z}\right)\\
&\implies 2\int_0^1 \frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left(\frac{1+x}{1-x}\right)\mathrm{d}x + 2\int_1^\infty \frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left(\frac{x+1}{x-1}\right)\mathrm{d}x + \pi^2\\
&\quad = 2\pi i\left(-\frac{i}{2a} - \frac{1}{2}\ln\left(\frac{i-a}{i+a}\right)\right)\\
&\implies 2\int_0^\infty \frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left|\frac{1+x}{1-x}\right|\mathrm{d}x + \pi^2 = \frac{\pi}{a} - 2\pi\tan^{-1}\frac{1}{a} + \pi^2\\
&\implies \int_0^\infty \frac{x(1-x^2)}{(x^2+a^2)^2}\ln\left|\frac{1+x}{1-x}\right|\mathrm{d}x = \pi\left(\frac{1}{2a} - \tan^{-1}\frac{1}{a}\right)
\end{aligned}$$

□

Exercise 1.2.53: Evaluate:

$$\int_0^\infty \frac{\left(\tan^{-1}x\right)^2\ln(x)}{x^{3/2}}\mathrm{d}x$$

Solution: Let

$$I_1(\alpha) = \int_0^\infty \frac{x^s\tan^{-1}\alpha x}{1+x^2}\mathrm{d}x$$

Then by differentiating with respect to α, we have

$$\begin{aligned}
I_1'(\alpha) &= \int_0^\infty \frac{x^{s+1}}{(1+x^2)(1+\alpha^2x^2)}\mathrm{d}x\\
&= \int_0^\infty x^{s+1}\left(\frac{1}{(1-\alpha^2)(1+x^2)} - \frac{\alpha^2}{(1-\alpha^2)(1+\alpha^2x^2)}\right)\mathrm{d}x\\
&= -\frac{\pi}{2\sin\left(\frac{\pi s}{2}\right)}\cdot\frac{1}{1-\alpha^2} + \frac{\pi}{2\sin\left(\frac{\pi s}{2}\right)}\cdot\frac{\alpha^{-s}}{1-\alpha^2}\\
&= -\frac{\pi}{2\sin\left(\frac{\pi s}{2}\right)}\left(\frac{1-\alpha^{-s}}{1-\alpha^2}\right)
\end{aligned}$$

Since $I_1(0) = 0$, we can write

$$\begin{aligned}
I_1(1) &= -\frac{\pi}{2\sin\left(\frac{\pi s}{2}\right)}\int_0^1\left(\frac{1-\alpha^{-s}}{1-\alpha^2}\right)\mathrm{d}\alpha\\
&= -\frac{\pi}{2\sin\left(\frac{\pi s}{2}\right)}\left\{\frac{\gamma+\psi_0\left(\frac{1-s}{2}\right)}{2} + \ln 2\right\}
\end{aligned}$$

Now let $I_2(\alpha) = \int_0^\infty x^{s-1}\left(\tan^{-1}\alpha x\right)^2\,\mathrm{d}x$

Then we have

$$\begin{aligned} I_2'(\alpha) &= 2\int_0^\infty \frac{x^s \tan^{-1}\alpha x}{1+\alpha^2 x^2}\,\mathrm{d}x \\ &= \frac{2}{\alpha^{s+1}}\int_0^\infty \frac{x^s \tan^{-1} x}{1+x^2}\mathrm{d}x \quad x\mapsto x/\alpha \\ &= \frac{2}{\alpha^{s+1}} I_1(1) \end{aligned}$$

So that

$$\begin{aligned} \int_0^\infty x^{s-1}\left(\tan^{-1}x\right)^2\,\mathrm{d}x = I_2(1) &= \int_0^1 \frac{2}{\alpha^{s+1}} I_1(1)\,\mathrm{d}\alpha \\ &= 2I_1(1)\int_0^1 \frac{1}{\alpha^{s+1}}\mathrm{d}\alpha = -\frac{2}{s} I_1(1) \\ &= \frac{\pi}{2s\sin\left(\frac{\pi s}{2}\right)}\left\{\gamma+\psi_0\left(\frac{1-s}{2}\right)+2\ln 2\right\} \end{aligned}$$

Now, we need to differentiate the above formula with respect to s:

$$\int_0^\infty x^{s-1}\left(\tan^{-1}x\right)^2\,\mathrm{d}x = \frac{\pi}{2s\sin\left(\frac{\pi s}{2}\right)}\left\{\gamma+\psi_0\left(\frac{1-s}{2}\right)+2\ln 2\right\}$$

$$\begin{aligned} \Longrightarrow \int_0^\infty x^{s-1}\left(\tan^{-1}x\right)^2\ln(x)\,\mathrm{d}x = &\left\{-\frac{\pi}{2s^2}\cdot\frac{1}{\sin\left(\frac{\pi s}{2}\right)} - \frac{\pi^2}{4s}\cdot\frac{\cos\left(\frac{\pi s}{2}\right)}{\sin^2\left(\frac{\pi s}{2}\right)}\right\} \\ &\cdot\left\{\gamma+\psi_0\left(\frac{1-s}{2}\right)+2\ln 2\right\} - \frac{\pi}{4s\sin\left(\frac{\pi s}{2}\right)}\psi_1\left(\frac{1-s}{2}\right) \end{aligned}$$

Put $s=-\dfrac{1}{2}$ to get the answer

$$\int_0^\infty \frac{\left(\tan^{-1}x\right)^2\ln(x)}{x^{3/2}}\mathrm{d}x = 4\sqrt{2}\pi\mathbf{G} - \sqrt{2}\pi^2 + \frac{\pi^3}{2\sqrt{2}} - 2\sqrt{2}\pi - \frac{\pi^2}{\sqrt{2}}\ln(2)$$

□

Exercise 1.2.54: Evaluate:

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2(1-x)}{x}\,\mathrm{d}x$$

Solution:

$$\begin{aligned} I &= \sum_{n=1}^\infty \frac{1}{n^2}\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x \\ &= \sum_{n=1}^\infty \frac{1}{n^2}\left(\frac{(H_n)^2+H_n^{(2)}}{n}\right) \\ &= \sum_{n=1}^\infty \frac{(H_n)^2}{n^3} + \sum_{n=1}^\infty \frac{H_n^{(2)}}{n^3} \end{aligned}$$

We can prove using residues that

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5)$$
$$\sum_{n=1}^{\infty} \frac{(H_n)^2}{n^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5)$$

Adding them gives

$$I = 2\zeta(2)\zeta(3) - \zeta(5)$$

□

Exercise 1.2.55: Prove:

$$\int_0^{\infty} x^{z-1}\mathrm{li}(e^{-x})\,\mathrm{d}x = -\frac{\Gamma(z)}{z}$$

where $\mathrm{li}(z)$ is the logarithmic integral defined by $\mathrm{li}(z) = \int_0^z \frac{1}{\ln t}\mathrm{d}t$

Proof: By parts we have

$$I = -\frac{1}{z}\int_0^{\infty} x^{z-1}e^{-x}\,\mathrm{d}x = \frac{-\Gamma(z)}{z}$$

□

Exercise 1.2.56: Evaluate:

$$\int_0^{\infty} \frac{\ln(1+x^2)}{x^2+2x\cos\theta+1}\mathrm{d}x$$

Solution: Let $f(z) = \dfrac{\ln(1+z^2)\ln(-z)}{z^2+2z\cos\theta+1}$, the logarithms being defined no their main sheets.

Since $z^2+2z\cos\theta+1 = (z+e^{i\theta})(z+e^{-i\theta})$, there are poles at $z = -e^{i\theta}, -e^{-i\theta}$.

We will integrate $f(z)$ around a keyhole contour about the positive real axis but with additional keyholes about the branch points $e^{i\pi/2}$ and $e^{3i\pi/2}$.

The usual calculation then gives

$$\begin{aligned}
&-2\pi i\int_0^{\infty}\frac{\ln(1+x^2)}{x^2+2\cos\theta x+1}\mathrm{d}x = 2\pi i\left(\mathrm{Res}_{z=-e^{i\theta}}f(z) + \mathrm{Res}_{z=-e^{-i\theta}}f(z)\right)\\
&+2\pi i\int_{e^{i\pi/2}}^{e^{i\pi/2}\infty}\frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x + 2\pi i\int_{e^{i3\pi/2}}^{e^{i3\pi/2}\infty}\frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x\\
&\Longrightarrow \int_0^{\infty}\frac{\ln(1+x^2)}{x^2+2\cos\theta x+1}\mathrm{d}x = -\left(\mathrm{Res}_{z=-e^{i\theta}}f(z) + \mathrm{Res}_{z=-e^{-i\theta}}f(z)\right)\\
&-\int_{e^{3i\pi/2}}^{e^{3i\pi/2}\infty}\frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x - \int_{e^{i\pi/2}}^{e^{i\pi/2}\infty}\frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x \quad (*)
\end{aligned}$$

The residue computation is relatively easy:

$$\begin{aligned}
\mathrm{Res}_{z=-e^{i\theta}}f(z) + \mathrm{Res}_{z=-e^{-i\theta}}f(z) &= -\frac{\theta}{2\sin\theta}\ln(1+e^{2i\theta}) - \frac{\theta}{2\sin\theta}\ln(1+e^{-2i\theta})\\
&= -\frac{\theta\ln|2\cos\theta|}{\sin\theta}
\end{aligned}$$

Substituting $x \mapsto 1/x$ gives

$$\begin{aligned}\int_{e^{i\pi/2}}^{e^{i\pi/2}\infty} \frac{\ln(-x)}{x^2+2x\cos\theta+1}\mathrm{d}x &= -\int_0^{e^{-i\pi/2}} \frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x \\ &= i\int_0^1 \frac{\ln|x|+\dfrac{i\pi}{2}}{(1-x^2)-2ix\cos\theta}\mathrm{d}x\end{aligned}$$

and Simliarly

$$\int_{e^{3i\pi/2}}^{e^{3i\pi/2}\infty} \frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x = -i\int_0^1 \frac{\ln|x|-\dfrac{i\pi}{2}}{(1-x^2)+2ix\cos\theta}\mathrm{d}x$$

Adding them both results in

$$\begin{aligned}&\int_{e^{i\pi/2}}^{e^{i\pi/2}\infty} \frac{\ln(-x)}{x^2+2x\cos\theta+1}\mathrm{d}x + \int_{e^{3i\pi/2}}^{e^{3i\pi/2}\infty} \frac{\ln(-x)}{x^2+2\cos\theta x+1}\mathrm{d}x \\ &= i\int_0^1 \frac{4ix\cos\theta\ln|x|+i\pi(1-x^2)}{(1-x^2)^2+4x^2\cos^2\theta}\mathrm{d}x \\ &= -4\cos\theta\int_0^1 \frac{x\ln x}{(1-x^2)^2+4x^2\cos^2\theta}\mathrm{d}x - \pi\int_0^1 \frac{1-x^2}{(1-x^2)^2+4x^2\cos^2\theta}\mathrm{d}x\end{aligned}$$

Using **Lemma 1** on this page
http://integralsandseries.prophpbb.com/topic197.html#p1399
we obtain

$$\begin{aligned}&4\cos\theta\int_0^1 \frac{x\ln(x)}{(1-x^2)^2+4x^2\cos^2\theta}\mathrm{d}x = \cos\theta\int_0^1 \frac{\ln(x)}{(1-x)^2+4x\cos^2\theta}\mathrm{d}x \quad (x\mapsto\sqrt{x}) \\ &= \cos\theta\int_0^1 \frac{\ln(x)}{x^2+2x(2\cos^2\theta-1)+1}\mathrm{d}x = \cos\theta\int_0^1 \frac{\ln(x)}{x^2+2\cos(2\theta)x+1}\mathrm{d}x \\ &= \frac{\cos\theta}{\sin(2\theta)}\int_0^1 \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(2n\theta)}{x^{n+1}}\ln(x)\,\mathrm{d}x = \frac{\cos\theta}{\sin(2\theta)}\sum_{n=1}^{\infty}(-1)^{n+1}\sin(2n\theta)\int_0^1\frac{\ln x}{x^{n+1}}\mathrm{d}x \\ &= \frac{\cos\theta}{\sin(2\theta)}\sum_{n=1}^{\infty}(-1)^n\frac{\sin(2n\theta)}{n^2} = \frac{1}{4i\sin(\theta)}\sum_{n=1}^{\infty}(-1)^n\frac{e^{2in\theta}-e^{-2in\theta}}{n^2} \\ &= \frac{\mathrm{Li}_2(-e^{2i\theta})-\mathrm{Li}_2(-e^{-2i\theta})}{4i\sin(\theta)}\end{aligned}$$

and similarly

$$\begin{aligned}
&\pi\int_0^1 \frac{1-x^2}{(1-x^2)^2+4x^2\cos^2\theta}\mathrm{d}x = \pi\int_0^1 \frac{1-x^2}{x^4+2\cos(2\theta)x^2+1}\mathrm{d}x\\
&= \frac{\pi}{\sin(2\theta)}\int_0^1 (1-x^2)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(2n\theta)}{x^{2n+2}}\,\mathrm{d}x\\
&= \frac{\pi}{\sin(2\theta)}\sum_{n=1}^{\infty}(-1)^{n+1}\sin(2n\theta)\int_0^1\left(\frac{1}{x^{2n+2}}-\frac{1}{x^{2n}}\right)\mathrm{d}x\\
&= \frac{\pi}{\sin(2\theta)}\sum_{n=1}^{\infty}(-1)^{n+1}\sin(2n\theta)\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\\
&= \frac{\pi}{\sin(2\theta)}\Im\left(e^{i\theta}\tan^{-1}(e^{i\theta})+e^{-i\theta}\tan^{-1}(e^{i\theta})-1\right)\\
&= \frac{\pi}{2\sin\theta}\Re\ln\left(\frac{1-ie^{i\theta}}{1+ie^{i\theta}}\right) = \frac{\pi}{2\sin\theta}\ln\left|\frac{1-ie^{i\theta}}{1+ie^{i\theta}}\right|\\
&= \frac{\pi}{4\sin(\theta)}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)
\end{aligned}$$

Finally putting these in $(*)$ we obtain

$$\int_0^\infty \frac{\ln(1+x^2)}{x^2+2x\cos\theta+1}\mathrm{d}x = \frac{1}{\sin\theta}\left\{\frac{\pi}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)+\theta\ln|2\cos\theta| + \frac{\mathrm{Li}_2(-e^{2i\theta})-\mathrm{Li}_2(-e^{-2i\theta})}{4i}\right\}$$

□

Exercise 1.2.57: Evaluate:

$$\int_0^{\pi/2}\sin(2nx)\sinh(a\sin x)\sin(a\cos x)\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int_0^{\pi/2}\sin(2nx)\sinh(a\sin x)\sin(a\cos x)\;\mathrm{d}x = \Im\int_0^{\frac{\pi}{2}}\sin(2nx)\cos(ae^{-ix})\;\mathrm{d}x\\
&= \Im\int_0^{\frac{\pi}{2}}\sin(2nx)\sum_{k=0}^{\infty}\frac{(-1)^k(ae^{-ix})^{2k}}{2k!}\mathrm{d}x = \Im\sum_{k=1}^{\infty}\frac{(-1)^n a^k}{2k!}\int_0^{\frac{\pi}{2}}\sin(2nx)e^{-2ikx}\;\mathrm{d}x\\
&= \Im\sum_{k=1}^{\infty}\frac{(-1)^k a^k}{(2k)!}\int_0^{\frac{\pi}{2}}\sin(2nx)\Big(\cos 2nx - i\sin 2nx\Big)\;\mathrm{d}x
\end{aligned}$$

Due to the orthogonality of sine and cosine functions, $\int_0^{\pi/2}\sin(2nx)\cos(2kx)\mathrm{d}x = 0$ for all k and $\int_0^{\pi/2}\sin(2nx)\sin(2kx)\mathrm{d}x = 0$ unless $k=n$. So

$$\begin{aligned}
&\int_0^{\pi/2}\sin(2nx)\sinh(a\sin x)\sin(a\cos x)\;\mathrm{d}x = -\Im\frac{(-1)^n a^{2n}}{(2n)!}\int_0^{\pi/2}\sin^2(2nx)\;\mathrm{d}x\\
&= \frac{(-1)^{n+1}a^{2n}}{2n!}\int_0^{\pi/2}\left(\frac{1}{2}-\frac{\cos 4nx}{2}\right)\mathrm{d}x = \frac{(-1)^{n+1}a^{2n}}{2n!}\left(\frac{\pi}{4}-0\right) = \frac{\pi(-1)^{n+1}a^{2n}}{4(2n)!}
\end{aligned}$$

□

Exercise 1.2.58: Evaluate:

$$\int_0^\infty \mathrm{Si}(x)\mathrm{Ci}(x)\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int \mathrm{Si}(x)\mathrm{Ci}(x)\,\mathrm{d}x \\
&= (x\mathrm{Ci}(x)-\sin(x))\,\mathrm{Si}(x) - \int \frac{x\mathrm{Ci}(x)-\sin(x)}{x}\sin(x)\,\mathrm{d}x \\
&= (x\mathrm{Ci}(x)-\sin(x))\,\mathrm{Si}(x) - \int \mathrm{Ci}(x)\sin(x)\,dx + \int \frac{\sin^2 x}{x}\mathrm{d}x \\
&= (x\mathrm{Ci}(x)-\sin(x))\,\mathrm{Si}(x) - \left(-\mathrm{Ci}(x)\cos(x) + \int \frac{\cos^2(x)}{x}\mathrm{d}x\right) + \int \frac{\sin^2 x}{x}\mathrm{d}x \\
&= (x\mathrm{Ci}(x)-\sin(x))\,\mathrm{Si}(x) + \mathrm{Ci}(x)\cos(x) - \int \frac{\cos^2 x-\sin^2 x}{x}\mathrm{d}x \\
&= (x\mathrm{Ci}(x)-\sin(x))\,\mathrm{Si}(x) + \mathrm{Ci}(x)\cos(x) - \int \frac{\cos(2x)}{x}\mathrm{d}x \\
&= (x\mathrm{Ci}(x)-\sin(x))\,\mathrm{Si}(x) + \mathrm{Ci}(x)\cos(x) - \mathrm{Ci}(2x)
\end{aligned}$$

And therefore

$$\begin{aligned}
&\int_0^\infty \mathrm{Si}(x)\mathrm{Ci}(x)\,\mathrm{d}x \\
&= \lim_{x\to\infty}\left\{\cos(x)\mathrm{Ci}(x)-\mathrm{Ci}(2x)+(x\mathrm{Ci}(x)-\sin(x))\mathrm{Si}(x)\right\} \\
&\quad - \lim_{x\to 0}\left\{\cos(x)\mathrm{Ci}(x)-\mathrm{Ci}(2x)+(x\mathrm{Ci}(x)-\sin(x))\mathrm{Si}(x)\right\}
\end{aligned}$$

The result follows since

$$\begin{aligned}
\lim_{x\to 0}\left\{\cos(x)\mathrm{Ci}(x)-\mathrm{Ci}(2x)+(x\mathrm{Ci}(x)-\sin(x))\mathrm{Si}(x)\right\} &= -\ln(2) \\
\lim_{x\to\infty}\left\{\cos(x)\mathrm{Ci}(x)-\mathrm{Ci}(2x)+(x\mathrm{Ci}(x)-\sin(x))\mathrm{Si}(x)\right\} &= 0
\end{aligned}$$

Hence

$$\int_0^\infty \mathrm{Si}(x)\mathrm{Ci}(x)\mathrm{d}x = \ln 2$$

□

Exercise 1.2.59: Evaluate:

$$\int_0^\infty x^2 e^{-x^2}\mathrm{erf}(x)\,\mathrm{d}x$$

Solution: Let $I(a) = \int_0^\infty e^{-a^2x^2}\mathrm{erf}(x)\,\mathrm{d}x$

$$\begin{aligned}
I(a) &= \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^x e^{-a^2x^2}e^{-t^2}\,\mathrm{d}t\,\mathrm{d}x = \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^1 xe^{-a^2x^2}e^{-x^2u^2}\,\mathrm{d}u\,\mathrm{d}x \\
&= \frac{2}{\sqrt{\pi}}\int_0^1\int_0^\infty xe^{-(a^2+u^2)x^2}\,\mathrm{d}x\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{a^2+u^2}\,\mathrm{d}u \\
&= \frac{1}{a\sqrt{\pi}}\arctan\left(\frac{1}{a}\right)
\end{aligned}$$

Then

$$I'(a) = -2a\int_0^\infty x^2 e^{-a^2x^2}\operatorname{erf}(x)\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\frac{-\frac{a}{1+a^2} - \arctan\frac{1}{a}}{a^2}$$

And

$$\int_0^\infty x^2 e^{-x^2}\operatorname{erf}(x)\,\mathrm{d}x = -\frac{1}{2\sqrt{\pi}}\left(-\frac{1}{2}-\frac{\pi}{4}\right) = \frac{1}{4\sqrt{\pi}} + \frac{\sqrt{\pi}}{8}$$

□

Exercise 1.2.60: Evaluate:

$$\int_0^\infty x^2\, e^{-x^2}\operatorname{erf}(x)\,\ln(x)\,\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int_0^\infty x^a e^{-x^2}\operatorname{erf}(x)\,\mathrm{d}x = \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^x x^a e^{-x^2}e^{-t^2}\,\mathrm{d}t\,\mathrm{d}x\\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^1 x^{a+1}e^{-x^2}e^{-x^2u^2}\,\mathrm{d}u\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}\int_0^1\int_0^\infty x^{a+1}e^{-(1+u^2)x^2}\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{1}{\sqrt{\pi}}\Gamma\left(1+\frac{a}{2}\right)\int_0^1\frac{1}{(1+u^2)^{a/2+1}}\,\mathrm{d}u\\
&= \frac{1}{2\sqrt{\pi}}\Gamma\left(1+\frac{a}{2}\right)\int_0^1\frac{w^{-1/2}}{(1+w)^{1+a/2}}\,\mathrm{d}w\\
&= \frac{1}{2\sqrt{\pi}}\Gamma\left(1+\frac{a}{2}\right)\beta\left(\frac{1}{2};\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)
\end{aligned}$$

Then

$$\int_0^\infty x^2e^{-x^2}\operatorname{erf}(x)\ln x\,\mathrm{d}x = \frac{1}{2\sqrt{\pi}}\left(\frac{1}{2}\Gamma'(2)\beta\left(\frac{1}{2};\frac{1}{2},\frac{3}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}a}\beta\left(\frac{1}{2};\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)\Big|_{a=2}\right)$$

where

$$\Gamma'(2) = 1-\gamma$$

And

$$\begin{aligned}
\beta\left(\frac{1}{2};\frac{1}{2},\frac{3}{2}\right) &= \int_0^1\frac{x^{-1/2}}{(1+x)^2}\,\mathrm{d}x = 2\int_0^1\frac{du}{(1+u^2)^2}\,\mathrm{d}u\\
&= 2\int_0^{\pi/4}\cos^2 w\,\mathrm{d}w = 2\int_0^{\pi/4}\left(\frac{1}{2}+\frac{\cos 2w}{w}\right)\mathrm{d}w\\
&= 2\left(\frac{\pi}{8}+\frac{1}{4}\right)
\end{aligned}$$

From the definition of the incomplete beta function, it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\beta\left(\frac{1}{2};\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)\Bigg|_{a=2} &= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\sqrt{1-x}}{\sqrt{x}}\ln(1-x)\,\mathrm{d}x \\
&= \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-x^2}\ln(1-x^2)\,\mathrm{d}x \quad (x\mapsto x^2) \\
&= 2\int_0^{\frac{\pi}{4}} \cos^2(x)\ln(\cos x)\,\mathrm{d}x \quad (x\mapsto \sin x) \\
&= 2\int_0^{\frac{\pi}{4}} \cos^2(x)\left(-\ln(2)-\sum_{k=1}^{\infty}(-1)^k\frac{\cos(2kx)}{k}\right)\mathrm{d}x \\
&= -\ln(2)\left(\frac{2+\pi}{4}\right)-2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{4}}\cos^2(x)\cos(2kx)\,\mathrm{d}x \\
&= -\ln(2)\left(\frac{2+\pi}{4}\right)+\frac{4+\pi}{8}-2\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{4}}\cos^2(x)\cos(2kx)\,\mathrm{d}x \\
&= -\ln(2)\left(\frac{2+\pi}{4}\right)+\frac{4+\pi}{8}+\frac{1}{2}\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\left\{\frac{\cos\frac{\pi k}{2}}{k^2-1}-\frac{\sin\frac{\pi k}{2}}{k}\right\} \\
&= -\ln(2)\left(\frac{2+\pi}{4}\right)+\frac{\pi}{8}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(4k^2-1)} \\
&= -\ln(2)\left(\frac{2+\pi}{4}\right)+\frac{\pi}{8}+\frac{\mathbf{G}}{2}+\frac{\ln(2)-1}{4} \\
&= -\frac{1}{4}+\frac{\mathbf{G}}{2}+\frac{\pi}{8}-\frac{\ln(2)}{4}-\frac{1}{4}\pi\ln(2)
\end{aligned}$$

Hence

$$\int_0^{\infty} x^2\,e^{-x^2}\,\mathrm{erf}(x)\,\ln(x)\,\mathrm{d}x = \frac{2-\ln(2)}{16}\sqrt{\pi}-\frac{\gamma+\ln(2)}{16\sqrt{\pi}}(\pi+2)+\frac{\mathbf{G}}{4\sqrt{\pi}}$$

□

Exercise 1.2.61: Evaluate:

$$\int_0^{\pi/2}\cos^a x\frac{\sin[(a+1)x]}{\sin x}\,\mathrm{d}x=\frac{\pi}{2}$$

Solution: Method One:

One may begin with the identity:

$$2^a\cos^a(x)\sin(a+1)x=\sum_{k=0}^{\infty}\binom{a}{k}\sin(2k+1)x$$

Divide both sides by $\sin(x)$:

$$\frac{2^a\cos^a(x)\sin(a+1)x}{\sin(x)}=\sum_{k=0}^{\infty}\binom{a}{k}\frac{\sin(2k+1)x}{\sin(x)}$$

Integrate both sides and note that $\int_0^{\frac{\pi}{2}}\frac{\sin(2k+1)x}{\sin(x)}\mathrm{d}x=\frac{\pi}{2}$ whatever the integer value of k is.

So, divide both sides by 2^a, we essentially have

$$\int_0^{\frac{\pi}{2}}\frac{\cos^a(x)\sin(a+1)x}{\sin(x)}\mathrm{d}x=\frac{1}{2^a}\sum_{k=0}^{\infty}\binom{a}{k}\frac{\pi}{2}$$

But,

$$\frac{\pi}{2}\sum_{k=0}^{\infty}\binom{a}{k}=\pi\cdot 2^{a-1}$$

So, putting the right side together gives $\frac{1}{2^a}\cdot\pi\cdot 2^{a-1}=\frac{\pi}{2}$ Thus,

$$\int_0^{\frac{\pi}{2}}\frac{\cos^a(x)\sin(a+1)x}{\sin(x)}\mathrm{d}x=\frac{\pi}{2}$$

Method Two:

$$\sum_{k=0}^{\infty}\binom{a}{k}\sin(2k+1)x=\Im\sum_{k=0}^{\infty}\binom{a}{k}e^{i(2k+1)x}=\Im\, e^{ix}\sum_{k=0}^{\infty}\binom{a}{k}(e^{2ix})^k$$
$$=\Im\, e^{ix}(1+e^{2ix})^a=e^{ix}\left(e^{ix}(e^{-ix}+e^{ix})\right)^a=\Im\, e^{ix(a+1)}(e^{-ix}+e^{ix})^a$$
$$=\Im\, e^{ix(a+1)}2^a\cos^a(x)=2^a\cos^a(x)\sin(a+1)x$$

$$\int_0^{\pi/2}\cos^a(x)\frac{\sin(a+1)x}{\sin x}\,\mathrm{d}x$$
$$=\frac{1}{2}\Im\int_{-\pi/2}^{\pi/2}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^a\frac{e^{i(a+1)x}}{\frac{e^{ix}-e^{-ix}}{2i}}\,\mathrm{d}x=\frac{1}{2^a}\Im\, i\int_{-\pi/2}^{\pi/2}\frac{(e^{2ix}+1)^a}{1-e^{-2ix}}\,\mathrm{d}x$$
$$=\frac{1}{2a}\Im\, i\int_{|z|=1}\frac{(z+1)^a}{1-\frac{1}{z}}\frac{\mathrm{d}z}{2iz}=\frac{1}{2^{a+1}}\Im\int_{|z|=1}\frac{(z+1)^a}{z-1}\,\mathrm{d}z$$
$$=\frac{1}{2^{a+1}}\Im\,\pi i\,\operatorname{Res}\left[\frac{(z+1)^a}{z-1},1\right]=\frac{\pi}{2^{a+1}}2^a=\frac{\pi}{2}$$

□

Exercise 1.2.62: Evaluate:

$$\int_0^{\infty}\frac{1-\cos x}{x(e^x-1)}\mathrm{d}x$$

Solution: Let

$$I(a)=\int_0^{\infty}\frac{1-\cos(ax)}{x(e^x-1)}\,\mathrm{d}x$$

$$I'(a)=\int_0^{\infty}\frac{\sin ax}{e^x-1}\,\mathrm{d}x=\int_0^{\infty}\sin ax\sum_{n=1}^{\infty}e^{-nx}\,\mathrm{d}x=\sum_{n=1}^{\infty}\int_0^{\infty}\sin(ax)e^{-nx}\,\mathrm{d}x$$
$$=\sum_{n=1}^{\infty}\frac{a}{a^2+n^2}=\frac{1}{2}\left(\pi\coth(\pi a)-\frac{1}{a}\right)$$
$$\Rightarrow I(a)=\frac{1}{2}\Big(\ln(\sinh\pi a)-\ln(a)\Big)+C=\frac{1}{2}\ln\left(\frac{\sinh\pi a}{a}\right)+C$$
$$\Rightarrow I(0)=0=\frac{1}{2}\lim_{a\to 0}\ln\left(\frac{\sinh\pi a}{a}\right)+C\implies C=-\frac{\ln(\pi)}{2}$$
$$\int_0^{\infty}\frac{1-\cos x}{x(e^x-1)}\,\mathrm{d}x=\frac{1}{2}\ln\left(\frac{\sinh\pi}{\pi}\right)$$

□

Exercise 1.2.63: Evaluate:

$$\int_0^\infty \ln\left(\frac{1+a\sin bx}{1-a\sin bx}\right)\frac{\mathrm{d}x}{x}$$

Solution: This integral can be solved by cleverly breaking up and making use of the expansion $\dfrac{1}{\sin x}=\sum_{n=-\infty}^{\infty}\dfrac{(-1)^n}{x+n\pi}$.

$$\begin{aligned}
\int_0^\infty \ln\left(\frac{1+a\sin bx}{1-a\sin bx}\right)\frac{\mathrm{d}x}{x} &= \frac{1}{2}\int_{-\infty}^{\infty}\ln\left(\frac{1+a\sin t}{1-a\sin t}\right)\frac{\mathrm{d}t}{t} \quad t=bx\\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi}\ln\left(\frac{1+a\sin t}{1-a\sin t}\right)\frac{1}{t}\mathrm{d}t\\
&= \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin t}{1-a\sin t}\right)\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{t+n\pi}\right)\mathrm{d}t\\
&= \int_0^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin t}{1-a\sin t}\right)\frac{1}{\sin t}\,\mathrm{d}t\\
&= \pi\arcsin(a)
\end{aligned}$$

□

Exercise 1.2.64: Evaluate:

$$\int_0^\infty \frac{1}{x}\left(\frac{\sinh ax}{\sinh x}-ae^{-2x}\right)\mathrm{d}x \quad |a|<1$$

✎ **Solution:**

$$\begin{aligned}
&\int_0^\infty \left(\frac{\sinh ax}{\sinh x} - ae^{-2x}\right) \frac{\mathrm{d}x}{x} = \int_0^\infty \int_0^\infty \left(\frac{\sinh ax}{\sinh x} - ae^{-2x}\right) e^{-xt}\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \left(\frac{e^{ax}-e^{-ax}}{e^x-e^{-x}} - ae^{-2x}\right) e^{-tx}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\infty \left(\int_0^\infty \frac{e^{ax}-e^{-ax}}{1-e^{-2x}} e^{-(t+1)x}\,\mathrm{d}x - a\int_0^\infty e^{-(t+2)x}\,\mathrm{d}x\right)\mathrm{d}t \\
&= \int_0^\infty \left(\int_0^\infty (e^{ax}-e^{-ax})e^{-(t+1)x} \sum_{n=0}^\infty e^{-2nx}\,\mathrm{d}x - \frac{a}{t+2}\right)\mathrm{d}t \\
&= \int_0^\infty \left[\sum_{n=0}^\infty \int_0^\infty \left(e^{-(t-a+1+2n)x} - e^{-(t+a+1+2n)x}\right)\mathrm{d}x - \frac{a}{t+2}\right]\mathrm{d}t \\
&= \int_0^\infty \left[\sum_{n=0}^\infty \left(\frac{1}{t-a+1+2n} - \frac{1}{t+a+1+2n}\right) - \frac{a}{t+2}\right]\mathrm{d}t \\
&= \int_0^\infty \left[-\frac{1}{2}\psi\left(\frac{t-a+1}{2}\right) + \frac{1}{2}\psi\left(\frac{t+a+1}{2}\right) - \frac{a}{t+2}\right]\mathrm{d}t \\
&= -\ln\Gamma\left(\frac{t-a+1}{2}\right) + \ln\Gamma\left(\frac{t+a+1}{2}\right) - a\ln(t+2)\Big|_0^\infty \\
&= \ln\left(\frac{\Gamma\left(\frac{t+a+1}{2}\right)}{\Gamma\left(\frac{t-a+1}{2}\right)(t+2)^a}\right)\Bigg|_0^\infty = -a\ln 2 - \ln\left(\frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{1-a}{2}\right)2^a}\right) \\
&= \ln\left(\frac{\Gamma\left(\frac{1-a}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)}\right) = \ln\left(\frac{\Gamma\left(\frac{1-a}{2}\right)\Gamma\left(\frac{1+a}{2}\right)}{\Gamma^2\left(\frac{1+a}{2}\right)}\right) = \ln\left(\frac{\pi}{\sin\left(\pi\frac{1+a}{2}\right)\Gamma^2\left(\frac{1+a}{2}\right)}\right) \\
&= \ln\left(\frac{\pi}{\cos\left(\frac{\pi a}{2}\right)\Gamma^2\left(\frac{1+a}{2}\right)}\right)
\end{aligned}$$

□

Exercise 1.2.65: Evaluate:

$$\int_0^\infty \ln\left(\frac{1+a\sin^2 bx}{1-a\sin^2 bx}\right)\frac{1}{x^2}\,\mathrm{d}x$$

✎ Solution:

$$\begin{aligned}
\int_0^\infty \ln\left(\frac{1+a\sin^2 bx}{1-a\sin^2 bx}\right)\frac{1}{x^2}\,\mathrm{d}x &= b\int_0^\infty \ln\left(\frac{1+a\sin^2 x}{1-a\sin^2 x}\right)\frac{1}{x^2}\,\mathrm{d}x \\
&= 2b\int_0^\infty \left(\sum_{n=0}^{\infty}\frac{a^{2n+1}\sin^{2(2n+1)}(x)}{2n+1}\right)\frac{\mathrm{d}x}{x^2} \\
&= 2b\sum_{n=0}^{\infty}\frac{a^{2n+1}}{2n+1}\int_0^\infty\left(\frac{\sin^{2n+1}(x)}{x}\right)^2\mathrm{d}x \\
&= b\sqrt{\pi}\sum_{n=0}^{\infty}\frac{a^{2n+1}}{2n+1}\cdot\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+1)} \qquad (1)\\
&= \frac{\pi ab\sqrt{2}}{\sqrt{1+\sqrt{1-a^2}}} \qquad (2)\\
&= \pi b\left(\sqrt{1+a}-\sqrt{1-a}\right)
\end{aligned}$$

Identities Used:

(1): $\displaystyle\int_0^\infty\left(\frac{\sin^{2n+1}(x)}{x}\right)^2\mathrm{d}x = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+1)}$

(2): $\displaystyle\sum_{n=0}^{\infty}\frac{a^{2n+1}}{2n+1}\cdot\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+1)} = \frac{a\sqrt{2\pi}}{\sqrt{1+\sqrt{1-a^2}}}$ □

⬎ Exercise 1.2.66: Prove:

$$\int_0^\infty \mathrm{Ci}(x)^3\,\mathrm{d}x = -\frac{3\pi}{2}\ln(2)$$

☞ Proof: First integrate by parts.

$$\int_0^\infty \mathrm{Ci}^3(x)\,\mathrm{d}x = x\mathrm{Ci}^3(x)\Big|_0^\infty - 3\int_0^\infty \mathrm{Ci}^2(x)\cos x\,\mathrm{d}x$$

The boundary term goes to zero because $\mathrm{Ci}(x)$ behaves like $\ln x$ near $x=0$ and like $\dfrac{\sin x}{x}$ near $x=\infty$.

Integrate by parts again.

$$= -6\mathrm{Ci}^2(x)\sin x\Big|_0^\infty + 6\int_0^\infty \frac{\mathrm{Ci}(x)\sin x\cos x}{x}\,\mathrm{d}x = 3\int_0^\infty\frac{\mathrm{Ci}(x)\sin 2x}{x}\,\mathrm{d}x$$

Notice that $\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,\mathrm{d}t = -\int_1^\infty \frac{\cos(xu)}{u}\,\mathrm{d}u$. Hence,

$$\begin{aligned}
\int_0^\infty \mathrm{Ci}^3(x)\,\mathrm{d}x &= -3\int_0^\infty\int_1^\infty \frac{\cos(xu)\sin 2x}{xu}\,\mathrm{d}u\,\mathrm{d}x \\
&= -3\int_1^\infty \frac{1}{u}\int_0^\infty \frac{\cos(ux)\sin 2x}{x}\,\mathrm{d}x\,\mathrm{d}u \\
&= -\frac{3}{2}\int_1^\infty \frac{1}{u}\int_0^\infty \left(\frac{\sin[(2-u)x]+\sin[(2+u)x]}{x}\right)\mathrm{d}x\,\mathrm{d}u \\
&= -\frac{3\pi}{4}\int_1^\infty \frac{1}{u}\Big(\operatorname{sgn}(2-u)+1\Big)\,\mathrm{d}u \\
&= -\frac{3\pi}{4}\int_1^2 \frac{2}{u}\,\mathrm{d}u = -\frac{3\pi\ln 2}{2}
\end{aligned}$$

□

Exercise 1.2.67: Evaluate:

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\frac{\mathrm{d}x}{1+x^2}$$

Solution: Let

$$f(a) = \int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\frac{1}{1+x^2}\mathrm{d}x$$

Then

$$f''(a) = \int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\frac{x^2}{1+x^2}\mathrm{d}x$$

Adding these two gives

$$\begin{aligned}
f(a)+f''(a) &= \int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\mathrm{d}x \\
\implies f(a)+f''(a) &= \frac{1}{2}\tan\left(\frac{a}{2}\right)
\end{aligned}$$

To solve this second order non homogenous differential equation, one can use **variation of parameters**.

https://en.wikipedia.org/wiki/Variation_of_parameters

Anyway, the general solution is

$$f(a) = \frac{\sin a}{2}\left\{2\ln\left(2\cos\frac{a}{2}\right) - a\cot(a)\right\} + C_1\cos(a) + C_2\sin(a)$$

To determine the value of the constants C_1, C_2 you can use $f(0)=0$ and $f(\pi)=\frac{\pi}{2}$. It turns out that $C_1, C_2 = 0$ so the final answer is

$$f(a) = \frac{\sin a}{2}\left\{2\ln\left(2\cos\frac{a}{2}\right) - a\cot(a)\right\}$$

□

Exercise 1.2.68: Evaluate:

$$\int_0^\infty \frac{\ln\left(\sqrt{x+1}-1\right)\ln\left(\sqrt{\frac{1}{x}+1}-1\right)}{(x+1)^{3/2}}\,\mathrm{d}x$$

✎ **Solution: Part I: A new hope**

$$\begin{aligned} I &= \int_0^\infty \frac{\ln\left(\sqrt{x+1}-1\right)\ln\left(\sqrt{\frac{1}{x}+1}-1\right)}{(x+1)^{3/2}}\mathrm{d}x \\ &= 2\int_0^1 \ln\left(\frac{1}{t}-1\right)\ln\left(\frac{1}{\sqrt{1-t^2}}-1\right)\mathrm{d}t \quad \text{where } t=\frac{1}{\sqrt{1+x}} \\ &= 2\Bigg\{\int_0^1 \ln(1-t)\ln(1-\sqrt{1-t^2})\mathrm{d}t - \int_0^1 \ln(t)\ln(1-\sqrt{1-t^2})\mathrm{d}t \\ &\quad + \int_0^1 \ln(t)\ln\sqrt{1-t^2}\mathrm{d}t - \int_0^1 \ln(1-t)\ln\sqrt{1-t^2}\mathrm{d}t\Bigg\} \end{aligned}$$

Now, it is a straightforward exercise using Taylor Series of Logarithms to show that

$$\int_0^1 \ln(1-t)\ln\sqrt{1-t^2}\mathrm{d}t = \frac{1}{2}\left(4-\frac{\pi^2}{6}+\ln^2(2)-\ln(4)\right)$$

$$\int_0^1 \ln(t)\ln\sqrt{1-t^2}\mathrm{d}t = 2-\frac{\pi^2}{8}-\ln(2)$$

Plugging these, we get

$$I = 2\left\{\int_0^1 \ln(1-t)\ln(1-\sqrt{1-t^2})\mathrm{d}t - \int_0^1 \ln(t)\ln(1-\sqrt{1-t^2})\mathrm{d}t - \frac{\pi^2}{24} - \frac{\ln^2(2)}{2}\right\}$$

Part II : Evaluation of $J = \int_0^1 \ln(t)\ln(1-\sqrt{1-t^2})\mathrm{d}t$

By parts we have

$$\begin{aligned} J &= -\int_0^1 \ln(1-\sqrt{1-t^2})\mathrm{d}t - \int_0^1 t^2\ln(t)\frac{1}{(1-\sqrt{1-t^2})\sqrt{1-t^2}}\mathrm{d}t \\ &= -\int_0^1 \ln(1-\sqrt{1-t^2})\mathrm{d}t - \int_0^1 \left(\frac{\ln t}{\sqrt{1-t^2}}+\ln t\right)\mathrm{d}t \end{aligned}$$

Apply integration by parts once more on the first integral:

$$\begin{aligned} J &= \int_0^1 \frac{t^2}{\left(1-\sqrt{1-t^2}\right)\sqrt{1-t^2}}\mathrm{d}t - \int_0^1 \left(\frac{\ln t}{\sqrt{1-t^2}}+\ln t\right)\mathrm{d}t \\ &= \int_0^1 \left(\frac{1}{\sqrt{1-t^2}}+1-\frac{\ln t}{\sqrt{1-t^2}}-\ln t\right)\mathrm{d}t \\ &= 2+\frac{\pi}{2}+\frac{\pi}{2}\ln(2) \end{aligned}$$

Where all integrals in the last step are elementary. So far we have

$$I = -4-\pi(1+\ln 2)-\frac{\pi^2}{12}-\ln^2(2)+2\int_0^1 \ln(1-t)\ln(1-\sqrt{1-t^2})\mathrm{d}t$$

Part III : Evaluation of $K = \int_0^1 \ln(1-t)\ln(1-\sqrt{1-t^2})\mathrm{d}t$

Integrate by parts to get

$$\begin{aligned} K &= \int_0^1 \frac{t}{1-t}\ln(1-\sqrt{1-t^2})\mathrm{d}t - \int_0^1 \ln(1-t)\left(1+\frac{1}{\sqrt{1-t^2}}\right)\mathrm{d}t \\ &= \int_0^1 \left(\frac{1}{1-t}-1\right)\ln(1-\sqrt{1-t^2})\mathrm{d}t - \int_0^1 \ln(1-t)\left(1+\frac{1}{\sqrt{1-t^2}}\right)\mathrm{d}t \end{aligned}$$

Apply by parts again on the first integral:

$$K = \int_0^1 \left(\frac{\ln(1-t)}{\sqrt{1-t^2}} \left(\frac{1}{t} - 1 \right) - \ln(1-t) + \frac{\ln(1-t)}{t} + \frac{1}{\sqrt{1-t^2}} + 1 \right) \mathrm{d}t$$
$$= 2 + \frac{\pi}{2}(1 + \ln 2) + 2\mathbf{G} - \frac{13\pi^2}{24}$$

where I have used previously established identities like

$$\int_0^1 \frac{\ln(1-t)}{t\sqrt{1-t^2}} \mathrm{d}t = -\frac{3\pi^2}{8}$$
$$\int_0^1 \frac{\ln(1-t)}{\sqrt{1-t^2}} \mathrm{d}t = -2\mathbf{G} - \frac{\pi}{2}\ln(2)$$

Part IV: The final answer

Putting the value of K, we get

$$I = -7\zeta(2) + 4\mathbf{G} - \ln^2(2)$$

□

Exercise 1.2.69: Evaluate:

$$\int_0^\infty e^{-px^2} \operatorname{erf}(bx) \operatorname{erf}(cx) \mathrm{d}x$$

Solution:

$$\int_0^\infty e^{-px^2} \operatorname{erf}(bx) \operatorname{erf}(cx) \mathrm{d}x = \frac{4}{\pi} \int_0^\infty \int_0^b \int_0^c x^2 e^{-px^2} e^{-x^2y^2} e^{-x^2z^2} \mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$= \frac{4}{\pi} \int_0^b \int_0^c \int_0^\infty x^2 e^{-(p+y^2+z^2)x^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{\sqrt{\pi}} \int_0^b \int_0^c \frac{1}{(p+y^2+z^2)^{3/2}} \mathrm{d}y\mathrm{d}z$$
$$= \frac{1}{\sqrt{\pi}} \int_0^b \int_0^{\arctan(c/\sqrt{p+z^2})} \frac{\cos\theta}{p+z^2} \mathrm{d}\theta\mathrm{d}z \qquad \text{Let } y = \sqrt{p+z^2}\tan\theta$$
$$= \frac{1}{\sqrt{\pi}} \int_0^b \frac{1}{p+z^2} \sin\left[\arctan\left(\frac{c}{\sqrt{p+z^2}} \right) \right] \mathrm{d}z = \frac{c}{\sqrt{\pi}} \int_0^b \frac{1}{p+z^2} \frac{1}{\sqrt{p+c^2+z^2}} \mathrm{d}z$$
$$\overset{t=\frac{1}{z}}{=} \int_{1/b}^\infty \frac{1}{1+pt^2} \frac{t}{\sqrt{1+c^2t^2+pt^2}} \mathrm{d}t$$
$$\overset{u^2 = 1+c^2t^2+pt^2}{=} \frac{c}{\sqrt{\pi}} \int_{\sqrt{p+b^2+c^2}/b}^\infty \frac{1}{c^2+pu^2} \mathrm{d}u$$
$$= \frac{1}{\sqrt{\pi p}} \int_{\sqrt{p(p+b^2+c^2)}/bc}^\infty \frac{\mathrm{d}w}{1+w^2} = \frac{1}{\sqrt{\pi p}} \left[\frac{\pi}{2} - \arctan\left(\frac{\sqrt{p(p+b^2+c^2)}}{bc} \right) \right]$$
$$= \frac{1}{\sqrt{\pi p}} \arctan\left(\frac{bc}{\sqrt{p(b^2+c^2+p)}} \right)$$

□

Exercise 1.2.70: Evaluate:

$$\int_0^1 \frac{\ln(t)}{\sqrt{t(2-t)}} \mathrm{d}t$$

✎ **Solution:** Let $t = 1 - x$ and turning it into $\displaystyle\int_0^1 \frac{\ln(1-x)}{\sqrt{1-x^2}}\mathrm{d}x$

$$\int_0^1 \frac{\ln(1-x)}{\sqrt{1-x^2}}\mathrm{d}x = \int_0^{\pi/2} \ln\left(1-\sin u\right)\mathrm{d}u$$

In general,

$$\begin{aligned}
&\int_0^\theta \ln(1-\sin x)\ \mathrm{d}x = \int_0^\theta \left(2\Re\ln(1+ie^{ix}) - \ln 2\right)\ \mathrm{d}x \\
&= 2\Re\int_0^\theta \sum_{n=1}^\infty (-1)^{n-1}\frac{(ie^{ix})^n}{n}\ \mathrm{d}x - \theta\ln 2 \\
&= 2\Re\sum_{n=1}^\infty \frac{(-1)^{n-1}i^n}{n}\int_0^\theta e^{inx}\ \mathrm{d}x - \theta\ln 2 \\
&= 2\Re i\sum_{n=1}^\infty \frac{(-ie^{i\theta})^n}{n^2} - 2\Re\sum_{n=1}^\infty \frac{(i)^{n-1}}{n^2} - \theta\ln 2 \\
&= 2\Re i\mathrm{Li}_2(-ie^{i\theta}) - 2\mathbf{G} - \theta\ln 2 \\
&= 2\sum_{n=0}^\infty \frac{(-1)^n\cos(2n+1)\theta}{(2n+1)^2} + \frac{1}{2}\sum_{n=1}^\infty \frac{(-1)^{n-1}\sin 2n\theta}{n^2} - 2\mathbf{G} - \theta\ln 2
\end{aligned}$$

Hence

$$\int_0^1 \frac{\ln(1-x)}{\sqrt{1-x^2}}\ \mathrm{d}x = -2\mathbf{G} - \frac{\pi\ln 2}{2}$$

□

⊸ **Exercise 1.2.71:** Evaluate:

$$\int_0^\infty \frac{(1-x^2)\tan^{-1}(x^2)}{x^4+4x^2+1}\mathrm{d}x$$

✎ **Solution:** For $\alpha > 0$ let

$$f(z) = \frac{\ln(1+iz^2)}{z^2+\alpha^2}\ ,\ I(\alpha) = \int_0^\infty \frac{\tan^{-1}(x^2)}{x^2+\alpha^2}\mathrm{d}x$$

Then using a sand dollar with with branch point at $e^{i\pi/4}$ we get

$$\begin{aligned}
I(\alpha) &= \frac{1}{2}\Im\int_{-\infty}^\infty f(x)\mathrm{d}x \\
&= \Im\left\{\pi i\left(\mathrm{Res}_{z=i\alpha}f(z)\right) + \pi i\int_{e^{i\pi/4}}^{e^{i\pi/4}\infty}\frac{1}{x^2+\alpha^2}\mathrm{d}x\right\} \\
&= -\frac{\pi}{2\alpha}\tan^{-1}\alpha^2 + \Im\left\{\pi i\int_{e^{i\pi/4}}^{e^{i\pi/4}\infty}\frac{1}{x^2+\alpha^2}\mathrm{d}x\right\} \\
&= \frac{\pi^2}{2\alpha} - \frac{\pi}{2\alpha}\tan^{-1}\alpha^2 - \Im\left\{\frac{\pi i}{\alpha}\tan^{-1}\left(\frac{e^{i\pi/4}}{\alpha}\right)\right\} \\
&= \frac{\pi^2}{2\alpha} - \frac{\pi}{2\alpha}\tan^{-1}\alpha^2 + \frac{\pi}{2\alpha}\arg\left(\frac{\alpha - ie^{i\pi/4}}{\alpha + ie^{i\pi/4}}\right) \\
&= \frac{\pi}{2\alpha}\left\{\pi - \tan^{-1}(\alpha^2) - \tan^{-1}\left(\frac{\sqrt{2}\alpha}{\alpha^2-1}\right)\right\}
\end{aligned}$$

Now use the partial fraction decomposition

$$\frac{1-x^2}{x^4+4x^2+1}=\frac{\sqrt{3}-1}{2}\frac{1}{x^2+2-\sqrt{3}}-\frac{\sqrt{3}+1}{2}\frac{1}{x^2+2+\sqrt{3}}$$

to get

$$\begin{aligned}\int_0^\infty \frac{(1-x^2)\tan^{-1}(x^2)}{x^4+4x^2+1}\mathrm{d}x &= \frac{\sqrt{3}-1}{2}I\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)-\frac{\sqrt{3}+1}{2}I\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\\ &=\frac{\pi}{2\sqrt{2}}\left\{\tan^{-1}\left(2+\sqrt{3}\right)-\tan^{-1}\left(2-\sqrt{3}\right)-\frac{\pi}{2}\right\}\\ &=\frac{\pi}{2\sqrt{2}}\left\{\frac{\pi}{3}-\frac{\pi}{2}\right\}\\ &=-\frac{\pi^2}{12\sqrt{2}}\end{aligned}$$

□

Exercise 1.2.72: Evaluate:

$$\int_0^1 \frac{\ln(1-x)}{\sqrt{x(1-x^2)}}\mathrm{d}x$$

Solution: Rewrite the integral as

$$\begin{aligned}I&=\int_0^1\frac{\ln(1-x)}{\sqrt{x}\sqrt{1-x^2}}\mathrm{d}x\\ &=\int_0^1\frac{\ln(1-x^2)-\ln(1+x)}{\sqrt{x}\sqrt{1-x^2}}\mathrm{d}x\\ &=\int_0^1\frac{\ln(1-x^2)}{\sqrt{x}\sqrt{1-x^2}}\mathrm{d}x-\int_0^1\frac{\ln(1+x)}{\sqrt{x}\sqrt{1-x^2}}\mathrm{d}x\end{aligned}$$

The first integral can be computed by calculating a derivative of the **Beta function**.

$$\begin{aligned}&\int_0^1\frac{\ln(1-x^2)}{\sqrt{x}\sqrt{1-x^2}}\mathrm{d}x\overset{y=x^2}{=}\frac{1}{2}\int_0^1\frac{\ln(1-y)}{y^{\frac{3}{4}}\sqrt{1-y}}\mathrm{d}y\\ &=\frac{1}{2}\frac{\partial}{\partial\alpha}\left\{B\left(\frac{1}{4},\alpha\right)\right\}_{\alpha=\frac{1}{2}}=\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}\left\{\psi_0\left(\frac{1}{2}\right)-\psi_0\left(\frac{3}{4}\right)\right\}\\ &=\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}}\left(2\ln 2-\pi\right)\end{aligned}$$

The other integral can be evaluated by using equation (3.22) of this paper.

http://120.52.73.75/arxiv.org/pdf/1005.2941v1.pdf

$$\begin{aligned}&\int_0^1\frac{\ln(1+x)}{\sqrt{x}\sqrt{1-x^2}}\mathrm{d}x\\ &\overset{x=\sin^2 t}{=}2\int_0^{\frac{\pi}{2}}\frac{\ln(1+\sin^2 t)}{\sqrt{1+\sin^2 t}}\mathrm{d}t\\ &=\ln(2)\mathbf{K}(\sqrt{-1})\\ &=\ln(2)\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}}\end{aligned}$$

where $\mathbf{K}(k)$ is the **complete elliptic integral of the first kind**. After combining everything, one gets

$$I = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}}(\ln 2 - \pi)$$

□

Exercise 1.2.73: Evaluate:

$$\int_0^1 \frac{\sqrt[3]{x^2(1-x)}}{(1+x)^3}\mathrm{d}x$$

Solution: **Method One:**

Let $f(z) = \dfrac{\sqrt[3]{z^2(z-1)}}{(1+z)^3}$ where $0 \le \arg(z), \arg(z-1) \le 2\pi$.

Then integrating around a dumbbell contour,

$$2\pi i a_{-1} + \int_0^1 \frac{\sqrt[3]{x^2(1-x)e^{i\pi}}}{(1+x)^3}\mathrm{d}x + \int_1^0 \frac{\sqrt[3]{(xe^{2\pi i})^2(1-x)e^{i\pi}}}{(1+x)^3}\mathrm{d}x = 2\pi i \mathrm{Res}[f(z), -1]$$

where a_{-1} is the coefficient of the $\dfrac{1}{z}$ term in the Laurent expansion at infinity, that is, $-\mathrm{Res}[f(z), \infty]$.

$$\begin{aligned} 0 + 2i\sin\left(\frac{\pi}{3}\right)\int_0^1 \frac{\sqrt[3]{x^2(1-x)}}{(1+x)^3}\,\mathrm{d}x &= -\frac{2\pi i}{9}\lim_{z\to -1} x^2\left(x^2(x-1)\right)^{-5/3} \\ &= -\frac{2\pi i}{9}\left((e^{\pi i})^2 2e^{\pi i}\right)^{-5/3} \\ &= -\frac{2\pi i}{9}2^{-5/3}e^{-5\pi i} = \frac{\pi i}{9}2^{-2/3} \end{aligned}$$

Which implies

$$\int_0^1 \frac{\sqrt[3]{x^2(1-x)}}{(1+x)^3}\mathrm{d}x = \frac{\pi\ 2^{-2/3}}{9\sqrt{3}} = \frac{\pi\sqrt[3]{2}}{18\sqrt{3}}$$

Method Two:

$$\begin{aligned} &\int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{(x+p)^{a+b}}\mathrm{d}x = \frac{1}{p^{a+b}}\int_0^1 x^{a-1}(1-x)^{b-1}\left(1+\frac{x}{p}\right)^{-(a+b)}\mathrm{d}x \\ &= \frac{\mathrm{B}(a,b)}{p^{a+b}}\,{}_2F_1\left(a+b, a; a+b; -\frac{1}{p}\right) = \frac{\mathrm{B}(a,b)}{p^{a+b}}\,{}_1F_0\left(a; -1; -\frac{1}{p}\right) \\ &= \frac{\mathrm{B}(a,b)}{p^{a+b}}\left(1+\frac{1}{p}\right)^{-a} = \frac{\mathrm{B}(a,b)}{p^b(1+p)^a} \end{aligned}$$

Put $a = \dfrac{5}{3}$, $b = \dfrac{4}{3}$ and $p = 1$:

$$\begin{aligned} \int_0^1 \frac{\sqrt[3]{x^2(1-x)}}{(1+x)^3}\,\mathrm{d}x &= \frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{4}{3}\right)}{2^{\frac{5}{3}}\cdot 2!} \\ &= \frac{\pi}{9\cdot 2^{\frac{5}{3}}\sin\left(\frac{\pi}{3}\right)} \\ &= \frac{\pi\sqrt[3]{2}}{18\sqrt{3}} \end{aligned}$$

□

Exercise 1.2.74: Evaluate:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{b^{2n}+x^{2n}}\mathrm{d}x$$

Solution: First we need to find the partial fraction decomposition of $\frac{1}{x^{2n}+b^{2n}}$.

$$\frac{1}{x^{2n}+b^{2n}} = \sum_{k=0}^{2n-1} \frac{A_k}{x - be^{i\pi(2k+1)/(2n)}}$$

where $A_k = \operatorname{Res}\left[\frac{1}{z^{2n}+b^{2n}}, be^{i\pi(2k+1)/(2n)}\right] = -\frac{1}{2nb^{2n-1}}e^{i\pi(2k+1)/(2n)}$.
denote $\frac{2k+1}{2n}\pi$ by λ_k, then

$$\begin{aligned}\frac{1}{x^{2n}+b^{2n}} &= -\frac{1}{2nb^{2n-1}}\sum_{k=0}^{2n-1}\frac{e^{i\lambda_k}}{x-be^{i\lambda_k}} = -\frac{1}{2nb^{2n-1}}\sum_{k=0}^{n-1}\left(\frac{e^{i\lambda_k}}{x-be^{i\lambda_k}}+\frac{e^{-i\lambda_k}}{x-be^{-i\lambda_k}}\right)\\ &= \frac{1}{nb^{2n-1}}\sum_{k=0}^{n-1}\frac{b-x\cos(\lambda_k)}{x^2-2bx\cos(\lambda_k)+b^2}\\ &= \frac{1}{nb^{2n-1}}\sum_{k=0}^{n-1}\frac{b-x\cos(\lambda_k)}{(x-b\cos\lambda_k)^2+b^2\sin^2(\lambda_k)}\end{aligned}$$

We also need the following two results which are fairly easy to derive using other known results.

$$\int_{-\infty}^{\infty}\frac{\cos rx}{(x-b)^2+a^2}\,\mathrm{d}x = \frac{\pi}{a}e^{-ar}\cos(br)$$

$$\int_{-\infty}^{\infty}\frac{x\cos(rx)}{(x-b)+a^2}\,\mathrm{d}x = \frac{\pi}{a}e^{-ar}\left(b\cos br - a\sin br\right)$$

Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^{2n}+b^{2n}}\,\mathrm{d}x &= \frac{1}{nb^{2n-1}}\sum_{k=0}^{n-1}\left[b\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x-b\cos\lambda_k)^2+b^2\sin^2(\lambda_k)}\,\mathrm{d}x \right.\\
&\quad \left. -\cos\lambda_k\int_{-\infty}^{\infty}\frac{x\cos(ax)}{(x-b\cos\lambda_k)^2+b^2\sin^2\lambda_k}\,\mathrm{d}x\right]\\
&= \frac{1}{nb^{2n-1}}\sum_{k=0}^{n-1}\left[\frac{\pi}{\sin\lambda_k}e^{-ab\sin\lambda_k}\cos(ab\cos\lambda_k)\right.\\
&\quad \left. -\frac{\pi\cos\lambda_k}{b\sin\lambda_k}e^{-ab\sin\lambda_k}\Big(b\cos(\lambda_k)\cos(ab\cos\lambda_k)-b\sin(\lambda_k)\sin(ab\cos\lambda_k)\Big)\right]\\
&= \frac{\pi}{nb^{2n-1}}\sum_{k=0}^{n-1}\frac{1}{\sin\lambda_k}e^{-ab\sin\lambda_k}\Big(\cos(ab\cos\lambda_k)-\cos(\lambda_k)\cos(ab\cos\lambda_k+\lambda_k)\Big)\\
&= \frac{\pi}{nb^{2n-1}}\sum_{k=0}^{n-1}\frac{1}{\sin\lambda_k}e^{-ab\sin\lambda_k}\Big(\cos(ab\cos\lambda_k)\\
&\quad -\cos^2(\lambda_k)\cos(ab\cos\lambda_k)+\cos(\lambda_k)\sin(\lambda_k)\sin(ab\cos\lambda_k)\Big)\\
&= \frac{\pi}{nb^{2n-1}}\sum_{k=0}^{n-1}e^{-ab\sin\lambda_k}\Big(\sin(\lambda_k)\cos(ab\cos\lambda_k)+\cos(\lambda_k)\sin(ab\cos\lambda_k)\Big)\\
&= \frac{\pi}{nb^{2n-1}}\sum_{k=0}^{n-1}e^{-ab\sin(\lambda_k)}\sin\left(ab\cos\lambda_k+\lambda_k\right)
\end{aligned}$$

□

Exercise 1.2.75: Evaluate:

$$\int\left(\frac{\arctan x}{x-\arctan x}\right)^2 dx$$

Solution:

$$\begin{aligned}
\int\left(\frac{\arctan x}{x-\arctan x}\right)^2\mathrm{d}x &= \int\frac{t^2}{(\tan t-t)^2}\sec^2 t\,\mathrm{d}t = \int\frac{t^2}{(\sin t-t\cos t)^2}\,\mathrm{d}t\\
&= \int\left(-\frac{t}{\sin t}\right)\left(-\frac{t\sin t}{(\sin t-t\cos t)^2}\right)\mathrm{d}t\\
&= -\frac{t}{\sin t}\frac{1}{\sin t-t\cos t}+\int\frac{\mathrm{d}t}{\sin^2 t}\\
&= -\frac{(1+\tan^2 t)t}{\tan t(\tan t-t)}-\frac{1}{\tan t}+C\\
&= -\frac{(1+x^2)\arctan x}{x(x-\arctan x)}-\frac{1}{x}+C\\
&= -\frac{1+x\arctan x}{x-\arctan x}+C
\end{aligned}$$

□

Exercise 1.2.76: Prove:

$$\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{2x}{3}\right)}{\tan x}\,\mathrm{d}x = \frac{\sqrt{3}}{4}(3-2\ln 2)$$

☞ **Proof:** $u=\sin(2x/3),\ \ \mathrm{d}v=\cot(x)\mathrm{d}x,\ \ \mathrm{d}u=2/3\cos(2x/3),\ \ v=\ln(\sin(x))$

$$\underbrace{\sin(2x/3)\ln(\sin(x))}_{0}-2/3\int_0^{\frac{\pi}{2}}\cos(2x/3)\ln(\sin(x))\mathrm{d}x$$

We can use the famous and handy sum identity for $\ln(\sin x)$: $\ln(2)+\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}$.

Note that $\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+a)}=\frac{-1}{2a}\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(\frac{a+2}{2}\right)+2\ln(2)\right]$, we have

$$\begin{aligned}
&=2/3\ln(2)\int_0^{\frac{\pi}{2}}\cos(2x/3)\mathrm{d}x+2/3\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{\cos(2x/3)\cos(2kx)}{k}\mathrm{d}x\\
&=\frac{\sqrt{3}}{2}\ln(2)+\frac{\sqrt{3}}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(3k+1)}-\frac{\sqrt{3}}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(3k-1)}\\
&=\frac{\sqrt{3}}{2}\ln(2)+\frac{\sqrt{3}}{12}\Phi\left(-1,1,\frac{1}{3}\right)-\frac{\sqrt{3}}{12}\Phi\left(-1,1,-\frac{1}{3}\right)\\
&=\frac{\sqrt{3}}{2}\ln(2)+\frac{\sqrt{3}}{12}(9-\sqrt{3}\pi-6\ln(2))-\frac{\sqrt{3}}{12}(6\ln(2)-\sqrt{3}\pi)\\
&=\frac{3\sqrt{3}}{4}-\frac{\sqrt{3}}{2}\ln(2)
\end{aligned}$$

□

Exercise 1.2.77: Evaluate:

$$\int_0^{\infty}\frac{\ln(1+x^3)}{x^3}\,\mathrm{d}x$$

✎ **Solution:** Let I denote the integral. If we apply the substitution $x=\tan^{2/3}\theta$, we have

$$I=-\frac{4}{3}\int_0^{\frac{\pi}{2}}\sin^{-7/3}\theta\cos^{1/3}\theta\ln\cos\theta\,\mathrm{d}\theta$$

Associated to this integral representation, let $I(s)$ denote

$$I(s)=-\frac{4}{3}\int_0^{\frac{\pi}{2}}\sin^{2s-1}\theta\cos^{1/3}\theta\ln\cos\theta\,\mathrm{d}\theta$$

It is easy to find that for large s, we have

$$I(s)=-\frac{1}{3}\frac{\partial\beta}{\partial w}(s,w)\Big|_{w=2/3}=\frac{1}{3}\frac{\Gamma(s)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}+s\right)}\left\{\psi_0\left(\frac{2}{3}+s\right)-\psi_0\left(\frac{2}{3}\right)\right\}$$

Then by the argument of analytic continuation, we find that this remains true for $s>1$. Therefore taking the limit $s\to-\frac{2}{3}$, we obtain

$$I=\lim_{s\to-\frac{2}{3}}I(s)=\frac{1}{2}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{\sqrt{3}}$$

Generalizing this solution, for any $\alpha>\beta>0$ we obtain

$$\int_0^{\infty}\frac{\ln(1+x^{\alpha})}{x^{\beta+1}}\,\mathrm{d}x=\frac{\pi}{\beta}\csc\left(\frac{\pi\beta}{\alpha}\right)$$

□

Exercise 1.2.78: Evaluate:

$$\int_0^\infty \frac{\ln^3(1+x^3)}{x^3}\mathrm{d}x$$

Solution: consider the following general case for $\alpha > 1, n \in \mathbb{Z} \geq 1$

$$\begin{aligned}\int_0^\infty \frac{\ln^n(1+x^\alpha)}{x^\alpha}\,\mathrm{d}x &= \frac{n\alpha}{\alpha-1}\int_0^\infty \frac{\ln^{n-1}(1+x^\alpha)}{1+x^\alpha}\,\mathrm{d}x \\ &= \frac{n}{(\alpha-1)}\int_0^\infty \frac{x^{\frac{1}{\alpha}-1}\,\ln^{n-1}(1+x)}{1+x}\,\mathrm{d}x\end{aligned}$$

Now consider the case

$$I(s) = \int_0^\infty x^{\frac{1}{\alpha}-1}(1+x)^{s-1}\mathrm{d}x = \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-s-\frac{1}{\alpha}\right)}{\Gamma\left(1-s\right)}$$

Hence,

$$\begin{aligned}\int_0^\infty \frac{\ln^n(1+x^\alpha)}{x^\alpha}\,\mathrm{d}x &= \frac{n}{\alpha-1}I^{(n-1)}(0) \\ \int_0^\infty \frac{\ln^3(1+x^3)}{x^3}\,\mathrm{d}x &= \frac{3}{2}I''(0) \\ &= \frac{\sqrt{3}\pi}{4}\left(9\ln^2(3)+4\psi'\left(\frac{2}{3}\right)\right) - \frac{\pi^3\sqrt{3}}{12} - \frac{3\pi^2}{2}\ln(3)\end{aligned}$$

□

Exercise 1.2.79: Evaluate:

$$\int_0^\infty e^{-ax}\left(\frac{1}{x}-\coth x\right)\,\mathrm{d}x$$

Solution: Begin with parts:

$$u = e^{-ax},\quad \mathrm{d}v = \left(\frac{1}{x}-\coth(x)\right)\mathrm{d}x,\quad \mathrm{d}u = -ae^{-ax}\mathrm{d}x,\quad v = (\ln(x)-\ln(\sinh(x)))$$

$$\begin{aligned}&e^{-ax}\left(\ln(x)-\ln(\sinh(x))\right)\Big|_0^\infty + a\int_0^\infty e^{-ax}\left(\ln(x)-\ln(\sinh(x))\right)\mathrm{d}x \\ &= a\int_0^\infty e^{-ax}\ln(x)\mathrm{d}x - a\int_0^\infty e^{-ax}\ln(\sinh(x))\mathrm{d}x \\ &= -\gamma-\ln(a)-a\int_0^\infty e^{-ax}\ln\left(\frac{e^x-e^{-x}}{2}\right)\mathrm{d}x \\ &= -\gamma-\ln(a)-a\int_0^\infty e^{-ax}\left[\ln(e^x(1-e^{-2x}))-\ln(2)\right]\mathrm{d}x \\ &= -\gamma-\ln(a)-a\int_0^\infty xe^{-ax}\mathrm{d}x - a\int_0^\infty e^{-ax}\ln(1-e^{-2x})\mathrm{d}x + a\ln(2)\int_0^\infty e^{-ax}\mathrm{d}x \\ &= -\gamma-\ln(a)-\frac{1}{a}+\ln(2)-a\int_0^\infty e^{-ax}\ln(1-e^{-2x})\mathrm{d}x\end{aligned}$$

Now, use the series for $\ln\left(1-e^{-2x}\right)$: $-\int_0^\infty \sum_{k=1}^\infty \frac{ae^{-2kx-ax}}{k}\mathrm{d}x$.

This leads to the series $\sum_{k=1}^\infty \frac{a}{k(a+2k)} = \psi\left(\frac{a}{2}+1\right)+\gamma$

Giving a final result of $-\ln(a)-\frac{1}{a}+\ln(2)+\psi\left(\frac{a}{2}+1\right)$ □

Exercise 1.2.80: Evaluate:

$$\int_{-\infty}^{\infty}\left(\frac{\coth(x)}{x^3}-\frac{1}{3x^2}-\frac{1}{x^4}\right)\mathrm{d}x$$

Solution: By considering a rectangular contour with vertices

$$\pm R,\quad \pm R+\left(N+\frac{1}{2}\right)\pi i$$

(with R a positive real number and N a positive integer) and letting $R, N\to\infty$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\coth x-\frac{1}{x}-\frac{x}{3}\right)\frac{\mathrm{d}x}{x^3} &= 2\pi i\sum_{n=1}^{\infty}\mathrm{Res}_{z=n\pi i}\left(\coth z-\frac{1}{z}-\frac{z}{3}\right)\frac{1}{z^3}\\ &= 2\pi i\sum_{n=1}^{\infty}\frac{1}{(n\pi i)^3}\\ &= -\frac{2\zeta(3)}{\pi^2}\end{aligned}$$

□

Exercise 1.2.81: Prove:

$$\int_0^1 \frac{\mathbf{K}(k)}{\sqrt{1-k}}\mathrm{d}k = \frac{\sqrt{2}\pi^3}{4\Gamma^2\left(\frac{5}{8}\right)\Gamma^2\left(\frac{7}{8}\right)}$$

$\mathbf{K}(k)$ is the **Complete Elliptic Integral of the First Kind**.

Proof: It requires the facts that

$$\mathbf{K}(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix};k^2\right)$$

and

$$\int_0^1 k^{2n}(1-k)^{-\frac{1}{2}}\mathrm{d}k = \mathrm{B}\left(\frac{1}{2},2n+1\right)$$

So we have

$$\begin{aligned}I = \int_0^1 \frac{\mathbf{K}(k)}{\sqrt{1-k}}\mathrm{d}k &= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(1)_n n!}\int_0^1 k^{2n}(1-k)^{-1/2}\mathrm{d}k\\ &= \frac{\sqrt{2}\pi^2}{4\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{\left(\frac{3}{4}\right)_n\left(\frac{5}{4}\right)_n n!} = \frac{\sqrt{2}\pi^2}{4\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\\frac{3}{4},\frac{5}{4}\end{matrix};1\right)\end{aligned}$$

Now by **Whipple's Theorem**, we get

$$I = \frac{\sqrt{2}\pi^3}{4\Gamma^2\left(\frac{5}{8}\right)\Gamma^2\left(\frac{7}{8}\right)}$$

□

Exercise 1.2.82: Prove:

$$\int_0^1 \frac{\ln\left(1+x+x^2+x^3+x^4\right)}{x}\mathrm{d}x = \frac{2\pi^2}{15}$$

Proof:

$$\begin{aligned}\int_0^1 \frac{\ln\left(1+x+x^2+x^3+x^4\right)}{x}\mathrm{d}x &= \int_0^1 \frac{\ln\left(\frac{1-x^5}{1-x}\right)}{x}\mathrm{d}x \\ &= \int_0^1 \frac{\ln\left(1-x^5\right)}{x}\mathrm{d}x - \int_0^1 \frac{\ln\left(1-x\right)}{x}\mathrm{d}x \\ &= \frac{1}{5}\int_0^1 \frac{\ln\left(1-x\right)}{x}\mathrm{d}x - \int_0^1 \frac{\ln\left(1-x\right)}{x}\mathrm{d}x \\ &= \left(1-\frac{1}{5}\right)\mathrm{Li}_2(1) = \frac{2\pi^2}{15}\end{aligned}$$

□

Exercise 1.2.83: Evaluate:

$$\int_0^1 \frac{\ln\left(1+x+x^2\right)}{1+x}\mathrm{d}x$$

Solution:

$$\begin{aligned}&\int_0^1 \frac{\ln(1+x+x^2)}{1+x}\,\mathrm{d}x = \int_1^2 \frac{\ln(x^2-x+1)}{x}\,\mathrm{d}x = \int_1^2 \frac{\ln\left(\frac{x^3+1}{x+1}\right)}{x}\,\mathrm{d}x \\ &= \int_1^2 \frac{\ln(x^3+1)-\ln(x+1)}{x}\,\mathrm{d}x = \int_1^2 \frac{\ln(x^3+1)}{x}\,\mathrm{d}x - \int_1^2 \frac{\ln(x+1)}{x}\,\mathrm{d}x \\ &= \frac{1}{3}\int_{-1}^{-8} \frac{\ln(1-x)}{x}\,\mathrm{d}x - \int_{-1}^{-2} \frac{\ln(1-x)}{x}\,\mathrm{d}x \\ &= \mathrm{Li}_2(-2) - \frac{\mathrm{Li}_2(-8)}{3} + \left(\frac{1}{3}-1\right)\mathrm{Li}_2(-1) \\ &= \mathrm{Li}_2(-2) - \frac{1}{3}\mathrm{Li}_2(-8) + \frac{\pi^2}{18}\end{aligned}$$

□

Exercise 1.2.84: Evaluate:

$$\int_0^\infty {}_2F_1\left(a,b;c,-x\right)x^{s-1}\mathrm{d}x$$

Solution:

$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+xt)^{-a}\,\mathrm{d}t$$

hence we get

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty x^{s-1}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+xt)^{-a}\,\mathrm{d}t\,\mathrm{d}x$$

Using funbini theorem we have

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\int_0^\infty x^{s-1}(1+xt)^{-a}\,\mathrm{d}x\,\mathrm{d}t$$

Hence we get

$$\frac{\Gamma(c)\Gamma(s)\Gamma(a-s)}{\Gamma(a)\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-s-1}(1-t)^{c-b-1}\,\mathrm{d}t=\frac{\Gamma(c)\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(a)\Gamma(b)\Gamma(c-s)}$$

□

- **Exercise 1.2.85:** Prove:

$$\int_0^1\frac{(1-x)^{x-1}}{x^x}\sin(\pi x)\mathrm{d}x=\frac{\pi}{e}$$

- **Proof:** By considering

$$I=\int_0^1\frac{e^{\pi ix}}{x^x(1-x)^{1-x}}\mathrm{d}x=\int_0^1\frac{e^{\pi ix}(1-x)^x}{x^x(1-x)}\mathrm{d}x=\int_0^1\frac{e^{(\pi i+\ln(1-x)-\ln(x))x}}{1-x}\mathrm{d}x$$

Now, let $x=\dfrac{1}{1+e^t}$, $\mathrm{d}x=\dfrac{-e^t}{(e^t+1)^2}\mathrm{d}t$

$$\int_{-\infty}^\infty\frac{e^{(\pi i+t)\frac{1}{1+e^t}}e^{-t}(1+e^t)e^t}{(e^t+1)^2}\mathrm{d}t=\int_{-\infty}^\infty\frac{e^{\frac{t+\pi i}{e^t+1}}}{e^t+1}\mathrm{d}t$$

Now, letting $t\to t+\pi i$ because of the rectangle contour. The vertical sides go to 0. The contour is centered at the origin.

$$\Rightarrow\int_{-\infty+\pi i}^{\infty+\pi i}\frac{e^{\frac{t}{1-e^t}}}{1-e^t}\mathrm{d}t$$

So, now consider the function $f(z)=\dfrac{e^{\frac{z}{1-e^z}}}{1-e^z}$. Now, it is rather straightforward using residues. The only pole is at $z=0$, so the residue there is $\lim\limits_{z\to0}e^{\frac{z}{1-e^z}}=\dfrac{1}{e}$. We have the segments consisting of the horizontal sides of the rectangle.

$$\underbrace{\int_{-\pi i-\infty}^{-\pi i+\infty}f(z)\mathrm{d}z}_{\text{lower}}+\underbrace{\int_{\infty+\pi i}^{-\infty+\pi i}f(z)\mathrm{d}z}_{\text{upper}}=\frac{2\pi i}{e}\Rightarrow\int_{-\infty}^\infty f(z)\mathrm{d}z=\frac{\pi i}{e}$$

Hence,

$$\int_0^1\frac{\sin(\pi x)}{x^x(1-x)^{1-x}}\mathrm{d}x=\frac{\pi}{e}$$

□

- **Exercise 1.2.86:** Evaluate:

$$\int_0^\infty\frac{t^2}{2e^t-1}\mathrm{d}t$$

✎ **Solution:** Here is the general solution to this class of integrals:

$$\begin{aligned}\int_0^\infty \frac{t^n}{2e^t-1}\mathrm{d}t &= \frac{1}{2}\int_0^\infty \frac{t^n e^{-t}}{1-\frac{e^{-t}}{2}}\mathrm{d}t \\ &= \frac{1}{2}\sum_{k=0}^\infty \frac{1}{2^k}\int_0^\infty t^n e^{-kt}\mathrm{d}t \\ &= \Gamma(n+1)\sum_{k=0}^\infty \frac{1}{2^{k+1}(k+1)^{n+1}} \\ &= \Gamma(n+1)\mathrm{Li}_{n+1}\left(\frac{1}{2}\right)\end{aligned}$$

So let $n=2$ we get

$$\int_0^\infty \frac{t^2}{2e^t-1}\mathrm{d}t = 2\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{4}\zeta(3)+\frac{1}{3}\ln^3 2-\frac{\pi^2}{6}\ln 2$$

□

Exercise 1.2.87: Evaluate:

$$\int_0^\infty \frac{\cos(tx)}{\cosh\left(\frac{\pi x}{2}\right)(1+x^2)}\mathrm{d}x$$

✎ **Solution:** Let

$$y(t) = \int_0^\infty \frac{\cos(tx)}{\cosh\left(\frac{\pi x}{2}\right)(1+x^2)}\mathrm{d}x$$

Then

$$y''(t)-y(t) = -\int_0^\infty \frac{\cos(tx)}{\cosh\left(\frac{\pi}{2}x\right)}\mathrm{d}x$$

The integral on the right side equals

$$\int_0^\infty \frac{\cos(tx)}{\cosh\left(\frac{\pi}{2}x\right)}\mathrm{d}x = \mathrm{sech}(t)$$

Thus, it suffices to solve the following differential equation:

$$y''(t)-y(t) = -\mathrm{sech}(t)$$

I will use **variation of parameters** to solve it.
First, we find the solutions to the homogenous differential equation:

$$y''-y=0$$

From the characteristic equation

$$\lambda^2-1=0 \to \lambda = 1,-1$$

So we get $u_1 = e^t$ and $u_2 = e^{-t}$. The Wronskian of these two functions is

$$W = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2$$

We seek functions $A(t)$ and $B(t)$ such that $A(t)u_1 + B(t)u_2$ is a general solution of the non-homogenous equation. That is

$$A(x) = -\frac{1}{2}\int e^{-t}\text{sech}(t)\mathrm{d}t = -\frac{t}{2} + \frac{1}{2}\ln(2\cosh t) + C_1$$

$$B(x) = \frac{1}{2}\int e^{t}\text{sech}(t)\mathrm{d}t = \frac{t}{2} + \frac{1}{2}\ln(2\cosh t) + C_2$$

So we get

$$y(t) = \cosh t \ln(2\cosh t) - t\sinh t + C_1 e^x + C_2 e^{-x}$$

It can be shown that $C_1 = C_2 = 0$.

$$y(t) = \cosh t \ln(2\cosh t) - t\sinh t$$

□

Exercise 1.2.88: Evaluate:

$$\int_0^\infty \ln\left(\frac{a^2+x^2}{x^2}\right)\cos(bx)\mathrm{d}x$$

Solution: Apply Integration by parts

$$I = \int_0^\infty \cos(bx)\ln\left(\frac{a^2+x^2}{x^2}\right)\mathrm{d}x = \frac{2}{b}\underbrace{\int_0^\infty \frac{\sin(bx)}{x}\mathrm{d}x}_{=\frac{\pi}{2}} - \frac{2}{b}\int_0^\infty \frac{x\sin(bx)}{a^2+x^2}\mathrm{d}x$$

$$= \frac{\pi}{b} - \frac{2}{b}\int_0^\infty \frac{x\sin(bx)}{a^2+x^2}\mathrm{d}x$$

The remaining integral can be calculated by integrating the function $f(z) = \dfrac{ze^{ibz}}{a^2+z^2}$ around a large semicircle in the upper half plane:

$$I = \frac{\pi}{b} - \Im\left(\frac{2\pi i}{b}\text{Res}_{z=ia}\frac{ze^{ibz}}{a^2+z^2}\right)$$

$$= \frac{\pi}{b} - \frac{\pi e^{-ab}}{b}$$

Note that $a, b > 0$. □

Exercise 1.2.89: Prove:

$$\int_0^\infty x^2\frac{\cos(\pi x^2)}{\cosh(\pi x)}\mathrm{d}x = \frac{1}{8\sqrt{2}} - \frac{1}{4\pi}$$

Proof: Let

$$\phi_w(t) = \int_0^\infty \frac{\cos(\pi t x)}{\cosh(\pi x)}e^{-\pi w x^2}\mathrm{d}x$$

Step 1. Some Properties of $\phi_w(t)$

It is well known that

$$\int_0^\infty \frac{\cos(2\pi xz)}{\cosh(\pi z)}\mathrm{d}z = \frac{1}{2\cosh(\pi x)}$$

Therefore, we have

$$\begin{aligned}\phi_w(t) &= 2\int_0^\infty \int_0^\infty \frac{\cos(2\pi xz)}{\cosh(\pi z)}\cos(\pi tx)e^{-\pi w x^2}\mathrm{d}x\ \mathrm{d}z \\ &= \frac{e^{-\frac{1}{4}\pi t^2 w'}}{\sqrt{w}}\int_0^\infty \frac{\cosh(\pi txw')}{\cosh(\pi x)}e^{-\pi x^2 w'}\mathrm{d}x\end{aligned}$$

where $w' = \dfrac{1}{w}$. It follows that

$$\phi_w(t) = \frac{1}{\sqrt{w}}e^{-\frac{1}{4}\pi t^2 w'}\phi_{w'}(itw') \tag{1}$$

It is obvious that

$$\phi_w(t) = \phi_w(-t) \tag{2}$$

Another property which can be proven easily is

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}}e^{-\frac{1}{4}\pi t^2/w} \tag{3}$$

By identities (1) and (3), we see that

$$e^{\frac{1}{4}\pi(t+w)^2/w}\phi_w(t+w) + e^{\frac{1}{4}\pi(t-w)^2/w}\phi_w(t-w) = e^{\frac{1}{4}t^2/w} \tag{4}$$

Suppose in (3) we replace t by $t+i$ we will get

$$\phi_w(t) + \phi_w(t+2i) = \frac{1}{\sqrt{w}}e^{-\frac{1}{4}\pi(t+i)^2/w} \tag{5}$$

Similarly in (4) replace t by $t+w$:

$$e^{\frac{1}{4}\pi t^2/w}\phi_w(t) + e^{\frac{1}{4}\pi(t+2w)^2/w}\phi_w(t+2w) = e^{\frac{\pi}{4}(t+w)^2/w} \tag{6}$$

Now, put $w = i$ in (5) and (6) and then solve for $\phi_i(t)$:

$$(1+e^{\pi t})\phi_i(t) = e^{i\pi/4}e^{\pi t/2} - ie^{\pi t/2}e^{\pi i t^2/4} \tag{7}$$

Separating real parts, we obtain the incredible result

$$\int_0^\infty \frac{\cos(\pi tx)}{\cosh(\pi x)}\cos\left(\pi x^2\right)\mathrm{d}x = \frac{1+\sqrt{2}\sin\left(\pi\dfrac{t^2}{4}\right)}{2\sqrt{2}\cosh\left(\dfrac{\pi t}{2}\right)} \tag{8}$$

Step 2. Differentiation

Equation (8) may be written in the form

$$\int_0^\infty \frac{\cos(2\pi tx)\cos\left(\pi x^2\right)}{\cosh(\pi x)}\mathrm{d}x = \frac{1+\sqrt{2}\sin\left(\pi t^2\right)}{2\sqrt{2}\cosh(\pi t)} \tag{9}$$

Now, differentiating (9) with respect to t twice and putting $t = 0$, we have

$$\int_0^\infty \frac{x^2 \cos(\pi x^2)}{\cosh(\pi x)} \mathrm{d}x = \frac{1}{8\sqrt{2}} - \frac{1}{4\pi}$$

□

Exercise 1.2.90: Evaluate:

$$\int_0^\infty \frac{x^2}{2 + \cosh(x)} \mathrm{d}x$$

Solution: Let $f(z) = \dfrac{z^3}{2 + \cos z}$ and C be a **rectangular contour** with vertices $0, 2\pi, 2\pi + i\infty$ and $i\infty$.

The only pole inside the contour is $\pi + i\mathrm{arcosh}(2)$.

$$\mathrm{Res}_{z=\pi+i\mathrm{arcosh}(2)} f(z) = \frac{(i\pi - \mathrm{arcosh}(2))^3}{\sqrt{3}}$$

So by the residue theorem we get

$$\oint_C f(z)\mathrm{d}z = 2\pi i \frac{(i\pi - \mathrm{arcosh}(2))^3}{\sqrt{3}}$$

$$\int_0^{2\pi} \frac{x^3}{2 + \cos x} \mathrm{d}x + \int_{2\pi}^{2\pi + i\infty} \frac{x^3}{2 + \cos x} \mathrm{d}x + \underbrace{\int_{2\pi + i\infty}^{i\infty} \frac{x^3}{2 + \cos x} \mathrm{d}x}_{=0}$$

$$+ \int_{i\infty}^0 \frac{x^3}{2 + \cos x} \mathrm{d}x = 2\pi i \frac{(i\pi - \mathrm{arcosh}(2))^3}{\sqrt{3}}$$

$$\int_0^{2\pi} \frac{x^3}{2 + \cos(x)} \mathrm{d}x + i \int_0^\infty \frac{(ix + 2\pi)^3 - (ix)^3}{2 + \cos(ix)} \mathrm{d}x = 2\pi i \frac{(i\pi - \mathrm{arcosh}(2))^3}{\sqrt{3}}$$

$$\int_0^{2\pi} \frac{x^3}{2 + \cos(x)} \mathrm{d}x + \int_0^\infty \frac{-6\pi i x^2 + 8\pi^3 i - 12\pi^2 x}{2 + \cosh(x)} \mathrm{d}x = 2\pi i \frac{(i\pi - \mathrm{arcosh}(2))^3}{\sqrt{3}}$$

Separating the imaginary parts, we obtain

$$\int_0^\infty \frac{-6\pi x^2 + 8\pi^3}{2 + \cosh(x)} \mathrm{d}x = \frac{-2\pi \mathrm{arcosh}^3(2) + 6\pi^3 \mathrm{arcosh}(2)}{\sqrt{3}}$$

$$\begin{aligned} \int_0^\infty \frac{x^2}{2 + \cosh(x)} \mathrm{d}x &= \frac{4}{3}\pi^2 \int_0^\infty \frac{1}{2 + \cosh(x)} \mathrm{d}x + \frac{\mathrm{arcosh}^3(2) - 3\pi^2 \mathrm{arcosh}(2)}{3\sqrt{3}} \\ &= \frac{4}{3}\pi^2 \left(\frac{\mathrm{arcosh}(2)}{\sqrt{3}} \right) + \frac{\mathrm{arcosh}^3(2) - 3\pi^2 \mathrm{arcosh}(2)}{3\sqrt{3}} \\ &= \frac{\mathrm{arcosh}(2) \left(\pi^2 + \mathrm{arcosh}^2(2) \right)}{3\sqrt{3}} \end{aligned}$$

Similarly, we can show that

$$\int_0^\infty \frac{x^4}{2 + \cosh(x)} \mathrm{d}x = \frac{\mathrm{arcosh}(2) \left(\pi^2 + \mathrm{arcosh}^2(2) \right) \left(7\pi^2 + 3\mathrm{arcosh}^2(2) \right)}{15\sqrt{3}}$$

□

Exercise 1.2.91: Evaluate:

$$\int_0^\infty \frac{x^{a-1}\sin x}{\cos x+\cosh x}\mathrm{d}x$$

Solution: We can use that

$$\frac{\sin(x)}{\cos(x)+\cosh(x)} = 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-xk}\sin(xk)$$

Exchanging integral and sum, we get

$$\begin{aligned}
I &= \int_0^\infty \frac{x^{a-1}\sin x}{\cos x+\cosh x}\mathrm{d}x \\
&= 2\sum_{k=1}^{\infty}(-1)^{k+1}\int_0^\infty x^{a-1}e^{-xk}\sin(xk)\mathrm{d}x \\
&= 2\sum_{k=1}^{\infty}(-1)^{k+1}\left\{2^{-\frac{a}{2}}\frac{\Gamma(a)\sin\left(\frac{\pi a}{4}\right)}{k^a}\right\} \\
&= 2^{1-\frac{a}{2}}\Gamma(a)\sin\left(\frac{\pi a}{4}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^a} \\
&= 2^{1-\frac{a}{2}}\Gamma(a)\sin\left(\frac{\pi a}{4}\right)\left(1-2^{1-a}\right)\zeta(a)
\end{aligned}$$

□

Exercise 1.2.92: Evaluate:

$$\int_0^1 \frac{\sqrt{x}\sqrt{1-x}}{x^2+1}\mathrm{d}x$$

Solution: **Method One:**

Let

$$I = \int_0^1 \frac{\sqrt{x(1-x)}}{1+x^2}\mathrm{d}x$$

By the substitution $y = \dfrac{x}{1-x}$, we get

$$I = \int_0^\infty \frac{\sqrt{y}}{(1+y)\left(2y^2+2y+1\right)}\mathrm{d}y$$

Let $f(z) = \dfrac{e^{\frac{1}{2}\ln z}}{(1+z)\left(2z^2+2z+1\right)}$ and use the branch of $\ln z$ where $0 < \arg(z) \le 2\pi$. Now, we integrate $f(z)$ around a key-hole contour which detours around the real axis.

$$\begin{aligned}
&\int_0^\infty \frac{e^{\frac{1}{2}\ln x}}{(1+x)\left(2x^2+2x+1\right)}\mathrm{d}x - \int_0^\infty \frac{e^{\frac{1}{2}\ln x+i\pi}}{(1+x)\left(2x^2+2x+1\right)}\mathrm{d}x \\
&= 2\pi i\left(\mathrm{Res}_{z=e^{i\pi}} f(z) + \mathrm{Res}_{z=\frac{1}{\sqrt{2}}e^{\frac{3i\pi}{4}}} f(z) + \mathrm{Res}_{z=\frac{1}{\sqrt{2}}e^{\frac{5i\pi}{4}}} f(z)\right) \\
2\int_0^\infty \frac{\sqrt{x}}{(1+x)\left(2x^2+2x+1\right)}\mathrm{d}x &= 2\pi i\left(i - \frac{i}{2}\sqrt[4]{2}\left(e^{\frac{i\pi}{8}}+e^{-\frac{i\pi}{8}}\right)\right) \\
I = \int_0^\infty \frac{\sqrt{x}}{(1+x)\left(2x^2+2x+1\right)}\mathrm{d}x &= \pi\left(\sqrt[4]{2}\cos\left(\frac{\pi}{8}\right)-1\right)
\end{aligned}$$

Note that $\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$, so this further simplifies to

$$I = \pi\left(\frac{\sqrt{2+\sqrt{2}}}{2^{\frac{3}{4}}} - 1\right)$$

Method Two:

Let

$$f(z) = \frac{\sqrt{z(z-1)}}{1+z^2} = \frac{\sqrt{|z|e^{i\arg z}|z-1|e^{i\arg(z-1)}}}{1+z^2}$$

where $0 \le \arg(z), \arg(z-1) < 2\pi$. Then there is a single branch cut on $[0,1]$. Let C_R be a circle of radius R where $R > 1$. Then

$$\begin{aligned}
&\lim_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z + \int_0^1 \frac{\sqrt{x(1-x)e^{\pi i}}}{1+x^2}\mathrm{d}x + \int_1^0 \frac{\sqrt{xe^{2\pi i}(1-x)e^{\pi i}}}{1+x^2}\mathrm{d}x \\
&= 2\pi i\Big(\mathrm{Res}[f(z),i] + \mathrm{Res}[f(z),-i]\Big) \\
&\implies 2\pi i a_{-1} + 2i\int_0^1 \frac{\sqrt{x(1-x)}}{1+x^2}\mathrm{d}x = 2\pi i\Big(\mathrm{Res}[f(z),i] + \mathrm{Res}[f(z),-i]\Big)
\end{aligned}$$

where a_1 is the coefficient of the $\frac{1}{z}$ term in the Laurent expansion of $f(z)$ at infinity

$$\begin{aligned}
\mathrm{Res}[f,i] &= \lim_{z\to i}\frac{\sqrt{|z|e^{i\arg z}|z-1|e^{i\arg(z-1)}}}{z+i} = \frac{\sqrt{e^{\frac{\pi i}{2}}\sqrt{2}ie^{\frac{3\pi i}{4}}}}{z+i} = \frac{2^{\frac{1}{4}}e^{\frac{5\pi i}{8}}}{2i} \\
\mathrm{Res}[f,-i] &= \lim_{z\to -i}\frac{\sqrt{|z|e^{i\arg z}|z-1|e^{i\arg(z-1)}}}{z-i} = \frac{\sqrt{e^{\frac{3\pi i}{2}}\sqrt{2}e^{\frac{5\pi i}{4}}}}{-2i} = -\frac{2^{\frac{1}{4}}e^{\frac{11\pi i}{8}}}{2i}
\end{aligned}$$

$$\begin{aligned}
f(z) &= \frac{\sqrt{z(z-1)}}{1+z^2} = \frac{\sqrt{z^2}}{z^2}\frac{(1-\frac{1}{z})^{\frac{1}{2}}}{(1+\frac{1}{z^2})} = \pm\frac{1}{z}\left(1-\frac{1}{2z}+\mathcal{O}(z^{-2})\right)\left(1-\frac{1}{z^2}+\mathcal{O}(z^{-4})\right) \\
&= \pm\frac{1}{z}\left(1-\frac{1}{2z}+\mathcal{O}(z^{-2})\right) = \pm\left(\frac{1}{z}-\frac{1}{2z^2}+\mathcal{O}(z^{-3})\right)
\end{aligned}$$

Due to the way I defined $f(z)$, $f(z)$ is real-valued and positive on the real axis for $x > 1$. Thus we must choose the plus sign in the Laurent expansion at infinity. So finally we have

$$\begin{aligned}
&2\pi i(1) + 2i\int_0^1 \frac{\sqrt{x(1-x)}}{1+x^2}\mathrm{d}x = 2\pi i\left(\frac{2^{\frac{1}{4}}e^{\frac{5\pi i}{8}}}{2i} - \frac{2^{\frac{1}{4}}e^{\frac{11\pi i}{8}}}{2i}\right) = 2\pi i\ 2^{\frac{1}{4}}\cos\left(\frac{\pi}{8}\right) \\
&\implies \int_0^1 \frac{\sqrt{x(1-x)}}{1+x^2}\mathrm{d}x = \pi\ 2^{\frac{1}{4}}\cos\left(\frac{\pi}{8}\right) - \pi
\end{aligned}$$

□

Exercise 1.2.93: Prove:

$$\int_0^\infty \frac{\ln x}{x^a(x-1)}\mathrm{d}x = \frac{2\pi^2}{1-\cos(2\pi a)}$$

Proof: Let $f(z) = \dfrac{\ln z}{z^a(z-1)}$ and integrate around a keyhole contour with the branch cut along the positive real axis. Then

$$\int_0^\infty \frac{\ln x}{x^a(z-1)}\mathrm{d}x - \int_0^\infty \frac{\ln x + 2\pi i}{x^a e^{2\pi i a}(x-1)}\mathrm{d}x - \pi i\ \mathrm{Res}[f(z), e^{2\pi i}] = 0$$

$$\text{Res}[f(z), e^{2\pi i}] = \lim_{z\to e^{2\pi i}} \frac{\ln z}{x^a} = \frac{2\pi i}{e^{2\pi i a}}$$

So

$$\begin{aligned}
&\left(1 - e^{-2\pi i a}\right) \int_0^\infty \frac{\ln x}{x^a(x-1)} \mathrm{d}x - 2\pi i e^{-2\pi i a} \int_0^\infty \frac{1}{x^a(x-1)} \mathrm{d}x \\
&= \pi i \left(2\pi i e^{-2\pi i a}\right) = -2\pi^2 e^{-2\pi i a} \\
&\implies \left(e^{2\pi i a} - 1\right) \int_0^\infty \frac{\ln x}{x^a(x-1)} \mathrm{d}x - 2\pi i \int_0^\infty \frac{1}{x^a(x-1)} \mathrm{d}x = -2\pi^2 \\
&\implies \left(\cos(2\pi a) + i \sin(2\pi a) - 1\right) \int_0^\infty \frac{\ln x}{x^a(x-1)} \mathrm{d}x - 2\pi i \int_0^\infty \frac{1}{x^a(x-1)} \mathrm{d}x = -2\pi^2
\end{aligned}$$

Now equate the real parts on both sides of the equation.

$$\int_0^\infty \frac{\ln x}{x^a(x-1)} \mathrm{d}x = \frac{2\pi^2}{1 - \cos(2\pi a)}$$

□

Exercise 1.2.94: Evaluate:

$$\int_0^\pi \frac{\sin(nt)\sin(t)}{1 - 2r\cos(t) + r^2} \mathrm{d}t$$

Solution:

$$\begin{aligned}
\int_0^\pi \frac{\sin(nx)\sin(x)}{1 - 2r\cos x + r^2}\, \mathrm{d}x &= \frac{1}{2} \int_0^{2\pi} \frac{\sin(nx)\sin(x)}{1 - 2r\cos x + r^2}\, \mathrm{d}x = \frac{1}{2} \Im \int_0^{2\pi} \frac{e^{inx}\sin(x)}{1 - 2r\cos x + r^2}\, \mathrm{d}x \\
&= \frac{1}{4i} \Im \int_0^{2\pi} \frac{e^{inx}(e^{ix} - e^{-ix})}{1 - r(e^{ix} + e^{-ix}) + r^2}\, \mathrm{d}x
\end{aligned}$$

Let $z = e^{ix}$.

$$\begin{aligned}
\Rightarrow \frac{1}{4i} \Im \int_{|z|=1} \frac{z^n(z - z^{-1})}{1 - r(z + z^{-1}) + r^2} \frac{\mathrm{d}z}{iz} &= -\frac{1}{4} \Im \int_{|z|=1} \frac{z^{n-1}(z^2 - 1)}{z - rz^2 - r + r^2 z}\, \mathrm{d}z \\
&= -\frac{1}{4} \Im \int_{|z|=1} \frac{z^{n-1}(z^2 - 1)}{(z - r)(1 - rz)}\, \mathrm{d}z
\end{aligned}$$

If $r < 1$,

$$\begin{aligned}
\int_0^\pi \frac{\sin(nx)\sin(x)}{1 - 2r\cos x + r^2}\, \mathrm{d}x &= -\frac{1}{4} \Im\, 2\pi i\, \text{Res}\left[\frac{z^{n-1}(z^2 - 1)}{(z - r)(1 - rz)}, r\right] \\
&= -\frac{1}{4} \Im\, 2\pi i \lim_{z\to r} \frac{z^{n-1}(z^2 - 1)}{1 - rz} = \frac{\pi}{2} r^{n-1}
\end{aligned}$$

And if $r > 1$,

$$\begin{aligned}
\int_0^\pi \frac{\sin(nx)\sin(x)}{1 - 2r\cos x + r^2}\, \mathrm{d}x &= -\frac{1}{4} \Im\, 2\pi i\, \text{Res}\left[\frac{z^{n-1}(z^2 - 1)}{(z - r)(1 - rz)}, \frac{1}{r}\right] \\
&= \frac{1}{4} \Im\, 2\pi i \lim_{z\to \frac{1}{r}} \frac{1}{r} \frac{z^{n-1}(z^2 - 1)}{z - r} = \frac{\pi}{2} \frac{1}{r^{n+1}}
\end{aligned}$$

Hence

$$\int_0^\pi \frac{\sin(nt)\sin(t)}{1-2r\cos(t)+r^2}\mathrm{d}t = \begin{cases} \frac{\pi}{2}r^{n-1} & (r<1) \\ \frac{\pi}{2}r^{-n-1} & (r>1) \end{cases}$$

□

Exercise 1.2.95: Evaluate:

$$\int_0^\infty x^{s-1}\frac{\psi(x+1)+\gamma}{x}\mathrm{d}x$$

Solution:

$$\frac{x^{s-1}(\psi(x+1)+\gamma)}{x} = \sum_{n=1}^\infty \frac{x^{s-1}}{n(n+x)}$$

Integrating both sides:

$$\begin{aligned} \int_0^\infty \frac{x^{s-1}(\psi(x+1)+\gamma)}{x}\mathrm{d}x &= \sum_{n=1}^\infty \int_0^\infty \frac{x^{s-1}}{n(n+x)}\mathrm{d}x = \sum_{n=1}^\infty \frac{1}{n^{2-s}}\int_0^\infty \frac{x^{s-1}}{1+x}\mathrm{d}x \\ &= \sum_{n=1}^\infty \frac{1}{n^{2-s}}\mathrm{B}(s,1-s) = \sum_{n=1}^\infty \frac{1}{n^{2-s}}\Gamma(s)\Gamma(1-s) \\ &= \sum_{n=1}^\infty \frac{1}{n^{2-s}}\pi\csc(\pi s) = \pi\csc(\pi s)\sum_{n=1}^\infty \frac{1}{n^{2-s}} \\ &= \pi\csc(\pi s)\zeta(2-s) \end{aligned}$$

□

Exercise 1.2.96: Prove:

$$\int_0^\infty \frac{e^{a\cos(bx)}\sin\left(a\sin(bx)\right)}{x}\mathrm{d}x = \frac{\pi}{2}\left(e^a-1\right)$$

Proof:

$$\int_0^\infty \frac{e^{a\cos bx}\sin(a\sin bx)}{x}\mathrm{d}x = \frac{1}{2}\Im\int_{-\infty}^\infty \frac{e^{a\cos bx}e^{ia\sin bx}}{x}\mathrm{d}x = \frac{1}{2}\Im\int_{-\infty}^\infty \frac{e^{ae^{ibx}}}{x}\mathrm{d}x$$

Let $f(z)=\dfrac{e^{ae^{ibz}}}{z}$ and integrate around the typical closed half-circle of radius R in the upper half plane that is indented at the origin.

$$\mathrm{PV}\int_{-\infty}^\infty \frac{e^{ae^{ibx}}}{x}\mathrm{d}x - \pi i\mathrm{Res}[f,0] + \lim_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z = 0$$

But don't assume that $\lim\limits_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z$ goes to zero. Because it doesn't.

$$\begin{aligned} \lim_{R\to\infty}\int_{C_r} f(z)\mathrm{d}z &= \lim_{R\to\infty}\int_{C_R}\left(\frac{1}{z}+\frac{ae^{ibz}}{z}+\ldots\right)\mathrm{d}z = \lim_{R\to\infty}\left(\int_{C_R}\frac{1}{z}\mathrm{d}z+\int_{C_R}\frac{ae^{ibz}}{z}\mathrm{d}z+\ldots\right) \\ &= \lim_{R\to\infty}\int_{C_R}\frac{1}{z}\mathrm{d}z = \lim_{R\to\infty}\int_0^\pi \frac{1}{Re^{it}}iRe^{it}\mathrm{d}t = i\pi \end{aligned}$$

So

$$\begin{aligned}
&\text{PV}\int_{-\infty}^{\infty}\frac{e^{ae^{ibx}}}{x}\mathrm{d}x-\pi i(e^a)+i\pi=0\\
\Longrightarrow\ &\text{PV}\int_{-\infty}^{\infty}\frac{e^{ae^{ibx}}}{x}\mathrm{d}x=\pi i(e^a-1)\\
\Longrightarrow\ &\int_0^{\infty}\frac{e^{a\cos bx}\sin(a\sin bx)}{x}\mathrm{d}x=\frac{\pi}{2}(e^a-1)
\end{aligned}$$

□

Exercise 1.2.97: Prove:

$$\int_0^{\infty}\frac{x\ e^{a\cos bx}\sin(a\sin bx)}{1+x^2}\mathrm{d}x=\frac{\pi}{2}(e^{ae^{-b}}-1)\ ,\ a,b>0$$

Proof:

$$\int_{-\infty}^{\infty}\frac{ze^{ae^{ibx}}}{1+z^2}\mathrm{d}z=2\pi i\text{Res}_{z=i}\frac{ze^{ae^{ibx}}}{1+z^2}-\lim_{R\to\infty}\int_{C_R}\frac{ze^{ae^{ibx}}}{1+z^2}\mathrm{d}z$$

where C_R is the upper half circle $|z|=R$ traversed in the counterclockwise direction.

$$\begin{aligned}
\lim_{R\to\infty}\int_{C_R}\frac{ze^{ae^{ibx}}}{1+z^2}\mathrm{d}z&=\lim_{R\to\infty}\int_{C_R}\frac{z+aze^{ibz}+\cdots}{1+z^2}\mathrm{d}z=\lim_{R\to\infty}\int_{C_R}\frac{z}{1+z^2}\mathrm{d}z\\
&=i\pi\lim_{z\to\infty}\frac{z^2}{1+z^2}=i\pi
\end{aligned}$$

and

$$\text{Res}_{z=i}\frac{ze^{ae^{ibx}}}{1+z^2}=\frac{e^{ae^{-b}}}{2}$$

So

$$\int_{-\infty}^{\infty}\frac{xe^{ae^{ibx}}}{1+x^2}\mathrm{d}x=i\pi\left(e^{ae^{-b}}-1\right)$$

Separating imaginary parts, we get

$$\int_0^{\infty}\frac{xe^{a\cos(bx)}\sin(a\sin(bx))}{1+x^2}\mathrm{d}x=\frac{\pi}{2}\left(e^{ae^{-b}}-1\right)$$

□

Exercise 1.2.98: Prove:

$$\int_0^{\infty}\frac{\cos\left(b\arctan\frac{x}{a}\right)}{(1+x^2)(a^2+x^2)^{\frac{b}{2}}}\mathrm{d}x=\frac{\pi}{2(1+a)^b}\ ,\ a>0,\ b>-1$$

Proof:

$$\begin{aligned}
(a-ix)=\left(a^2+x^2\right)^{\frac{1}{2}}\ e^{-i\arctan\frac{x}{a}}\ &\Longrightarrow\ (a-ix)^{-b}=\left(a^2+x^2\right)^{\frac{-b}{2}}\ e^{ib\arctan\frac{x}{a}}\\
&\Longrightarrow\ \Re\ (a-ix)^{-b}=\frac{\cos\left(b\arctan\frac{x}{a}\right)}{(a^2+x^2)^{\frac{b}{2}}}
\end{aligned}$$

So

$$\int_0^\infty \frac{\cos\left(b\arctan\dfrac{x}{a}\right)}{(1+x^2)(a^2+x^2)^{\frac{b}{2}}}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos\left(b\arctan\dfrac{x}{a}\right)}{(1+x^2)(a^2+x^2)^{\frac{b}{2}}}\mathrm{d}x = \frac{1}{2}\,\Re\int_{-\infty}^\infty \frac{1}{(1+x^2)(a-ix)^b}\mathrm{d}x$$
$$= \Re\; 2\pi i\;\mathrm{Res}\left[\frac{1}{(1+z^2)(a-iz)}, i\right] = \Re\;\pi i\;\lim_{z\to i}\frac{1}{(z+i)(a-iz)^b}$$
$$= \Re\;\pi i\;\frac{1}{2i(a+1)^b} = \frac{\pi}{2(1+a)^b}$$

□

Exercise 1.2.99: Evaluate:

$$\int_0^{\frac{1}{2}} \frac{\ln\left((1-x)^2+x^2z\right)}{x}\mathrm{d}x$$

Solution: Let

$$\varphi(z) = \int_0^{\frac{1}{2}} \frac{\ln\left((1-x)^2+x^2z\right)}{x}\mathrm{d}x$$

$$\begin{aligned}\varphi'(z) &= \int_0^{\frac{1}{2}} \frac{x}{(1-x)^2+x^2z}\mathrm{d}x \\ &= \frac{\arctan\left(\dfrac{z-1}{2\sqrt{z}}\right)}{(1+z)\sqrt{z}} + \frac{\ln\left(\dfrac{1+z}{4}\right)}{2(1+z)} + \frac{\arctan\left(\dfrac{1}{\sqrt{z}}\right)}{(1+z)\sqrt{z}} \\ &= \frac{\arctan\left(\sqrt{z}\right)}{(1+z)\sqrt{z}} + \frac{\ln\left(\dfrac{1+z}{4}\right)}{2(1+z)}\end{aligned}$$

$$\begin{aligned}\varphi(z) &= \int\left\{\frac{\arctan\left(\sqrt{z}\right)}{(1+z)\sqrt{z}} + \frac{\ln\left(\dfrac{1+z}{4}\right)}{2(1+z)}\right\}\mathrm{d}z \\ &= \arctan^2\left(\sqrt{z}\right) + \frac{1}{4}\ln^2\left(\frac{1+z}{4}\right) + C\end{aligned}$$

When $z=0$,

$$\varphi(0) = -2\mathrm{Li}_2\left(\frac{1}{2}\right) = -\zeta(2) + \ln^2(2)$$

So we get

$$C = -\zeta(2)$$

Therefore

$$\varphi(z) = \frac{1}{4}\ln^2\left(\frac{1+z}{4}\right) + \arctan^2\left(\sqrt{z}\right) - \zeta(2)$$

□

Exercise 1.2.100: Evaluate:

$$\int_0^\infty \frac{\cos\left(ax^2+bx\right) - e^{-bx}\cos\left(ax^2\right)}{x}\frac{p+qx^4}{m^4+n^4x^4}\mathrm{d}x$$

✎ **Solution:** Let $f(z)=\dfrac{e^{iaz^2}(e^{ibz}-e^{-bz})(p+qz^4)}{z(m^4+n^4z^4)}$ and integrate around a quarter circle. The singularity at $z=0$ is removable. So

$$\int_0^\infty \frac{e^{iax^2}(e^{ibx}-e^{-bx})(p+qx^4)}{x(m^4+n^4x^4)}\mathrm{d}x+\int_\infty^0 \frac{e^{-iat^2}(e^{-bt}-e^{-ibt})(p+qt^4)}{t(m^4+n^4t^4)}\mathrm{d}t=2\pi i\,\mathrm{Res}\left[f(z),\frac{m}{n}e^{\frac{i\pi}{4}}\right]$$

Or

$$2\int_0^\infty \frac{\cos\left(ax^2+bx\right)-e^{-bx}\cos\left(ax^2\right)}{x}\frac{p+qx^4}{m^4+n^4x^4}\mathrm{d}x=\Re\ 2\pi i\ \mathrm{Res}\left[f(z),\frac{m}{n}e^{\frac{i\pi}{4}}\right]$$

$$\begin{aligned}\mathrm{Res}\left[f(z),\frac{m}{n}e^{\frac{i\pi}{4}}\right]&=\lim_{z\to\frac{m}{n}e^{\frac{i\pi}{4}}}\frac{e^{iaz^2}(e^{ibz}-e^{-bz})(p+qz^4)}{(m^4+n^4z^4)+4n^4z^4}\\&=\frac{e^{-a\frac{m^2}{n^2}}e^{-\frac{bm}{n\sqrt{2}}}(e^{i\frac{bm}{n\sqrt{2}}}-e^{-i\frac{bm}{n\sqrt{2}}})(pn^4-qm^4)}{-4n^4m^4}\\&=-2i\ e^{-a\frac{m^2}{n^2}}e^{-\frac{bm}{n\sqrt{2}}}\sin\left(\frac{bm}{n\sqrt{2}}\right)\frac{pn^4-qm^4}{4n^4m^4}\end{aligned}$$

So

$$\int_0^\infty \frac{\cos\left(ax^2+bx\right)-e^{-bx}\cos\left(ax^2\right)}{x}\frac{p+qx^4}{m^4+n^4x^4}\mathrm{d}x=\frac{\pi}{2}\frac{pn^4-qm^4}{m^4n^4}e^{-a\frac{m^2}{n^2}}e^{-\frac{bm}{n\sqrt{2}}}\sin\left(\frac{bm}{n\sqrt{2}}\right)$$

□

Exercise 1.2.101: Prove:

$$\int_0^\infty \frac{x^2}{\cosh x+\cos a}\mathrm{d}x=\frac{a\left(\pi^2-a^2\right)}{3\sin a}\ ,\ (0<a<\pi)$$

Proof: Started with:

$$\int_0^\infty \frac{e^{-sx}}{\cosh(x)+\cos(a)}\mathrm{d}x=\frac{2}{\sin(a)}\sum_{k=1}^\infty \frac{(-1)^{k-1}\sin(ak)}{s+k} \tag{1}$$

To show this, write

$$\begin{aligned}\frac{e^{-sx}}{\cosh(x)+\cos(a)}&=\frac{e^{-sx}}{\sin(a)}\left(\frac{1}{1+e^{-(x+ia)}}-\frac{1}{1+e^{-(x-ia)}}\right)\\&=\frac{2e^{sx}}{\sin(a)}\sum_{k=0}^\infty(-1)^{k-1}e^{-xk}\sin(ak)\end{aligned}$$

Integrating gives the above formula in (1).

Now, take (1) and diff twice, then let $s=0$, this gives:

$$\int_0^\infty \frac{x^2}{\cosh(x)+\cos(a)}\mathrm{d}x=\frac{4}{\sin(a)}\sum_{k=1}^\infty \frac{(-1)^{k-1}\sin(ka)}{k^3}$$

If the $\sin(ak)$ on the right is written in terms of its exponential, then we have a trilog series.

$$\int_0^\infty \frac{x^2}{\cosh(x)+\cos(a)}\mathrm{d}x=\frac{2i}{\sin(a)}\left[\mathrm{Li}_3(-e^{ia})-\mathrm{Li}_3(-e^{-ia})\right]$$

This is the same as $\frac{a\left(\pi^2-a^2\right)}{3\sin(a)}$ by virtue of the identity.

$$\mathrm{Li}_3(z)-\mathrm{Li}_3\left(\frac{1}{z}\right)=-\frac{1}{6}\ln^3(-z)+\frac{\pi^2}{6}\ln(-z)$$

$$\frac{2i}{\sin(a)}\left(-\frac{1}{6}\ln^3(e^{ia})+\frac{\pi^2}{6}\ln(e^{-ia})\right)=\frac{a(\pi^2-a^2)}{3\sin(a)}$$

□

Exercise 1.2.102: Prove:

$$\int_0^1 \mathbf{K}(k)\mathrm{d}k=2\mathbf{G}$$

Proof:

$$\begin{aligned}
I&=\int_0^1\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\mathrm{d}\theta\mathrm{d}k=\int_0^{\frac{\pi}{2}}\int_0^1\frac{1}{\sqrt{1-k^2\sin^2\theta}}\mathrm{d}k\mathrm{d}\theta\\
&=\int_0^{\frac{\pi}{2}}\frac{\theta}{\sin\theta}\mathrm{d}\theta=-\int_0^{\frac{\pi}{2}}\ln\tan\frac{\theta}{2}\mathrm{d}\theta=-2\int_0^{\frac{\pi}{4}}\ln\tan\theta\mathrm{d}\theta\\
&=-2\int_0^1\frac{\ln x}{1+x^2}\mathrm{d}x=-2\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)\mathrm{d}x\\
&=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=2\mathbf{G}
\end{aligned}$$

□

Exercise 1.2.103: Evaluate:

$$\int_0^{\ln\phi}x^2\coth x\mathrm{d}x$$

✎ **Solution:**

$$\int_0^{\ln\phi} x^2 \coth x dx = \int_0^{\ln\phi} x^2 \frac{1+e^{-2x}}{1-e^{-2x}} \mathrm{d}x = \int_0^{\ln\phi} x^2 \left(\frac{1}{1-e^{-2x}} + \frac{e^{-2x}}{1-e^{-2x}} \right) \mathrm{d}x$$

$$= \int_0^{\ln\phi} x^2 \left(\sum_{k=0}^{\infty} e^{-2nx} + \sum_{k=1}^{\infty} e^{-2nx} \right) \mathrm{d}x = \int_0^{\ln\phi} \left(x^2 + 2x^2 \sum_{n=1}^{\infty} e^{-2nx} \right) \mathrm{d}x$$

$$= \frac{\ln^3\phi}{3} + 2\sum_{n=1}^{\infty} \int_0^{\ln\phi} x^2 e^{-2nx} \mathrm{d}x = \frac{\ln^3\phi}{3} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{-2nt}(2n^2t^2+2nt+1)}{n^3} \Bigg|_0^{\ln\phi}$$

$$= \frac{\ln^3\phi}{3} - t^2 \sum_{n=1}^{\infty} \frac{e^{-2nt}}{n} - t\sum_{n=1}^{\infty} \frac{e^{-2nt}}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{-2nt}}{n^3} \Bigg|_0^{\ln\phi}$$

$$= \frac{\ln^3\phi}{3} + t^2 \ln\left(1-e^{-2t}\right) - t\,\mathrm{Li}_2\left(e^{-2t}\right) - \frac{1}{2}\mathrm{Li}_3\left(e^{-2t}\right) \Bigg|_0^{\ln\phi}$$

$$= \frac{\ln^3\phi}{3} + \ln^2(\phi)\ln\left(1-\phi^{-2}\right) - \ln(\phi)\,\mathrm{Li}_2\left(\phi^{-2}\right) - \frac{1}{2}\mathrm{Li}_3\left(\phi^{-2}\right) + \frac{1}{2}\zeta(3)$$

$$= \frac{\ln^3\phi}{3} + \ln^2(\phi)\ln\left(1-\phi^{-2}\right) - \ln(\phi)\left(\frac{\pi^2}{15} - \ln^2(\phi)\right)$$

$$- \frac{1}{2}\left(\frac{4}{5}\zeta(3) + \frac{2}{3}\ln^3(\phi) - \frac{2\pi^2}{15}\ln(\phi)\right) + \frac{1}{2}\zeta(3)$$

$$= \ln^2(\phi)\ln\left(1-\phi^{-2}\right) + \ln^3(\phi) + \frac{\zeta(3)}{10}$$

$$= \ln^2(\phi)\ln\left(\frac{1}{\phi}\right) + \ln^3(\phi) + \frac{\zeta(3)}{10} = -\ln^3(\phi) + \ln^3(\phi) + \frac{\zeta(3)}{10}$$

$$= \frac{\zeta(3)}{10}$$

□

Exercise 1.2.104: Evaluate:

$$\int_0^{\frac{\pi}{4}} 2\ln(\sin\theta)\ln(\cos\theta)\ \mathrm{d}\theta$$

✎ **Solution:**

$$\int_0^{\frac{\pi}{4}} 2\ln(\sin\theta)\ln(\cos\theta)\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \ln(\sin\theta)\ln(\cos\theta)\ \mathrm{d}\theta$$

$$= \frac{1}{8}\frac{\partial^2\beta}{\partial p\partial q}\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{8\Gamma(1)}\left[(-\gamma+2\ln 2+\gamma)^2 - \frac{\pi^2}{6}\right]$$

$$= \frac{\pi}{2}\ln^2 2 - \frac{\pi^3}{48}$$

□

Exercise 1.2.105: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\ln^2\left(\frac{1+\cos x-\sin x}{1+\cos x+\sin x}\right)\mathrm{d}x$$

✎ **Solution:** Let

$$f(z)=\frac{\pi\csc(\pi z)}{2z+1}\psi_2(-z)$$

The residue at $z=-\frac{1}{2}$ is equal to $-\frac{\pi}{2}\psi_2\left(\frac{1}{2}\right)$. At a positive integer, $n\geq 0$ the residue is

$$-\frac{16(-1)^n}{(2n+1)^4}-\frac{2\pi^2}{3}\frac{(-1)^n}{(2n+1)^2}+\frac{(-1)^n}{2n+1}\psi_2(n+1)$$

At a negative integer, n the pole is simple and the residue is equal to

$$(-1)^{n-1}\frac{\psi_2(n)}{2n-1}$$

The sum of all residues is equal to 0, so we get

$$-\frac{\pi}{2}\psi_2\left(\frac{1}{2}\right)+2\sum_{n=0}^{\infty}\frac{(-1)^n\psi_2(n+1)}{2n+1}-16\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}-\frac{2\pi^2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0$$

$$\Longrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n\psi_2(n+1)}{2n+1}=8\beta(4)+\frac{\pi^2}{3}\mathbf{G}-\frac{7}{2}\pi\zeta(3)$$

The value of the original integral is

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{x}{\sin x}\ln^2\left(\frac{1+\cos x-\sin x}{1+\cos x+\sin x}\right)\mathrm{d}x&=\frac{7\pi}{4}\zeta(3)-4\int_0^1\frac{\arctan z\ln^2 z}{1-z^2}\mathrm{d}z\\&=\frac{7\pi}{4}\zeta(3)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\psi_2(n+1)}{2n+1}\\&=\frac{7\pi}{4}\zeta(3)+\frac{1}{2}\left(8\beta(4)+\frac{\pi^2}{3}\mathbf{G}-\frac{7}{2}\pi\zeta(3)\right)\\&=4\beta(4)+\frac{\pi^2}{6}\mathbf{G}\\&=\frac{1}{192}\psi_3\left(\frac{1}{4}\right)-\frac{\pi^4}{24}+\frac{\pi^2}{6}\mathbf{G}\end{aligned}$$

□

Exercise 1.2.106: Prove:

$$\int_0^{\frac{\pi}{2}}\frac{[\theta-\tan^{-1}(n\tan\theta)]\sin(2\theta)}{1+2n\cos(2\theta)+n^2}\mathrm{d}\theta=\frac{\pi}{4n}\ln\left(\frac{1+n}{1+n^2}\right),\quad n<1$$

☞ **Proof:** Substitute $t=\tan\theta$

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}\frac{(\theta-\arctan(n\tan\theta)\sin(2\theta)}{1+2n\cos(2\theta)+n^2}\mathrm{d}t\\&=\int_0^{\infty}\left(\frac{2t}{t^2+1}\right)\frac{\arctan(t)-\arctan(nt)}{(n+1)^2+(n-1)^2t^2}\mathrm{d}t\end{aligned}$$

Use $\arctan(t)-\arctan(nt)=\int_n^1\frac{t}{t^2x^2+1}\mathrm{d}x$ and switch integration order

$$I=\int_n^1\int_0^{\infty}\frac{2t^2}{(t^2+1)(t^2x^2+1)((n+1)^2+(n-1)^2t^2)}\mathrm{d}t\,\mathrm{d}x$$

This integral doesn't look very nice to deal with, but notice that it is of the form

$$I(a,b,c)=\int_0^\infty \frac{t^2}{(t^2+a^2)(t^2+b^2)(t^2+c^2)}\mathrm{d}t$$

where $a,b,c>0$. Splitting the integrand into partial fractions gives

$$I(a,b,c)=\int_0^\infty \frac{a^2}{(a^2-b^2)\,(c^2-a^2)\,(a^2+t^2)}+\frac{b^2}{(a^2-b^2)\,(b^2-c^2)\,(b^2+t^2)}+\frac{c^2}{(c^2-a^2)\,(b^2-c^2)\,(c^2+t^2)}\mathrm{d}t$$

So after doing the integration we get that

$$I(a,b,c)=\frac{\pi}{2(a+b)(b+c)(c+a)}$$

which is a pretty nice formula! When $n<1$ we have $\dfrac{n+1}{n-1}<0$ so we have to use $-\dfrac{n+1}{n-1}$ as the square root of $\left(\dfrac{n+1}{n-1}\right)^2$.

$$\begin{aligned}
I&=2\int_n^1\frac{1}{x^2(n-1)^2}\int_0^\infty\frac{t^2}{(t^2+1)\left(t^2+\frac{1}{x^2}\right)\left(t^2+\left(\frac{n+1}{n-1}\right)^2\right)}\mathrm{d}t\,\mathrm{d}x\\
&=2\int_n^1\frac{1}{x^2(n-1)^2}I\left(1,\frac{1}{x},-\frac{n+1}{n-1}\right)\mathrm{d}x\\
&=\pi\int_n^1\frac{1}{x^2(n-1)^2\left(1+\frac{1}{x}\right)\left(\frac{1}{x}-\frac{n+1}{n-1}\right)\left(-\frac{n+1}{n-1}+1\right)}\mathrm{d}x\\
&=\frac{\pi}{2}\int_n^1\frac{\mathrm{d}x}{(x+1)((n+1)x-(n-1))}\\
&=\frac{\pi}{4n}\ln\left(\frac{n+1}{n^2+1}\right)
\end{aligned}$$

When $n>1$, we can do the same thing but use $\dfrac{n+1}{n-1}$ as the square root to get

$$I=\frac{\pi}{4n}\ln\left(\frac{n^2+n}{n^2+2n-1}\right)$$

□

Exercise 1.2.107: Let H_x denote the extended harmonic number. Prove that:

$$\int_0^1 H_x\,\mathrm{d}x=\gamma$$

Proof: Use $\psi(x+1)=H_x-\gamma$, then

$$\begin{aligned}
\int_0^1 H_x\mathrm{d}x&=\int_0^1\psi(x+1)+\gamma\,\mathrm{d}x\\
&=\ln\Gamma(2)-\ln\Gamma(1)+\gamma\\
&=\gamma
\end{aligned}$$

□

Exercise 1.2.108: Prove:

$$\int_0^1 \frac{\cosh(a\ln(x))\ln(1+x)}{x}\mathrm{d}x = \frac{1}{2a}\left(\frac{\pi}{\sin(\pi a)} - \frac{1}{a}\right), \quad a < 1$$

Proof: Method One:

If $1 < a < 1$,

$$\begin{aligned}\int_0^1 \frac{\cosh(a\ln x)\ln(1+x)}{x}\,\mathrm{d}x &= \int_0^\infty \cosh(au)\ln(1+e^{-u})\,\mathrm{d}u \\ &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\int_0^\infty \cosh(au)e^{-nu}\,\mathrm{d}u \\ &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2-a^2} \\ &= \frac{1}{2}\left(\mathrm{Res}\left[\frac{\pi\csc(\pi z)}{z^2-a^2}, a\right] + \mathrm{Res}\left[\frac{\pi\csc(\pi z)}{z^2-a^2}, -a\right] - \frac{1}{a^2}\right) \\ &= \frac{1}{2}\left(\frac{\pi\csc(\pi a)}{a} - \frac{1}{a^2}\right)\end{aligned}$$

Method Two:

Write as:

$$\begin{aligned}\int_0^1 \frac{e^{\ln(x^a)} + e^{-\ln(x^a)}}{2}\cdot\frac{\ln(1+x)}{x}\mathrm{d}x &= \int_0^1 \frac{x^a + x^{-a}}{x}\cdot\frac{\ln(1+x)}{2}\mathrm{d}x \\ &= \int_0^1 \frac{\left(x^{a-1} + x^{-(a+1)}\right)}{2}\ln(1+x)\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 x^{a-1}\ln(1+x)\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\ln(1+x)}{x^{a+1}}\mathrm{d}x\end{aligned}$$

These now look familiar. But, using parts with $u = \ln(1+x)$ leads to:

$$\frac{1}{2a}\int_0^1 \frac{x^{-a} - x^a}{1+x}\mathrm{d}x = \frac{1}{2a}\left(\frac{\pi}{\sin(\pi a)} - \frac{1}{a}\right)$$

□

Exercise 1.2.109: Prove:

$$\int_{-\infty}^\infty \frac{\mathrm{sech}(x)}{x^2+\pi^2}\mathrm{d}x = \frac{4}{\pi} - 1$$

Proof: Method One:

Rewrite the integral as

$$\begin{aligned}I = \int_{-\infty}^\infty \frac{\mathrm{sech}(x)}{\pi^2+x^2}\mathrm{d}x &= 2\int_0^\infty \frac{1}{(\pi^2+x^2)\cosh(x)}\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{(1+x^2)\cosh(\pi x)}\mathrm{d}x\end{aligned}$$

Now, we make use of the following series identity:

$$\frac{1}{\cosh(\pi x)} = \frac{4}{\pi}\sum_{n=0}^\infty \frac{(-1)^n(2n+1)}{(2n+1)^2+4x^2}$$

Hence

$$\begin{aligned} I &= \frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_0^{\infty}\frac{1}{((2n+1)^2+4x^2)\,(1+x^2)}\mathrm{d}x \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3} = \frac{4}{\pi}\left(1-\frac{\pi}{4}\right) \\ &= \frac{4}{\pi}-1 \end{aligned}$$

Method Two:

Using the geometric series

$$\frac{1}{2}\mathrm{sech}(x) = \sum_{k=0}^{\infty}(-1)^k e^{-(2k+1)x}$$

to rewrite the integral as

$$I = \int_{-\infty}^{\infty}\frac{\mathrm{sech}(x)}{x^2+\pi^2}\mathrm{d}x = 4\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}\frac{e^{-(2k+1)x}}{x^2+\pi^2}\mathrm{d}x$$

Now let n be an odd integer and define

$$\begin{aligned} I(n) &= \int_0^{\infty}\frac{e^{-nx}}{x^2+\pi^2}\mathrm{d}x = \int_0^{\infty}\int_0^{\infty}e^{-nx}e^{-xy}\frac{\sin(\pi y)}{\pi}\mathrm{d}x\mathrm{d}y \\ &= \int_0^{\infty}\frac{\sin(\pi y)}{\pi(n+y)}\mathrm{d}y = \int_n^{\infty}\frac{\sin(\pi(y-n))}{\pi y}\mathrm{d}y \\ &= \int_n^{\infty}-\frac{\sin(\pi y)}{\pi y}\mathrm{d}y = \frac{\mathrm{Si}(n\pi)}{\pi}-\frac{1}{2} \end{aligned}$$

Now we have

$$\begin{aligned} I &= 4\sum_{k=0}^{\infty}(-1)^k I(2k+1) = 4\sum_{k=0}^{\infty}I(4k+1)-I(4k+3) \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}\mathrm{Si}((4k+1)\pi)-\mathrm{Si}((4k+3)\pi) \end{aligned}$$

To evaluate this, use the Fourier series of $\mathrm{sinc}(x)$ for $-\pi < x < \pi$.

$$\begin{aligned} \mathrm{sinc}(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{sinc}(t)\mathrm{d}t + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\cos(kx)\int_{-\pi}^{\pi}\mathrm{sinc}(t)\cos(kt)\mathrm{d}t\right) \\ &= \frac{\mathrm{Si}(\pi)}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos(kx)(\mathrm{Si}((k+1)\pi)-\mathrm{Si}((k-1)\pi)) \end{aligned}$$

Now evaluate the series at $x = \dfrac{\pi}{2}$ to get

$$\frac{2}{\pi} = \frac{\mathrm{Si}(\pi)}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{2}\right)(\mathrm{Si}((k+1)\pi)-\mathrm{Si}((k-1)\pi))$$

Note that every coefficient of $\mathrm{Si}(n\pi)$ in this series is $\pm\dfrac{2}{\pi}$ apart from the last one which is $-\dfrac{1}{\pi}$, so we can rewrite it as

$$\begin{aligned} \frac{2}{\pi} &= \lim_{N\to\infty}\frac{2}{\pi}\sum_{k=0}^{N}\mathrm{Si}((4k+1)\pi)-\mathrm{Si}((4k+3)\pi)+\frac{\mathrm{Si}((4N+3)\pi)}{\pi} \\ &= \lim_{N\to\infty}\frac{2}{\pi}\sum_{k=0}^{N}\mathrm{Si}((4k+1)\pi)-\mathrm{Si}((4k+3)\pi)+\frac{1}{2} \end{aligned}$$

Hence,

$$I=\frac{4}{\pi}\sum_{k=0}^{\infty}\operatorname{Si}((4k+1)\pi)-\operatorname{Si}((4k+3)\pi)=2\left(\frac{2}{\pi}-\frac{1}{2}\right)=\frac{4}{\pi}-1$$

□

Exercise 1.2.110: Evaluate:

$$y(t)=\int_0^{\infty}\frac{\cos(tx)}{(1+x^2)\cosh(\pi x)}\mathrm{d}x$$

Solution: In the above calculation, we have shown that $y(0)=2-\frac{\pi}{2}$. Differentiate $y(t)$ twice to get

$$y''(t)=-\int_0^{\infty}\frac{x^2\cos(tx)}{(1+x^2)\cosh(\pi x)}\mathrm{d}x$$

and note that

$$\begin{aligned}y(t)-y''(t)&=\int_0^{\infty}\frac{\cos(tx)}{\cosh(\pi x)}\mathrm{d}x=2\int_0^{\infty}\cos(tx)\left(\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)\pi x}\right)\mathrm{d}x\\&=2\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}\cos(tx)e^{-(2n+1)\pi x}\,\mathrm{d}x\\&=2\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2+t^2}=2\pi\left(\frac{\operatorname{sech}\left(\frac{t}{2}\right)}{4\pi}\right)\\&=\frac{1}{2}\operatorname{sech}\left(\frac{t}{2}\right)\end{aligned}$$

The differential equation $y''(t)-y(t)=-\frac{1}{2}\operatorname{sech}\left(\frac{t}{2}\right)$ can be solved using variation of parameters or any other technique.

$$y(t)=2\cosh\left(\frac{t}{2}\right)-e^t\tan^{-1}\left(e^{-t/2}\right)-e^{-t}\tan^{-1}\left(e^{t/2}\right)+e^tC_1+e^{-t}C_2$$

To determine the constants C_1 and C_2, we can use the conditions $y(0)=2-\frac{\pi}{2}$ and $y'(0)=0$. This gives $C_1=C_2=0.$, hence

$$y(t)=2\cosh\left(\frac{t}{2}\right)-e^t\tan^{-1}\left(e^{-t/2}\right)-e^{-t}\tan^{-1}\left(e^{t/2}\right)$$

□

Exercise 1.2.111: Evaluate:

$$\int_0^1\frac{\ln(1+x)\ln\left(1+x^3\right)}{x}\mathrm{d}x$$

Solution: Let

$$f(z)=\frac{\ln(1+z)\ln(1+z^3)}{z},\ J=\int_0^1 f(z)\,\mathrm{d}z$$

The main idea is to integrate $f(z)$ along three paths and obtain an algebraic equation with J.

The first path is simply the interval $[1,1]$. Since the contributions from the indentations around the branch points tend to 0, we may change the path to $e^{\pi i[0,1]}$.

$$\begin{aligned}\int_{-1}^{1} f(z)\,\mathrm{d}z &= J - \int_0^1 \frac{\ln(1-x)\ln(1-x^3)}{x}\mathrm{d}x \\ &= \Im\int_0^{\pi} \ln\left(1+e^{i\theta}\right)\ln\left(1+e^{3i\theta}\right)\,\mathrm{d}\theta \\ &= \frac{3}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta + \frac{1}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{3\theta}{2}\right)\mathrm{d}\theta \\ &\quad + \int_{\frac{\pi}{3}}^{\pi}\left(\frac{3}{2}\theta-\pi\right)\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta\end{aligned}$$

The second path:

$$\begin{aligned}\Re\int_{e^{\pi i/3}[0,1]} f(z)\,\mathrm{d}z &= \Re\int_0^1 \frac{\ln(1+xe^{\pi i/3})\ln(1-x^3)}{x}\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \frac{\ln(1+x+x^2)\ln(1-x^3)}{x}\mathrm{d}x \\ &= \frac{\zeta(3)}{3} - \frac{1}{2}\int_0^1 \frac{\ln(1-x)\ln(1-x^3)}{x}\mathrm{d}x \\ &= \frac{\zeta(3)}{3} + \frac{1}{2}\int_{-1}^{1} f(z)\,\mathrm{d}z - \frac{1}{2}J\end{aligned}$$

The third path:

$$\begin{aligned}\Re\int_{e^{\pi i[1/3,0]}} f(z)\,\mathrm{d}z &= \Im\int_0^{\frac{\pi}{3}} \ln(1+e^{i\theta})\ln(1+e^{3i\theta})\,\mathrm{d}\theta \\ &= \frac{3}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta + \frac{1}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{3\theta}{2}\right)\mathrm{d}\theta\end{aligned}$$

Let $\gamma = [0,1]\cup e^{\pi i[0,1/3]}\cup e^{\pi i/3}[1,0]$. By Cauchy's integral theorem,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Therefore,

$$\begin{aligned}I = \frac{2}{9}\zeta(3) + \frac{3}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta + \frac{1}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{3\theta}{2}\right)\mathrm{d}\theta \\ + \frac{1}{3}\int_{\frac{\pi}{3}}^{\pi}\left(\frac{3}{2}\theta-\pi\right)\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta\end{aligned}$$

The red integral:

$$\begin{aligned}\frac{3}{2}\int_0^{\frac{\pi}{3}} \theta\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta &= -\frac{3}{2}\Re\int_1^{e^{\pi i/3}} \frac{\ln z\ln(1+z)}{z}\mathrm{d}z \\ &= -\frac{3}{2}\Re\Big[\mathrm{Li}_3(-z) - \mathrm{Li}_2(-z)\ln z\Big]_1^{e^{\pi i/3}} \\ &= -\frac{9}{8}\zeta(3) - \frac{3}{2}\mathrm{Cl}_3\left(\frac{2\pi}{3}\right) + \frac{\pi}{2}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\end{aligned}$$

The blue integral:

$$\begin{aligned}\frac{1}{2}\int_0^{\frac{\pi}{3}}\theta\ln\left(2\cos\frac{3\theta}{2}\right)\mathrm{d}\theta &= \frac{1}{18}\Re\int_{-1}^{1}\frac{\ln z\ln(1+z)}{z}\mathrm{d}z\\ &= \frac{1}{18}\Re\Big[\mathrm{Li}_3(-z)-\mathrm{Li}_2(-z)\ln z\Big]_{-1}^{1}\\ &= -\frac{7}{72}\zeta(3)\end{aligned}$$

The green integral:

$$\begin{aligned}\frac{1}{3}\int_{\frac{\pi}{3}}^{\pi}\left(\frac{3}{2}\theta-\pi\right)\ln\left(2\cos\frac{\theta}{2}\right)\mathrm{d}\theta &= 2\Re\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\theta\ln(1+e^{2i\theta})\,\mathrm{d}\theta-\frac{2\pi}{3}\Re\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\ln(1+e^{2i\theta})\,\mathrm{d}\theta\\ &= \frac{1}{2}\Re\int_{-1}^{e^{\pi i/3}}\frac{\ln z\ln(1+z)}{z}\mathrm{d}z+\frac{\pi}{3}\Im\int_{-1}^{e^{\pi i/3}}\frac{\ln(1+z)}{z}\mathrm{d}z\\ &= \frac{1}{2}\Re\Big[\mathrm{Li}_3(-z)-\mathrm{Li}_2(-z)\ln z\Big]_{-1}^{e^{\pi i/3}}-\frac{\pi}{3}\Im\,\mathrm{Li}_2(-e^{\pi i/3})\\ &= -\frac{1}{2}\zeta(3)+\frac{1}{2}\mathrm{Cl}_3\left(\frac{2\pi}{3}\right)+\frac{\pi}{6}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\end{aligned}$$

Finally,

$$J=\int_0^1\frac{\ln(1+x)\ln(1+x^3)}{x}\mathrm{d}x=-\frac{3}{2}\zeta(3)-\mathrm{Cl}_3\left(\frac{2\pi}{3}\right)+\frac{2\pi}{3}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)$$

Hence we can calculate

$$\mathcal{M}=\int_0^1\frac{\ln^2\left(1-x+x^2\right)}{x}\mathrm{d}x\ ,\ \mathcal{N}=\int_0^1\frac{\ln\left(1+x\right)\ln\left(1-x+x^2\right)}{x}\mathrm{d}x$$

Let us first write $1-x+x^2=\dfrac{1+x^3}{1+x}$, so the first integral becomes

$$\begin{aligned}\mathcal{M} &= \int_0^1\frac{\ln^2(1+x^3)+\ln^2(1+x)-2\ln(1+x)\ln(1+x^3)}{x}\mathrm{d}x\\ &= \left(\frac{1}{3}+1\right)\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x-2\int_0^1\frac{\ln(1+x)\ln(1+x^3)}{x}\mathrm{d}x\\ &= \frac{\zeta(3)}{3}-2J\end{aligned}$$

Similarly the second integral equals

$$\mathcal{N}=J-\frac{\zeta(3)}{4}$$

□

Exercise 1.2.112: Prove:

$$\int_{-\infty}^{\infty}\left(\frac{\arctan(x)}{x}\right)^2\mathrm{d}x=2\pi\ln(2)$$

☞ **Proof:** Just use integration by parts,

$$I=\int_{-\infty}^{\infty}\left(\frac{\tan^{-1}x}{x}\right)^2\mathrm{d}x=2\int_0^{\infty}\left(\frac{\tan^{-1}x}{x}\right)^2\mathrm{d}x$$
$$=2\left[-\frac{(\tan^{-1}x)^2}{x}\bigg|_0^{\infty}+2\int_0^{\infty}\frac{\tan^{-1}x}{x(1+x^2)}\mathrm{d}x\right]$$
$$=4\int_0^{\infty}\frac{\tan^{-1}x}{x(1+x^2)}\mathrm{d}x$$

Substitute $x=\tan\theta$ and use integration by parts once again:

$$I=4\int_0^{\pi/2}\frac{\theta}{\tan\theta}\mathrm{d}\theta=4\theta\ln(\sin\theta)\bigg|_0^{\pi/2}-4\int_0^{\pi/2}\ln(\sin\theta)\mathrm{d}\theta$$
$$=-4\int_0^{\pi/2}\ln(\sin\theta)\mathrm{d}\theta$$

That last integral is very famous and it's value is $-\frac{\pi}{2}\ln(2)$, hence

$$\int_{-\infty}^{\infty}\left(\frac{\arctan(x)}{x}\right)^2\mathrm{d}x=2\pi\ln(2)$$

□

• **Exercise 1.2.113:** Prove:

$$\int_0^{\frac{\pi}{2}}\frac{1}{(\cos^{1/n}(x)+\sin^{1/n}(x))^{2n}}\mathrm{d}x=\frac{2}{\binom{2n}{n}}$$

☞ **Proof:**

$$\int_0^{\frac{\pi}{2}}\frac{1}{\left(\sin^{\frac{1}{n}}(x)+\cos^{\frac{1}{n}}(x)\right)^{2n}}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{1}{\cos^2(x)\left(\tan^{\frac{1}{n}}(x)+1\right)^{2n}}\mathrm{d}x$$
$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2(x)}{\left(\tan^{\frac{1}{n}}(x)+1\right)^{2n}}\mathrm{d}x$$

Substitution: $\tan(x)\to x^n$

$$\int_0^{\frac{\pi}{2}}\frac{\sec^2(x)}{\left(\tan^{\frac{1}{n}}(x)+1\right)^{2n}}\mathrm{d}x=n\int_0^{\infty}\frac{x^{n-1}}{(x+1)^{2n}}\mathrm{d}x=n\cdot\mathrm{B}(n,n)$$
$$=n\cdot\frac{(n-1)!(n-1)!}{(2n-1)!}=2\cdot\frac{n!n!}{(2n)!}=\frac{2}{\binom{2n}{n}}$$

□

• **Exercise 1.2.114:** Evaluate:

$$\int_0^1\ln\Gamma(x+\alpha)\mathrm{d}x,\ \alpha\geq 0$$

✎ **Solution:** Differentiating w.r.t α hence:

$$\begin{aligned} I'(\alpha) = \int_0^1 (\ln\Gamma(x+\alpha))' \mathrm{d}x &= \ln\Gamma(x+\alpha)\Big|_0^1 \\ &= \ln\Gamma(1+\alpha) - \ln\Gamma(0+\alpha) \\ &= \ln(\alpha\Gamma(\alpha)) - \ln\Gamma(\alpha) \\ &= \ln\alpha + \ln\Gamma(\alpha) - \ln\Gamma(\alpha) \\ &= \ln\alpha \end{aligned}$$

where C is just the integral $\int_0^1 \ln\Gamma(x)\mathrm{d}x = \dfrac{\ln 2\pi}{2}$, hence

$$\int_0^1 \ln\Gamma\left(x+\alpha\right)\,\mathrm{d}x = \frac{\ln\left(2\pi\right)}{2} + \alpha\ln\left(\alpha\right) - \alpha$$

□

Exercise 1.2.115: Evaluate:

$$\int_0^\infty \frac{1}{e^x\sqrt{\sinh(2x)}}\mathrm{d}x$$

✎ **Solution:** In general,

$$\begin{aligned} \int_0^\infty \frac{1}{e^x\sqrt{\sinh ax}}\,\mathrm{d}x &= \sqrt{2}\int_0^1 \frac{u^{a/2}}{\sqrt{1-u^{2a}}}\,\mathrm{d}u \\ &= \frac{\sqrt{2}}{2a}\int_0^1 \frac{w^{1/(2a)-3/4}}{\sqrt{1-w}}\,\mathrm{d}w \\ &= \frac{\sqrt{2}}{2a}\mathrm{B}\left(\frac{1}{2a}+\frac{1}{4},\frac{1}{2}\right) \end{aligned}$$

put $a = 2$ we have

$$\int_0^\infty \frac{1}{e^x\sqrt{\sinh(2x)}}\mathrm{d}x = \frac{\pi}{2\sqrt{2}}$$

□

Exercise 1.2.116: Evaluate:

$$\int_0^\infty \frac{(1-\sin(ax))(1-\cos(bx))}{x^2}\mathrm{d}x$$

where a and b are Real.

✎ **Solution:**

$$\begin{aligned} I(b) &= \int_0^\infty \frac{(1-\sin(ax))(1-\cos(bx))}{x^2}\mathrm{d}x,\ I(0) = 0 \\ I'(b) &= \int_0^\infty \frac{(1-\sin(ax))\sin(bx)}{x}\mathrm{d}x = \int_0^\infty\int_0^\infty e^{-xy}(1-\sin(ax))\sin(bx)\mathrm{d}y\mathrm{d}x \\ I'(b) &= \int_0^\infty\int_0^\infty e^{-xy}(\sin(bx)-\sin(ax)\sin(bx))\mathrm{d}x\mathrm{d}y \\ I'(b) &= \int_0^\infty\int_0^\infty e^{-xy}\left(\sin(bx)+\frac{1}{2}\cos((a+b)x)-\frac{1}{2}\cos((a-b)x)\right)\mathrm{d}x\mathrm{d}y \\ I'(b) &= \int_0^\infty \left(\frac{b}{y^2+b^2}+\frac{1}{2}\cdot\frac{y}{y^2+(a+b)^2}-\frac{1}{2}\cdot\frac{y}{y^2+(a-b)^2}\right)\mathrm{d}y \end{aligned}$$

$$\Rightarrow I'(b) = \begin{cases} 0 & \text{if } b = 0 \\ \left(\arctan\left(\frac{y}{b}\right) + \frac{1}{4}\ln\left(\frac{y^2+(a+b)^2}{y^2+(a-b)^2}\right)\right)\Big|_0^\infty & \text{if } b \neq 0 \end{cases}$$

$$I'(b) = \begin{cases} 0 & \text{if } b = 0 \\ \frac{b}{|b|}\cdot\frac{\pi}{2} + \frac{1}{4}\ln(a-b)^2 - \frac{1}{4}\ln(a+b)^2 & \text{if } b \neq 0 \end{cases}$$

$$I(b) = \begin{cases} C_1 & \text{if } b = 0 \\ \frac{|b|\pi}{2} - \frac{a-b}{2}\ln|a-b| - \frac{a+b}{2}\ln|a+b| + C_2 & \text{if } b \neq 0 \end{cases}$$

Evaluating $I(0) = 0$ we get $C_1 = 0$, $\lim_{b\to 0} I(b) = 0 \Rightarrow C_2 = a\ln|a|$.

$$\Rightarrow I(b) = \frac{|b|\pi}{2} + a\ln|a| - \frac{a-b}{2}\ln|a-b| - \frac{a+b}{2}\ln|a+b|,\ b \in \mathbb{R}$$

For the case $a = b$: $\lim_{a\to b} I(b) = \frac{|b|\pi}{2} - b\ln 2$

For the case $a = -b$: $\lim_{a\to -b} I(b) = \frac{|b|\pi}{2} + b\ln 2$

For the case $a = 0$: $\lim_{a\to 0} I(b) = \frac{|b|\pi}{2}$

These cases can be joined to obtain $\frac{|b|\pi}{2} - a\ln 2$.

Therefore

$$\int_0^\infty \frac{(1-\sin(ax))(1-\cos(bx))}{x^2}\mathrm{d}x$$
$$= \begin{cases} \frac{|b|\pi}{2} - a\ln 2 & \text{if } a = 0 \vee a = \pm b \\ \frac{|b|\pi}{2} + a\ln|a| - \frac{a-b}{2}\ln|a-b| - \frac{a+b}{2}\ln|a+b| & \text{if } a \neq 0 \vee a \neq \pm b \end{cases}$$

□

Exercise 1.2.117: Evaluate:

$$\int_0^\infty \frac{(x+1)\ln^2(1+x)}{(4x^2+8x+5)^{3/2}}\mathrm{d}x$$

Solution:

$$\int_0^\infty \frac{(x+1)\ln^2(x+1)}{(4x^2+8x+5)^{3/2}}\mathrm{d}x = \int_0^\infty \frac{(x+1)\ln^2(x+1)}{(4(x+1)^2+1)^{3/2}}\mathrm{d}x$$

Using the substitution $x+1 \to \frac{1}{x}$ we have:

$$\int_0^\infty \frac{(x+1)\ln^2(x+1)}{(4(x+1)^2+1)^{3/2}}\mathrm{d}x = \int_0^1 \frac{\ln^2 x}{(4+x^2)^{3/2}}\mathrm{d}x$$

Integrating by parts:

$$\int_0^1 \frac{\ln^2 x}{(4+x^2)^{3/2}}\mathrm{d}x = \frac{x\ln^2 x}{4\sqrt{x^2+4}}\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{\ln x}{\sqrt{x^2+4}}\mathrm{d}x = -\frac{1}{2}\int_0^1 \frac{\ln x}{\sqrt{x^2+4}}\mathrm{d}x$$

Integrating by parts again:

$$-\frac{1}{2}\int_0^1 \frac{\ln x}{\sqrt{x^2+4}}\mathrm{d}x = -\frac{1}{2}\ln x\cdot\sinh^{-1}\left(\frac{x}{2}\right)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{\sinh^{-1}\left(\frac{x}{2}\right)}{x}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \frac{\sinh^{-1}\left(\frac{x}{2}\right)}{x}\mathrm{d}x$$

Using the substitution $\frac{x}{2}\to\sinh(x)$ we have:

$$\frac{1}{2}\int_0^1 \frac{\sinh^{-1}\left(\frac{x}{2}\right)}{x}\mathrm{d}x = \frac{1}{2}\int_0^{\ln\phi}\frac{x\cosh(x)}{\sinh(x)}\mathrm{d}x$$

where ϕ is the golden ratio.

$$\frac{1}{2}\int_0^{\ln\phi}\frac{x\cosh(x)}{\sinh(x)}\mathrm{d}x = \frac{1}{2}\int_0^{\ln\phi}\frac{x\left(e^x+e^{-x}\right)}{e^x-e^{-x}}\mathrm{d}x = \frac{1}{2}\int_0^{\ln\phi}\left(x+\frac{2xe^{-x}}{e^x-e^{-x}}\right)\mathrm{d}x$$
$$= \frac{\ln^2\phi}{4}+\int_0^{\ln\phi}\frac{xe^{-x}}{e^x-e^{-x}}\mathrm{d}x = \frac{\ln^2\phi}{4}+\int_0^{\ln\phi}\frac{xe^{-2x}}{1-e^{-2x}}\mathrm{d}x = \frac{\ln^2\phi}{4}+\sum_{n=1}^{\infty}\int_0^{\ln\phi}xe^{-2nx}\mathrm{d}x$$
$$= \frac{\ln^2\phi}{4}-\sum_{n=1}^{\infty}\frac{\phi^{-2n}(2n\ln\phi+1)-1}{4n^2} = \frac{\ln^2\phi}{4}+\frac{\pi^2}{24}-\frac{1}{4}\mathrm{Li}_2\left(\phi^{-2}\right)-\frac{\ln\phi}{2}\mathrm{Li}_1\left(\phi^{-2}\right)$$
$$= \frac{\ln^2\phi}{4}+\frac{\pi^2}{24}+\frac{\ln^2\phi}{4}-\frac{\pi^2}{60}-\frac{\ln^2\phi}{2} = \frac{\pi^2}{40}$$

□

Exercise 1.2.118: Evaluate:

$$\int_0^{\infty}\frac{\ln(1+x)\ln x}{1+x^2}\mathrm{d}x$$

Solution:

$$\int_0^{\infty}\frac{\ln(1+x)\ln(x)}{1+x^2}\mathrm{d}x = \int_0^1\frac{\ln(1+x)\ln(x)}{1+x^2}\mathrm{d}x+\int_1^{\infty}\frac{\ln(1+x)\ln(x)}{1+x^2}\mathrm{d}x$$
$$= \int_0^1\frac{\ln(1+x)\ln(x)}{1+x^2}\mathrm{d}x-\int_0^1\frac{\ln\left(1+\frac{1}{x}\right)\ln(x)}{1+x^2}\mathrm{d}x$$
$$= \int_0^1\frac{\ln^2(x)}{1+x^2}\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\int_0^1\ln^2(x)\,x^{2n}\mathrm{d}x$$
$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = 2\beta(3) = \frac{\pi^3}{16}$$

□

Exercise 1.2.119: Evaluate:

$$\int_0^1\frac{\ln(1+x)}{x\left(x^2+1\right)}\mathrm{d}x$$

Solution: Take

$$I(a) = \int_0^1\frac{\ln(1+ax)}{x(1+x^2)}\,\mathrm{d}x \implies I'(a) = \int_0^1\frac{1}{(1+ax)(1+x^2)}\,\mathrm{d}x$$

Since,

$$\int_0^1 \frac{1}{(1+ax)(1+x^2)}\,\mathrm{d}x = \int_0^1 \frac{1}{(1+a^2)(1+x^2)} - \frac{a}{1+a^2}\left(\frac{x}{1+x^2} - \frac{a}{1+ax}\right)\,\mathrm{d}x$$
$$= \frac{\pi}{4(1+a^2)} - \frac{a\ln 2}{2(1+a^2)} + \frac{a\ln(1+a)}{1+a^2}$$

Hence,

$$I(1) = \frac{\pi^2}{16} - \frac{1}{4}\ln^2 2 + \int_0^1 \frac{\ln(1+a)}{a}\,\mathrm{d}a - \int_0^1 \frac{\ln(1+a)}{a(1+a^2)}\,\mathrm{d}a$$
$$\implies 2I(1) = \frac{\pi^2}{16} - \frac{1}{4}\ln^2 2 + \int_0^1 \frac{\ln(1+a)}{a}\,\mathrm{d}a$$
$$\implies I(1) = \frac{\pi^2}{32} - \frac{1}{8}\ln^2 2 + \frac{\pi^2}{24} = \frac{7\pi^2}{96} - \frac{1}{8}\ln^2 2$$

□

Exercise 1.2.120: Evaluate:

$$\int_0^\infty J_v\left(bx\right)\mathrm{d}x$$

Solution: For $-v < s < \frac{3}{2}$ and $b > 0$,

$$\int_0^\infty x^{s-1}J_v(bx)\,\mathrm{d}x = \int_0^\infty x^{s-1}\frac{1}{\Gamma(v+1)}\left(\frac{bx}{2}\right)^v \sum_{n=0}^\infty \frac{\Gamma(v+1)}{\Gamma(n+v+1)}\frac{\left(-\frac{b^2x^2}{4}\right)^n}{n!}\,\mathrm{d}x$$
$$= \frac{b^v}{2^v\Gamma(v+1)}\int_0^\infty \left(\frac{2\sqrt{u}}{b}\right)^{s+v-1}\sum_{n=0}^\infty \frac{\Gamma(v+1)}{\Gamma(n+v+1)}\frac{(-u)^n}{n!}\frac{2}{b}\frac{1}{2\sqrt{u}}\,\mathrm{d}u$$
$$= \frac{2^{s-1}}{b^s\Gamma(v+1)}\int_0^\infty u^{(s+v)/2-1}\sum_{n=0}^\infty \frac{\Gamma(v+1)}{\Gamma(n+v+1)}\frac{(-u)^n}{n!}\mathrm{d}u$$
$$= \frac{2^{s-1}}{b^s\Gamma(v+1)}\Gamma\left(\frac{s+v}{2}\right)\frac{\Gamma(v+1)}{\Gamma\left(-\frac{s+v}{2}+v+1\right)}$$
$$= \frac{2^{s-1}}{b^s}\frac{\Gamma\left(\frac{s+v}{2}\right)}{\Gamma\left(\frac{2-s+v}{2}\right)}$$

On the first line I simply wrote the series definition of the Bessel function of the first kind in a form that would allow me to use the theorem, and on the second line I made the substitution $u = \frac{b^2x^2}{4}$.
If we let $s = 1$, it reduces to

$$\int_0^\infty J_v(bx)\,\mathrm{d}x = \frac{1}{b}\ ,\ v > -1$$

□

Exercise 1.2.121: Evaluate:

$$\int_0^{2\pi} \ln\Gamma\left(\frac{x}{2\pi}\right)e^{\cos(x)}\sin(x+\sin(x))\mathrm{d}x$$

✎ **Solution:** Using a keyhole contour with unit radius,

$$\int_0^{2\pi} \ln\Gamma\left(\frac{x}{2\pi}\right) e^{\cos x}\sin(x+\sin x)\mathrm{d}x = \Im\int_0^{2\pi} \ln\Gamma\left(\frac{x}{2\pi}\right) e^{e^{ix}+ix}\mathrm{d}x = -\Re\oint_{|z|=1} e^z \ln\Gamma\left(\frac{\ln z}{2\pi i}\right)\mathrm{d}z$$

$$= \Re\int_0^1 e^x \ln\left(\frac{\Gamma\left(\frac{\ln x}{2\pi i}\right)}{\Gamma\left(\frac{\ln x+2\pi i}{2\pi i}\right)}\right)\mathrm{d}x = \ln(2\pi)\int_0^1 e^x\mathrm{d}x - \int_0^1 e^x\ln(-\ln x)\ \mathrm{d}x$$

$$= (e-1)\ln(2\pi) - \sum_{k=0}^{\infty}\frac{1}{k!}\int_0^{\infty} e^{-(k+1)x}\ln x\ \mathrm{d}x = (e-1)\ln(2\pi) - \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial}{\partial s}\left[\frac{\Gamma(s)}{(k+1)^s}\right]_{s=1}$$

$$= (e-1)\ln(2\pi) - \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\Gamma(s)\psi_0(s)}{(k+1)^s} - \frac{\Gamma(s)\ln(k+1)}{(k+1)^s}\right]_{s=1}$$

$$= (e-1)\ln(2\pi) + \gamma\sum_{k=0}^{\infty}\frac{1}{(k+1)!} + \sum_{k=0}^{\infty}\frac{\ln(k+1)}{(k+1)!} = (e-1)(\gamma+\ln(2\pi)) + \sum_{k=2}^{\infty}\frac{\ln k}{k!}$$

□

Exercise 1.2.122: Prove:

$$\int_0^1 \frac{\cosh(a\ln(x))\ln(1+x)}{x}\mathrm{d}x = \frac{1}{2a}\left(\pi\csc(\pi a) - \frac{1}{a}\right), \quad a<1$$

☞ **Proof:**

$$\int_0^1 \frac{\cosh(a\ln x)\ln(1+x)}{x}\mathrm{d}x = \frac{1}{2}\int_0^1 \left(x^{a-1}+x^{-a-1}\right)\ln(1+x)\mathrm{d}x$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{n+a}+\frac{1}{n-a}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2}$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2} - \frac{1}{a^2}\right) = \frac{1}{2}\left(\sum_{\pm}\operatorname*{Res}_{z=\pm a}\frac{\pi\csc(\pi z)}{z^2-a^2} - \frac{1}{a^2}\right)$$

$$= \frac{1}{2a}\left(\pi\csc(\pi a) - \frac{1}{a}\right)$$

□

Exercise 1.2.123: Evaluate:

$$\int_0^{\frac{\pi}{6}} \ln^2(\cos x)\ \mathrm{d}x$$

✎ Solution:

$$\begin{aligned}
&\int_0^{\frac{\pi}{6}} \ln^2(\cos x)\,\mathrm{d}x = \Re\int_0^{\frac{\pi}{6}} \ln^2(1+e^{2ix})\,\mathrm{d}x - 2\ln 2\int_0^{\frac{\pi}{6}} \ln(2\cos x)\,\mathrm{d}x + \frac{\pi^3}{648} + \frac{\pi\ln^2 2}{6}\\
=&\frac{1}{2}\Im\int_C \frac{\ln^2(1+z)}{z}\mathrm{d}z - 2\ln 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{6}}\cos(2nx)\,\mathrm{d}x + \frac{\pi^3}{648} + \frac{\pi\ln^2 2}{6}\\
=&\frac{1}{2}\Im\int_1^{e^{\pi i/3}} \frac{\ln^2(1+z)}{z}\mathrm{d}z + \ln 2\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\pi n/3)}{n^2} + \frac{\pi^3}{648} + \frac{\ln^2 2}{6}\\
=&\Im\left[-\mathrm{Li}_3(1+z) + \mathrm{Li}_2(1+z)\ln(1+z) + \frac{1}{2}\ln(-z)\ln^2(1+z)\right]_1^{e^{\pi i/3}}\\
&-\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\ln 2 + \frac{\pi^3}{648} + \frac{\pi\ln^2 2}{6}\\
=&-\Im\mathrm{Li}_3\left(1+e^{\frac{\pi i}{3}}\right) + \frac{\ln 3}{2}\Im\mathrm{Li}_2\left(1+e^{\frac{\pi i}{3}}\right) + \frac{\pi}{6}\Re\mathrm{Li}_2\left(1+e^{\frac{\pi i}{3}}\right) - \frac{\pi\ln 3}{6} - \frac{5\pi\ln^2 2}{2}\\
&-\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\ln 2 + \frac{\pi^3}{648} + \frac{\pi\ln^2 2}{6}
\end{aligned}$$

□

⊷ Exercise 1.2.124: Prove:

$$\int_{-1}^{1}\frac{\ln(1-x^2)}{\sqrt{1-x^2}(a+bx)}\mathrm{d}x = \frac{2\pi}{\sqrt{a^2-b^2}}\ln\left(\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}\right),\quad a>b$$

☞ Proof:

$$\begin{aligned}
&\int_{-1}^{1}\frac{\ln(1-x^2)}{\sqrt{1-x^2}(a+bx)}\mathrm{d}x = \int_0^1\frac{\ln(1-x^2)}{\sqrt{1-x^2}}\left(\frac{1}{a+bx}+\frac{1}{a-bx}\right)\mathrm{d}x\\
&= 2a\int_0^1\frac{\ln(1-x^2)}{(a^2-b^2x^2)\sqrt{1-x^2}}\mathrm{d}x = \frac{2}{a}\int_0^1\frac{\ln(1-x^2)}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}\left(\frac{b^2}{a^2}x^2\right)^n\mathrm{d}x\\
&= \frac{2}{a}\sum_{n=0}^{\infty}\left(\frac{b^2}{a^2}\right)^n\int_0^1\frac{x^{2n}\ln(1-x^2)}{\sqrt{1-x^2}}\mathrm{d}x = -\frac{\pi}{a}\sum_{n=0}^{\infty}\left(\frac{b^2}{a^2}\right)^n\left\{\binom{2n}{n}\frac{H_n+2\ln 2}{4^n}\right\}\\
&= -\frac{\pi}{a}\left\{\sum_{n=0}^{\infty}H_n\binom{2n}{n}\left(\frac{b}{2a}\right)^{2n} + 2\ln 2\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{b}{2a}\right)^{2n}\right\}
\end{aligned}$$

For $|x|<1$, it's known that

$$\sum_{n=0}^{\infty}\binom{2n}{n}H_n\,x^n = \frac{2}{\sqrt{1-4x}}\ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right)$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$$

Hence we obtain

$$\begin{aligned}
\int_{-1}^{1}\frac{\ln(1-x^2)}{\sqrt{1-x^2}(a+bx)}\mathrm{d}x &= -\frac{\pi}{a}\left\{\frac{2}{\sqrt{1-\frac{b^2}{a^2}}}\ln\left(\frac{1+\sqrt{1-\frac{b^2}{a^2}}}{2\sqrt{1-\frac{b^2}{a^2}}}\right) + \frac{2\ln 2}{\sqrt{1-\frac{b^2}{a^2}}}\right\}\\
&= \frac{2\pi}{\sqrt{a^2-b^2}}\ln\left(\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}\right)
\end{aligned}$$

□

Exercise 1.2.125: Evaluate:

$$\int_0^{+\infty} \operatorname{sech}^2\left(x^2\right) \mathrm{d}x$$

Solution: We have

$$\int_0^{\infty} \frac{x^a}{\cosh^2 x}\, \mathrm{d}x = \frac{2\Gamma(a+1)\eta(a)}{2^a} \qquad , \qquad \text{for } a > -1$$

where $\eta(a)$ is the **Dirichlet eta function**. Proof can be seen here.
Making substitution $x^2 \mapsto x$ and using $\operatorname{sech}^2 x = \dfrac{1}{\cosh^2 x}$, then the considered integral can be rewritten as

$$\frac{1}{2}\int_0^{\infty} \frac{1}{\cosh^2 x}\, \frac{\mathrm{d}x}{\sqrt{x}}$$

Using Hardy's formula to evaluate the negative argument of Dirichlet eta function

$$\eta(-s) = \frac{s}{\pi^{1+s}} \frac{2^{1+s}-1}{2^s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(s)\eta(s+1)$$

and the relation of the Dirichlet eta function and the Riemann zeta function

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s)$$

then we have

$$\int_0^{+\infty} \operatorname{sech}^2\left(x^2\right) \mathrm{d}x = \sqrt{2}\,\Gamma\left(\frac{1}{2}\right)\eta\left(-\frac{1}{2}\right) = \frac{\left(2\sqrt{2}-1\right)}{2\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)$$

□

Exercise 1.2.126: Evaluate:

$$\mathcal{R} = \int_0^{\frac{\pi}{2}} \sin^2 x \ln\left(\sin^2(\tan x)\right)\, \mathrm{d}x$$

Solution: After the substitution $\tan x \mapsto x$, we get

$$\begin{aligned}\mathcal{R} &= \int_0^{\infty} \frac{x^2}{(1+x^2)^2} \ln(\sin^2 x)\, \mathrm{d}x \\ &= \Re \int_{-\infty}^{\infty} \frac{x^2 \ln(1-e^{i2x})}{(1+x^2)^2} \mathrm{d}x - \ln 2 \underbrace{\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2} \mathrm{d}x}_{\frac{\pi}{2}}\end{aligned}$$

Even though the function $f(z) = \dfrac{z^2 \ln(1-e^{i2z})}{(1+z^2)^2}$ has infinitely many branch points at $z = n\pi$, once we close the contour along the upper half of $|R|$ and make semicircular bumps around the branch points, one may check (by letting $z = n\pi + \epsilon e^{i\theta}$) that the contribution along the bumps vanishes. The integral along the big arc also tends to 0. Hence

$$\mathcal{R} = 2\pi i \operatorname{Res}(f, i) - \frac{\pi}{2}\ln 2$$

Using WolframAlpha to compute the residue and simplify terms,

$$\mathcal{R} = \frac{\pi}{e^2-1} - \pi - \frac{\pi}{2}\ln 2 + \frac{\pi}{2}\ln(e^2-1)$$

□

Exercise 1.2.127: Evaluate:

$$\int_0^1 \ln(x)\cdot(e^x - e^{-x})\,\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^1 e^{-x}\ln x\ \mathrm{d}x &= \int_0^\infty e^{-x}\ln x\ \mathrm{d}x - \int_1^\infty e^{-x}\ln x\ \mathrm{d}x\\ &= -\gamma + \mathrm{Ei}(-1)\end{aligned}$$

$$\begin{aligned}\int_0^1 e^x\ln x\mathrm{d}x &= e^x\ln x|_0^1 - \int_0^1 \frac{1}{x}e^x\mathrm{d}x\\ &= e^x\ln x|_0^1 + \int_{-1}^0 \frac{1}{x}e^{-x}\mathrm{d}x\\ &= e^x\ln x|_0^1 + \int_{-1}^\infty \frac{1}{x}e^{-x}\mathrm{d}x - \int_0^\infty \frac{1}{x}e^{-x}\mathrm{d}x\\ &= e^x\ln x|_0^1 - \mathrm{Ei}(1) - e^{-x}\ln x|_0^1 - \int_0^\infty e^{-x}\ln x\mathrm{d}x\\ &= \gamma - \mathrm{Ei}(1)\end{aligned}$$

Their difference gives

$$\int_0^1 (e^x - e^{-x})\ln x\mathrm{d}x = 2\gamma - \mathrm{Ei}(1) - \mathrm{Ei}(-1)$$

□

Exercise 1.2.128: Evaluate:

$$\int_0^\infty \frac{x}{\sqrt{e^x-1}}\mathrm{d}x$$

Solution: Notice that, with the substitution $\tan\theta = \sqrt{e^x-1}$ we have

$$I = \int_0^\infty \frac{x}{\sqrt{e^x-1}}\,\mathrm{d}x = -4\int_0^{\frac{\pi}{2}} \ln\cos\theta\,\mathrm{d}\theta = -4\int_0^{\frac{\pi}{2}} \ln\sin\theta\,\mathrm{d}\theta$$

Now by the symmetry of the sine function followed by the substitution $\theta \mapsto 2\theta$, we get

$$\begin{aligned}I &= -2\int_0^{\pi} \ln\sin\theta\,\mathrm{d}\theta = -4\int_0^{\frac{\pi}{2}} \ln\sin 2\theta\,\mathrm{d}\theta\\ &= -4\int_0^{\frac{\pi}{2}} (\ln 2 + \ln\sin\theta + \ln\cos\theta)\,\mathrm{d}\theta\\ &= -2\pi\ln 2 + 2I\end{aligned}$$

Therefore

$$I = 2\pi\ln 2$$

□

Exercise 1.2.129: Evaluate:

$$\int_0^\infty \frac{1}{\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{(a+1)^2}\right)\left(1+\frac{x^2}{(a+2)^2}\right)\cdots}\mathrm{d}x$$

Solution: The product admits the closed form

$$\prod_{i=0}^{\infty}\left(1+\frac{x^2}{(a+i)^2}\right)=\frac{\Gamma(a)^2}{\Gamma(a-i\,x)\Gamma(a+i\,x)}$$

which implies we need to evaluate the integral

$$\begin{aligned} I &= \int_0^\infty \frac{\Gamma(a-i\,x)\Gamma(a+i\,x)}{\Gamma(a)^2}\mathrm{d}x \\ &= \frac{1}{2\,\Gamma(a)^2}\int_{-\infty}^{\infty}\Gamma(a-i\,x)\Gamma(a+i\,x)\mathrm{d}x \quad a>0 \end{aligned}$$

To evaluate the above integral one can use contour integration by considering the integral

$$F=\int_C \Gamma(a-i\,z)\Gamma(a+i\,z)\mathrm{d}z$$

with a suitable contour C. The function $\Gamma(a+i\,z)$ has the simple poles $p_n=i(a+n)$ with corresponding residues $r_n=\dfrac{(-1)^{n+1}}{n!}i\,\Gamma(2a+n)$. So by the residue theorem the integral F evaluates to

$$F=2\pi\,i\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}i\,\Gamma(2a+n)=2\pi 4^{-n}\Gamma(2a)$$

which implies

$$I=\frac{1}{2}\frac{2\pi 4^{-n}\Gamma(2a)}{\Gamma(a)^2}=\frac{\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}{2\,\Gamma(a)}$$

□

Exercise 1.2.130: Evaluate:

$$\int_0^{\pi/2}\frac{x}{\sin x}\,\mathrm{d}x\;,\quad \int_0^{\pi/2}\frac{x^2}{\sin^2 x}\,\mathrm{d}x\;,\quad \int_0^{\pi/2}\frac{x^4}{\sin^4 x}\,\mathrm{d}x$$

Solution: 1.Integrating by parts, we have

$$\int\frac{x}{\sin x}\,\mathrm{d}x=x\ln\tan\frac{x}{2}-\int\ln\tan\frac{x}{2}\,\mathrm{d}x$$

Evaluating between 0 and $\pi/2$ yields

$$\begin{aligned} -2\int_0^{\pi/4}\ln\tan x\,\mathrm{d}x &= -2\int_{-\infty}^{0}x\frac{e^x}{1+e^{2x}}\,\mathrm{d}x=2\sum_{k\geq 0}(-1)^k\int_0^\infty xe^{-(2k+1)x}\,\mathrm{d}x \\ &= 2\sum_{k\geq 0}(-1)^k\frac{1}{(2k+1)^2}=2\mathbf{G} \end{aligned}$$

2.Integrating by parts, we have

$$\begin{aligned}\int \frac{x^2}{\sin^2 x}\,\mathrm{d}x &= -x^2\cot x + \int 2x\cot x\,\mathrm{d}x\\ &= -x^2\cot x + 2x\ln\sin x - \int 2\ln\sin x\,\mathrm{d}x\end{aligned}$$

Evaluating this between 0 and $\pi/2$, we find that the boundary terms vanish (by taking the appropriate limits), so we are left with the well-known integral

$$-2\int_0^{\pi/2}\ln\sin x\,\mathrm{d}x = \pi\ln 2$$

3. Integrating by parts 3 times,

$$\begin{aligned}\int_0^{\pi/2}\frac{x^4}{\sin^4 x}\mathrm{d}x &= -\frac{x^4}{3}\cot(x)\left(\csc^2(x)+2\right)\Big|_0^{\pi/2} + \frac{4}{3}\int_0^{\pi/2}x^3\cot(x)\left(\csc^2(x)+2\right)\mathrm{d}x\\ &= \frac{4}{3}\int_0^{\pi/2}x^3\cot(x)\left(\csc^2(x)+2\right)\mathrm{d}x\\ &= \frac{8}{3}\int_0^{\pi/2}x^3\cot(x)\mathrm{d}x + \frac{4}{3}\int_0^{\pi/2}x^3\cot(x)\csc^2(x)\mathrm{d}x\\ &= \frac{8}{3}\int_0^{\pi/2}x^3\cot(x)\mathrm{d}x - \frac{2}{3}x^3\cot^2(x)\Big|_0^{\pi/2} + 2\int_0^{\pi/2}x^2\cot^2(x)\mathrm{d}x\\ &= \frac{8}{3}\int_0^{\pi/2}x^3\cot(x)\mathrm{d}x + 2\int_0^{\pi/2}x^2\cot^2(x)\mathrm{d}x\\ &= \frac{8}{3}\int_0^{\pi/2}x^3\cot(x)\mathrm{d}x - 2x^2\Big(x+\cot(x)\Big)\Big|_0^{\pi/2} + 4\int_0^{\pi/2}x\Big(x+\cot(x)\Big)\mathrm{d}x\\ &= \frac{8}{3}\int_0^{\pi/2}x^3\cot(x)\mathrm{d}x - \frac{\pi^3}{4} + 4\int_0^{\pi/2}x^2\mathrm{d}x + 4\int_0^{\pi/2}x\cot(x)\mathrm{d}x\\ &= \frac{8}{3}\int_0^{\pi/2}x^3\cot(x)\mathrm{d}x - \frac{\pi^3}{12} + 4\int_0^{\pi/2}x\cot(x)\mathrm{d}x\end{aligned}$$

In general,

$$\int_a^b f(x)\cot(x)\,\mathrm{d}x = 2\sum_{n=1}^{\infty}\int_a^b f(x)\sin(2nx)\,\mathrm{d}x$$

see here.

So

$$\begin{aligned}\int_0^{\pi/2}x^3\cot(x)\,\mathrm{d}x &= 2\sum_{n=1}^{\infty}\int_0^{\pi/2}x^3\sin(2nx)\,\mathrm{d}x\\ &= 2\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}\pi^3}{16n} - \frac{(-1)^{n-1}3\pi}{8n^3}\right)\\ &= \frac{\pi^3}{8}\ln(2) - \frac{3\pi}{4}\eta(3)\\ &= \frac{\pi^3}{8}\ln(2) - \frac{9\pi}{16}\zeta(3)\end{aligned}$$

And

$$\begin{aligned}\int_0^{\pi/2} x\cot(x)\,\mathrm{d}x &= 2\sum_{n=1}^{\infty}\int_0^{\pi/2} x\sin(2nx)\,\mathrm{d}x\\ &= -\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\ &= \frac{\pi\ln 2}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{\pi/2}\frac{x^4}{\sin^4 x}\,\mathrm{d}x &= \frac{8}{3}\left(\frac{\pi^3}{8}\ln(2)-\frac{9\pi}{16}\zeta(3)\right)-\frac{\pi^3}{12}+4\left(\frac{\pi\ln 2}{2}\right)\\ &= -\frac{\pi^3}{12}+2\pi\ln(2)+\frac{\pi^3}{3}\ln(2)-\frac{3\pi}{2}\zeta(3)\end{aligned}$$

□

Exercise 1.2.131: Evaluate:

$$\int_0^{\infty}\frac{\ln x}{x^2+2x+4}\mathrm{d}x$$

Solution: Method One:

Let $f(z)=\dfrac{\ln^2 z}{z^2+2z+4}$ and integrate around a keyhole contour where the branch cut for $\ln z$ is along the positive real axis. Then

$$\begin{aligned}&\int_0^{\infty}\frac{\ln^2 x}{x^2+2x+4}\,\mathrm{d}x-\int_0^{\infty}\frac{(\ln x+2\pi i)^2}{x^2+2x+4}\,\mathrm{d}x\\ &=2\pi i\Big(\mathrm{Res}[f(z),-1+i\sqrt{3}]+\mathrm{Res}[f(z),-1-i\sqrt{3}]\Big)\end{aligned}$$

where

$$\begin{aligned}\mathrm{Res}\left[f(z),-1+i\sqrt{3}\right] &= \frac{\ln^2(-1+i\sqrt{3})}{2\sqrt{3}i}=\frac{\left(\ln 2+\frac{2\pi i}{3}\right)^2}{2\sqrt{3}i}\\ &= \frac{2\pi\ln 2}{3\sqrt{3}}+i\left(\frac{2\pi^2}{9\sqrt{3}}-\frac{\ln^2(2)}{2\sqrt{3}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}\left[f(z),-1-i\sqrt{3}\right] &= -\frac{\ln^2(-1-i\sqrt{3})}{2\sqrt{3}i}=-\frac{\left(\ln 2+\frac{4\pi i}{3}\right)^2}{2\sqrt{3}i}\\ &= -\frac{4\pi\ln 2}{3\sqrt{3}}+i\left(\frac{\ln^2(2)}{2\sqrt{3}}-\frac{8\pi^2}{9\sqrt{3}}\right)\end{aligned}$$

Equating the imaginary parts on both sides of the equation,

$$\int_0^{\infty}\frac{\ln x}{x^2+2x+4}\mathrm{d}x=\frac{\pi\ln 2}{3\sqrt{3}}$$

Method Two:
With the change of variable $x \to \dfrac{4}{x}$,

$$\begin{aligned} I &= \int_0^\infty \frac{\ln x}{x^2+2x+4}\,\mathrm{d}x = \int_0^\infty \frac{\ln 4 - \ln x}{x^2+2x+4}\,\mathrm{d}x \\ &= 2\ln 2 \int_0^\infty \frac{\mathrm{d}x}{x^2+2x+4} - I \\ \Rightarrow I &= \ln 2 \int_0^\infty \frac{\mathrm{d}x}{x^2+2x+4} = \frac{\pi \ln 2}{3\sqrt{3}} \end{aligned}$$

□

Exercise 1.2.132: Evaluate:

$$\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{1+x}\mathrm{d}x$$

Solution:

$$\begin{aligned} I = \int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{1+x}\mathrm{d}x &= \int_1^2 \frac{\ln^2(2-x)\ln(x)}{x}\mathrm{d}x \quad x \mapsto x-1 \\ &= -\int_{1/2}^1 \ln^2\left(2-\frac{1}{x}\right)\frac{\ln(x)}{x}\mathrm{d}x \quad x \mapsto \frac{1}{x} \end{aligned}$$

Note that

$$\ln^2\left(2-\frac{1}{x}\right) = \ln^2(2x-1) + \ln^2(x) - 2\ln(x)\ln(2x-1)$$

So we get

$$I = \frac{\ln^4(2)}{4} - \left\{\int_{1/2}^1 \frac{\ln^2(2x-1)\ln(x)}{x}\mathrm{d}x - 2\int_{1/2}^1 \frac{\ln(2x-1)\ln^2(x)}{x}\mathrm{d}x\right\} \tag{1}$$

Step I:

$$\begin{aligned} \int_{1/2}^1 \frac{\ln(2x-1)\ln^2(x)}{x}\mathrm{d}x &= \int_{1/2}^1 \frac{\left(\ln(2x)+\ln\left(1-\frac{1}{2x}\right)\right)\ln^2(x)}{x}\mathrm{d}x \\ &= \frac{\ln^4(2)}{12} - \sum_{n=1}^\infty \frac{1}{2^n n}\int_{1/2}^1 \frac{\ln^2(x)}{x^{n+1}}\mathrm{d}x \\ &= \frac{\ln^4(2)}{12} - \sum_{n=1}^\infty \frac{1}{2^n n}\left\{\frac{2^n\ln^2(2)}{n} - 2\frac{2^n\ln(2)}{n^2} + 2\frac{2^n}{n^3} - \frac{2}{n^3}\right\} \\ &= 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \zeta(3)\ln(4) - \frac{\pi^4}{45} + \frac{\ln^4(2)}{12} - \frac{1}{6}\pi^2\ln^2(2) \end{aligned} \tag{2}$$

Step II:

$$\begin{aligned}
&\int_{1/2}^{1}\frac{\ln^2(2x-1)\ln(x)}{x}\mathrm{d}x\\
&=\int_{1/2}^{1}\frac{\left(-\ln^2(2x)+\ln^2\left(1-\frac{1}{2x}\right)+2\ln(2x)\ln\left(2x-1\right)\right)\ln(x)}{x}\mathrm{d}x\\
&=\frac{\ln^4(2)}{12}+2\ln(2)\int_{1/2}^{1}\frac{\ln(x)\ln(2x-1)}{x}\mathrm{d}x+2\int_{1/2}^{1}\frac{\ln(2x-1)\ln^2(x)}{x}\mathrm{d}x\\
&\quad+\int_{1/2}^{1}\frac{\ln^2\left(1-\frac{1}{2x}\right)\ln(x)}{x}\mathrm{d}x\\
&=\frac{\ln^4(2)}{12}+\frac{\ln(2)}{12}\left(2\pi^2\ln(2)-3\zeta(3)\right)+2\Bigg(2\mathrm{Li}_4\left(\frac{1}{2}\right)+\zeta(3)\ln(4)-\frac{\ln^4}{45}\\
&\quad+\frac{\ln^4(2)}{12}-\frac{1}{6}\pi^2\ln^2(2)\Bigg)+\int_{1/2}^{1}\frac{\ln^2\left(1-\frac{1}{2x}\right)\ln(x)}{x}\mathrm{d}x\\
&=\frac{1}{4}\left(16\mathrm{Li}_4\left(\frac{1}{2}\right)+15\zeta(3)\ln(2)+\ln^4(2)\right)-\frac{2\pi^4}{45}-\frac{1}{6}\pi^2\ln^2(2)+\int_{1/2}^{1}\frac{\ln^2\left(1-\frac{1}{2x}\right)\ln(x)}{x}\mathrm{d}x\\
&=\frac{1}{4}\left(16\mathrm{Li}_4\left(\frac{1}{2}\right)+15\zeta(3)\ln(2)+\ln^4(2)\right)-\frac{2\pi^4}{45}-\frac{1}{6}\pi^2\ln^2(2)\\
&\quad+\sum_{n,m=1}^{\infty}\frac{1}{nm2^{n+m}}\int_{1/2}^{1}\frac{\ln(x)}{x^{n+m+1}}\mathrm{d}x\\
&=\frac{1}{4}\left(16\mathrm{Li}_4\left(\frac{1}{2}\right)+15\zeta(3)\ln(2)+\ln^4(2)\right)-\frac{2\pi^4}{45}-\frac{1}{6}\pi^2\ln^2(2)\\
&\quad+\sum_{n,m=1}^{\infty}\frac{1}{nm2^{n+m}}\frac{2^{n+m}-2^{n+m}(n+m)\ln(2)-1}{(n+m)^2}\\
&=\frac{1}{4}\left(16\mathrm{Li}_4\left(\frac{1}{2}\right)+15\zeta(3)\ln(2)+\ln^4(2)\right)-\frac{2\pi^4}{45}-\frac{1}{6}\pi^2\ln^2(2)\\
&\quad+\sum_{n,m=1}^{\infty}\frac{1}{nm(n+m)^2}-\ln(2)\sum_{n,m=1}^{\infty}\frac{1}{nm(n+m)}-\sum_{n,m=1}^{\infty}\frac{1}{nm(n+m)^2 2^{n+m}}
\end{aligned}\tag{3}$$

Note that

$$\begin{aligned}
\sum_{m,n=1}^{\infty}\frac{1}{nm(m+n)}&=2\sum_{r=1}^{\infty}\frac{H_r}{(r+1)^2}=2\zeta(3)\\
\sum_{m,n=1}^{\infty}\frac{1}{nm(m+n)^2}&=2\sum_{r=1}^{\infty}\frac{H_r}{(r+1)^3}=\frac{\pi^4}{180}\\
\sum_{m,n=1}^{\infty}\frac{1}{nm(m+n)^2 2^{m+n}}&=\sum_{r=1}^{\infty}\frac{H_r}{2^r(r+1)^3}=\frac{\pi^4}{360}+\frac{\ln^4(2)}{12}-\frac{\ln(2)\zeta(3)}{4}
\end{aligned}$$

Put these values in equation (3):

$$\int_{1/2}^{1}\frac{\ln^2(2x-1)\ln(x)}{x}\mathrm{d}x=4\mathrm{Li}_4\left(\frac{1}{2}\right)+2\zeta(3)\ln(2)-\frac{\pi^4}{24}+\frac{\ln^4(2)}{6}-\frac{1}{6}\pi^2\ln^2(2)\tag{4}$$

Final Answer:

Put the results of equation (2) and (4) in (1) to get

$$\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{1+x}\mathrm{d}x = 2\zeta(3)\ln 2 - \frac{\pi^4}{360} + \frac{\ln^4 2}{4} - \frac{1}{6}\pi^2\ln^2 2$$

□

Exercise 1.2.133: Prove:

$$\int_0^{\pi/2} \frac{\arctan\left(\sqrt{3}\,\sin\theta\right)\cdot\sin\theta}{\left(1+3\sin^2\theta\right)^2}\mathrm{d}\theta = \frac{\pi\sqrt{3}}{192}(3+\ln 4)$$

Proof: Differentiate under the integral sign to obtain

$$\begin{aligned}\mathcal{I} &= \int_0^{\pi/2} \frac{\arctan\left(\sqrt{3}\sin\theta\right)\sin\theta}{(1+3\sin^2\theta)^2}\mathrm{d}\theta \\ &= \int_0^{\sqrt{3}}\int_0^{\pi/2} \frac{\sin^2\theta}{(1+\alpha^2\sin^2\theta)(1+3\sin^2\theta)^2}\,\mathrm{d}\theta\,\mathrm{d}\alpha\end{aligned}$$

We first evaluate the inner integral which we will call $\mathcal{J}(\alpha)$. Applying the double angle formula followed by the change of variables $t=\tan\theta$, we get

$$\begin{aligned}\mathcal{J}(\alpha) &= 4\int_0^{\pi/2} \frac{1-\cos 2\theta}{(\alpha^2+2-\alpha^2\cos 2\theta)(5-3\cos 2\theta)^2}\mathrm{d}\theta \\ &= \int_0^{\infty} \frac{t^2(1+t^2)}{(1+4t^2)^2(1+(1+\alpha^2)t^2)}\mathrm{d}t \\ &= i\pi\left[\mathrm{Res}\,(f,i/2) + \mathrm{Res}\left(f, i(1+\alpha^2)^{-1/2}\right)\right] \\ &= i\pi\left[-\frac{i(5\alpha^2+9)}{32(\alpha^2-3)^2} + \frac{i\alpha^2}{2(\alpha^2-3)^2\sqrt{1+\alpha^2}}\right] \\ &= \frac{\pi(5\alpha^2+9)}{32(\alpha^2-3)^2} - \frac{\pi\alpha^2}{2(\alpha^2-3)^2\sqrt{1+\alpha^2}}\end{aligned}$$

Integrating back, we get

$$\begin{aligned}\mathcal{I} &= \int_0^{\sqrt{3}} \frac{\pi(5\alpha^2+9)}{32(\alpha^2-3)^2} - \frac{\pi\alpha^2}{2(\alpha^2-3)^2\sqrt{1+\alpha^2}}\mathrm{d}\alpha \\ &= \frac{\pi}{192}\left[\frac{12\alpha\sqrt{1+\alpha^2}-24\alpha}{\alpha^2-3} + \sqrt{3}\ln\left(\frac{7\alpha^2+4\sqrt{3}\alpha\sqrt{1+\alpha^2}+3}{(\alpha+\sqrt{3})^2}\right)\right]_0^{\sqrt{3}} \\ &= \frac{\pi}{192}\left[\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(12\sqrt{\alpha+3}\sqrt{\alpha+4}-24\sqrt{\alpha+3}\right)\Big|_{\alpha=0} + \sqrt{3}\ln 4\right] \\ &= \frac{\pi\sqrt{3}}{192}(3+2\ln 2)\end{aligned}$$

□

Exercise 1.2.134: Evaluate:

$$\int_0^1 \frac{\arctan^3 x}{x}\mathrm{d}x$$

✎ **Solution:** Let $u = \tan^{-1}(x)$, $\mathrm{d}x = \sec^2(u)\mathrm{d}u$

$$2\int_0^{\frac{\pi}{4}} u^3 \csc(2u)\mathrm{d}u$$

Let $t = 2u$, $\mathrm{d}u = \frac{1}{2}\mathrm{d}t$

$$\frac{1}{8}\int_0^{\frac{\pi}{2}} t^3 \csc(t)\mathrm{d}t$$

Now, a handy identity when integrating a polynomial times csc is:

$$\int_a^b p(x)\csc(x)\mathrm{d}x = 2\sum_{n=0}^{\infty}\int_a^b p(x)\sin[(2n+1)x]\mathrm{d}x$$

So, we have

$$\begin{aligned}\frac{1}{4}\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}} t^3\sin[(2n+1)t]\mathrm{d}t &= \frac{3\pi^2}{16}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}\\ &= \frac{3\pi^2}{16}\mathbf{G} - \frac{1}{1024}\left(\psi_3\left(\frac{1}{4}\right) - \psi_3\left(\frac{3}{4}\right)\right)\end{aligned}$$

□

Exercise 1.2.135: Evaluate:

$$\int_0^1 (\arctan x)^3\mathrm{d}x$$

✎ **Solution:** Make the obvious sub $u = \tan^{-1}(x)$, $x = \tan(u)$, $\mathrm{d}x = \sec^2(u)\mathrm{d}u$

$$\int_0^{\frac{\pi}{4}} u^3\sec^2(u)\mathrm{d}u$$

Parts: $u = u^3$, $\mathrm{d}u = 3u^2\mathrm{d}u$, $\mathrm{d}v = \sec^2(u)\mathrm{d}u$, $v = \tan(u)$

$$\begin{aligned}&u^3\tan(u)\Big|_0^{\frac{\pi}{4}} - 3\int_0^{\frac{\pi}{4}} u^2\tan(u)\mathrm{d}u\\ &\Rightarrow \frac{\pi^3}{64} - 3\int_0^{\frac{\pi}{4}} u^2\tan(u)\mathrm{d}u\end{aligned}$$

parts again: $u = u^2$, $\mathrm{d}v = \tan(u)\mathrm{d}u$, $v = \ln(\cos(u))$, $\mathrm{d}u = 2u\mathrm{d}u$

$$\begin{aligned}&-u^2\ln(\cos(u))\Big|_0^{\frac{\pi}{4}} + 2\int_0^{\frac{\pi}{4}} u\ln(\cos(u))\mathrm{d}u\\ &\Rightarrow \frac{\pi^3}{64} + \frac{3\pi^2}{32}\ln(2) + 2\int_0^{\frac{\pi}{4}} u\ln(\cos(u))\mathrm{d}u\end{aligned}$$

Now, a good way to go about evaluating the lncos integral is to use the famous Fourier series:

$$\int_0^{\frac{\pi}{4}} u\ln(\cos(u))\mathrm{d}u = -\int_0^{\frac{\pi}{4}}\sum_{k=1}^{\infty}\frac{(-1)^k u\cos(2ku)}{k}\mathrm{d}u - \ln(2)\int_0^{\frac{\pi}{4}} u\mathrm{d}u$$

Swapping sum and integral will give sums in terms of alternating $\zeta(3)$ and $\mathbf{G}$.

Hence we have

$$\int_0^1 (\arctan x)^3\mathrm{d}x = \frac{\pi^3}{64} + \frac{63}{64}\zeta(3) - \frac{3\pi}{4}\mathbf{G} + \frac{3\pi^2}{32}\ln(2)$$

□

Exercise 1.2.136: Prove:

$$\int_0^\infty \frac{x\,\arctan x}{\cosh\left(\frac{\pi\,x}{2}\right)}\mathrm{d}x = \ln\left(\frac{\mathbf{A}^{12}}{\pi\,e\,\sqrt[3]{2}}\right)$$

Proof: Let $t = \frac{\pi x}{2}$, $\mathrm{d}x = \frac{2}{\pi}\mathrm{d}t$, and use the exponential for $\cosh(t)$.

$$\frac{8}{\pi^2}\int_0^\infty \frac{t\cdot\tan^{-1}\left(\frac{2t}{\pi}\right)e^t}{1+e^{2t}}\mathrm{d}t$$

Using the useful relation $\tan^{-1}\left(\frac{2t}{\pi}\right) = \int_0^\infty \frac{e^{-\pi y/2}\sin(ty)}{y}\mathrm{d}y$, we have:

$$\int_0^\infty\int_0^\infty \frac{e^{-\pi y/2}\sin(ty)}{y(e^{2t}+1)}\mathrm{d}t\mathrm{d}y = \int_0^\infty \frac{e^{-\pi y/2}}{y}dy\int_0^\infty \frac{te^t\sin(ty)}{e^{2t}+1}\mathrm{d}t$$

since $\frac{e^t}{1+e^{2t}} = \sum_{n=1}^{\infty}(-1)^{n+1}e^{-(2n-1)t}$

$$\int_0^\infty \frac{te^t\sin(ty)}{1+e^{2t}}\mathrm{d}t = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}y(2n-1)}{(y^2+(2n-1)^2)^2}$$

To save time and work, I just used tech to do the sum. I have also included the $\frac{8}{\pi^2}$ from up top. This could be a separate problem. This sums evaluates to:

$$\Rightarrow \frac{y^4\tanh(\pi y/2)}{(y^2+1)^2\cosh(\pi y/2)} + \frac{\tanh(\pi y/2)}{(y^2+1)^2\cosh(\pi y/2)} + \frac{2y^2\tanh(\pi y/2)}{(y^2+1)^2\cosh(\pi y/2)}$$

Multiply by $\frac{e^{-\pi y/2}}{y}$ and then do the second integration. Write in terms of exponentials if one likes. which may be easier. It whittles down quite a bit.

$$\int_0^\infty \frac{e^{-\pi y/2}}{y}\left(\frac{y^4\tanh(\pi y/2)}{(y^2+1)^2\cosh(\pi y/2)} + \frac{\tanh(\pi y/2)}{(y^2+1)^2\cosh(\pi y/2)} + \frac{2y^2\tanh(\pi y/2)}{(y^2+1)^2\cosh(\pi y/2)}\right)\mathrm{d}y$$

$$= 2\int_0^\infty \frac{e^{\pi y}-1}{y(e^{\pi y}+1)^2}\mathrm{d}y$$

This has an equivalent series which is derived from Euler's famous transformation

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_{k+1}$$

$$2\int_0^\infty \frac{e^t-1}{t(e^t+1)^2}\mathrm{d}t = \sum_{n=0}^{\infty}\frac{2}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}k\ln\left(\frac{k+1}{k}\right)$$
$$= -2\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k\ln(k+1) + 2\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k\ln(k)$$

It can be shown that

$$\Rightarrow -2\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k\ln(k+1)+2\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k\ln(k)$$
$$=-2\int_0^1\ln\left(\frac{\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)}{\Gamma\left(\frac{x}{2}\right)}\right)dx-\ln(\pi)-1 \tag{1}$$

On the log-gamma integral, break it up and let $1+t=\dfrac{x+1}{2}$ in the left integral and $t+1=x/2$ in the right one.

$$\int_0^1\ln\Gamma\left(\frac{1+x}{2}\right)\mathrm{d}x-\int_0^1\ln\Gamma\left(\frac{x}{2}\right)\mathrm{d}x$$

left one, let $x=2y+1$

$$-4\int_0^{-1/2}\ln\Gamma(y+1)\mathrm{d}y$$

the right one, let $x=2y+2$

$$2\int_{-1}^{-1/2}\ln(y+1)\mathrm{d}y$$

the first one, use Alexiewesky formula and let $t=1/2$. The right one requires a little more finesse. Let $t=1$ and $t=1/2$, then subtract them.

$$\int_0^t\ln\Gamma(x+1)\mathrm{d}x=\frac{1}{2}(\ln(2\pi)-1)t+\frac{1}{2}t^2+t\ln\Gamma(t+1)-\ln[G(t+1)]$$

i.e.

$$\begin{aligned}\int_0^{-1/2}\ln\Gamma(1+x)\mathrm{d}x&=\frac{1}{2}\left(\ln(2\pi)-1\right)\left(-\frac{1}{2}\right)+1/2\left(-\frac{1}{2}\right)^2-\frac{1}{2}\ln\left(\sqrt{\pi}\right)\\&\quad-\left(-\frac{3}{2}\ln(\mathbf{A})+\frac{1}{4}\ln(\pi)+\frac{1}{8}+\frac{1}{24}\ln(2)\right)\\&=\ln\left(\frac{2^{7/24}}{e^{1/4}\mathbf{A}^{3/2}\pi^{1/4}}\right)\end{aligned}$$

we ultimately get:

$$\int_0^1\ln\left(\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x}{2}\right)}\right)\mathrm{d}x=\ln\left(\frac{2^{1/6}}{\mathbf{A}^6}\right)$$

From the right side of (1), we finally obtain:

$$\int_0^{\infty}\frac{x\arctan x}{\cosh\left(\frac{\pi x}{2}\right)}\mathrm{d}x=-2\ln\left(\frac{2^{1/6}}{\mathbf{A}^6}\right)-\ln(\pi)-1=\ln\left(\frac{\mathbf{A}^{12}}{2^{1/3}\pi e}\right)$$

□

Exercise 1.2.137: Evaluate:

$$\int_{-1}^{1}\frac{\sin\left(e^x\sin\sqrt{1-x^2}\right)\exp\left(e^x\cos\sqrt{1-x^2}\right)}{5-4x}\mathrm{d}x$$

✎ **Solution:** This integral is related to one of the formulas in **Exercise 1.2.23**. Specifically,

$$\int_0^\pi \frac{f(a+e^{i\theta})-f(a+e^{-i\theta})}{1-2b\cos\theta+b^2}\sin\theta\,\mathrm{d}\theta = \frac{i\pi}{b}\Big(f(a+b)-f(a)\Big)$$

If you let $f(z)=e^{e^z}$, $a=0$, and $b=1/2$, you get

$$\begin{aligned}2i\int_0^\pi \frac{e^{e^{\cos\theta}\cos(\sin\theta)}\sin\big(e^{\cos\theta}\sin(\sin\theta)\big)}{\frac{5}{4}-\cos\theta}\sin\theta\,\mathrm{d}\theta &= 8i\,\Im\int_0^\pi \frac{e^{e^{e^{i\theta}}}}{5-4\cos\theta}\sin\theta\,\mathrm{d}\theta\\ &= 2\pi i\left(e^{e^{1/2}}-e\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{-1}^1 \frac{\sin\left(e^x\sin\sqrt{1-x^2}\right)\exp\left(e^x\cos\sqrt{1-x^2}\right)}{5-4\,x}\mathrm{d}x &= \Im\int_0^\pi \frac{e^{e^{e^{i\theta}}}}{5-4\cos\theta}\sin\theta\,\mathrm{d}\theta\\ &= \frac{\pi}{4}\left(e^{e^{1/2}}-e\right)\end{aligned}$$

□

Exercise 1.2.138: Evaluate:

$$\int_0^\infty \frac{1}{x^2+4}\arctan^2\left(\frac{3x}{4}+\frac{1}{x}\right)\mathrm{d}x$$

✎ **Solution:** With the sub $x=2\tan\theta$, the integral is:

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2}\arctan^2\left(\frac{3}{2}\tan\theta+\frac{1}{2\tan\theta}\right)\mathrm{d}\theta &= \frac{1}{2}\int_0^{\pi/2}\arctan^2\left(\frac{2-\cos(2\theta)}{\sin(2\theta)}\right)\mathrm{d}\theta\\ &= \frac{1}{4}\int_0^{\pi}\arctan^2\left(\frac{2-\cos t}{\sin t}\right)\mathrm{d}t\end{aligned}$$

Consider

$$I(a)=\frac{1}{4}\int_0^\pi \arctan^2\left(\frac{a-\cos t}{\sin t}\right)\mathrm{d}t \qquad (a>1)$$

$$\begin{aligned}\Rightarrow I'(a) &= \frac{1}{2}\int_0^\pi \arctan\left(\frac{a-\cos t}{\sin t}\right)\frac{\sin t}{a^2-2a\cos t+1}\,\mathrm{d}t\\ &= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{a^{n+1}}\int_0^\pi \sin(nt)\arctan\left(\frac{a-\cos t}{\sin t}\right)\mathrm{d}t\\ &= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{a^{n+1}}\left(\left(-\frac{\cos(nt)}{n}\arctan\left(\frac{a-\cos t}{\sin t}\right)\right)\Big|_0^\pi+\frac{1}{n}\int_0^\pi\cos(nt)\frac{1-a\cos t}{a^2-2a\cos t+1}\,\mathrm{d}t\right)\\ &= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{a^{n+1}}\left(\frac{\pi}{2n}(1-(-1)^n)-\frac{1}{n}\sum_{m=1}^\infty\frac{1}{a^m}\int_0^\pi\cos(nt)\cos(mt)\,\mathrm{d}t\right)\\ &= \frac{\pi}{2}\sum_{n=1}^\infty\frac{1}{(2n-1)a^{2n}}-\frac{\pi}{4}\sum_{n=1}^\infty\frac{1}{na^{2n+1}}\\ \Rightarrow I(a) &= \frac{\pi}{8}\sum_{n=1}^\infty\frac{1}{n^2a^{2n}}-\frac{\pi}{2}\sum_{n=1}^\infty\frac{1}{(2n-1)^2a^{2n-1}}+C\end{aligned}$$

With $I(1)=\pi^3/48$,

$$\Rightarrow \frac{\pi^3}{48}=\frac{\pi}{8}\sum_{n=1}^\infty\frac{1}{n^2}-\frac{\pi}{2}\sum_{n=1}^\infty\frac{1}{(2n-1)^2}+C \Rightarrow C=\frac{\pi^3}{16}$$

For the given problem, we need $I(2)$ i.e

$$I(2)=\frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{4}\right)-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2 2^{2n-1}}+\frac{\pi^3}{16}$$

The sum can be written as

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2 2^{2n-2}}=\frac{\pi^2}{3}-2\ln^2(2)-4\mathrm{Li}_2\left(-\frac{1}{2}\right)$$

Now, sub this into the above formula:

$$\begin{aligned}&\frac{\pi^3}{16}-\frac{\pi}{16}\left(\frac{\pi^2}{3}-2\ln^2(2)-4\mathrm{Li}_2\left(-\frac{1}{2}\right)\right)+\frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{4}\right)\\&=\frac{\pi}{4}\mathrm{Li}_2\left(-\frac{1}{2}\right)+\frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{4}\right)+\frac{\pi}{8}\ln^2(2)+\frac{\pi^3}{24}\end{aligned}$$

But,

$$\frac{\pi}{4}\mathrm{Li}_2\left(-\frac{1}{2}\right)=\frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{4}\right)-\frac{\pi^3}{48}+\frac{\pi}{8}\ln^2(2)$$

Hence we obtain

$$\begin{aligned}\int_0^{\infty}\frac{1}{x^2+4}\arctan^2\left(\frac{3x}{4}+\frac{1}{x}\right)\mathrm{d}x&=\frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{4}\right)-\frac{\pi^3}{48}+\frac{\pi}{8}\ln^2(2)\\&\quad+\frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{4}\right)+\frac{\pi}{8}\ln^2(2)+\frac{\pi^3}{24}\\&=\frac{\pi^3}{48}+\frac{\pi}{4}\ln^2 2+\frac{\pi}{4}\mathrm{Li}_2\left(\frac{1}{4}\right)\end{aligned}$$

□

Exercise 1.2.139: Prove:

$$\int_0^{\infty}\frac{x^2\,\ln\sinh(x)}{\cosh(3x)}\mathrm{d}x=\frac{\pi^2}{9}\mathbf{G}-\frac{5\pi^3}{108}\ln 2$$

Proof: Begin with the sub $t=3x,\quad x=t/3,\quad \mathrm{d}x=1/3\mathrm{d}t$

$$\begin{aligned}&\frac{1}{3}\int_0^{\infty}\frac{\left(\frac{t}{3}\right)^2\ln\left(\sinh\left(\frac{t}{3}\right)\right)}{\cosh(t)}\mathrm{d}t=\frac{1}{27}\int_0^{\infty}\frac{t^2\ln\left(\sinh\left(\frac{t}{3}\right)\right)}{\cosh(t)}\mathrm{d}t\\&=\frac{1}{27}\int_0^{\infty}\frac{t^2\ln\left(\frac{1}{2}(e^{t/3}-e^{-t/3})\right)}{\cosh(t)}\mathrm{d}t=\frac{1}{27}\int_0^{\infty}\frac{t^2\ln\left(\frac{1}{2}e^{t/3}(1-e^{-2t/3})\right)}{\cosh(t)}\mathrm{d}t\\&=\frac{1}{27}\int_0^{\infty}\frac{t^2(-\ln(2)+t/3+\ln(1-e^{-2t/3})}{\cosh(t)}\mathrm{d}t\\&=-\frac{1}{27}\ln(2)\int_0^{\infty}\frac{t^2}{\cosh(t)}\mathrm{d}t+\frac{1}{81}\int_0^{\infty}\frac{t^3}{\cosh(t)}\mathrm{d}t+\frac{1}{27}\int_0^{\infty}\frac{t^2\ln(1-e^{-2t/3})}{\cosh(t)}\mathrm{d}t\end{aligned}$$

The left two are rather cliche and have general forms. Play on

$$\int_0^{\infty}\frac{t^a}{\cosh(t)}\mathrm{d}t=2\Gamma(a+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{a+1}}$$

So, when $a=2$, the leftmost integral is $-\frac{1}{27}\cdot\ln(2)\cdot\frac{\pi^3}{8}$.
and when $a=3$, the middle one is $\frac{1}{81}\cdot\frac{1}{128}\left(\psi_3\left(\frac{1}{4}\right)-\psi_3\left(\frac{3}{4}\right)\right)$.
The right one:

$$\frac{1}{27}\int_0^\infty \frac{t^2\ln(1-e^{-2t/3})}{\cosh(t)}\mathrm{d}t$$

Let $u=e^{-t/3}$, $-3\ln(u)=t$, $\mathrm{d}t=-\frac{3}{u}\mathrm{d}u$ and $\cosh(-3\ln(u))=\frac{u^6+1}{2u^3}$

$$\Rightarrow -2\int_0^1 \frac{u^2\ln^2(u)\ln(1-u^2)}{u^6+1}\mathrm{d}u$$

which can be written equivalently as:

$$-\frac{1}{4}\int_0^1 \frac{\sqrt{u}\ln^2(u)\ln(1-u)}{u^3+1}\mathrm{d}u=-\frac{1}{4}\int_0^1 \frac{u^{n+1/2}\ln^2(u)}{u^3+1}\sum_{n=1}^\infty\frac{1}{n}\mathrm{d}u$$

Make the sub $y=u^3$ and begin with

$$\int_0^1\frac{y^{a-1}}{1+y}\mathrm{d}y=\frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(\frac{a}{2}\right)\right]$$

Diffing twice w.r.t a, to introduce the log-square term, gives a tetragamma series:

$$-\frac{1}{864}\sum_{n=1}^\infty\left[\frac{\psi_2\left(\frac{n}{6}+\frac{3}{4}\right)-\psi_2\left(\frac{n}{6}+\frac{1}{4}\right)}{n}\right]=\boxed{-\frac{1}{10368}\left[\psi_3\left(\frac{1}{4}\right)-\psi_3\left(\frac{3}{4}\right)\right]+\frac{\pi^2}{9}\mathbf{G}-\frac{\pi^3\ln(2)}{24}}$$

Sum the boxed portions and obtain:

$$\int_0^\infty\frac{x^2\,\ln\sinh(x)}{\cosh(3x)}\mathrm{d}x=\frac{\pi^2}{9}\mathbf{G}-\frac{5\pi^3}{108}\ln 2$$

□

Exercise 1.2.140: Prove:

$$\int_0^{\frac{\pi}{2}}\ln(1+\sin(x))\ln(1+\cos(x))\tan(x)\mathrm{d}x=\frac{\pi^2}{8}\ln(2)-\frac{5}{16}\zeta(3)$$

Proof: The integral equals

$$\begin{aligned}&\int_0^1\ln\left(1+\sqrt{1-x^2}\right)\frac{\ln(1+x)}{x}\mathrm{d}x\\&=-\mathrm{Li}_2(-x)\ln\left(1+\sqrt{1-x^2}\right)\Big|_0^1+\int_0^1\mathrm{Li}_2(-x)\frac{1}{x}\left[1-\frac{1}{\sqrt{1-x^2}}\right]\mathrm{d}x\\&=\mathrm{Li}_3(-1)-\int_0^1\frac{\mathrm{Li}_2(-x)}{x\sqrt{1-x^2}}\mathrm{d}x\end{aligned}$$

Now, $\mathrm{Li}_3(-1)=-\beta(3)=-\left(1-2^{1-3}\right)\zeta(3)=-\frac{3}{4}\zeta(3)$. In the second term, write $\mathrm{Li}_2(-x)$ as a series and interchange summation and integration:

$$-\int_0^1\frac{\mathrm{Li}_2(-x)}{x\sqrt{1-x^2}}\mathrm{d}x=\sum_{k\geq1}\frac{(-1)^{k-1}}{k^2}\int_0^1 x^{k-1}\left(1-x^2\right)^{-1/2}\mathrm{d}x=\sum_{k\geq1}\frac{(-1)^{k-1}}{2k^2}\mathrm{B}\left(\frac{k}{2},\frac{1}{2}\right)$$

We calculate this sum, to start with

$$\sum_{k\geq 1}(-1)^{k-1}\mathrm{B}\left(\frac{k}{2},\frac{1}{2}\right)x^{k-1}=\frac{\pi-2\arcsin x}{\sqrt{1-x^2}}$$

Integrating once from 0 to x yields

$$\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\mathrm{B}\left(\frac{k}{2},\frac{1}{2}\right)x^{k}=\pi\arcsin x-\arcsin^2 x$$

Dividing by x and integrating from 0 to 1 yields

$$\begin{aligned}\sum_{k\geq 1}\frac{(-1)^{k-1}}{k^2}\mathrm{B}\left(\frac{k}{2},\frac{1}{2}\right)&=\int_0^{\pi/2}\frac{\pi x\cos x}{\sin x}-\frac{x^2\cos x}{\sin x}\mathrm{d}x\\&=-\int_0^{\pi/2}\pi\ln\sin x-2x\ln\sin x\mathrm{d}x\\&=\frac{\pi^2}{2}\ln 2+2\int_0^{\pi/2}x\ln\sin x\mathrm{d}x\end{aligned}$$

The latter integral can be calculated as follows: use the identity

$$\ln\sin x=-\ln 2-\sum_{k\geq 0}\cos(2kx)$$

and interchange summation and integration. This gives the value

$$2\int_0^{\pi/2}x\ln\sin x\mathrm{d}x=-\frac{\pi^2}{4}\ln 2+\beta(3)=-\frac{\pi^2}{4}\ln 2+\frac{7}{8}\zeta(3)$$

Adding everything up yields the correct value of the integral:

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\ln(1+\sin(x))\ln(1+\cos(x))\tan(x)\mathrm{d}x&=-\frac{3}{4}\zeta(3)+\frac{1}{2}\left[\frac{\pi^2}{2}\ln 2-\frac{\pi^2}{4}\ln 2+\frac{7}{8}\zeta(3)\right]\\&=\frac{\pi^2}{8}\ln 2-\frac{5}{16}\zeta(3)\end{aligned}$$

□

Exercise 1.2.141: Prove:

$$\int_0^1\operatorname{arsech}(x)\operatorname{arsinh}(x)\,\mathrm{d}x=\mathbf{G}+\frac{\ln 2}{2}-\frac{\pi}{4}$$

Proof: Integrating by parts yields

$$\int_0^1\operatorname{asech}(x)\operatorname{asinh}(x)\mathrm{d}x=x\operatorname{asech}(x)\operatorname{asinh}(x)\Big|_0^1-\int_0^1\frac{x\operatorname{asech}(x)}{\sqrt{1+x^2}}\mathrm{d}x+\int_0^1\frac{\operatorname{asinh}(x)}{\sqrt{1-x^2}}\mathrm{d}x$$

We have

$$\int_0^1\frac{\operatorname{asinh}(x)}{\sqrt{1-x^2}}\mathrm{d}x=\int_0^{\pi/2}\operatorname{asinh}\sin x\mathrm{d}x=\mathbf{G}$$

This is a known result, see for example

http://math.stackexchange.com/questions/750095/

Furthermore,

$$\int_0^1\frac{x\operatorname{asech}(x)}{\sqrt{1+x^2}}\mathrm{d}x=\lim_{a\to 0}\left[\sqrt{1+x^2}\operatorname{asech}x\Big|_a^1-\int_a^1\frac{1}{x}\sqrt{\frac{1+x^2}{1-x^2}}\mathrm{d}x\right]$$

Note that asech diverges logarithmically at zero.

Substituting $x = \sqrt{\dfrac{1-y^2}{1+y^2}}$ in the integral yields, after some algebra,

$$\int_{\sqrt{\frac{1-a^2}{1+a^2}}}^{1} \frac{1}{1+t^2} + \frac{1}{1-t^2}\mathrm{d}t = \frac{\pi}{4} + O(a) + \frac{1}{2}\ln\frac{1+\sqrt{\dfrac{1-a^2}{1+a^2}}}{1-\sqrt{\dfrac{1-a^2}{1+a^2}}}$$

To lowest order in a, we have $\sqrt{1+a^2}\operatorname{asech} a = \ln 2 - \ln a$

and $\dfrac{1}{2}\ln\dfrac{1+\sqrt{\frac{1-a^2}{1+a^2}}}{1-\sqrt{\frac{1-a^2}{1+a^2}}} = \dfrac{1}{2}\ln 2 - \ln a.$

Therefore,

$$\int_0^1 \frac{x\operatorname{asech}(x)}{\sqrt{1+x^2}}\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Adding the terms gives the result

$$\int_0^1 \operatorname{arsech}(x)\operatorname{arsinh}(x)\,\mathrm{d}x = \mathbf{G} + \frac{\ln 2}{2} - \frac{\pi}{4}$$

□

Exercise 1.2.142: Evaluate:

$$\int_0^{\pi} \ln\left(\sin(x) + \sqrt{1+\sin^2(x)}\right)\,\mathrm{d}x$$

Solution: Using the Taylor expansion of $\operatorname{arcsinh}(x)$ at $x = 0$,

$$\begin{aligned}
&\int_0^{\pi} \ln(\sin x + \sqrt{1+\sin^2 x})\,\mathrm{d}x = \int_0^{\pi} \operatorname{arcsinh}(\sin x)\,\mathrm{d}x \\
&= 2\int_0^{\pi/2} \operatorname{arcsinh}(\sin x)\,\mathrm{d}x = 2\int_0^{\pi/2} \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} \frac{\sin^{2n+1} x}{2^{2n}(2n+1)}\,\mathrm{d}x \\
&= 2\sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} \frac{1}{2^{2n}(2n+1)} \int_0^{\pi/2} \sin^{2n+1}(x)\,\mathrm{d}x \\
&= 2\sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} \frac{1}{2^{2n}(2n+1)} \frac{1}{2n+1} \frac{2^{2n}}{\binom{2n}{n}} \qquad (1) \\
&= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 2\mathbf{G}
\end{aligned}$$

(1) The integral $\displaystyle\int_0^{\pi/2} \sin^{2n+1} x\,\mathrm{d}x$ can be evaluated by relating it to the beta function and then using the gamma duplication formula.

Specifically,

$$\int_0^{\pi/2} \sin^{2n+1} x \, \mathrm{d}x = \int_0^{\pi/2} \sin^{2(n+1)-1}(x) \cos^{2(1/2)-1}(x) \, \mathrm{d}x$$
$$= \frac{1}{2}\mathrm{B}\left(n+1, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{1}{2}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}$$
$$= \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2n+1}\frac{2^{2n-1}\Gamma(n)}{\Gamma(2n)\Gamma\left(\frac{1}{2}\right)}\frac{2n}{2n} = \frac{1}{2n+1}2^{2n}\frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+1)}$$
$$= \frac{1}{2n+1}\frac{2^{2n}}{\binom{2n}{n}}$$

□

Exercise 1.2.143: Evaluate:

$$\int_0^{\infty} \frac{\sin x \cdot \operatorname{arccot} x}{1+8\sin^2\left(\frac{x}{2}\right)}\mathrm{d}x$$

Solution: Let

$$I(a) = \int_0^{\infty} \frac{\sin x \cdot \operatorname{arccot}\left(\frac{x}{a}\right)}{1+8\sin^2\left(\frac{x}{2}\right)}\mathrm{d}x$$

$$I'(a) = \int_{-\infty}^{\infty} \frac{\sin x}{1+8\sin^2\frac{x}{2}}\frac{x}{a^2+x^2}\mathrm{d}x$$
$$= \int_0^{2\pi} \frac{\sin x}{1+8\sin^2\frac{x}{2}}\left(\frac{x}{a^2+x^2}+\sum_{n=1}^{\infty}\frac{x+2n\pi}{a^2+(x+2n\pi)^2}+\sum_{n=1}^{\infty}\frac{x-2n\pi}{a^2+(x-2n\pi)^2}\right)\mathrm{d}x$$
$$= \int_0^{2\pi} \frac{\sin x}{1+8\sin^2\frac{x}{2}}\frac{\sin x}{2\cosh a-2\cos x}\mathrm{d}x = \frac{\pi}{4e^a-2}$$

Hence

$$\int_0^{\infty} \frac{\sin x}{1+8\sin^2\frac{x}{2}}\cot^{-1}\frac{x}{a}\mathrm{d}x = \frac{\pi}{4}\ln\left(2e^a-1\right)-\frac{a\pi}{4}$$

now let $a=1$ we obtain

$$\int_0^{\infty} \frac{\sin x \cdot \operatorname{arccot} x}{1+8\sin^2\left(\frac{x}{2}\right)}\mathrm{d}x = \frac{\pi}{4}\ln\left(2e-1\right)-\frac{\pi}{4}$$

□

Exercise 1.2.144: Evaluate:

$$\int_0^{\infty} \frac{x\cos(ax)\coth(\pi x)}{1+x^2}\mathrm{d}x$$

✎ **Solution:** Let's assume $a > 0$. Now plug the series $\coth(\pi x) = \frac{x}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+x^2}$ and interchange sum and integral:

$$\begin{aligned}
I &= \int_0^\infty \frac{x\cos(ax)\coth(\pi x)}{1+x^2}\mathrm{d}x = \frac{1}{\pi}\int_0^\infty \frac{x^2\cos(ax)}{1+x^2}\sum_{n=-\infty}^{\infty}\frac{1}{x^2+n^2}\mathrm{d}x \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{x^2\cos(ax)}{(x^2+1)(x^2+n^2)}\mathrm{d}x \\
&= \frac{1}{\pi}\left\{\frac{\pi}{2}\sum_{n=-\infty\ ,\ n\neq\pm1}^{\infty}\frac{|n|e^{-a|n|}-e^{-a}}{n^2-1} + 2\int_0^\infty\left(\frac{x}{1+x^2}\right)^2\cos(ax)\ \mathrm{d}x\right\} \\
&= \frac{1}{2}\sum_{n=-\infty\ ,\ n\neq\pm1}^{\infty}\frac{|n|e^{-a|n|}-e^{-a}}{n^2-1} - \frac{1}{2}(a-1)e^{-a} \qquad (*)
\end{aligned}$$

Evaluation of the sum is straightforward using the taylor series of $\ln(1x)$. I will solve it separately.

$$\begin{aligned}
\sum_{n=-\infty\ ,\ n\neq\pm1}^{\infty}\frac{|n|e^{-a|n|}-e^{-a}}{n^2-1} &= e^{-a} + 2\sum_{n=2}^{\infty}\frac{ne^{-na}-e^{-a}}{n^2-1} = -\frac{1}{2}e^{-a} + 2\sum_{n=2}^{\infty}\frac{ne^{-na}}{n^2-1} \\
&= -\frac{1}{2}e^{-a} + \sum_{n=2}^{\infty}\frac{e^{-an}}{n-1} + \sum_{n=2}^{\infty}\frac{e^{-an}}{n+1} \\
&= -\frac{1}{2}e^{-a} - e^{-a}\ln(1-e^{-a}) - \frac{e^{-a}}{2} - e^{a}\ln(1-e^{-a}) - 1 \\
&= -e^{-a} - 2\cosh(a)\ln(1-e^{-a}) - 1 \qquad (**)
\end{aligned}$$

Put this in $(*)$:

$$\int_0^\infty \frac{x\cos(ax)\coth(\pi x)}{1+x^2}\mathrm{d}x = -\frac{a}{2}e^{-a} - \cosh(a)\ln(1-e^{-a}) - \frac{1}{2}$$

□

Exercise 1.2.145: Evaluate:

$$\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\mathrm{d}x = \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\left(\zeta(2) + x\int_0^1\frac{\ln(ux)}{1-ux}du\right)\mathrm{d}x \\
&= \zeta(2)\zeta(3) + \int_0^1 \mathrm{Li}_2(x)\int_0^1\frac{\ln(ux)}{1-ux}\mathrm{d}u\ \mathrm{d}x \\
&= \zeta(2)\zeta(3) + \sum_{n=0}^{\infty}\int_0^1 u^n\int_0^1 x^n\mathrm{Li}_2(x)\ln(ux)\ \mathrm{d}x\ \mathrm{d}u \\
&= \zeta(2)\zeta(3) + \sum_{n=0}^{\infty}\int_0^1 u^n\left\{\ln u\int_0^1 x^n\mathrm{Li}_2(x)\ \mathrm{d}x + \int_0^1 x^n\ln(x)\mathrm{Li}_2(x)\ \mathrm{d}x\right\}\mathrm{d}u \\
&= \zeta(2)\zeta(3) + \sum_{n=0}^{\infty}\int_0^1 u^n\left\{\ln u\left(\frac{\zeta(2)}{1+n} - \frac{H_{n+1}}{(1+n)^2}\right) + \frac{2H_{n+1}}{(1+n)^3} - \frac{2\zeta(2)}{(n+1)^2} + \frac{H_{n+1}^{(2)}}{(n+1)^2}\right\}\mathrm{d}u \\
&= -2\zeta(2)\zeta(3) + 3\sum_{n=1}^{\infty}\frac{H_n}{n^4} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}
\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=3\zeta(2)\zeta(3)-\frac{9\zeta(5)}{2}\ ,\ \sum_{n=1}^{\infty}\frac{H_n}{n^4}=3\zeta(5)-\zeta(2)\zeta(3)$$

So the final answer is

$$\int_0^1\frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\mathrm{d}x=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3)$$

There is a formula for Euler sums of the form:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{2p+1}}=\zeta(2)\zeta(2p+1)-\frac{(p+2)(2p+1)}{2}\zeta(2p+3)+2\sum_{k=2}^{p+1}(k-2)\zeta(2k-1)\zeta(2p+4-2k)$$

□

Exercise 1.2.146: Evaluate:

$$\int_0^{\infty}\mathrm{si}(x)\tan^{-1}\left(\frac{\sin x}{2-\cos x}\right)\mathrm{d}x$$

where $\mathrm{si}(x)=-\int_x^{\infty}\frac{\sin t}{t}\mathrm{d}t$

Solution: Consider

$$I(a)=\int_0^{\infty}\mathrm{si}(x)\tan^{-1}\left(\frac{\sin x}{a-\cos x}\right)\mathrm{d}x$$

$$\Rightarrow\frac{\mathrm{d}I}{\mathrm{d}a}=-\int_0^{\infty}\mathrm{si}(x)\frac{\sin x}{a^2-2a\cos x+1}\mathrm{d}x=-\sum_{n=1}^{\infty}\frac{1}{a^{n+1}}\int_0^{\infty}\mathrm{si}(x)\sin(nx)\,\mathrm{d}x$$

From integration by parts:

$$\begin{aligned}\int_0^{\infty}\mathrm{si}(x)\sin(nx)\,\mathrm{d}x&=\mathrm{si}(x)\frac{-\cos(nx)}{n}\bigg|_0^{\infty}+\frac{1}{n}\int_0^{\infty}\frac{\sin x\cos(nx)}{x}\mathrm{d}x\\&=-\frac{\pi}{2n}+\frac{1}{2n}\int_0^{\infty}\frac{\sin((n+1)x)-\sin((n-1)x)}{x}\mathrm{d}x\\&=-\frac{\pi}{2n}+\frac{\pi}{4n}\left(\mathrm{sgn}(1-n)+\mathrm{sgn}(1+n)\right)\end{aligned}$$

Notice that the second term is always zero except $n=1$. Hence

$$-\sum_{n=1}^{\infty}\frac{1}{a^{n+1}}\int_0^{\infty}\mathrm{si}(x)\sin(nx)\,\mathrm{d}x=\frac{\pi}{2}\sum_{n=2}^{\infty}\frac{1}{na^{n+1}}+\frac{\pi}{4a^2}$$

$$\Rightarrow I(a)=-\frac{\pi}{2}\sum_{n=2}^{\infty}\frac{1}{n^2a^n}-\frac{\pi}{4a}+C$$

As $a\to\infty$, $I(a)\to 0$, hence $C=0$. Therefore,

$$\begin{aligned}I(2)&=-\frac{\pi}{2}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n^2 2^n}\right)-\frac{\pi}{8}=\frac{\pi}{8}-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2 2^n}\\&=-\frac{\pi}{2}\frac{1}{12}\left(\pi^2-6\ln^2 2\right)+\frac{\pi}{8}=-\frac{\pi^3}{24}+\frac{\pi}{4}\ln^2 2+\frac{\pi}{8}\end{aligned}$$

□

Exercise 1.2.147: Evaluate:

$$\int_0^\infty \frac{e^{p\cos(ax)}\sin\left(p\sin(ax)\right)}{\sin(ax)}\frac{\mathrm{d}x}{b^2+x^2}$$

Solution:

$$I=\int_0^\infty \frac{e^{p\cos(ax)}\sin(p\sin(ax))}{\sin(ax)}\frac{\mathrm{d}x}{b^2+x^2}=\frac{1}{2}\Im\ \mathrm{PV}\int_{-\infty}^\infty \frac{e^{pe^{iax}}}{\sin(ax)}\frac{\mathrm{d}x}{b^2+x^2}$$

Let $f(z)=\dfrac{e^{pe^{iaz}}}{\sin(az)}\dfrac{1}{b^2+z^2}$ and integrate it around a contour which consists of the following:

1. Real axis having circular indents around each singularity and
2. C_R, a semicircle in the upper half plane transversed in the counter clockwise direction.

This will give us

$$\begin{aligned}
&2I+\Im\lim_{R\to\infty}\int_{C_R}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}\mathrm{d}z\\
&=\Im\left(2\pi i\ \mathrm{Res}_{z=ib}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}+i\pi\sum_{k=-\infty}^{\infty}\mathrm{Res}_{z=\pi k/a}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}\right)\\
\implies I&=\Im\left(\pi i\ \mathrm{Res}_{z=ib}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}+\frac{\pi i}{2}\sum_{k=-\infty}^{\infty}\mathrm{Res}_{z=\pi k/a}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}\right.\\
&\qquad\left.-\frac{1}{2}\lim_{R\to\infty}\int_{C_R}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}\mathrm{d}z\right)
\end{aligned}$$

Now,

1. $\mathrm{Res}_{z=ib}\dfrac{e^{pe^{iaz}}}{\sin(az)}\dfrac{1}{b^2+z^2}=-\dfrac{e^{pe^{-ab}}}{2b\sinh(ab)}$
2. $\lim\limits_{R\to\infty}\displaystyle\int_{C_R}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}\mathrm{d}z=0$

$$\sum_{k=-\infty}^{\infty}\mathrm{Res}_{z=\pi k/a}\frac{e^{pe^{iaz}}}{\sin(az)}\frac{1}{b^2+z^2}=\sum_{k=-\infty}^{\infty}\frac{e^{pe^{i\pi k}}}{b^2+\dfrac{\pi^2k^2}{a^2}}\frac{1}{a\cos(\pi k)}=a\sum_{k=-\infty}^{\infty}\frac{(-1)^ke^{p(-1)^k}}{b^2+\pi^2k^2}$$

$$=a\sum_{k=-\infty}^{\infty}\left(\frac{e^p}{a^2b^2+4\pi^2k^2}-\frac{e^{-p}}{a^2b^2+(2k+1)^2\pi^2}\right)=e^p\frac{\coth\left(\dfrac{ab}{2}\right)}{2b}-e^{-p}\frac{\tanh\left(\dfrac{ab}{2}\right)}{2b}$$

Therefore

$$I=\frac{\pi}{b}\left(e^p\frac{\coth\left(\dfrac{ab}{2}\right)}{4}-e^{-p}\frac{\tanh\left(\dfrac{ab}{2}\right)}{4}-\frac{e^{pe^{-ab}}}{2\sinh(ab)}\right)$$

□

Exercise 1.2.148: Evaluate:

✎ **Solution:**

$$\int_0^{\frac{\pi}{2}} \frac{xe^{-\tan^2 x}\sin(4x)}{\cos^8(x)}\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \frac{xe^{-\tan^2(x)}2\sin(x)\cos(2x)}{\cos^3(x)\cos^4(x)}\mathrm{d}x$$

let

$$u = \tan^2(x), \quad \mathrm{d}u = \frac{2\sin(x)}{\cos^3(x)}\mathrm{d}x$$

this results in:

$$\begin{aligned} 2\int_0^{\infty} e^{-u}(1-u^2)\tan^{-1}(\sqrt{u})\mathrm{d}u = 2\int_0^{\infty} e^{-u}\tan^{-1}(\sqrt{u})\mathrm{d}u \\ -2\int_0^{\infty} u^2 e^{-u}\tan^{-1}(\sqrt{u})\mathrm{d}u \end{aligned} \tag{1}$$

let $t = \sqrt{u}$, $\mathrm{d}u = 2t\mathrm{d}t$,the left one becomes:

$$\begin{aligned} 4\int_0^{\infty} te^{-t^2}\tan^{-1}(t)\mathrm{d}t = 4\int_0^{\infty}\int_0^{t} \frac{te^{-t^2}}{u^2+1}\mathrm{d}u\mathrm{d}t = 4\int_0^{\infty}\int_u^{\infty} \frac{te^{-t^2}}{u^2+1}\mathrm{d}t\mathrm{d}u \\ = 2\int_0^{\infty} \frac{e^{-u^2}}{u^2+1}\mathrm{d}u \end{aligned} \tag{2}$$

by making the same sub, we get:

$$\begin{aligned} &4\int_0^{\infty} t^5 e^{-t^2}\tan^{-1}(t)\mathrm{d}t = 4\int_0^{\infty}\int_0^{t} \frac{t^5e^{-t^2}}{u^2+1}\mathrm{d}u\mathrm{d}t = 4\int_0^{\infty}\int_u^{\infty} \frac{t^5e^{-t^2}}{u^2+1}\mathrm{d}t\mathrm{d}u \\ &= 2\int_0^{\infty} \frac{e^{-u^2}}{u^2+1}\mathrm{d}u + 2\int_0^{\infty} u^2e^{-u^2}\mathrm{d}u + 2\int_0^{\infty} e^{-u^2}\mathrm{d}u = 2\int_0^{\infty} \frac{e^{-u^2}}{u^2+1}\mathrm{d}u + \frac{3}{2}\sqrt{\pi} \end{aligned} \tag{3}$$

Hence we have

$$\int_0^{\frac{\pi}{2}} \frac{xe^{-\tan^2 x}\sin(4x)}{\cos^8(x)}\mathrm{d}x = -\frac{3}{2}\sqrt{\pi}$$

□

Exercise 1.2.149: Evaluate:

$$\int_0^1 \mathrm{li}(x)\sin(a\ln x)x^{p-1}\,\mathrm{d}x \quad a,p>0$$

$$\int_0^1 \mathrm{li}(x)\frac{x^{p-1}}{\sqrt{\ln\frac{1}{x}}} \quad p>0$$

where $\mathrm{li}(x) = \int_0^x \frac{1}{\ln t}\mathrm{d}t$

✎ **Solution:** 1.With the substitution $x = e^{-t} \Rightarrow \mathrm{d}x = -e^{-t}\,\mathrm{d}t$, the integral is

$$I(p) = -\int_0^{\infty} \mathrm{li}(e^{-t})\sin(at)e^{-pt}\,\mathrm{d}t = -\Im\left(\int_0^{\infty} \mathrm{li}(e^{-t})e^{-t(p-ia)}\,\mathrm{d}t\right)$$

Consider the following integral:

$$J(k) = \int_0^{\infty} \mathrm{li}(e^{-t})e^{-kt}\,\mathrm{d}t$$

From integration by parts:

$$J(k) = t\mathrm{li}(e^{-t})e^{-kt}\Big|_0^\infty - \int_0^\infty t\left(\mathrm{li}(e^{-t})(-ke^{-kt}) + e^{-kt}\frac{-e^{-t}}{\ln(e^{-t})}\right)\mathrm{d}t$$
$$\Rightarrow J(k) = k\int_0^\infty t\mathrm{li}(e^{-t})e^{-kt}\,\mathrm{d}t - \int_0^\infty e^{-(k+1)t}\,\mathrm{d}t$$
$$\Rightarrow J(k) = -kJ'(k) - \frac{1}{k+1} \Rightarrow J'(k) + \frac{J(k)}{k} = -\frac{1}{k(k+1)} \Rightarrow J(k) = -\frac{\ln(1+k)}{k}$$

To solve the given problem, we need $-\Im(J(p-ia))$.

$$\begin{aligned}-\Im(J(p-ia)) &= -\Im\left(-\frac{\ln(1+p-ia)}{p-ia}\right) = \Im\left(\frac{\ln(1+p-ia)}{p-ia}\right)\\ &= \frac{1}{p^2+a^2}\Im\left(\ln\left(\sqrt{(p+1)^2+a^2}e^{i\theta}\right)(p+ia)\right)\\ &= \frac{1}{p^2+a^2}\Im\left((\frac{1}{2}\ln((p+1)^2+a^2)+i\theta)(p+ia)\right)\\ &= \frac{1}{p^2+a^2}\left(\frac{a}{2}\ln((p+1)^2+a^2)+p\arctan\left(\frac{-a}{p+1}\right)\right)\end{aligned}$$
$$\Rightarrow I(p) = \frac{1}{p^2+a^2}\left(\frac{a}{2}\ln((p+1)^2+a^2) - p\arctan\left(\frac{a}{p+1}\right)\right)$$

2.Use the substitution $x = e^{-u^2} \Rightarrow \mathrm{d}x = -2ue^{-u^2}\,\mathrm{d}u$.

$$\Rightarrow I(p) = 2\int_0^\infty \mathrm{li}(e^{-u^2})e^{-pu^2}\,\mathrm{d}u$$

Divide both the sides by two and use integration by parts to obtain:

$$\frac{I(p)}{2} = u\mathrm{li}(e^{-u^2})e^{-pu^2}\Big|_0^\infty - \int_0^\infty u\left(\mathrm{li}(e^{-u^2})(-2pue^{-pu^2}) + \frac{e^{-pu^2}(-2ue^{-u^2})}{\ln(e^{-u^2})}\right)\mathrm{d}u$$
$$\Rightarrow \frac{I(p)}{4} = p\int_0^\infty u^2\mathrm{li}(e^{-u^2})e^{-pu^2}\,\mathrm{d}u - \int_0^\infty e^{-(p+1)u^2}\,\mathrm{d}u$$
$$\Rightarrow \frac{I(p)}{4} = -p\frac{I'(p)}{2} - \frac{\sqrt{\pi}}{2\sqrt{p+1}} \Rightarrow I'(p) + \frac{I(p)}{2p} = -\frac{\sqrt{\pi}}{2p\sqrt{p+1}}$$

The above is a linear differential equation and its solution is:

$$I(p) = -2\sqrt{\frac{\pi}{p}}\ln\left(\sqrt{p}+\sqrt{p+1}\right) + C$$

As p tends to infinity, $I(p)$ tends to zero, hence $C = 0$.

$$\Rightarrow I(p) = -2\sqrt{\frac{\pi}{p}}\ln\left(\sqrt{p}+\sqrt{p+1}\right)$$

□

Exercise 1.2.150: Evaluate:

$$\int_0^\infty \frac{x}{x^2+b^2}\ln\left(\frac{x^2+2ax\cos(t)+a^2}{x^2-2ax\cos(t)+a^2}\right)\mathrm{d}x$$

Solution: Let

$$I(t) = \int_0^\infty \frac{x}{x^2+b^2}\ln\left(\frac{x^2+2ax\cos(t)+a^2}{x^2-2ax\cos(t)+a^2}\right)\mathrm{d}x$$

$$I'(t)=\int_0^\infty \frac{x}{x^2+b^2}\left(\frac{-2ax\sin(t)}{x^2+2ax\cos(t)+a^2}-\frac{2ax\sin(t)}{x^2-2ax\cos(t)+a^2}\right)\mathrm{d}x$$

$$\begin{aligned}\Rightarrow I'(t) &= -2a\int_0^\infty \frac{x}{x^2+b^2}\left(\frac{x\sin(t)}{x^2+2ax\cos(t)+a^2}+\frac{x\sin(t)}{x^2-2ax\cos(t)+a^2}\right)\mathrm{d}x\\ &= -2a\,\Im\left(\int_0^\infty \frac{x}{x^2+b^2}\left(\frac{e^{it}}{x+ae^{it}}+\frac{e^{it}}{x-ae^{it}}\right)\mathrm{d}x\right) \qquad (1)\\ &= -2a\,\Im\left(2e^{it}\int_0^\infty \frac{x^2}{(x^2+b^2)(x^2-a^2e^{2it})}\,\mathrm{d}x\right)=-2a\,\Im\left(2e^{it}\frac{\pi}{2(b-iae^{it})}\right) \qquad (2)\\ &= -2\pi a\,\Im\left(\frac{e^{it}}{b-iae^{it}}\right)=-2\pi a\,\frac{b\sin t+a}{b^2+2ab\sin t+a^2} \qquad (3)\\ &= -\pi\left(1+\frac{a^2-b^2}{b^2+2ab\sin(t)+a^2}\right)\end{aligned}$$

$$\Rightarrow I(t)=-\pi t-2\pi\arctan\left(\frac{(a^2+b^2)\tan(t/2)+2ab}{a^2-b^2}\right)+C \qquad (4)$$

Since $I(\pi/2)=0$, $C=\dfrac{\pi^2}{2}+2\pi\arctan\left(\dfrac{a+b}{a-b}\right)$

$$\Rightarrow I(t)=\frac{\pi^2}{2}-\pi t-2\pi\arctan\left(\frac{(a^2+b^2)\tan(t/2)+2ab}{a^2-b^2}\right)+2\pi\arctan\left(\frac{a+b}{a-b}\right)$$

The above expression can be shown equivalent to as shown in OP by simplifying arctangents:

$$\begin{aligned}&-2\pi\left(\arctan\left(\frac{(a^2+b^2)\tan(t/2)+2ab}{a^2-b^2}\right)-\arctan\left(\frac{a+b}{a-b}\right)\right)\\ &=2\pi\arctan\left(\frac{a-b}{a+b}\frac{1-\sin(t)}{\cos t}\right)=\pi\arctan\left(\frac{(a^2-b^2)\cos(t)}{(a^2+b^2)\sin(t)+2ab}\right)\end{aligned}$$

Hence

$$I(t)=\frac{\pi^2}{2}-\pi t+\pi\arctan\left(\frac{(a^2-b^2)\cos(t)}{(a^2+b^2)\sin(t)+2ab}\right)$$

Proof of (1):

$$\begin{aligned}\frac{x\sin t}{x^2+2ax\cos(t)+a^2}&=\frac{1}{2i}\frac{x(e^{it}-e^{-it})}{x^2+axe^{it}+axe^{-it}+a^2}=\frac{1}{2i}\left(\frac{e^{it}}{x+ae^{it}}-\frac{e^{-it}}{x+ae^{-it}}\right)\\ &=\Im\left(\frac{e^{it}}{x+ae^{it}}\right)\quad\left(\Im(z)=\frac{1}{2i}(z-\bar z)\right)\end{aligned}$$

Similarly it can be shown that:

$$\frac{x\sin t}{x^2-2ax\cos(t)+a^2}=\Im\left(\frac{e^{it}}{x-ae^{it}}\right)$$

Proof for (2):

$$\begin{aligned}&\int_0^\infty \frac{x^2}{(x^2+b^2)(x^2-a^2e^{2it})}\,\mathrm{d}x=\frac{1}{b^2+a^2e^{2it}}\int_0^\infty\left(\frac{x^2}{x^2+b^2}-\frac{x^2}{x^2-a^2e^{2it}}\right)\mathrm{d}x\\ &=-\frac{1}{b^2+a^2e^{2it}}\int_0^\infty\left(\frac{b^2}{x^2+b^2}+\frac{a^2e^{2it}}{x^2-a^2e^{2it}}\right)\mathrm{d}t=\frac{1}{b^2+a^2e^{2it}}\left(\frac{\pi b}{2}+\frac{\pi iae^{it}}{2}\right)\\ &=\frac{\pi}{2}\frac{(b+iae^{it})}{(b+iae^{it})(b-ia^{it})}=\frac{\pi}{2(b-iae^{it})}\end{aligned}$$

Proof for (3):

$$\begin{aligned}\Im\left(\frac{e^{it}}{b-iae^{it}}\right) &= \Im\left(\frac{\cos(t)+i\sin(t)}{(b+a\sin(t))-ia\cos(t)}\right)\\ &= \Im\left(\frac{(cos(t)+i\sin(t))(b+a\sin(t)+ia\cos(t))}{(b+\sin(t))^2+a^2\cos^2(t)}\right)\\ &= \frac{b\sin(t)+a}{b^2+2ab\sin(t)+a^2}\end{aligned}$$

Proof for (4):

$$\begin{aligned}&\int\frac{a^2-b^2}{b^2+2ab\sin(t)+a^2}\,\mathrm{d}t=(a^2-b^2)\int\frac{\sec^2(t/2)}{(a^2+b^2)(1+\tan^2(t/2))+4ab\tan(t/2)}\,\mathrm{d}t\\ &=2(a^2-b^2)\int\frac{\mathrm{d}p}{(a^2+b^2)(1+p^2)+4abp}=2\frac{a^2-b^2}{a^2+b^2}\int\frac{\mathrm{d}p}{p^2+\frac{4abp}{a^2+b^2}+1}\\ &=2\frac{a^2-b^2}{a^2+b^2}\int\frac{\mathrm{d}p}{\left(p+\frac{2ab}{a^2+b^2}\right)^2+\left(\frac{a^2-b^2}{a^2+b^2}\right)^2}=2\arctan\left(\frac{(a^2+b^2)\tan(t/2)+2ab}{a^2-b^2}\right)+C\end{aligned}$$

□

Exercise 1.2.151: Prove:

$$\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{\sin^{2n+2}(x)}\frac{1}{e^{2\pi\cot x}-1}\mathrm{d}x=(-1)^{n-1}\frac{2n-1}{4(2n+1)}$$

Proof: Notice that

$$\frac{\sin(2nx)}{\sin^{2n}(x)}=\sum_{r=1}^{n}\binom{2n}{2r-1}(-1)^{r-1}\cot^{2n-2r+1}(x)$$

Let I denote the integral. Now use the above identity:

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{\sin^{2n+2}(x)}\frac{1}{e^{2\pi\cot x}-1}\mathrm{d}x\\ &=\int_0^{\frac{\pi}{2}}\left(\sum_{r=1}^{n}\binom{2n}{2r-1}(-1)^{r-1}\cot^{2n-2r+1}(x)\right)\frac{\csc^2(x)}{e^{2\pi\cot x}-1}\mathrm{d}x\\ &=\sum_{r=1}^{n}\binom{2n}{2r-1}(-1)^{r-1}\int_0^{\frac{\pi}{2}}\cot^{2n-2r+1}(x)\frac{\csc^2(x)}{e^{2\pi\cot x}-1}\mathrm{d}x\\ &=\sum_{r=1}^{n}\binom{2n}{2r-1}(-1)^{r-1}\int_0^{\infty}\frac{t^{2n-2r+1}}{e^{2\pi t}-1}\mathrm{d}t\\ &=\sum_{r=1}^{n}\binom{2n}{2r-1}(-1)^{r-1}\left\{\frac{(2n-2r+1)!\zeta(2n-2r+2)}{(2\pi)^{2n-2r+2}}\right\}\\ &=\sum_{r=1}^{n}\binom{2n}{2r-1}(-1)^{r-1}\left\{(-1)^{n-r}\frac{B_{2n-2r+2}}{n-r+1}\right\}\\ &=\frac{(-1)^{n-1}}{4}\sum_{r=1}^{n}\binom{2n}{2r-1}\frac{B_{2n-2r+2}}{n-r+1}=\frac{(-1)^{n-1}}{4}\sum_{r=1}^{n}\binom{2n}{2r-1}\frac{B_{2r}}{r}\end{aligned}\qquad(*)$$

B_{2r} is a Bernoulli number. So the problem is reduced to proving

$$\sum_{r=1}^{n}\binom{2n}{2r-1}\frac{B_{2r}}{r}=\frac{2n-1}{2n+1}$$

$$\begin{aligned}
&\sum_{r=1}^{n}\binom{2n}{2r-1}\frac{B_{2r}}{r}=\sum_{r=1}^{n}\frac{(2n)!}{(2n-2r+1)!(2r-1)!}\frac{B_{2r}}{r}=\frac{2}{2n+1}\sum_{r=1}^{n}\frac{(2n+1)!}{(2n-2r+1)!(2r)!}B_{2r}\\
&=\frac{2}{2n+1}\sum_{r=1}^{n}\binom{2n+1}{2r}B_{2r}=\frac{2}{2n+1}\left\{\sum_{r=0}^{n}\binom{2n+1}{2r}B_{2r}-\binom{2n+1}{0}B_0\right\}\\
&=\frac{2}{2n+1}\left\{\sum_{r=0}^{n}\left(\binom{2n+1}{2r}B_{2r}+\binom{2n+1}{2r+1}B_{2r+1}\right)-\binom{2n+1}{0}B_0-\binom{2n+1}{1}B_1\right\}\\
&=\frac{2}{2n+1}\left\{\sum_{r=0}^{2n}\binom{2n+1}{r}B_r-\binom{2n+1}{0}B_0-\binom{2n+1}{1}B_1\right\}\\
&=\frac{2}{2n+1}\left\{0-\binom{2n+1}{0}B_0-\binom{2n+1}{1}B_1\right\}=\frac{2}{2n+1}\left\{-1+\frac{2n+1}{2}\right\}\\
&=\frac{2n-1}{2n+1}
\end{aligned}$$

Put this result back in $(*)$ to get the final answer:

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{\sin^{2n+2}(x)}\frac{1}{e^{2\pi\cot x}-1}\mathrm{d}x=\frac{(-1)^{n-1}}{4}\cdot\frac{2n-1}{2n+1}$$

□

Exercise 1.2.152: Evaluate:

$$\int_0^{\pi/2}\frac{\sin^2(ax)}{\sin^2(x)}\mathrm{d}x$$

Solution: Let $a\notin\mathbb{Z}$. Start by applying "integration by parts":

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\left(\frac{\sin(ax)}{\sin(x)}\right)^2\mathrm{d}x=\Big\{-\cot(x)\sin^2(ax)\Big\}_0^{\frac{\pi}{2}}+2a\int_0^{\frac{\pi}{2}}\cot(x)\sin(ax)\cos(ax)\mathrm{d}x\\
&=a\int_0^{\frac{\pi}{2}}\cot(x)\sin(2ax)\,\mathrm{d}x=\Big\{a\ln(\sin x)\sin(2ax)\Big\}_0^{\frac{\pi}{2}}-2a^2\int_0^{\frac{\pi}{2}}\ln(\sin x)\cos(2ax)\,\mathrm{d}x\\
&=-2a^2\int_0^{\frac{\pi}{2}}\ln(\sin x)\cos(2ax)\,\mathrm{d}x
\end{aligned}$$

Now use the following series:

$$\ln(\sin(x))=-\ln(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}$$

This gives us

$$\int_0^{\frac{\pi}{2}} \left(\frac{\sin(ax)}{\sin(x)}\right)^2 \mathrm{d}x = -2a^2 \int_0^{\frac{\pi}{2}} \left(-\ln(2) - \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k}\right) \cos(2ax)\,\mathrm{d}x$$
$$= a\sin(a\pi)\ln(2) + 2a^2 \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\frac{\pi}{2}} \cos(2kx)\cos(2ax)\,\mathrm{d}x$$
$$= a\sin(a\pi)\ln(2) + a^3 \sin(a\pi) \sum_{k=1}^{\infty} \frac{1}{k} \frac{(-1)^k}{a^2-k^2}$$
$$= a\sin(a\pi)\ln(2) + \left(-a\sin(a\pi)\ln(2) + \frac{a}{2}\sin(a\pi) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{-a+k} + \frac{a}{2}\sin(a\pi) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{a+k}\right)$$
$$= \frac{a}{2}\sin(a\pi) \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{-a+k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{a+k}\right)$$
$$= -\frac{a\sin(a\pi)}{4} \left\{\psi_0\left(\frac{1+a}{2}\right) - \psi_0\left(\frac{a}{2}\right) + \psi_0\left(\frac{1-a}{2}\right) - \psi_0\left(-\frac{a}{2}\right)\right\}$$
$$= \frac{\pi a}{2} - \frac{\sin(\pi a)}{2} - \frac{a\sin(\pi a)}{2} \left[\psi\left(\frac{1+a}{2}\right) - \psi\left(1+\frac{a}{2}\right)\right]$$

□

Exercise 1.2.153: Evaluate:

$$\int_0^{\infty} \sin(x^2)\mathrm{erfc}^2(x)\,\mathrm{d}x$$

Solution: Let I denote the integral. Then

$$I = \int_0^{\infty}\int_1^{\infty} \frac{2x}{\sqrt{\pi}} \sin(x^2) e^{-u^2x^2} \mathrm{erfc}(x)\,\mathrm{d}u\mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_1^{\infty}\int_0^{\infty} 2x\sin(x^2) e^{-u^2x^2} \mathrm{erfc}(x)\,\mathrm{d}x\mathrm{d}u$$
$$= \frac{1}{\sqrt{\pi}} \int_1^{\infty} \left[-\frac{e^{-u^2x^2}}{u^4+1}(\cos(x^2)+u^2\sin(x^2))\cdot\mathrm{erfc}(x)\right]_0^{\infty} \mathrm{d}u$$
$$- \frac{2}{\pi}\int_1^{\infty}\int_0^{\infty} \frac{e^{-(u^2+1)x^2}}{u^4+1}(\cos(x^2)+u^2\sin(x^2))\,\mathrm{d}x\mathrm{d}u$$
$$= I_1 - I_2$$

Surprisingly, the former integral evaluates as

$$I_1 = \frac{1}{\sqrt{\pi}} \int_1^{\infty} \frac{\mathrm{d}u}{u^4+1} = \frac{\pi - 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}} = \frac{\pi - \cosh^{-1}(3)}{4\sqrt{2\pi}}$$

which is nothing but the value of the first integral. For the second integral, notice that

$$I_2 = \frac{2}{\pi}\Im \int_1^{\infty}\int_0^{\infty} \frac{e^{-(u^2-i+1)x^2}}{u^2-i}\,\mathrm{d}x\mathrm{d}u = \frac{1}{\pi}\Im \int_{1-i}^{\infty}\int_0^{\infty} \frac{e^{-(z+1)x^2}}{z\sqrt{z+i}}\,\mathrm{d}x\mathrm{d}z$$
$$= \frac{1}{2\sqrt{\pi}}\Im \int_{1-i}^{\infty} \frac{\mathrm{d}z}{z\sqrt{z+i}\sqrt{z+1}}$$

Since the last one has antiderivative

$$\int \frac{\mathrm{d}z}{z\sqrt{z+i}\sqrt{z+1}} = -\sqrt{2}(1-i)\tanh^{-1}\left(\frac{z+\sqrt{(z+1)(z+i)}}{\sqrt{2}z+\sqrt{i}}\right) + C$$

(where both are analytic near the path of integration $z = u^2 - i$, $(u \geq 1)$), the calculation is doable, and using the power of symbolic calculation software (aka Mathematica), we obtain

$$I = \frac{1}{4\sqrt{2\pi}} \left\{ \pi - 2\cosh^{-1}(3) + \cosh^{-1}(2+\sqrt{5}) - \tan^{-1}\left(2\sqrt{2+\sqrt{5}}\right) \right\}$$

□

Exercise 1.2.154: Evaluate:

$$\int_0^\infty \arctan\left(\frac{2\theta\,(1+\theta x)}{x^2(1+\theta^2)+2\theta x+1-\theta^2}\right)\frac{\mathrm{d}x}{1+x^2}$$

Solution: **Step 1.** Let $\theta = \tan(\alpha/2)$ with $0 < \alpha < \pi$ Then the given integral reduces to

$$I(\alpha) = \int_0^\infty \arctan\left(\frac{(1-\cos\alpha)x+\sin\alpha}{x^2+x\sin\alpha+\cos\alpha}\right)\frac{\mathrm{d}x}{x^2+1}$$

We first analyze the property of the integrand.

- For $0 < \alpha < \pi/2$, the argument of the arctan is always non-negative and thus is smooth.
- For $\pi/2 < \alpha < \pi$, $\cos\alpha < 0$ and thus the denominator of the argument has exactly one positive zero, namely $x(\alpha)$. Then the arctan part has jump of size π at $x = x(\alpha)$.

So if we set

$$x(\alpha) = \begin{cases} 0, & \alpha \in [0, \pi/2] \\ \text{unique positive zero of } x^2 + x\sin\alpha + \cos\alpha = 0, & \alpha \in (\pi/2, \pi) \end{cases}$$

then we may write

$$\begin{aligned} I(\alpha) &= \int_0^\infty \left\{ \arctan\left(\frac{(1-\cos\alpha)x+\sin\alpha}{x^2+x\sin\alpha+\cos\alpha}\right) - \pi \mathbf{1}_{[0,x(\alpha))}(x) \right\} \frac{\mathrm{d}x}{x^2+1} \\ &= J(\alpha) - \pi\arctan(x(\alpha)) \end{aligned}$$

where

$$J(\alpha) = \int_0^\infty \arctan\left(\frac{(1-\cos\alpha)x+\sin\alpha}{x^2+x\sin\alpha+\cos\alpha}\right)\frac{\mathrm{d}x}{x^2+1}$$

and arctan is the inverse of the tangent function on $(0, \pi)$
so that the function $x \mapsto \arctan\left(\frac{(1-\cos\alpha)x+\sin\alpha}{x^2+x\sin\alpha+\cos\alpha}\right)$ is continuous and vanishes as $x \to \infty$.
Step 2. Now, in order to calculate $J(\alpha)$ we differentiate it. Then

$$J'(\alpha) = \int_0^\infty \frac{x\sin\alpha+1}{x^2+2x\sin\alpha+1}\frac{\mathrm{d}x}{x^2+1}$$

Applying partial fraction and integrating, we obtain a surprisingly simple answer $J'(\alpha) = \frac{\pi}{4}$. Since $J(0) = 0$, this implies $J(\alpha) = \frac{\pi\alpha}{4}$. Therefore we have

$$I(\alpha) = \frac{\pi\alpha}{4} - \pi\arctan(x(\alpha))$$

Step 3. Plugging the substitution $\theta = \tan(\alpha/2)$ and express everything in terms of θ, the final answer is

$$\int_0^\infty \arctan\left(\frac{2\theta\,(1+\theta x)}{x^2(1+\theta^2)+2\theta x+1-\theta^2}\right)\frac{\mathrm{d}x}{1+x^2}$$
$$= \begin{cases} \dfrac{\pi}{2}\arctan\theta, & 0\le\theta\le 1, \\ \dfrac{\pi}{2}\arctan\theta-\pi\arctan\left(\dfrac{\sqrt{\theta^4+\theta^2-1}-\theta}{\theta^2+1}\right), & \theta>1 \end{cases}$$

□

Exercise 1.2.155: Evaluate:

$$\int_0^1 \ln\left(\frac{1+2x\cos(a)+x^2}{1-2x\cos(a)+x^2}\right)\cdot\frac{1}{x}\mathrm{d}x$$

Solution: You may write

$$I=\int_0^1 \ln\left(\frac{x^2+2x\cos\alpha+1}{x^2-2x\cos\alpha+1}\right)\cdot\frac{\mathrm{d}x}{x}=\int_{-1}^1 \ln(x^2+2x\cos\alpha+1)\cdot\frac{\mathrm{d}x}{x}$$
$$=2\Re\int_{-1}^1 \ln(1+xe^{i\alpha})\cdot\frac{\mathrm{d}x}{x}$$

Now shifting the contour to the upper semicircular arc $x=e^{i(\pi-t)}$, $(0\le t\le\pi)$, we have

$$I=2\Re\int_0^\pi -i\ln(1-e^{i(\alpha-t)})\,\mathrm{d}t=\int_{\alpha-\pi}^{\alpha}2\Im\ln(1-e^{it})\,\mathrm{d}t$$

By noting that

$$2\Im\ln(1-e^{it})=\begin{cases} t+\pi, & -\pi<t<0 \\ t-\pi, & 0<t<\pi \end{cases}$$

for $0<\alpha<\pi$ we obtain the desired result. For general α, we can also prove that

$$\int_0^1 \ln\left(\frac{x^2+2x\cos\alpha+1}{x^2-2x\cos\alpha+1}\right)\cdot\frac{\mathrm{d}x}{x}=\frac{\pi^2}{2}-\pi|\alpha|\ ,\ \text{for } |\alpha|\le\pi \text{ and extended by periodicity}$$

□

Exercise 1.2.156: Evaluate:

$$\int_0^{\frac{\pi}{2}} x^2\sqrt{\tan x}\sin(2x)\,\mathrm{d}x$$

Solution: **Method One:**

$$\int_0^{\frac{\pi}{2}} x^2\sqrt{\tan x}\sin(2x)\,\mathrm{d}x=2\Im\int_0^{\frac{\pi}{2}} x^2e^{ix}\sqrt{\sin x\cos x}\,\mathrm{d}x$$

So consider the following integral:

$$J(\alpha)=\int_0^{\frac{\pi}{2}} e^{i\alpha x}\sqrt{\sin x\cos x}\,\mathrm{d}x$$

Note that the integral we seek is

$$\int_0^{\frac{\pi}{2}} x^2\sqrt{\tan x}\sin(2x)\,\mathrm{d}x=-2\Im\left[\frac{\partial^2}{\partial\alpha^2}J(\alpha)\right]_{\alpha=1}$$

It turns out that there is, in fact, a closed for for $J(\alpha)$:

$$J(\alpha) = -\frac{\left(\frac{1}{16}+\frac{i}{16}\right)\sqrt{\frac{\pi}{2}}\left(e^{\frac{i\pi a}{2}}-i\right)\Gamma\left(\frac{a-1}{4}\right)}{\Gamma\left(\frac{a+5}{4}\right)}$$

EDIT: The integral is even nicer when we consider

$$J(\alpha) = \int_0^{\frac{\pi}{2}} \sin\alpha x\sqrt{\sin x\cos x}\,\mathrm{d}x$$

Then

$$J(\alpha) = \frac{\pi^{3/2}\sin\left(\frac{\pi a}{4}\right)}{8\Gamma\left(\frac{5-a}{4}\right)\Gamma\left(\frac{5+a}{4}\right)}$$

and

$$\left[\frac{\partial^2}{\partial\alpha^2}J(\alpha)\right]_{\alpha=1} = \frac{\pi\left(-5\pi^2+6\pi(\ln(4)-2)+3(8+(\ln(4)-4)\ln(4))\right)}{192\sqrt{2}}$$

The integral we seek is then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2\sqrt{\tan x}\sin(2x)\,\mathrm{d}x &= -2\left[\frac{\partial^2}{\partial\alpha^2}J(\alpha)\right]_{\alpha=1}\\ &= \frac{\pi\left(-24+12\pi+5\pi^2-3\ln^2(4)+12\ln(4)-6\pi\ln(4)\right)}{96\sqrt{2}}\end{aligned}$$

Method Two:

Let $f(z) = \ln^2(z)\left(\frac{z-1}{z+1}\right)^{1/2}$ where $-\pi < \arg(z), \arg(z-1), \arg(z+1) \le \pi$. Now integrate around a contour that consists of the segment just above $[1,1]$ and the upper half of the unit circle, and is indented around the branch points at $z=0, z=1$, and $z=1$. The contributions from the indentations around the branch points can be shown to be vanishing small.

Then going around the contour counterclockwise,

$$\begin{aligned}\int_{-1}^{0}\ln^2(|x|+i\pi)\left(\frac{(1-x)e^{\pi i}}{1+x}\right)^{\frac{1}{2}}\mathrm{d}x\ &+\int_0^1\ln^2(x)\left(\frac{(1-x)e^{\pi i}}{1+x}\right)^{1/2}\mathrm{d}x\\ &+i\int_0^{\pi}\ln^2(e^{it})\left(\frac{e^{it}-1}{1+e^{it}}\right)^{1/2}e^{it}\,\mathrm{d}t = 0\end{aligned}$$

Looking at each piece separately,

$$\begin{aligned}\int_{-1}^{0}\ln^2(|x|+i\pi)\left(\frac{(1-x)e^{i\pi}}{1+x}\right)^{\frac{1}{2}}\mathrm{d}x &= i\int_0^1\left(\ln x+i\pi\right)^2\left(\frac{1+x}{1-x}\right)^{1/2}\mathrm{d}x\\ &= i\int_0^1\left(\ln^2(x)+2\pi i\ln x-\pi^2\right)\frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x\end{aligned}$$

$$\int_0^1\ln^2(x)\left(\frac{(1-x)e^{\pi i}}{1+x}\right)^{1/2}\mathrm{d}x = i\int_0^1\ln^2(x)\frac{1-x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$i\int_0^{\pi}\ln^2(e^{it})\left(\frac{e^{it}-1}{1-e^{it}}\right)^{1/2}e^{it}\,\mathrm{d}t = -i\int_0^{\pi}t^2\sqrt{i\tan t/2}\;e^{it}\,\mathrm{d}t = -ie^{i\pi/4}\int_0^{\pi}t^2\sqrt{\tan t/2}\;e^{it}\,\mathrm{d}t$$

So we have

$$e^{i\pi/4}\int_0^\pi t^2\sqrt{\tan t/2}\;e^{it}\;\mathrm{d}t = 2\int_0^1 \frac{\ln^2(x)}{\sqrt{1-x^2}}\;\mathrm{d}x + 2\pi i\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\;\mathrm{d}x$$
$$+ 2\pi i\int_0^1 \frac{x\ln x}{\sqrt{1-x^2}}\mathrm{d}x - \pi^2\int_0^1 \frac{1}{\sqrt{1-x^2}}\;\mathrm{d}x - \pi^2\int_0^1 \frac{x}{\sqrt{1-x^2}}\;\mathrm{d}x$$

Now we can use the fact that

$$\int_0^1 \frac{x^a}{\sqrt{1-x^2}}\;\mathrm{d}x = \frac{1}{2}\int_0^1 u^{a/2+1/2-1}(1-u)^{1/2-1}\;\mathrm{d}u = \frac{1}{2}B\left(\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)$$

to evaluate the integrals on the right.

$$\int_0^1 \frac{\ln^2(x)}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\mathrm{d}^2}{\mathrm{d}a^2}\frac{1}{2}B\left(\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)\Big|_{a=0}$$
$$= \frac{1}{8}B\left(\frac{1}{2},\frac{1}{2}\right)\left[\left(\psi\left(\frac{1}{2}\right)-\psi(1)\right)^2+\psi_1\left(\frac{1}{2}\right)-\psi_1(1)\right] = \frac{\pi^3}{24}+\frac{\ln^2 4}{8}$$

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\;\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{2}B\left(\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)\Big|_{a=0}$$
$$= \frac{1}{4}B\left(\frac{1}{2},\frac{1}{2}\right)\left(\psi\left(\frac{1}{2}\right)-\psi(1)\right) = -\frac{\pi\ln 2}{2}$$

$$\int_0^1 \frac{x\ln x}{\sqrt{1-x^2}}\;\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{2}B\left(\frac{1}{2},\frac{a}{2}+\frac{1}{2}\right)\Big|_{a=1}$$
$$= \frac{1}{4}B\left(\frac{1}{2},1\right)\left(\psi(1)-\psi\left(\frac{3}{2}\right)\right) = \ln 2 - 1$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\;\mathrm{d}x = \frac{\pi}{2}\;,\quad \int_0^1 \frac{x}{\sqrt{1-x^2}}\;\mathrm{d}x = 1$$

Therefore,

$$\int_0^\pi t^2\sqrt{\tan\frac{t}{2}}\;e^{it}\;\mathrm{d}t = e^{-i\pi/4}\left(\frac{\pi^3}{12}+\frac{\pi\ln^2 4}{4}-i\pi^2\ln 2+2\pi i\ln 2-2\pi i-\frac{\pi^3}{2}-\pi^2\right)$$

or

$$\int_0^{\pi/2} t^2\sqrt{\tan t}\;e^{2it}\;\mathrm{d}t = \frac{1}{8}e^{-i\pi/4}\left(\frac{\pi^3}{12}+\frac{\pi\ln^2 4}{4}-i\pi^2\ln 2+2\pi i\ln 2-2\pi i-\frac{\pi^3}{2}-\pi^2\right)$$

Then finally equating the imaginary parts on both sides of the equation,

$$\int_0^{\pi/2} t^2\sqrt{\tan t}\sin(2t)\;\mathrm{d}t = \frac{1}{8\sqrt{2}}\left(-\pi^2\ln 2+2\pi\ln 2-2\pi-\frac{\pi^3}{12}-\frac{\pi\ln^2 4}{4}+\frac{\pi^3}{2}+\pi^2\right)$$
$$= \frac{\pi}{96\sqrt{2}}\left(5\pi^2+12\pi-12\pi\ln 2+24\ln 2-3\ln^2 4-24\right)$$

□

Exercise 1.2.157: Evaluate:

$$\int_0^\infty \frac{x}{x^2+1}\frac{\mathrm{d}x}{\sinh(ax)}$$

✎ **Solution:** We can use the following expansion

$$\frac{1}{\sinh(ax)} = \frac{1}{ax} - 2ax\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a^2x^2+n^2\pi^2}$$

Now term by term integration gives

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^2}\frac{1}{\sinh(ax)}\mathrm{d}x &= \int_0^\infty \frac{x}{1+x^2}\left(\frac{1}{ax} - 2ax\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a^2x^2+n^2\pi^2}\right)\mathrm{d}x\\ &= \frac{\pi}{2a} - 2a\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty \frac{x^2}{(1+x^2)(a^2x^2+n^2\pi^2)}\mathrm{d}x = \frac{\pi}{2a} - \pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a+n\pi}\\ &= -\frac{\pi}{2a} + \pi\sum_{n=0}^{\infty}\frac{(-1)^n}{a+n\pi} = -\frac{\pi}{2a} + \frac{1}{2}\left\{\psi_0\left(\frac{\pi+a}{2\pi}\right) - \psi_0\left(\frac{a}{2\pi}\right)\right\}\end{aligned}$$

We can get some more interesting identities by differentiation:

$$\int_0^\infty \frac{x^2}{(1+x^2)\sinh(ax)\tanh(ax)}\mathrm{d}x = -\frac{\pi}{2a^2} - \frac{1}{4\pi}\left\{\psi_1\left(\frac{\pi+a}{2\pi}\right) - \psi_1\left(\frac{a}{2\pi}\right)\right\}$$

Upon putting $a=\pi$ gives:

$$\int_0^\infty \frac{x^2}{(1+x^2)\sinh(\pi x)\tanh(\pi x)}\mathrm{d}x = -\frac{1}{2\pi} + \frac{\pi}{12}$$

□

Exercise 1.2.158: Evaluate:

$$\int_0^1 \frac{\ln^2 x\ln(1-x)}{1+x}\mathrm{d}x$$

✎ **Solution:** Let I denote the integral in the question. Then we can write

$$\begin{aligned}I &= \int_0^1 \frac{\ln^2(x)\ln(1-x^2) - \ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x\\ &= \int_0^1 \frac{\ln^2(x)\ln(1-x^2)}{1+x}\mathrm{d}x - \int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x \qquad (1)\end{aligned}$$

The first integral can be manipulated as follows:

$$\begin{aligned}\int_0^1 \frac{\ln^2(x)\ln(1-x^2)}{1+x}\mathrm{d}x &= \frac{1}{8}\int_0^1 \frac{\ln^2(x)\ln(1-x)}{(1+\sqrt{x})\sqrt{x}}\mathrm{d}x\\ &= \frac{1}{8}\left(\int_0^1 \frac{\ln^2(x)\ln(1-x)}{(1-x)\sqrt{x}}\mathrm{d}x - \int_0^1 \frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x\right)\end{aligned}$$

Calculation of these two integrals is done by differentiating the beta function. The end result is

$$\int_0^1 \frac{\ln^2(x)\ln(1-x^2)}{1+x}\mathrm{d}x = \frac{7}{2}\zeta(3)\ln(2) - \frac{11\pi^4}{360}$$

Now we turn our attention to the second integral of equation (1):

$$\begin{aligned}\int_0^1 \frac{\ln^2(x)\ln(1+x)}{1+x}\mathrm{d}x &= \sum_{n=1}^{\infty}(-1)^{n+1}H_n\int_0^1 x^n\ln^2(x)\,\mathrm{d}x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{(n+1)^3}\\ &= 4\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{2}\zeta(3)\ln(2) - \frac{\pi^4}{24} + \frac{\ln^4(2)}{6} - \frac{1}{6}\pi^2\ln^2(2)\end{aligned}$$

Refer to this thread for the evaluation of the Euler Sum.

Substitute everything in equation (1) to get

$$I=-4\text{Li}_4\left(\frac{1}{2}\right)+\frac{\pi^4}{90}-\frac{\ln^4(2)}{6}+\frac{1}{6}\pi^2\ln^2(2)$$

□

Exercise 1.2.159: Evaluate:

$$\int_0^\infty \frac{\ln(1+x^2)}{\sqrt{(1+x^2)(a^2+x^2)}}\mathrm{d}x \quad a>1$$

Solution: Let

$$I(a)=\int_0^\infty \frac{\ln(1+x^2)}{\sqrt{(1+x^2)(a^2+x^2)}}\mathrm{d}x$$

for $a>1$. Then rewrite the integral as

$$\begin{aligned}I(a)&=\frac{1}{2}\int_1^\infty \frac{\ln x}{\sqrt{x(x-1)(x+a^2-1)}}\mathrm{d}x=\frac{1}{2}\Re\int_0^\infty \frac{\ln x}{\sqrt{x(x-1)(x+a^2-1)}}\mathrm{d}x\\&=2\Re\int_0^\infty \frac{\ln x}{\sqrt{(x^2-1)(x^2+a^2-1)}}\mathrm{d}x=\Re\left(-i\mathbf{K}(a)\ln\left(i\sqrt{a^2-1}\right)\right)\\&=\Re\left(\mathbf{K}(a)\left(\frac{\pi}{2}-\frac{i}{2}\ln(a^2-1)\right)\right)\end{aligned}$$

where $\mathbf{K}(k)$ denotes the complete elliptic integral of the first kind. Furthermore, it is known that

$$\mathbf{K}(k)=\frac{\mathbf{K}\left(\frac{1}{k}\right)+i\mathbf{K}'\left(\frac{1}{k}\right)}{k} \quad k>1$$

Thus, we obtain

$$I(a)=\Re\left\{\frac{\mathbf{K}\left(\frac{1}{a}\right)+i\mathbf{K}'\left(\frac{1}{a}\right)}{a}\left(\frac{\pi}{2}-\frac{i}{2}\ln(a^2-1)\right)\right\}=\frac{\pi}{2a}\mathbf{K}\left(\frac{1}{a}\right)+\frac{\ln(a^2-1)}{2a}\mathbf{K}'\left(\frac{1}{a}\right)$$

□

Exercise 1.2.160: Evaluate:

$$I=\int_0^\infty \frac{1}{x}\left(\frac{1}{e^x+1}-\frac{1}{e^{2x}+1}\right)\mathrm{d}x$$

Solution:

$$I(s)=\int_0^\infty \frac{x^{s-1}}{1+e^x}\,\mathrm{d}x-\frac{1}{2^s}\int_0^\infty \frac{x^{s-1}}{1+e^x}\,\mathrm{d}x=\Gamma(s)\eta(s)-\frac{\Gamma(s)\eta(s)}{2^s}=\left(1-\frac{1}{2^s}\right)\eta(s)\Gamma(s)$$

So

$$I=\lim_{s\to0}I(s)=\lim_{s\to0}\frac{(1-2^{1-s})\left(1-\frac{1}{2^s}\right)\zeta(s)}{s}=\frac{1}{2}\lim_{s\to0}\frac{1-2^{-s}}{s}=\frac{\ln(2)}{2}$$

□

Exercise 1.2.161: Evaluate:

$$\int_0^1 \frac{1-\sqrt{1-x^2}}{1+\sqrt{1+x^2}}\mathrm{d}x$$

Solution: Notice that

$$\int_0^1 \frac{1-\sqrt{1-x^2}}{1+\sqrt{1+x^2}}\mathrm{d}x = \int_0^1 \frac{\sqrt{1+x^2}-1+\sqrt{1-x^2}-\sqrt{1-x^4}}{x^2}\,\mathrm{d}x$$

$$\int_0^1 \frac{\sqrt{1+x^2}-1}{x^2}\,\mathrm{d}x = 1-\sqrt{2}+\int_0^1 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = 1-\sqrt{2}+\sinh^{-1}(1)$$

$$\int_0^1 \frac{\sqrt{1-x^2}-\sqrt{1-x^4}}{x^2}\,\mathrm{d}x = \int_0^1 \frac{1}{x}\left(\frac{-2x}{2\sqrt{1-x^2}}+\frac{4x^3}{2\sqrt{1-x^4}}\right)\mathrm{d}x$$

$$\int_0^1 \frac{2x^2}{\sqrt{1-x^4}}-\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{(2\pi)^{\frac{3}{2}}}{\Gamma^2\left(\frac{1}{4}\right)}-\frac{\pi}{2}$$

Hence

$$\int_0^1 \frac{1-\sqrt{1-x^2}}{1+\sqrt{1+x^2}}\mathrm{d}x = 1-\sqrt{2}+\sinh^{-1}(1)+\frac{\sqrt{8\pi^3}}{\Gamma^2\left(\frac{1}{4}\right)}-\frac{\pi}{2}$$

□

Exercise 1.2.162: Evaluate:

$$\int_0^\infty \frac{1}{\sinh^{1/4}(x)\cosh(x)}\mathrm{d}x$$

Solution: For $b>a>1$,

$$\int_0^\infty \frac{\sinh^a x}{\cosh^b x}\,\mathrm{d}x = \int_0^\infty \frac{(\cosh^2(x)-1)^{a/2}}{\cosh^b x}\,\mathrm{d}x$$

Let $u=\cosh x$.

$$=\int_1^\infty \frac{(u^2-1)^{a/2}}{u^b}\frac{\mathrm{d}u}{\sqrt{u^2-1}} = \int_1^\infty u^{-b}(u^2-1)^{a/2-1/2}\,\mathrm{d}u = \int_1^\infty u^{a-b-1}\left(1-\frac{1}{u^2}\right)^{a/2-1/2}\,\mathrm{d}u$$

Let $w=\frac{1}{u^2}$.

$$=\frac{1}{2}\int_0^1 w^{b/2-a/2+1/2}(1-w)^{a/2+1/2}\frac{\mathrm{d}w}{w^{3/2}} = \frac{1}{2}\int_0^1 w^{b/2-a/2-1}(1-w)^{a/2-1/2}\,\mathrm{d}w$$
$$=\frac{1}{2}\frac{\Gamma\left(\frac{b}{2}-\frac{a}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}$$

So

$$\int_0^\infty \frac{1}{\sinh^{1/4}(x)\cosh x}\,\mathrm{d}x = \frac{1}{2}\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{3}{8}\right) = \frac{\pi\csc\left(\frac{3\pi}{8}\right)}{2} = \frac{\pi\sqrt{4-2\sqrt{2}}}{2}$$

□

Exercise 1.2.163: Prove:

$$\int_0^\infty \frac{\cos(2nx)}{(1+x^2)^{m+1}}\mathrm{d}x = \frac{\pi n^m e^{-2n}}{2m!}\sum_{k=0}^{m}\frac{(m+k)!}{(4n)^k(m-k)!k!}$$

where $n>0$ and m is a non negative integer.

Proof: Consider

$$f(z)=\frac{e^{2niz}}{(z+i)^{n+1}(z-i)^{n+1}}$$

use a typical semicircle in the UHP with $z=i$ the only pole in the contour. Of course, it has order $n+1$. So, we take n derivatives to find the residue at $z=i$.

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\frac{e^{2niz}}{(z+i)^{m+1}}\right] = \frac{(2ni)^m e^{2niz}}{(z+i)^{m+1}} - m(2ni)^{m-1}\frac{e^{2niz}(m+1)}{(z+i)^{m+1}}+\cdots + \frac{e^{2niz}(-1)^m(m+1)(m+2)\cdots(2m)}{(z+i)^{2m+1}}$$

This is the tedious part, using the binomial expansion. multiply by $\frac{2\pi i}{m!}$ and let $z=i$ due to the pole at i.

$$\frac{2\pi e^{-2n}}{m!}\left[\frac{(2n)^m}{2^{m+1}}+\frac{m(m+1)}{1}\cdot\frac{(2n)^{m-1}}{2^{m+2}}+\frac{(m+2)(m+1)m(m-1)}{2!}\cdot\frac{(2n)^{m-2}}{2^{m+3}}+\cdots+\frac{(2m)!}{m!}\cdot\frac{1}{2^{2m+1}}\right]$$

along the contour we have:

$$\int_{-\infty}^0\frac{e^{2nix}}{(x^2+1)^{m+1}}\mathrm{d}x+\int_0^\infty\frac{e^{2nix}}{(x^2+1)^{m+1}}\mathrm{d}x+\int_0^\pi\frac{e^{2niRe^{i\theta}}iRe^{i\theta}}{(1+R^2e^{2i\theta})^{m+1}}\mathrm{d}\theta$$

The third one goes to 0 as $R\to\infty$ due to the usual $e^{-2nR\sin\theta}$ thing. Let $x\to -x$ in the first one and add to the second and get:

$$\begin{aligned}\int_0^\infty\frac{\cos(2nx)}{(x^2+1)^{m+1}}\mathrm{d}x &= \frac{\pi}{m!}\cdot\frac{e^{-2n}}{2\cdot 4^m}\left[(4n)^m+\frac{m(m+1)}{1}\cdot(4n)^{m-1}\right.\\ &\quad\left.+\frac{(m+2)(m+1)m}{2!}\cdot(4n)^{m-2}+\cdots+\frac{(2m)!}{m!}\right]\\ &= \frac{\pi n^m e^{-2n}}{2m!}\sum_{k=0}^{m}\frac{(m+k)!}{k!(m-k)!(4n)^k}\end{aligned}$$

□

Exercise 1.2.164: Evaluate:

$$\int_{-\infty}^\infty \frac{\exp\left(\frac{ax}{1+x^2}\right)\sin\left(\frac{a}{1+x^2}\right)}{1+x^2}\,\mathrm{d}x\quad(a>0)$$

✎ **Solution:** Anyway, note that

$$I = \int_{-\infty}^{\infty} \frac{\exp\left(\dfrac{ax}{x^2+1}\right)\sin\left(\dfrac{a}{x^2+1}\right)}{x^2+1}\,\mathrm{d}x = -\Im\int_{-\infty}^{\infty} \frac{\exp\left(a\dfrac{x-i}{x^2+1}\right)}{x^2+1}\,\mathrm{d}x$$
$$= -\Im\int_{-\infty}^{\infty} \frac{\exp\left(\dfrac{a}{x+i}\right)}{x^2+1}\,\mathrm{d}x$$

Now apply the substitution $\dfrac{1}{x+i} = \dfrac{z+1}{2i}$, or equivalently, $x = i\dfrac{1-z}{1+z}$. This maps the real line $\mathbb{R}$ to the unit circle with counter-clockwise orientation. So by the Cauchy integration formula, we have

$$I = \Im\frac{i}{2}\oint_{|z|=1} \frac{e^{-\frac{1}{2}ia(z+1)}}{z}\,\mathrm{d}z = \Im\left(-\pi e^{-\frac{ia}{2}}\right) = \pi\sin\frac{a}{2}$$

□

Exercise 1.2.165: Evaluate:

$$\int_0^{+\infty} \Gamma(a+ix)\,\Gamma(a-ix)\,\mathrm{d}x$$

✎ **Solution:** There is a theorem similar to Parseval's theorem for the Fourier transform that states that if $f(x)$ is square integrable, then

$$\int_0^{\infty} \frac{|f(x)|^2}{x}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(it)|^2\,\mathrm{d}t$$

where $F(s)$ is the Mellin transform of $f(x)$.
Recognizing that if $f(x) = x^a e^{-x}$ then $F(s) = \Gamma(a+s)$,

$$\int_0^{\infty} x^{2a-1}e^{-2x}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Gamma(a+it)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+it)\Gamma(a-it)\,\mathrm{d}t$$

which implies

$$\int_0^{\infty} \Gamma(a+it)\Gamma(a-it)\,\mathrm{d}t = \pi\int_0^{\infty} x^{2a-1}e^{-2x}\,\mathrm{d}x$$
$$= \pi\frac{\Gamma(2a)}{2^{2a}} = \pi\frac{2^{2a-1}\Gamma(a)\Gamma\left(a+\dfrac{1}{2}\right)}{2^{2a}\sqrt{\pi}} = \frac{\sqrt{\pi}}{2}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)$$

□

Exercise 1.2.166: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\cos^2 x}\mathrm{d}x$$

✎ **Solution:** Notice that

$$1 + 2\sum_{k=1}^{\infty} a^k\cos kx = \frac{1-a^2}{1-2a\cos x+a^2} \qquad |a|<1$$

Therefore

$$\int_0^\pi \frac{x^2}{1-2a\cos x+a^2}\mathrm{d}x = \frac{1}{1-a^2}\left(\int_0^\pi x^2\mathrm{d}x + 2\int_0^\pi x^2\sum_{k=1}^{\infty}\cos kx\mathrm{d}x\right)$$
$$= \frac{1}{1-a^2}\left(\frac{\pi^3}{3} + 2\sum_{k=1}^{\infty}a^k\int_0^\pi x^2\cos kx\mathrm{d}x\right) = \frac{1}{1-a^2}\left(\frac{\pi^3}{3} + 2\sum_{k=1}^{\infty}a^k\frac{2\pi(-1)^k}{k^2}\right)$$
$$= \frac{1}{1-a^2}\left(\frac{\pi^3}{3} + 4\pi\mathrm{Li}_2(-a)\right)$$

And

$$\int_0^\pi \frac{x^2}{1-2a\cos x+a^2}\mathrm{d}x = \frac{1}{1+a^2}\int_0^\pi \frac{x^2}{1-\frac{2a}{1+a^2}\cos x}\mathrm{d}x$$

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\cos^2 x}\mathrm{d}x = \frac{1}{12}\int_0^\pi \frac{x^2}{1+\frac{1}{3}\cos x}\mathrm{d}x$$

So let $\frac{-2a}{1+a^2} = \frac{1}{3}$ $|a|<1$, gives $a = 2\sqrt{2}-3$, that we obtain

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\cos^2 x}\mathrm{d}x = \frac{\pi}{2\sqrt{2}}\mathrm{Li}_2(3-2\sqrt{2}) + \frac{\pi^3}{24\sqrt{2}}$$

□

Exercise 1.2.167: Evaluate:

$$\int_0^\infty \frac{(\tan^{-1}(x))^2}{x^4-6x^2+25}\mathrm{d}x$$

Solution: Since the integrand is even, we can write

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(\tan^{-1}(x))^2}{x^4-6x^2+25}\mathrm{d}x$$

Using the equation (1) here, the integral can be written as

$$I = -\frac{1}{8}\int_{-\infty}^{\infty} \frac{\ln^2(1+ix)+\ln^2(1-ix)-2\ln(1+ix)\ln(1-ix)}{x^4-6x^2+25}\mathrm{d}x$$

The logarithms are defined on their main sheets. Now, each of these integrals can be evaluated using the sand dollar contours. (For more details see here). We have

$$\int_{-\infty}^{\infty} \frac{\ln^2(1+ix)}{x^4-6x^2+25}\mathrm{d}x = 2\pi i\left(\mathrm{res}_{z=-2+i}\frac{\ln^2(1+iz)}{z^4-6z^2+25} + \mathrm{res}_{z=2+i}\frac{\ln^2(1+iz)}{z^4-6z^2+25}\right)$$
$$+\int_{e^{i\pi/2}}^{\infty e^{i\pi/2}} \frac{(\ln|1+ix|+i\pi)^2-(\ln|1+ix|-i\pi)^2}{x^4-6x^2+25}\mathrm{d}x$$
$$= 2\pi i\left(\frac{1}{80}i\left(\pi^2-4\ln^2(2)-2\pi\ln(2)\right)\right) + 4\pi i\int_{e^{i\pi/2}}^{\infty e^{i\pi/2}} \frac{\ln|1+ix|}{x^4-6x^2+25}\mathrm{d}x$$
$$= -\frac{\pi\left(\pi^2-4\ln^2(2)-2\pi\ln(2)\right)}{40} - 4\pi\int_1^\infty \frac{\ln(x-1)}{x^4+6x^2+25}\mathrm{d}x \qquad (1)$$

The integral $\int_1^\infty \frac{\ln(x-1)}{x^4+6x^2+25}\mathrm{d}x$ can be evaluated using Polylogarithms. It is rather tedious, so I will not show the entire calculation.

$$\int_1^\infty \frac{\ln(x-1)}{x^4+6x^2+25}\mathrm{d}x = -\frac{3\pi^2}{640}+\frac{7}{320}\pi\ln(2)-\frac{\ln^2(2)}{32}$$

Putting this in equation (1), we get

$$\int_{-\infty}^\infty \frac{\ln^2(1+ix)}{x^4-6x^2+25}\mathrm{d}x = -\frac{\pi^3}{160}+\frac{9}{40}\pi\ln^2(2)-\frac{3}{80}\pi^2\ln(2)$$

The second integral is clearly equal to the first one.

$$\int_{-\infty}^\infty \frac{\ln^2(1-ix)}{x^4-6x^2+25}\mathrm{d}x = \int_{-\infty}^\infty \frac{\ln^2(1+ix)}{x^4-6x^2+25}\mathrm{d}x = -\frac{\pi^3}{160}+\frac{9}{40}\pi\ln^2(2)-\frac{3}{80}\pi^2\ln(2)$$

To evaluate the third integral we run the same trick again:

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{\ln(1+ix)\ln(1-ix)}{x^4-6x^2+25}\mathrm{d}x \\
&= 2\pi i\left(\mathrm{res}_{z=-2+i}\frac{\ln(1+iz)\ln(1-iz)}{z^4-6z^2+25}+\mathrm{res}_{z=2+i}\frac{\ln(1+iz)\ln(1-iz)}{z^4-6z^2+25}\right) \\
&\quad+\int_{e^{i\pi/2}}^{\infty e^{i\pi/2}}\frac{(\ln|1+ix|+i\pi)-(\ln|1+ix|-i\pi)}{x^4-6x^2+25}\ln(1-ix)\mathrm{d}x \\
&= 2\pi\left(\frac{\pi^2}{160}+\frac{3}{40}\ln^2(2)+\frac{\pi\ln(2)}{80}\right)+2\pi i\int_{e^{i\pi/2}}^{\infty e^{i\pi/2}}\frac{\ln(1-ix)}{x^4-6x^2+25}\mathrm{d}x \\
&= \pi\left(\frac{\pi^2}{80}+\frac{3}{20}\ln^2(2)+\frac{\pi\ln(2)}{40}\right)-2\pi\int_1^\infty\frac{\ln(1+x)}{x^4+6x^2+25}\mathrm{d}x
\end{aligned}$$

Since,

$$\int_1^\infty \frac{\ln(1+x)}{x^4+6x^2+25}\mathrm{d}x = \frac{\mathbf{G}}{20}-\frac{\pi^2}{240}-\frac{\ln^2(2)}{40}+\frac{\pi\ln(2)}{40}$$

we get

$$\int_{-\infty}^\infty \frac{\ln(1+ix)\ln(1-ix)}{x^4-6x^2+25}\mathrm{d}x = -\frac{\mathbf{G}\pi}{10}+\frac{\pi^3}{48}-\frac{\pi^2\ln(2)}{40}+\frac{\pi}{5}\ln^2(2)$$

Combine all this crap to get

$$\begin{aligned}
I &= \frac{1}{4}\left(-\frac{\mathbf{G}\pi}{10}+\frac{\pi^3}{48}-\frac{\pi^2\ln(2)}{40}+\frac{\pi}{5}\ln^2(2)+\frac{\pi^3}{160}-\frac{9}{40}\pi\ln^2(2)+\frac{3}{80}\pi^2\ln(2)\right) \\
&= \frac{13\pi^3}{1920}-\frac{\pi\ln^2(2)}{160}+\frac{\pi^2\ln(2)}{320}-\frac{\pi}{40}\mathbf{G}
\end{aligned}$$

□

Exercise 1.2.168: Evaluate:

$$\int_{-\infty}^\infty \frac{\sin(ax)}{\sinh(x)}\mathrm{d}x$$

✎ Solution:

$$\int_0^\infty \frac{\sin(ax)}{\sinh(x)}\mathrm{d}x = 2\int_0^\infty \frac{e^{-x}\sin(ax)}{1-e^{-2x}}\mathrm{d}x = 2\sum_{n=0}^{\infty}\int_0^\infty e^{-(2n+1)x}\sin(ax)\,\mathrm{d}x$$
$$= 2a\sum_{n=0}^{\infty}\frac{1}{a^2+(2n+1)^2} = \frac{a}{2}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{ia+1}{2}\right)\left(n+\frac{1-ia}{2}\right)}$$
$$= \frac{i}{2}\left\{\psi_0\left(\frac{1-ia}{2}\right)-\psi_0\left(\frac{1+ia}{2}\right)\right\} = -\frac{i}{2}\pi\cot\left(\pi\left(\frac{1-ia}{2}\right)\right)$$
$$= -\frac{i\pi}{2}\tan\frac{i\pi a}{2} = \frac{\pi}{2}\tanh\frac{\pi a}{2}$$

□

☞ Exercise 1.2.169: Prove:

$$\int_0^\infty \frac{\cos(x)}{x}\left(\int_0^x \frac{\sin t}{t}\mathrm{d}t\right)^2\mathrm{d}x = -\frac{7}{6}\zeta(3)$$

☞ Proof: 1. Reduction to Logarithmic Integrals

$$I = \int_0^\infty \frac{\cos(x)}{x}\mathrm{Si}(x)^2\,\mathrm{d}x = \int_0^\infty\int_0^1\int_0^1 \frac{\cos(x)\sin(yx)\sin(zx)}{xyz}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$
$$= \int_0^1\int_0^1\frac{1}{yz}\int_0^\infty \frac{\cos(x)\sin(yx)\sin(zx)}{x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \frac{1}{4}\int_0^1\int_0^1\frac{1}{yz}\int_0^\infty \frac{\begin{pmatrix}\cos(x(y-z+1))+\cos(x(y-z-1))\\-\cos(x(y+z+1))-\cos(x(y+z-1))\end{pmatrix}}{x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Using the formula $\int_0^\infty \frac{\cos(bx)-\cos(ax)}{x}\mathrm{d}x = \ln\left|\frac{a}{b}\right|$ $a,b\neq 0$ we get

$$I = \frac{1}{4}\int_0^1\int_0^1\frac{1}{yz}\ln\left|\frac{(y+z+1)(y+z-1)}{(y-z+1)(y-z-1)}\right|\mathrm{d}y\,\mathrm{d}z$$

Now use the fact that $\ln|z| = \Re\ln z$ and compute the inner integral in terms of dilogarithms:

$$I = \frac{1}{4}\Re\int_0^1 \frac{\mathrm{Li}_2\left(\frac{1}{1+z}\right)+\mathrm{Li}_2\left(\frac{1}{z-1}\right)-\mathrm{Li}_2\left(-\frac{1}{1+z}\right)-\mathrm{Li}_2\left(\frac{1}{1-z}\right)}{z}\mathrm{d}z \tag{1}$$

2. Computation of Logarithmic Integrals

We need to prove

$$\int_0^1 \frac{\mathrm{Li}_2\left(\frac{1}{z-1}\right)-\mathrm{Li}_2\left(-\frac{1}{1+z}\right)}{z}\mathrm{d}z$$
$$= \mathrm{Li}_3\left(-\frac{1}{3}\right)-2\mathrm{Li}_3\left(\frac{1}{3}\right)+\left\{\mathrm{Li}_2\left(-\frac{1}{3}\right)-2\mathrm{Li}_2\left(\frac{1}{3}\right)\right\}\ln(3)+\frac{\zeta(3)}{8}-\frac{1}{3}\ln^3(3) \tag{2}$$

$$\Re\int_0^1 \frac{\mathrm{Li}_2\left(\frac{1}{1+z}\right)-\mathrm{Li}_2\left(\frac{1}{1-z}\right)}{z}\mathrm{d}z = -\frac{21}{8}\zeta(3) \tag{3}$$

3. Final Answer

Putting the above results in equation (1), we get

$$I=\frac{1}{4}\left\{\mathrm{Li}_3\left(-\frac{1}{3}\right)-2\mathrm{Li}_3\left(\frac{1}{3}\right)+\mathrm{Li}_2\left(-\frac{1}{3}\right)\ln(3)-2\mathrm{Li}_2\left(\frac{1}{3}\right)\ln(3)-\frac{5\zeta(3)}{2}-\frac{1}{3}\ln^3(3)\right\} \quad (4)$$

Using the identities,

$$\mathrm{Li}_2\left(-\frac{1}{3}\right)-2\mathrm{Li}_2\left(\frac{1}{3}\right)=\frac{\ln^2 3}{2}-\frac{\pi^2}{6}$$
$$\mathrm{Li}_3\left(-\frac{1}{3}\right)-2\mathrm{Li}_3\left(\frac{1}{3}\right)=-\frac{\ln^3 3}{6}+\frac{\pi^2}{6}\ln(3)-\frac{13\zeta(3)}{6}$$

equation (4) can be simplified to

$$I=-\frac{7}{6}\zeta(3)$$

□

Exercise 1.2.170: Prove:

$$\int_0^\infty \ln x\cdot\ln\left(1+\frac{1}{2\cosh x}\right)\mathrm{d}x=\frac{\pi^2}{72}\ln\left(\frac{2592\,\pi^5}{\mathbf{A}^{60}}\right)$$

Proof: Define

$$I(\alpha)=\int_0^\infty \ln x\ln(1-e^{-\alpha x})\,\mathrm{d}x$$

Integrating by parts, followed by the substitution $\alpha x\mapsto x$, we have

$$I(\alpha)=\alpha\int_0^\infty\frac{x-x\ln x}{e^{\alpha x}-1}\,\mathrm{d}x=\frac{1}{\alpha}\int_0^\infty\frac{(1+\ln\alpha)x-x\ln x}{e^x-1}\,\mathrm{d}x$$
$$=\frac{1}{\alpha}\left\{(1+\ln\alpha)\zeta(2)-\frac{\mathrm{d}\zeta(s)\Gamma(s)}{s}\bigg|_{s=2}\right\}=\frac{1}{\alpha}\left\{(\gamma+\ln\alpha)\zeta(2)-\zeta'(2)\right\}$$

Then it follows that

$$\int_0^\infty \ln x\ln\left(1+\frac{1}{2\cosh x}\right)\mathrm{d}x=\int_0^\infty\ln x\ln\left(\frac{1-e^{-3x}}{1-e^{-x}}\cdot\frac{1-e^{-2x}}{1-e^{-4x}}\right)\mathrm{d}x$$
$$=I(2)+I(3)-I(1)-I(4)=\frac{5}{12}\zeta'(2)-\frac{5}{72}\gamma\pi^2+\frac{1}{18}\pi^2\ln(3)$$

Plugging some identities relating $\zeta'(2)$ and the Glaisher-Kinkelin constant $\mathbf{A}$,

$$\zeta'(2)=\frac{\pi^2}{6}\left[\gamma+\ln(2\pi)-12\ln\mathbf{A}\right]$$

Hence we have the final answer

$$\int_0^\infty \ln x\cdot\ln\left(1+\frac{1}{2\cosh x}\right)\mathrm{d}x=\frac{\pi^2}{72}\ln\left(\frac{2592\,\pi^5}{\mathbf{A}^{60}}\right)$$

□

Exercise 1.2.171: Evaluate:

$$\int_0^1\ln\Gamma\left(\frac{x}{2}\right)\mathrm{d}x$$

✎ **Solution:**

$$\int_0^1 \ln\Gamma\left(\frac{x}{2}\right)\mathrm{d}x = 2\int_0^{1/2} \ln\Gamma(x)\mathrm{d}x = \frac{1}{2}\ln(2\pi) + \frac{1}{2}\left(1-\frac{1}{2}\right) - 2\zeta'(-1) + 2\zeta'\left(-1,\frac{1}{2}\right)$$
$$= \frac{\ln(2\pi)}{2} + \frac{1}{12} + 2\ln\mathbf{A} + 2\zeta'\left(-1,\frac{1}{2}\right)$$

$$\zeta\left(z,\frac{1}{2}\right) = (2^z-1)\zeta(z) \implies \zeta'\left(z,\frac{1}{2}\right) = \ln(2)2^z\zeta(z) + (2^z-1)\zeta'(z)$$
$$\Rightarrow \zeta'\left(-1,\frac{1}{2}\right) = \frac{\ln 2}{2}\zeta(-1) - \frac{1}{2}\zeta'(-1) = -\frac{\ln 2}{24} + \frac{\ln\mathbf{A}}{2} - \frac{1}{24}$$

Therefore,

$$\int_0^1 \ln\Gamma\left(\frac{x}{2}\right)\mathrm{d}x = \frac{\ln 2\pi}{2} - \frac{\ln 2}{12} + 3\ln\mathbf{A} = \frac{5}{12}\ln 2 + \frac{\ln\pi}{2} + 3\ln\mathbf{A}$$

□

Exercise 1.2.172: Prove:

$$\int_0^{\frac{1}{2}} x\ln\Gamma(x)\mathrm{d}x = \frac{1}{96}\ln\left(16\pi^6\mathbf{A}^{24}\right) - \frac{7\zeta(3)}{32\pi^2}$$

☞ **Proof:** Here's another solution using the Kummer's Fourier Series of $\ln\Gamma(x)$:

$$\int_0^{\frac{1}{2}} x\ln\Gamma(x)\mathrm{d}x = \frac{1}{16}\ln(2\pi) + (\gamma+\ln(2\pi))\int_0^{\frac{1}{2}}\left(\frac{x}{2}-x^2\right)\mathrm{d}x - \frac{1}{2}\int_0^{\frac{1}{2}} x\ln(2\sin\pi x)\mathrm{d}x$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln(n)}{n}\int_0^{\frac{1}{2}} x\sin(2\pi x)\mathrm{d}x$$
$$= \frac{\ln(2\pi)}{16} + \frac{\gamma+\ln(2\pi)}{48} - \frac{1}{2}\int_0^{\frac{1}{2}} x\ln(2\sin\pi x)\mathrm{d}x + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n}\left(\frac{-\cos(n\pi)}{4n\pi}\right)$$
$$= \frac{\gamma+4\ln(2\pi)}{48} - \frac{1}{2}\int_0^{\frac{1}{2}} x\ln(2\sin\pi x)\mathrm{d}x + \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}(-1)^{n+1}$$

The result now follows since

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}(-1)^{n+1} = -\frac{\gamma\pi^2}{12} - \frac{1}{6}\pi^2\ln(2) + \pi^2\ln(\mathbf{A}) - \frac{1}{12}\pi^2\ln(\pi)$$

and

$$\int_0^{\frac{1}{2}} x\ln(2\sin\pi x)\mathrm{d}x = \frac{7\zeta(3)}{16\pi^2}$$

Hence we have

$$\int_0^{\frac{1}{2}} x\ln\Gamma(x)\mathrm{d}x = \frac{1}{96}\ln\left(16\pi^6\mathbf{A}^{24}\right) - \frac{7\zeta(3)}{32\pi^2}$$

□

Exercise 1.2.173: Evaluate:

$$\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{\sin^3 x} + \sqrt{\cos^3 x}\right)\mathrm{d}x$$

✎ **Solution:** Note that we can write

$$I = \int_0^{\frac{\pi}{4}} \ln\left(\sqrt{\sin^3 x} + \sqrt{\cos^3 x}\right)\mathrm{d}x = \frac{3}{4}\mathbf{G} - \frac{3\pi}{8}\ln(2) + 2\int_0^1 \frac{x\ln\left(1+x^3\right)}{1+x^4}\mathrm{d}x$$

Let $J=\int_0^1 \frac{x\ln\left(1+x^3\right)}{1+x^4}\mathrm{d}x$, make the substitution $x\mapsto 1/x$ and average out both the integrals:

$$J=\frac{1}{2}\int_0^\infty \frac{x\ln\left(1+x^3\right)}{1+x^4}\mathrm{d}x-\frac{3}{2}\int_1^\infty \frac{x\ln\left(x\right)}{1+x^4}\mathrm{d}x$$

The second integral is simply

$$\begin{aligned}\int_1^\infty \frac{x\ln\left(x\right)}{1+x^4}\mathrm{d}x&=-\int_0^1 \frac{x\ln\left(x\right)}{1+x^4}\mathrm{d}x=-\sum_{n=0}^\infty(-1)^n\int_0^1 x^{4n+1}\ln(x)\mathrm{d}x\\&=\sum_{n=0}^\infty\frac{(-1)^n}{(4n+2)^2}=\frac{\mathbf{G}}{4}\end{aligned}$$

So we are left with

$$J=-\frac{3}{8}\mathbf{G}+\frac{1}{2}\int_0^\infty \frac{x\ln(1+x^3)}{1+x^4}\mathrm{d}x \tag{1}$$

Let $f(z)=\frac{z\ln\left(1+z^3\right)\ln(-z)}{1+z^4}$ and integrate it around a keyhole contour about the positive real axis but with additional keyholes about the branch points $z=e^{i\pi/3},e^{i\pi}$ and $e^{5i\pi/3}$:

$$\begin{aligned}-2\pi i\int_0^\infty \frac{x\ln(1+x^3)}{1+x^4}\mathrm{d}x&=2\pi i\left(\sum_{n=0}^3\operatorname{res}[z=e^{(2n+1)i\pi/4},\ f(z)]\right)+2\pi i\int_{e^{i\pi/3}}^{e^{i\pi/3}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x\\&\quad+2\pi i\int_{e^{i\pi}}^{e^{i\pi}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x+2\pi i\int_{e^{5i\pi/3}}^{e^{5i\pi/3}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x\\\implies\int_0^\infty \frac{x\ln(1+x^3)}{1+x^4}\mathrm{d}x&=-\frac{\pi}{16}\ln\left(\frac{17}{2}+6\sqrt{2}\right)-\int_{e^{i\pi/3}}^{e^{i\pi/3}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x-\int_{e^{i\pi}}^{e^{i\pi}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x\\&\quad-\int_{e^{5i\pi/3}}^{e^{5i\pi/3}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x\end{aligned} \tag{2}$$

I calculated these three integrals separately so I will give only the result:

$$\begin{aligned}&\int_{e^{i\pi/3}}^{e^{i\pi/3}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x=-\frac{\mathbf{G}}{6}-\frac{\pi}{6}\ln\left(2+\sqrt{3}\right)+\frac{\pi^2}{16}i\quad,\quad\int_{e^{i\pi}}^{e^{i\pi}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x=\frac{\mathbf{G}}{4}\\&\int_{e^{5i\pi/3}}^{e^{5i\pi/3}\infty}\frac{x\ln(-x)}{1+x^4}\mathrm{d}x=-\frac{\mathbf{G}}{6}-\frac{\pi}{6}\ln\left(2+\sqrt{3}\right)-\frac{\pi^2}{16}i\end{aligned}$$

Putting these in equation (2), gives

$$\int_0^\infty \frac{x\ln(1+x^3)}{1+x^4}\mathrm{d}x=\frac{\mathbf{G}}{12}+\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)-\frac{\pi}{16}\ln\left(\frac{17}{2}+6\sqrt{2}\right) \tag{3}$$

Substituting (3) in (1) gives

$$J=\int_0^1 \frac{x\ln\left(1+x^3\right)}{1+x^4}\mathrm{d}x=-\frac{\mathbf{G}}{3}+\frac{\pi}{6}\ln\left(2+\sqrt{3}\right)-\frac{\pi}{32}\ln\left(\frac{17}{2}+6\sqrt{2}\right)$$

Hence we get the **FINAL ANSWER**

$$\int_0^{\frac{\pi}{4}}\ln\left(\sqrt{\sin^3 x}+\sqrt{\cos^3 x}\right)\mathrm{d}x=\frac{\mathbf{G}}{12}-\frac{5\pi}{16}\ln 2+\frac{\pi}{8}\ln(3-2\sqrt{2})+\frac{\pi}{3}\ln(2+\sqrt{3})$$

□

Exercise 1.2.174: Prove:

$$\int_{-\infty}^{\infty}\left(\frac{\coth(x)}{x^{2n-1}}-\sum_{j=0}^{n-1}\frac{2^{2j}B_{2j}}{(2j)!}\frac{1}{x^{2n-2j}}\right)\mathrm{d}x=(-1)^{n+1}\frac{2\zeta(2n-1)}{\pi^{2n-2}}$$

where B_{2j} is a Bernoulli Number.

Proof: Integrating around a rectangle or circle in the upper-half complex plane that avoids the poles on the imaginary axis

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\frac{\coth(x)}{x^{2n-1}}-\sum_{k=0}^{n-1}\frac{2^{2k}B_{2k}}{(2k)!}x^{2k-2n}\right)\mathrm{d}x=2\pi i\sum_{m=1}^{\infty}\operatorname{Res}\left[\frac{\coth z}{z^{2n-1}},m\pi i\right]\\&=2\pi i\sum_{m=1}^{\infty}\operatorname{Res}\left[\frac{\coth(z+m\pi i)}{(z+m\pi i)^{2n-1}},0\right]=2\pi i\sum_{m=1}^{\infty}\operatorname{Res}\left[\frac{\coth(z)}{(z+m\pi i)^{2n-1}},0\right]\\&=2\pi i\sum_{m=1}^{\infty}\frac{1}{(m\pi i)^{2n-1}}=(-1)^{n-1}\frac{2\zeta(2n-1)}{\pi^{2n-2}}\end{aligned}$$

□

Exercise 1.2.175: Prove:

$$\int_0^1\frac{\left(\frac{1}{2}-x\right)\ln(1-x)}{x^2-x+1}\mathrm{d}x=\frac{\pi^2}{36}$$

Proof:

$$\begin{aligned}&\int_0^1\frac{\left(\frac{1}{2}-x\right)\ln(1-x)}{1-x+x^2}\mathrm{d}x=\int_0^1\frac{\left(\frac{1}{2}-x\right)\ln(1-x)}{\left(\frac{1}{2}-x\right)^2+\frac{3}{4}}\mathrm{d}x=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\ln\left(\frac{1}{2}+x\right)}{\frac{3}{4}+x^2}\mathrm{d}x\\&=\Re\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\ln\left(\frac{1}{2}+x\right)}{x+\frac{\sqrt{3}}{2}i}\mathrm{d}x=\Re\left[\ln\left(x+\frac{1}{2}\right)\ln\left(1-\left(\frac{1}{2}+x\right)e^{\frac{i\pi}{3}}\right)+\mathrm{Li}_2\left(\left(x+\frac{1}{2}\right)e^{\frac{i\pi}{3}}\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}\\&=\Re\mathrm{Li}_2\left(e^{\frac{i\pi}{3}}\right)=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n}{3}\right)}{n^2}=\frac{\pi^2}{36}\end{aligned}$$

□

Exercise 1.2.176: Evaluate:

$$\int_0^1\frac{\ln^2(1+x)\ln(x)}{1-x}\mathrm{d}x$$

Solution: Part I : Reduction

Let $I=\int_0^1\frac{\ln^2(1+x)\ln(x)}{1-x}\mathrm{d}x$. Note that

$$\begin{aligned}I=\int_0^1\frac{\ln^2(1+x)\ln(x)}{1-x}\mathrm{d}x&=\int_0^1\frac{\ln^2(1-x^2)\ln(x)}{1-x}\mathrm{d}x-\int_0^1\frac{\ln^2(1-x)\ln(x)}{1-x}\mathrm{d}x\\&\quad-2\int_0^1\frac{\ln(1+x)\ln(1-x)\ln(x)}{1-x}\mathrm{d}x\end{aligned}\tag{1}$$

Now,

$$\int_0^1 \frac{\ln^2(1-x^2)\ln(x)}{1-x}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\ln^2(1-x)\ln(x)}{\sqrt{x}(1-\sqrt{x})}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\ln^2(1-x)\ln(x)}{(1-x)\sqrt{x}}(1+\sqrt{x})\mathrm{d}x$$
$$= \frac{1}{4}\int_0^1 \frac{\ln^2(1-x)\ln(x)}{(1-x)\sqrt{x}}\mathrm{d}x + \frac{1}{4}\int_0^1 \frac{\ln^2(1-x)\ln(x)}{1-x}\mathrm{d}x$$

Both these integrals can be evaluated easily by differentiating the beta function. The answer is

$$\int_0^1 \frac{\ln^2(1-x^2)\ln(x)}{1-x}\mathrm{d}x = -\frac{17\pi^4}{360} - \frac{\pi^2}{2}\ln^2(2) + 7\ln(2)\zeta(3)$$

Also,

$$\int_0^1 \frac{\ln^2(1-x)\ln(x)}{1-x}\mathrm{d}x = -\frac{\pi^4}{45}$$

Plugging these into equation (1), we get

$$I = -\frac{\pi^4}{40} - \frac{\pi^2}{2}\ln^2(2) + 7\ln(2)\zeta(3) - 2I_0 \tag{2}$$

where $I_0 = \displaystyle\int_0^1 \frac{\ln(1+x)\ln(1-x)\ln(x)}{1-x}\mathrm{d}x$.

Part II : Evaluation of I_0

We have

$$I_0 = \int_0^1 \frac{\ln(1+x)\ln(1-x)\ln(x)}{1-x}\mathrm{d}x = \int_0^1 \frac{\ln(2-x)\ln(1-x)\ln(x)}{x}\mathrm{d}x$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln(2-x)\ln(x)\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 \frac{1}{2-x}\left\{\frac{x^n\ln x}{n} - \frac{x^n}{n^2}\right\}\mathrm{d}x$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 \frac{x^n\ln(x)}{2-x}\mathrm{d}x + \sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 \frac{x^n}{2-x}\mathrm{d}x$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{m=1}^{\infty}\frac{1}{2^m(m+n)^2}\right) + \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{m=1}^{\infty}\frac{1}{2^m(m+n)}\right)$$
$$= \sum_{m=1}^{\infty}\frac{1}{m^2 2^m}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+m}\right)^2 + \sum_{m=1}^{\infty}\frac{1}{m2^m}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{n} - \frac{1}{n+m}\right)$$
$$= \sum_{m=1}^{\infty}\frac{2\zeta(2) - H_m^{(2)} - \dfrac{2}{m}H_m}{m^2 2^m} + \sum_{m=1}^{\infty}\frac{\zeta(3) - \dfrac{\zeta(2)}{m} + \dfrac{H_m}{m^2}}{m2^m}$$
$$= \zeta(2)\mathrm{Li}_2\left(\frac{1}{2}\right) + \zeta(3)\ln(2) - \left\{\sum_{m=1}^{\infty}\frac{H_m}{m^3 2^m} + \sum_{m=1}^{\infty}\frac{H_m^{(2)}}{m^2 2^m}\right\} \tag{3}$$

It is known that

$$\sum_{m=1}^{\infty}\frac{H_m}{m^3 2^m} = \frac{\zeta(4)}{8} - \frac{\zeta(3)\ln(2)}{8} + \frac{\ln^4(2)}{24} + \mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$\sum_{m=1}^{\infty}\frac{H_m^{(2)}}{m^2 2^m} = \frac{\zeta(2)\ln^2(2)}{4} + \frac{37}{16}\zeta(4) - \frac{7}{4}\zeta(3)\ln(2) - \frac{1}{24}\ln^4(2) + \mathrm{Li}_4\left(\frac{1}{2}\right)$$

Putting into (3), we obtain

$$I_0 = \frac{17\pi^4}{1440} - \frac{\pi^2}{24}\ln^2(2) - \frac{\ln^4(2)}{12} + \frac{7}{8}\zeta(3)\ln(2) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

Refer to this thread for the evaluation of euler sums.

Part III : The Final Answer

Now, substitute the value of I_0 into equation (2), to get

$$I = -\frac{7\pi^4}{144} - \frac{5}{12}\pi^2\ln^2(2) + \frac{\ln^4(2)}{2} + \frac{21}{4}\zeta(3)\ln(2) + 4\mathrm{Li}_4\left(\frac{1}{2}\right)$$

□

Exercise 1.2.177: Evaluate:

$$\int_0^\infty \frac{x}{\sinh(\pi x)}\ln\left(\alpha^2+x^2\right)\mathrm{d}x$$

Solution: Let $\mathcal{I}(z) = \displaystyle\int_0^\infty \frac{x}{\sinh(\pi x)}\ln\left(z^2+x^2\right)\mathrm{d}x$, then

$$\begin{aligned}
\mathcal{I}'(z) &= 2z\int_0^\infty \frac{x}{\sinh(\pi x)\left(z^2+x^2\right)}\mathrm{d}x = \frac{2z}{\pi}\int_0^\infty \frac{x}{x^2+z^2}\left(\frac{1}{x} - 2x\sum_{n=1}^\infty \frac{(-1)^{n+1}}{x^2+n^2}\right)\mathrm{d}x \\
&= 1 - \frac{4z}{\pi}\sum_{n=1}^\infty (-1)^{n+1}\int_0^\infty \frac{x^2}{(x^2+z^2)(x^2+n^2)}\mathrm{d}x = 1 - 2z\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n+z} \\
&= 1 - 2z\left(\frac{1}{z} + \frac{1}{2}\psi_0\left(\frac{z}{2}\right) - \frac{1}{2}\psi_0\left(\frac{z+1}{2}\right)\right) = -1 + z\left(\psi_0\left(\frac{z+1}{2}\right) - \psi_0\left(\frac{z}{2}\right)\right) \qquad (1)
\end{aligned}$$

Integrating (1) gives

$$\begin{aligned}
\mathcal{I}(\alpha) - \mathcal{I}(0) &= -\alpha + \int_0^\alpha z\left(\psi_0\left(\frac{z+1}{2}\right) - \psi_0\left(\frac{z}{2}\right)\right)\mathrm{d}z \\
&= -\alpha + \left[2z\ln\frac{\Gamma\left(\frac{1+z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\right]_{z=0}^{z=\alpha} - 2\int_0^\alpha \ln\frac{\Gamma\left(\frac{1+z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\mathrm{d}z \\
&= -\alpha + 2\alpha\ln\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} - 2\int_0^\alpha \ln\frac{\Gamma\left(\frac{1+z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\mathrm{d}z \\
&= -\frac{1}{2} - \frac{\ln 2}{6} + 6\ln\mathbf{A} + 2\ln\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma^2\left(\frac{\alpha}{2}\right)} + 4\ln\frac{\mathrm{G}\left(\frac{1+\alpha}{2}\right)}{\mathrm{G}\left(\frac{\alpha}{2}\right)} \qquad (2)
\end{aligned}$$

where $\mathrm{G}(z)$ is the Barnes G Function. The last integral was evaluated using DLMF 5.17.4. It remains to calculate $I(0)$.

$$\begin{aligned}
\mathcal{I}(0) &= \int_0^\infty \frac{x\ln x^2}{\sinh(\pi x)}\mathrm{d}x = 2\frac{\mathrm{d}}{\mathrm{d}a}\left[\int_0^\infty \frac{x^{a-1}}{\sinh(\pi x)}\mathrm{d}x\right]_{a=2} = 2\frac{\mathrm{d}}{\mathrm{d}a}\left[\frac{2^a-1}{2^{a-1}\pi^a}\Gamma(a)\zeta(a)\right]_{a=2} \\
&= \frac{1}{2} + \frac{2}{3}\ln 2 - 6\ln\mathbf{A}
\end{aligned}$$

Putting this in (2), we get

$$\mathcal{I}(\alpha) = \frac{\ln 2}{2} + 2\ln\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma^2\left(\frac{\alpha}{2}\right)} + 4\ln\frac{\mathrm{G}\left(\frac{1+\alpha}{2}\right)}{\mathrm{G}\left(\frac{\alpha}{2}\right)}$$

Putting $\alpha = 1$, gives

$$\mathcal{I}(1) = \frac{\ln 2}{3} - \ln(\pi) + 6\ln(\mathbf{A}) - \frac{1}{2}$$

We also get

$$\mathcal{I}\left(\frac{1}{2}\right) = \frac{2\mathbf{G}}{\pi} + \ln(2\pi) - 2\ln\Gamma\left(\frac{1}{4}\right)$$

□

Exercise 1.2.178: Evaluate:

$$\int_0^1 \frac{\ln x \ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x$$

Solution: Step 1. Let I be the integral in question:

$$I = \int_0^1 \frac{\ln x \ln(1-x)\ln^2(1+x)}{x}\mathrm{d}x$$

By the simple algebraic formula $(a+b)^3 + (a-b)^3 - 2a^3 = 6ab^2$, it follows that

$$\begin{aligned} I &= \frac{1}{6}\int_0^1 \frac{\ln x}{x}\left\{\ln^3(1-x^2) + \ln^3\left(\frac{1-x}{1+x}\right) - 2\ln^3(1-x)\right\}\mathrm{d}x \\ &= \frac{1}{6}\int_0^1 \frac{\ln x \ln^3(1-x^2)}{x}\mathrm{d}x + \frac{1}{6}\int_0^1 \frac{\ln x \ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x - \frac{1}{3}\int_0^1 \frac{\ln x \ln^3(1-x)}{x}\mathrm{d}x \qquad (1) \end{aligned}$$

Applying the substitution $x^2 \mapsto x$, the first integral reduces to

$$\begin{aligned} \frac{1}{6}\int_0^1 \frac{\ln x \ln^3(1-x^2)}{x}\mathrm{d}x = \frac{1}{24}\int_0^1 \frac{\ln x \ln^3(1-x)}{x}\mathrm{d}x &= \frac{1}{24}\frac{\partial^4 \beta}{\partial z \partial w^3}(0^+, 1) \\ &= \frac{1}{2}\zeta(5) - \frac{1}{4}\zeta(2)\zeta(3) \end{aligned}$$

So (1) can be written as

$$I = \frac{7}{4}\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5) + I_2 \qquad (2)$$

where I_2 is given by

$$I_2 = \frac{1}{6}\int_0^1 \frac{\ln x \ln^3\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x$$

Step 2. Now we calculate I_2. Applying integrating by parts, followed by the substitution $x \mapsto \dfrac{1-y}{1+y}$, we have

$$\begin{aligned} I_2 &= \left[\frac{1}{12}\ln^2 x \ln^3\left(\frac{1-x}{1+x}\right)\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\ln^2 x \ln^2\left(\frac{1-x}{1+x}\right)}{1-x^2}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\ln^2 y \ln^2\left(\frac{1-y}{1+y}\right)}{y}\mathrm{d}y \\ &= \int_0^1 \frac{\ln^2 y}{y}\left\{\frac{1}{2}\ln^2\left(\frac{1+y}{1-y}\right)\right\}^2 \mathrm{d}y = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2m+1)(2n+1)}\int_0^1 y^{2m+2n+1}\ln^2 y \mathrm{d}y \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2}{(2m+1)(2n+1)(2m+2n+2)^3} \end{aligned}$$

By the following partial fraction decomposition

$$\frac{1}{ab(a+b)^3} = \frac{a+b}{ab(a+b)^4} = \frac{1}{a(a+b)^4} + \frac{1}{b(a+b)^4}$$

together with the symmetry in the role of m and n, it follows that

$$\begin{aligned} I_2 &= 4\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2m+2n+2)^4} = \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n+1)(m+n+1)^4} \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{2n+1}\cdot\frac{1}{m^4} = \frac{1}{4}\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{1}{2n-1}\cdot\frac{1}{m^4} = \frac{1}{4}\sum_{m=1}^{\infty}\left(\sum_{n=1}^{m}\frac{1}{2n-1}\right)\frac{1}{m^4} \end{aligned}$$

Now by exploiting the following identity

$$\int_0^1 \frac{1-x^k}{1-x}\mathrm{d}x = 1+\frac{1}{2}+\cdots+\frac{1}{k} \Longrightarrow \quad \int_0^1 \frac{1+x^k-2x^{2k}}{2(1-x)}\mathrm{d}x = 1+\frac{1}{3}+\cdots+\frac{1}{2k-1}$$

we can rephrase I_2 as follows:

$$I_2 = \frac{1}{8}\sum_{m=1}^{\infty}\frac{1}{m^4}\int_0^1 \frac{1+x^m-2x^{2m}}{1-x}\mathrm{d}x = \frac{1}{8}\int_0^1 \frac{1}{1-x}\left(\zeta(4)+\mathrm{Li}_4(x)-2\mathrm{Li}_4(x^2)\right)\mathrm{d}x$$

Writing $\dfrac{1}{1-x} = \dfrac{\mathrm{Li}_0(x)}{x}$ and applying the simple differentiation rule $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_s(x^p) = \dfrac{p\mathrm{Li}_{s-1}(x^p)}{x}$, by applying integrating parts 3 times,

$$\begin{aligned} I_2 &= \frac{1}{8}\left[\mathrm{Li}_1(x)\left(\zeta(4)+\mathrm{Li}_4(x)-2\mathrm{Li}_4(x^2)\right)\right]_0^1 - \frac{1}{8}\int_0^1 \frac{\mathrm{Li}_1(x)}{x}\left(\mathrm{Li}_3(x)-4\mathrm{Li}_3(x^2)\right)\mathrm{d}x \\ &= -\frac{1}{8}\left[\mathrm{Li}_2(x)\left(\mathrm{Li}_3(x)-4\mathrm{Li}_3(x^2)\right)\right]_0^1 + \frac{1}{8}\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\left(\mathrm{Li}_2(x)-8\mathrm{Li}_2(x^2)\right)\mathrm{d}x \\ &= \frac{3}{8}\zeta(2)\zeta(3) + \frac{1}{8}\left[\mathrm{Li}_3(x)\left(\mathrm{Li}_2(x)-8\mathrm{Li}_2(x^2)\right)\right]_0^1 - \frac{1}{8}\int_0^1 \frac{\mathrm{Li}_3(x)}{x}\left(\mathrm{Li}_1(x)-16\mathrm{Li}_1(x^2)\right)\mathrm{d}x \\ &= -\frac{1}{2}\zeta(2)\zeta(3) + \frac{1}{8}\int_0^1 \frac{\mathrm{Li}_3(x)}{x}\left(\ln(1-x)-16\ln(1-x^2)\right)\mathrm{d}x \\ &= -\frac{1}{2}\zeta(2)\zeta(3) + \frac{1}{8}\int_0^1 \frac{\mathrm{Li}_3(x)\ln(1-x)}{x}\,\mathrm{d}x - 2\int_0^1 \frac{\mathrm{Li}_3(x)\ln(1-x^2)}{x}\,\mathrm{d}x \end{aligned}$$

Now applying the substitution $x \mapsto x^2$ to the first integral, from the identity $\mathrm{Li}_3(x^2) = 4\mathrm{Li}_3(x) + 4\mathrm{Li}_3(-x)$ it follows that

$$\frac{1}{8}\int_0^1 \frac{\mathrm{Li}_3(x)\ln(1-x)}{x}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\mathrm{Li}_3(x^2)\ln(1-x^2)}{x}\mathrm{d}x = \int_0^1 \frac{\{\mathrm{Li}_3(x)+\mathrm{Li}_3(-x)\}\ln(1-x^2)}{x}\mathrm{d}x$$

This finally gives

$$I_2 = -\frac{1}{2}\zeta(2)\zeta(3) - \int_{-1}^1 \frac{\mathrm{Li}_3(x)\ln(1-x^2)}{x}\mathrm{d}x$$

Plugging this back to (2), we obtain

$$I_1 = \frac{5}{4}\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5) + I_3 \tag{3}$$

where I_3 is given by

$$I_3 = -\int_{-1}^1 \frac{\mathrm{Li}_3(x)\ln(1-x^2)}{x}\mathrm{d}x$$

Step 3. Finally, we evaluate I_3 into a closed form and thus complete the calculation. Since the integrand is holomorphic on the unit disc, we can shift up the contour of integration to the clockwise semicircular arc:

$$I_3 = -i\int_{\pi}^{0} \mathrm{Li}_3(e^{i\theta})\ln(1-e^{2i\theta})\mathrm{d}\theta = -\frac{1}{i}\int_0^{\pi} \mathrm{Li}_3(e^{i\theta})\ln(1-e^{2i\theta})\mathrm{d}\theta$$

Now expanding with Taylor series and taking advantage of the fact that I_3 is real, we have

$$\begin{aligned} I_3 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^3}\frac{1}{m}\int_0^{\pi}\{\cos(n\theta)\sin(2m\theta)+\sin(n\theta)\cos(2m\theta)\}\mathrm{d}\theta \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^3}\frac{1}{m}\int_0^{\pi}\cos(n\theta)\sin(2m\theta)\mathrm{d}\theta + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^3}\frac{1}{m}\int_0^{\pi}\sin(n\theta)\cos(2m\theta)\mathrm{d}\theta \\ &= S_1 + S_2 \end{aligned}$$

To evaluate S_1, we note the following simple identity

$$\int_0^{\pi}\cos(n\theta)\sin(2m\theta)\mathrm{d}\theta = \frac{1+(-1)^{n-1}}{2}\int_0^{\pi}\cos\left(\frac{1}{2}n\theta\right)\sin(m\theta)\mathrm{d}\theta$$

Then we can write

$$\begin{aligned} S_1 &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\int_0^{\pi}\left\{\sum_{m=1}^{\infty}\frac{\sin(m\theta)}{m}\right\}\cos\left(\left(n+\frac{1}{2}\right)\theta\right)\mathrm{d}\theta \\ S_2 &= \sum_{m=1}^{\infty}\frac{1}{m}\int_0^{\pi}\left\{\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^3}\right\}\cos(2m\theta)\mathrm{d}\theta \end{aligned}$$

But it is not hard to check that the following Fourier expansion holds:

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n} = \Im\sum_{n=1}^{\infty}\frac{e^{i\theta}}{n} = -\Im\ln(1-e^{i\theta}) = \frac{\pi-\theta}{2}, \quad 0<\theta<\pi$$

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^3} = \frac{\theta^3}{12} - \frac{\pi\theta^2}{4} + \frac{\pi^2\theta}{6}, \quad 0<\theta<\pi$$

Plugging these to S_1, S_2 and performing some tedious calculation, we obtain

$$S_1 = \sum_{n=0}^{\infty}\frac{2}{(2n+1)^5} = \frac{31}{16}\zeta(5) \quad \text{and} \quad S_2 = -\frac{\pi^2}{16}\sum_{m=1}^{\infty}\frac{1}{m^3} = -\frac{3}{8}\zeta(2)\zeta(3)$$

Therefore we have

$$I = \frac{7}{8}\zeta(2)\zeta(3) - \frac{25}{16}\zeta(5)$$

□

Exercise 1.2.179: Prove:

$$\int_0^{\infty}\frac{\sqrt[5]{\sin x}}{x}\mathrm{d}x = \sqrt[5]{2}\frac{\Gamma^2\left(\frac{1}{10}\right)}{4\Gamma\left(\frac{1}{5}\right)}$$

☞ **Proof:** I will begin with DreamWeaver's reference.

For $s > 0$, define $f_s(x) = \operatorname{sgn}(x)|x|^s$ as the odd extension of the function $x \mapsto x^s$. Then by Lemma 3.1 of the reference,

$$\int_0^\infty \frac{f_s(\sin x)}{x}\,\mathrm{d}x = \frac{1}{2}\int_0^\pi \frac{\sin^s x}{\tan(x/2)}\,\mathrm{d}x = 2^s \int_0^{\frac{\pi}{2}} \sin^{s-1}\theta\cos^{s+1}\theta\,\mathrm{d}\theta$$
$$= 2^{s-1}\mathrm{B}\left(\frac{s}{2},\frac{s}{2}+1\right) = 2^{s-1}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}{\Gamma(s+1)}$$

Plugging $s = \frac{1}{5}$, the result follows. (Of course, it is not hard to prove Lemma 3.1 itself.) □

Exercise 1.2.180: Prove:

$$\int_{-\infty}^\infty \frac{e^{ax}-e^{bx}}{1-e^x}\mathrm{d}x = \pi(\cot(\pi a)-\cot(\pi b))$$

$a > 0, \quad b < 1$

☞ **Proof:** Let us begin with the sub $t = e^x$:

$$I = \int_0^\infty \frac{t^{a-1}-t^{b-1}}{1-t}\mathrm{d}t = \int_0^1 \frac{t^{a-1}-t^{b-1}}{1-t}\mathrm{d}t + \int_1^\infty \frac{t^{a-1}-t^{b-1}}{1-t}\mathrm{d}t$$
$$= \int_0^1 \frac{t^{a-1}-t^{b-1}}{1-t}\mathrm{d}t + \int_0^1 \frac{t^{-b}-t^{-a}}{1-t}\mathrm{d}t \quad t \mapsto 1/t$$
$$= -\psi_0(a)+\psi_0(b)-\psi_0(1-b)+\psi_0(1-a)$$
$$= \pi\cot(\pi a) - \pi\cot(\pi b)$$

Here, I used that $\psi_0(s) = -\gamma + \int_0^1 \frac{1-x^{s-1}}{1-x}\mathrm{d}x$ □

Exercise 1.2.181: Prove:

$$\int_0^1 \frac{\ln(1-x+x^2)}{x-x^2}\mathrm{d}x = -\frac{\pi^2}{9}$$

☞ **Proof:** Writing as $\int_0^1 \frac{\ln(x^2-x+1)\,\mathrm{d}x}{1-x} + \int_0^1 \frac{\ln(x^2-x+1)\,\mathrm{d}x}{x}$ and letting $x \mapsto 1-x$ in the first reveals that we seek:

$$2\int_0^1 \frac{\ln(x^2-x+1)\,\mathrm{d}x}{x} = 2\int_0^1 \frac{\ln(1+x^3)\,\mathrm{d}x}{x} - 2\int_0^1 \frac{\ln(1+x)\,\mathrm{d}x}{x}$$
$$= 2\int_0^1 \sum_{k\geq 1}\frac{(-x)^{3k-1}}{k}\,\mathrm{d}x - 2\int_0^1 \sum_{k\geq 1}\frac{(-x)^{k-1}}{k}\,\mathrm{d}x = -\frac{4}{3}\sum_{k\geq 1}\frac{(-1)^{k+1}}{k^2} = -\frac{\pi^2}{9}$$

□

Exercise 1.2.182: Prove:

$$\int_0^1 \frac{\mathbf{K}(k)}{1+k}\mathrm{d}k = \frac{\pi^2}{8}$$

☞ **Proof:** Let $I = \int_0^1 \frac{\mathbf{K}(k)}{1+k}\mathrm{d}k$. Substitute $k = \frac{1-x}{1+x}$ and $\mathrm{d}k = \frac{-2}{(x+1)^2}\mathrm{d}x$ and use **quadratic transformation**.

$$I = \int_0^1 \frac{1}{1+x}\mathbf{K}\left(\frac{1-x}{1+x}\right)\mathrm{d}x = \frac{1}{2}\int_0^1 \mathbf{K}'(x)\,\mathrm{d}x$$

Since, $\mathbf{K}'(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos^2\theta + k^2\sin^2\theta}} \mathrm{d}\theta$ we can write

$$
\begin{aligned}
I &= \frac{1}{2}\int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos^2\theta + x^2\sin^2\theta}} \mathrm{d}\theta \, \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{\sqrt{\cos^2\theta + x^2\sin^2\theta}} \mathrm{d}x \, \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \csc(\theta)\ln\left(\tan\theta + \sec\theta\right) \mathrm{d}\theta = \frac{1}{2}\int_0^{\infty} \frac{t}{\sinh t} \mathrm{d}t = \sum_{n=0}^{\infty} \int_0^{\infty} t e^{-(2n+1)t} \, \mathrm{d}t \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}
\end{aligned}
$$

□

Exercise 1.2.183: Prove:

$$\int_0^{\infty} \frac{x\arctan(x)}{1+x^4} \mathrm{d}x = \frac{\pi^2}{16}$$

Proof: Let $x = \frac{1}{u}$, then $\int_0^{\infty} \frac{x\arctan(x)}{1+x^4} \, \mathrm{d}x = \int_0^{\infty} \frac{u\arctan\frac{1}{u}}{1+u^4} \, \mathrm{d}u$ So

$$
\begin{aligned}
\int_0^{\infty} \frac{x\arctan(x)}{1+x^4} \, \mathrm{d}x &= \frac{1}{2}\int_0^{\infty} \frac{x\left(\arctan x + \arctan\frac{1}{x}\right)}{1+x^4} \, \mathrm{d}x = \frac{\pi}{4}\int_0^{\infty} \frac{x}{1+x^4} \, \mathrm{d}x \\
&= \frac{\pi}{8}\int_0^{\infty} \frac{1}{1+v^2} \, \mathrm{d}v = \frac{\pi}{8}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{16}
\end{aligned}
$$

□

Exercise 1.2.184: Evaluate:

$$\int_1^{\infty} \frac{\ln\left(x^2-1\right)}{1+x^2} \mathrm{d}x$$

Solution:

$$
\begin{aligned}
\int_1^{\infty} \frac{\ln\left(x^2-1\right)}{1+x^2} \mathrm{d}x &= \int_0^1 \frac{\ln\left(\frac{1-x^2}{x^2}\right)}{1+x^2} \mathrm{d}x = \int_0^1 \frac{\ln(1-x^2) - 2\ln(x)}{1+x^2} \mathrm{d}x \\
&= \int_0^1 \frac{\ln(1-x^2)}{1+x^2} \mathrm{d}x + 2\mathbf{G} = 2\mathbf{G} + \int_0^{\frac{\pi}{4}} \left(\ln(\cos 2\theta) - 2\ln(\cos\theta)\right) \mathrm{d}\theta \\
&= 2\mathbf{G} - \frac{\pi}{4}\ln(2) - 2\int_0^{\frac{\pi}{4}} \ln(\cos\theta) \mathrm{d}\theta \\
&= 2\mathbf{G} - \frac{\pi}{4}\ln(2) - 2\left(-\frac{\pi}{4}\ln(2) - \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \int_0^{\frac{\pi}{4}} \cos(2jx) \mathrm{d}x\right) \\
&= 2\mathbf{G} + \frac{\pi}{4}\ln(2) + \sum_{j=1}^{\infty} \frac{(-1)^j \sin\left(\frac{\pi j}{2}\right)}{j^2} = 2\mathbf{G} + \frac{\pi}{4}\ln(2) - \mathbf{G} = \mathbf{G} + \frac{\pi}{4}\ln(2)
\end{aligned}
$$

□

Exercise 1.2.185: Evaluate:

$$\int_{-\infty}^{\infty} \frac{1}{\left(t^2+\pi^2\right)^2 \cosh(t)} \mathrm{d}t$$

✎ **Solution:** Let $f(z) = \dfrac{1}{(z^2+\pi^2)\cosh(z)}$. We will integrate $f(z)$ around the famous semicircle contour. Note that

$$\operatorname{res}_{z=i\pi} f(z) = \frac{i}{4\pi^3} \quad , \quad \sum_{n=0}^{\infty} \operatorname{res}_{z=\frac{(2n+1)i\pi}{2}} f(z) = \frac{16i}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-3)^2(2n+1)^2}$$

We get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{(\pi^2+x^2)^2\cosh(x)}\mathrm{d}x &= 2\pi i\left(\frac{i}{4\pi^3} + \frac{16i}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-3)^2(2n+1)^2}\right)\\ &= \frac{-1}{2\pi^2} + \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-3)^2(2n+1)^2}\end{aligned}$$

The sum can be evaluated by the use of partial fractions and then using polygamma function:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-3)^2(2n+1)^2} = \frac{3-2\mathbf{G}}{16}$$

The final answer is

$$\int_{-\infty}^{\infty} \frac{1}{(\pi^2+x^2)^2\cosh(x)}\mathrm{d}x = -\frac{1}{2\pi^2} + \frac{6-4\mathbf{G}}{\pi^3}$$

□

Exercise 1.2.186: Evaluate:

$$\int_{-\infty}^{\infty} \frac{\ln(1+e^{ax})}{1+e^{bx}}\,\mathrm{d}x \quad a,b>0$$

✎ **Solution:** With the substitution $t = e^{bx}$, we have

$$I(a,b) = \int_{-\infty}^{\infty}\frac{\ln(1+e^{ax})}{1+e^{bx}}\,\mathrm{d}x = \frac{1}{b}\int_0^{\infty}\frac{\ln(1+t^{a/b})}{t(1+t)}\,\mathrm{d}t$$

With the substitution $t \mapsto \dfrac{1}{t}$, it follows that

$$I(a,b) = \frac{1}{b}\int_0^{\infty}\frac{\ln(1+t^{a/b}) - \frac{a}{b}\ln t}{1+t}\,\mathrm{d}t$$

Summing these two identities, we have

$$2I(a,b) = \frac{1}{b}\int_0^{\infty}\left(\frac{\ln(1+t^{a/b})}{t} - \frac{a}{b}\frac{\ln t}{1+t}\right)\mathrm{d}t = \lim_{R\to\infty}\frac{1}{b}\int_0^{R}\left(\frac{\ln(1+t^{a/b})}{t} - \frac{a}{b}\frac{\ln t}{1+t}\right)\mathrm{d}t$$

Now we focus on the integral inside the limit.

$$\begin{aligned}&\frac{1}{b}\int_0^{R}\left(\frac{\ln(1+t^{a/b})}{t} - \frac{a}{b}\frac{\ln t}{1+t}\right)\mathrm{d}t = \frac{1}{a}\int_0^{R^{a/b}}\frac{\ln(1+t)}{t}\,dt - \frac{a}{b^2}\int_0^{R}\frac{\ln t}{1+t}\,\mathrm{d}t\\ &= \frac{1}{a}\int_0^{R^{a/b}}\frac{\ln(1+t)}{t}\,\mathrm{d}t - \frac{a}{b^2}\ln R\ln(R+1) + \frac{a}{b^2}\int_0^{R}\frac{\ln(1+t)}{t}\,\mathrm{d}t\\ &= -\frac{1}{a}\mathrm{Li}_2(-R^{a/b}) - \frac{a}{b^2}\mathrm{Li}_2(-R) - \frac{a}{b^2}\ln R\ln(R+1)\end{aligned}$$

But we know that the dilogarithm satisfies

$$-\mathrm{Li}_2(-x)=\mathrm{Li}_2\left(\frac{x}{1+x}\right)+\frac{1}{2}\ln^2(x+1)=\zeta(2)+\frac{1}{2}\ln^2 x+o(1)$$

as $x\to\infty$. Plugging this identity, it follows that

$$\frac{1}{b}\int_0^R\left(\frac{\ln(1+t^{a/b})}{t}-\frac{a}{b}\frac{\ln t}{1+t}\right)\mathrm{d}t=\frac{a^2+b^2}{ab^2}\zeta(2)+o(1)$$

as $R\to\infty$, and therefore we obtain

$$I(a,b)=\int_{-\infty}^{\infty}\frac{\ln(1+e^{ax})}{1+e^{bx}}\,\mathrm{d}x=\frac{\pi^2}{12}\frac{a^2+b^2}{ab^2}$$

□

Exercise 1.2.187: Prove:

$$\int_0^1\frac{\tanh^{-1}(x)\ln(x)}{x(1-x)(1+x)}\mathrm{d}x=-\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)$$

Proof: 1. Reduction to Euler Integrals

$$\begin{aligned}I&=\int_0^1\frac{\ln(x)\mathrm{artanh}(x)}{x(1-x)(1+x)}\mathrm{d}x=\frac{1}{2}\int_0^1\frac{\ln(1+x)\ln(x)-\ln(1-x)\ln(x)}{x(1-x)(1+x)}\mathrm{d}x\\&=\frac{1}{4}\int_0^1\frac{\ln(1+x)\ln(x)}{1-x}\mathrm{d}x+\frac{1}{2}\int_0^1\frac{\ln(1+x)\ln(x)}{x}\mathrm{d}x-\frac{1}{4}\int_0^1\frac{\ln(1+x)\ln(x)}{1+x}\mathrm{d}x\\&\quad-\frac{1}{4}\int_0^1\frac{\ln(1-x)\ln(x)}{1-x}\mathrm{d}x-\frac{1}{2}\int_0^1\frac{\ln(1-x)\ln(x)}{x}\mathrm{d}x+\frac{1}{4}\int_0^1\frac{\ln(1-x)\ln(x)}{1+x}\mathrm{d}x\end{aligned}$$

2. Evaluation of Euler Integrals

1. Two of these Euler integrals have been evaluated before:Three Logarithmic Integrals. For example integration by parts gives

$$\begin{aligned}\int_0^1\frac{\ln(1+x)\ln(x)}{1+x}\mathrm{d}x&=\frac{1}{2}\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x=\frac{\zeta(3)}{8}\\\int_0^1\frac{\ln(1-x)\ln(x)}{1-x}\mathrm{d}x&=-\frac{1}{2}\int_0^1\frac{\ln^2(1-x)}{x}\mathrm{d}x=-\zeta(3)\end{aligned}$$

2. Mathematica gives the value

$$\int_0^1\frac{\ln(1+x)\ln(x)}{1-x}\mathrm{d}x+\int_0^1\frac{\ln(1-x)\ln(x)}{1+x}\mathrm{d}x=-\frac{\pi^2}{2}\ln(2)+\frac{21}{8}\zeta(3)$$

These two integrals are twins i.e. we may form system of equations to solve each of them. To evaluate the first one note that

$$\begin{aligned}&\int_0^1\frac{\ln(1+x)\ln(x)}{1-x}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1\frac{x^n\ln(x)}{1-x}\mathrm{d}x\\&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(H_n^{(2)}-\frac{\pi^2}{6}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}H_n^{(2)}-\frac{\pi^2}{6}\ln(2)\end{aligned}$$

The euler sum is evaluated as follows

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}H_n^{(2)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=1}^{n}\frac{1}{k^2}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_0^1(-r)^{n-1}\,\mathrm{d}r\int_0^1\int_0^1(st)^{k-1}\,\mathrm{d}s\,\mathrm{d}t$$
$$=\sum_{k=1}^{\infty}\int_0^1\left(\sum_{n=k}^{\infty}(-r)^{n-1}\right)\mathrm{d}r\int_0^1\int_0^1(st)^{k-1}\,\mathrm{d}s\,\mathrm{d}t=\sum_{k=1}^{\infty}\int_0^1\frac{(-r)^{k-1}}{1+r}\mathrm{d}r\int_0^1\int_0^1(st)^{k-1}\,\mathrm{d}s\,\mathrm{d}t$$
$$=\int_0^1\int_0^1\int_0^1\frac{1}{(1+r)(1+rst)}\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t=\int_0^1\int_0^1\frac{\ln(1+rs)}{(1+r)(rs)}\mathrm{d}s\,\mathrm{d}r=-\int_0^1\frac{\mathrm{Li}_2(-r)}{(1+r)r}\mathrm{d}r$$
$$=\int_0^1\mathrm{Li}_2(-r)\left(\frac{1}{1+r}-\frac{1}{r}\right)\mathrm{d}r=-\ln(2)\frac{\zeta(2)}{2}+\int_0^1\frac{\ln^2(1+r)}{r}\mathrm{d}r+\frac{\zeta(3)}{4}$$
$$=\zeta(3)-\frac{\zeta(2)\ln(2)}{2}$$

$$\int_0^1\frac{\ln(x)\ln(1+x)}{1-x}\mathrm{d}x=\left(\zeta(3)-\frac{\zeta(2)\ln(2)}{2}\right)-\frac{\pi^2}{6}\ln(2)$$
$$=\zeta(3)-\frac{\pi^2}{12}\ln(2)-\frac{\pi^2}{6}\ln(2)=\zeta(3)-\frac{\pi^2}{4}\ln(2)$$

The result now follows from integration by parts and the third integral of "Three Logarithmic Integrals".

3. The remaining two integrals are evaluated as follows:

$$\int_0^1\frac{\ln(x)\ln(1+x)}{x}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1x^{n-1}\ln(x)\mathrm{d}x=-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}=-\frac{3\zeta(3)}{4}$$
$$\int_0^1\frac{\ln(x)\ln(1-x)}{x}\mathrm{d}x=-\sum_{n=1}^{\infty}\int_0^1x^{n-1}\ln(x)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(3)$$

3. Final Answer

Combine all the results and we get

$$\int_0^1\frac{\tanh^{-1}(x)\ln(x)}{x(1-x)(1+x)}\mathrm{d}x=-\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)$$

□

Exercise 1.2.188: Evaluate:

$$\int_0^{\infty}e^{-at}\sin(bt)\frac{\ln t}{t}\,\mathrm{d}t$$

Solution: We can start by the following integral

$$I(s)=\int_0^{\infty}t^{s-1}e^{-at}\sin(bt)\mathrm{d}t=\int_0^{\infty}t^{s-1}e^{-at}\sum_{n\geq0}\frac{(-1)^n(bt)^{2n+1}}{\Gamma(2n+2)}\mathrm{d}t$$
$$=\sum_{n\geq0}\frac{(-1)^n(b)^{2n+1}}{\Gamma(2n+2)}\int_0^{\infty}t^{s+2n}e^{-at}\mathrm{d}t=\frac{1}{a^s}\sum_{n\geq0}\frac{(-1)^n(b)^{2n+1}\Gamma(s+2n+1)}{\Gamma(2n+2)a^{2n+1}}$$

$$I'(0)=\int_0^{\infty}e^{-at}\sin(bt)\frac{\ln t}{t}\,\mathrm{d}t=\sum_{n\geq0}\frac{(-1)^n\psi_0(2n+1)}{2n+1}\left(\frac{b}{a}\right)^{2n+1}$$
$$-\ln(a)\sum_{n\geq0}\frac{(-1)^n}{2n+1}\left(\frac{b}{a}\right)^{2n+1}=\sum_{n\geq0}\frac{(-1)^nH_{2n}-\gamma}{2n+1}\left(\frac{b}{a}\right)^{2n+1}-\ln(a)\arctan\left(\frac{b}{a}\right)$$
$$=\sum_{n\geq0}\frac{(-1)^nH_{2n}}{2n+1}\left(\frac{b}{a}\right)^{2n+1}-(\gamma+\ln(a))\arctan\left(\frac{b}{a}\right)$$

We first solve the following

$$\sum_{k\geq 0}(-1)^k H_{2k}x^{2k} = \sum_{k\geq 0}(-1)^k x^{2k}\int_0^1 \frac{1-t^{2k}}{1-t}\,\mathrm{d}t = \int_0^1 \frac{1}{1-t}\sum_{k\geq 0}(-1)^k x^{2k}\left(1-t^{2k}\right)\,\mathrm{d}t$$
$$= \int_0^1 \frac{1}{1-t}\sum_{k\geq 0}(-1)^k\left(x^{2k}-(xt)^{2k}\right)\,\mathrm{d}t = \int_0^1 \frac{1}{1-t}\left(\frac{1}{1+x^2}-\frac{1}{1+t^2x^2}\right)\,\mathrm{d}t$$
$$= \frac{1}{1+x^2}\int_0^1 \frac{1+t^2x^2-1-x^2}{(1-t)(1+t^2x^2)}\,\mathrm{d}t = \frac{-x^2}{1+x^2}\int_0^1 \frac{(1-t^2)}{(1-t)(1+t^2x^2)}\,\mathrm{d}t$$
$$= \frac{-x^2}{1+x^2}\int_0^1 \frac{1+t}{(1+t^2x^2)}\,\mathrm{d}t = \frac{-x^2}{1+x^2}\left(\int_0^1 \frac{1}{1+t^2x^2}+\frac{t}{1+t^2x^2}\,\mathrm{d}t\right)$$
$$= \frac{-1}{2(1+x^2)}\left(2x\arctan(x)+\ln(1+x^2)\right)$$

Using this we conclude by integrating

$$\sum_{k\geq 0}\frac{(-1)^k H_{2k}}{2k+1}x^{2k} = -\frac{1}{2}\ln(1+x^2)\arctan(x)$$

$$\sum_{k\geq 0}\frac{(-1)^k H_{2k}}{2k+1}\left(\frac{b}{a}\right)^{2k+1} = -\frac{1}{2}\ln\left(\frac{a^2+b^2}{a^2}\right)\arctan\left(\frac{b}{a}\right)$$

Hence the following

$$\int_0^\infty e^{-at}\sin(bt)\frac{\ln t}{t}\,\mathrm{d}t = -\left(\frac{1}{2}\ln\left(\frac{a^2+b^2}{a^2}\right)+\gamma+\ln(a)\right)\arctan\left(\frac{b}{a}\right)$$
$$= -\left(\frac{\ln(a^2+b^2)}{2}+\gamma\right)\arctan\left(\frac{b}{a}\right)$$

□

Exercise 1.2.189: Prove:

$$\int_0^\infty \frac{x^2+1}{x^4+x^2+1}\frac{\ln(1-x+x^2)}{\ln(x)}\mathrm{d}x = \frac{\pi}{\sqrt{3}}$$

Proof: By observing that under the transformation $x\mapsto x^{-1}$,

$$\int_0^\infty \frac{x^2+1}{x^4+x^2+1}\frac{\ln(x^2-x+1)}{\ln x}\,\mathrm{d}x = I = -\int_0^\infty \frac{x^2+1}{x^4+x^2+1}\frac{\ln(x^2-x+1)-2\ln x}{\ln x}\,\mathrm{d}x$$
$$= -I+2\int_0^\infty \frac{x^2+1}{x^4+x^2+1}\,\mathrm{d}x$$

This shows that

$$I = \int_0^\infty \frac{x^2+1}{x^4+x^2+1}\,\mathrm{d}x = \int_0^\infty \frac{1}{(x-x^{-1})^2+3}\cdot\left(1+\frac{1}{x^2}\right)\,\mathrm{d}x = \int_{-\infty}^\infty \frac{1}{u^2+3}\,\mathrm{d}u$$
$$= \frac{\pi}{\sqrt{3}}$$

□

Exercise 1.2.190: Prove:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3(x)}{2+\sin(x)}\mathrm{d}x = \left(\frac{9}{4}-\frac{8\sqrt{3}}{9}\right)\pi-2$$

☞ **Proof:** By noting that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{2+\sin x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \left(\sin^2 x - 2\sin x + 4 - \frac{8}{2+\sin x}\right)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \left(\sin^2 x - 2\sin x + 4\right)\,\mathrm{d}x - 8\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin x} \\ &= \left[\frac{9x}{2} + 2\cos x - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} - 8\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin x}\end{aligned}$$

it is sufficient to show that

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3(x)}{2+\sin(x)}\mathrm{d}x = \left(\frac{9}{4} - \frac{8\sqrt{3}}{9}\right)\pi - 2$$

Of course, we can evaluate the integral using real analysis technique in many ways, one of which is as follows:

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin x} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin 2x} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\sin x\cos x} \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,\mathrm{d}x}{\sec^2 x + \tan x} = \frac{1}{2}\int_0^{\infty} \frac{\mathrm{d}t}{t^2+t+1} = \int_0^1 \frac{\mathrm{d}t}{t^2+t+1}\end{aligned}$$

But here we exploit some contour integration technique. Since $2+\sin z = 0$ if and only if $\Re z \in \frac{3\pi}{2} + 2\pi\mathbb{Z}$ and $\Im z = \operatorname{arcosh} 2$, the function $(2+\sin z)^{-1}$ is holomorphic on $\left[0, \frac{\pi}{2}\right] \times [0,\infty)$ and vanishes as $\Im z \to \infty$. Thus by replacing the contour of integration,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin x} = \Re\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\sin x} = \Re\left(\int_0^{i\infty} \frac{\mathrm{d}z}{2+\sin z} + \int_{\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}} \frac{\mathrm{d}z}{2+\sin z}\right) \\ &= \Re\left(\int_0^{\infty} \frac{i\mathrm{d}x}{2+i\sinh x} - \int_0^{\infty} \frac{i\mathrm{d}x}{2+\cosh x}\right) = \Re\int_0^{\infty} \frac{\mathrm{d}x}{\sinh x - 2i} \\ &= \int_0^{\infty} \frac{\sinh x}{\sinh^2 x + 4}\,\mathrm{d}x = \int_0^{\infty} \frac{\sinh x}{\cosh^2 x + 3}\,\mathrm{d}x = \int_1^{\infty} \frac{\mathrm{d}u}{u^2+3}\end{aligned}$$

and the result follows. □

Exercise 1.2.191: Evaluate:

$$\int_0^{\infty} \sin\left(x^n\right)\mathrm{d}x$$

Solution:

$$\int_0^\infty \sin\left(x^n\right) \mathrm{d}x = \frac{1}{n}\int_0^\infty x^{\frac{1}{n}-1}\sin(x)\ \mathrm{d}x = \frac{1}{n\Gamma\left(1-\frac{1}{n}\right)}\int_0^\infty \left(\int_0^\infty u^{-\frac{1}{n}}e^{-xu}du\right)\sin(x)\ \mathrm{d}x$$

$$= \frac{1}{n\Gamma\left(1-\frac{1}{n}\right)}\int_0^\infty u^{-\frac{1}{n}}\left(\int_0^\infty e^{-xu}\sin(x)\ dx\right)\mathrm{d}u = \frac{1}{n\Gamma\left(1-\frac{1}{n}\right)}\int_0^\infty \frac{u^{-\frac{1}{n}}}{1+u^2}\mathrm{d}u$$

$$= \frac{1}{n\Gamma\left(1-\frac{1}{n}\right)}\int_0^{\frac{\pi}{2}} \tan^{-\frac{1}{n}}(\theta)\mathrm{d}\theta = \frac{1}{n\Gamma\left(1-\frac{1}{n}\right)}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{n}}(\theta)\cos^{\frac{1}{n}}(\theta)\mathrm{d}\theta$$

$$= \frac{1}{2n\Gamma\left(1-\frac{1}{n}\right)}\mathrm{B}\left(\frac{1-n}{2},\frac{1+n}{2}\right) = \frac{1}{2n\Gamma\left(1-\frac{1}{n}\right)}\Gamma\left(\frac{n-1}{2n}\right)\Gamma\left(\frac{n+1}{2n}\right)$$

$$= \frac{\sin\left(\frac{\pi}{n}\right)}{2n\cos\left(\frac{\pi}{2n}\right)}\Gamma\left(\frac{1}{n}\right) = \frac{1}{n}\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{n}\right)$$

□

Exercise 1.2.192: Evaluate:

$$\int_0^\infty \frac{\ln\left(1+\frac{\pi^2}{4x}\right)}{e^{\sqrt{x}}-1}\mathrm{d}x$$

Solution: Step 1

Let $I = \int_0^\infty \frac{\ln\left(1+\frac{\pi^2}{4x}\right)}{e^{\sqrt{x}}-1}\mathrm{d}x$. Now put $x \mapsto x^2$

$$I = 2\int_0^\infty \frac{x\ln\left(1+\frac{\pi^2}{4x^2}\right)}{e^x-1}\mathrm{d}x = 2\int_0^\infty \frac{x\ln\left(4x^2+\pi^2\right)-x\ln(4)-2x\ln(x)}{e^x-1}\mathrm{d}x$$

$$= 2\underbrace{\int_0^\infty \frac{x\ln\left(4x^2+\pi^2\right)}{e^x-1}\mathrm{d}x}_{=I_1} - 4\ln(2)\underbrace{\int_0^\infty \frac{x}{e^x-1}\mathrm{d}x}_{=I_2} - 4\underbrace{\int_0^\infty \frac{x\ln(x)}{e^x-1}\mathrm{d}x}_{=I_3} \tag{1}$$

Step 2 - Evaluation of I_3

Note that for $\Re(z) > 1$, we have

$$\int_0^\infty \frac{x^{z-1}}{e^x-1}\mathrm{d}x = \Gamma(z)\zeta(z) \tag{2}$$

Differentiate both sides with respect to z.

$$\int_0^\infty \frac{x^{z-1}\ln(x)}{e^x-1}\mathrm{d}x = \psi_0(z)\Gamma(z)\zeta(z) + \Gamma(z)\zeta'(z) \tag{3}$$

Put $z = 2$ to obtain

$$\int_0^\infty \frac{x\ln(x)}{e^x-1}\mathrm{d}x = \frac{\pi^2}{6}(1-\gamma) - \frac{\pi^2}{6}\left(12\ln(\mathbf{A}) - \gamma - \ln(2\pi)\right)$$

$$= \frac{\pi^2}{6}\left(1+\ln(2\pi) - 12\ln(\mathbf{A})\right) \tag{4}$$

Step 3 - Evaluation of I_2

I_2 can be evaluated easily by means of equation (2).

$$\int_0^\infty \frac{x}{e^x-1}\mathrm{d}x = \frac{\pi^2}{6} \tag{5}$$

Step 4 - Evaluation of I_1

From here begins the dirty job. We require equation (4) of Adamchik's paper

$$\int_0^\infty \frac{x\ln(x^2+z^2)}{e^{2\pi x}-1}\mathrm{d}x = \zeta'(-1,z) - \frac{z^2}{2}\ln z + \frac{z^2}{4} + \frac{z}{2}\ln z - 2z\int_0^\infty \frac{\arctan\left(\frac{x}{z}\right)}{e^{2\pi x}-1}\mathrm{d}x \quad \Re(z)>0$$

Now, from **Binet's second formula** we have

$$\int_0^\infty \frac{x\ln(x^2+z^2)}{e^{2\pi x}-1}\mathrm{d}x = \zeta'(-1,z) - \frac{z^2}{2}\ln z + \frac{z^2}{4} + \frac{z}{2}\ln z - z\left\{\ln\Gamma(z) - \left(z-\frac{1}{2}\right)\ln z + z - \frac{1}{2}\ln(2\pi)\right\} \tag{6}$$

Set $x \mapsto \frac{x}{2\pi}$ and put $z = \frac{1}{4}$.

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_0^\infty \frac{x\ln\left(4x^2+\pi^2\right)}{e^x-1}\mathrm{d}x - \frac{\ln 2+\ln(2\pi)}{12}\\
=&\zeta'\left(-1,\frac{1}{4}\right) + \frac{1}{16}\ln(2) + \frac{1}{64} - \frac{1}{4}\ln(2) - \frac{1}{4}\left\{\ln\Gamma\left(\frac{1}{4}\right) - \frac{\ln(2)}{2} + \frac{1}{4} - \frac{1}{2}\ln(2\pi)\right\}\\
&\int_0^\infty \frac{x\ln\left(4x^2+\pi^2\right)}{e^x-1}\mathrm{d}x\\
=&4\pi^2\zeta'\left(-1,\frac{1}{4}\right) + \frac{\pi^2}{12}\ln(2) + \frac{5}{6}\pi^2\ln(2\pi) - \frac{3}{16}\pi^2 - \pi^2\ln\Gamma\left(\frac{1}{4}\right)
\end{aligned} \tag{7}$$

Now, from equations (3) and (11) of Adamchik's paper we get

$$\begin{aligned}
\zeta'\left(-1,\frac{1}{4}\right) &= \zeta'(-1) - \frac{3}{4}\ln\Gamma\left(\frac{1}{4}\right) - \ln \mathrm{G}\left(\frac{1}{4}\right)\\
&= \frac{1}{12} - \ln(\mathbf{A}) - \frac{3}{4}\ln\Gamma\left(\frac{1}{4}\right) - \left(\frac{3}{32} - \frac{\mathbf{G}}{4\pi} - \frac{3}{4}\ln\Gamma\left(\frac{1}{4}\right) - \frac{9}{8}\ln(\mathbf{A})\right)\\
&= \frac{\ln(\mathbf{A})}{8} + \frac{\mathbf{G}}{4\pi} - \frac{1}{96}
\end{aligned} \tag{8}$$

where $\mathrm{G}(z)$ denotes the Barnes G Function.

Using this result in (7), we get

$$\begin{aligned}
&\int_0^\infty \frac{x\ln\left(4x^2+\pi^2\right)}{e^x-1}\mathrm{d}x\\
&= 4\pi^2\left(\frac{\ln(\mathbf{A})}{8} + \frac{\mathbf{G}}{4\pi} - \frac{1}{96}\right) + \frac{\pi^2}{12}\ln(2) + \frac{5}{6}\pi^2\ln(2\pi) - \frac{3}{16}\pi^2 - \pi^2\ln\Gamma\left(\frac{1}{4}\right)\\
&= \frac{\pi^2}{2}\ln(\mathbf{A}) + \frac{\pi^2}{12}\ln(2) - \frac{11}{48}\pi^2 + \frac{5\pi^2}{6}\ln(2\pi) + \pi\mathbf{G} - \pi^2\ln\Gamma\left(\frac{1}{4}\right)
\end{aligned} \tag{9}$$

Step 5 - Final Answer

Combining everything with the help of equation (1), we get

$$I = 2\left(\frac{\pi^2}{2}\ln(\mathbf{A}) + \frac{\pi^2}{12}\ln(2) - \frac{11}{48}\pi^2 + \frac{5\pi^2}{6}\ln(2\pi) + \pi\mathbf{G} - \pi^2\ln\Gamma\left(\frac{1}{4}\right)\right)$$
$$- 4\ln(2)\left(\frac{\pi^2}{6}\right) - 4\left(\frac{\pi^2}{6}\left(1 + \ln(2\pi) - 12\ln(\mathbf{A})\right)\right)$$
$$= \pi^2\left\{\ln\left(\frac{\pi\mathbf{A}^9\sqrt{2}}{\Gamma^2\left(\frac{1}{4}\right)}\right) - \frac{9}{8}\right\} + 2\pi\mathbf{G}$$

□

Exercise 1.2.193: Evaluate:

$$\int_0^\infty \frac{\ln\left(x^{4n}+1\right)}{x^2+1}\mathrm{d}x$$

Solution: Let $f(t) = \int_0^\infty \frac{\ln(1+tx^{4n})}{1+x^2}\mathrm{d}x \quad n \in \mathbb{N}$

1. Calculation of $f'(t)$

$$f'(t) = \int_0^\infty \frac{x^{4n}}{(1+x^2)(1+tx^{4n})}\mathrm{d}x$$

Here we can use the partial fraction decomposition

$$\frac{x^{4n}}{(1+x^2)(1+tx^{4n})} = \frac{1}{(1+t)(1+x^2)} + \frac{1}{(1+t)(1+x^{4n})}\sum_{r=0}^{n-1}\left(-x^{4r} + x^{4r+2}\right)$$

Thus, integrating term by term gives

$$f'(t) = \frac{\pi}{2(1+t)} + \frac{1}{1+t}\sum_{r=0}^{n-1}\left(-\int_0^\infty \frac{x^{4r}}{1+tx^{4n}}\mathrm{d}x + \int_0^\infty \frac{x^{4r+2}}{1+x^{4n}}\mathrm{d}x\right)$$
$$= \frac{\pi}{2(1+t)} + \frac{\pi}{1+t}\sum_{r=0}^{n-1}\left(-\frac{t^{-\frac{1+4r}{4n}}}{4n\sin\left(\frac{\pi+4\pi r}{4n}\right)} + \frac{t^{-\frac{3+4r}{4n}}}{4n\sin\left(\frac{3\pi+4\pi r}{4n}\right)}\right)$$

2. Calculating $f(1)$

Since $f(0) = 0$, we have

$$f(1) = \int_0^1 f'(t)\mathrm{d}t = \frac{\pi}{2}\ln(2) + \pi\int_0^1 \frac{1}{1+t}\sum_{r=0}^{n-1}\left(-\frac{t^{-\frac{1+4r}{4n}}}{4n\sin\left(\frac{\pi+4\pi r}{4n}\right)} + \frac{t^{-\frac{3+4r}{4n}}}{4n\sin\left(\frac{3\pi+4\pi r}{4n}\right)}\right)\mathrm{d}t$$
$$= \frac{\pi}{2}\ln(2) - \pi\sum_{r=0}^{n-1}\frac{1}{4n\sin\left(\frac{\pi+4\pi r}{4n}\right)}\int_0^1 \frac{t^{-\frac{1+4r}{4n}}}{1+t}\mathrm{d}t + \pi\sum_{r=0}^{n-1}\frac{1}{4n\sin\left(\frac{3\pi+4\pi r}{4n}\right)}\int_0^1 \frac{t^{-\frac{3+4r}{4n}}}{1+t}\mathrm{d}t$$
$$= \frac{\pi}{2}\ln(2) + \frac{\pi}{8}\sum_{r=0}^{n-1}\left[\frac{1}{n\sin\left(\frac{\pi+4\pi r}{4n}\right)}\left(\psi_0\left(\frac{4n-4r-1}{8n}\right) - \psi_0\left(\frac{8n-4r-1}{8n}\right)\right)\right.$$
$$\left. + \frac{1}{n\sin\left(\frac{3\pi+4\pi r}{4n}\right)}\left(-\psi_0\left(\frac{4n-4r-3}{8n}\right) + \psi_0\left(\frac{8n-4r-3}{8n}\right)\right)\right]$$

3. Evaluation of the Integral for $n = 1, 2, 3, \cdots$

For $n = 1$, we have

$$\begin{aligned}\int_0^\infty \frac{\ln(1+x^4)}{1+x^2}\mathrm{d}x &= \frac{\pi}{2}\ln(2) + \frac{\pi}{4\sqrt{2}}\left[\psi_0\left(\frac{3}{8}\right) - \psi_0\left(\frac{7}{8}\right) - \psi_0\left(\frac{1}{8}\right) + \psi_0\left(\frac{5}{8}\right)\right] \\ &= \frac{\pi}{2}\ln\left(6+4\sqrt{2}\right)\end{aligned}$$

and for $n = 2$

$$\begin{aligned}\int_0^\infty \frac{\ln\left(1+x^8\right)}{1+x^2}\mathrm{d}x = \frac{\pi}{2}\ln(2) &+ \frac{\pi}{16\sin\left(\frac{\pi}{8}\right)}\left[\psi_0\left(\frac{7}{16}\right) - \psi_0\left(\frac{15}{16}\right) - \psi_0\left(\frac{1}{16}\right) + \psi_0\left(\frac{9}{16}\right)\right] \\ &+ \frac{\pi}{16\cos\left(\frac{\pi}{8}\right)}\left[-\psi_0\left(\frac{5}{16}\right) + \psi_0\left(\frac{13}{16}\right) + \psi_0\left(\frac{3}{16}\right) - \psi_0\left(\frac{11}{16}\right)\right]\end{aligned}$$

□

Exercise 1.2.194: Evaluate:

$$\int_0^1\int_0^1 \frac{\ln\left(2-xy\right)}{1-xy}\mathrm{d}x\mathrm{d}y$$

Solution:

$$\begin{aligned}&\int_0^1\int_0^1 \frac{\ln(2-xy)}{1-xy}\mathrm{d}x\ \mathrm{d}y = \int_0^1\int_0^1 \frac{\ln\left(1+(1-xy)\right)}{1-xy}\mathrm{d}x\ \mathrm{d}y \\ &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\int_0^1\int_0^1 (1-xy)^{n-1}\mathrm{d}x\ \mathrm{d}y = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}\int_0^1\left(\frac{1-(1-y)^n}{y}\right)\mathrm{d}y \\ &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}H_n = \frac{5}{8}\zeta(3)\end{aligned}$$

The last Euler Sum's proof can be seen at **Exercise 2.1.24**. □

Exercise 1.2.195: Prove:

$$\int_0^1 \frac{\ln(1+x^2)}{1+x}\mathrm{d}x = \frac{3}{4}\ln^2(2) - \frac{\pi^2}{48}$$

Proof: 1. Reduction to Euler Sums

$$\begin{aligned}I &= \int_0^1 \frac{\ln(1+x^2)}{1+x}\mathrm{d}x = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\int_0^1 \frac{x^{2n}}{1+x}\mathrm{d}x \\ &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n}\left(\psi_0(n+1) - \psi_0\left(n+\frac{1}{2}\right)\right) \\ &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n}\left(\sum_{k=1}^n \frac{1}{k} + 2\ln(2) - 2\sum_{k=1}^n \frac{1}{2k-1}\right) \\ &= \ln^2(2) + \frac{1}{2}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sum_{k=1}^n \frac{1}{k} - \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sum_{k=1}^n \frac{1}{2k-1}\end{aligned}$$

2. Evaluation of Euler Sums Let's first evaluate $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sum_{k=1}^n \frac{1}{2k-1}$.

I am going to use $\displaystyle\int_0^1 (-s)^{n-1}\mathrm{d}s = \frac{(-1)^{n+1}}{n}$ and $\displaystyle\int_0^1 r^{2k-2}\mathrm{d}r = \frac{1}{2k-1}$.

Plugging these into the Euler Sum, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=1}^{n}\frac{1}{2k-1}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_0^1(-s)^{n-1}\mathrm{d}s\int_0^1 r^{2k-2}\mathrm{d}r\\
&=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\int_0^1(-s)^{n-1}\mathrm{d}s\int_0^1 r^{2k-2}\mathrm{d}r=\sum_{k=1}^{\infty}\int_0^1\frac{(-s)^{k-1}}{1+s}\mathrm{d}s\int_0^1 r^{2k-2}\mathrm{d}r\\
&=\int_0^1\int_0^1\sum_{k=1}^{\infty}\frac{(-sr^2)^{k-1}}{1+s}\mathrm{d}s\mathrm{d}r=\int_0^1\int_0^1\frac{1}{(1+s)(1+sr^2)}\mathrm{d}s\,\mathrm{d}r\\
&=\int_0^1\frac{1}{1+s}\frac{\arctan\sqrt{s}}{\sqrt{s}}\mathrm{d}s=2\int_0^1\frac{\arctan t}{1+t^2}\mathrm{d}t=\frac{\pi^2}{16}
\end{aligned}$$

In a similar way the other Euler Sum is evaluated easily.

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{n=1}^{k}\frac{1}{n}=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_0^1(-x)^{k-1}\mathrm{d}x\int_0^1 y^{n-1}\mathrm{d}y\\
&=\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\int_0^1(-x)^{k-1}\mathrm{d}x\int_0^1 y^{n-1}\mathrm{d}y=\sum_{n=1}^{\infty}\int_0^1\frac{(-x)^{n-1}}{1+x}\mathrm{d}x\int_0^1 y^{n-1}\mathrm{d}y\\
&=\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{(-xy)^{n-1}}{1+x}\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^1\frac{1}{(1+x)(1+xy)}\mathrm{d}x\mathrm{d}y\\
&=\int_0^1\int_0^1\frac{1}{(1+x)(1+xy)}\mathrm{d}y\mathrm{d}x=\int_0^1\frac{\ln(1+x)}{x(1+x)}\mathrm{d}x\\
&=\int_0^1\frac{\ln(1+x)}{x}\mathrm{d}x-\int_0^1\frac{\ln(1+x)}{(1+x)}\mathrm{d}x=\frac{\zeta(2)}{2}-\frac{\ln^2 2}{2}
\end{aligned}$$

Here, I used the calculation

$$\int_0^1\frac{\ln(1+x)}{x}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{n-1}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\zeta(2)}{2}$$

3. Final Answer

Now, we can combine all these efforts to give the final answer.

$$I=\ln^2(2)+\frac{1}{2}\left(\frac{\pi^2}{12}-\frac{\ln^2(2)}{2}\right)-\frac{\pi^2}{16}=\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}$$

□

Exercise 1.2.196: Evaluate:

$$\int_0^1\ln(1+x)\ln(x)\mathrm{d}x\quad,\quad\int_0^1 x\ln(1+x)\ln(x)\mathrm{d}x$$

Solution:

$$\begin{aligned}
&\int_0^1 x\ln(1+x)\ln(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{n+1}\ln(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)^2}\\
&=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{4n}-\frac{1}{2(n+2)^2}-\frac{1}{4(n+2)}\right)=-\frac{1}{4}\ln(2)-\frac{9}{24}+\frac{\pi^2}{24}-\frac{1}{8}+\frac{1}{4}\ln(2)\\
&=\frac{\pi^2}{24}-\frac{1}{2}
\end{aligned}$$

$$\int_0^1 \ln(x)\ln(1+x)dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^n\ln(x)\,dx = \sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)^2}$$
$$= \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{(1+n)^2}-\frac{1}{1+n}\right) = -\ln(2)+1-\frac{\pi^2}{12}+1-\ln(2)$$
$$= 2-\frac{\pi^2}{12}-2\ln(2)$$

□

Exercise 1.2.197: Prove:

$$\int_0^{\infty}\frac{x^5}{\sinh(\pi x)(1+x^6)}dx = \frac{1}{3}\left(\ln(2)-1+\pi\text{sech}\left(\frac{\sqrt{3}\pi}{2}\right)\right)$$

Proof: We can prove using residues/digamma that

$$\frac{\pi\text{csch}(\pi z)}{2} = \frac{1}{2z}+\sum_{n=1}^{\infty}\frac{(-1)^n z}{z^2+n^2}$$

Plugging this series into the integral we obtain

$$I = \frac{2}{\pi}\int_0^{\infty}\frac{x^5}{1+x^6}\left(\frac{1}{2x}+\sum_{n=1}^{\infty}\frac{(-1)^n x}{x^2+n^2}\right)dx$$
$$= \frac{1}{\pi}\int_0^{\infty}\frac{x^4}{1+x^6}dx+\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty}\frac{x^6}{(1+x^6)(x^2+n^2)}dx$$

Again using residues it can be shown that

$$\frac{1}{\pi}\int_0^{\infty}\frac{x^4}{1+x^6}dx = \frac{1}{3}\quad,\quad \frac{2}{\pi}\int_0^{\infty}\frac{x^6}{(1+x^6)(x^2+n^2)}dx = \frac{1}{3}\left(\frac{1}{1+n}+\frac{2n+1}{n^2+n+1}\right)$$

So we obtain

$$I = \frac{1}{3}+\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{1+n}+\frac{2n+1}{n^2+n+1}\right)$$

The first series is simply

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n} = -1+\ln(2)$$

The second series is evaluated using digamma/residues.

$$\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n^2+n+1} = -1+\pi\text{sech}\left(\frac{\sqrt{3}\pi}{2}\right)$$

So after combining all our results we get

$$I = \int_0^{\infty}\frac{x^5}{\sinh(\pi x)(1+x^6)}dx = \frac{1}{3}\left(\ln(2)-1+\pi\text{sech}\left(\frac{\sqrt{3}\pi}{2}\right)\right)$$

□

Exercise 1.2.198: Prove:

$$\int_0^{\infty}\left\{(2C(x)-1)^2+(2S(x)-1)^2\right\}dx = \frac{4}{\pi}$$

☞ **Proof:** **Method One:** Let

$$f(x)=\left(\frac{1}{2}-S(x)\right)\cos\left(\frac{\pi}{2}x^2\right)-\left(\frac{1}{2}-C(x)\right)\sin\left(\frac{\pi}{2}x^2\right)$$
$$g(x)=\left(\frac{1}{2}-C(x)\right)\cos\left(\frac{\pi}{2}x^2\right)+\left(\frac{1}{2}-S(x)\right)\sin\left(\frac{\pi}{2}x^2\right)$$

Note that

$$g(x)+if(x)=e^{-\frac{\pi}{2}ix^2}\int_x^\infty e^{\frac{i\pi t^2}{2}}\mathrm{d}t=xe^{-\frac{\pi}{2}ix^2}\int_1^\infty e^{\frac{i\pi x^2y^2}{2}}\mathrm{d}y$$
$$g(x)-if(x)=e^{\frac{\pi}{2}ix^2}\int_x^\infty e^{\frac{-i\pi t^2}{2}}\mathrm{d}t=xe^{\frac{\pi}{2}ix^2}\int_1^\infty e^{\frac{-i\pi x^2z^2}{2}}\mathrm{d}z$$

and

$$(2S(x)-1)^2+(2C(x)-1)^2=4(f(x)^2+g(x)^2)=4x^2\int_1^\infty\int_1^\infty e^{-\frac{i\pi}{2}(y^2-z^2)}\mathrm{d}z\ \mathrm{d}y$$

After the substitutions $s=y+z$ and $r=yz$ we get

$$(2S(x)-1)^2+(2C(x)-1)^2=2x^2\int_2^\infty\int_{-(s+2)}^{s+2}e^{i\pi x^2rs/2}\mathrm{d}r\mathrm{d}s$$

This can be further simplified to

$$(2S(x)-1)^2+(2C(x)-1)^2=\frac{8}{\pi}\int_1^\infty\frac{\sin(2\pi x^2s(s-1))}{s}\mathrm{d}s$$

Finally we have

$$\int_0^\infty\left\{(2S(x)-1)^2+(2C(x)-1)^2\right\}\mathrm{d}x=\frac{8}{\pi}\int_1^\infty\frac{1}{s}\int_0^\infty\sin(2\pi x^2s(s-1))\mathrm{d}x\ \mathrm{d}s$$
$$=\frac{2}{\pi}\int_1^\infty\frac{1}{s^{\frac{3}{2}}(s-1)^{\frac{1}{2}}}\mathrm{d}s=\frac{2}{\pi}\int_1^\infty\frac{1}{s^{\frac{3}{2}}(s-1)^{\frac{1}{2}}}\mathrm{d}s=\frac{2}{\pi}\int_0^1\frac{1}{(1-u)^{\frac{1}{2}}}\mathrm{d}u=\frac{4}{\pi}$$

Method Two: Let

$$f(x)=\int_x^\infty\frac{\cos t}{\sqrt{t}}\,\mathrm{d}t=\sqrt{\frac{\pi}{2}}\left[1-2S\left(\sqrt{\frac{2x}{\pi}}\right)\right]$$
$$g(x)=\int_x^\infty\frac{\sin t}{\sqrt{t}}\,\mathrm{d}t=\sqrt{\frac{\pi}{2}}\left[1-2C\left(\sqrt{\frac{2x}{\pi}}\right)\right]$$

Then the integral in question reduces to

$$I=\frac{\sqrt{2}}{\pi\sqrt{\pi}}\int_0^\infty\frac{1}{\sqrt{x}}\left(f(x)^2+g(x)^2\right)\,\mathrm{d}x$$

Since both $f(x)$ and $g(x)$ has decay of order $x^{1/2}$,

$$\int_0^\infty\frac{1}{\sqrt{x}}f(x)^2\,\mathrm{d}x=\left[2\sqrt{x}f(x)^2\right]_0^\infty-\int_0^\infty4\sqrt{x}f'(x)f(x)\,\mathrm{d}x=4\int_0^\infty f(x)\cos x\,\mathrm{d}x$$
$$=\left[4f(x)\sin x\right]_0^\infty-4\int_0^\infty f'(x)\sin x\,\mathrm{d}x=4\int_0^\infty\frac{\sin x\cos x}{\sqrt{x}}\,\mathrm{d}x$$

Similarly,

$$\int_0^\infty \frac{1}{\sqrt{x}} g(x)^2\,\mathrm{d}x = \Big[2\sqrt{x}g(x)^2\Big]_0^\infty - \int_0^\infty 4\sqrt{x}g'(x)g(x)\,\mathrm{d}x = 4\int_0^\infty g(x)\sin x\,\mathrm{d}x$$
$$= \Big[4g(x)(1-\cos x)\Big]_0^\infty - 4\int_0^\infty g'(x)(1-\cos x)\,\mathrm{d}x = 4\int_0^\infty \frac{\sin x(1-\cos x)}{\sqrt{x}}\,\mathrm{d}x$$

Summing, we have

$$I = \frac{4\sqrt{2}}{\pi\sqrt{\pi}}\int_0^\infty \frac{\sin x}{\sqrt{x}}\,\mathrm{d}x = \frac{4}{\pi}$$

□

Exercise 1.2.199: Prove that

$$\int_0^{\frac{1}{2}} \frac{\ln^2(1-x)}{x}\mathrm{d}x = -\frac{1}{3}\ln^3(2) + \frac{\zeta(3)}{4}$$
$$\int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x = \frac{\zeta(3)}{4}$$
$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = -\frac{5}{8}\zeta(3)$$

Proof: **1. The first integral**

Let

$$I(t) = \int_0^t \frac{\ln^2(1-x)}{x}\mathrm{d}x$$

Repeated integration by parts gives

$$I(t) = \ln(t)\ln(1-t) + \int_0^t \frac{2\ln(x)\ln(1-x)}{1-x}\mathrm{d}x$$
$$= \ln(t)\ln(1-t) + 2\ln(1-t)\mathrm{Li}_2(1-t) + 2\int_0^t \frac{\mathrm{Li}_2(1-x)}{1-x}\mathrm{d}x$$
$$= \ln(t)\ln(1-t) + 2\ln(1-t)\mathrm{Li}_2(1-t) - 2\mathrm{Li}_3(1-t) + 2\zeta(3)$$

Put $t = \frac{1}{2}$

$$I\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{\ln^2(1-x)}{x}\mathrm{d}x = -\ln^3(2) - 2\ln(2)\mathrm{Li}_2\left(\frac{1}{2}\right) - 2\mathrm{Li}_3\left(\frac{1}{2}\right) + 2\zeta(3)$$
$$= -\ln^3(2) - 2\ln(2)\left(\frac{\pi^2}{12} - \frac{\ln^2(2)}{2}\right) - \frac{1}{12}\left(-2\pi^2\ln(2) + 4\ln(2)^3 + 21\zeta(3)\right) + 2\zeta(3)$$
$$= -\frac{1}{3}\ln^2(2) + \frac{\zeta(3)}{4}$$

2. The Second Integral

Method One: Let

$$I(t) = \int_0^t \frac{\ln^2(1+x)}{x}\mathrm{d}x$$

By the substitution $u=\ln(1+x)$, we have

$$\begin{aligned}I(t)&=\int_0^{\ln(1+t)}\frac{u^2}{1-e^{-u}}\mathrm{d}u=\sum_{n=0}^{\infty}\int_0^{\ln(1+t)}u^2e^{-nu}\mathrm{d}u\\&=\frac{\ln^3(1+t)}{3}+\sum_{n=1}^{\infty}\left(\frac{2}{n^3}-\frac{2}{n^3(1+t)^n}-\frac{2\ln(1+t)}{n^2(1+t)^n}-\frac{\ln^2(1+t)}{n(1+t)^n}\right)\\&=\frac{\ln^3(1+t)}{3}+2\zeta(3)-2\mathrm{Li}_3\left(\frac{1}{1+t}\right)-2\ln(1+t)\mathrm{Li}_2\left(\frac{1}{1+t}\right)\\&\quad+\ln^2(1+t)\ln\left(1-\frac{1}{1+t}\right)\end{aligned}$$

Putting $t=1$ gives

$$I\left(1\right)=\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x=\frac{\ln^3(2)}{3}+2\zeta(3)-2\mathrm{Li}_3\left(\frac{1}{2}\right)-2\ln(2)\mathrm{Li}_2\left(\frac{1}{2}\right)-\ln^3(2)=\frac{\zeta(3)}{4}$$

Method Two: Use integration by parts repeatedly

$$\begin{aligned}I(t)&=\int_0^t\frac{\ln^2(1+x)}{x}\mathrm{d}x=\ln(t)\ln^2(1+t)-2\int_0^t\frac{\ln(x)\ln(1+x)}{1+x}\mathrm{d}x\\&=\ln^2(t)\ln(1+t)-2\int_1^{1+t}\frac{\ln(y)}{y}\left(\ln(y)+\ln\left(1-\frac{1}{y}\right)\right)\mathrm{d}y\quad(y=x+1)\\&=\ln(t)\ln^2(1+t)-\frac{2}{3}\ln^3(1+t)-2\ln(1+t)\mathrm{Li}_2\left(\frac{1}{1+t}\right)-2\mathrm{Li}_3\left(\frac{1}{1+t}\right)+2\mathrm{Li}_3(1)\end{aligned}$$

3. The Third Integral

This one can be reduced easily into a Euler Sum.

$$\begin{aligned}\int_0^1\frac{\ln(1+x)\ln(1-x)}{x}\mathrm{d}x&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1x^{n-1}\ln(1-x)\,\mathrm{d}x\\&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{-H_n}{n}\right)=-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n^2}=-\frac{5}{8}\zeta(3)\end{aligned}$$

□

Exercise 1.2.200: For $a>0$, evaluate:

$$\int_0^{\infty}e^{-ax}\frac{e^{-x}}{(1+e^{-x})^2}\mathrm{d}x$$

Solution: Let

$$I=\int_0^{\infty}e^{-ax}\frac{e^{-x}}{(1+e^{-x})^2}\mathrm{d}x$$

By, the substitution, $u=e^x$ we have

$$I=\int_0^1\frac{u^a}{(1+u)^2}\mathrm{d}u$$

Now, use integration by parts

$$I=-\frac{u^a}{1+u}\bigg|_0^1+a\int_0^1\frac{u^{a-1}}{1+u}\mathrm{d}u=-\frac{1}{2}+a\int_0^1\frac{u^{a-1}}{1+u}\mathrm{d}u$$

Expand the denominator into a binomial series:

$$I=-\frac{1}{2}+a\int_0^1 u^{a-1}\sum_{n=0}^{\infty}(-1)^n u^n \mathrm{d}u=-\frac{1}{2}+a\sum_{n=0}^{\infty}\frac{(-1)^n}{a+n}$$

This sum can be expressed in terms of the Digamma Function.

$$I=-\frac{1}{2}+\frac{a}{2}\left[\psi_0\left(\frac{a+1}{2}\right)-\psi_0\left(\frac{a}{2}\right)\right]$$

□

Exercise 1.2.201: Prove:

$$\int_0^1 \frac{\ln(1+x)\ln(1-x)}{x^2}\mathrm{d}x=-\frac{\pi^2}{12}-\ln^2(2)$$

Proof:

$$\begin{aligned}\int_0^1 \frac{\ln(1+x)\ln(1-x)}{x^2}\mathrm{d}x&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{n-2}\ln(1-x)\mathrm{d}x\\&=-\zeta(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n(n+1)}=-\zeta(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}\\&=-\frac{\pi^2}{6}+\left(\frac{\pi^2}{12}-\frac{1}{2}\ln^2(2)\right)-\frac{\ln^2(2)}{2}=-\frac{\pi^2}{12}-\ln^2(2)\end{aligned}$$

□

Exercise 1.2.202: Evaluate:

$$\int_0^{\frac{\pi}{2}}\frac{\cos\left\{(1-2n)\arcsin\left(\frac{\sin x}{\sqrt{2}}\right)\right\}}{\sqrt{1-\frac{\sin^2 x}{2}}}\mathrm{d}x$$

Solution: Let $I(n)$ denote the integral. By the substitution $\frac{\sin x}{\sqrt{2}}=\sin\theta$, we have

$$\begin{aligned}I(n)&=\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{\cos((2n-1)\theta)}{\sqrt{1-2\sin^2\theta}}\,\mathrm{d}\theta=2\int_0^{\frac{\pi}{4}}\frac{\cos((2n-1)\theta)}{\sqrt{2\cos 2\theta}}\,\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{2}}\frac{\cos\left(\left(n-\frac{1}{2}\right)\theta\right)}{\sqrt{2\cos\theta}}\,\mathrm{d}\theta=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{i(n-\frac{1}{2})\theta}}{\sqrt{2\cos\theta}}\,\mathrm{d}\theta\end{aligned}$$

Now let C be the counter-clockwise semicircular contour of radius 1 which is centered at the origin and begins from i to i. Then with the substitution $z=e^{i\theta}$, we have

$$I(n)=\frac{1}{2i}\int_C\frac{z^{n-1}}{\sqrt{z^2+1}}\,\mathrm{d}z$$

Since both $\pm i$ are logarithmic singularities of the integrand, any contour integration taken near these singularities vanish as the contour shrinks to these points. Thus we can neglect

the effect of the presence of singularities and we can choose a straight line from i to i as a new contour. This gives

$$\begin{aligned} I(n) &= \frac{1}{2i}\int_{-i}^{i}\frac{z^{n-1}}{\sqrt{z^2+1}}\,\mathrm{d}z = \frac{i^{n-1}+(-i)^{n-1}}{2}\int_0^1\frac{t^{n-1}}{\sqrt{1-t^2}}\,\mathrm{d}t \\ &= \cos\left(\frac{(n-1)\pi}{2}\right)\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,\mathrm{d}\theta = \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}}{2}\sin\left(\frac{n\pi}{2}\right)\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \end{aligned}$$

□

Exercise 1.2.203: Prove:

$$\int_0^{\frac{\pi}{3}}\arccos\left(\frac{\cos\theta-2}{4\cos\theta+1}\right)\mathrm{d}\theta = 2\pi\left(\arctan\sqrt{\frac{3}{5}}-\frac{\pi}{9}\right)$$

Proof: 1. Generalized Ahmed integrals

Let us define $B(x)=\dfrac{\arctan\sqrt{x}}{\sqrt{x}}=\displaystyle\int_0^1\frac{\mathrm{d}t}{1+xt^2}$, and define the **Ahmed integral** of parameter (p,q,r) by

$$A(p,q,r)=\int_0^1\frac{B(q(px^2+1))}{(r+1)px^2+1}\,\mathrm{d}x$$

The **classical Ahmed integral** then reduces to

$$\int_0^1\frac{\tan^{-1}\sqrt{x^2+\alpha}}{(x^2+1)\sqrt{x^2+\alpha}}\,\mathrm{d}x = A\left(\frac{1}{\alpha},\alpha,\alpha-1\right)$$

We also introduce an associated form

$$\tilde{A}(p,q,r)=A(p,q,r)+B(r(q+1))B\left(\frac{pq}{q+1}\right)$$

2. Two identities involving Ahmed integrals

The following two identities hold for p,q,r:

(1): $\tilde{A}(p,q,r)=\tilde{A}(q,r,p)=\tilde{A}(r,p,q)$

(2):

$$\begin{aligned} &\sqrt{pqr}A(p,q,r)+\frac{1}{\sqrt{pqr}}A\left(\frac{1}{q},\frac{1}{p},\frac{1}{r}\right) \\ &=\frac{\pi}{2}\left(\arctan\sqrt{\frac{1}{r(q+1)}}+\arctan\sqrt{\frac{pr}{p+1}}\right)-\arctan\sqrt{p(r+1)}\cdot\arctan\sqrt{\frac{r+1}{qr}} \end{aligned}$$

In particular, we have

(3): $A\left(\dfrac{1}{\alpha},\alpha,1\right)=\dfrac{\pi}{2}\arctan\sqrt{\dfrac{1}{\alpha+1}}-\dfrac{1}{2}\left(\arctan\sqrt{\dfrac{2}{\alpha}}\right)^2$

3. Transforming Coxeter integrals into Ahmed integrals

Let

$$p=\frac{a-b}{2}\frac{1-\cos\theta_0}{a\cos\theta_0+b},\quad q=\frac{2}{a+b},\quad r=\frac{a+b}{a-b}$$

Then the following identity holds:

(4): $\int_0^{\theta_0} \arctan\sqrt{\dfrac{\cos\theta+1}{a\cos\theta+b}}\,\mathrm{d}\theta = 2\sqrt{pqr}A\,(p,q,r)$

4. Application to this problem

Put $a=\dfrac{5}{3}$, $b=-\dfrac{1}{3}$ and $\theta_0=\dfrac{\pi}{3}$. Then $(p,q,r)=\left(1,\dfrac{3}{2},\dfrac{2}{3}\right)$. Then by putting the formulas together, we have

$$\int_0^{\frac{\pi}{3}} \arccos\left(\frac{\cos\theta-2}{4\cos\theta+1}\right)\mathrm{d}\theta = 2\int_0^{\theta_0}\arctan\sqrt{\frac{\cos\theta+1}{a\cos\theta+b}}\,\mathrm{d}\theta$$
$$= 4A\left(1,\frac{3}{2},\frac{2}{3}\right) = 4A\left(\frac{3}{2},\frac{2}{3},1\right) = 2\pi\left(\arctan\sqrt{\frac{3}{5}}-\frac{\pi}{9}\right)$$

□

Exercise 1.2.204: Prove:

$$\int_0^\infty \frac{\cos(\pi x)}{e^{2\pi\sqrt{x}}-1}\,\mathrm{d}x = \frac{2-\sqrt{2}}{8}$$

Proof: Applying integration by parts,

$$\int_0^\infty \frac{\cos(\pi x)}{e^{2\pi\sqrt{x}}-1}\,\mathrm{d}x = \int_0^\infty \frac{2x\cos(\pi x^2)}{e^{2\pi x}-1}\,\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x = \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x$$

Now, shifting the line of integration by i, we have

$$\int_{-\infty}^\infty \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x = \pi i\mathrm{Res}_{z=i}\frac{\sin(\pi z^2)}{\sinh^2(\pi z)} + \mathrm{PV}\int_{-\infty+i}^{\infty+i}\frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x$$

Calculating the residue, we have $\pi i\mathrm{Res}_{z=i}\dfrac{\sin(\pi z^2)}{\sinh^2(\pi z)} = 2$. And for the PV integral,

$$\mathrm{PV}\int_{-\infty+i}^{\infty+i}\frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x = -\mathrm{PV}\int_{-\infty}^{\infty}\frac{\sin(\pi x^2+2\pi i x)}{\sinh^2(\pi x)}\,\mathrm{d}x = -\int_{-\infty}^{\infty}\frac{\sin(\pi x^2)\cosh(2\pi x)}{\sinh^2(\pi x)}\,\mathrm{d}x$$
$$= -\int_{-\infty}^{\infty}\frac{\sin(\pi x^2)(1+2\sinh^2(\pi x))}{\sinh^2(\pi x)}\,\mathrm{d}x = -\int_{-\infty}^{\infty}\frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x - 2\int_{-\infty}^{\infty}\sin(\pi x^2)\,\mathrm{d}x$$

Since $\displaystyle\int_{-\infty}^{\infty}\sin(\pi x^2)\,\mathrm{d}x = \frac{1}{\sqrt{2}}$, we have

$$\int_{-\infty}^\infty \frac{\sin(\pi x^2)}{\sinh^2(\pi x)}\,\mathrm{d}x = \frac{2-\sqrt{2}}{2}$$

Therefore we have

$$\int_0^\infty \frac{\cos(\pi x)}{e^{2\pi\sqrt{x}}-1}\,\mathrm{d}x = \frac{2-\sqrt{2}}{8}$$

□

Exercise 1.2.205: Prove:

$$\int_0^\infty x^{a-1}\cos(x^b)\mathrm{d}x = \frac{1}{b}\cos\left(\frac{\pi a}{2b}\right)\Gamma\left(\frac{a}{b}\right),\quad b>a,\quad a>1$$

☞ **Proof:** Let $r = a/b \in (0,1)$. Then

$$\int_0^\infty x^{a-1}\cos(x^b)\,\mathrm{d}x = \frac{1}{b}\int_0^\infty \frac{\cos t}{t^{1-r}}\,\mathrm{d}t = \frac{1}{b\Gamma(1-r)}\int_0^\infty \left(\int_0^\infty u^{-r}e^{-tu}\,\mathrm{d}u\right)\cos t\,\mathrm{d}t$$
$$= \frac{1}{b\Gamma(1-r)}\int_0^\infty \left(\int_0^\infty e^{-ut}\cos t\,\mathrm{d}t\right)u^{-r}\,\mathrm{d}u = \frac{1}{b\Gamma(1-r)}\int_0^\infty \frac{u^{1-r}}{u^2+1}\,\mathrm{d}u$$
$$= \frac{1}{b\Gamma(1-r)}\int_0^{\frac{\pi}{2}} \tan^{1-r}\theta \mathrm{d}\theta$$

Simplifying using the beta function identity, we have

$$\int_0^\infty x^{a-1}\cos(x^b)\,\mathrm{d}x = \frac{\Gamma\left(1-\frac{r}{2}\right)\Gamma\left(\frac{r}{2}\right)}{2b\Gamma(1-r)\Gamma(r)}\Gamma(r) = \frac{\sin(\pi r)}{2b\sin\left(\frac{\pi r}{2}\right)}\Gamma(r) = \frac{1}{b}\Gamma(r)\cos\left(\frac{\pi r}{2}\right)$$
$$= \frac{1}{b}\cos\left(\frac{\pi a}{2b}\right)\Gamma\left(\frac{a}{b}\right)$$

□

Exercise 1.2.206: Prove:

$$\int_0^\infty \frac{\ln(x)\sin(x^n)}{x}\mathrm{d}x = -\frac{\gamma\pi}{2n^2} \quad , \quad n > 0$$

☞ **Proof:** We have

$$\int_0^\infty x^{a-1}\sin(x^n)\mathrm{d}x = \frac{\sqrt{\pi}2^{\frac{a}{n}-1}\Gamma\left(\frac{a}{2n}+\frac{1}{2}\right)}{n\Gamma\left(1-\frac{a}{2n}\right)}$$

Differentiating with respect to a gives

$$\int_0^\infty x^{a-1}\sin(x^n)\ln(x)\mathrm{d}x = \frac{\sqrt{\pi}2^{\frac{a}{n}-1}\Gamma\left(\frac{a}{2n}+\frac{1}{2}\right)}{n^2\Gamma\left(1-\frac{a}{2n}\right)}\ln(2)$$
$$+\frac{\sqrt{\pi}2^{\frac{a}{2n}-1}}{2n^2}\frac{\Gamma\left(\frac{a}{2n}+\frac{1}{2}\right)}{\Gamma\left(1-\frac{a}{2n}\right)}\left[\psi\left(\frac{1}{2}+\frac{a}{2n}\right)+\psi\left(1-\frac{a}{2n}\right)\right]$$

Putting $a = 0$ gives

$$\int_0^\infty \frac{\ln(x)\sin(x^n)}{x}\mathrm{d}x = \frac{\pi}{2n^2}\ln(2) - \frac{\pi}{2n^2}\ln(2) - \frac{\pi\gamma}{2n^2} = -\frac{\gamma\pi}{2n^2}$$

A fun consequence of this result is

$$\int_0^\infty \frac{\ln(x)\sin(\sqrt[3]{x})}{x}\mathrm{d}x = -\frac{9\gamma\pi}{2}$$

□

Exercise 1.2.207: Evaluate:

$$\int_0^1 \frac{\arccos^4(x^2)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

✎ **Solution:** Let I denote the integral. Using the substitution $x = \sqrt{\cos t}$, we get

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{t^4 \sin(t)}{\sqrt{\cos t - \cos^2 t}}\mathrm{d}t = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{t^4 \cos\left(\frac{t}{2}\right)}{\sqrt{1-2\sin^2\left(\frac{t}{2}\right)}}\mathrm{d}t$$

Next, substitute $2\sin^2\left(\frac{t}{2}\right) = \sin^2\theta$ to get

$$I = 16\int_0^{\frac{\pi}{2}} \left[\sin^{-1}\left(\frac{\sin\theta}{\sqrt{2}}\right)\right]^4 \mathrm{d}\theta$$

Now, plug in the series $(\sin^{-1} z)^4 = \sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}(2z)^{2k}}{k^2\binom{2k}{k}}$ and integrate termwise:

$$I = 24\int_0^{\frac{\pi}{2}} \left(\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}2^k}{k^2\binom{2k}{k}}\sin^{2k}(\theta)\right)\mathrm{d}\theta = 24\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}2^k}{k^2\binom{2k}{k}}\int_0^{\frac{\pi}{2}}\sin^{2k}(\theta)\mathrm{d}\theta$$

$$= 12\pi\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}2^k}{k^2\binom{2k}{k}}\left(\frac{\binom{2k}{k}}{2^{2k}}\right) = 12\pi\sum_{k=1}^{\infty}\frac{H_{k-1}^{(2)}}{k^2 2^k} = 12\pi\sum_{k=1}^{\infty}\frac{\zeta(2)-\psi_1(k)}{k^2 2^k}$$

$$= 12\pi\zeta(2)\mathrm{Li}_2\left(\frac{1}{2}\right) - 12\pi\sum_{n=1}^{\infty}\frac{\psi_1(k)}{k^2 2^k}$$

The final result follows from **Exercise 2.2.77**

$$\int_0^1 \frac{\arccos^4(x^2)}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi^5}{120} + \frac{\pi}{2}\ln^4 2 + 3\pi\zeta(3)\ln 2 - \frac{\pi^3}{2}\ln^2 2$$

□

Exercise 1.2.208: Prove:

$$\int_0^1 \frac{\mathrm{Li}_2\left(\frac{x}{4}\right)}{4-x}\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)\mathrm{d}x = \frac{\pi^4}{1944}$$

☞ **Proof:**

$$\int_0^1 \frac{\mathrm{Li}_2\left(\frac{x}{4}\right)}{4-x}\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)\mathrm{d}x = \frac{1}{4}\int_0^1\left(\sum_{n=1}^{\infty}\frac{H_n^{(2)}x^n}{2^{2n}}\right)\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{2^{2(n+1)}}\int_0^1 x^n\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)2^{2(n+1)}}\int_0^1\frac{x^n}{\sqrt{1-x}}\mathrm{d}x$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)(2n+1)\binom{2n}{n}} = \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2\binom{2(n+1)}{n+1}} = \frac{2}{3}\left(\sin^{-1}\frac{1}{2}\right)^4$$

$$= \frac{\pi^4}{1994}$$

□

Exercise 1.2.209: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{x(1+\sin^2 x)\cos x}{(1+3\sin^2 x)(\sin^2 x+3)}\mathrm{d}x$$

Solution:

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{x(1+\sin^2 x)\cos x}{(3+\sin^2 x)(1+3\sin^2 x)}\,\mathrm{d}x = \frac{1}{4}\int_0^{\pi/2} \frac{x\cos x}{1+3\sin^2 x}\,\mathrm{d}x + \frac{1}{4}\int_0^{\pi/2} \frac{x\cos x}{3+\sin^2 x}\,\mathrm{d}x \\ &= \frac{1}{4\sqrt{3}}\int_0^{\pi/2} x\,\mathrm{d}\left(\arctan\left(\sqrt{3}\sin x\right)+\arctan\left(\frac{1}{\sqrt{3}}\sin x\right)\right) \\ &= \frac{\pi^2}{16\sqrt{3}} - \frac{1}{4\sqrt{3}}\int_0^{\pi/2}\left[\arctan\left(\sqrt{3}\sin x\right)+\arctan\left(\frac{1}{\sqrt{3}}\sin x\right)\right]\mathrm{d}x \end{aligned}$$

Now, let

$$\begin{aligned} J(a) &= \int_0^{\pi/2} \arctan\left(a\sin x\right)\,\mathrm{d}x \\ J'(a) &= \int_0^{\pi/2} \frac{\sin x}{(1+a^2)-a^2\cos^2 x}\,\mathrm{d}x = \frac{\sinh^{-1} a}{a\sqrt{1+a^2}} \end{aligned}$$

Thus,

$$\begin{aligned} J(a) &= \int_0^a \frac{\sinh^{-1} t}{t\sqrt{1+t^2}}\,\mathrm{d}t = -\left[\sinh^{-1}(t)\sinh^{-1}\left(\frac{1}{t}\right)\right]_0^a + \int_0^a \frac{\sinh^{-1}\frac{1}{t}}{\sqrt{1+t^2}}\,\mathrm{d}t \\ &= -\left[\sinh^{-1}(a)\sinh^{-1}\left(\frac{1}{a}\right)\right] + \int_{1/a}^{\infty} \frac{\sinh^{-1} t}{t\sqrt{1+t^2}}\,\mathrm{d}t \\ J(a)+J\left(\frac{1}{a}\right) &= -\sinh^{-1}(a)\sinh^{-1}\left(\frac{1}{a}\right) + \int_0^{\infty} \frac{\sinh^{-1} t}{t\sqrt{1+t^2}}\,\mathrm{d}t \\ &= -\sinh^{-1}(a)\sinh^{-1}\left(\frac{1}{a}\right) + \int_0^{\infty} \frac{t}{\sinh t}\,\mathrm{d}t \\ &= \frac{\pi^2}{4} - \sinh^{-1}(a)\sinh^{-1}\left(\frac{1}{a}\right) \end{aligned}$$

Hence,

$$I = \frac{1}{4\sqrt{3}}\sinh^{-1}(\sqrt{3})\sinh^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{8\sqrt{3}}\ln(3)\ln(2-\sqrt{3})$$

□

Exercise 1.2.210: Prove:

$$\int_0^1 \frac{\arctan x\,\mathrm{arctanh}\,x\ln x}{x}\mathrm{d}x = \frac{\pi^2}{16}\mathbf{G} - \frac{7\pi}{32}\zeta(3)$$

Proof: Let $n = 0, 1, 2, \cdots$, We define $I, I_{1,n}, I_{2,n}$ and I_n as follows:

$$\begin{aligned} I &= \int_0^1 \frac{\ln(x)\tan^{-1}(x)\tanh^{-1}(x)}{x}\mathrm{d}x \quad , \quad I_{1,n} = \int_0^1 x^{2n}\ln(x)\ln(1-x)\,\mathrm{d}x \\ I_{2,n} &= \int_0^1 x^{2n}\ln(x)\ln(1+x)\,\mathrm{d}x \quad , \quad I_n = \int_0^1 x^{2n}\ln(x)\tanh^{-1}(x)\,\mathrm{d}x \end{aligned}$$

Part I : Evaluation of $I_{1,n}$, $I_{2,n}$ and I_n

$$\begin{aligned}
I_{1,n} &= \int_0^1 x^{2n}\ln(x)\ln(1-x)\mathrm{d}x = \int_0^1 x^{2n}\ln(x)\left(-\sum_{j=1}^{\infty}\frac{x^j}{j}\right)\mathrm{d}x \\
&= -\sum_{j=1}^{\infty}\frac{1}{j}\int_0^1 x^{2n+j}\ln(x)\mathrm{d}x = \sum_{j=1}^{\infty}\frac{1}{j\,(2n+1+j)^2} \\
&= \frac{1}{(2n+1)^2}\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{j+2n+1}\right) - \frac{1}{(2n+1)}\sum_{j=1}^{\infty}\frac{1}{(j+2n+1)^2} \\
&= \frac{\gamma+\psi_0(2n+2)}{(2n+1)^2} - \frac{\psi_1(2n+2)}{2n+1}
\end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned}
I_{2,n} &= \int_0^1 x^{2n}\ln(x)\ln(1+x)\mathrm{d}x = \int_0^1 x^{2n}\ln(x)\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}x^j}{j}\right)\mathrm{d}x \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\int_0^1 x^{2n+j}\ln(x)\mathrm{d}x = \sum_{j=1}^{\infty}\frac{(-1)^j}{j\,(2n+1+j)^2} \\
&= \frac{1}{(2n+1)^2}\sum_{j=1}^{\infty}\frac{(-1)^j}{j} - \frac{1}{(2n+1)^2}\sum_{j=1}^{\infty}\frac{(-1)^j}{j+2n+1} - \frac{1}{2n+1}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+2n+1)^2} \\
&= -\frac{\ln(2)}{(2n+1)^2} + \frac{\psi_0\left(n+\frac{3}{2}\right)-\psi_0(n+1)}{2(2n+1)^2} + \frac{\psi_1(n+1)-\psi_1\left(n+\frac{3}{2}\right)}{4(2n+1)}
\end{aligned}$$

We can make some simplifications using the following identities:

$$\begin{aligned}
\psi_0\left(n+\frac{3}{2}\right) &= 2\psi_0(2n+2)-\psi_0(n+1)-2\ln(2) \\
\psi_1\left(n+\frac{3}{2}\right) &= 4\psi_1(2n+2)-\psi_1(n+1)
\end{aligned}$$

So, $I_{2,n}$ can be written as:

$$I_{2,n} = -\frac{2\ln(2)}{(2n+1)^2} + \frac{\psi_0(2n+2)-\psi_0(n+1)}{(2n+1)^2} + \frac{2\psi_1(n+1)-4\psi_1(2n+2)}{4(2n+1)} \tag{2}$$

Also note that $I_n = \dfrac{I_{2,n}-I_{1,n}}{2}$. Therefore,

$$I_n = -\frac{\ln(2)}{(2n+1)^2} - \frac{\gamma+\psi_0(n+1)}{2(2n+1)^2} + \frac{\psi_1(n+1)}{4(2n+1)} \tag{3}$$

Part II : Expressing I in terms of Euler Sums

$$\begin{aligned}
I &= \int_0^1 \frac{\ln(x)\tan^{-1}(x)\tanh^{-1}(x)}{x}\mathrm{d}x = \int_0^1 \frac{\ln(x)\tanh^{-1}(x)}{x}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\right)\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n}\ln(x)\tanh^{-1}(x)\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}I_n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(-\frac{\ln(2)}{(2n+1)^2} - \frac{\gamma+\psi_0(n+1)}{2(2n+1)^2} + \frac{\psi_1(n+1)}{4(2n+1)}\right) \\
&= -\ln(2)\frac{\pi^3}{32} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\left(\gamma+\psi_0(n+1)\right) + \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\psi_1(n+1)}{(2n+1)^2}
\end{aligned}$$

Let us denote the euler sums by E_1 and E_2:

$$E_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(\gamma + \psi_0(n+1)\right) \quad , \quad E_2 = \sum_{n=0}^{\infty} \frac{(-1)^n \psi_1(n+1)}{(2n+1)^2}$$

Part III : Evaluation of E_1

We use the FS-contour method. Let $f(z) = \dfrac{\pi \csc(\pi z)\left(\gamma + \psi_0(-z)\right)}{(2z+1)^3}$. Then the sum of all residues of $f(z)$ is zero.

The sum of the residues at the negative integers is equal to:

$$\sum_{n=1}^{\infty} \text{Res}_{z=-n} f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\gamma + \psi_0(n)\right)}{(2n-1)^3} = E_1$$

At $z = -\dfrac{1}{2}$, the residue is

$$\text{Res}_{z=-1/2} f(z) = \frac{\pi^3}{8}\ln(2) + \frac{7\pi}{8}\zeta(3)$$

The sum of the residues at the positive integers is:

$$\sum_{n=0}^{\infty} \text{Res}_{z=n} f(z) = \sum_{n=0}^{\infty} \left(-6\frac{(-1)^n}{(2n+1)^4} + \frac{(-1)^n H_n}{(2n+1)^3}\right) = -6\beta(4) + E_1$$

Therefore,

$$E_1 + \frac{\pi^3}{8}\ln(2) + \frac{7\pi}{8}\zeta(3) + E_1 - 6\beta(4) = 0 \implies E_1 = \boxed{3\beta(4) - \frac{7\pi}{16}\zeta(3) - \frac{\pi^3}{16}\ln(2)}$$

Part IV : Evaluation of E_2

This time we use FS contour method to the function $g(z) = \dfrac{\pi \csc(\pi z)\psi_1(-z)}{(2z+1)^2}$.

The sum of the residues at the negative integers is:

$$\sum_{n=1}^{\infty} \text{Res}_{z=-n} g(z) = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\psi_1(n)}{(2n-1)^2} = -E_2$$

The residue at $z = -\dfrac{1}{2}$ is :

$$\text{Res}_{z=-1/2} g(z) = -\frac{7\pi}{2}\zeta(3)$$

The sum of the residues at the positive integers is:

$$\begin{aligned}\sum_{n=0}^{\infty} \text{Res}_{z=n} g(z) &= \sum_{n=0}^{\infty} \left(12\frac{(-1)^n}{(2n+1)^4} + \frac{\pi^2(-1)^n}{2(2n+1)^2} - \frac{(-1)^n\psi_1(n+1)}{(2n+1)^2}\right) \\ &= 12\beta(4) + \frac{\pi^2}{2}\mathbf{G} - E_2\end{aligned}$$

The sum of all the residues is zero. Therefore,

$$-E_2 - \frac{7\pi}{2}\zeta(3) + 12\beta(4) + \frac{\pi^2}{2}\mathbf{G} - E_2 = 0 \implies E_2 = \boxed{6\beta(4) - \frac{7\pi}{4}\zeta(3) + \frac{\pi^2}{4}\mathbf{G}}$$

Part V : The Final Answer

$$\begin{aligned} I &= -\frac{\pi^3\ln(2)}{32} - \frac{E_1}{2} + \frac{E_2}{4} \\ &= -\frac{\pi^3\ln(2)}{32} - \frac{1}{2}\left(3\beta(4) - \frac{7\pi}{16}\zeta(3) - \frac{\pi^3}{16}\ln(2)\right) + \frac{1}{4}\left(6\beta(4) - \frac{7\pi}{4}\zeta(3) + \frac{\pi^2}{4}\mathbf{G}\right) \\ &= \frac{\pi^2}{16}\mathbf{G} - \frac{7\pi\zeta(3)}{32} \end{aligned}$$

□

Exercise 1.2.211: Prove:

$$\int_0^\infty \frac{\cos(4nx)\tanh(x)}{x}\mathrm{d}x = \ln(\coth(\pi n))$$

Proof: Using the familiar integral, $\int_0^\infty \frac{\cos(px)}{1+x^2}\mathrm{d}x = \frac{\pi}{2}e^{-p}$, and the familiar logarithmic derivative of the Weierstrass product of the hyperbolic cosine,

$$\frac{\pi}{8x}\tanh(\pi x) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2+(2x)^2}$$

we have

$$\begin{aligned} \int_0^\infty \frac{\pi}{8x}\tanh(\pi x)\cos(4n\pi x)\mathrm{d}x &= \sum_{k=0}^{\infty}\int_0^\infty \frac{\cos(4n\pi x)}{(2k+1)^2+(2x)^2}\,\mathrm{d}x \\ = \sum_{k=0}^{\infty}\frac{\pi}{4}\frac{e^{-2\pi n(2k+1)}}{2k+1} &= \frac{\pi}{8}\ln\left(\frac{1+e^{-2\pi n}}{1-e^{-2\pi n}}\right) = \frac{\pi}{8}\ln\coth(\pi n) \end{aligned}$$

Hence

$$\int_0^\infty \frac{\cos(4nx)\tanh(x)}{x}\mathrm{d}x = \ln(\coth(\pi n))$$

□

Exercise 1.2.212: Prove that:

$$\Im(\mathrm{Li}_2(iz)) = \mathrm{Ti}_2(z)$$

Proof: It follows pretty much from the definition for $z \in \mathbb{R}$:

$$\begin{aligned} \mathrm{Li}_2(iz) &= -\int_0^z \frac{\ln(1-ix)}{x}\,\mathrm{d}x = -\int_0^z \frac{\ln\left((1+x^2)^{1/2}e^{-i\arctan x}\right)}{x}\,\mathrm{d}x \\ &= -\frac{1}{2}\int_0^z \frac{\ln(1+x^2)}{x}\,\mathrm{d}x + i\int_0^z \frac{\arctan x}{x}\,\mathrm{d}x = -\frac{1}{4}\int_0^{-z^2}\frac{\ln(1-x)}{x}\,\mathrm{d}x + i\int_0^z \frac{\arctan x}{x}\,\mathrm{d}x \\ &= \frac{1}{4}\mathrm{Li}_2(-z^2) + i\,\mathrm{Ti}_2(z) \end{aligned}$$

□

Exercise 1.2.213:

Solution:

□

Exercise 1.2.214:

✎ **Solution:**

□

➻ **Exercise 1.2.215:**

✎ **Solution:**

□

➻ **Exercise 1.2.216:**

✎ **Solution:**

□

➻ **Exercise 1.2.217:**

✎ **Solution:**

□

➻ **Exercise 1.2.218:**

✎ **Solution:**

□

➻ **Exercise 1.2.219:**

✎ **Solution:**

□

➻ **Exercise 1.2.220:**

✎ **Solution:**

□

➻ **Exercise 1.2.221:**

✎ **Solution:**

□

➻ **Exercise 1.2.222:**

✎ **Solution:**

□

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1.3 贴吧

✎ *

➻ **Exercise 1.3.1:** 计算：

$$\int_0^{\frac{\pi}{4}} \frac{x}{\cos x \left(\sin x + \cos x\right)} \, \mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \frac{x}{\cos x\left(\sin x+\cos x\right)}\,\mathrm{d}x \\
&=\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,\mathrm{d}x \\
&=x\ln\left(1+\tan x\right)\Big|_0^{\frac{\pi}{4}}-\boxed{\int_0^{\frac{\pi}{4}}\ln\left(1+\tan x\right)\mathrm{d}x} \\
&=\frac{\pi}{4}\ln 2-\frac{\pi}{8}\ln 2 \\
&=\frac{\pi}{8}\ln 2
\end{aligned}$$

框中积分的多种计算方法在 此贴. □

Exercise 1.3.2: 计算:

$$\int_0^1 \left\{\frac{1}{x}\right\}\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
\int_0^1 \left\{\frac{1}{x}\right\}\mathrm{d}x=\int_1^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t&=\sum_{k=1}^{\infty}\int_k^{k+1}\frac{t-k}{t^2}\mathrm{d}t \\
&=\sum_{k=1}^{\infty}\left(\ln\frac{k+1}{k}-\frac{1}{k+1}\right) \\
&=1-\gamma
\end{aligned}$$

□

Exercise 1.3.3: 计算:

$$\int_0^1 \left\{\frac{1}{x}\right\}^2\mathrm{d}x$$

✎ **Solution:** 我们计算比较一般的情况

$$\int_0^1 \left\{\frac{k}{x}\right\}^2\mathrm{d}x=k\int_k^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t=k\sum_{m=k}^{\infty}\left(2-2m\ln\frac{m+1}{m}-\frac{1}{m+1}\right)$$

令 $S_n=\sum_{m=k}^{n}\left(2-2m\ln\frac{m+1}{m}-\frac{1}{m+1}\right)$，我们有

$$\begin{aligned}
S_n&=\sum_{m=k}^{n}\left(2-2m\ln\frac{m+1}{m}-\frac{1}{m+1}\right) \\
&=2\left(n-k+1\right)-\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+n}\right) \\
&\quad-2n\ln\left(n+1\right)+2k\ln k+2\ln n!-2\ln k!
\end{aligned}$$

由 $2\ln n!\sim\ln\left(2\pi\right)+\left(2n+1\right)\ln n-2n$，得

$$\begin{aligned}
S_n\sim\ &2\left(1-k\right)+\ln\left(2\pi\right)+2k\ln k-2\ln k!-2n\ln\frac{n+1}{n} \\
&-\left(\frac{1}{k+1}+\cdots+\frac{1}{n+1}-\ln n\right)
\end{aligned}$$

所以

$$\lim_{n\to\infty} S_n = \ln(2\pi) - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + 2k\ln k - 2k - 2\ln k!$$

因此，当 $k=1$，则有

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 \mathrm{d}x = \ln(2\pi) - \gamma - 1$$

□

Exercise 1.3.4: 计算：

$$\int_0^1 \left\{\frac{1}{x}\right\}^3 \mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^1 \left\{\frac{1}{x}\right\}^3 \mathrm{d}x &= \int_1^\infty \frac{\{t\}^3}{t^2}\mathrm{d}t = \sum_{k=1}^\infty \int_k^{k+1} \frac{(t-k)^3}{t^2}\mathrm{d}t \\ &= \sum_{k=1}^\infty \left(3k^2 \ln\frac{k+1}{k} + \frac{3}{2} - 3k - \frac{1}{k+1}\right)\end{aligned}$$

令 $S_n = \sum_{k=1}^n \left(3k^2 \ln\frac{k+1}{k} + \frac{3}{2} - 3k - \frac{1}{k+1}\right)$，简单计算得

$$S_n = 1 - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1} - \ln n\right) - \frac{3}{2}n^2 - \ln n + 3\sum_{k=1}^n k^2 \ln\frac{k+1}{k}$$

其中

$$\begin{aligned}\sum_{k=1}^n k^2 \ln\frac{k+1}{k} &= \ln\prod_{k=1}^n \left(\frac{k+1}{k}\right)^{k^2} = \ln\left[\frac{(n+1)^{n^2}\cdot n!}{(2^2\cdot 3^3\cdots n^n)^2}\right] \\ \Rightarrow -\frac{3}{2}n^2 - \ln n + 3\sum_{k=1}^n k^2 \ln\frac{k+1}{k} &= \ln\left[\frac{(n+1)^{3n^2}\cdot (n!)^3}{(2^2\cdot 3^3\cdots n^n)^6 \cdot e^{\frac{3n^2}{2}}\cdot n}\right]\end{aligned}$$

令

$$a_n = \frac{(n+1)^{3n^2}\cdot (n!)^3}{(2^2\cdot 3^3\cdots n^n)^6 \cdot e^{\frac{3n^2}{2}}\cdot n} = \frac{n^{3n^2+3n+\frac{1}{2}}\cdot e^{-\frac{3n^2}{2}}}{(2^2\cdot 3^3\cdots n^n)^6}\cdot\frac{(n+1)^{3n^2}\cdot (n!)^3}{n^{3n^2+3n+\frac{3}{2}}}$$

易知第一部分的极限为 $\frac{1}{\mathbf{A}^6}$，对于第二部分，使用 **Stirling's formula** $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ 有

$$\frac{(n+1)^{3n^2}\cdot (n!)^3}{n^{3n^2+3n+\frac{3}{2}}} \sim (2\pi)^{\frac{3}{2}}\left[\left(\frac{n+1}{n}\right)^n \frac{1}{e}\right]^{3n} \to (2\pi)^{\frac{3}{2}} e^{-\frac{3}{2}}$$

所以 $x_n \to \frac{(2\pi)^{\frac{3}{2}} e^{-\frac{3}{2}}}{\mathbf{A}^6}$，由此可得

$$\int_0^1 \left\{\frac{1}{x}\right\}^3 \mathrm{d}x = \lim_{n\to\infty} S_n = -\frac{1}{2} - \gamma + \frac{3}{2}\ln(2\pi) - 6\ln\mathbf{A}$$

□

Exercise 1.3.5: 计算积分：

$$\int_0^\infty \frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}\int_0^{\infty}\frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x &= 4\int_0^{\infty}\frac{e^{-x^2}}{\left(2x^2+1\right)^2}\mathrm{d}x = \frac{4}{\sqrt{2}}\int_0^{\infty}\frac{e^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2}\mathrm{d}x\\ &= \frac{4}{\sqrt{2}}\frac{1}{\Gamma(2)}\int_0^{\infty}t\cdot e^{-t}\mathrm{d}t\int_0^{\infty}e^{-\frac{x^2}{2}-tx^2}\mathrm{d}x \qquad (1)\\ &= \frac{4\sqrt{\pi}}{\sqrt{2}}\int_0^{\infty}\frac{te^{-t}}{\sqrt{4t+2}}\mathrm{d}t = \frac{4\sqrt{\pi}}{\sqrt{2}}\cdot\frac{1}{2\sqrt{2}} = \sqrt{\pi}\end{aligned}$$

(1): $\dfrac{1}{(1+x)^{\alpha}} = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\infty}t^{\alpha-1}e^{-xt}\mathrm{d}t$ □

Exercise 1.3.6:

✎ **Solution:**

□

Exercise 1.3.7:

✎ **Solution:**

□

Exercise 1.3.8:

✎ **Solution:**

□

Exercise 1.3.9:

✎ **Solution:**

□

Exercise 1.3.10:

✎ **Solution:**

□

Exercise 1.3.11:

✎ **Solution:**

□

Exercise 1.3.12:

✎ **Solution:**

□

Exercise 1.3.13:

✎ **Solution:**

□

Exercise 1.3.14:

✎ **Solution:**

□

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1.4 QQ 群

✎ *

Exercise 1.4.1: 计算积分:

$$\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\mathrm{d}x &= \int_0^{\pi/2} \frac{\arctan\left(\sin x\right)}{\sin x}\mathrm{d}x\\ I\left(\alpha\right) &= \int_0^{\pi/2} \frac{\arctan\left(\alpha\sin x\right)}{\sin x}\mathrm{d}x\\ I'\left(\alpha\right) &= \int_0^{\pi/2} \frac{1}{1+\alpha^2\sin^2 x}\mathrm{d}x = \frac{\pi}{2\sqrt{1+\alpha^2}}\end{aligned}$$

$$\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\mathrm{d}x = I\left(1\right) = \int_0^1 \frac{\pi}{2\sqrt{1+\alpha^2}}\mathrm{d}\alpha = \frac{1}{2}\pi\sinh^{-1}\left(1\right)$$

□

Exercise 1.4.2: 计算积分:

$$\int_0^{\pi/2} \sqrt{\tan x}\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}\int_0^{\pi/2} \sqrt{\tan x}\mathrm{d}x &\overset{\tan x=u^2}{\longrightarrow} \int_0^{+\infty} \frac{2u^2}{1+u^4}\mathrm{d}u\\ &= \int_0^{+\infty} \frac{u^2+1}{1+u^4}\mathrm{d}u + \int_0^{+\infty} \frac{u^2-1}{1+u^4}\mathrm{d}u\\ &= \int_0^{+\infty} \frac{1+\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\mathrm{d}u + \int_0^{+\infty} \frac{1-\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\mathrm{d}u\\ &= \left[\frac{\sqrt{2}}{2}\arctan\left(\frac{u-\frac{1}{u}}{\sqrt{2}}\right)\right]_0^{+\infty} + \left[\frac{1}{2\sqrt{2}}\ln\left|\frac{\frac{u+\frac{1}{u}}{\sqrt{2}}-1}{\frac{u+\frac{1}{u}}{\sqrt{2}}+1}\right|\right]_0^{+\infty}\\ &= \frac{\pi}{\sqrt{2}}\end{aligned}$$

□

Exercise 1.4.3: 计算积分:

$$\int_0^{\pi/4} \sqrt{1+\tan x}\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
\int_0^{\pi/4}\sqrt{1+\tan x}\mathrm{d}x &\xrightarrow{\sqrt{1+\tan x}=t} \int_1^{\sqrt{2}}\frac{2t^2}{t^4-2t^2+2}\mathrm{d}t=\int_1^{\sqrt{2}}\frac{\sqrt{2}+t^2+\left(t^2-\sqrt{2}\right)}{t^4-2t^2+2}\mathrm{d}t\\
&=\int_0^{\sqrt{2}}\frac{\frac{\sqrt{2}}{t^2}+1}{t^2-2+\frac{2}{t^2}}\mathrm{d}t+\int_0^{\sqrt{2}}\frac{1-\frac{\sqrt{2}}{t^2}}{t^2-2+\frac{2}{t^2}}\mathrm{d}t\\
&=\frac{1}{\sqrt{2\sqrt{2}-2}}\int_0^{\sqrt{2}}\frac{1}{\left(\frac{t-\frac{\sqrt{2}}{t}}{\sqrt{2\sqrt{2}-2}}\right)^2+1}\mathrm{d}\left(\frac{t-\frac{\sqrt{2}}{t}}{\sqrt{2\sqrt{2}-2}}\right)\\
&+\int_0^{\sqrt{2}}\frac{1}{\left(t+\frac{\sqrt{2}}{t}-\sqrt{2\sqrt{2}+2}\right)\left(t+\frac{\sqrt{2}}{t}+\sqrt{2\sqrt{2}+2}\right)}\mathrm{d}\left(t+\frac{\sqrt{2}}{t}\right)\\
&=\left[\frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left(\frac{t-\frac{\sqrt{2}}{t}}{\sqrt{2\sqrt{2}-2}}\right)\right]_0^{\sqrt{2}}\\
&+\left[\frac{1}{2\sqrt{2\sqrt{2}+2}}\ln\left|\frac{t+\frac{\sqrt{2}}{t}-\sqrt{2\sqrt{2}+2}}{t+\frac{\sqrt{2}}{t}+\sqrt{2\sqrt{2}+2}}\right|\right]_0^{\sqrt{2}}\\
&=\frac{2}{\sqrt{2\sqrt{2}-2}}\arctan\left(\frac{\sqrt{2}-1}{\sqrt{2\sqrt{2}-2}}\right)
\end{aligned}$$

□

Exercise 1.4.4: 计算积分：

$$\int_0^1\frac{\arcsin\sqrt{x}}{x^4-2x^3+2x^2-x+1}\mathrm{d}x$$

✎ **Solution:**

$$\begin{aligned}
&\int_0^1\frac{\arcsin\sqrt{x}}{x^4-2x^3+2x^2-x+1}\mathrm{d}x=\int_0^1\frac{\arcsin\sqrt{1-x}}{x^4-2x^3+2x^2-x+1}\mathrm{d}x\\
&=\frac{1}{2}\int_0^1\frac{\arcsin\sqrt{x}+\arcsin\sqrt{1-x}}{x^4-2x^3+2x^2-x+1}\mathrm{d}x=\frac{\pi}{4}\int_0^1\frac{1}{x^4-2x^3+2x^2-x+1}\mathrm{d}x\\
&=2\pi\int_{-1}^1\frac{1}{u^4+2u^2+13}\mathrm{d}u\,(u=2x-1)=4\pi\int_0^1\frac{1}{u^4+2u^2+13}\mathrm{d}u\\
&=4\pi\int_0^1\frac{1}{\left(u^2-\sqrt{2\sqrt{13}-2}u+\sqrt{13}\right)\left(u^2+\sqrt{2\sqrt{13}-2}u+\sqrt{13}\right)}\mathrm{d}u\\
&=\frac{\sqrt{2}\pi}{\sqrt{13\sqrt{13}-13}}\left[\ln\frac{1+\sqrt{2\sqrt{13}-2}+\sqrt{13}}{4}+\frac{\sqrt{39}-\sqrt{3}}{6}\arctan\frac{\sqrt{2\sqrt{13}+2}}{\sqrt{13}-1}\right]
\end{aligned}$$

□

Exercise 1.4.5: 计算积分：

$$\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x}\mathrm{d}x$$

Solution: 令

$$f\left(t\right)=\int_0^1 \frac{\ln\left(1+tx^2\right)}{1+x}\mathrm{d}x$$

$$\begin{aligned}
f'\left(t\right)&=\int_0^1 \frac{x^2}{\left(1+x\right)\left(1+tx^2\right)}\mathrm{d}x\\
&=\frac{1}{t+1}\int_0^1 \frac{x-1}{1+tx^2}\mathrm{d}x+\frac{1}{t+1}\int_0^1 \frac{\mathrm{d}x}{x+1}\\
&=\frac{1}{t+1}\left[\frac{1}{2t}\ln\left(1+tx^2\right)-\frac{1}{\sqrt{t}}\arctan\left(\sqrt{t}x\right)+\ln\left(x+1\right)\right]_0^1\\
&=\frac{1}{t+1}\left[\frac{1}{2t}\ln\left(1+t\right)-\frac{1}{\sqrt{t}}\arctan\left(\sqrt{t}\right)+\ln 2\right]
\end{aligned}$$

$$\Rightarrow f\left(t\right)=\frac{1}{2}\left[-\mathrm{Li}_2\left(-t\right)-\frac{1}{2}\ln^2\left(t+1\right)\right]-\arctan^2\sqrt{t}+\ln 2\ln\left(t+1\right)$$

所以

$$\begin{aligned}
\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x}\mathrm{d}x&=f\left(1\right)\\
&=\frac{1}{2}\left[-\mathrm{Li}_2\left(-1\right)-\frac{1}{2}\ln^2 2\right]-\frac{\pi^2}{16}+\ln^2 2\\
&=\frac{3}{4}\ln^2 2-\frac{\pi^2}{48}
\end{aligned}$$

□

Exercise 1.4.6: 计算积分：

$$\int_{-1}^1 x^3\ln\left(1+e^x\right)\mathrm{d}x$$

Solution: 事实上，我们有以下推论，

Corollary 1.4.1

$$\int_{-a}^a f\left(x\right)g\left(x\right)\mathrm{d}x=\begin{cases}\displaystyle\int_0^a g\left(x\right)\left[f\left(x\right)-f\left(-x\right)\right]\mathrm{d}x & g\left(x\right)\text{ 为奇函数}\\ \displaystyle\int_0^a g\left(x\right)\left[f\left(x\right)+f\left(-x\right)\right]\mathrm{d}x & g\left(x\right)\text{ 为偶函数}\end{cases}$$

♣

所以

$$\int_{-1}^1 x^3\ln\left(1+e^x\right)\mathrm{d}x=\int_0^1 x^3\ln\left(\frac{1+e^x}{1+e^{-x}}\right)\mathrm{d}x=\int_0^1 x^4\mathrm{d}x=\frac{1}{5}$$

□

Exercise 1.4.7: 计算积分:

$$\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\mathrm{d}x$$

Solution: 类似 Exercise 1.1.14 我们有:

$$\begin{aligned}
&\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\mathrm{d}x\\
&=\frac{1}{2}\ln^2 x\ln^2(1-x)\Big|_0^1+\int_0^1\frac{\ln^2 x\ln(1-x)}{1-x}\mathrm{d}x\\
&=\int_0^1\sum_{k=1}^{\infty}(-1)^{2k-1}H_k x^k\ln^2 x\mathrm{d}x=\sum_{k=1}^{\infty}(-1)^{2k-1}H_k\int_0^1 x^k\ln^2 x\mathrm{d}x\\
&=2\sum_{k=1}^{\infty}(-1)^{2k-1}\frac{H_k}{(k+1)^3}\\
&=2\sum_{k=1}^{\infty}(-1)^{2k-1}\left[\frac{H_{k+1}}{(k+1)^3}-\frac{1}{(k+1)^4}\right]\\
&=2\sum_{k=1}^{\infty}(-1)^{2k-1}\left[\frac{H_k}{k^3}-\frac{1}{k^4}\right]\\
&=-\frac{\pi^4}{36}+\frac{\pi^4}{45}=-\frac{\pi^4}{180}
\end{aligned}$$

□

Exercise 1.4.8: 计算积分:

$$\int_0^{\infty}\frac{\ln x}{1+e^x}\,\mathrm{d}x$$

Solution:

$$\begin{aligned}
\int_0^{\infty}\frac{\ln x}{1+e^x}\,\mathrm{d}x&=\int_0^{\infty}\frac{e^{-x}\ln x}{1+e^{-x}}\,\mathrm{d}x\\
&=\int_0^{\infty}e^{-x}\ln x\sum_{n=0}^{\infty}(-1)^n e^{-nx}\,\mathrm{d}x\\
&=\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-(n+1)x}\ln x\,\mathrm{d}x\\
&=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\ln(n+1)+\gamma}{n+1}\qquad(1)\\
&=-\frac{1}{2}\ln^2 2
\end{aligned}$$

其中 (1) 处用到了

$$\int_0^{\infty}e^{-nx}\ln x\,\mathrm{d}x=-\frac{\gamma+\ln n}{n}$$

□

Exercise 1.4.9: 计算积分:

$$\int_0^{\infty}e^{-ax}\sin^n x\,\mathrm{d}x$$

✎ **Solution:** 利用分部积分，我们有

$$\begin{aligned}
I_n &= \frac{-1}{a}e^{-ax}\sin^n(x)\Big|_0^\infty + \frac{n}{a}\int_0^\infty \sin^{n-1}(x)\cos(x)e^{-ax}\mathrm{d}x \\
\Rightarrow I_n &= \frac{n}{a}\int_0^\infty e^{-ax}\sin^{n-1}(x)\cos(x)\mathrm{d}x \\
\Rightarrow I_n &= \frac{n}{a}\left[\frac{-1}{a}e^{-ax}\sin^{n-1}(x)\cos(x)\Big|_0^\infty\right. \\
&\qquad \left. + \int_0^\infty \frac{e^{-ax}}{a}\cdot\left((n-1)\sin^{n-2}(x)\cos^2(x) - \sin^n(x)\right)\mathrm{d}x\right] \\
\Rightarrow I_n &= \frac{n(n-1)}{a^2}\int_0^\infty e^{-ax}\sin^{n-2}(x)\mathrm{d}x - \frac{n^2}{a^2}\int_0^\infty e^{-ax}\sin^n(x)\mathrm{d}x \\
\Rightarrow I_n &= \frac{n(n-1)}{a^2}I_{n-2} - \frac{n^2}{a^2}I_n
\end{aligned}$$

解出 I_n

$$I_n\left(1+\frac{n^2}{a^2}\right) = \frac{n(n-1)}{a^2}I_{n-2} \Rightarrow I_n = \frac{n(n-1)}{n^2+a^2}I_{n-2}$$

所以，当 n 为偶数时

$$I_n = \frac{n(n-1)}{n^2+a^2}\cdot\frac{(n-2)(n-3)}{(n-2)^2+a^2}\cdots\frac{2\cdot 1}{2^2+a^2}\cdot\underbrace{\frac{1}{a}}_{I_0}$$

当 n 为奇数时

$$I_n = \frac{n(n-1)}{n^2+a^2}\cdot\frac{(n-2)(n-3)}{(n-2)^2+a^2}\cdots\frac{3\cdot 2}{3^2+a^2}\cdot\underbrace{\frac{1}{a^2+1}}_{I_1}$$

综上我们有

$$\int_0^\infty e^{-ax}\sin^n x\ \mathrm{d}x = \begin{cases} \dfrac{(2m)!}{a\cdot\prod\limits_{k=1}^{m}\left(4k^2+a^2\right)} = \dfrac{\pi\cdot\operatorname{csch}\left(\frac{\pi a}{2}\right)\cdot(2m)!}{2^{2m+1}\Gamma\left(m-\frac{ai}{2}+1\right)\Gamma\left(m+\frac{ai}{2}+1\right)} \\ \dfrac{(2m+1)!}{\prod\limits_{k=0}^{m}\left((2k+1)^2+a^2\right)} = \dfrac{\pi\cdot\operatorname{sech}\left(\frac{\pi a}{2}\right)\cdot(2m+1)!}{4^{m+1}\Gamma\left(m-\frac{ai}{2}+\frac{3}{2}\right)\Gamma\left(m+\frac{ai}{2}+\frac{3}{2}\right)} \end{cases}$$

□

Exercise 1.4.10: 计算积分：

$$\int_0^\pi \ln(1+\sin x)\,\mathrm{d}x$$

✎ **Solution:** 注意到

$$\ln(1+\sin x) = \ln 2 + 2\ln\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

令 $\frac{x}{2}+\frac{\pi}{4}\to y$，则有

$$\int_0^{\pi}\ln\left(1+\sin x\right)\mathrm{d}x=\pi\ln 2+4\underbrace{\int_{\pi/4}^{3\pi/4}\ln\sin y\mathrm{d}y}_{J}$$

利用 $\ln\sin x$ 傅里叶级数我们有

$$\begin{aligned}J&=-\ln 2\int_{\pi/4}^{3\pi/4}\mathrm{d}y-\sum_{n=1}^{\infty}\frac{1}{k}\int_{\pi/4}^{3\pi/4}\cos\left(2ky\right)\mathrm{d}y\\&=-\frac{\pi}{2}\ln 2+\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{\left(2k+1\right)^2}\\&=-\frac{\pi}{2}\ln 2+\mathbf{G}\end{aligned}$$

因此

$$\int_0^{\pi}\ln\left(1+\sin x\right)\mathrm{d}x=\pi\ln 2+4J=-\pi\ln 2+4\mathbf{G}$$

同样也有

$$\int_0^{\pi}\ln\left(1-\sin x\right)\mathrm{d}x=\pi\ln 2-4\mathbf{G}$$

□

Exercise 1.4.11: 计算积分：

$$\int_0^{\infty}\frac{\sin x-\sin x^2}{x}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^{\infty}\frac{\sin x-\sin x^2}{x}\mathrm{d}x&=\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x-\int_0^{\infty}\frac{\sin x^2}{x}\mathrm{d}x\\&=\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x-\frac{1}{2}\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x\\&=\frac{1}{2}\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x=\frac{\pi}{4}\end{aligned}$$

□

Exercise 1.4.12: 计算积分：

$$\int_0^{\infty}\frac{\cos x-\cos x^2}{x}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^{\infty}\frac{\cos x-\cos x^2}{x}\mathrm{d}x&=\lim_{\alpha\to\infty}\int_0^{\alpha}\frac{\cos x-\cos x^2}{x}\mathrm{d}x=\lim_{\alpha\to\infty}-\int_0^{\alpha}\frac{1-\cos x+\cos x^2-1}{x}\mathrm{d}x\\&=\lim_{\alpha\to\infty}\left(-\int_0^{\alpha}\frac{1-\cos x}{x}\mathrm{d}x+\int_0^{\alpha}\frac{1-\cos x^2}{x}\mathrm{d}x\right)\\&=\lim_{\alpha\to\infty}\left(-\int_0^{\alpha}\frac{1-\cos x}{x}\mathrm{d}x+\frac{1}{2}\int_0^{\alpha^2}\frac{1-\cos x}{x}\mathrm{d}x\right)\\&=\lim_{\alpha\to\infty}\left\{\mathrm{Ci}\left(\alpha\right)-\gamma-\ln\alpha+\frac{1}{2}\left[\gamma+\ln\alpha^2-\mathrm{Ci}\left(\alpha^2\right)\right]\right\}\\&=\lim_{\alpha\to\infty}\left[-\frac{\gamma}{2}+\mathrm{Ci}\left(\alpha\right)-\frac{1}{2}\mathrm{Ci}\left(\alpha^2\right)\right]\\&=-\frac{\gamma}{2}\end{aligned}$$

□

Exercise 1.4.13: 计算下面两个积分的比值:

$$\int_0^1 \frac{1}{\sqrt{1+t^4}}\mathrm{d}t\ ,\ \int_0^1 \frac{1}{\sqrt{1-t^4}}\mathrm{d}t$$

Solution:

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1+t^4}}\mathrm{d}t &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1+t^2)^2-2t^2}}\\ &= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{2t}{1+t^2}\right)}}\frac{2}{1+t^2}\mathrm{d}t\\ \left(t=\tan\frac{\theta}{2}\right) &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{1}{2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-t^4}}\mathrm{d}t &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1+\cos^2\theta}} \quad (t=\cos\theta)\\ &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{2-\sin^2\theta}}\\ &= \frac{1}{\sqrt{2}}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)\end{aligned}$$

所以

$$\frac{\int_0^1 \frac{1}{\sqrt{1-x^4}}\mathrm{d}x}{\int_0^1 \frac{1}{\sqrt{1+x^4}}\mathrm{d}x} = \frac{\frac{1}{\sqrt{2}}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}$$

□

Exercise 1.4.14: 计算积分:

$$\int_0^\infty \frac{x^4e^{-2x^2}}{(1+x^2)^4}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^\infty \frac{x^4e^{-2x^2}}{(1+x^2)^4}\mathrm{d}x &= \frac{1}{\Gamma(4)}\int_0^\infty y^3e^{-y}\mathrm{d}y\int_0^\infty x^4e^{-2x^2-yx^2}\mathrm{d}x\\ &= \frac{3\sqrt{\pi}}{8\Gamma(4)}\int_0^\infty \frac{y^3e^{-y}}{(y+2)^{5/2}}\mathrm{d}y\\ &= \frac{3\sqrt{\pi}e^2}{8\Gamma(4)}\int_2^\infty \frac{(t-2)^3e^{-t}}{t^{5/2}}\mathrm{d}t\\ &= \frac{17}{16}\sqrt{2\pi}-\frac{241e^2\pi}{96}\operatorname{erfc}\left(\sqrt{2}\right)\\ &\approx 0.0117502\cdots\end{aligned}$$

□

Exercise 1.4.15: 计算积分：

$$\int_0^1 \frac{x\ln x}{1+x^2}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^1 \frac{x\ln x}{1+x^2}\mathrm{d}x &= \int_0^1 x\ln x\sum_{n=0}^{\infty}(-1)^n x^{2n}\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln x\mathrm{d}x\\ &= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{1}{4}\cdot\left(-\frac{\pi^2}{12}\right)\\ &= -\frac{\pi^2}{48}\end{aligned}$$

□

Exercise 1.4.16: 计算积分：

$$\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x^2}\mathrm{d}x$$

Solution: 方法一：考虑含参积分

$$\mathcal{I}(\alpha) = \int_0^1 \frac{\ln\left(1+\alpha x^2\right)}{1+x^2}\mathrm{d}x$$

则

$$\mathcal{I}'(\alpha) = \int_0^1 \frac{x^2}{\left(1+x^2\right)\left(1+\alpha x^2\right)}\mathrm{d}x$$

往下积分有理积分形式.

方法二：

$$\begin{aligned}\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x^2}\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1\frac{x^{2n}}{1+x^2}\mathrm{d}x\\ &= \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\psi_0\left(\frac{n}{2}+\frac{3}{4}\right)-\psi_0\left(\frac{n}{2}+\frac{1}{4}\right)\right)\end{aligned}$$

接下来就涉及到了 Euler Sum 的计算. 上述两种方法都过于麻烦，现在我们看一种更为简单的方法.

方法三： 做代换 $x=\tan t$，我们有

$$\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x^2}\mathrm{d}x = \int_0^{\frac{\pi}{4}}\ln\left(1+\tan^2 t\right)\mathrm{d}t = -2\int_0^{\frac{\pi}{4}}\ln\cos t\mathrm{d}t$$

由 **Exercise 1.1.156** 我们有

$$\int_0^{\frac{\pi}{4}}\ln\cos t\mathrm{d}t = \frac{1}{2}\left(\mathbf{G}-\frac{\pi}{2}\ln 2\right)$$

所以

$$\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x^2}\mathrm{d}x = -2\int_0^{\frac{\pi}{4}}\ln\cos t\mathrm{d}t = \frac{\pi}{2}\ln 2-\mathbf{G}$$

□

Exercise 1.4.17:

✎ **Solution:**

□

Exercise 1.4.18:

✎ **Solution:**

□

Exercise 1.4.19:

✎ **Solution:**

□

Exercise 1.4.20:

✎ **Solution:**

□

Exercise 1.4.21:

✎ **Solution:**

□

Exercise 1.4.22:

✎ **Solution:**

□

*

1.5 其他

✎ *

Exercise 1.5.1: 计算积分：

$$\int_0^\infty e^{-x^2-x^{-2}}\mathrm{d}x$$

✎ **Solution:** 事实上，我们可以计算更一般的情况：

$$I=\int_0^\infty e^{-\alpha^2\left(x^2+x^{-2}\right)}\mathrm{d}x$$

令 $x=x^{-1}$ 得

$$I=\int_0^\infty e^{-\alpha^2\left(x^2+x^{-2}\right)}\mathrm{d}x=\int_0^\infty x^{-2}e^{-\alpha^2\left(x^2+x^{-2}\right)}\mathrm{d}x$$

所以我们有

$$\begin{aligned}I=\frac{1}{2}\int_0^\infty\left(1+x^{-2}\right)e^{-\alpha^2\left(x^2+x^{-2}\right)}\mathrm{d}x&=\frac{1}{2}\int_{-\infty}^\infty e^{-\alpha^2\left(x^2+x^{-2}\right)}\mathrm{d}\left(x-x^{-1}\right)\\&=\frac{1}{2}\int_{-\infty}^\infty e^{-\alpha^2\left[\left(x-x^{-1}\right)^2-2\right]}\mathrm{d}\left(x-x^{-1}\right)\\&=\frac{1}{2}e^{-2\alpha^2}\int_{-\infty}^\infty e^{-\alpha^2t^2}\mathrm{d}t\\&=e^{-2\alpha^2}\cdot\frac{\sqrt{\pi}}{2\alpha}\end{aligned}$$

令 $\alpha=1$，即得原题：

$$\int_0^{\infty} e^{-x^2-x^{-2}}\mathrm{d}x=\frac{\sqrt{\pi}}{2e^2}$$

□

Exercise 1.5.2: 计算积分：

$$\int_0^1 \frac{\sqrt[4]{x\left(1-x\right)^3}}{\left(1+x\right)^3}\mathrm{d}x$$

$$\int_0^1 \frac{\sqrt[3]{x\left(1-x\right)^2}}{\left(1+x\right)^3}\mathrm{d}x$$

Solution:

Definition 1.5.1 Beta function

$$B\left(p,q\right)=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)}=\int_0^1 x^{p-1}\left(1-x\right)^{q-1}\mathrm{d}x\qquad\left(\Re q>0,\Re p>0\right)$$

同样，我们来计算更一般的情况：

$$I\left(m,n\right)=\int_0^1 \frac{\sqrt[n]{x^m\left(1-x\right)^{n-m}}}{\left(1+x\right)^3}\mathrm{d}x$$

为了化为 B 函数，我们作如下变换

$$t=\frac{2x}{1+x},\quad 1-t=\frac{1-x}{1+x},\quad \mathrm{d}t=\frac{2\mathrm{d}x}{\left(1+x\right)^2}$$

于是直接计算得：

$$\begin{aligned}\int_0^1 \frac{\sqrt[n]{x^m\left(1-x\right)^{n-m}}}{\left(1+x\right)^3}\mathrm{d}x&=\int_0^1\left(\frac{x}{1+x}\right)^{\frac{m}{n}}\left(\frac{1-x}{1+x}\right)^{\frac{n-m}{n}}\frac{\mathrm{d}x}{\left(1+x\right)^2}\\&=2^{-\frac{n+m}{n}}\int_0^1 t^{\frac{m}{n}}\left(1-t\right)^{\frac{n-m}{n}}\mathrm{d}t\\&=\frac{2^{-\frac{n+m}{n}}}{\Gamma\left(3\right)}\Gamma\left(\frac{m+n}{n}\right)\Gamma\left(\frac{2n-m}{n}\right)\\&=2^{-\frac{2n+m}{n}}\cdot\frac{m}{n}\cdot\frac{n-m}{n}\cdot\Gamma\left(\frac{m}{n}\right)\cdot\Gamma\left(1-\frac{m}{n}\right)\\&=2^{-\frac{2n+m}{n}}\cdot\frac{m\left(n-m\right)}{n^2}\cdot\frac{\pi}{\sin\left(\frac{m\pi}{n}\right)}\end{aligned}$$

所以原题即为：

$$I\left(1,4\right)=\frac{3\sqrt[4]{2}}{64}\pi$$

$$I\left(1,3\right)=\frac{\pi}{18}\frac{\sqrt[3]{4}}{\sqrt{3}}$$

□

Exercise 1.5.3: 计算积分：

$$\int_0^\pi \arctan\frac{r\sin\theta}{1+r\cos\theta}\frac{\mathrm{d}\theta}{\sin\theta},\quad 0\le r<1$$

$$\int_0^\pi \arctan\frac{r\sin\theta}{1+r\cos\theta}\frac{\mathrm{d}\theta}{\tan\theta},\quad 0\le r<1$$

Solution:

Corollary 1.5.1

$$\int_0^\pi \frac{\sin n\theta}{\sin\theta}\mathrm{d}\theta=\begin{cases}0 & n=2k\\ \pi & n=2k-1\end{cases}\qquad (k=1,2,\cdots n)$$

♣

所以我们有

$$\begin{aligned}\int_0^\pi \arctan\frac{r\sin\theta}{1+r\cos\theta}\frac{\mathrm{d}\theta}{\sin\theta}&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}r^n\int_0^\pi\frac{\sin n\theta}{\sin\theta}\mathrm{d}\theta\\&=\sum_{n=0}^{\infty}\frac{\pi}{2n+1}r^{2n+1}=\frac{\pi}{2}\ln\frac{1+r}{1-r}\end{aligned}$$

$$\begin{aligned}\int_0^\pi \arctan\frac{r\sin\theta}{1+r\cos\theta}\frac{\mathrm{d}\theta}{\tan\theta}&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}r^n\int_0^\pi\frac{\sin n\theta\cos\theta}{\sin\theta}\mathrm{d}\theta\\&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}r^n\frac{1}{2}\int_0^\pi\frac{\sin(n+1)\theta+\sin(n-1)\theta}{\sin\theta}\mathrm{d}\theta\\&=-\sum_{n=0}^{\infty}\frac{\pi}{2n}r^{2n}=-\frac{\pi}{2}\ln\frac{1}{1-r^2}\end{aligned}$$

□

Exercise 1.5.4: 计算积分：

$$\int_0^\infty\frac{\ln x}{x^2-1}\mathrm{d}x$$

Solution: 显然

$$\int_0^\infty\frac{\ln x}{x^2-1}\mathrm{d}x=\int_0^1\frac{\ln x}{x^2-1}\mathrm{d}x+\int_1^\infty\frac{\ln x}{x^2-1}\mathrm{d}x$$

又有

$$\int_0^1\frac{\ln x}{x^2-1}\mathrm{d}x=\int_1^\infty\frac{\ln x}{x^2-1}\mathrm{d}x$$

所以只需计算其中一个

$$\begin{aligned}\int_0^1 \frac{\ln x}{x^2-1}\mathrm{d}x &= -\int_0^1 \ln x \sum_{n=0}^{\infty} x^{2n}\mathrm{d}x \\ &= -\sum_{n=0}^{\infty}\int_0^1 x^{2n}\ln x\mathrm{d}x \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\ &= \frac{\pi^2}{8}\end{aligned}$$

所以原积分为

$$\int_0^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x = \frac{\pi^2}{4}$$

□

Exercise 1.5.5: 计算积分:

$$\int_0^{\pi} \theta \ln\tan\frac{\theta}{2}\mathrm{d}\theta$$

Solution: 显然

$$\int_0^{\pi} \theta \ln\tan\frac{\theta}{2}\mathrm{d}\theta = 4\int_0^{\pi/2} x\ln\tan x\mathrm{d}x$$

Definition 1.5.2 Lobachevskiy 函数

$$\mathrm{L}(x) = -\int_0^x \ln\cos x\mathrm{d}x, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

所以

$$\begin{aligned}\int_0^{\pi/2} x\ln\tan x\mathrm{d}x &= x\left[\mathrm{L}(x)+\mathrm{L}\left(\frac{\pi}{2}-x\right)\right]_0^{\pi/2} - \int_0^{\pi/2}\left[\mathrm{L}(x)+\mathrm{L}\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\ &= \left(\frac{\pi}{2}\right)^2\ln 2 - 2\int_0^{\pi/2}\mathrm{L}(x)\,\mathrm{d}x\end{aligned}$$

再利用

$$\mathrm{L}(x) = x\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\sin 2kx$$

就能计算得

$$\begin{aligned}\int_0^{\pi/2}\mathrm{L}(x)\,\mathrm{d}x &= \frac{1}{2}\left(\frac{\pi}{2}\right)^2\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\int_0^{\pi/2}\sin 2kx\mathrm{d}x \\ &= \frac{\pi^2}{8}\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\end{aligned}$$

所以

$$\begin{aligned}\int_0^{\pi/2} x\ln\tan x\mathrm{d}x &= \frac{\pi^2}{4}\ln 2-2\left[\frac{\pi^2}{8}\ln 2-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\right]\\ &=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\\ &=\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{k=1}^{\infty}\frac{1}{(2k)^3}=\frac{7}{8}\zeta(3)\end{aligned}$$

亦即

$$\int_0^{\pi}\theta\ln\tan\frac{\theta}{2}\mathrm{d}\theta=\frac{7}{2}\zeta(3)$$

另外还可以得到

$$\int_0^{\pi}\theta\ln\tan\frac{\theta}{2}\mathrm{d}\theta=\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\psi\left(n+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)\right]=\frac{7}{2}\zeta(3)$$

或

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\psi\left(n+\frac{1}{2}\right)=\frac{7}{2}\zeta(3)-(\gamma+2\ln 2)\frac{\pi^2}{6}$$

□

Exercise 1.5.6: 计算:

$$\int_0^1\ln^2\left(\frac{1-x}{1+x}\right)\frac{\mathrm{d}x}{x}$$

Solution: 令 $x=\dfrac{1-t}{1+t}$, $\mathrm{d}x=-\dfrac{2}{(t+1)^2}\mathrm{d}t$, 则

$$\begin{aligned}\int_0^1\ln^2\left(\frac{1-x}{1+x}\right)\frac{\mathrm{d}x}{x} &= -2\int_0^1\frac{\ln^2 t}{t^2-1}\mathrm{d}t=-2\int_0^1\frac{\ln^2 t}{t^2}\sum_{n=0}^{\infty}t^{-2n}\mathrm{d}t\\ &=-2\int_0^1\ln^2 t\sum_{n=0}^{\infty}t^{-2(2+1)}\mathrm{d}t=-2\sum_{n=0}^{\infty}\int_0^1 t^{-2(n+1)}\ln^2 t\mathrm{d}t\\ &=4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}=4\left(\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{1}{(2n)^3}\right)\\ &=4\left(\zeta(3)-\frac{1}{8}\zeta(3)\right)\\ &=\frac{7}{2}\zeta(3)\end{aligned}$$

□

Exercise 1.5.7: 计算积分:

$$\int_0^1\frac{\arctan x}{1+x^2}\ln(1+x)\,\mathrm{d}x$$

Solution: 易知

$$\int_0^1\frac{\arctan x}{1+x^2}\ln(1+x)\,\mathrm{d}x=\int_0^{\frac{\pi}{4}}x\ln(1+\tan x)\,\mathrm{d}x$$

下面来计算后者，

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} x\ln\left(1+\tan x\right)\mathrm{d}x=\int_0^{\frac{\pi}{4}} x\ln\left(\sin x+\cos x\right)\mathrm{d}x-\int_0^{\frac{\pi}{4}} x\ln\left(\cos x\right)\mathrm{d}x\\
&=\int_0^{\frac{\pi}{4}} x\ln\left[\sqrt{2}\left(\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x\right)\right]\mathrm{d}x\\
&\quad-\left[\frac{3}{16}\zeta\left(3\right)+\frac{\pi^2}{32}\ln 2-\frac{\pi}{8}\mathrm{Cl}_2\left(\frac{\pi}{2}\right)+\frac{1}{4}\mathrm{Cl}_3\left(\frac{\pi}{2}\right)\right]\\
&=-\frac{\pi^2}{64}\ln 2+\int_0^{\frac{\pi}{4}} x\ln\left[2\sin\left(x+\frac{\pi}{4}\right)\right]\mathrm{d}x\\
&\quad-\left[\frac{3}{16}\zeta\left(3\right)+\frac{\pi^2}{32}\ln 2-\frac{\pi}{8}\mathrm{Cl}_2\left(\frac{\pi}{2}\right)+\frac{1}{4}\mathrm{Cl}_3\left(\frac{\pi}{2}\right)\right]\\
&=-\frac{\pi^2}{64}\ln 2+\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\frac{y}{2}-\frac{\pi}{4}\right)\ln\left(2\sin\frac{y}{2}\right)\mathrm{d}y\\
&\quad-\left[\frac{3}{16}\zeta\left(3\right)+\frac{\pi^2}{32}\ln 2-\frac{\pi}{8}\mathrm{Cl}_2\left(\frac{\pi}{2}\right)+\frac{1}{4}\mathrm{Cl}_3\left(\frac{\pi}{2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\pi}\left(\frac{y}{2}-\frac{\pi}{4}\right)\ln\left(2\sin\frac{y}{2}\right)\mathrm{d}y=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} y\ln\left(2\sin\frac{y}{2}\right)\mathrm{d}y-\frac{\pi}{4}\int_{\frac{\pi}{2}}^{\pi}\ln\left(2\sin\frac{y}{2}\right)\mathrm{d}y\\
&=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} y\left[-\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Cl}_2\left(y\right)\right]\mathrm{d}y+\frac{\pi}{4}\mathrm{Cl}_2\left(y\right)\Big|_{\frac{\pi}{2}}^{\pi}=-\frac{\pi}{4}\mathrm{Cl}_2\left(\pi\right)+\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\mathrm{Cl}_2\left(y\right)\mathrm{d}y\\
&=-\frac{\pi}{4}\mathrm{Cl}_2\left(\pi\right)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_{\frac{\pi}{2}}^{\pi}\sin kx\mathrm{d}x=-\frac{\pi}{4}\mathrm{Cl}_2\left(\pi\right)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos kx}{k^3}\Big|_{\frac{\pi}{2}}^{\pi}\\
&=-\frac{\pi}{4}\mathrm{Cl}_2\left(\pi\right)-\frac{1}{2}\mathrm{Cl}_3\left(\pi\right)-\frac{1}{2}\mathrm{Cl}_3\left(\frac{\pi}{2}\right)
\end{aligned}$$

所以，

$$\begin{aligned}
&\int_0^1\frac{\arctan x}{1+x^2}\ln\left(1+x\right)\mathrm{d}x=\int_0^{\frac{\pi}{4}} x\ln\left(1+\tan x\right)\mathrm{d}x\\
&=-\frac{\pi^2}{64}\ln 2-\frac{\pi}{8}\mathrm{Cl}_2\left(\pi\right)-\frac{1}{4}\mathrm{Cl}_3\left(\pi\right)-\frac{1}{4}\mathrm{Cl}_3\left(\frac{\pi}{2}\right)\\
&\quad-\left[\frac{3}{16}\zeta\left(3\right)+\frac{\pi^2}{32}\ln 2-\frac{\pi}{8}\mathrm{Cl}_2\left(\frac{\pi}{2}\right)+\frac{1}{4}\mathrm{Cl}_3\left(\frac{\pi}{2}\right)\right]\\
&=-\frac{\pi\mathbf{G}}{8}+\frac{21}{64}\zeta\left(3\right)+\frac{\pi^2}{64}\ln 2
\end{aligned}$$

□

Exercise 1.5.8: 计算积分：

$$\int_0^1\frac{x\arctan x}{1+x^2}\mathrm{d}x$$

Solution:

$$\begin{aligned}
\int_0^1\frac{x\arctan x}{1+x^2}\mathrm{d}x&=\int_0^{\frac{\pi}{4}} x\tan x\mathrm{d}x=\frac{\pi}{8}\ln 2+\int_0^{\frac{\pi}{4}}\ln\cos x\mathrm{d}x\\
&=\frac{\pi}{8}\ln 2+\frac{\mathbf{G}}{2}-\frac{\pi}{4}\ln 2=\frac{\mathbf{G}}{2}-\frac{\pi\ln 2}{8}
\end{aligned}$$

□

Exercise 1.5.9: 计算积分：

$$\int_0^{\infty} \frac{\ln(2x+1)}{x(1+x)}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^{\infty} \frac{\ln(2x+1)}{x(1+x)}\mathrm{d}x &= 2\int_1^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x = -2\int_0^1 \frac{\ln x}{1-x^2}\mathrm{d}x \\ &= -2\int_0^1 \ln x \sum_{n=0}^{\infty} x^{2n}\mathrm{d}x = -2\sum_{n=0}^{\infty}\int_0^1 x^{2n}\ln x\mathrm{d}x \\ &= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 2\cdot\frac{\pi^2}{8} = \frac{\pi^2}{4}\end{aligned}$$

利用 WA 还可以

$$\begin{aligned}\int_0^{\infty} \frac{\ln(2x+1)}{x(1+x)}\mathrm{d}x &= \Big[-\ln(2(1+x))\ln(1+2x) - \mathrm{Li}_2(-1-2x) - \mathrm{Li}_2(-2x)\Big]\Big|_0^{\infty} \\ &= \frac{\pi^2}{3} - \frac{\pi^2}{12} = \frac{\pi^2}{4}\end{aligned}$$

□

Exercise 1.5.10: 计算积分：

$$\int_0^{\infty} \frac{\sin x}{\sinh x}\mathrm{d}x$$

Solution:

$$\begin{aligned}\int_0^{\infty} \frac{\sin x}{\sinh x}\mathrm{d}x &= \int_0^{\infty} \sin x\left(\frac{e^{-x}}{1-e^{-x}} + \frac{e^{-x}}{1+e^{-x}}\right)\mathrm{d}x \\ &= \sum_{n=1}^{\infty}\left(\int_0^{\infty} e^{-nx}\sin x\mathrm{d}x - (-1)^n e^{-nx}\sin x\mathrm{d}x\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{1+(-1)^{n-1}}{1+n^2}\right) = \frac{\pi}{2}\tanh\frac{\pi}{2}\end{aligned}$$

□

Exercise 1.5.11: 计算积分：

$$\int_0^{\infty} \frac{\cos\lambda x}{1+\lambda^2}\mathrm{d}\lambda\ ,\quad \int_0^{\infty} \frac{(u^2+2)\cos ux}{u^4+4}\mathrm{d}u$$

Solution:

$$\begin{aligned}e^{-x} &= \frac{2}{\pi}\int_0^{\infty}\cos\lambda x\mathrm{d}\lambda\int_0^{\infty} e^{-u}\cos\lambda u\mathrm{d}u = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\lambda x}{1+\lambda^2}\mathrm{d}\lambda \\ e^{-x}\cos x &= \frac{2}{\pi}\int_0^{\infty}\cos ux\mathrm{d}u\int_0^{\infty} e^{-t}\cos t\cos ut\mathrm{d}t = \frac{2}{\pi}\int_0^{\infty}\frac{(u^2+2)\cos ux}{u^4+4}\mathrm{d}u\end{aligned}$$

所以

$$\int_0^{\infty} \frac{\cos\lambda x}{1+\lambda^2}\mathrm{d}\lambda = \frac{\pi}{2}e^{-x}\ ,\quad \int_0^{\infty} \frac{(u^2+2)\cos ux}{u^4+4}\mathrm{d}u = \frac{\pi}{2}e^{-x}\cos x$$

□

Exercise 1.5.12: 计算积分：

$$\int_0^1 x\sqrt{1+x^3}\mathrm{d}x$$

Solution: 易知

$$\int_0^1 x\sqrt{1+x^3}\mathrm{d}x=\frac{1}{3}\int_0^1 x^{-\frac{1}{3}}\left(1+x\right)^{\frac{1}{2}}\mathrm{d}x$$

下面我们来看这个一般形式

$$\int_0^u y^{b-1}\left(u-y\right)^{c-b-1}\left(y+\frac{u}{x}\right)^{-a}\mathrm{d}t$$

首先我们引入 Beta 函数

$$\mathrm{B}\left(a,b\right)=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}\mathrm{d}t$$

然后引入超几何函数 ${}_2F_1$ 的定义

$${}_2F_1\left(a,b;c;x\right)=\frac{1}{\mathrm{B}\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-tx\right)^{-a}\mathrm{d}t$$

简单调整之后我们可以得到

$$\int_0^1 t^b\left(1-t\right)^c\left(1-tx\right)^a\mathrm{d}t=\mathrm{B}\left(b+1,c+1\right)\ {}_2F_1\left(-a,b+1;b+c+2;x\right)$$

做代换 $y=tu$, $x\to -x$ 后，我们有

$$\begin{aligned}&\int_0^1 t^b\left(1-t\right)^c\left(1-tx\right)^a\mathrm{d}t=\int_0^1\left(\frac{y}{u}\right)^b\left(1-\frac{y}{u}\right)^c\left(1+\frac{yx}{u}\right)^a\frac{1}{u}\mathrm{d}y\\&=\left(\frac{u}{x}\right)^{-a}u^{-b-c-1}\int_0^u y^b\left(u-y\right)^c\left(y+\frac{u}{x}\right)^a\mathrm{d}y\end{aligned}$$

然后做代换 $b+1\to b$, $c+1\to c-b$, $a\to -a$ 我们有

$$\int_0^u y^{b-1}\left(u-y\right)^{c-b-1}\left(y+\frac{u}{x}\right)^{-a}\mathrm{d}t=\left(\frac{u}{x}\right)^a u^{c-1}\mathrm{B}\left(b,c-b\right)\ {}_2F_1\left(a,b;c;-x\right)$$

所以我们令 $u=1$, $x=1$, $b=\frac{2}{3}$, $c=\frac{5}{3}$, $a=-\frac{1}{2}$，可以得到

$$\int_0^1 x^{-\frac{1}{3}}\left(1+x\right)^{\frac{1}{2}}\mathrm{d}x=\frac{3}{2}\,{}_2F_1\left(-\frac{1}{2},\frac{2}{3};\frac{5}{3};-1\right)$$

所以

$$\int_0^1 x\sqrt{1+x^3}\mathrm{d}x=\frac{1}{2}\,{}_2F_1\left(-\frac{1}{2},\frac{2}{3};\frac{5}{3};-1\right)$$

□

Exercise 1.5.13: 计算积分：

$$\int_0^1\left[\frac{1+\sqrt{1-x}}{x}+\frac{2}{\ln\left(1-x\right)}\right]\mathrm{d}x$$

✎ **Solution:** 方法一： 令 $\ln(1-x)=-t$，有

$$\int_0^1\left[\frac{1+\sqrt{1-x}}{x}+\frac{2}{\ln(1-x)}\right]\mathrm{d}x=\int_0^\infty e^{-t}\left(\frac{1}{1-e^{-\frac{t}{2}}}-\frac{2}{t}\right)\mathrm{d}t$$
$$=\int_0^\infty\left[-2\left(\frac{e^{-t}}{t}-\frac{1}{e^t-1}\right)-\frac{e^{-\frac{t}{2}}}{1+e^{\frac{t}{2}}}\right]\mathrm{d}t=-2\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{1}{e^t-1}\right)\mathrm{d}t-\int_0^\infty\frac{e^{-\frac{t}{2}}}{1+e^{\frac{t}{2}}}\mathrm{d}t$$
$$=2\gamma-\int_0^\infty\left(e^{-\frac{t}{2}}-\frac{e^{-\frac{t}{2}}}{1+e^{-\frac{t}{2}}}\right)\mathrm{d}t=2\gamma-2+2\ln 2$$

方法二： 令 $1-x\to x$，有

$$\int_0^1\left[\frac{1+\sqrt{1-x}}{x}+\frac{2}{\ln(1-x)}\right]\mathrm{d}x=\int_0^1\left(\frac{1+\sqrt{x}}{1-x}-\frac{2}{\ln x}\right)\mathrm{d}x$$
$$=\int_\infty^0\int_0^1\frac{x^t\ln x}{1-x}+\frac{x^{t+\frac{1}{2}}\ln x}{1-x}+2x^t\mathrm{d}x\mathrm{d}t=-\int_0^\infty\left[\psi_1(t+1)+\psi_1\left(t+\frac{2}{3}\right)-\frac{2}{t+1}\right]\mathrm{d}t$$
$$=-\psi_0(1)-\psi_0\left(\frac{3}{2}\right)=2\gamma-2+2\ln 2$$

□

⊸ **Exercise 1.5.14:** 证明：

$$\int_0^{[x]}(t-[t])\,\mathrm{d}t=\frac{[x]}{2}$$

☞ **Proof:** 我们来看更一般的形式，令 $m=\lfloor a\rfloor$，有

$$\int_a^{a+k}(t-[t])\,\mathrm{d}t=\int_a^{a+k}\{t\}\,\mathrm{d}t=\int_a^{m+1}\{t\}\,\mathrm{d}t+\sum_{j=m+1}^{k+m-1}\left(\int_j^{j+1}\{t\}\,\mathrm{d}t\right)+\int_{k+m}^{k+a}\{t\}\,\mathrm{d}t$$
$$=\int_a^{m+1}(t-m)\,\mathrm{d}t+\sum_{j=m+1}^{k+m-1}\left(\int_j^{j+1}(t-j)\,\mathrm{d}t\right)+\int_{k+m}^{k+a}(t-k-m)\,\mathrm{d}t=\frac{k}{2}$$

所以令 $a=0$, $k=[x]$，即得

$$\int_0^{[x]}(t-[t])\,\mathrm{d}t=\frac{[x]}{2}$$

□

⊸ **Exercise 1.5.15:**

✎ **Solution:**

□

⊸ **Exercise 1.5.16:**

✎ **Solution:**

□

⊸ **Exercise 1.5.17:**

✎ **Solution:**

□

⊸ **Exercise 1.5.18:**

✎ **Solution:**

□

Exercise 1.5.19:

Solution:

□

Exercise 1.5.20:

Solution:

□

Exercise 1.5.21:

Solution:

□

Exercise 1.5.22:

Solution:

□

Exercise 1.5.23:

Solution:

□

*

第 2 章 Series 级数

2.1 Stack Exchange

✎ *

Exercise 2.1.1: Compute the following series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^3}$$

✎ **Solution:** Note Landen's identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\ln^2(1-z) \tag{1}$$

which can be easily derived by differentiating $\mathrm{Li}_2\left(\frac{z}{z-1}\right)$ then integrating back. Plucking in $z = 0.5$ yields

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2$$

since $\mathrm{Li}_2(-1) = -\eta(1) = -\frac{\pi^2}{12}$. We will use the same idea to evaluate $\mathrm{Li}_3\left(\frac{1}{2}\right)$.

Observe $D_z\mathrm{Li}_{s+1}\left(\frac{z}{z-1}\right) = \frac{1}{z(1-z)}\mathrm{Li}_s\left(\frac{z}{z-1}\right)$. So dividing (1) throughout by $z(1-z)$ then integrating gives us

$$\mathrm{Li}_3(z) + \int \frac{\mathrm{Li}_2(z)}{1-z}\mathrm{d}z + \mathrm{Li}_3\left(\frac{z}{z-1}\right) = \frac{1}{6}\ln^3(1-z) - \frac{1}{2}\int \frac{\ln^2(1-z)}{z}\mathrm{d}z \tag{2}$$

The integral on the LHS evaluates to

$$\int \frac{\mathrm{Li}_2(z)}{1-z}\mathrm{d}z = -\mathrm{Li}_2(z)\ln(1-z) - \int \frac{\ln^2(1-z)}{z}\mathrm{d}z$$

so we can rewrite (2) as

$$\mathrm{Li}_3(z) + \mathrm{Li}_3\left(\frac{z}{z-1}\right) = \mathrm{Li}_2(z)\ln(1-z) + \frac{1}{6}\ln^3(1-z) + \frac{1}{2}\int \frac{\ln^2(1-z)}{z}\mathrm{d}z \tag{3}$$

and the integral on the RHS of (3) is

$$\begin{aligned}
\frac{1}{2}\int \frac{\ln^2(1-z)}{z}\mathrm{d}z &= \frac{1}{2}\ln z\ln^2(1-z) + \int \frac{\ln z\ln(1-z)}{1-z}\mathrm{d}z \\
&= \frac{1}{2}\ln z\ln^2(1-z) + \mathrm{Li}_2(1-z)\ln(1-z) + \int \frac{\mathrm{Li}_2(1-z)}{1-z}\mathrm{d}z \\
&= \frac{1}{2}\ln z\ln^2(1-z) + \mathrm{Li}_2(1-z)\ln(1-z) - \mathrm{Li}_3(1-z) + C
\end{aligned}$$

Plucking it into (3) then moving all the trilog terms to one side gives us

$$\begin{aligned}&\mathrm{Li}_3(z)+\mathrm{Li}_3(1-z)+\mathrm{Li}_3\left(\frac{z}{z-1}\right)\\=&\frac{1}{6}\ln^3(1-z)+\Big[\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)\Big]\ln(1-z)+\frac{1}{2}\ln z\ln^2(1-z)+C\\=&\frac{1}{6}\ln^3(1-z)-\frac{1}{2}\ln z\ln^2(1-z)+\frac{\pi^2}{6}\ln(1-z)+C\end{aligned}$$

where we applied the dilogarithm reflection formula

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)=\frac{\pi^2}{6}-\ln z\ln(1-z)$$

to get to it. This formula can be easily derived by differentiating $\mathrm{Li}_2(1-z)$ then integrating back. Now, by letting $z=0$, it is evident that $C=\mathrm{Li}_3(1)=\zeta(3)$. Therefore,

$$\begin{aligned}&\mathrm{Li}_3(z)+\mathrm{Li}_3(1-z)+\mathrm{Li}_3\left(\frac{z}{z-1}\right)\\=&\frac{1}{6}\ln^3(1-z)-\frac{1}{2}\ln z\ln^2(1-z)+\frac{\pi^2}{6}\ln(1-z)+\zeta(3)\end{aligned}$$

Finally, letting $z=0.5$, we get

$$2\mathrm{Li}_3\left(\frac{1}{2}\right)-\frac{3}{4}\zeta(3)=-\frac{1}{6}\ln^3 2+\frac{1}{2}\ln^3 2-\frac{\pi^2}{6}\ln 2+\zeta(3)$$

which implies

$$\mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln 2+\frac{1}{6}\ln^3 2$$

□

Exercise 2.1.2: Compute the following series:

$$\sum_{n=1}^{\infty}\frac{H_n}{2^n n^4}$$

Solution: Let

$$\mathcal{S}=\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}$$

We first consider a slightly different yet related sum. The main idea is to solve this sum with two different methods, one of which involves the sum in question. This then allows us to determine the value of the desired sum.

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^4}=\frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}H_n\int_0^1 x^{n-1}\ln^3 x\,\mathrm{d}x=\frac{1}{6}\int_0^1\frac{\ln^3 x\ln(1+x)}{x(1+x)}\mathrm{d}x\\=&\frac{1}{6}\int_0^1\frac{\ln^3 x\ln(1+x)}{x}\mathrm{d}x-\frac{1}{6}\int_0^1\frac{\ln^3 x\ln(1+x)}{1+x}\mathrm{d}x=\frac{1}{6}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{n-1}\ln^3 x\,\mathrm{d}x\\&-\frac{1}{6}\int_1^2\frac{\ln x\ln^3(x-1)}{x}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}+\int_{\frac{1}{2}}^1\frac{\ln x\ln^3(1-x)}{6x}-\int_{\frac{1}{2}}^1\frac{\ln^2 x\ln^2(1-x)}{2x}\mathrm{d}x\\&+\int_{\frac{1}{2}}^1\frac{\ln^3 x\ln(1-x)}{2x}\mathrm{d}x-\int_{\frac{1}{2}}^1\frac{\ln^4 x}{6x}\mathrm{d}x=-\frac{15}{16}\zeta(5)+\mathcal{I}_1-\mathcal{I}_2+\mathcal{I}_3-\mathcal{I}_4\end{aligned}$$

Starting with the easiest integral,

$$\mathcal{I}_4 = \frac{1}{30}\ln^5 2$$

For $\mathcal{I}_3$,

$$\begin{aligned}
\mathcal{I}_3 =& -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^{1} x^{n-1}\ln^3 x\,\mathrm{d}x = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial^3}{\partial n^3}\left(\frac{1}{n}-\frac{1}{n2^n}\right)\\
=&\sum_{n=1}^{\infty}\left(\frac{3}{n^5}-\frac{3}{n^5 2^n}-\frac{3\ln 2}{n^4 2^n}-\frac{3\ln^2 2}{n^3 2^{n+1}}-\frac{\ln^3 2}{n^2 2^{n+1}}\right)\\
=&3\zeta(5)-3\mathrm{Li}_5\left(\frac{1}{2}\right)-3\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2-\frac{3}{2}\ln^2 2\left(\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln 2+\frac{1}{6}\ln^3 2\right)\\
&-\frac{1}{2}\ln^3 2\left(\frac{\pi^2}{12}-\frac{1}{2}\ln^2 2\right)\\
=&3\zeta(5)-3\mathrm{Li}_5\left(\frac{1}{2}\right)-3\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2-\frac{21}{16}\zeta(3)\ln^2 2+\frac{\pi^2}{12}\ln^3 2
\end{aligned}$$

For $\mathcal{I}_2$,

$$\begin{aligned}
\mathcal{I}_2 =&\frac{1}{6}\ln^5 2+\frac{1}{3}\int_{\frac{1}{2}}^{1}\frac{\ln^3 x\ln(1-x)}{1-x}\mathrm{d}x = \frac{1}{6}\ln^5 2-\frac{1}{3}\sum_{n=1}^{\infty}H_n\frac{\partial^3}{\partial n^3}\left(\frac{1}{n+1}-\frac{1}{(n+1)2^{n+1}}\right)\\
=&\frac{1}{6}\ln^5 2+\sum_{n=1}^{\infty}\frac{2H_n}{(n+1)^4}-\sum_{n=1}^{\infty}\frac{2H_n}{(n+1)^4 2^{n+1}}-\sum_{n=1}^{\infty}\frac{2\ln 2H_n}{(n+1)^3 2^{n+1}}\\
&-\sum_{n=1}^{\infty}\frac{\ln^2 2H_n}{(n+1)^2 2^{n+1}}-\sum_{n=1}^{\infty}\frac{\ln^3 2H_n}{3(n+1)2^{n+1}}\\
=&\frac{1}{6}\ln^5 2+4\zeta(5)-\frac{\pi^2}{3}\zeta(3)-2\mathcal{S}+2\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{\pi^4}{360}\ln 2+\frac{1}{4}\zeta(3)\ln^2 2-\frac{1}{12}\ln^5 2\\
&-\frac{1}{8}\zeta(3)\ln^2 2+\frac{1}{6}\ln^5 2-\frac{1}{6}\ln^5 2\\
=&-2\mathcal{S}+2\mathrm{Li}_5\left(\frac{1}{2}\right)+4\zeta(5)-\frac{\pi^4}{360}\ln 2+\frac{1}{8}\zeta(3)\ln^2 2-\frac{\pi^2}{3}\zeta(3)+\frac{1}{12}\ln^5 2
\end{aligned}$$

For $\mathcal{I}_1$,

$$\begin{aligned}
\mathcal{I}_1 =&\frac{1}{6}\int_0^{\frac{1}{2}}\frac{\ln^3 x\ln(1-x)}{1-x}\mathrm{d}x = -\frac{1}{6}\sum_{n=1}^{\infty}H_n\frac{\partial^3}{\partial n^3}\left(\frac{1}{(n+1)2^{n+1}}\right)\\
=&\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4 2^{n+1}}+\sum_{n=1}^{\infty}\frac{\ln 2H_n}{(n+1)^3 2^{n+1}}+\sum_{n=1}^{\infty}\frac{\ln^2 2H_n}{2(n+1)^2 2^{n+1}}+\sum_{n=1}^{\infty}\frac{\ln^3 2H_n}{6(n+1)2^{n+1}}\\
=&\mathcal{S}-\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{\pi^4}{720}\ln 2-\frac{1}{16}\zeta(3)\ln^2 2+\frac{1}{24}\ln^5 2
\end{aligned}$$

Combining these four integrals as $\mathcal{I}_1-\mathcal{I}_2+\mathcal{I}_3-\mathcal{I}_4$ and $-\frac{15}{16}\zeta(5)$ gives

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^4} =&3\mathcal{S}-6\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{31}{16}\zeta(5)-3\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2+\frac{\pi^4}{240}\ln 2\\
&-\frac{3}{2}\zeta(3)\ln^2 2+\frac{\pi^2}{3}\zeta(3)+\frac{\pi^2}{12}\ln^3 2-\frac{3}{40}\ln^5 2
\end{aligned}$$

But consider $f(z)=\dfrac{\pi\csc(\pi z)(\gamma+\psi_0(-z))}{z^4}$. At the positive integers,

$$\sum_{n=1}^{\infty}\operatorname{Res}(f,n)=\sum_{n=1}^{\infty}\operatorname*{Res}_{z=n}\left[\frac{(-1)^n}{z^4(z-n)^2}+\frac{(-1)^nH_n}{z^4(z-n)}\right]=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^4}+\frac{15}{4}\zeta(5)$$

At $z=0$,

$$\begin{aligned}\operatorname{Res}(f,0)&=[z^3]\left(\frac{1}{z}+\frac{\pi^2}{6}z+\frac{7\pi^4}{360}z^3\right)\left(\frac{1}{z}-\frac{\pi^2}{6}z-\zeta(3)z^2-\frac{\pi^4}{90}z^3-\zeta(5)z^4\right)\\&=-\zeta(5)-\frac{\pi^2}{6}\zeta(3)\end{aligned}$$

At the negative integers,

$$\sum_{n=1}^{\infty}\operatorname{Res}(f,-n)=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^4}+\frac{15}{16}\zeta(5)$$

Since the sum of the residues is zero,

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^4}=-\frac{59}{32}\zeta(5)+\frac{\pi^2}{12}\zeta(3)$$

Hence,

$$\begin{aligned}-\frac{59}{32}\zeta(5)+\frac{\pi^2}{12}\zeta(3)=&3\mathcal{S}-6\operatorname{Li}_5\left(\frac{1}{2}\right)-\frac{31}{16}\zeta(5)-3\operatorname{Li}_4\left(\frac{1}{2}\right)\ln 2+\frac{\pi^4}{240}\ln 2\\&-\frac{3}{2}\zeta(3)\ln^2 2+\frac{\pi^2}{3}\zeta(3)+\frac{\pi^2}{12}\ln^3 2-\frac{3}{40}\ln^5 2\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}=&2\operatorname{Li}_5\left(\frac{1}{2}\right)+\frac{1}{32}\zeta(5)+\operatorname{Li}_4\left(\frac{1}{2}\right)\ln 2-\frac{\pi^4}{720}\ln 2+\frac{1}{2}\zeta(3)\ln^2 2\\&-\frac{\pi^2}{12}\zeta(3)-\frac{\pi^2}{36}\ln^3 2+\frac{1}{40}\ln^5 2\end{aligned}$$

□

Exercise 2.1.3: Compute the following series:

$$\mathscr{S}=\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\zeta(k)}{(k+2)2^{k+2}}$$

Solution: From the classic identity verified by the digamma function $\psi:=\Gamma'/\Gamma$, wich may be obtained from the Euler product giving $\Gamma(x+1)$, you have

$$\psi(x+1)=-\gamma+\sum_{k=1}^{\infty}\frac{x}{k(k+x)}\quad x\neq 0,-1,-2,-3,\dots$$

then you easily obtain, for $|x|<1$,

$$\begin{aligned}\psi(x+1)&=-\gamma+\sum_{k=1}^{\infty}\frac{x}{k^2}\frac{1}{1+\dfrac{x}{k}}=-\gamma+\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{k^n}x^{n+1}\\&=-\gamma-\sum_{n=0}^{\infty}(-1)^{n-1}\zeta(n+2)x^{n+1}=-\gamma-\sum_{k=2}^{\infty}(-1)^{k-1}\zeta(k)x^{k-1}\end{aligned}$$

and

$$-\gamma x^2 - x^2\psi(x+1) = \sum_{k=2}^{\infty}(-1)^{k-1}\zeta(k)x^{k+1}. \tag{1}$$

Using (1) gives

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\zeta(k)}{(k+2)2^{k+2}} &= \sum_{k=2}^{\infty}(-1)^{k-1}\zeta(k)\int_0^{1/2}x^{k+1}\mathrm{d}x = \int_0^{1/2}\sum_{k=2}^{\infty}(-1)^{k-1}\zeta(k)x^{k+1}\,\mathrm{d}x \\ &= -\gamma\int_0^{1/2}x^2\mathrm{d}x - \int_0^{1/2}x^2\psi(x+1)\,\mathrm{d}x = -\frac{\gamma}{24} - \int_0^{1/2}x^2\psi(x+1)\,\mathrm{d}x\end{aligned}$$

then integrating by parts twice leads to

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\zeta(k)}{(k+2)2^{k+2}} &= -\frac{\gamma}{24} - \frac{1}{4}\ln\Gamma\left(\frac{3}{2}\right) + 2\int_0^{1/2}x\ln\Gamma(x+1)\,dx \\ &= -\frac{\gamma}{24} - \frac{1}{8}\ln\pi - \frac{1}{4}\ln 2 + \zeta'\left(-1,\frac{3}{2}\right) + 2\int_0^{1/2}\zeta'(-1,x+1)\,\mathrm{d}x \\ &= -\frac{\gamma}{24} - \frac{1}{8} + \frac{\ln 2}{12} + \frac{\ln \mathbf{A}}{2} - \frac{7\zeta(3)}{16\pi^2}\end{aligned}$$

□

Exercise 2.1.4: Closed form of:

$$\sum_{n=1}^{\infty}\frac{J_0(2n)}{n^2}$$

Solution: Recall that

$$J_0(2n) = \frac{1}{\pi}\int_0^{\pi}\cos(2n\sin x)\,\mathrm{d}x$$

http://dlmf.nist.gov/10.9

and that

$$\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} = \frac{\pi^2}{6} - \frac{\pi t}{2} + \frac{t^2}{4}, \quad 0 \le t \le 2\pi.$$

Then, due to normal convergence of the series on $[0, 2\pi]$, we may write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{J_0(2n)}{n^2} &= \frac{1}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n\sin x)}{n^2}\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{6} - \pi\sin x + \sin^2 x\right)\mathrm{d}x \\ &= \frac{\pi^2}{6} - \frac{3}{2} = 0.144934066848226436...\end{aligned}$$

thus

$$\sum_{n=1}^{\infty}\frac{J_0(2n)}{n^2} = \frac{\pi^2}{6} - \frac{3}{2}$$

Using a similar technique, one may obtain the following result.

$$\sum_{n=1}^{\infty}\frac{J_0(2n\alpha)}{n^2} = \frac{\pi^2}{6} - 2\alpha + \frac{\alpha^2}{2}, \qquad \alpha \in [0,\pi)$$

□

Exercise 2.1.5: Evaluate:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$$

Solution: You may write

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{3n} \mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{3n} \mathrm{d}x = \int_0^1 \frac{1}{1+x^3} \mathrm{d}x$$
$$= \int_0^1 \left(\frac{1}{3(1+x)} + \frac{2-x}{3\left(1-x+x^2\right)} \right) \mathrm{d}x = \frac{1}{3} \ln 2 + \frac{\pi\sqrt{3}}{9}$$

Some details.

$$\int_0^1 \frac{1}{1+x^3} \mathrm{d}x = \frac{1}{3} \int_0^1 \frac{\mathrm{d}x}{(1+x)} + \frac{1}{3} \int_0^1 \frac{2-x}{(x-1/2)^2+3/4} \mathrm{d}x = \frac{1}{3} \ln 2 + \frac{1}{3} \int_{-1/2}^{1/2} \frac{3/2-u}{u^2+3/4} \mathrm{d}u$$
$$= \frac{1}{3} \ln 2 + \frac{1}{3} \int_{-1/2}^{1/2} \frac{3/2}{u^2+3/4} \mathrm{d}u = \frac{1}{3} \ln 2 + \int_0^{1/2} \frac{1}{u^2+3/4} \mathrm{d}u$$
$$= \frac{1}{3} \ln 2 + \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}} u \right) \Big|_0^{1/2} = \frac{1}{3} \ln 2 + \frac{\pi\sqrt{3}}{9}$$

where we have used $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. □

Exercise 2.1.6: Prove:

$$\sum_{n=1}^{+\infty} \left(H_n - \ln n - \gamma - \frac{1}{2n} \right) = \frac{1}{2} \left(1 - \ln(2\pi) + \gamma \right)$$

Proof: Adding the four finite sums,

$$\sum_{k=1}^{n} H_k = (n+1) H_n - n \ , \quad \sum_{k=1}^{n} \ln n = \ln n! \ , \quad \sum_{k=1}^{n} \gamma = \gamma\, n \ , \quad \sum_{k=1}^{n} \frac{1}{2k} = \frac{1}{2} H_n$$

gives us a representation of the n-th partial sum for the infinite series. Writing the infinite series as the limit of partial sums, we get:

$$S = \sum_{k=1}^{\infty} \left(H_k - \ln n - \gamma - \frac{1}{2k} \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(H_k - \ln n - \gamma - \frac{1}{2k} \right)$$
$$= \lim_{n\to\infty} \left((n+1) H_n - n - \ln n! - \gamma\, n - \frac{1}{2} H_n \right) = \lim_{n\to\infty} \left(\left(n + \frac{1}{2} \right) H_n - (1+\gamma) n - \ln n! \right)$$

Use Stirling's approximation for the factorial to obtain an asymptotic formula for the log-factorial term in the series:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \implies \ln n! \sim \ln \left(\sqrt{2\pi n} \left(\frac{n}{e} \right)^n \right) = \left(n + \frac{1}{2} \right) \ln n - n + \frac{1}{2} \ln(2\pi)$$

Then,

$$S = \lim_{n\to\infty} \left(\left(n + \frac{1}{2} \right) H_n - (1+\gamma) n - \ln n! \right)$$
$$= \lim_{n\to\infty} \left(\left(n + \frac{1}{2} \right) H_n - (1+\gamma) n - \left(n + \frac{1}{2} \right) \ln n + n - \frac{1}{2} \ln(2\pi) \right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(\left(n+\frac{1}{2}\right)H_n-\gamma\, n-\left(n+\frac{1}{2}\right)\ln n\right)-\frac{1}{2}\ln(2\pi)\\
&= \lim_{n\to\infty}\left(n\left(H_n-\gamma-\ln n\right)+\frac{1}{2}\left(H_n-\ln n\right)\right)-\frac{1}{2}\ln(2\pi)\\
&= \lim_{n\to\infty} n\left(H_n-\gamma-\ln n\right)+\frac{1}{2}\lim_{n\to\infty}\left(H_n-\ln n\right)-\frac{1}{2}\ln(2\pi)\\
&= \lim_{n\to\infty} n\left(H_n-\gamma-\ln n\right)+\frac{1}{2}\gamma-\frac{1}{2}\ln(2\pi)\\
&= \frac{1}{2}+\frac{1}{2}\gamma-\frac{1}{2}\ln(2\pi)
\end{aligned}
$$

Appendix:

Using the asymptotic series for the digamma function given by Eq.16 on this Wolfram Mathworld page,

http://mathworld.wolfram.com/DigammaFunction.html

$$
\begin{aligned}
\lim_{n\to\infty} n\left(H_n-\gamma-\ln n\right) &= \lim_{n\to\infty} n\left(\Psi(n+1)-\ln n\right) = \lim_{n\to\infty} n\left(\frac{1}{2n}-\sum_{\ell=1}^{\infty}\frac{B_{2\ell}}{2\ell n^{2\ell}}\right)\\
&= \frac{1}{2}-\lim_{n\to\infty}\sum_{\ell=1}^{\infty}\frac{B_{2\ell}}{2\ell n^{2\ell-1}} = \frac{1}{2}
\end{aligned}
$$

□

Exercise 2.1.7: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(n)!(n-1)!}{(2n+1)!}2^{n+1}$$

Solution: One may recall that, by the **Wallis' integrals**, we have :

$$\frac{1}{2^{n-1}\,n}\int_0^{\pi/2}\sin^{2n+1}x\,\mathrm{d}x=\frac{(n)!(n-1)!}{(2n+1)!}2^{n+1},\quad n=0,1,2,\ldots, \tag{1}$$

then summing (1) from $n=1$ to ∞, we get

$$-2\int_0^{\pi/2}\sin x\ln\left(1-\frac{\sin^2 x}{2}\right)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(n)!(n-1)!}{(2n+1)!}2^{n+1}.$$

The integral is easily evaluated

$$
\begin{aligned}
&-2\int_0^{\pi/2}\sin x\ln\left(1-\frac{\sin^2 x}{2}\right)\mathrm{d}x=-2\int_0^{\pi/2}\sin x\ln\left(\frac{1+\cos^2 x}{2}\right)\mathrm{d}x\\
&=-2\int_0^1\ln\left(\frac{1+u^2}{2}\right)\mathrm{d}u=4\int_0^1\frac{u^2}{1+u^2}\,\mathrm{d}u=4-\pi
\end{aligned}
$$

□

Exercise 2.1.8: Evaluate:

$$\sum_{n=1}^{\infty}\psi'(n)^2$$

Solution: **Method One:** First, we have

$$\psi'(n)=\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}=\sum_{k=n}^{\infty}\frac{1}{k^2} \tag{1}$$

Then

$$\sum_{n=1}^{\infty}\psi'(n)^2 = \sum_{n=1}^{\infty}\sum_{j=n}^{\infty}\frac{1}{j^2}\sum_{k=n}^{\infty}\frac{1}{k^2} \tag{2}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{j=n}^{\infty}\frac{1}{j^4} + 2\sum_{j=n}^{\infty}\sum_{m=1}^{\infty}\frac{1}{j^2}\frac{1}{(j+m)^2}\right) \tag{3}$$

$$= \sum_{j=1}^{\infty}\sum_{n=1}^{j}\frac{1}{j^4} + 2\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{j}\frac{1}{j^2}\frac{1}{(j+m)^2} \tag{4}$$

$$= \sum_{j=1}^{\infty}\frac{1}{j^3} + 2\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{j(j+m)^2} \tag{5}$$

$$= \zeta(3) + 2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2} \tag{6}$$

$$= \zeta(3) - 2\zeta(3) + 2\sum_{n=1}^{\infty}\frac{H_n}{n^2} \tag{7}$$

$$= \zeta(3) - 2\zeta(3) + 4\zeta(3) \tag{8}$$

$$= 3\zeta(3)$$

Explanation:

(2): use (1)

(3): first sum covers $j = k$ the other $j < k$ and $j > k$

(4): change order of summation

(5): sum in n

(6): $n = j + m$ and $j = nm$ and sum in m

(7): $H_{n-1} = H_n - \dfrac{1}{n}$

(8): equation (14) of this answer here

http://math.stackexchange.com/a/469785

with $q = 2$

Method Two:

$$\sum_{k=1}^{\infty}\left(\psi^{(1)}(k)\right)^2 = \int_0^1\int_0^1 \frac{\ln u \ln v}{(1-uv)(1-u)(1-v)}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^1 \frac{\ln v}{(1-v)^2}\int_0^1\left(\frac{\ln u}{1-u} - \frac{v\ln u}{1-vu}\right)\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^1 \frac{\ln v}{(1-v)^2}\left(\int_0^1\frac{\ln u}{1-u}\,\mathrm{d}u - v\int_0^1\frac{\ln u}{1-vu}\,\mathrm{d}u\right)\mathrm{d}v$$

where

$$\int_0^1\frac{\ln u}{1-u}\,\mathrm{d}u = \int_0^1\frac{\ln(1-w)}{w}\mathrm{d}w = -\mathrm{Li}_2(1) = -\zeta(2)$$

and

$$\int_0^1\frac{\ln u}{1-vu}\,\mathrm{d}u = -\frac{1}{v}\ln(1-vu)\ln u\Big|_0^1 + \frac{1}{v}\int_0^1\frac{\ln(1-vu)}{u}\,\mathrm{d}u = -\frac{\mathrm{Li}_2(v)}{v}.$$

Therefore,

$$\sum_{k=1}^{\infty}\left(\psi^{(1)}(k)\right)^2 = \int_0^1\frac{\ln v}{(1-v)^2}\Big(\mathrm{Li}_2(v) - \zeta(2)\Big)\,\mathrm{d}v.$$

Then integrating by parts,

$$=\Big(\mathrm{Li}_2(v)-\zeta(2)\Big)\left(\ln(1-v)+\frac{v\ln v}{1-v}\right)\Bigg|_0^1+\int_0^1\frac{\ln^2(1-v)}{v}\,\mathrm{d}v+\int_0^1\frac{\ln(1-v)\ln v}{1-v}\,\mathrm{d}v$$
$$=\int_0^1\frac{\ln^2(1-v)}{v}\,\mathrm{d}v+\int_0^1\frac{\ln(1-v)\ln v}{1-v}\,\mathrm{d}v$$

where

$$\int_0^1\frac{\ln^2(1-v)}{v}\,\mathrm{d}t=\ln^2(1-v)\ln v\Big|_0^1+2\int_0^1\frac{\ln(1-v)\ln v}{1-v}\,\mathrm{d}v=2\int_0^1\frac{\ln(1-v)\ln v}{1-v}\,\mathrm{d}v$$

Therefore,

$$\sum_{k=1}^{\infty}\Big(\psi^{(1)}(k)\Big)^2=3\int_0^1\frac{\ln(1-v)\ln v}{1-v}\,\mathrm{d}v=-3\int_0^1\ln v\sum_{n=1}^{\infty}H_nv^n\,\mathrm{d}v=-3\sum_{n=1}^{\infty}H_n\int_0^1v^n\ln v\,\mathrm{d}v$$
$$=3\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}=3\left(\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}\right)=3\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2}-1-\zeta(3)+1\right)$$
$$=3\Big(2\zeta(3)-\zeta(3)\Big)=3\zeta(3)\tag{1}$$

(1) Generalized Euler sum $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^q}$

http://math.stackexchange.com/questions/469023/ □

Exercise 2.1.9: Evaluate:

$$\sum_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{1}{n!}\int_0^1x(x+1)\cdots(x+n-1)\,\mathrm{d}x$$

Solution: Let u be a complex number such that $|u|<1$. Then

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n!}\left(\int_0^1x(x+1)\cdots(x+n-1)\,\mathrm{d}x\right)u^n$$
$$=-1+\frac{(1+i)u}{2(1+iu)\ln(1+iu)}+\frac{(1-i)u}{2(1-iu)\ln(1-iu)},\tag{1}$$

where $i^2=-1$.

Proof. Recall that

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k\tag{2}$$

where $\begin{bmatrix}n\\k\end{bmatrix}$ are the **unsigned Stirling numbers of the first kind** counting the numbers of permutations of n letters that have exactly k cycles, giving

$$\int_0^1x(x+1)\cdots(x+n-1)\,\mathrm{d}x=\sum_{k=1}^{n}\frac{\begin{bmatrix}n\\k\end{bmatrix}}{k+1}.\tag{3}$$

Using (3), we have proved that (see here, identity (3))

http://math.stackexchange.com/questions/870036/

for $|u|<1$,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\int_0^1 x(x+1)\cdots(x+n-1)\,\mathrm{d}x\right)u^n=-1+\frac{u}{(1+u)\ln(1+u)} \tag{4}$$

and that (see here)

http://math.stackexchange.com/questions/872042/

$$\frac{1}{n!}\int_0^1 x(x+1)\cdots(x+n-1)\,\mathrm{d}x=\mathcal{O}\left(\frac{1}{\ln n}\right),\quad \text{as}\quad n\to\infty.$$

Consequently, setting $a_n:=\displaystyle\int_0^1 x(x+1)\cdots(x+n-1)\,\mathrm{d}x$ for notational convenience, with $|u|<1$, we are allowed to write

$$\sum_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{a_n}{n!}u^n=\sum_{p=1}^{\infty}(-1)^p\frac{a_{2p}}{(2p)!}u^{2p}+\sum_{p=1}^{\infty}(-1)^p\frac{a_{2p-1}}{(2p-1)!}u^{2p-1} \tag{5}$$

$$=\frac{1}{2}\left(g(iu)+g(-iu)\right)-\frac{i}{2}\left(g(iu)-g(-iu)\right) \tag{6}$$

where $g(\cdot)$ is the right hand side member of (4). This gives (1).

The series

$$\sum_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{a_n}{n!} \tag{7}$$

is convergent by the Dirichlet test, then we using

$$-1+\frac{1}{2\ln(1-i)}+\frac{1}{2\ln(1+i)}=-1+\frac{8\ln 2}{\pi^2+4\ln^2 2}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n!}\int_0^1 x(x+1)\cdots(x+n-1)\,\mathrm{d}x=\frac{8\ln 2}{\pi^2+4\ln^2 2}-1$$

□

Exercise 2.1.10: Evaluate:

$$\sum_{r=1}^{\infty}\frac{1}{(r+2)r!}$$

Solution: Using

$$\int_0^1 x^{r+1}\mathrm{d}x=\frac{1}{r+2}$$

you may write

$$\sum_{r=1}^{\infty}\frac{1}{(r+2)r!}=\sum_{r=1}^{\infty}\frac{1}{r!}\int_0^1 x^{r+1}\mathrm{d}x=\int_0^1\sum_{r=1}^{\infty}\frac{1}{r!}x^{r+1}\mathrm{d}x=\int_0^1 x(e^x-1)\mathrm{d}x$$

giving

$$\sum_{r=1}^{\infty}\frac{1}{(r+2)r!}=\frac{1}{2}$$

□

Exercise 2.1.11: Prove:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = \frac{17}{36}\zeta(4)$$

Proof: **Step 1. Reduction to an integral representation**

Let S denote the summation in question. Then by the successive application of integration by parts, we obtain

$$S = 8\int_0^1 \frac{1}{x}\int_0^{x/2} \frac{\arcsin^2 t}{t}\,\mathrm{d}t\mathrm{d}x = -8\int_0^1 \frac{\arcsin^2(x/2)}{x}\ln x\,\mathrm{d}x = 8\int_0^1 \frac{\arcsin(x/2)}{\sqrt{1-(x/2)^2}}\ln^2 x\,\frac{\mathrm{d}x}{2}$$

Thus with the substitution $x = 2\sin\theta$, we have

$$S = 8\int_0^{\frac{\pi}{6}} \theta\ln^2(2\sin\theta)\,\mathrm{d}\theta.$$

To evaluate this integral, note that for $0 < \theta < \dfrac{\pi}{6}$ we have

$$e^{i\theta}\cdot 2\sin\theta = i\cdot(1-e^{2i\theta}).$$

Taking logarithm (with the branch cut $(-\infty, 0]$ as usual) to both sides, it follows that

$$i\theta + \ln(2\sin\theta) = \frac{i\pi}{2} + \ln(1-e^{2i\theta}).$$

Cubing both sides and integrating on $\left(0, \dfrac{\pi}{6}\right)$ and taking imaginary parts only,

$$S = \frac{2}{3}\left(\frac{\pi}{6}\right)^4 + \frac{8}{3}\Im\int_0^{\frac{\pi}{6}} \left(\frac{i\pi}{2} + \ln(1-e^{2i\theta})\right)^3 \mathrm{d}\theta. \tag{1}$$

Step 2. Some complex-analysis techniques

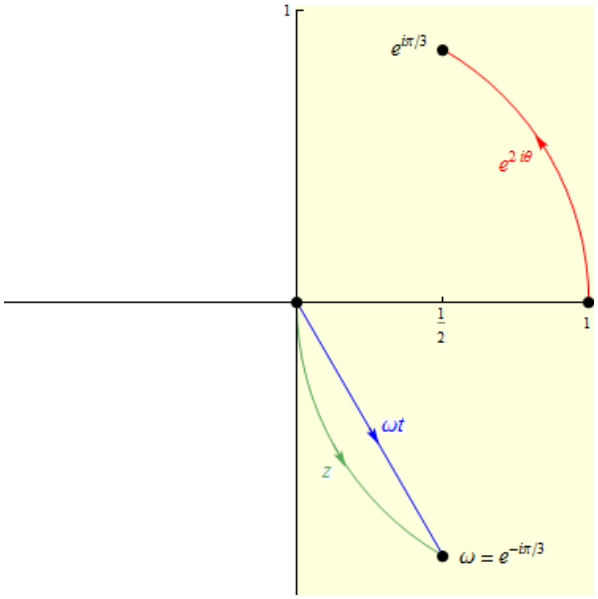
Now we focus on the integral in the imaginary part:

$$I := \int_0^{\frac{\pi}{6}} \left(\frac{i\pi}{2} + \ln(1-e^{2i\theta})\right)^3 \mathrm{d}\theta. \tag{2}$$

Once we evaluate the imaginary part of II, the identity (1) immediately gives us the answer. We first make the substitution $z = 1-e^{2i\theta}$ and $\omega = e^{-i\pi/3}$ to obtain

$$I = \int_0^{\omega} \left(\frac{i\pi}{2} + \ln z\right)^3 \frac{\mathrm{d}z}{2i(z-1)}.$$

Here, the path of integration is a circular arc joining from 0 to ω centered at 1 (green-colored path in the figure below).



But since the integrand is analytic for $0 < \Re z < 1$, we may change the path of integration as $z = \omega t$ for $0 \le t \le 1$ (blue-colored path in the figure above). This gives

$$I = \frac{1}{2i}\int_0^1 \left(\frac{i\pi}{6} + \ln t\right)^3 \frac{\omega\,\mathrm{d}t}{\omega t - 1}.$$

Plugging $t = e^{-x}$, I reduces to

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \left(\frac{\pi}{6} + ix\right)^3 \frac{\omega e^{-x}}{1-\omega e^{-x}}\,\mathrm{d}x = \frac{1}{2}\sum_{n=1}^\infty \omega^n \int_0^\infty \left(\frac{\pi}{6} + ix\right)^3 e^{-nx}\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{n=1}^\infty \omega^n \left(-\frac{6i}{n^4} - \frac{\pi}{n^3} + \frac{i\pi^2}{12n^2} + \frac{\pi^3}{216n}\right). \end{aligned}$$

Taking the imaginary part,

$$\Im I = -3\sum_{n=1}^\infty \frac{\cos(n\pi/3)}{n^4} + \frac{\pi}{2}\sum_{n=1}^\infty \frac{\sin(n\pi/3)}{n^3} + \frac{\pi^2}{24}\sum_{n=1}^\infty \frac{\cos(n\pi/3)}{n^2} - \frac{\pi^3}{432}\sum_{n=1}^\infty \frac{\sin(n\pi/3)}{n} \tag{3}$$

Step 3. Evaluation of a series

Note that for $0 < \theta < \pi$, we have

$$\sum_{n=1}^\infty \frac{\sin n\theta}{n} = \Im \sum_{n=1}^\infty \frac{e^{in\theta}}{n} = -\Im \ln(1 - e^{i\theta}) = \frac{\pi - \theta}{2}.$$

Integrating both sides, we obtain

$$\sum_{n=1}^\infty \frac{1-\cos n\theta}{n^2} = \frac{\theta(2\pi-\theta)}{4} \quad\Longrightarrow\quad \sum_{n=1}^\infty \frac{\cos n\theta}{n^2} = \frac{\theta^2}{4} - \frac{\pi\theta}{2} + \frac{\pi^2}{6}.$$

Repeating this procedure, we obtain

$$\sum_{n=1}^\infty \frac{\sin n\theta}{n^3} = \frac{\theta^3}{12} - \frac{\pi\theta^2}{4} + \frac{\pi^2\theta}{6}$$

and

$$\sum_{n=1}^\infty \frac{\cos n\theta}{n^4} = -\frac{\theta^4}{48} + \frac{\pi\theta^3}{12} - \frac{\pi^2\theta^2}{12} + \frac{\pi^4}{90}.$$

Plugging $\theta = \dfrac{\pi}{3}$, we have

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n^2} = \frac{\pi}{3} \quad , \quad \sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^2} = \frac{\pi^3}{36}$$
$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n^3} = \frac{5\pi^3}{162} \, , \, \sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^4} = \frac{91\pi^4}{19440}$$

Plugging these to (3), we have

$$\Im I = \frac{23\pi^4}{12960} \quad \Longrightarrow \quad S = \frac{17\pi^4}{3240} = \frac{17}{36}\zeta(4)$$

□

Exercise 2.1.12: Evaluate:

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k}$$

Solution: One may first observe that your initial sum may be rewritten as

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} &= \sum_{k=0}^{n} (-1)^k \left(\int_0^1 x^{2k} \, \mathrm{d}x \right) \binom{n}{k} = \int_0^1 \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k} \, \mathrm{d}x \\ &= \int_0^1 (1-x^2)^n \mathrm{d}x \end{aligned}$$

Then, integrating by parts, we have

$$\begin{aligned} \int_0^1 (1-x^2)^n \mathrm{d}x &= x(1-x^2)^n \Big|_0^1 + 2n \int_0^1 x^2 (1-x^2)^{n-1} \mathrm{d}x \\ &= 0 - 2n \int_0^1 (1-x^2-1)(1-x^2)^{n-1} \mathrm{d}x \\ &= -2n \int_0^1 (1-x^2)^n \mathrm{d}x + 2n \int_0^1 (1-x^2)^{n-1} \mathrm{d}x. \end{aligned}$$

which is equivalent to

$$\int_0^1 (1-x^2)^n \mathrm{d}x = \frac{2n}{2n+1} \int_0^1 (1-x^2)^{n-1} \mathrm{d}x$$

giving

$$\int_0^1 (1-x^2)^n \mathrm{d}x = \frac{2}{3}\frac{4}{5} \cdots \frac{2n}{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)(2n)!}$$

Finally

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} = \frac{2^{2n}(n!)^2}{(2n+1)(2n)!}$$

□

Exercise 2.1.13: Evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(3n+2)}$$

✎ **Solution: A General Trick**

A General Trick for summing this series is to use **Telescoping Series**:

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(3n+2)}=\frac{1}{3}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)=\frac{1}{3}\lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{3n-1}-\sum_{n=1}^{N}\frac{1}{3n+2}\right]$$
$$=\frac{1}{3}\lim_{N\to\infty}\left[\sum_{n=0}^{N-1}\frac{1}{3n+2}-\sum_{n=1}^{N}\frac{1}{3n+2}\right]=\frac{1}{3}\lim_{N\to\infty}\left[\frac{1}{2}-\frac{1}{3N+2}\right]=\frac{1}{6}$$

An Integral Trick

Since

$$\int_0^{\infty}e^{-nt}\,\mathrm{d}t=\frac{1}{n}$$

for $n>0$, we can write

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(3n+2)}=\sum_{n=1}^{\infty}\frac{1}{3}\int_0^{\infty}\left(e^{-(3n-1)t}-e^{-(3n+2)t}\right)\mathrm{d}t=\frac{1}{3}\int_0^{\infty}\frac{e^{-2t}-e^{-5t}}{1-e^{-3t}}\mathrm{d}t$$
$$=\frac{1}{3}\int_0^{\infty}e^{-2t}\,\mathrm{d}t=\frac{1}{6}$$

□

Exercise 2.1.14: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}H_n$$

✎ **Solution:**

$$\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n}H_n=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}+\sum_{n=2}^{N}\frac{(-1)^{n-1}}{n}H_{n-1}=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}$$
$$+\frac{1}{2}\sum_{n=2}^{N}\sum_{k=1}^{n-1}\frac{(-1)^{n-1}}{n}\left(\frac{1}{k}+\frac{1}{n-k}\right)=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}+\frac{1}{2}\sum_{n=2}^{N}\sum_{k=1}^{n-1}\frac{(-1)^{n-1}}{k(n-k)}$$
$$=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}+\frac{1}{2}\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\frac{(-1)^{n-1}}{k(n-k)}=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}+\frac{1}{2}\sum_{k=1}^{N-1}\sum_{n=1}^{N-k}\frac{(-1)^{n+k-1}}{kn}$$
$$=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}-\frac{1}{2}\sum_{k=1}^{N-1}\frac{(-1)^{k-1}}{k}\sum_{n=1}^{N-1}\frac{(-1)^{n-1}}{n}+\frac{1}{2}\sum_{k=1}^{N-1}\frac{(-1)^{k-1}}{k}\sum_{n=N-k+1}^{N-1}\frac{(-1)^{n-1}}{n}\tag{1}$$

where, using the Alternating Series Test, we have

$$\frac{1}{2}\left|\sum_{k=1}^{N-1}\frac{(-1)^{k-1}}{k}\sum_{n=N-k+1}^{N-1}\frac{(-1)^{n-1}}{n}\right|$$
$$\le\frac{1}{2}\left|\sum_{k=1}^{N/2}\frac{(-1)^{k-1}}{k}\sum_{n=N-k+1}^{N-1}\frac{(-1)^{n-1}}{n}\right|+\frac{1}{2}\left|\sum_{k=N/2}^{N-1}\frac{(-1)^{k-1}}{k}\sum_{n=N-k+1}^{N-1}\frac{(-1)^{n-1}}{n}\right|$$
$$\le\frac{1}{2}\cdot 1\cdot\frac{2}{N}+\frac{1}{2}\cdot\frac{2}{N}\cdot 1=\frac{2}{N}\tag{2}$$

Applying (2) to (1) and letting $N\to\infty$, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}H_n=\frac{1}{2}\zeta(2)-\frac{1}{2}\ln(2)^2$$

□

Exercise 2.1.15: Find a closed form of this sum:

$$\sum_{n=1}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)^4\, 4^n\, n!}$$

Solution: First, in view of Legrende's duplication formula,

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)^4\, 4^n\, n!} = 2\sqrt{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(2n)}{\Gamma(n)\, n!\, (2n+1)^4\, 16^n} \\ &= -\frac{\sqrt{\pi}}{3} \int_0^1 \ln^3(x) \sum_{n=1}^{\infty} \frac{\Gamma(2n)}{\Gamma(n)\, n!} \left(\frac{x^2}{16}\right)^n \mathrm{d}x \\ &= -\sqrt{\pi} - \frac{\sqrt{\pi}}{6} \int_0^1 \frac{\ln^3(x)}{\sqrt{1-x^2/4}} \mathrm{d}x = -\sqrt{\pi} - \frac{\sqrt{\pi}}{3} \int_0^{\frac{\pi}{6}} \ln^3(2\sin x) \mathrm{d}x \end{aligned}$$

Claim: for $0 < a \le \frac{\pi}{2}$,

$$\begin{aligned} \int_0^a \ln^3\left(\frac{\sin x}{\sin a}\right) \mathrm{d}x = {} & \frac{4a-3\pi}{2} a^2 \ln(2\sin a) - \frac{3\pi}{4}\zeta(3) \\ & + 3\left(\frac{\pi}{2}-a\right) \Re\left(\frac{1}{2}\operatorname{Li}_3(e^{2ia}) + \operatorname{Li}_3(1-e^{2ia})\right) \\ & + 3\Im\left(\frac{1}{4}\operatorname{Li}_4(e^{2ia}) + \operatorname{Li}_4(1-e^{2ia})\right) \end{aligned}$$

Proof. The idea is exactly identical to the proof displayed in this question.

http://math.stackexchange.com/questions/1438381/

The proof is rather tedious (and obviously inefficient), and ends with a somewhat of a cancellation (implying the existence of a shortcut) , so I omit the boring algebra and outline the main ideas, which can be repeated systematically to obtain closed forms for even higher powers of logsine.

Things to know:

$$\ln(2\sin x) = \ln(1-e^{2ix}) + i\left(\frac{\pi}{2}-x\right) \tag{1}$$

$$\begin{aligned} \int \frac{\ln^3(1-x)}{x} \mathrm{d}x = {} & \ln^3(1-x)\ln(x) + 3\ln^2(1-x)\operatorname{Li}_2(1-x) \\ & - 6\ln(1-x)\operatorname{Li}_3(1-x) + 6\operatorname{Li}_4(1-x) \end{aligned} \tag{2}$$

$$\int_0^a x \ln(2\sin x)\mathrm{d}x = -\frac{a}{2}\operatorname{Cl}_2(2a) - \frac{1}{4}\Re\operatorname{Li}_3(e^{2ia}) + \frac{\zeta(3)}{4} \tag{3}$$

$$\int_0^a x^2 \ln(2\sin x)\mathrm{d}x = -\frac{a^2}{2}\operatorname{Cl}_2(2a) - \frac{a}{2}\Re\operatorname{Li}_3(e^{2ia}) + \frac{1}{4}\Im\operatorname{Li}_4(e^{2ia}) \tag{4}$$

$$\int_0^a \ln(\sin x)\mathrm{d}x = -a\ln 2 - \frac{1}{2}\operatorname{Cl}_2(2a) \tag{5}$$

$$\int_0^a \ln^2(\sin x)\mathrm{d}x = \frac{a^3}{3} + a\ln^2 2 - a\ln^2(2\sin a) - \ln(\sin a)\mathrm{Cl}_2(2a) - \Im\mathrm{Li}_3(1-e^{2ia}) \tag{6}$$

(1) is trivial, (2) is not too hard to find, (5) and (6) are shown in the linked answer, and (3), (4) are easily found using $\ln(2\sin x) = -\sum_{n\geq 1}\frac{\cos(2xn)}{n}$.

It is obvious that since we have (5) and (6), the claim (0) depends on a closed form for $\int_0^a \ln^3(\sin x)\mathrm{d}x$, and the latter may be evaluated in terms of $\int_0^a \ln^3(2\sin x)\mathrm{d}x$.
But, with the help of (1),

$$\begin{aligned}\int_0^a \ln^3(2\sin x)\mathrm{d}x &= \Re\int_0^a \ln^3(1-e^{2ix})\mathrm{d}x + 3\int_0^a \ln(2\sin x)\left(\frac{\pi}{2}-x\right)^2\mathrm{d}x \\ &= \frac{1}{2}\Im\int_1^{e^{2ia}}\frac{\ln^3(1-x)}{x}\mathrm{d}x + 3\int_0^a \ln(2\sin x)\left(\frac{\pi}{2}-x\right)^2\mathrm{d}x\end{aligned}$$

(Same idea **RandomVariable** had in this answer.)

http://math.stackexchange.com/questions/917154/

Now we employ (2),(3),(4), and (5). Some expressions cancel and claim follows.
This result, together with the fact that $e^{i\pi/3}$ and $1-e^{i\pi/3}$ are conjugates, yields

$$\int_0^{\frac{\pi}{6}} \ln^3(2\sin x)\mathrm{d}x = -\frac{\pi}{4}\zeta(3) - \frac{9}{4}\Im\mathrm{Li}_4(e^{i\pi/3})$$

and use

$$\frac{2}{\sqrt{3}}\Im\mathrm{Li}_4(e^{i\pi/3}) = \sum_{n\geq 0}\frac{(-1)^n}{(3n+1)^4} + \sum_{n\geq 0}\frac{(-1)^n}{(3n+2)^4} = \frac{\psi^{(3)}\left(\frac{1}{3}\right)}{216} - \frac{\pi^4}{81}$$

Hence we have

$$\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)^4\,4^n\,n!} = \sqrt{\pi}\left(\frac{\pi}{12}\zeta(3) + \frac{\psi^{(3)}\left(\frac{1}{3}\right)}{192\sqrt{3}} - \frac{\pi^4}{72\sqrt{3}} - 1\right)$$

It can also be expressed in **generalized hypergeometric function** with the help of **WolframAlpha**:

$$\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)^4\,4^n\,n!} = \frac{\sqrt{\pi}}{648}{}_6F_5\left(\begin{matrix}1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\end{matrix}\middle|\,\frac{1}{4}\right)$$

□

Exercise 2.1.16: Prove:

$$\frac{\pi}{2} = \sum_{k=0}^{\infty}\frac{k!}{(2k+1)!!}$$

Proof:

$$\int_0^{\frac{\pi}{2}}\sin^{2k+1}(x)\,\mathrm{d}x = \frac{2k}{2k+1}\frac{2k-2}{2k-1}\cdots\frac{2}{3} = \frac{1}{2k+1}\frac{4^k}{\binom{2k}{k}} \tag{1}$$

Using (1),we have

$$\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!!}=\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)\binom{2k}{k}}=\sum_{k=0}^{\infty}\int_0^{\frac{\pi}{2}}\sqrt{2}\left(\frac{\sin x}{\sqrt{2}}\right)^{2k+1}\mathrm{d}x=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\sin x}{\sqrt{2}}\right)}{1-\left(\frac{\sin x}{\sqrt{2}}\right)^2}\,\mathrm{d}x$$
$$=\int_0^{\frac{\pi}{2}}\frac{2\sin x}{2-\sin^2 x}\,\mathrm{d}x=\int_{\frac{\pi}{2}}^{0}\frac{2\,\mathrm{d}\cos x}{1+\cos^2 x}=\frac{\pi}{2}$$

□

Exercise 2.1.17: Computing $\zeta(6)=\sum_{k=1}^{\infty}\frac{1}{k^6}$ with Fourier series.

Solution: One method is to consider the generating function of $\zeta(2k)$:

$$f(x)=\sum_{k=1}^{\infty}\zeta(2k)\,x^{2k}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{2k}}{n^{2k}}=\sum_{n=1}^{\infty}\frac{x^2/n^2}{1-x^2/n^2}=\sum_{n=1}^{\infty}\frac{x^2}{n^2-x^2}$$
$$=-\frac{x}{2}\sum_{n=1}^{\infty}\left(\frac{1}{x-n}+\frac{1}{x+n}\right)=-\frac{x}{2}\left(\pi\cot(\pi x)-\frac{1}{x}\right)=\frac{1}{2}(1-\pi x\cot(\pi x)) \tag{1}$$

In light of equation (1), find the power series of

$$x\cot(x)=\sum_{k=0}^{\infty}a_k x^{2k}\ ,\ \ \cos(x)=\frac{\sin(x)}{x}\sum_{k=0}^{\infty}a_k x^{2k}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}\ \ \sum_{k=0}^{\infty}a_k x^{2k}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\frac{a_{n-k}}{(2k+1)!}\right)x^{2n} \tag{2}$$

Comparing the coefficients of the powers of x in (2) yields

$$a_n=\frac{(-1)^n}{(2n)!}-\sum_{k=1}^{n}(-1)^k\frac{a_{n-k}}{(2k+1)!}=\frac{(-1)^n 2n}{(2n+1)!}-\sum_{k=1}^{n-1}(-1)^k\frac{a_{n-k}}{(2k+1)!} \tag{3}$$

Since $a_n=-2\frac{\zeta(2n)}{\pi^{2n}}$ for positive n, (3) becomes

$$\zeta(2n)=\frac{(-1)^{n-1}\pi^{2n}}{(2n+1)!}n+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\pi^{2k}}{(2k+1)!}\zeta(2n-2k) \tag{4}$$

Equation (4) gives $\zeta(2k)$ recursively for positive n:

$$\zeta(2)=\frac{\pi^2}{3!}=\frac{\pi^2}{6}$$
$$\zeta(4)=-\frac{\pi^4}{5!}2+\frac{\pi^2}{3!}\zeta(2)=\frac{\pi^4}{90}$$
$$\zeta(6)=\frac{\pi^6}{7!}3-\frac{\pi^4}{5!}\zeta(2)+\frac{\pi^2}{3!}\zeta(4)=\frac{\pi^6}{945}$$
$$\zeta(8)=-\frac{\pi^8}{9!}4+\frac{\pi^6}{7!}\zeta(2)-\frac{\pi^4}{5!}\zeta(4)+\frac{\pi^2}{3!}\zeta(6)=\frac{\pi^8}{9450}$$

□

Exercise 2.1.18: Prove this identity:

$$\frac{\zeta(2)}{2}+\frac{\zeta(4)}{2^3}+\frac{\zeta(6)}{2^5}+\frac{\zeta(8)}{2^7}+\cdots=1$$

Proof: Method One:

Since

$$\zeta(2n)=\frac{1}{(2n-1)!}\int_0^{+\infty}\frac{x^{2n-1}}{e^x-1}\,\mathrm{d}x$$

we have:

$$\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{2^{2n-1}}=\int_0^{+\infty}\frac{\sinh(x/2)}{e^x-1}\,\mathrm{d}x=\frac{1}{2}\int_0^{+\infty}e^{-x/2}\,\mathrm{d}x=1$$

Method Two:

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n-1}}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{2}{k^{2n}2^{2n}} \tag{1}$$

$$=2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(4k^2)^n} \tag{2}$$

$$=2\sum_{k=1}^{\infty}\frac{1}{4k^2-1} \tag{3}$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right) \tag{4}$$

$$=1 \tag{5}$$

Explanation: (1): expand $\zeta(2n)=\sum\limits_{k=1}^{\infty}\frac{1}{k^{2n}}$.

(2): change the order of summation.

(3): sum of a geometric series.

(4): partial fractions.

(5): telescoping sum.

Method Three:

The **Laurent expansion** of $\cot(z)$ at the origin in terms of the Riemann zeta function is

$$\cot(z)=\frac{1}{z}-2\sum_{k=1}^{\infty}\zeta(2k)\frac{z^{2k-1}}{\pi^{2k}}\ ,\ 0<|z|<\pi.$$

Letting $z=\frac{\pi}{2}$,

$$\cot\left(\frac{\pi}{2}\right)=\frac{2}{\pi}-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2^{2k-1}}.$$

But $\cot\left(\frac{\pi}{2}\right)=0$.

Hence,

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2^{2k-1}}=1$$

□

Exercise 2.1.19: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{3n(3n-1)}$$

✎ Solution: Extended Harmonic Numbers

Normally, we think of Harmonic Numbers as

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{1}$$

However, an alternate definition is often useful:

$$H_n = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+n} \right) \tag{2}$$

For integer $n \geq 1$, it is not too difficult to see that the two definitions agree. However, (2) is easily extendible to all $n \in \mathbb{R}$ (actually, to all $n \in \mathbb{C}$). We can say some things about H_n for some $n \in \mathbb{Q} \setminus \mathbb{Z}$.

Note that for $m, n \in \mathbb{Z}$,

$$\begin{aligned} H_{mn} - H_n &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+mn} \right) - H_n = \sum_{k=1}^{\infty} \sum_{j=0}^{m-1} \left(\frac{1}{km-j} - \frac{1}{km-j+mn} \right) - H_n \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=1}^{\infty} \left(\left(\frac{1}{k} - \frac{1}{k-j/m+n} \right) - \left(\frac{1}{k} - \frac{1}{k-j/m} \right) \right) - H_n \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \left((H_{n-j/m} - H_n) - H_{-j/m} \right) \end{aligned} \tag{3}$$

Since $H_0 = 0$ and $H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right)$, where γ is the **Euler-Mascheroni Constant**, if we let $n \to \infty$ in (3), we get

$$\sum_{j=1}^{m-1} H_{-j/m} = -m \ln(m) \tag{4}$$

Using identity (7) from this answer,

http://math.stackexchange.com/a/143179

$$\begin{aligned} \pi \cot(\pi z) &= \sum_{k \in \mathbb{Z}} \frac{1}{k+z} = \sum_{k=1}^{\infty} \left(\frac{1}{k-1+z} - \frac{1}{k-z} \right) \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k-z} \right) - \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z-1} \right) H_{-z} - H_{z-1} \end{aligned} \tag{5}$$

which implies

$$H_{-j/m} - H_{-(m-j)/m} = \pi \cot\left(\frac{\pi j}{m} \right) \tag{6}$$

Using (4) and (6) for $m = 3$ yields

$$H_{-1/3} + H_{-2/3} = -3 \ln(3) \tag{7}$$

and

$$H_{-1/3} - H_{-2/3} = \pi \cot\left(\frac{\pi}{3} \right) \tag{8}$$

Averaging (7) and (8) yields

$$H_{-1/3} = -\frac{3}{2}\ln(3) + \frac{\pi}{2\sqrt{3}} \tag{9}$$

Finally,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{3n(3n-1)} &= -\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n-1/3}\right) = -\frac{1}{3}H_{-1/3} \\ &= \frac{1}{2}\ln(3) - \frac{\pi}{6\sqrt{3}}\end{aligned} \tag{10}$$

□

Exercise 2.1.20: Evaluate:

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{k^n}$$

Solution: We remark that

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{u^{z-1}}{e^u - 1}\mathrm{d}u \ , \ \sum_{n=2}^{\infty} \frac{t^n}{(n-1)!} = (e^t - 1)t \ , \ \psi_0(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x}\mathrm{d}x$$

where $\psi_0(s)$ is **Digamma Function** and γ is the **Euler's Constant**.
Then

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{\zeta(n)}{k^n} &= \sum_{n=2}^{\infty} \frac{1}{k^n\Gamma(n)} \int_0^{\infty} \frac{u^{n-1}}{e^u - 1}\mathrm{d}u = \int_0^{\infty} \frac{1}{u(e^u-1)}\left(\sum_{n=2}^{\infty} \frac{1}{(n-1)!}\left(\frac{u}{k}\right)^n\right)\mathrm{d}u \\ &= \frac{1}{k}\int_0^{\infty} \frac{e^{\frac{u}{k}} - 1}{e^u - 1}\mathrm{d}u\end{aligned}$$

Substituting $t = e^{-u}$, we get

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{k^n} = -\frac{1}{k}\int_0^1 \frac{1 - t^{-1/k}}{1-t}\mathrm{d}t = \frac{-1}{k}\left\{\gamma + \psi_0\left(1 - \frac{1}{k}\right)\right\} \tag{1}$$

If $k(\geq 2)$ is an integer, equation (1) can be further simplified using **Gauss' Digamma Theorem**.

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{k^n} = -\frac{\pi}{2k}\cot\left(\frac{\pi}{k}\right) + \frac{\ln k}{k} - \frac{1}{k}\sum_{m=1}^{k-1} \cos\left(\frac{2\pi m}{k}\right)\log\left(2\sin\frac{\pi m}{2}\right)$$

□

Exercise 2.1.21: Prove:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \left(1 + \frac{q}{2}\right)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2} \zeta(k+1)\zeta(q-k)$$

☞ **Proof:**

$$\sum_{j=0}^{k}\zeta(k+2-j)\zeta(j+2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{k}\frac{1}{m^{k+2-j}n^{j+2}} \tag{1}$$

$$=(k+1)\zeta(k+4)+\sum_{\substack{m,n=1\\m\neq n}}^{\infty}\frac{1}{m^2n^2}\frac{\frac{1}{m^{k+1}}-\frac{1}{n^{k+1}}}{\frac{1}{m}-\frac{1}{n}} \tag{2}$$

$$=(k+1)\zeta(k+4)+\sum_{\substack{m,n=1\\m\neq n}}^{\infty}\frac{1}{nm^{k+2}(n-m)}-\frac{1}{mn^{k+2}(n-m)} \tag{3}$$

$$=(k+1)\zeta(k+4)+2\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\frac{1}{nm^{k+2}(n-m)}-\frac{1}{mn^{k+2}(n-m)} \tag{4}$$

$$=(k+1)\zeta(k+4)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n+m)m^{k+2}n}-\frac{1}{m(n+m)^{k+2}n} \tag{5}$$

$$=(k+1)\zeta(k+4)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{k+3}n}-\frac{1}{(m+n)m^{k+3}}$$

$$-2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m(n+m)^{k+3}}+\frac{1}{n(n+m)^{k+3}} \tag{6}$$

$$=(k+1)\zeta(k+4)+2\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}}-4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n(n+m)^{k+3}} \tag{7}$$

$$=(k+1)\zeta(k+4)+2\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}}-4\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{nm^{k+3}} \tag{8}$$

$$=(k+1)\zeta(k+4)+2\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}}-4\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{1}{nm^{k+3}}+4\zeta(k+4) \tag{9}$$

$$=(k+5)\zeta(k+4)+2\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}}-4\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{1}{nm^{k+3}} \tag{10}$$

$$=(k+5)\zeta(k+4)+2\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}}-4\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}} \tag{11}$$

$$=(k+5)\zeta(k+4)-2\sum_{m=1}^{\infty}\frac{H_m}{m^{k+3}} \tag{12}$$

Letting $q=k+3$ and reindexing $j\mapsto j-1$ yields

$$\sum_{j=1}^{q-2}\zeta(q-j)\zeta(j+1)=(q+2)\zeta(q+1)-2\sum_{m=1}^{\infty}\frac{H_m}{m^q}$$

and finally

$$\sum_{m=1}^{\infty}\frac{H_m}{m^q}=\frac{q+2}{2}\zeta(q+1)-\frac{1}{2}\sum_{j=1}^{q-2}\zeta(q-j)\zeta(j+1)$$

Explanation:

(1):expand ζ.

(2):pull out the terms for $m=n$ and use the formula for finite geometric sums on the rest.

(3):simplify terms.

(4):utilize the symmetry of $\dfrac{1}{nm^{k+2}(n-m)}+\dfrac{1}{mn^{k+2}(m-n)}$.
(5):$n\mapsto n+m$ and change the order of summation.
(6):$\dfrac{1}{mn}=\dfrac{1}{m(m+n)}+\dfrac{1}{n(m+n)}$.
(7):$H_m=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+m}$ and use the symmetry of $\dfrac{1}{m(n+m)^{k+3}}+\dfrac{1}{n(n+m)^{k+3}}$.
(8):$m\mapsto m-n$.
(9):subtract and add the terms for $m=n$.
(10):combine $\zeta(k+4)$ and change the order of summation.
(11):$H_m=\displaystyle\sum_{n=1}^{m}\frac{1}{n}$.
(12):combine sums. □

Exercise 2.1.22: Evaluate:

$$\sum_{n=1}^{\infty}(n\ \operatorname{arccot}\ n-1)$$

Solution:

$$\begin{aligned}n\cot^{-1}(n)-1&=n\tan^{-1}\left(\frac{1}{n}\right)-1=n\int_0^{1/n}\frac{\mathrm{d}x}{1+x^2}-1\\&=-n\int_0^{1/n}\frac{x^2\,\mathrm{d}x}{1+x^2}=-\int_0^1\frac{x^2\,\mathrm{d}x}{n^2+x^2}\end{aligned}\tag{1}$$

Using formula (9) from this answer

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and substituting $z\mapsto ix$, we get

$$\sum_{n=1}^{\infty}\frac{1}{n^2+x^2}=\frac{\pi\coth(\pi x)}{2x}-\frac{1}{2x^2}\tag{2}$$

Combining (1) and (2) yields

$$\begin{aligned}&\sum_{n=1}^{\infty}(n\cot^{-1}(n)-1)=\frac{1}{2}\int_0^1(1-\pi x\coth(\pi x))\,\mathrm{d}x=\frac{1}{2}\int_0^1\left(1-\pi x\left(1+\frac{2e^{-2\pi x}}{1-e^{-2\pi x}}\right)\right)\,\mathrm{d}x\\&=\frac{2-\pi}{4}-\pi\int_0^1\frac{xe^{-2\pi x}}{1-e^{-2\pi x}}\,\mathrm{d}x=\frac{2-\pi}{4}-\pi\int_0^1 x\left(\sum_{n=1}^{\infty}e^{-2\pi nx}\right)\,\mathrm{d}x\\&=\frac{2-\pi}{4}-\pi\sum_{n=1}^{\infty}\left(\frac{1}{(2\pi n)^2}-\left(\frac{1}{2\pi n}+\frac{1}{(2\pi n)^2}\right)e^{-2\pi n}\right)\\&=\frac{2-\pi}{4}-\frac{\pi}{24}-\frac{1}{2}\ln\left(1-e^{-2\pi}\right)+\frac{1}{4\pi}\mathrm{Li}_2\left(e^{-2\pi}\right)=\frac{1}{2}+\frac{17\pi}{24}-\frac{1}{2}\ln\left(e^{2\pi}-1\right)+\frac{1}{4\pi}\mathrm{Li}_2\left(e^{-2\pi}\right)\end{aligned}$$

□

Exercise 2.1.23: Evaluate:

$$\sum_{n=0}^{\infty}\frac{2n+1}{e^{(2n+1)\pi}+1}$$

Solution: Let's start with

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\tag{1}$$

Differentiating (1) and multiplying by x, we get

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \tag{2}$$

Taking the odd part of (2) yields

$$\sum_{n=0}^{\infty} (2n+1)x^{2n+1} = \frac{x(1+x^2)}{(1-x^2)^2} \tag{3}$$

Using (3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2n+1}{e^{(2n+1)\pi}+1} &= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} (-1)^{k-1}(2n+1)e^{-(2n+1)k\pi} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} (-1)^{k-1}(2n+1)e^{-(2n+1)k\pi} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{-k\pi}\left(1+e^{-2k\pi}\right)}{\left(1-e^{-2k\pi}\right)^2} = \frac{1}{2}\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cosh(k\pi)}{\sinh^2(k\pi)}\end{aligned} \tag{4}$$

We can use the formula proven in this answer

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$$\pi\cot(\pi z) = \sum_{k\in\mathbb{Z}} \frac{1}{z+k} \tag{5}$$

to derive

$$\begin{aligned}\pi\csc(\pi z) = \pi\cot(\pi z/2) - \pi\cot(\pi z) &= \sum_{k\in\mathbb{Z}} \frac{2}{z+2k} - \sum_{k\in\mathbb{Z}} \frac{1}{z+k} = \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{z+k} \\ \pi^2 \frac{\cos(\pi z)}{\sin^2(\pi z)} &= \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(z+k)^2}\end{aligned} \tag{6}$$

then rotate coordinates with $z \mapsto iz$ to get

$$\pi^2 \frac{\cosh(\pi z)}{\sinh^2(\pi z)} = \sum_{j\in\mathbb{Z}} \frac{(-1)^j}{(z+ij)^2} \tag{7}$$

Now plug (7) into (4):

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{2n+1}{e^{(2n+1)\pi}+1} \\ &= \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\sum_{j\in\mathbb{Z}} (-1)^{j+k-1}\frac{1}{(k+ij)^2} = \frac{1}{2\pi^2}\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k^2} \\ &+ \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (-1)^{j+k-1}\left(\frac{1}{(k+ij)^2} + \frac{1}{(k-ij)^2}\right) \\ &= \frac{1}{2\pi^2}\frac{\pi^2}{12} + \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (-1)^{j+k-1}\frac{2(k^2-j^2)}{(k^2+j^2)^2} \\ &= \frac{1}{24}\end{aligned} \tag{8}$$

□

Exercise 2.1.24: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}H_n$$

Solution:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}H_n=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{k}-\frac{1}{k+n}\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(k+n)}$$
$$=\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{1}{nk(n-k)}=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{nk(n-k)}=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n^2}\left(\frac{1}{k}+\frac{1}{n-k}\right)$$
$$=2\sum_{n=1}^{\infty}\frac{1}{n^2}H_{n-1}=2\sum_{n=1}^{\infty}\frac{1}{n^2}H_n-2\zeta(3)\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}H_n=2\zeta(3)$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{k}-\frac{1}{k+n}\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{nk(k+n)}$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n=-\frac{3}{4}\zeta(3)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_{n-1}$$
$$=-\frac{3}{4}\zeta(3)+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{(-1)^n}{n^2}\left(\frac{1}{k}+\frac{1}{n-k}\right)$$
$$=-\frac{3}{4}\zeta(3)+\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{(-1)^n}{nk(n-k)}=-\frac{3}{4}\zeta(3)+\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+k}}{(n+k)kn}$$

With the fact that $1+(-1)^k+(-1)^n+(-1)^{n+k}=4$ if k and n are both even and 0 otherwise:

$$\zeta(3)=\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{nk(n+k)}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1+(-1)^k+(-1)^n+(-1)^{n+k}}{nk(n+k)}$$
$$=2\zeta(3)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n+\frac{3}{2}\zeta(3)$$
$$-\frac{5}{8}\zeta(3)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}H_n$$

That is,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}H_n=\frac{5}{8}\zeta(3)$$

□

Exercise 2.1.25: Evaluate:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n}$$

Solution:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{n-1}^{(2)}}{n}\tag{1}$$
$$=\frac{3}{4}\zeta(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n-1}\frac{1}{k^2}\tag{2}$$

$$= \frac{3}{4}\zeta(3) + \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{(-1)^{n-1}}{nk^2} \tag{3}$$

$$= \frac{3}{4}\zeta(3) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{k+n-1}}{(k+n)k^2} \tag{4}$$

$$= \frac{3}{4}\zeta(3) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n-1}\left(\frac{1}{k^2n} - \frac{1}{kn(k+n)}\right) \tag{5}$$

$$= \frac{3}{4}\zeta(3) - \frac{1}{2}\zeta(2)\ln(2) + \frac{1}{4}\zeta(3) \tag{6}$$

$$= \zeta(3) - \frac{1}{2}\zeta(2)\ln(2)$$

Explanation:

(1): $H_n^{(2)} = \dfrac{1}{n^3} + H_{n-1}^{(2)}$.

(2): expand $H_{n-1}^{(2)}$.

(3): change order of summation.

(4): reindex $n \mapsto k+n$.

(5): $\dfrac{1}{(k+n)k^2} = \dfrac{1}{k^2n} - \dfrac{1}{kn(k+n)}$.

(6): $\displaystyle\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{k+n}}{kn(k+n)} = \frac{1}{4}\zeta(3)$ from **Exercise 2.1.24**. □

Exercise 2.1.26: Prove:

$$\phi = \frac{13}{8} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!}{(n+2)!n!4^{2n+3}}$$

Proof: Method One:

Using the series

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$$

we get

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!}{(n+2)!\,n!}x^{n+2} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\binom{2n}{n}\frac{x^{n+1}}{n+1}$$
$$= \frac{1}{2}\int_0^x\left[(1+4t)^{-1/2} - 1\right]\,\mathrm{d}t = \frac{1}{4}(1+4x)^{1/2} - \frac{x}{2} - \frac{1}{4}$$

Therefore,

$$\frac{13}{8} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!}{(n+2)!\,n!4^{2n+3}} = \frac{13}{8} + 4f\left(\frac{1}{16}\right) = \frac{13}{8} + 4\left(\frac{1}{4}\left(1+\frac{1}{4}\right)^{1/2} - \frac{1}{32} - \frac{1}{4}\right)$$
$$= \frac{1+\sqrt{5}}{2} = \phi$$

Method Two:Catalan Numbers

Some straightforward manipulations of the sum brings it to the form

$$S = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2(n+1))!}{(n+2)!(n+1)!4^{2n+3}}$$

Using the **definition of the Catalan numbers** this nicely rewrites as

$$S = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}C_{n+1}}{4^{2n+3}} = \frac{1}{8}\sum_{n=0}^{\infty}C_{n+1}\left(\frac{-1}{16}\right)^{n+1}$$

Using the generating function of the Catalan numbers,

$$\sum_{n=0}^{\infty}C_{n+1}x^{n+1} = \frac{1-\sqrt{1-4x}}{2x} - 1$$

setting $x = -\frac{1}{16}$ we may conclude that

$$S = \frac{1}{8}\left(4\sqrt{5}-9\right)$$

Furthermore

$$S + \frac{13}{8} = \frac{\sqrt{5}}{2} - \frac{9}{8} + \frac{13}{8} = \frac{1+\sqrt{5}}{2} = \phi$$

Method Three:Contour Integration

We can show by an straightforward application of Cauchy's integral formula, that

$$\binom{n}{k} = \frac{1}{2\pi i}\oint_C \frac{(1+z)^n}{z^{k+1}}\mathrm{d}z \tag{1}$$

where C is a circle traversed counter-clockwise (with radius $< 1/4$ in our case).

Furthermore we may observe that (C_n are again Catalan numbers)

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \tag{2}$$

Now let's apply (1) and (2) to the function $q(x) = \sum_{n=0}^{\infty}C_{n+1}x^{n+1}$,we get

$$q(x) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty}x^{n+1}\left(\frac{(1+z)^{2n+2}}{z^{n+2}} - \frac{(1+z)^{2n+2}}{z^{n+3}}\right)\mathrm{d}z$$

where the exchange of summation and integration is justified as long as the poles lie not on the contour of integration (which will be the case) the sums are now usual geometric series and yield

$$q(x) = \frac{1}{2\pi i}\oint_C \left(-\frac{1}{z}\frac{x(z+1)^2}{x(z+1)^2 - z} + \frac{1}{z^2}\frac{x(z+1)^2}{x(z+1)^2 - z}\right)\mathrm{d}z$$

It is easy to show that the zero's of the denominator are given by $z_{\pm} = \frac{-2x \pm \sqrt{1-4x}+1}{2x}$ and only z_- lies in the unit circle if $x < 1/4$ which is important because we want to set $x = -\frac{1}{16}$ in the end.

Applying the residue theorem, we get

$$q(x) = \mathrm{Res}(z=0) + \mathrm{Res}(z=z_-) = -\frac{x-1}{x} + \frac{\sqrt{1-4x}+1}{2x} = \frac{2x+\sqrt{1-4x}-1}{2x}$$

Recalling the sum is

$$S = \frac{1}{8}q\left(\frac{-1}{16}\right)$$

we obtain

$$S + \frac{13}{8} = \phi$$

□

Exercise 2.1.27: Prove:

$$\sum_{l=1}^{n}\sum_{k=1}^{n-1} \tan\frac{lk\pi}{2n+1}\tan\frac{l(k+1)\pi}{2n+1} = 0$$

Proof: Starting with

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \tag{1}$$

we get

$$\tan(x)\tan(y) = \frac{\tan(x)-\tan(y)}{\tan(x-y)} - 1 \tag{2}$$

Thus,

$$\tan\left(\frac{l(k+1)\pi}{2n+1}\right)\tan\left(\frac{lk\pi}{2n+1}\right) = \frac{\tan\left(\frac{l(k+1)\pi}{2n+1}\right)-\tan\left(\frac{lk\pi}{2n+1}\right)}{\tan\left(\frac{l\pi}{2n+1}\right)} - 1 \tag{3}$$

Therefore, because of the telescoping sum,

$$\begin{aligned}\sum_{k=1}^{n-1}\tan\left(\frac{l(k+1)\pi}{2n+1}\right)\tan\left(\frac{lk\pi}{2n+1}\right) &= \frac{\tan\left(\frac{ln\pi}{2n+1}\right)-\tan\left(\frac{l\pi}{2n+1}\right)}{\tan\left(\frac{l\pi}{2n+1}\right)} - (n-1)\\ &= \frac{\tan\left(\frac{ln\pi}{2n+1}\right)}{\tan\left(\frac{l\pi}{2n+1}\right)} - n\end{aligned} \tag{4}$$

Note that

$$\frac{\tan\left(\frac{(2n+1-l)n\pi}{2n+1}\right)}{\tan\left(\frac{(2n+1-l)\pi}{2n+1}\right)} = \frac{\tan\left(\frac{ln\pi}{2n+1}\right)}{\tan\left(\frac{l\pi}{2n+1}\right)} \tag{5}$$

so that by replacing the odd ls with even $2n+1l$s and using $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$, we get

$$\begin{aligned}\sum_{l=1}^{n}\frac{\tan\left(\frac{ln\pi}{2n+1}\right)}{\tan\left(\frac{l\pi}{2n+1}\right)} &= \sum_{l=1}^{n}\frac{\tan\left(\frac{2ln\pi}{2n+1}\right)}{\tan\left(\frac{2l\pi}{2n+1}\right)} = -\sum_{l=1}^{n}\frac{\tan\left(\frac{l\pi}{2n+1}\right)}{\tan\left(\frac{2l\pi}{2n+1}\right)}\\ &= \frac{1}{2}\sum_{l=1}^{n}\left(\tan^2\left(\frac{l\pi}{2n+1}\right)-1\right)\end{aligned} \tag{6}$$

Using contour integration, we will compute

$$\sum_{l=1}^{n} \tan^2\left(\frac{\pi l}{2n+1}\right) = n(2n+1) \tag{7}$$

Note that

$$\frac{(2n+1)/z}{z^{2n+1}-1} \tag{8}$$

has simple poles. It has residue 1 at $z = e^{\frac{2\pi li}{2n+1}}$ for each l and residue $(2n+1)$ at $z = 0$. Furthermore, at $z = e^{i\theta}$,

$$-\left(\frac{z-1}{z+1}\right)^2 = \tan^2(\theta/2) \tag{9}$$

Because the total residue of

$$f(z) = \left(\frac{z-1}{z+1}\right)^2 \frac{(2n+1)/z}{z^{2n+1}-1} \tag{10}$$

is 0, we get that the sum of its residues at $z = 0$ and $z = 1$ equals

$$\sum_{l=1}^{2n} \tan^2\left(\frac{\pi l}{2n+1}\right) = 2\sum_{l=1}^{n} \tan^2\left(\frac{\pi l}{2n+1}\right) \tag{11}$$

First,

$$\mathrm{Res}_{z=0} f(z) = -(2n+1) \tag{12}$$

Next,

$$\begin{aligned}
&\mathrm{Res}_{z=-1} f(z) = \mathrm{Res}_{z=0} f(z-1) = \mathrm{Res}_{z=0}(2n+1)\left(\frac{z-2}{z}\right)^2 \frac{1}{1-z}\frac{1}{1+(1-z)^{2n+1}} \\
&= \mathrm{Res}_{z=0}(2n+1)\left(1-\frac{4}{z}+\frac{4}{z^2}\right)(1+z+\dots)\frac{1}{2}\left(1+\frac{2n+1}{2}z+\dots\right) \\
&= \mathrm{Res}_{z=0}\frac{2n+1}{2}\left(1-\frac{4}{z}+\frac{4}{z^2}\right)\left(1+\frac{2n+3}{2}z+\dots\right) \\
&= \mathrm{Res}_{z=0}\frac{4n+2}{z^2}+\frac{(2n+1)^2}{z}+\dots = (2n+1)^2
\end{aligned} \tag{13}$$

Combining ((11), ((12), and ((13), yields (7).
Combining (4), (6), and (7) yields

$$\sum_{l=1}^{n}\sum_{k=1}^{n-1} \tan\frac{lk\pi}{2n+1}\tan\frac{l(k+1)\pi}{2n+1} = 0$$

□

Exercise 2.1.28: Prove:

$$1+\sum_{n=0}^{\infty}\frac{1}{\pi\left(2n+\frac{3}{4}\right)\left(2n+\frac{5}{4}\right)} = \sqrt{2}$$

☞ **Proof:**

$$1+\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{3}{8})(n+\frac{5}{8})}=1+\frac{1}{\pi}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{3}{8}}-\frac{1}{n+\frac{5}{8}}\right)$$
$$=1+\frac{1}{\pi}\sum_{n=0}^{\infty}\Bigg(\underbrace{\frac{1}{n+\frac{3}{8}}}_{n+\frac{3}{8}\text{ for }n\geq 0}+\underbrace{\frac{1}{-n-1+\frac{3}{8}}}_{n+\frac{3}{8}\text{ for }n\leq -1}\Bigg)$$
$$=1+\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n+\frac{3}{8}}=1+\frac{1}{\pi}\pi\cot\left(\frac{3}{8}\pi\right)=1+\sqrt{2}-1=\sqrt{2} \tag{1}$$

Explanation:

(1)use (7) from this answer.

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Exercise 2.1.29: Evaluate:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\frac{4^k}{\binom{2k}{k}}$$

✎ **Solution: Determine a Related Generating Function**

Using the Beta function, we get the following identity:

$$\frac{1}{\binom{2n}{n}}=(2n+1)\int_0^1 t^n(1-t)^n\mathrm{d}t \tag{1}$$

Thus,

$$\sum_{n=0}^{\infty}\frac{(-4)^n x^{2n}}{(2n+1)\binom{2n}{n}}=\int_0^1\frac{1}{1+4x^2t(1-t)}\mathrm{d}t \tag{2a}$$
$$=\int_0^1\frac{1}{1+x^2-x^2(2t-1)^2}\mathrm{d}t \tag{2b}$$
$$=\frac{1}{1+x^2}\int_0^1\frac{1}{1-\frac{x^2}{1+x^2}(2t-1)^2}\mathrm{d}t \tag{2c}$$
$$=\frac{1}{1+x^2}\int_{-1}^1\frac{1}{1-\frac{x^2}{1+x^2}t^2}\frac{1}{2}\mathrm{d}t \tag{2d}$$
$$=\frac{1}{2x\sqrt{1+x^2}}\int_{-x/\sqrt{1+x^2}}^{x/\sqrt{1+x^2}}\frac{1}{1-t^2}\mathrm{d}t \tag{2e}$$
$$=\frac{1}{x\sqrt{1+x^2}}\operatorname{arctanh}\left(\frac{x}{\sqrt{1+x^2}}\right) \tag{2f}$$
$$=\frac{1}{x\sqrt{1+x^2}}\operatorname{arcsinh}(x) \tag{2g}$$

Explanation:

(2a): multiply (1) by $\frac{(-4)^n x^{2n}}{2n+1}$ and sum using the formula for the sum of a geometric series.

(2b): rearrange the denominator of the integrand.

(2c): divide numerator and denominator by $1+x^2$.

(2d): substitute $t \mapsto (t+1)/2$.

(2e): substitute $t \mapsto \dfrac{t\sqrt{1+x^2}}{x}$.

(2f): integrate.

(2g): $\tanh(x) = \dfrac{\sinh(x)}{\sqrt{1+\sinh^2(x)}}$.

Integrate the Generating Function to Evaluate the Sum

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)^2\binom{2n}{n}} = \int_0^1 \frac{1}{x\sqrt{1+x^2}}\operatorname{arcsinh}(x)\,\mathrm{d}x \tag{3a}$$

$$= -\int_0^1 \operatorname{arcsinh}(x)\frac{1}{\sqrt{1+1/x^2}}\mathrm{d}\frac{1}{x} \tag{3b}$$

$$= -\int_0^1 \operatorname{arcsinh}(x)\,\mathrm{d}\operatorname{arcsinh}\left(\frac{1}{x}\right) \tag{3c}$$

$$= -\operatorname{arcsinh}^2(1) + \int_0^1 \operatorname{arcsinh}\left(\frac{1}{x}\right)\,\mathrm{d}\operatorname{arcsinh}(x) \tag{3d}$$

$$= -\operatorname{arcsinh}^2(1) - \int_1^\infty \operatorname{arcsinh}(x)\,\mathrm{d}\operatorname{arcsinh}\left(\frac{1}{x}\right) \tag{3e}$$

$$= -\operatorname{arcsinh}^2(1) + \int_1^\infty \frac{1}{x\sqrt{1+x^2}}\operatorname{arcsinh}(x)\,\mathrm{d}x \tag{3f}$$

$$= -\frac{1}{2}\operatorname{arcsinh}^2(1) + \frac{1}{2}\int_0^\infty \frac{1}{x\sqrt{1+x^2}}\operatorname{arcsinh}(x)\,\mathrm{d}x \tag{3g}$$

$$= -\frac{1}{2}\operatorname{arcsinh}^2(1) + \frac{1}{2}\int_0^\infty \frac{t\,\mathrm{d}t}{\sinh(t)} \tag{3h}$$

Explanation:

(3a): integrate (2g).

(3b): pull a factor of x from the square root.

(3c): prepare to integrate by parts.

(3d): integrate by parts.

(3e): substitute $x \mapsto 1/x$.

(3f): $\mathrm{d}\operatorname{arcsinh}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x$.

(3g): average (3a) and (3f).

(3h): substitute $x = \sinh(t)$.

Expand $\dfrac{1}{\sinh(x)}$ as a series in e^{-kt}:

$$\int_0^\infty \frac{t\,\mathrm{d}t}{\sinh(t)} = \int_0^\infty \sum_{k=0}^{\infty} 2t\,e^{-(2k+1)t}\,\mathrm{d}t = \sum_{k=0}^{\infty}\frac{2}{(2k+1)^2} = \frac{\pi^2}{4} \tag{4}$$

Combining (3) and (4) yields

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)^2\binom{2n}{n}} = \frac{\pi^2}{8} - \frac{1}{2}\operatorname{arcsinh}^2(1)$$

□

Exercise 2.1.30: Evaluate:

$$\sum_{n=0}^{\infty}(-1)^n\frac{\left(2-\sqrt{3}\right)^{2n+1}}{(2n+1)^2}$$

Solution: We can look at a related generating function.

$$\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{2k+1}=\arctan(t) \tag{1}$$

Dividing (1) by t and integrating gives

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k\frac{(2-\sqrt{3})^{2k+1}}{(2k+1)^2} &= \int_0^{2-\sqrt{3}}\frac{\arctan(t)}{t}\mathrm{d}t && \text{(2a)}\\
&= \int_0^{\pi/12} x\,\mathrm{d}\ln(\tan(x)) && \text{(2b)}\\
&= \frac{\pi}{12}\ln(2-\sqrt{3})-\int_0^{\pi/12}\ln(\tan(x))\,\mathrm{d}x && \text{(2c)}\\
&= \frac{\pi}{12}\ln(2-\sqrt{3})+2\int_0^{\pi/12}\sum_{k=0}^{\infty}\frac{\cos((4k+2)x)}{2k+1}\,\mathrm{d}x && \text{(2d)}\\
&= \frac{\pi}{12}\ln(2-\sqrt{3})+\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi/6)}{(2k+1)^2} && \text{(2e)}\\
&= \frac{\pi}{12}\ln(2-\sqrt{3})+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}+\frac{3}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(6k+3)^2} && \text{(2f)}\\
&= \frac{\pi}{12}\ln(2-\sqrt{3})+\frac{2}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} && \text{(2g)}\\
&= \frac{\pi}{12}\ln(2-\sqrt{3})+\frac{2}{3}\mathbf{G} && \text{(2h)}
\end{aligned}$$

Explanation:

(2a): divide (1) by t and integrate.

(2b): substitute $t=\tan(x)$.

(2c): integrate by parts.

(2d): $\ln(\tan(x))=-2\sum_{k=0}^{\infty}\frac{\cos((4k+2)x)}{2k+1}$.

(2e): integrate.

(2f): details below.

(2g): $\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{9}=\frac{2}{3}$.

(2h): $\mathbf{G}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$ is **Catalan's Constant**.

Details of (2f)

$$\begin{aligned}\sin((2k+1)\pi/6) &= \left(\frac{1}{2},\quad 1\ ,\frac{1}{2},-\frac{1}{2},-1\,,-\frac{1}{2},\frac{1}{2},\quad 1\ ,\frac{1}{2},\dots\right)\\ &= \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\ \ \frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\cdots\right)\\ &+\left(\ 0\ ,\quad \frac{3}{2},\ 0\ ,\quad 0\ ,-\frac{3}{2},\quad 0\ ,\ 0\ ,\quad \frac{3}{2},\ 0\ ,\dots\right)\\ 2k+1 &= (\ \ 1\,,\quad 3\,,\ 5\,,\quad 7\,,\quad 9\,,\quad 11,13,\quad 15\,,17\,,\dots\,)\end{aligned}$$

Note that the $\pm\frac{3}{2}$ corrections are at positions where $2k+1$ is 3 times an odd number. That is,

$$\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi/6)}{(2k+1)^2} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}+\frac{3}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(6k+3)^2}$$

which justifies (2f). □

Exercise 2.1.31: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2+n^4}$$

Solution: First, we have

$$\frac{1}{z^4+z^2+1} = \frac{1}{12}\left(\frac{-3-i\sqrt{3}}{z-e^{\pi i/3}}+\frac{3+i\sqrt{3}}{z-e^{4\pi i/3}}+\frac{3-i\sqrt{3}}{z-e^{2\pi i/3}}+\frac{-3+i\sqrt{3}}{z-e^{5\pi i/3}}\right) \tag{1}$$

Let γ be the rectangle

$$[-1-i,1-i]\cup[1-i,1+i]\cup[1+i,-1+i]\cup[-1+i,-1-i] \tag{2}$$

then the integral

$$\frac{1}{2\pi i}\int_{(n+\frac{1}{2})\gamma}\frac{\pi\cot(\pi z)}{z^4+z^2+1}\,\mathrm{d}z \tag{3}$$

tends to 0 since along the horizontal paths, $|\pi\cot(\pi z)|\to\pi$ and along the vertical paths, $|\pi\cot(\pi z)|<\pi$.

Since $\pi\cot(\pi z)$ has residue 1 at each integer, we get that (3) is the sum of the residues of (1) times $\pi\cot(\pi z)$ at the singularities of (1) plus

$$1+2\sum_{n=1}^{\infty}\frac{1}{n^4+n^2+1} \tag{4}$$

The sum of the residues of (1) times $\pi\cot(\pi z)$ at the singularities of (1) is

$$-4\,\Re\left(\frac{3+i\sqrt{3}}{12}\pi\cot\left(\pi\frac{1+i\sqrt{3}}{2}\right)\right) \tag{5}$$

Since the sum of (4) and (5) is 0, we have

$$\begin{aligned}1+2\sum_{n=1}^{\infty}\frac{1}{n^4+n^2+1} &= 4\,\Re\left(\frac{3+i\sqrt{3}}{12}\pi\cot\left(\pi\frac{1+i\sqrt{3}}{2}\right)\right) = -4\,\Re\left(\frac{3+i\sqrt{3}}{12}\pi\tan\left(\pi\frac{i\sqrt{3}}{2}\right)\right)\\ &= \frac{\pi\sqrt{3}}{3}\tanh\left(\pi\frac{\sqrt{3}}{2}\right)\end{aligned} \tag{6}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^4+n^2+1}=\frac{\pi\sqrt{3}}{6}\tanh\left(\pi\frac{\sqrt{3}}{2}\right)-\frac{1}{2}$$

□

Exercise 2.1.32: Prove:

$$\sum_{n=1}^{\infty}\frac{\coth n\pi}{n^7}=\frac{19\pi^7}{56700}$$

Proof: Since (7) from this answer

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is valid for any $z\in\mathbb{C}$, we have

$$\pi\coth(\pi n)=\sum_{k\in\mathbb{Z}}\frac{1}{n+ik}=\frac{1}{n}+2n\sum_{k=1}^{\infty}\frac{1}{n^2+k^2}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\pi\coth(\pi n)}{n^7}=\sum_{n=1}^{\infty}\frac{1}{n^8}+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^6(n^2+k^2)}=\zeta(8)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{n^6}-\frac{1}{n^4(n^2+k^2)}\right)$$

$$=\zeta(8)+2\zeta(2)\zeta(6)-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{n^4(n^2+k^2)}$$

$$=\zeta(8)+2\zeta(2)\zeta(6)-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^4}\left(\frac{1}{n^4}-\frac{1}{n^2(n^2+k^2)}\right)$$

$$=\zeta(8)+2\zeta(2)\zeta(6)-2\zeta(4)\zeta(4)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^4n^2(n^2+k^2)}$$

$$=\zeta(8)+2\zeta(2)\zeta(6)-\zeta(4)\zeta(4)=\frac{19\pi^8}{56700}$$

Hence

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7}=\frac{19\pi^7}{56700}$$

□

Exercise 2.1.33: How to find $\zeta(0)=-\frac{1}{2}$ by definition?

Proof: Consider the integral

$$\int_0^{\infty}\frac{xt^{x-1}}{e^t+1}\mathrm{d}t=\int_0^{\infty}\frac{xt^{x-1}}{1+e^{-t}}e^{-t}\,\mathrm{d}t=x\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty}t^{x-1}e^{-kt}\,\mathrm{d}t$$

$$=x\sum_{k=1}^{\infty}(-1)^{k-1}k^{-x}\int_0^{\infty}t^{x-1}e^{-t}\,\mathrm{d}t=x\eta(x)\Gamma(x)=(1-2^{1-x})\zeta(x)\Gamma(x+1)\tag{1}$$

Integrate by parts to get

$$\lim_{x\to0^+}\int_0^{\infty}\frac{xt^{x-1}}{e^t+1}\mathrm{d}t=\lim_{x\to0^+}\int_0^{\infty}\frac{t^xe^t}{(e^t+1)^2}\mathrm{d}t=\int_1^{\infty}\frac{\mathrm{d}u}{(u+1)^2}=\frac{1}{2}\tag{2}$$

Sending xx to 0 in (1) and combining with (2), we get $\zeta(0)=-\frac{1}{2}$. □

Exercise 2.1.34: Evaluate:

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2$$

✎ Solution: Method One:

1. Basic facts on the dilogarithm. Let $\mathrm{Li}_2(z)$ be the dilogarithm function defined by

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-x)}{x}\,\mathrm{d}x.$$

Here the branch cut of ln is chosen to be $(-\infty, 0]$ so that Li_2 defines a holomorphic function on the region $\mathbb{C} \setminus [1, \infty)$. Also, it is easy to check (by differentiating both sides) that the following identities hold

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\mathrm{Li}_2(z) - \frac{1}{2}\ln^2(1-z); \quad z \notin [1,\infty) \tag{1}$$

$$\mathrm{Li}_2\left(\frac{1}{1-z}\right) = \boxed{\mathrm{Li}_2(z) + \zeta(2) - \frac{1}{2}\ln^2(1-z)} + \boxed{\ln(-z)\ln(1-z)}; \quad z \notin [0,\infty) \tag{2}$$

Notice that in (2), the blue-colored part is holomorphic on $|z| < 1$ while the red-colored part induces the branch cut $[1, 0]$.

2. A useful power series. Now let us consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{H_n}{n} z^n.$$

Then $f(z)$ is automatically holomorphic inside the disc $|z| < 1$. Moreover, it is easy to check that

$$\sum_{n=1}^{\infty} H_n z^{n-1} = \frac{1}{z}\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right)\left(\sum_{n=0}^{\infty} z^n\right) = -\frac{\ln(1-z)}{z(1-z)}.$$

thus integrating both sides, together with the identity (1), we obtain the following representation of $f(z)$.

$$f(z) = \mathrm{Li}_2(z) + \frac{1}{2}\ln^2(1-z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right). \tag{3}$$

3. Integral representation and the result. By the Parseval's identity, we have

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}) f(e^{-it})\,\mathrm{d}t = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z} f\left(\frac{1}{z}\right)\mathrm{d}z \tag{4}$$

Since $\dfrac{1}{z}f(z)$ is holomorphic inside $|z| = 1$, the failure of holomorphy of the integrand stems from the branch cut of

$$f\left(\frac{1}{z}\right) = -\mathrm{Li}_2\left(\frac{1}{1-z}\right) = -\left(\mathrm{Li}_2(z) + \zeta(2) - \frac{1}{2}\ln^2(1-z)\right) - \ln(-z)\ln(1-z)$$

which is $[0, 1]$. To resolve this, we utilize the identity (2). Note that the blue-colored portion does not contributes to the the integral (4), since it remains holomorphic inside $|z| < 1$. That is, only the red-colored portion gives contribution to the integral. Consequently we have

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = -\frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z}\ln(-z)\ln(1-z)\,\mathrm{d}z. \tag{5}$$

Since the integrand is holomorphic on $\mathbb{C}\setminus[0,\infty)$, we can utilize the keyhole contour wrapping around $[0,1]$ to reduce (5) to

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}&=\frac{1}{2\pi i}\left\{\int_{0^-i}^{1+0^-i}\frac{f(z)\ln(-z)\ln(1-z)}{z}\,\mathrm{d}z+\int_{1+0^+i}^{+0^+i}\frac{f(z)\ln(-z)\ln(1-z)}{z}\,\mathrm{d}z\right\}\\&=\frac{1}{2\pi i}\left\{\int_0^1\frac{f(x)(\ln x+i\pi)\ln(1-x)}{x}\,\mathrm{d}x-\int_0^1\frac{f(x)(\ln x-i\pi)\ln(1-x)}{x}\,\mathrm{d}x\right\}\\&=\int_0^1\frac{f(x)\ln(1-x)}{x}\,\mathrm{d}x.\end{aligned}\tag{5}$$

Plugging (3) to the last integral and simplifying a little bit, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}&=-\int_0^1\frac{\mathrm{Li}_2(x)\ln(1-x)}{x}\,\mathrm{d}x-\frac{1}{2}\int_0^1\frac{\ln^3(1-x)}{x}\,\mathrm{d}x\\&=\left[\frac{1}{2}\mathrm{Li}_2(x)^2\right]_0^1-\frac{1}{2}\int_0^1\frac{\ln^3 x}{1-x}\,\mathrm{d}x=\frac{1}{2}\zeta(2)^2+\frac{1}{2}\Gamma(4)\zeta(4)=\frac{17\pi^4}{360}\end{aligned}$$

Method Two:

Compute the generating function of the harmonic numbers:

$$\sum_{n=1}^{\infty}H_nx^n=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{x^n}{k}=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{x^n}{k}=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{x^{n+k}}{k}=-\frac{\ln(1-x)}{1-x}\tag{1}$$

Integrating (1) yields

$$\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^n=\frac{1}{2}\ln(1-x)^2\tag{2}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}e^{\pm2\pi inx}=\frac{1}{2}\ln(1-e^{\pm2\pi ix})^2\tag{3}$$

Multiplying and integrating gives

$$\sum_{n=1}^{\infty}\frac{H_{n-1}^2}{n^2}=\frac{1}{4}\int_0^1\ln(1-e^{2\pi ix})^2\ln(1-e^{-2\pi ix})^2\,\mathrm{d}x\tag{4a}$$

$$=\frac{1}{8\pi i}\oint\ln(1-z)^2\ln(1-1/z)^2\frac{\mathrm{d}z}{z}\tag{4b}$$

$$\begin{aligned}&=\frac{1}{8\pi i}\int_0^1\ln(1-z)^2\left[-\pi i+\ln(1-z)-\ln(z)\right]^2\frac{\mathrm{d}z}{z}\\&\quad-\frac{1}{8\pi i}\int_0^1\ln(1-z)^2\left[\pi i+\ln(1-z)-\ln(z)\right]^2\frac{\mathrm{d}z}{z}\end{aligned}\tag{4c}$$

$$=-\frac{1}{2}\int_0^1\ln(1-z)^2\left[\ln(1-z)-\ln(z)\right]\frac{\mathrm{d}z}{z}\tag{4d}$$

Explanation:

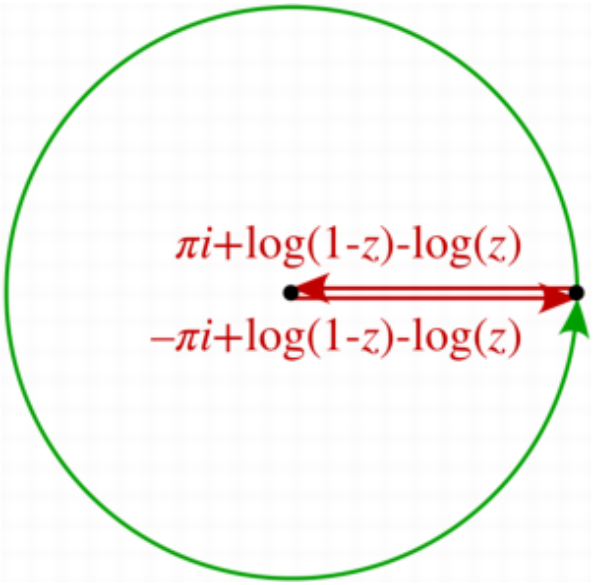
(4a): multiply the conjugates of (3) and integrate.

(4b): convert to contour integral with $z=e^{2\pi ix}$.

(4c): deflate the contour to lines above and below $[0,1]$.

(4d): algebra.

Contour *before* and *after* (4c):



Using $\ln(1-z) = -u$, we get

$$\int_0^1 \ln(1-z)^3 \frac{\mathrm{d}z}{z} = -\int_0^\infty u^3 \frac{\mathrm{d}u}{e^u - 1} = -\Gamma(4)\zeta(4) = -\frac{\pi^4}{15} \tag{5}$$

Using $\ln(z) = -u$ and , we get

$$\begin{aligned}
\int_0^1 \ln(1-z)^2 \ln(z) \frac{\mathrm{d}z}{z} &= -\int_0^\infty \ln(1-e^{-u})^2 u \,\mathrm{d}u && \text{(6a)}\\
&= -2\sum_{n=1}^\infty \int_0^\infty \frac{H_{n-1}}{n} e^{-nu} u \,\mathrm{d}u && \text{(6b)}\\
&= -2\sum_{n=1}^\infty \frac{H_{n-1}}{n^3} && \text{(6c)}\\
&= \zeta(2)^2 - 3\zeta(4) && \text{(6d)}\\
&= -\frac{\pi^4}{180} && \text{(6e)}
\end{aligned}$$

Explanation:

(6a): substitute $z = e^{-u}$.

(6b): apply (2).

(6c): integrate.

(6d): use this answer.

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(6e): evaluate.

Combining (4), (5), and (6) yields

$$\sum_{n=1}^\infty \frac{H_{n-1}^2}{n^2} = \frac{11\pi^4}{360}$$

Noting that

$$\sum_{n=1}^{\infty}\frac{H_{n-1}^2}{n^2}=\sum_{n=1}^{\infty}\frac{\left(H_n-\frac{1}{n}\right)^2}{n^2}=\sum_{n=1}^{\infty}\left(\frac{H_n^2}{n^2}-2\frac{H_n}{n^3}+\frac{1}{n^4}\right)$$

we get, again using the answer in the website above, that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\sum_{n=1}^{\infty}\frac{H_{n-1}^2}{n^2}+2\sum_{n=1}^{\infty}\frac{H_n}{n^3}-\zeta(4)=\frac{11\pi^4}{360}+5\zeta(4)-\zeta(2)^2-\zeta(4)=\frac{17\pi^4}{360}$$

□

Exercise 2.1.35: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{2^n\left(1+\sqrt[2^n]{2}\right)}$$

Solution: Notice that

$$\frac{1}{2^n(\sqrt[2^n]{2}-1)}-\frac{1}{2^n(\sqrt[2^n]{2}+1)}=\frac{1}{2^{n-1}(\sqrt[2^{n-1}]{2}-1)}$$

We can rearrange this to

$$\left(\frac{1}{2^n(\sqrt[2^n]{2}-1)}-1\right)=\frac{1}{2^n(\sqrt[2^n]{2}+1)}+\left(\frac{1}{2^{n-1}(\sqrt[2^{n-1}]{2}-1)}-1\right)$$

and for $n=1$,

$$\frac{1}{2^{n-1}(\sqrt[2^{n-1}]{2}-1)}-1=0$$

therefore, the partial sum is

$$\sum_{n=1}^{m}\frac{1}{2^n(\sqrt[2^n]{2}+1)}=\frac{1}{2^m(\sqrt[2^m]{2}-1)}-1$$

Taking the limit as $m\to\infty$, we get

$$\sum_{n=1}^{\infty}\frac{1}{2^n(\sqrt[2^n]{2}+1)}=\frac{1}{\ln(2)}-1$$

□

Exercise 2.1.36: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{4^n\cos^2\frac{x}{2^n}}=\frac{1}{\sin^2 x}-\frac{1}{x^2}$$

Proof: Often when $\sin(x)$ and x appear in a formula such as this, it is based on the fact that

$$\lim_{a\to 0}\frac{\sin(ax)}{a}=x$$

Since the summands are $\dfrac{4^{-k}}{\cos^2(2^{-k}x)}$, it appears that an identity involving a double angle formula is being exploited. Therefore, with an eye toward a telescoping sum, I started with

$$\frac{4}{\sin^2(2x)}-\frac{1}{\sin^2(x)}=\frac{4-4\cos^2(x)}{4\sin^2(x)\cos^2(x)}=\frac{4\sin^2(x)}{4\sin^2(x)\cos^2(x)}=\frac{1}{\cos^2(x)}$$

Substitute $x \mapsto 2^{-k}x$ and multiply by 4^{-k} and sum (and telescope)

$$\sum_{k=1}^{n} \frac{4^{-k}}{\cos^2(2^{-k}x)} = \sum_{k=1}^{n} \left(\frac{4^{-k+1}}{\sin^2(2^{-k+1}x)} - \frac{4^{-k}}{\sin^2(2^{-k}x)} \right) = \frac{1}{\sin^2(x)} - \frac{4^{-n}}{\sin^2(2^{-n}x)} \to \frac{1}{\sin^2(x)} - \frac{1}{x^2}$$

□

Exercise 2.1.37: Find a closed form expression for

$$\sum_{r=0}^{n} \frac{(-1)^r}{\binom{n}{r}}$$

Solution: Method One:Brute force with the Beta function

$$\begin{aligned}
\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} &= \sum_{k=0}^{n} (-1)^k \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+1)} = (n+1) \sum_{k=0}^{n} (-1)^k \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \\
&= (n+1) \sum_{k=0}^{n} (-1)^k \, \mathrm{B}(k+1, n-k+1) = (n+1) \int_0^1 \sum_{k=0}^{n} (-1)^k t^k (1-t)^{n-k} \, \mathrm{d}t \\
&= (n+1) \int_0^1 (1-t)^n \frac{1 + \left(\frac{t}{1-t} \right)^{n+1}}{1 + \frac{t}{1-t}} \, \mathrm{d}t = (n+1) \int_0^1 \left[(1-t)^{n+1} + t^{n+1} \right] \mathrm{d}t \\
&= 2\frac{n+1}{n+2}
\end{aligned}$$

Method Two:A more elementary approach

For $k > 0$, partial fractions gives

$$\frac{1}{\binom{n}{k}} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{n-j+1} \tag{2}$$

Therefore,

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{\binom{n}{k}} \tag{3a}$$

$$= 1 + \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{j}{n-j+1} \tag{3b}$$

$$= 1 + \sum_{j=1}^{n}\sum_{k=j}^{n}(-1)^j\binom{k}{j}\frac{j}{n-j+1} \tag{3c}$$

$$= 1 + \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{j}{n-j+1} \tag{3d}$$

$$= 1 + \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(n+1)\,j}{(j+1)(n-j+1)} \tag{3e}$$

$$= 1 + \frac{n+1}{n+2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(\frac{j}{j+1}+\frac{j}{n-j+1}\right) \tag{3f}$$

$$= 1 + \frac{n+1}{n+2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{n}{j+1} \tag{3g}$$

$$= 1 + \frac{n+1}{n+2}\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{n}{n+1} \tag{3h}$$

$$= 1 + \frac{n}{n+2} \tag{3i}$$

$$= 2\frac{n+1}{n+2}$$

Explanation:

(3a): separate the $k = 0$ term.

(3b): apply (2).

(3c): change order of summation.

(3d): $\sum_{k=j}^{n}\binom{k}{j} = \binom{n+1}{j+1}$.

(3e): $\binom{n+1}{j+1} = \binom{n}{j}\frac{n+1}{j+1}$.

(3f): partial fractions.

(3g): substitute $j \mapsto n-j$ in the right summand.

(3h): $\binom{n}{j}\frac{1}{j+1} = \binom{n+1}{j+1}\frac{1}{n+1}$.

(3i): $\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1} = 1$. □

Exercise 2.1.38: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^{n+1}}$$

Solution: Method One:

The Taylor series for $\ln(1+x)$ is

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$

Plugging in $x = \frac{1}{2}$, we get

$$\ln\left(\frac{3}{2}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k2^k}$$

Multiplying by $-\frac{1}{2}$ yields

$$-\frac{1}{2}\ln\left(\frac{3}{2}\right)=\sum_{k=1}^{\infty}(-1)^k\frac{1}{k2^{k+1}}$$

Method Two:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^{n+1}}&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{q^n}{n}\bigg|_{q=-1/2}=\frac{1}{2}\sum_{n=1}^{\infty}\int_0^q t^{n-1}\mathrm{d}t\bigg|_{q=-1/2}=\frac{1}{2}\int_0^q\sum_{n=1}^{\infty}t^{n-1}\mathrm{d}t\bigg|_{q=-1/2}\\&=\frac{1}{2}\int_0^q\frac{1}{1-t}\mathrm{d}t\bigg|_{q=-1/2}=\frac{1}{2}\Big[-\ln(1-t)\Big]_0^q\bigg|_{q=-1/2}=\frac{1}{2}\left(-\ln(1-q)\right)\bigg|_{q=-1/2}\\&=-\frac{1}{2}\ln\frac{3}{2}\end{aligned}$$

□

Exercise 2.1.39: Evaluate:

$$\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{k^2}\right)$$

Solution: Note that

$$\ln(1+ix)=\frac{1}{2}\ln(1+x^2)+i\tan^{-1}(x)\tag{1}$$

So, applying (1) yields

$$\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{k^2}\right)=\Im\left(\ln\left(\prod_{k=1}^{\infty}\left(1-\frac{\alpha^2}{k^2}\right)\right)\right)\tag{2}$$

where $\alpha=\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$, so that $\alpha^2=-i$. Furthermore, the **sinc function** has the following product expansion

$$\prod_{k=1}^{\infty}\left(1-\frac{\alpha^2}{k^2}\right)=\frac{\sin(\pi\alpha)}{\pi\alpha}\tag{3}$$

We also have

$$\begin{aligned}\frac{\sin(x+iy)}{x+iy}&=\frac{\sin(x)\cosh(y)+i\cos(x)\sinh(y)}{x+iy}\\&=\frac{x\sin(x)\cosh(y)+y\cos(x)\sinh(y)}{x^2+y^2}+i\frac{x\cos(x)\sinh(y)-y\sin(x)\cosh(y)}{x^2+y^2}\end{aligned}\tag{4}$$

Combining (2), (3), (4), and using the identity $\tan\left(\frac{\pi}{4}-x\right)=\frac{1-\tan(x)}{1+\tan(x)}$ yields

$$\begin{aligned}\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{k^2}\right)&=\Im\left(\ln\left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)\right)=\tan^{-1}\left(\frac{x\cos(x)\sinh(y)-y\sin(x)\cosh(y)}{x\sin(x)\cosh(y)+y\cos(x)\sinh(y)}\right)\\&=\tan^{-1}\left(\frac{\tan(\frac{\pi}{\sqrt{2}})-\tanh(\frac{\pi}{\sqrt{2}})}{\tan(\frac{\pi}{\sqrt{2}})+\tanh(\frac{\pi}{\sqrt{2}})}\right)=\frac{\pi}{4}-\tan^{-1}\left(\frac{\tanh(\frac{\pi}{\sqrt{2}})}{\tan(\frac{\pi}{\sqrt{2}})}\right)\end{aligned}$$

□

Exercise 2.1.40: Evaluate:

$$\sum_{k=0}^{n} \cos(k\theta)$$

Solution:

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2}\sum_{k=0}^{n}\left(e^{ik\theta} + e^{-ik\theta}\right) \tag{1}$$

$$= \frac{1}{2}\left(\frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1} + \frac{e^{-i(n+1)\theta}-1}{e^{-i\theta}-1}\right) \tag{2}$$

$$= \frac{1}{2}\left(\frac{e^{i(n+1/2)\theta}-e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} + \frac{e^{i\theta/2}-e^{-i(n+1/2)\theta}}{e^{i\theta/2}-e^{-i\theta/2}}\right) \tag{3}$$

$$= \frac{1}{2}\left(\frac{e^{i(n+1/2)\theta}-e^{-i(n+1/2)\theta}}{e^{i\theta/2}-e^{-i\theta/2}} + \frac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}\right) \tag{4}$$

$$= \frac{1}{2}\left(\frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)} + 1\right) \tag{5}$$

Explanation:

(1): rewrite cosine: $\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2}$.

(2): sum of geometric series: $\sum_{k=0}^{n} r^k = \dfrac{r^{n+1}-1}{r-1}$.

(3): multiply the left fraction by $\dfrac{e^{-i\theta/2}}{e^{-i\theta/2}}$ and the right fraction by $\dfrac{-e^{i\theta/2}}{-e^{i\theta/2}}$.

(4): shuffle terms .

(5): rewrite sine: $\sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$. □

Exercise 2.1.41: Evaluate:

$$\sum_{k=1}^{\infty} \frac{(k-1)!^2}{(2k)!}$$

Solution: Let's compute a slightly simpler sum

$$\sum_{k=1}^{\infty} \frac{\Gamma(k)^2}{\Gamma(2k)}(2x)^{2k-1} = \sum_{k=1}^{\infty} \mathrm{B}(k,k)(2x)^{2k-1} = \sum_{k=1}^{\infty}\int_0^1 t^{k-1}(1-t)^{k-1}(2x)^{2k-1}\mathrm{d}t$$

$$= \int_0^1 \frac{2x}{1-4x^2t(1-t)}\mathrm{d}t = \frac{1}{2x}\int_0^1 \frac{1}{t^2-t+\frac{1}{4x^2}}\mathrm{d}t$$

$$= \frac{1}{2x}\int_0^1 \frac{1}{(t-\alpha)(t-1+\alpha)}\mathrm{d}t \text{ where } 2\alpha-1 = \sqrt{1-\frac{1}{x^2}}$$

$$= \frac{1}{2x}\frac{1}{2\alpha-1}\int_0^1\left(\frac{1}{t-\alpha}-\frac{1}{t-1+\alpha}\right)\mathrm{d}t = \frac{1}{2\sqrt{x^2-1}}\left[\ln\left(\frac{\alpha-1}{\alpha}\right)-\ln\left(\frac{\alpha}{\alpha-1}\right)\right]$$

$$= \frac{1}{\sqrt{x^2-1}}\ln\left(\frac{\alpha-1}{\alpha}\right) \text{ where } \frac{\alpha-1}{\alpha} = \left(\sqrt{1-x^2}+ix\right)^2$$

$$= \frac{-2i}{\sqrt{1-x^2}} i\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{2}{\sqrt{1-x^2}}\sin^{-1}(x) \tag{1}$$

Integrating both sides of (1) yields

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{\Gamma(k)^2(2x)^{2k}}{\Gamma(2k+1)} = \left[\sin^{-1}(x)\right]^2 \tag{2}$$

Plugging $x = \dfrac{1}{2}$ into (2) gives

$$\sum_{k=1}^{\infty} \frac{(k-1)!^2}{(2k)!} = \frac{\pi^2}{18}$$

□

Exercise 2.1.42: Prove:

$$\left(\frac{\pi}{3}\right)^2 = 1 + 2\sum_{k=1}^{\infty} \frac{(2k+1)\zeta(2k+2)}{3^{2k+2}}$$

Proof: Exactly the same idea as in the linked similar identity by user "Grigory M" works for this.

http://math.stackexchange.com/questions/611869/

$$\pi \cot(\pi x) = \frac{1}{x} - 2\sum_{n=1}^{\infty} \zeta(2n) x^{2n-1}.$$

Taking derivatives:

$$\frac{\pi^2}{\sin^2(\pi x)} = x^{-2} + 2\sum_{n=1}^{\infty} (2n-1)\zeta(2n) x^{2n-2}$$

Change the index of summation to $2n = 2k + 2$. Thus, write the sum

$$\sum_{n=1}^{\infty} (2n-1)\zeta(2n) x^{2n-2} = \zeta(2) + \sum_{k=1}^{\infty} (2k+1)\zeta(2k+2) x^{2k}.$$

Then $\zeta(2) = \pi^2/6$, and evaluate both sides at $x = 1/3$. You have:

$$\frac{4\pi^2}{3} = 9 + \frac{\pi^2}{3} + 2\sum_{k=1}^{\infty} \frac{(2k+1)\zeta(2k+2)}{3^{2k}}$$

Carrying over the $\pi^2/3$ and dividing both sides by 9 gives:

$$\left(\frac{\pi}{3}\right)^2 = 1 + 2\sum_{n=1}^{\infty} \frac{(2k+1)\zeta(2k+2)}{3^{2k+2}}$$

□

Exercise 2.1.43: Prove the following identity:

$$\sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{120k^2 + 151k + 47}{512k^4 + 1024k^3 + 712k^2 + 194k + 15} \right) = \pi$$

Proof: To begin they note the following definite integrals as summations, $n = 1, \ldots, 7$:

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^{n-1}}{1-x^8} \mathrm{d}x = \int_0^{\frac{1}{\sqrt{2}}} \sum_{k=0}^{\infty} x^{n-1+8k} \mathrm{d}x = \frac{1}{2^{n/2}} \sum_{k=0}^{\infty} \frac{1}{16^k (8k+n)}$$

If the fractional factor of the summation in the Question is expanded by partial fractions:

$$\frac{120k^2 + 151k + 47}{512k^4 + 1024k^3 + 712k^2 + 194k + 15} = \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}$$

then the integrals above can be applied to give:

$$\sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)=\int_0^{\frac{1}{\sqrt{2}}}\frac{4\sqrt{2}-8x^3-4\sqrt{2}x^4-8x^5}{1-x^8}\mathrm{d}x$$

At this point the authors claim a substitution of $y=x\sqrt{2}$, making:

$$\int_0^{\frac{1}{\sqrt{2}}}\frac{4\sqrt{2}-8x^3-4\sqrt{2}x^4-8x^5}{1-x^8}\mathrm{d}x=\int_0^1\frac{16y-16}{y^4-2y^3+4y-4}\mathrm{d}y$$

Finally the last integral may be expanded by partial fractions to give:

$$\int_0^1\frac{4y}{y^2-2}\mathrm{d}y-\int_0^1\frac{4y-8}{y^2-2y+2}\mathrm{d}y=\pi$$

□

Exercise 2.1.44: Evaluate:

$$\sum_{k=1}^{\infty}\left(\zeta(2k)-\beta(2k)\right)$$

Solution:

$$\sum_{k=1}^{\infty}\left(\zeta(2k)-\beta(2k)\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k}}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=2}^{\infty}\frac{1}{n^{2k}}-\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k}}\right)$$

$$=\sum_{n=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^{2k}}-\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k}}\right)=\sum_{n=2}^{\infty}\left(\frac{1}{n^2}\frac{1}{1-\frac{1}{n^2}}+\frac{(-1)^n}{(2n-1)^2}\frac{1}{1-\frac{1}{(2n-1)^2}}\right)$$

$$=\sum_{n=2}^{\infty}\frac{1}{(n+1)(n-1)}+\sum_{n=2}^{\infty}\frac{1}{8n(2n-1)}-\sum_{n=2}^{\infty}\frac{1}{8(2n-1)(n-1)}=\frac{1}{2}+\frac{1}{2}\ln 2$$

□

Exercise 2.1.45: Prove:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1}=-\gamma+\ln(2)$$

Proof: First we note that

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1}=\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(k+1)n^{2k+1}}=\sum_{n=2}^{\infty}\left(-\frac{1}{n}-n\ln\left(1-\frac{1}{n^2}\right)\right)$$

Then

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1}=\sum_{n=2}^{\infty}\left(-\frac{1}{n}-n\ln\left(1-\frac{1}{n^2}\right)\right)$$

$$=\lim_{N\to\infty}\sum_{n=2}^{N}\left(-\frac{1}{n}-n\ln\left(1-\frac{1}{n^2}\right)\right)=\lim_{N\to\infty}\left[-H_N+1-\sum_{n=2}^{N}n\ln(n^2-1)+2\sum_{n=2}^{N}n\ln(n)\right]$$

$$=\lim_{N\to\infty}\left[-H_N+1-\sum_{n=2}^{N}\left(n\ln(n+1)+n\ln(n-1)-2n\ln(n)\right)\right]$$

$$= \lim_{N\to\infty}\left[-H_N+1+\ln(2)-\sum_{n=3}^{N+1}(n-1)\ln(n)-\sum_{n=3}^{N-1}(n+1)\ln(n)+\sum_{n=3}^{N}2n\ln(n)\right]$$
$$= \lim_{N\to\infty}\left[-H_N-N\ln(N+1)-(N-1)\ln(N)+2N\ln(N)+1+\ln(2)\right]$$
$$= \lim_{N\to\infty}\left(-\left(H_N-\ln N\right)+\ln(2)+1-N\ln\left(1+\frac{1}{N}\right)\right)$$
$$= \lim_{N\to\infty}\left(-\left(H_N-\ln N\right)+\ln(2)+\mathcal{O}(N^{-1})\right)$$

Since $\gamma=\lim_{N\to\infty}(H_N-\ln(N))$, we get

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1}=-\gamma+\ln(2)$$

□

Exercise 2.1.46: Evaluate:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+1)(2k+1)}$$

Solution: Start with

$$\sum_{k=1}^{\infty}\frac{x^{2k-1}}{\Gamma(2k)(k+1)(2k+1)}=\sum_{k=1}^{\infty}\frac{4kx^{2k-1}}{(2k+2)!}=2\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{k=1}^{\infty}\frac{x^{2k}}{(2k+2)!}\right]$$
$$=2\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cosh x-1-\frac{x^2}{2}}{x^2}\right]=\frac{e^x-e^{-x}}{x^2}-2\frac{e^x+e^{-x}-2}{x^3}$$

Divide both sides by e^x-1, integrate and use the identity for $\Re(s)>1$:

$$\zeta(s)\Gamma(s)=\int_0^{\infty}\frac{x^{s-1}}{e^x-1}\mathrm{d}x$$

We get:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+1)(2k+1)}=\int_0^{\infty}\left[\frac{1+e^{-x}}{x^2}-2\frac{1-e^{-x}}{x^3}\right]\mathrm{d}x=\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{x}+\frac{1-e^{-x}}{x^2}\right]\mathrm{d}x$$
$$=-\lim_{x\to0}\left[-\frac{1}{x}+\frac{1-e^{-x}}{x^2}\right]=\frac{1}{2}$$

□

Exercise 2.1.47: Prove:

$$\sum_{n=1}^{\infty}\frac{H_{2n+1}}{n^2}=\frac{11}{4}\zeta(3)+\zeta(2)+4\ln(2)-4$$

Proof: In this previous answer

http://math.stackexchange.com/questions/604316/

proved that

$$S(x)=\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n=\zeta(3)+\frac{\ln(1-x)^2\ln(x)}{2}+\ln(1-x)\operatorname{Li}_2(1-x)+\operatorname{Li}_3(x)-\operatorname{Li}_3(1-x)$$

Taking the limit as $x \to 1$ and $x \to -1$ we get :

$$S(1) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\,\zeta(3), \quad S(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n\, H_n}{n^2} = -\frac{5}{8}\zeta(3)$$

To obtain the limit for $S(1)$ use $\mathrm{Li}_n(\epsilon) \sim \epsilon$ as $|\epsilon| \to 0$ and $\mathrm{Li}_n(1) = \zeta(n)$.
For $S(1)$ the method is rather long $(*)$ so let's come back to the integral expression from the previous answer applied to $x = 1$ (using $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$ from $(*)$) :

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2}(-1)^n = \int_0^{-1} \frac{\ln(1-x)^2}{2\,x}\mathrm{d}x + \mathrm{Li}_3(-1) = \int_0^{1} \frac{\ln(1+x)^2}{2\,x}\mathrm{d}x - \frac{3}{4}\zeta(3)$$

Integrals like $a_k := \displaystyle\int_0^1 \frac{\ln(1+x)^k}{x}\mathrm{d}x$ were studied by Nielsen and **Ramanujan** who found that $a_1 = \dfrac{\zeta(2)}{2}$, $a_2 = \dfrac{\zeta(3)}{4}$ so that $S(-1) = \dfrac{1}{2}\dfrac{\zeta(3)}{4} - \dfrac{\zeta(3)}{4} = -\dfrac{5}{8}\zeta(3)$.
We conclude that :

$$S(1) + S(-1) = 2\sum_{m=1}^{\infty} \frac{H_{2m}}{(2m)^2} = 2\,\zeta(3) - \frac{5}{8}\zeta(3) = \frac{11}{8}\zeta(3)$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{2n+1}}{n^2} &= \sum_{n=1}^{\infty} \frac{H_{2n} + \dfrac{1}{2n+1}}{n^2} = \frac{11}{4}\zeta(3) + \sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} \\ &= \frac{11}{4}\zeta(3) + \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{4}{2n} + \frac{4}{2n+1} = \frac{11}{4}\zeta(3) + \sum_{n=1}^{\infty} \frac{1}{n^2} + 4\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \\ &= \frac{11}{4}\zeta(3) + \zeta(2) + 4(\ln(2) - 1)\end{aligned}$$

$(*)$ Alternative method to evaluate $S(1)$ (complicated, left for the record only) :

- $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$ from the second formula $\mathrm{Li}_n(-1) = -\left(1 - \dfrac{1}{2^{n-1}}\right)\zeta(n)$
  http://en.wikipedia.org/wiki/Polylogarithm
- the inversion formula $\mathrm{Li}_3(z) = \mathrm{Li}_3(1/z) + 2\zeta(2)\ln(z) - \dfrac{i\pi}{2}\ln(z)^2 - \dfrac{1}{6}\ln(z)^3$ for $z > 1$ applied to $z = 2$.
- the formula $\mathrm{Li}_3(1/2) = \dfrac{7}{8}\zeta(3) - \dfrac{\zeta(2)\ln(2)}{2} + \dfrac{1}{6}\ln(2)^3$ to conclude that $\mathrm{Li}_3(2) = \dfrac{7}{8}\zeta(3) - \dfrac{i\pi}{2}\ln(2)^2 + \dfrac{\pi^2}{4}\ln(2)$.
  http://en.wikipedia.org/wiki/Polylogarithm
- the same way obtain $\mathrm{Li}_2(2) = -\dfrac{\pi^2}{4} - \dfrac{1}{2}(\ln(2) + i\pi)^2 + \dfrac{1}{2}\ln(2)^2$ from $\mathrm{Li}_2(z) = -\mathrm{Li}_2(1/z) + 2\zeta(2) - i\pi\ln(z) - \dfrac{1}{2}\ln(z)^2$ and the formula for $\mathrm{Li}_2(1/2)$.

□

Exercise 2.1.48: Prove:

$$\frac{\pi^3}{32} = 1 - \sum_{k=1}^{\infty} \frac{2k(2k+1)\zeta(2k+2)}{4^{2k+2}}$$

☞ **Proof:** Using the formula for a geometric series,

$$\sum_{k=1}^{\infty}\frac{1}{x^{2k}}=\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right) \tag{1}$$

Differentiating (1) twice,

$$\sum_{k=1}^{\infty}\frac{2k(2k+1)}{x^{2k+2}}=\frac{1}{(x-1)^3}-\frac{1}{(x+1)^3} \tag{2}$$

Changing the order of summation and applying (2),

$$\begin{aligned}1-\sum_{k=1}^{\infty}\frac{2k(2k+1)\zeta(2k+2)}{4^{2k+2}}&=1-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{2k(2k+1)}{(4j)^{2k+2}}=1-\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{2k(2k+1)}{(4j)^{2k+2}}\\&=1-\sum_{j=1}^{\infty}\left(\frac{1}{(4j-1)^3}-\frac{1}{(4j+1)^3}\right)=1-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)^3}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\end{aligned} \tag{3}$$

The sum in (3) can be generalized as

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s} \tag{4}$$

and is known as the **Dirichlet beta function**. As shown below, $\beta(3)=\frac{\pi^3}{32}$.
We can develop a recurrence for $\beta(2k+1)$. First, the generating function for $\beta(2k+1)$ is

$$\begin{aligned}f(x)&=\sum_{k=0}^{\infty}\beta(2k+1)x^{2k+1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^n\left(\frac{x}{2n+1}\right)^{2k+1}\\&=\sum_{n=0}^{\infty}(-1)^n\frac{\frac{x}{2n+1}}{1-\left(\frac{x}{2n+1}\right)^2}=\frac{x}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2n+1+x}+\frac{1}{2n+1-x}\right)\\&=\frac{x}{2}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{1}{2n+1+x}=\frac{x}{4}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{1}{n+\frac{1+x}{2}}=\frac{x}{4}\pi\csc\left(\pi\frac{1+x}{2}\right)=\frac{\pi}{4}x\sec\left(\frac{\pi}{2}x\right)\end{aligned} \tag{5}$$

where we use (7) from this answer

http://math.stackexchange.com/a/143179

to get

$$\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{n+z}=\sum_{n=-\infty}^{+\infty}\frac{2}{2n+z}-\sum_{n=-\infty}^{+\infty}\frac{1}{n+z}=\pi\cot(\pi z/2)-\pi\cot(\pi z)=\pi\csc(\pi z) \tag{6}$$

We can use equation (5) to develop a recurrence relation:

$$\begin{aligned}\frac{\pi}{4}x=\cos\left(\frac{\pi}{2}x\right)f(x)&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{\left(\frac{\pi}{2}x\right)^{2k}}{(2k)!}\beta(2n-2k+1)x^{2n-2k+1}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(-\pi^2/4\right)^k}{(2k)!}\beta(2n-2k+1)x^{2n+1}\end{aligned} \tag{7}$$

For $n = 0$, we can use the arctan series to get

$$\beta(1) = \frac{\pi}{4} \tag{8}$$

and for $n > 0$, (7) gives

$$\beta(2n+1) = -\sum_{k=1}^{n} \frac{(-\pi^2/4)^k}{(2k)!}\, \beta(2n-2k+1) \tag{9}$$

Recursion (9) yields

$$\begin{aligned}
\beta(1) &= \frac{\pi}{4} \\
\beta(3) &= \frac{\pi^3}{32} \\
\beta(5) &= \frac{5\pi^5}{1536} \\
\beta(7) &= \frac{61\pi^7}{184320} \\
\beta(9) &= \frac{277\pi^9}{8257536} \\
\beta(11) &= \frac{50521\pi^{11}}{14863564800} \\
\beta(13) &= \frac{540553\pi^{13}}{1569592442880} \\
\beta(15) &= \frac{199360981\pi^{15}}{5713316492083200} \\
\beta(17) &= \frac{3878302429\pi^{17}}{1096956766479974400} \\
\beta(19) &= \frac{2404879675441\pi^{19}}{6713375410857443328000}
\end{aligned}$$

□

Exercise 2.1.49: Prove:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n x) + x^2 \coth(\pi n/x)}{n^3} = \frac{\pi^3}{90x}\left(1 + 5x^2 + x^4\right)$$

Proof: Using contour integration is to consider

$$f(z) = \frac{\pi \cot(\pi z)\coth(\pi x z)}{z^3}$$

Around circles centered at the origin that avoid the poles on the real and imaginary axes, $\int f(z)\, dz$ will vanish as the radii of such circles go to ∞.
Summing the residues,

$$2\sum_{n=1}^{\infty} \frac{\coth(n\pi x)}{n^3} + \sum_{n=1}^{\infty} \frac{\cot\left(\dfrac{in\pi}{x}\right)}{x\left(\dfrac{in}{x}\right)^3} + \sum_{n=1}^{\infty} \frac{\cot\left(-\dfrac{in\pi}{x}\right)}{x\left(-\dfrac{in}{x}\right)^3} + \mathrm{Res}[f(z), 0] = 0,$$

which implies

$$\sum_{n=1}^{\infty}\frac{\coth(n\pi x)}{n^3}+x^2\sum_{n=1}^{\infty}\frac{\coth\left(\frac{n\pi}{x}\right)}{n^3}=-\frac{1}{2}\,\mathrm{Res}[f(z),0].$$

Expanding at the origin,

$$f(z)=\frac{1}{\pi x}\frac{1}{z^5}+\frac{\pi(x^2-1)}{3x}\frac{1}{z^3}-\frac{\pi^3(x^4+5x^2+1)}{45x}\frac{1}{z}+\mathcal{O}(z).$$

Therefore,

$$\mathrm{Res}[f(z),0]=-\frac{\pi^3}{45x}\left(x^4+5x^2+1\right).$$

And

$$\sum_{n=1}^{\infty}\frac{\coth(\pi nx)+x^2\coth(\pi n/x)}{n^3}=\frac{\pi^3}{90x}\left(1+5x^2+x^4\right)$$

□

Exercise 2.1.50: Prove the following identity:

$$\sum_{n=1}^{\infty}\frac{(m-1)^n-1}{m^n}\zeta(n+1)=\pi\cot\left(\frac{\pi}{m}\right)$$

Proof: Using the identity (7) proven in this answer,

http://math.stackexchange.com/a/143179

we get

$$\sum_{n=1}^{\infty}\frac{(m-1)^n-1}{m^n}\zeta(n+1)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(m-1)^n-1}{m^n}\frac{1}{k^{n+1}}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{(m-1)^n}{m^nk^n}-\frac{1}{m^nk^n}\right)=\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\frac{m-1}{mk}}{1-\frac{m-1}{mk}}-\frac{\frac{1}{mk}}{1-\frac{1}{mk}}\right)$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{\frac{1}{m}-k}+\frac{1}{\frac{1}{m}+k-1}\right)=\sum_{k\in\mathbb{Z}}\frac{1}{\frac{1}{m}+k}=\pi\cot\left(\frac{\pi}{m}\right)$$

or we can use the **series for digamma** and the **reflection formula** to get

http://en.wikipedia.org/wiki/Digamma_function

$$=\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{\left(\frac{m-1}{m\,k}\right)}{1-\left(\frac{m-1}{m\,k}\right)}-\frac{\left(\frac{1}{m\,k}\right)}{1-\left(\frac{1}{m\,k}\right)}\right]=\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{m-1}{mk-m+1}-\frac{1}{m\,k-1}\right]$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1-\frac{1}{m}}{k-\left(1-\frac{1}{m}\right)}-\frac{\frac{1}{m}}{k-\frac{1}{m}}\right]=\psi\left(1-\frac{1}{m}\right)-\psi\left(\frac{1}{m}\right)=\pi\cot\left(\frac{\pi}{m}\right)$$

□

Exercise 2.1.51: Prove:

$$\sum_{n=1}^{\infty}\arctan\frac{10n}{(3n^2+2)\,(9n^2-1)}=\ln 3-\frac{\pi}{4}$$

☞ **Proof:** Since $\arctan(x) = \arg(1+ix)$ and we can factor

$$1+\frac{10in}{(3n^2+2)(9n^2-1)} = \frac{\left(1-\frac{i}{n}\right)\left(1+\frac{i}{3n-1}\right)\left(1+\frac{i}{3n+1}\right)\left(1+\frac{i}{3n}\right)}{1+\frac{2}{3n^2}}$$

we have that

$$\arctan\left(\frac{10n}{(3n^2+2)(9n^2-1)}\right) = \arctan\left(\frac{1}{3n-1}\right)+\arctan\left(\frac{1}{3n}\right)+\arctan\left(\frac{1}{3n+1}\right)-\arctan\left(\frac{1}{n}\right)$$

Now this becomes a telescoping series:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\arctan\left(\frac{10n}{(3n^2+2)(9n^2-1)}\right)\\
&=\lim_{m\to\infty}\sum_{n=1}^{m}\left[\arctan\left(\frac{1}{3n-1}\right)+\arctan\left(\frac{1}{3n}\right)+\arctan\left(\frac{1}{3n+1}\right)-\arctan\left(\frac{1}{n}\right)\right]\\
&=-\arctan(1)+\lim_{m\to\infty}\sum_{n=m+1}^{3m+1}\arctan\left(\frac{1}{n}\right)=-\arctan(1)+\lim_{m\to\infty}\sum_{n=m+1}^{3m+1}\left[\frac{1}{n}+O\left(\frac{1}{n^3}\right)\right]\\
&=\ln(3)-\frac{\pi}{4}
\end{aligned}$$

□

Exercise 2.1.52:

✎ **Solution:**

□

Exercise 2.1.53:

✎ **Solution:**

□

Exercise 2.1.54:

✎ **Solution:**

□

Exercise 2.1.55:

✎ **Solution:**

□

Exercise 2.1.56:

✎ **Solution:**

□

Exercise 2.1.57:

✎ **Solution:**

□

Exercise 2.1.58:

✎ **Solution:**

□

Exercise 2.1.59:

✎ **Solution:**

□

Exercise 2.1.60:

✎ **Solution:**

□

*

2.2 Integrals and Series

✎ *

Exercise 2.2.1: Evaluate:

$$\sum_{n=1}^{\infty} \psi_1\left(1+\frac{n}{2}\right) t^n$$

✎ **Solution:**

$$\begin{aligned}\sum_{n=1}^{\infty} \psi_1\left(1+\frac{n}{2}\right) t^n &= -\sum_{n=1}^{\infty} t^n \int_0^1 \frac{(\sqrt{x})^n \ln(x)}{1-x}\mathrm{d}x \\ &= -\int_0^1 \frac{\ln(x)}{1-x}\frac{t\sqrt{x}}{1-t\sqrt{x}}\mathrm{d}x\end{aligned}$$

Now substitute $u=\sqrt{x}$ and use partial fractions to get

$$\begin{aligned}\sum_{n=1}^{\infty} \psi_1\left(1+\frac{n}{2}\right) t^n &= -4t\int_0^1 \frac{u^2\ln(u)}{(1-u^2)(1-tu)}\mathrm{d}u \\ &= -\frac{4t}{1-t^2}\int_0^1 \ln u\left(\frac{1+ut}{1-u^2}-\frac{1}{1-tu}\right)\mathrm{d}u\end{aligned}$$

Now since $\displaystyle\int_0^1 \frac{\ln u}{1-tu}\mathrm{d}u = -\frac{\mathrm{Li}_2(t)}{t}$

$$\begin{aligned}\sum_{n=1}^{\infty} \psi_1\left(1+\frac{n}{2}\right) t^n &= -\frac{4t}{1-t^2}\int_0^1 \frac{(1+ut)\ln u}{1-u^2}\mathrm{d}u - \frac{4\mathrm{Li}_2(t)}{1-t^2} \\ &= -\frac{4t}{1-t^2}\sum_{j=0}^{\infty}\int_0^1 u^{2j}(1+ut)\ln(u)\,\mathrm{d}u - \frac{4\mathrm{Li}_2(t)}{1-t^2} \\ &= \frac{4t}{1-t^2}\sum_{j=0}^{\infty}\left\{\frac{1}{(2j+1)^2}+\frac{t}{4}\frac{1}{(j+1)^2}\right\} - \frac{4\mathrm{Li}_2(t)}{1-t^2} \\ &= \frac{4t}{1-t^2}\left(\frac{\pi^2}{8}+\frac{\pi^2}{6}\frac{t}{4}\right) - \frac{4\mathrm{Li}_2(t)}{1-t^2} \\ &= \frac{\pi^2}{6}\frac{t(t+3)}{1-t^2} - \frac{4\mathrm{Li}_2(t)}{1-t^2}\end{aligned}$$

□

Exercise 2.2.2: Evaluate:

$$\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{1+n}{2}\right)-\psi_1\left(1+\frac{n}{2}\right)\right\}\frac{(-1)^n}{n2^n}$$

✎ Solution:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{1+n}{2}\right)-\psi_1\left(1+\frac{n}{2}\right)\right\}\frac{t^n}{n}\\
&=4\int_0^t\frac{\mathrm{Li}_2(x)}{x(1+x)}\mathrm{d}x-\frac{\pi^2}{3}\int_0^t\frac{1}{1+x}\mathrm{d}x\\
&=-\frac{\pi^2}{3}\ln(1+t)+4\mathrm{Li}_3(t)-4\int_0^t\frac{\mathrm{Li}_2(x)}{1+x}\mathrm{d}x\\
&=-\frac{\pi^2}{3}\ln(1+t)+4\mathrm{Li}_3(t)-4\ln(1+t)\mathrm{Li}_2(t)-4\int_0^t\frac{\ln(1+x)\ln(1-x)}{x}\mathrm{d}x
\end{aligned}\tag{1}$$

According to Lewin's book(Page no 270),

$$\begin{aligned}
&\int_0^x\frac{\ln(1+t)\ln(1-t)}{t}\mathrm{d}t\\
&=\mathrm{Li}_3\left(\frac{1+x}{1-x}\right)+\mathrm{Li}_3(-1)+\mathrm{Li}_3(1)-\mathrm{Li}_3(1+x)-\mathrm{Li}_3(1-x)-\mathrm{Li}_3\left(-\frac{1+x}{1-x}\right)\\
&\quad+\ln(1+x)\left\{\mathrm{Li}_2(-1)-\mathrm{Li}_2(x)\right\}+\ln(1-x)\left\{\mathrm{Li}_2(1+x)-\mathrm{Li}_2(-1)+\mathrm{Li}_2(1)\right\}\\
&\quad-\frac{i\pi}{2}\ln^2(1-x)
\end{aligned}\tag{2}$$

Put $x=-\frac{1}{2}$:

$$\begin{aligned}
&-\int_{-1/2}^0\frac{\ln(1+t)\ln(1-t)}{t}\mathrm{d}t\\
&=-\mathrm{Li}_3\left(-\frac{1}{3}\right)+\mathrm{Li}_3\left(\frac{1}{3}\right)-\mathrm{Li}_3\left(\frac{3}{2}\right)+\mathrm{Li}_2\left(-\frac{1}{2}\right)\ln(2)\\
&\quad-\frac{5\zeta(3)}{8}-\frac{1}{2}i\pi\ln^2\left(\frac{3}{2}\right)-\frac{1}{6}\ln^2(2)\ln\left(\frac{27}{4}\right)+\frac{1}{6}\pi^2\ln\left(\frac{9}{2}\right)
\end{aligned}$$

At this point, we can make some simplifications using the identities

$$\mathrm{Li}_3\left(\frac{3}{2}\right)+\mathrm{Li}_3\left(-\frac{1}{2}\right)+\mathrm{Li}_3\left(\frac{1}{3}\right)=\zeta(3)+\frac{1}{6}\ln\left(\frac{3}{2}\right)\left(\pi^2-3i\pi\ln\left(\frac{3}{2}\right)+\ln\left(\frac{3}{2}\right)\ln(12)\right)$$

$$\mathrm{Li}_3\left(-\frac{1}{3}\right)-2\mathrm{Li}_3\left(\frac{1}{3}\right)=-\frac{\ln^3 3}{6}+\frac{\pi^2}{6}\ln 3-\frac{13\zeta(3)}{6}$$

giving us:

$$\begin{aligned}
&-\int_{-1/2}^0\frac{\ln(1+t)\ln(1-t)}{t}\mathrm{d}t\\
&=2\mathrm{Li}_3\left(\frac{1}{3}\right)-\mathrm{Li}_3\left(-\frac{1}{3}\right)+\mathrm{Li}_2\left(-\frac{1}{2}\right)\ln(2)+\mathrm{Li}_3\left(-\frac{1}{2}\right)+\frac{\pi^2}{6}\ln 3-\frac{13}{8}\zeta(3)\\
&\quad-\frac{\ln^3 3}{6}\\
&=\frac{13}{24}\zeta(3)+\mathrm{Li}_2\left(-\frac{1}{2}\right)\ln(2)+\mathrm{Li}_3\left(-\frac{1}{2}\right)
\end{aligned}\tag{3}$$

Put $t=-\frac{1}{2}$ in equation (1) to get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{1+n}{2}\right)-\psi_1\left(1+\frac{n}{2}\right)\right\}\frac{(-1)^n}{n2^n}\\
&=\frac{\pi^2}{3}\ln(2)+4\mathrm{Li}_2\left(-\frac{1}{2}\right)\ln(2)+4\mathrm{Li}_3\left(-\frac{1}{2}\right)+4\int_{-1/2}^0\frac{\ln(1+t)\ln(1-t)}{t}\mathrm{d}t
\end{aligned}\tag{4}$$

Combining this result with equation (3), we finally get

$$\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{1+n}{2}\right)-\psi_1\left(1+\frac{n}{2}\right)\right\}\frac{(-1)^n}{n2^n}=\frac{\pi^2}{3}\ln(2)-\frac{13}{6}\zeta(3)$$

□

Exercise 2.2.3: Prove:

$$\sum_{n=0}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}=\frac{21}{16}\zeta(3)$$

Proof: Let $S_1=\sum_{n=1}^{\infty}\frac{H_n}{n^2}$ and $S_2=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n^2}$. Then, our sum can be written as

$$\sum_{n=0}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}=\frac{S_1+S_2}{2}$$

We need to find S_1 and S_2.

1. Calculation of S_1

Note that

$$\frac{1}{k^2}=\int_0^1\int_0^1(xy)^{k-1}\mathrm{d}x\mathrm{d}y\ ,\ \frac{1}{n}=\int_0^1 z^{n-1}\mathrm{d}z$$

With the help of these, S_1 can be calculated.

$$\begin{aligned}
S_1&=\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{n=1}^{k}\frac{1}{n}=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_0^1\int_0^1(xy)^{k-1}\mathrm{d}x\ \mathrm{d}y\int_0^1 z^{n-1}\mathrm{d}z\\
&=\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\int_0^1\int_0^1(xy)^{k-1}\mathrm{d}x\ \mathrm{d}y\int_0^1 z^{n-1}\mathrm{d}z=\int_0^1\int_0^1\int_0^1\left(\sum_{n=1}^{\infty}\frac{(xyz)^{n-1}}{1-xy}\right)\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\\
&=\int_0^1\int_0^1\int_0^1\frac{1}{(1-xy)(1-xyz)}\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z=\int_0^1\int_0^1\frac{\ln(1-xy)}{xy(xy-1)}\mathrm{d}x\ \mathrm{d}y\\
&=-\int_0^1\int_0^1\frac{\ln(1-xy)}{xy}\mathrm{d}x\mathrm{d}y-\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}\mathrm{d}x\ \mathrm{d}y\\
&=-\int_0^1\int_0^1\frac{\ln(1-xy)}{xy}\mathrm{d}x\mathrm{d}y+\int_0^1\frac{\ln^2(1-y)}{2y}\mathrm{d}y
\end{aligned}$$

We have

$$\begin{aligned}
\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}\mathrm{d}x\ \mathrm{d}y&=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1\int_0^1(xy)^{n-1}\mathrm{d}x\ \mathrm{d}y\\
&=-\sum_{n=1}^{\infty}\frac{1}{n^3}=-\zeta(3)
\end{aligned}$$

and

$$\int_0^1\frac{\ln^2(1-y)}{2y}\mathrm{d}y=\zeta(3)$$

So

$$S_1=2\zeta(3)$$

2. Calculation of S_2

This can be done in the same way as the previous one. We will use

$$\frac{(-1)^{k-1}}{k^2}=\int_0^1(-x)^{k-1}\mathrm{d}x\int_0^1 z^{k-1}\mathrm{d}z=(-1)^{k-1}\int_0^1\int_0^1(xz)^{k-1}\mathrm{d}x\mathrm{d}z$$

Proceeding like the previous one we have

$$\begin{aligned}S_2 &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\sum_{n=1}^{k}\frac{1}{n} = \sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_0^1\int_0^1(-1)^{k-1}(xz)^{k-1}\mathrm{d}x\mathrm{d}z\int_0^1 y^{n-1}\mathrm{d}y\\ &= \int_0^1\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{(-xyz)^{n-1}}{1+xz}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1\int_0^1\int_0^1\frac{1}{(1+xz)(1+xyz)}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\int_0^1\frac{\ln(1+xz)}{xz(1+xz)}\mathrm{d}x\mathrm{d}z = \int_0^1\int_0^1\frac{\ln(1+xz)}{xz}\mathrm{d}x\mathrm{d}z - \int_0^1\int_0^1\frac{\ln(1+xz)}{1+xz}\mathrm{d}x\mathrm{d}z\\ &= \int_0^1\int_0^1\frac{\ln(1+xz)}{xz}\mathrm{d}x\mathrm{d}z - \int_0^1\frac{\ln^2(1+z)}{2z}\mathrm{d}z\end{aligned}$$

Here

$$\begin{aligned}\int_0^1\int_0^1\frac{\ln(1+xz)}{xz}\mathrm{d}x\mathrm{d}z &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1\int_0^1(xz)^{n-1}\mathrm{d}x\mathrm{d}z\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = \frac{3}{4}\zeta(3)\end{aligned}$$

and

$$\int_0^1\frac{\ln^2(1+z)}{2z}\mathrm{d}z = \frac{\zeta(3)}{8}$$

So

$$S_2 = \frac{5}{8}\zeta(3)$$

3. Final answer

$$\sum_{n=0}^{\infty}\frac{H_{2n+1}}{(2n+1)^2} = \frac{S_1+S_2}{2} = \frac{2\zeta(3)+\frac{5}{8}\zeta(3)}{2} = \frac{21}{16}\zeta(3)$$

Note the identity

$$\psi(2n+2)+\gamma = H_{2n+1}$$

This gives us:

$$\sum_{n=0}^{\infty}\frac{\psi(2n+2)+\gamma}{(2n+1)^2} = \sum_{n=0}^{\infty}\frac{\psi(2n+2)}{(2n+1)^2} + \gamma\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

The rightmost sum is rather famous, and evaluates to $\frac{\gamma\pi^2}{8}$.
The left sum with the digamma term evaluates to

$$\sum_{n=0}^{\infty}\frac{\psi(2n+2)}{(2n+1)^2} = \frac{21}{16}\zeta(3) - \frac{\gamma\pi^2}{8}$$

□

Exercise 2.2.4: Prove:

$$\sum_{n=1}^{\infty}\frac{H_n}{2^n n^2} = -\frac{\pi^2}{12}\ln(2)+\zeta(3)$$

Proof:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n = \int_0^x\frac{\mathrm{Li}_2(t)+\frac{1}{2}\ln^2(1-t)}{t}\mathrm{d}t$$

By letting $x = 1/2$ and integrating we get:

$$\int_0^x \frac{\mathrm{Li}_2(t) + \frac{1}{2}\ln^2(1-t)}{t}\mathrm{d}t = -\mathrm{Li}_3(1-t) + \mathrm{Li}_3(t) + \mathrm{Li}_2(1-t)\ln(1-t) + \frac{1}{2}\ln(t)\ln^2(1-t)\Big|_0^{1/2}$$

Note that

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{1}{6}\ln^3(2) - \frac{\pi^2}{12}\ln(2)$$
$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2}$$

So, we arrive at:

$$\sum_{n=1}^{\infty}\frac{H_n}{2^n n^2} = -\frac{\pi^2}{12}\ln(2) + \zeta(3)$$

□

Exercise 2.2.5: Prove:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\left(\frac{\pi(2n+1)}{2}\right)} = \frac{\pi}{8}$$

Proof: Since the series is alternating, use $\pi\csc(\pi z)$.

And we have $\oint \frac{\pi\csc(\pi z)}{(2z+1)\cosh\left(\frac{\pi}{2}(2z+1)\right)}\mathrm{d}z$.

The poles are at $z = -1/2,\ \ z = n,\ \ z = \frac{(2n+1)i}{2} - \frac{1}{2}$.

$$\mathrm{Res}\left(z = -\frac{1}{2}\right) = \frac{-\pi}{2}$$
$$\mathrm{Res}(z = n) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\left(\frac{\pi}{2}(2n+1)\right)}$$
$$\mathrm{Res}\left(z = \frac{(2n+1)i}{2} - \frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\left(\frac{\pi}{2}(2n+1)\right)}$$

Hence:

$$\oint \frac{\pi\csc(\pi z)}{(2z+1)\cosh\left(\frac{\pi}{2}(2z+1)\right)}\mathrm{d}z = \frac{-\pi}{2} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\left(\frac{\pi}{2}(2n+1)\right)}$$

As $N \to \infty$, then the integral on the left $\to 0$, so, we have $0 = -\frac{\pi}{2} + 4S$. Solving for S gives

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\left(\frac{\pi(2n+1)}{2}\right)} = \frac{\pi}{8}$$

□

Exercise 2.2.6: Prove:

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1}=\mathbf{G}-\frac{\pi}{2}\ln(2)$$

Proof: Method One:

Using the relation $H_n=\int_0^1\frac{1-x^n}{1-x}\mathrm{d}x$, we find that the series reduces to

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1}&=\int_0^1\frac{2}{1-x^2}\left(\frac{\pi x}{4}-\arctan x\right)\mathrm{d}x\\&=\int_0^1\frac{2}{1-x^2}\left(\arctan\left(\frac{1-x}{1+x}\right)-\frac{\pi(1-x)}{4}\right)\mathrm{d}x\\&=\int_0^1\frac{2}{1-x^2}\arctan\left(\frac{1-x}{1+x}\right)\mathrm{d}x-\int_0^1\frac{\pi}{2(1+x)}\mathrm{d}x\end{aligned}$$

For the former one, we use the substitution $t=\frac{1-x}{1+x}$ to obtain that

$$\int_0^1\frac{2}{1-x^2}\arctan\left(\frac{1-x}{1+x}\right)\mathrm{d}x=\int_0^1\frac{\arctan t}{t}\mathrm{d}t=\mathbf{G}$$

The latter one reduces to $-\frac{\pi}{2}\ln 2$, so the conclusion follows.

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1}=\mathbf{G}-\frac{\pi}{2}\ln(2)$$

Method Two:

$$\sum_{n=1}^{\infty}(-1)^nH_nt^n=\frac{-\ln(1+t)}{1+t}$$

Let $t=x^2$

$$\sum_{n=1}^{\infty}(-1)^nH_nx^{2n}=\frac{-\ln(1+x^2)}{1+x^2}$$

Integrating with respect to x and interchanging integral and summation we get

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1x^{2n}\mathrm{d}x&=-\int_0^1\frac{\ln(1+x^2)}{1+x^2}\mathrm{d}x\\\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1}&=-\int_0^1\frac{\ln(1+x^2)}{1+x^2}\mathrm{d}x\end{aligned}$$

In the integral, substitute $x=\tan(\theta)$:

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1}=2\int_0^{\frac{\pi}{4}}\ln(\cos\theta)\mathrm{d}\theta$$

The remaining integral is evaluated using a fourier series:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1}&=2\int_0^{\frac{\pi}{4}}\left(-\ln(2)-\sum_{j=1}^{\infty}\frac{(-1)^j\cos(2j\theta)}{j}\right)\mathrm{d}\theta\\&=-\frac{\pi}{2}\ln(2)-2\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\int_0^{\frac{\pi}{4}}\cos(2j\theta)\mathrm{d}\theta\\&=-\frac{\pi}{2}\ln(2)+\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^2}\sin\left(\frac{\pi j}{2}\right)\\&=-\frac{\pi}{2}\ln(2)+\mathbf{G}\end{aligned}$$

□

Exercise 2.2.7: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}=\frac{\pi^2}{18}$$

Proof: It requires the identity

$$\frac{2x\arcsin x}{\sqrt{1-x^2}}=\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n\binom{2n}{n}}$$

Divide by x and integrate both sides:

$$2(\arcsin x)^2=\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}}$$

Put $x=\frac{1}{2}$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}=\frac{\pi^2}{18}$$

□

Exercise 2.2.8: Prove:

$$\sum_{k=1}^{\infty}\frac{4^k}{\binom{2k}{k}(4k^2-1)}=\mathbf{G}+\frac{1}{2}$$

Proof:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{4^k}{\binom{2k}{k}(4k^2-1)}&=\sum_{k=1}^{\infty}\frac{4^k}{2k-1}\int_0^1(y-y^2)^k\mathrm{d}y\\&=\int_0^1\left(\sum_{k=1}^{\infty}\frac{(4y-4y^2)^k}{2k-1}\right)\mathrm{d}y\\&=\int_0^1\sqrt{4y-4y^2}\operatorname{artanh}(\sqrt{4y-4y^2})\mathrm{d}y\end{aligned}$$

Let S denote the sum. Plugging $y=\sin\theta$ followed by the substitution $t=\tan\theta$, we obtain

$$\begin{aligned}S&=\int_0^{\frac{\pi}{2}}\sin^2 2\theta\operatorname{artanh}\sin 2\theta\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\sin^2\theta\operatorname{artanh}\sin\theta\,\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{2}}\sin^2\theta\ln(\tan\theta+\sec\theta)\,\mathrm{d}\theta=\int_0^{\infty}\frac{t^2\ln(t+\sqrt{t^2+1})}{(t^2+1)^2}\,\mathrm{d}t\end{aligned}$$

Now with the substitution $t=\sinh s$, we find that

$$\begin{aligned}S&=\int_0^{\infty}\frac{s\sinh^2 s}{\cosh^3 s}\mathrm{d}s\\&=\left[-\frac{1}{2}\frac{s\sinh s}{\cosh^2 s}\right]_0^{\infty}+\frac{1}{2}\int_0^{\infty}\frac{\sinh s}{\cosh^2 s}\mathrm{d}s+\frac{1}{2}\int_0^{\infty}\frac{s}{\cosh s}\mathrm{d}s\\&=\frac{1}{2}+\mathbf{G}\end{aligned}$$

Here, we utilized nearly trivial calculations

$$\frac{1}{2}\int_0^\infty \frac{\sinh s}{\cosh^2 s}\mathrm{d}s = \left[-\frac{1}{2\cosh s}\right]_0^\infty = \frac{1}{2},$$
$$\frac{1}{2}\int_0^\infty \frac{s}{\cosh s}\mathrm{d}s = \frac{1}{2}\int_0^\infty \frac{se^{-s}}{1+e^{-2s}}\mathrm{d}s = \sum_{n=0}^\infty (-1)^n \int_0^\infty se^{-(2n+1)s}\mathrm{d}s = \mathbf{G}$$

□

Exercise 2.2.9: Evaluate:

$$\sum_{m,n\geq 1}\frac{1}{m^2(m^2+n^2)}$$

Solution: **Method One:**

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{m^2(m^2+n^2)} = \sum_{m=1}^\infty\left(\frac{-1}{2m^4}+\frac{\pi\coth(\pi m)}{2m^3}\right)$$
$$= -\frac{1}{2}\left(\frac{\pi^4}{90}\right)+\frac{\pi}{2}\sum_{m=1}^\infty\frac{\coth(\pi m)}{m^3}$$

The Series $\sum_{m=1}^\infty \frac{\coth(\pi m)}{m^3}$ is famous and can be evaluated using residues. I am sure you are familiar with it. It turns out that

$$\sum_{m=1}^\infty\frac{\coth(\pi m)}{m^3} = \frac{7\pi^3}{180}$$

We get

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{m^2(m^2+n^2)} = \frac{5\pi^4}{360} = \frac{\pi^4}{72}$$

Method Two:

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{m^2(m^2+n^2)} = \frac{1}{2}\sum_{m=1}^\infty\sum_{n=1}^\infty\left(\frac{1}{m^2(m^2+n^2)}+\frac{1}{n^2(m^2+n^2)}\right)$$
$$= \frac{1}{2}\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{1}{m^2n^2} = \frac{1}{2}\zeta(2)^2$$

□

Exercise 2.2.10: Evaluate:

$$\sum_{n=0}^\infty\binom{3n}{n}\frac{1}{8^n}$$

Solution: $\binom{3n}{n}$ is the coefficient of z^n in $(1+z)^{3n}$. So, it's the residue at $z=0$ of $\frac{(1+z)^{3n}}{z^{n+1}}$. So, what we can do is write

$$\sum_{n=0}^\infty\binom{3n}{n}\frac{1}{8^n} = \frac{1}{2\pi i}\int_C\sum_{n=0}^\infty\frac{(1+z)^{3n}}{(8z)^n}\cdot\frac{1}{z}\mathrm{d}z$$

where

$$\frac{1}{1-\left(\frac{(1+z)^3}{8z}\right)}\cdot\frac{1}{z} = \frac{-8}{z^3+3z^2-5z+1} = \frac{-8}{2\pi i}\int_C\frac{1}{z^3+3z^2-5z+1}\mathrm{d}z$$

The zeros are $z=\sqrt{5}-2,\quad -\sqrt{5}-2,\quad z=1$. Only one lies inside the contour, $\sqrt{5}-2$. So, the residue is $\dfrac{-8}{(z-1)(z+(-2-\sqrt{5}))}=\dfrac{3\sqrt{5}+5}{5}$, hence

$$\sum_{n=0}^{\infty}\binom{3n}{n}\frac{1}{8^n}=\frac{3\sqrt{5}+5}{5}$$

A symbolic evaluation through mathematica gives

$$\sum_{n=0}^{\infty}\binom{3n}{n}x^n=\frac{2\cos\left(\frac{1}{3}\arcsin\left(\frac{3\sqrt{3x}}{2}\right)\right)}{\sqrt{4-27x}}$$

□

Exercise 2.2.11: Evaluate:

$$\sum_{n=1}^{\infty}\frac{H_n}{\binom{2n}{n}}$$

Solution: Let us denote the sum as S. Then by the beta function identity, we have

$$S=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{n\mathrm{B}(n,n+1)}{m}=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{n}{m}\int_0^1 x^{n-1}(1-x)^n\mathrm{d}x$$

Now switching the order of summation and utilizing some power series identities, we obtain

$$\begin{aligned}
S&=\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{n}{m}\int_0^1 x^n(1-x)^n\,\frac{\mathrm{d}x}{x}\\
&=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{n+m}{m}\int_0^1 x^m(1-x)^m x^n(1-x)^n\,\frac{\mathrm{d}x}{x}\\
&=\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 x^m(1-x)^m\left(\frac{x(1-x)}{(1-x+x^2)^2}+\frac{m}{1-x+x^2}\right)\frac{\mathrm{d}x}{x}\\
&=\int_0^1\left(-\frac{x(1-x)}{(1-x+x^2)^2}\ln(1-x+x^2)+\frac{x(1-x)}{(1-x+x^2)^2}\right)\frac{\mathrm{d}x}{x}\\
&=\int_0^1\frac{1-x}{(1-x+x^2)^2}\left\{1-\ln(1-x+x^2)\right\}\mathrm{d}x
\end{aligned}$$

Splitting the integrand into the symmetric part and the anti-symmetric part with respect to the transform $x\mapsto 1-x$, we find that

$$S=\frac{1}{2}\int_0^1\frac{1-\ln(1-x+x^2)}{(1-x+x^2)^2}\mathrm{d}x$$

Now, we use the substitution $x-\dfrac{1}{2}=\dfrac{\sqrt{3}}{2}\tan\theta$. Then S reduces to

$$S=\frac{8}{3\sqrt{3}}\int_0^{\frac{\pi}{6}}\left\{1+2\ln\left(\frac{2\cos\theta}{\sqrt{3}}\right)\right\}\cos^2\theta\mathrm{d}\theta$$

By some tedious calculation (integration by parts and cosine double angle formulas are sufficient), it easily follows that

$$\begin{aligned}
&\int\left\{1+2\ln\left(\frac{2\cos\theta}{\sqrt{3}}\right)\right\}\cos^2\theta\,\mathrm{d}\theta\\
&=\theta(1-\ln 2\cos\theta)+\left(\theta+\frac{\sin 2\theta}{2}\right)\ln\left(\frac{2\cos\theta}{\sqrt{3}}\right)+\int\ln(2\cos\theta)\,\mathrm{d}\theta
\end{aligned}$$

yielding

$$S=\frac{8}{3\sqrt{3}}\left[\frac{\pi}{6}\left(1-\frac{1}{2}\ln 3\right)+\int_0^{\frac{\pi}{6}}\ln(2\cos\theta)\,\mathrm{d}\theta\right]$$

Now, from the identity

$$\ln(2\cos\theta)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos 2n\theta$$

we have

$$\int_0^{\frac{\pi}{6}}\ln(2\cos\theta)\,\mathrm{d}\theta=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n^2}\sin\left(\frac{\pi n}{3}\right)=\frac{1}{12\sqrt{3}}\left(\psi_1\left(\frac{1}{3}\right)-\psi_1\left(\frac{2}{3}\right)\right)$$

Putting together, we obtain

$$S=\frac{2}{27}\left\{\sqrt{3}\pi(2-\ln 3)+\psi_1\left(\frac{1}{3}\right)-\psi_1\left(\frac{2}{3}\right)\right\}$$

□

Exercise 2.2.12: Evaluate:

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)^2}$$

Solution:

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)^2}=4\sum_{n=1}^{\infty}\frac{1}{2n(2n-1)\binom{2n}{n}}=\int_0^{1/2}\frac{1}{x^2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n\binom{2n}{n}}\mathrm{d}x$$
$$=\int_0^{1/2}\frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}}\cdot\frac{1}{x^2}\mathrm{d}x=2\int_0^{\frac{\pi}{6}}\frac{t}{\sin(t)}\mathrm{d}t$$

Notice

$$\frac{2}{3}\mathbf{G}-\frac{\pi}{12}\ln(2+\sqrt{3})=\int_0^{2-\sqrt{3}}\frac{\tan^{-1}(x)}{x}\mathrm{d}x$$

Let $x=\tan\left(\frac{t}{2}\right)$

$$\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{t\sec^2\left(\frac{t}{2}\right)}{\tan\left(\frac{t}{2}\right)}\mathrm{d}t=\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{t}{\sin(t)}\mathrm{d}t$$

So,

$$\frac{4}{3}\mathbf{G}-\frac{\pi}{6}\ln(2+\sqrt{3})=\int_0^{\frac{\pi}{6}}\frac{t}{\sin(t)}\mathrm{d}t$$

Hence we get

$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)^2}=\frac{8}{3}\mathbf{G}-\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)$$

□

Exercise 2.2.13: Evaluate:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2^n3^k(k+n)}$$

✎ **Solution:**

$$I=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1\sum_{k=1}^{\infty}\frac{x^{n+k-1}}{3^k}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1\frac{x^n}{3-x}\mathrm{d}x$$
$$=\int_0^1\frac{x}{(3-x)(2-x)}\mathrm{d}x=\ln\left(\frac{32}{27}\right)$$

□

Exercise 2.2.14: Evaluate:

$$\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{4}\right)}{n\Gamma\left(n+\frac{3}{4}\right)}$$

✎ **Solution:**

$$\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{4}\right)}{n\Gamma\left(n+\frac{3}{4}\right)}=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-\frac{3}{4}}(1-x)^{-\frac{1}{2}}\mathrm{d}x=\frac{1}{\sqrt{\pi}}\int_0^1\frac{1}{\sqrt{1-x}}\sum_{n=1}^{\infty}\frac{x^{n-\frac{3}{4}}}{n}\,\mathrm{d}x$$

$$=-\frac{1}{\sqrt{\pi}}\int_0^1 x^{-\frac{3}{4}}\frac{\ln(1-x)}{\sqrt{1-x}}\mathrm{d}x=-\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(\psi_0\left(\frac{1}{2}\right)-\psi_0\left(\frac{1}{4}\right)\right)$$

$$=-\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(-2\ln(2)-\frac{\pi}{2}+3\ln(2)\right)=\frac{\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}\left(\pi-2\ln(2)\right)$$

□

Exercise 2.2.15: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}$$

✎ **Solution:** This can be done by using the identity

$$\prod_{k=1}^{\infty}(1-e^{2k\pi})=\frac{\pi^{1/4}e^{\frac{\pi}{12}}}{\sqrt{2}\Gamma\left(\frac{3}{4}\right)}$$

$$\frac{e^{-2\pi k}}{n\left(1-e^{-2\pi k}\right)}=\sum_{n=1}^{\infty}\frac{e^{-2k\pi n}}{n}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-2\pi kn}}{n}=-\sum_{k=1}^{\infty}\ln\left(1-e^{-2\pi k}\right)$$

Now, use the Theta identity mentioned above and take the ln of both sides we get

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}=-\ln\left(\frac{\pi^{1/4}e^{\frac{\pi}{12}}}{\sqrt{2}\Gamma\left(\frac{3}{4}\right)}\right)$$

□

Exercise 2.2.16: Prove:

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^4}\right)=\frac{\sinh(\pi)}{4\pi}$$

☞ **Proof:** We may rewrite the product as

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)\left(1+\frac{1}{k^2}\right)$$

The first product is simply

$$\begin{aligned}\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right) &= \lim_{N\to\infty}\exp\sum_{k=2}^{N}\ln\left(\frac{k^2-1}{k^2}\right)\\ &= \lim_{N\to\infty}\exp\left(-\ln(2)+\ln\left(1+\frac{1}{N}\right)\right)\\ &= \frac{1}{2}\end{aligned}$$

The second product can be evaluated using **Weierstrass's Product**

$$\begin{aligned}\prod_{k=2}^{\infty}\left(1+\frac{1}{k^2}\right) &= \frac{1}{2}\prod_{k=1}^{\infty}\left(1+\frac{i}{k}\right)\left(1-\frac{i}{k}\right)\\ &= \frac{1}{2}\left(\frac{1}{i\Gamma(i)\Gamma(1-i)}\right)\\ &= \frac{\sinh(\pi)}{2\pi}\end{aligned}$$

Upon combining, we obtain

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^4}\right)=\frac{\sinh(\pi)}{4\pi}$$

□

Exercise 2.2.17: Prove:

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^3}\right)=\frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}$$

☞ **Proof:**

$$\begin{aligned}\prod_{k=2}^{\infty}\left(1-\frac{1}{k^3}\right) &= \lim_{z\to1}\frac{1}{1-z^3}\prod_{k=1}^{\infty}\left(1-\frac{z^3}{k^3}\right) = \lim_{z\to1}\frac{1}{1-z^3}\frac{1}{\Gamma(1-z)}\frac{1}{\Gamma(1-e^{2\pi i/3}z)}\frac{1}{\Gamma(1-e^{4\pi i/3}z)}\\ &= \frac{1}{\Gamma(1-e^{2\pi i/3})}\frac{1}{\Gamma(1-e^{4\pi i/3})}\lim_{z\to1}\frac{1}{1-z^3}\frac{1}{\Gamma(1-z)}\end{aligned}$$

$$\Gamma\left(1-e^{2\pi i/3}\right)=\Gamma\left(\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)=\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\Gamma\left(1-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)$$

$$\Gamma\left(1-e^{4\pi i/3}\right)=\Gamma\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\Gamma\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

So

$$\begin{aligned}
\prod_{k=2}^{\infty}\left(1-\frac{1}{k^3}\right) &= \frac{1}{\pi}\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\lim_{z\to 1}\frac{1}{1-z^3}\frac{1}{\Gamma(1-z)}\\
&= \frac{1}{\pi}\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\lim_{z\to 0}\frac{1}{1-(z+1)^3}\frac{1}{\Gamma(-z)}\\
&= \frac{1}{\pi}\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\lim_{z\to 0}\frac{1}{-3z-3z^2-z^3}\frac{1}{\frac{-1}{z}+\mathcal{O}(1)}\\
&= -\frac{1}{3\pi}\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\lim_{z\to 0}\frac{1}{z}\frac{1}{1+z+\mathcal{O}(z^2)}\Big(-z+\mathcal{O}(\ddagger^{\in})\Big)\\
&= -\frac{1}{3\pi}\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\lim_{z\to 0}\Big(1-z+\mathcal{O}(z^2)\Big)\Big(-z+\mathcal{O}(z^2)\Big)\\
&= \frac{1}{3\pi}\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\lim_{z\to 0}\Big(1+\mathcal{O}(z^2)\Big) = \frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}
\end{aligned}$$

□

Exercise 2.2.18: Prove:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1} = -\gamma+\ln 2$$

Proof:

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1} = \sum_{k=2}^{\infty}\frac{\zeta(2k-1)-1}{k} = \sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{km^{2k-1}} = \sum_{m=2}^{\infty}\sum_{k=2}^{\infty}\frac{m}{km^{2k}}\\
&= \sum_{m=2}^{\infty}\left(-m\ln\left(1-\frac{1}{m^2}\right)-\frac{1}{m}\right) = \sum_{m=2}^{\infty}\left(m\ln\left(\frac{m^2}{m^2-1}\right)-\frac{1}{m}\right)\\
&= \sum_{m=2}^{\infty}\left(m\Big(\ln(m^2)-\ln(m^2-1)\Big)-\frac{1}{m}\right)\\
&= \lim_{M\to\infty}\sum_{m=2}^{M}\left(2m\ln(m)-m\ln(m+1)-m\ln(m-1)-\frac{1}{m}\right)\\
&= \lim_{M\to\infty}\Big(\ln 2+(M+1)\ln(M)-M\ln(M+1)-H_M+1\Big)\\
&= \lim_{M\to\infty}\left(\ln 2-H_M+\ln(M)+1-M\ln\left(1+\frac{1}{M}\right)\right)\\
&= \lim_{M\to\infty}\left(\ln 2-H_M+\ln(M)+1-M\left(\frac{1}{M}+\mathcal{O}(M^{-2})\right)\right)\\
&= \ln 2-\gamma
\end{aligned}$$

□

Exercise 2.2.19: Prove:

$$\sum_{k=2}^{\infty} \frac{\left\{2\left(\frac{3}{2}\right)^k - 3\right\}}{k} [\zeta(k) - 1] = \ln(\pi)$$

Proof: Using the formula:

$$\sum_{k=2}^{\infty} \frac{\zeta(k,a)t^{k-1}}{k} = \frac{\ln\Gamma(a-t)}{t} - \frac{\ln\Gamma(a)}{t} + \psi_0(a)$$

Putting $t = 1$ and $t = \frac{3}{2}$, we obtain

$$\sum_{k=2}^{\infty} \frac{\zeta(k,a)}{k} = \psi_0(a) + \ln\frac{\Gamma(a-1)}{\Gamma(a)} \tag{1}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k,a)}{k}\left(\frac{3}{2}\right)^{k-1} = \psi_0(a) + \frac{2}{3}\ln\frac{\Gamma\left(a-\frac{3}{2}\right)}{\Gamma(a)} \tag{2}$$

Subtract (1) from (2) and put $a = 2$:

$$\sum_{k=2}^{\infty} \frac{\left(\frac{3}{2}\right)^{k-1} - 1}{k} [\zeta(k) - 1] = \frac{1}{3}\ln\pi$$

Multiplying by 3 and simplifying gives us the required answer.

$$\sum_{k=2}^{\infty} \frac{\left\{2\left(\frac{3}{2}\right)^k - 3\right\}}{k} [\zeta(k) - 1] = \ln(\pi)$$

□

Exercise 2.2.20: Evaluate:

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n} H_n$$

Solution: We remark that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \quad |x| < \frac{1}{4}$$

$$H_n = \int_0^1 \frac{1-t^n}{1-t} \mathrm{d}t$$

Using these identities, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n}{n}H_n x^n &= \sum_{n=0}^{\infty}\binom{2n}{n}x^n\int_0^1\frac{1-t^n}{1-t}\mathrm{d}t\\ &=\int_0^1\frac{\dfrac{1}{\sqrt{1-4x}}-\dfrac{1}{\sqrt{1-4xt}}}{1-t}\mathrm{d}t\\ &=\frac{1}{\sqrt{1-4x}}\int_0^1\frac{\sqrt{1-4xt}-\sqrt{1-4x}}{(1-t)\sqrt{1-4xt}}\mathrm{d}t\\ &=\frac{1}{\sqrt{1-4x}}\left[-\ln(1-t)+\ln\left|\frac{\sqrt{1-4xt}-\sqrt{1-4x}}{\sqrt{1-4xt}+\sqrt{1-4x}}\right|\right]_{t=0}^{t=1}\\ &=\frac{2}{\sqrt{1-4x}}\ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right)\end{aligned}$$

Putting $x=1/4$, we arrive at

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{n-1}}{4^n}H_n=\sqrt{2}\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)$$

□

Exercise 2.2.21: Prove:

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_n}{4^n n}=\frac{\pi^2}{3}$$

Proof:

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_n}{4^n n}=2\int_0^{\frac14}\frac{1}{x\sqrt{1-4x}}\ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right)\mathrm{d}x$$

Substitute $y=\sqrt{1-4x}$ and $-\dfrac{\mathrm{d}y}{2}=\dfrac{\mathrm{d}x}{\sqrt{1-4x}}$:

$$\begin{aligned}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_n}{4^n n}&=4\int_0^1\frac{1}{1-y^2}\ln\left(\frac{1+y}{2y}\right)\mathrm{d}y\\ &=2\int_0^1\left\{\frac{\ln(1+y)-\ln(2y)}{1-y}+\frac{\ln(1+y)-\ln(2y)}{1+y}\right\}\mathrm{d}y\end{aligned}$$

Note that

$$\int_0^1\frac{\ln(1+y)}{1+y}\mathrm{d}y=\left[\frac{\ln^2(1+y)}{2}\right]_{y=0}^{y=1}=\frac{\ln^2(2)}{2}$$

$$\int_0^1\frac{\ln(2y)}{1+y}\mathrm{d}y=[\ln(2y)\ln(1+y)+\mathrm{Li}_2(-y)]_{y=0}^{y=1}=\ln^2(2)-\frac{\pi^2}{12}$$

$$\begin{aligned}&\int_0^1\frac{\ln(1+y)-\ln(2y)}{1-y}\mathrm{d}y\\ &=\left[\ln(1-y)\ln(2y)-\ln(1-\frac12(1+y))\ln(1+y)+\mathrm{Li}_2(y)-\mathrm{Li}_2\left(\frac{1+y}{2}\right)\right]_{y=0}^{y=1}\\ &=\frac{\pi^2}{12}+\frac{\ln^2(2)}{2}\end{aligned}$$

Putting these we obtain

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_n}{4^n n}=\frac{\pi^2}{3}$$

□

Exercise 2.2.22: Prove:

$$\sum_{k=1}^{n}\cot^2\left(\frac{k\pi}{2n+1}\right)=\frac{n(2n-1)}{3}$$

Proof:

$$\sum_{k=1}^{n}\cot^2\left(\frac{\pi k}{2n+1}\right)=\frac{1}{2}\sum_{k=1}^{2n}\cot^2\left(\frac{\pi k}{2n+1}\right)$$

Let $f(z)=\left(\frac{z+1}{z-1}\right)^2\frac{2n+1}{z(z^{2n+1}-1)}$, then since $f(z)$ has a finite number of isolated singularities on the complex plane,

$$\sum_{k=1}^{2n}\mathrm{Res}[f,e^{2\pi ki/(2n+1)}]+\mathrm{Res}[f,0]+\mathrm{Res}[f,1]+\mathrm{Res}[f,\infty]=0$$

Notice that $\frac{2n+1}{z(z^{2n+1}-1)}$ has residue 1 at the $2n$th roots of unity.

$$\mathrm{Res}[f,e^{2\pi ki/(2n+1)}]=\left(\frac{e^{2\pi ki/(2n+1)}+1}{e^{2\pi ki/(2n+1)}-1}\right)=-\cot^2\left(\frac{\pi k}{2n+1}\right)$$

$$\mathrm{Res}[f,0]=-(2n+1)$$

$$\mathrm{Res}[f,\infty]=0$$

$$\mathrm{Res}[f(z),1]=\mathrm{Res}[f(z+1),0]=\mathrm{Res}\left[\left(\frac{z+2}{z}\right)^2\frac{1}{1+z}\frac{2n+1}{(z+1)^{2n+1}-1},0\right]$$

I'm going to find the residue from the Laurent expansion.

$$\begin{aligned}
f(z+1)&=\left(1+\frac{4}{z}+\frac{4}{z^2}\right)(1-z+z^2+\ldots)\frac{2n+1}{(2n+1)z+\binom{2n+1}{2}z^2+\binom{2n+1}{3}z^3+\ldots}\\
&=\left(1+\frac{4}{z}+\frac{4}{z^2}\right)(1-z+z^2+\ldots)\frac{1}{z}\frac{1}{1+\binom{2n+1}{2}\frac{z}{2n+1}+\binom{2n+1}{3}\frac{z^2}{2n+1}+\ldots}\\
&=\left(1+\frac{4}{z}+\frac{4}{z^2}\right)(1-z+z^2+\ldots)\frac{1}{z}\frac{1}{1+nz+\frac{n(2n-1)}{3}z^2+\ldots}\\
&=\left(1+\frac{4}{z}+\frac{4}{z^2}\right)(1-z+z^2+\ldots)\frac{1}{z}\left(1-nz-\frac{n(2n-1)}{3}z^2+n^2z^2+\ldots\right)\\
&=\frac{4}{z^3}-\frac{4n}{z^2}+\frac{4n^2+4n+3}{3}\frac{1}{z}+\ldots
\end{aligned}$$

So

$$\sum_{k=1}^{n}\cot^2\left(\frac{\pi k}{2n+1}\right)=\frac{1}{2}\left(\frac{4n^2+4n+3}{3}-2n-1\right)=\frac{n(2n-1)}{3}$$

□

Exercise 2.2.23: Prove:

$$\lim_{n\to\infty}\left(\frac{\displaystyle\sum_{k=1}^{n}\sin\left(\frac{2k}{2n}\right)}{\displaystyle\sum_{k=1}^{n}\sin\left(\frac{2k-1}{2n}\right)}\right)^{n}=\exp\left(\frac{1}{2}\cot\left(\frac{1}{2}\right)\right)$$

Proof: Let $f:[0,1]\to[0,\infty)$ be of the class C_1 and not identically zero. Then by Mean Value Theorem, we have

$$\sum_{k=1}^{n}f\left(\frac{2k}{2n}\right)=\sum_{k=1}^{n}\left(f\left(\frac{2k-1}{2n}\right)+f'(x_{n,k})\frac{1}{2n}\right)$$

for some $x_{n,k}\in\left(\frac{2k-1}{2n},\frac{2k}{2n}\right)$. Letting

$$I_n=\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{2k-1}{2n}\right)\quad\text{and}\quad J_n=\frac{1}{n}\sum_{k=1}^{n}f'(x_{n,k}),$$

We have

$$I_n\to I:=\int_0^1 f(x)\,\mathrm{d}x\quad\text{and}\quad J_n\to J:=\int_0^1 f'(x)\,\mathrm{d}x.$$

Therefore we obtain

$$\left[\frac{\displaystyle\sum_{k=1}^{n}f\left(\frac{2k}{2n}\right)}{\displaystyle\sum_{k=1}^{n}f\left(\frac{2k-1}{2n}\right)}\right]^{n}=\left(\frac{nI_n+\frac{1}{2}J_n}{nI_n}\right)^{n}=\left(1+\frac{1}{n}\frac{J_n}{2I_n}\right)^{n}\xrightarrow[n\to\infty]{}\exp\left(\frac{J}{2I}\right).$$

Now plugging $f(x)=\sin x$, the corresponding limit is $\exp\left(\frac{1}{2}\cot\left(\frac{1}{2}\right)\right)$. □

Exercise 2.2.24: Prove:

$$\frac{4\mathbf{K}^2}{\pi^2}=\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3}(2kk')^{2j}={}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(2kk')^2\right)$$

$$\frac{4}{\pi}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3}\frac{6n+1}{4^n}$$

where $(a)_n=\dfrac{\Gamma(a+n)}{\Gamma(a)}$.

Proof:

$$\mathbf{K}(k)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$$

Using Kummer's transformation, we have

$$\mathbf{K}(k)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{4},\frac{1}{4};1;4k^2k'^2\right)$$

Now, comparing what is required to prove, they both look very familiar, quite familiar indeed. It is a matter of sheer luck that I recognized one is the Clausen transform of the other,

$$\left[{}_2F_1\left(\frac{1}{4},\frac{1}{4};1;4k^2k'^2\right)\right]^2 = {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;4k^2k'^2\right)$$

Differentiating the original statement of the first function with respect to k using the identity

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathbf{K}(k) = \frac{\mathbf{E}(k)-(1-k^2)\mathbf{K}(k)}{k(1-k^2)}$$

we get

$$4\frac{1-2k^2}{\sqrt{1-k^2}}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3}(2kk')^{2j-1} = \frac{8\mathbf{K}(k)}{\pi^2}\left(\frac{\mathbf{E}(k)-(1-k^2)\,\mathbf{K}(k)}{k\,(1-k^2)}\right)$$

$$\Longrightarrow \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3}(2kk')^{2j} = \frac{4\mathbf{K}(k)}{\pi^2}\left\{\frac{\mathbf{E}(k)-(1-k^2)\,\mathbf{K}(k)}{(1-2k^2)}\right\} \tag{1}$$

Multiply equation (1) by 6 and add it to the first function:

$$\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j^3(6j+1)}{(j!)^3}(2kk')^{2j} = \frac{4\mathbf{K}(k)^2}{\pi^2}+\frac{24\mathbf{K}(k)}{\pi^2}\left\{\frac{\mathbf{E}(k)-(1-k^2)\,\mathbf{K}(k)}{(1-2k^2)}\right\}$$

Put $k=k_3=\dfrac{\sqrt{3}-1}{2}$ and observe that $2k_3k_3^{'}=\dfrac{1}{2}$ and $\mathbf{K}(k_3)=\dfrac{3^{1/4}\Gamma^3\left(\frac{1}{3}\right)}{2^{7/3}\pi}$. After some laborious simplifications we get

$$\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3}\frac{6j+1}{4^j} = \frac{4}{\pi}$$

□

Exercise 2.2.25: Prove:

$$\sum_{k=1}^{\infty}\mathrm{Ci}(2\pi kn) = \frac{1}{2}\left\{\psi_0(n)-\ln(n)+\frac{1}{2n}\right\}$$

Proof:

$$\sum_{k=1}^{\infty}\mathrm{Ci}(2\pi kn) = -\sum_{k=1}^{\infty}\int_{2\pi kn}^{\infty}\frac{\cos t}{t}\mathrm{d}t = -\sum_{k=1}^{\infty}\int_{2\pi kn}^{\infty}\int_0^{\infty}\cos t e^{-tx}\,\mathrm{d}x\,\mathrm{d}t$$

$$= -\sum_{k=0}^{\infty}\int_0^{\infty}\int_{2\pi kn}^{\infty}\cos t e^{-xt}\mathrm{d}t\,\mathrm{d}x = -\sum_{k=1}^{\infty}\int_0^{\infty}e^{-2\pi knx}\frac{x}{1+x^2}\,\mathrm{d}x$$

$$= -\int_0^{\infty}\frac{x}{1+x^2}\sum_{k=1}^{\infty}e^{-2\pi knx}\,\mathrm{d}x = -\int_0^{\infty}\frac{x}{1+x^2}\frac{e^{-2\pi n}}{1-e^{-2\pi n}}\,\mathrm{d}x$$

$$= -\int_0^{\infty}\frac{x}{1+x^2}\frac{1}{e^{2\pi nx}-1}\,\mathrm{d}x = -\int_0^{\infty}\frac{u}{n^2+u^2}\frac{1}{e^{2\pi u}-1}\,\mathrm{d}u$$

And differentiating Binet's formula,

$$\psi(z) = \ln z + \left(z - \frac{1}{2}\right)\frac{1}{z} - 1 - 2\int_0^\infty \frac{t}{z^2+t^2}\frac{1}{e^{2\pi t}-1}\,\mathrm{d}t$$

Therefore,

$$\sum_{k=1}^{\infty} \mathrm{Ci}(2\pi k n) = \frac{1}{2}\left(\psi(n) - \ln(n) + \frac{1}{2n}\right)$$

□

Exercise 2.2.26: Prove:

$$\sum_{n=0}^{\infty} \frac{1}{\binom{4n}{2n}} = \frac{16}{15} + \frac{\sqrt{3}\pi}{27} - \frac{2\sqrt{5}}{25}\ln(\phi)$$

Proof: Let's begin with the well known identity:

$$\frac{2x\sin^{-1}x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}}$$

Now differentiate the above equation with respect to x:

$$\frac{2x}{1-x^2} + \frac{2x^2\sin^{-1}(x)}{(1-x^2)^{3/2}} + \frac{2\sin^{-1}(x)}{\sqrt{1-x^2}} = 4\sum_{n=1}^{\infty} \frac{(2x)^{2n-1}}{\binom{2n}{n}}$$

$$\implies \frac{x^2}{1-x^2} + \frac{x^3\sin^{-1}(x)}{(1-x^2)^{3/2}} + \frac{x\sin^{-1}(x)}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n}} \tag{1}$$

In equation (1), replace x by $xe^{i\pi/2}$:

$$-\frac{x^2}{1+x^2} - \frac{ix^3\sin^{-1}(ix)}{(1+x^2)^{3/2}} + \frac{ix\sin^{-1}(ix)}{\sqrt{1+x^2}} = \sum_{n=1}^{\infty}(-1)^n\frac{(2x)^{2n}}{\binom{2n}{n}} \tag{2}$$

Add equations (1) and (2) to get

$$2\sum_{n=1}^{\infty} \frac{(2x)^{4n}}{\binom{4n}{2n}} = \frac{x^2}{1-x^2} + \frac{x^3\sin^{-1}(x)}{(1-x^2)^{3/2}} + \frac{x\sin^{-1}(x)}{\sqrt{1-x^2}}$$
$$- \frac{x^2}{1+x^2} - \frac{ix^3\sin^{-1}(ix)}{(1+x^2)^{3/2}} + \frac{ix\sin^{-1}(ix)}{\sqrt{1+x^2}}$$

Using the formula $\sin^{-1}(ix) = -i\ln\left(\sqrt{1+x^2}-x\right)$,

$$2\sum_{n=1}^{\infty} \frac{(2x)^{4n}}{\binom{4n}{2n}} = \frac{x^2}{1-x^2} + \frac{x^3\sin^{-1}(x)}{(1-x^2)^{3/2}} + \frac{x\sin^{-1}(x)}{\sqrt{1-x^2}}$$
$$- \frac{x^2}{1+x^2} - \frac{x^3\ln\left(\sqrt{1+x^2}-x\right)}{(1+x^2)^{3/2}} + \frac{x\ln\left(\sqrt{1+x^2}-x\right)}{\sqrt{1+x^2}} \tag{3}$$

Putting $x = \frac{1}{2}$ in equation (3) and simplifying should give the required answer,

$$\sum_{n=0}^{\infty} \frac{1}{\binom{4n}{2n}} = \frac{16}{15} + \frac{\sqrt{3}\pi}{27} - \frac{2\sqrt{5}}{25}\ln(\phi)$$

□

Exercise 2.2.27: Prove:

$$\sum_{n=1}^{\infty} \frac{1}{2^n \binom{3n}{n}} = \frac{2}{25} - \frac{6}{125}\ln 2 + \frac{11\pi}{250}$$

Proof: We can use

$$\frac{1}{\binom{3n}{n}} = n\int_0^1 x^{n-1}(1-x)^{2n}\,\mathrm{d}x$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{2^n \binom{3n}{n}} = \sum_{n=1}^{\infty} \frac{n}{2^n}\int_0^1 x^{n-1}(1-x)^{2n}\mathrm{d}x = \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty} n\left(\frac{x(1-x)^2}{2}\right)^n \mathrm{d}x$$

$$= \int_0^1 \frac{(1-x)^2}{2}\cdot\left(1-\frac{x(1-x)^2}{2}\right)^{-2}\mathrm{d}x = 2\int_0^1 \left[\frac{(1-x)}{(2-x)(1+x^2)}\right]^2 \mathrm{d}x$$

$$= 2\left[\frac{1}{125}\left(\frac{5(4x+3)}{x^2+1} - \ln\left(x^2+1\right) - \frac{5}{x-2} + 2\ln|x-2| + 11\tan^{-1}(x)\right)\right]_{x=0}^{x=1}$$

$$= \frac{1}{250}(20 + 11\pi - 12\ln 2)$$

□

Exercise 2.2.28: Prove:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)}\frac{1}{2^{2k}} = 2 - \gamma + \frac{7}{3}\ln 2 - 12\ln \mathbf{A}$$

Proof: Begin with

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1}x^{2k+1} = \frac{1}{2}\ln\Gamma(1-x) - \frac{1}{2}\ln\Gamma(1+x) - \gamma\cdot x$$

Now, integrating this over 0 to say t (then let $t = 1/2$), the left side is:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2(k+1)(2k+1)}t^{2k+2} = \frac{1}{2}\int_0^t \ln\Gamma(1-x)\mathrm{d}x - \frac{1}{2}\int_0^t \ln\Gamma(1+x)\mathrm{d}x - \frac{\gamma t^2}{2}$$

Note that a famous identity is **Alexeiewsky's theorm**:

$$\int_0^t \ln[\Gamma(1+x)]\mathrm{d}x = \frac{1}{2}t(t-1) + \frac{1}{2}\ln(2\pi) - \ln[G(t+1)] + t\ln\Gamma(t)$$

Now, the integrals of log gamma can be expressed in terms of the Barnes G. which makes it kind of interesting. I will skip the intricate details, but integrating and multiplying by 2 results in:

$$(1-\gamma)t^2 + t\ln\frac{\Gamma(1-t)}{\Gamma(1+t)} + \ln[G(1+t)] + \ln[G(1-t)] \tag{1}$$

Now, let $t = 1/2$ and note the following identities:

$$\ln G\left(\frac{1}{2}\right) = \ln\left(\frac{2^{1/24}e^{1/8}}{\pi^{1/4}\mathbf{A}^{3/2}}\right)$$

and

$$\ln G\left(\frac{3}{2}\right) = \ln\left(\frac{2^{1/24}e^{1/8}\pi^{1/4}}{\mathbf{A}^{3/2}}\right)$$

let $t = 1/2$ in (1):

$$\begin{aligned}
&(1-\gamma)\left(\frac{1}{2}\right)^2 + \frac{1}{2}\ln\left(\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\right) + \ln G\left(\frac{3}{2}\right) + \ln G\left(\frac{1}{2}\right)\\
&= \frac{1}{4} - \frac{\gamma}{4} + \frac{1}{2}\ln\left(\frac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}}\right) + \ln\left(\frac{2^{1/24}e^{1/8}\pi^{1/4}}{\mathbf{A}^{3/2}}\right) + \ln\left(\frac{2^{1/24}e^{1/8}}{\pi^{1/4}\mathbf{A}^{3/2}}\right)\\
&= \frac{1}{2} - \frac{\gamma}{4} + \frac{7}{12}\ln(2) - 3\ln\mathbf{A}
\end{aligned}$$

multiply by 4 from up top and get:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(k+1)(2k+1)}\frac{1}{2^{2k}} = 2 - \gamma + \frac{7}{3}\ln 2 - 12\ln\mathbf{A}$$

□

Exercise 2.2.29: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)} = \frac{1}{6} - \frac{1}{2\pi}$$

Proof: Here's a solution using elliptic functions:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{2n}}{1-q^{2n}} &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=0}^{\infty}q^{2mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q^{2mn}}{n}\\
&= -\sum_{m=1}^{\infty}\ln(1-q^{2m}) = -\ln\prod_{m=1}^{\infty}(1-q^{2m})\\
&= -\frac{1}{6}\ln\left(2\pi^{-3}q^{-1/2}kk'\mathbf{K}(k)^3\right) \qquad (1)
\end{aligned}$$

where k and q are related by

$$\mathbf{K}(k) = \frac{\pi}{2}\varphi^2(q) \;\Leftrightarrow\; q = e^{-\pi\mathbf{K}'(k)/\mathbf{K}(k)}$$

Also note that Legendre's relation yields

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}k} &= -\pi e^{-\pi\mathbf{K}'(k)/\mathbf{K}(k)}\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\mathbf{K}'(k)}{\mathbf{K}(k)}\right)\\ &= -\pi q\frac{\frac{\mathrm{d}\mathbf{K}'(k)}{\mathrm{d}k}\mathbf{K}(k)-\frac{\mathrm{d}\mathbf{K}(k)}{\mathrm{d}k}\mathbf{K}'(k)}{\mathbf{K}(k)^2}\\ &= -\pi q\frac{k^2\mathbf{K}'(k)\mathbf{K}(k)-\mathbf{K}(k)\mathbf{E}'(k)-\mathbf{K}'(k)\mathbf{E}(k)+k'^2\mathbf{K}(k)\mathbf{K}'(k)}{\mathbf{K}(k)^2kk'^2}\\ &= q\frac{\pi^2}{2\mathbf{K}(k)^2kk'^2}\end{aligned}$$

Differentiate equation (1) with respect to $1/q$:

$$\begin{aligned}-2q\sum_{n=1}^{\infty}\frac{1}{(q^{-n}-q^n)^2} &= \frac{1}{6}\frac{2q\mathbf{K}(k)^2kk'^2}{\pi^2}\frac{\mathrm{d}}{\mathrm{d}k}\ln\left(2\pi^{-3}q^{-1/2}kk'\mathbf{K}(k)^3\right)\\ &= \frac{1}{6}\frac{2q\mathbf{K}(k)^2kk'^2}{\pi^2}\left\{\frac{1}{k}-\frac{k}{k'^2}+\frac{3}{\mathbf{K}(k)}\frac{\mathbf{E}(k)-k'^2\mathbf{K}(k)}{kk'^2}-\frac{1}{2q}\left(q\frac{\pi^2}{2\mathbf{K}(k)^2kk'^2}\right)\right\}\\ \Longrightarrow\ \sum_{n=1}^{\infty}\frac{1}{(q^{-n}-q^n)^2} &= -\frac{1}{12}\frac{2\mathbf{K}(k)^2kk'^2}{\pi^2}\left\{\frac{1}{k}-\frac{k}{k'^2}+\frac{3}{\mathbf{K}(k)}\frac{\mathbf{E}(k)-k'^2\mathbf{K}(k)}{kk'^2}-\frac{\pi^2}{4\mathbf{K}(k)^2kk'^2}\right\}\end{aligned}$$

Now put $q=e^{-\pi}\ \Leftrightarrow\ k=\dfrac{1}{\sqrt{2}}$:

$$\begin{aligned}\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)} &= \frac{1}{24}-\frac{\Gamma^2\left(\frac{1}{4}\right)}{8\pi^2\sqrt{\pi}}\left(\pi^{3/2}\Gamma^{-2}\left(\frac{1}{4}\right)+\frac{1}{8\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)-\frac{1}{8\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)\right)\\ \Longrightarrow\ \sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)} &= \frac{1}{6}-\frac{1}{2\pi}\end{aligned}$$

□

Exercise 2.2.30: Prove:

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\tan\left(\frac{2n+1}{2}\pi\sqrt{15}\right)=-\frac{\pi^3}{32\sqrt{15}}$$

Proof: Let $f(z)=\dfrac{\pi\tan(\pi z)\tan(\pi\theta z)}{z^3}$ (where $\theta\neq 1$) and let C_N be a square with vertices at $N(1+i)$, $N(1+i)$, $N(1i)$ and $N(1+i)$. It can be shown that

$$\lim_{N\to\infty}\int_{C_N}f(z)\,\mathrm{d}z=0$$

So the sum of residues of $f(z)$ equals 0.

An easy computation shows that

$$\begin{aligned}\mathrm{Res}_{z=0}\,f(z) &= \pi^3\theta\\ \mathrm{Res}_{z=(2n+1)\pi/2}\,f(z) &= -\frac{\tan\left(\frac{2n+1}{2}\pi\theta\right)}{\left(\frac{2n+1}{2}\right)^3}\\ \mathrm{Res}_{z=(2n+1)\pi/2\theta}\,f(z) &= -\frac{\tan\left(\frac{2n+1}{2\theta}\pi\right)}{\theta\left(\frac{2n+1}{2\theta}\right)^3}\end{aligned}$$

$$-8\sum_{n=-\infty}^{\infty}\frac{\tan\left(\frac{2n+1}{2}\pi\theta\right)}{(2n+1)^3}-8\theta^2\sum_{n=-\infty}^{\infty}\frac{\tan\left(\frac{2n+1}{2\theta}\pi\right)}{(2n+1)^3}+\pi^3\theta=0$$

$$\Longrightarrow\sum_{n=0}^{\infty}\frac{\tan\left(\frac{2n+1}{2}\pi\theta\right)}{(2n+1)^3}+\theta^2\sum_{n=0}^{\infty}\frac{\tan\left(\frac{2n+1}{2\theta}\pi\right)}{(2n+1)^3}=\frac{\pi^3}{16}\theta \tag{1}$$

Now put $\theta=4-\sqrt{15}\Leftrightarrow\dfrac{1}{\theta}=4+\sqrt{15}$ in equation (1) and use periodicity properties of trig functions:

$$-\sum_{n=0}^{\infty}\frac{\tan\left(\frac{2n+1}{2}\pi\sqrt{15}\right)}{(2n+1)^3}+(4-\sqrt{15})^2\sum_{n=0}^{\infty}\frac{\tan\left(\frac{2n+1}{2}\pi\sqrt{15}\right)}{(2n+1)^3}=\frac{\pi^3}{16}(4-\sqrt{15})$$

$$\Longrightarrow\sum_{n=0}^{\infty}\frac{\tan\left(\frac{2n+1}{2}\pi\sqrt{15}\right)}{(2n+1)^3}=\frac{4-\sqrt{15}}{(4-\sqrt{15})^2-1}\frac{\pi^3}{16}$$

$$\Longrightarrow\sum_{n=0}^{\infty}\frac{\tan\left(\frac{2n+1}{2}\pi\sqrt{15}\right)}{(2n+1)^3}=-\frac{\pi^3}{32\sqrt{15}}$$

□

Exercise 2.2.31: Prove:

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\ln\left(1+\frac{\left(\frac{2\ln(2+\sqrt{3})}{\pi}\right)^2}{(2n+1)^2}\right)=\frac{4\mathbf{G}}{3\pi}$$

Proof: Let

$$S(x)=\sum_{n=0}^{\infty}(-1)^n(2n+1)\ln\left(1+\frac{x^2}{(2n+1)^2}\right)$$

then

$$S(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^{x^2}\frac{1}{1+\frac{t}{(2n+1)^2}}\mathrm{d}t=\int_0^{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2+t}\mathrm{d}t$$

$$=\frac{\pi}{4}\int_0^{x^2}\operatorname{sech}\left(\frac{\pi\sqrt{t}}{2}\right)\mathrm{d}t=\frac{\pi}{2}\int_0^{x}t\operatorname{sech}\left(\frac{\pi t}{2}\right)\mathrm{d}t$$

$$=\pi\int_0^x t\frac{e^{-\pi t/2}}{1+e^{-\pi t}}\mathrm{d}t=-\frac{4}{\pi}\int_{e^{-\pi x/2}}^1\frac{\ln u}{1+u^2}\mathrm{d}u\quad(u=e^{-\pi t/2})$$

Now use integration by parts:

$$S(x)=-\frac{4}{\pi}\left\{\frac{\pi x}{2}\tan^{-1}\left(e^{-\pi x/2}\right)-\int_{e^{-\pi x/2}}^1\frac{\tan^{-1}(u)}{u}\mathrm{d}u\right\}$$

$$=-2x\tan^{-1}\left(e^{-\pi x/2}\right)+\frac{4}{\pi}\left(\mathrm{Ti}_2(1)-\mathrm{Ti}_2\left(e^{-\pi x/2}\right)\right)$$

Put $x=\frac{2}{\pi}\ln(2+\sqrt{3})$ and simplify to get the final answer:

$$\begin{aligned}S\left(\frac{2}{\pi}\ln\left(2+\sqrt{3}\right)\right)&=-\frac{1}{3}\ln\left(2-\sqrt{3}\right)+\frac{4}{\pi}\left(\mathbf{G}-\mathrm{Ti}_2\left(2-\sqrt{3}\right)\right)\\&=-\frac{1}{3}\ln\left(2-\sqrt{3}\right)+\frac{4}{\pi}\left\{\mathbf{G}-\left(\frac{2\mathbf{G}}{3}-\frac{\pi}{12}\ln\left(2-\sqrt{3}\right)\right)\right\}\\&=\frac{4\mathbf{G}}{3\pi}\end{aligned}$$

□

Exercise 2.2.32: Prove:

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{H_n}{2^{5n}}=\frac{\sqrt{\pi}}{2\Gamma^2\left(\frac{3}{4}\right)}(\pi-4\ln 2)$$

Proof: Begin with the identity

$${}_2F_1\left(a,1-a;c;\frac{1}{2}\right)=\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{(c)_n}\left(\frac{1}{2}\right)^n=\frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1+c}{2}\right)}{\Gamma\left(\frac{a+c}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)}$$

Differentiate the above with respect to c:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{(c)_n}\left(\frac{1}{2}\right)^n\{\psi_0(c)-\psi_0(c+n)\}\\&=\frac{1}{2}\frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1+c}{2}\right)}{\Gamma\left(\frac{a+c}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)}\left\{\psi_0\left(\frac{c}{2}\right)+\psi_0\left(\frac{1+c}{2}\right)-\psi_0\left(\frac{a+c}{2}\right)-\psi_0\left(\frac{1+c-a}{2}\right)\right\}\end{aligned}$$

Put $a=\frac{1}{2}$ and $c=1$:

$$\begin{aligned}&\Longrightarrow-\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(1)_n}\frac{H_n}{2^n}=\frac{\sqrt{\pi}}{2\Gamma\left(\frac{3}{4}\right)^2}\left\{-\gamma+\psi_0\left(\frac{1}{2}\right)-2\psi_0\left(\frac{3}{4}\right)\right\}\\&\Longrightarrow\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{H_n}{2^{5n}}=\frac{\sqrt{\pi}}{2\Gamma^2\left(\frac{3}{4}\right)}(\pi-4\ln 2)\end{aligned}$$

□

Exercise 2.2.33: Prove:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{\sinh(\pi k)}=\frac{1}{4\pi}$$

Proof: The Mellin transform of $\frac{x}{\sinh\pi x}$ is

$$\begin{aligned}\left\{\mathcal{M}\ \frac{x}{\sinh\pi x}\right\}(s)&=2\pi^{-s-1}\Gamma(s+1)\chi_{s+1}(1)\\&=\pi^{-s-1}\Gamma(s+1)\Big(\zeta(s+1)+2^{-s}\zeta(s+1)\Big)\end{aligned}$$

Then using the inverse Mellin transform,

$$\frac{x}{\sinh \pi x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \pi^{-s-1}\Gamma(s+1)\Big(\zeta(s+1)+2^{-s}\zeta(s+1)\Big)x^{-s}\mathrm{d}s$$

where c is any value greater than 0.

Replacing x with n and summing both sides,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{\sinh \pi k} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \pi^{-s-1}\Gamma(s+1)\Big(\zeta(s+1)+2^{-s}\zeta(s+1)\Big)(1-2^{1-s})\zeta(s)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\mathrm{d}s\end{aligned}$$

The integrand has a simple pole at the origin with residue $\frac{1}{2\pi}$. The integrand is also odd along the imaginary axis. I used Wolfram Alpha to confirm this. So we can shift the contour to the imaginary axis. Then

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{\sinh \pi k} = \frac{1}{2\pi i}\pi i\ \mathrm{Res}[f(s),0] = \frac{1}{4\pi}$$

□

Exercise 2.2.34: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n} = \gamma\ln(2)-\frac{\ln^2(2)}{2}$$

Proof:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n} &= \eta'(1) \\ &= \lim_{s\to 1}\Big(\ln(2)2^{1-s}\zeta(s)+(1-2^{1-s})\zeta'(s)\Big) \\ &= \lim_{s\to 1}\left(\ln(2)2^{1-s}\Big(\frac{1}{s-1}+\gamma+\mathcal{O}(s)\Big)+(1-2^{1-s})\Big(-\frac{1}{(s-1)^2}+\mathcal{O}(1)\Big)\right) \\ &= \gamma\ln(2)+\lim_{s\to 1}\frac{\ln(2)2^{1-s}(s-1)-(1-2^{1-s})}{(s-1)^2} \\ &= \gamma\ln(2)+\lim_{s\to 1}\frac{-\ln^2(2)2^{s-1}(s-1)+\ln(2)2^{s-1}-\ln(2)2^{s-1}}{2(s-1)} \\ &= \gamma\ln(2)-\frac{\ln^2(2)}{2}\end{aligned}$$

And since

$$\int_0^{\infty}\frac{x^{s-1}}{1+e^x}\mathrm{d}x = \Gamma(s)\eta(s)$$

$$\begin{aligned}\int_0^{\infty}\frac{\ln x}{1+e^x}\mathrm{d}x &= \Gamma'(1)\eta(1)+\Gamma(1)\eta'(1) \\ &= -\gamma\ln(2)+\gamma\ln(2)-\frac{\ln^2(2)}{2} \\ &= -\frac{\ln^2(2)}{2}\end{aligned}$$

□

Exercise 2.2.35: Prove:

$$\sum_{n=1}^{\infty} \frac{1}{n(16n^2-1)^2} = 3 - 3\ln 2 - \mathbf{G}$$

Proof: Break up the summand as

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{n(4n-1)(4n+1)}$$

The two let sums are, by definition, the series for trigamma.

That is,

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)^2} = \frac{1}{16}\psi_1\left(-\frac{1}{4}\right) = -\frac{\mathbf{G}}{2} + \frac{\pi^2}{16}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} = \frac{1}{16}\psi_1\left(\frac{1}{4}\right) - 1 = \frac{\mathbf{G}}{2} + \frac{\pi^2}{16} - 1$$

For the rightmost sum, I just used the geometric series

$$\sum_{n=1}^{\infty} x^{4n} = \frac{x^4}{1-x^4}$$

then integrated and divided by the appropriate x until I hammered it into the required form. You all know what I mean, so I will not write out all the minute. Doing the last integration finally gave

$$\sum_{n=1}^{\infty} \frac{x^{4n}}{n(4n+1)(4n-1)} = \ln\left(1-x^4\right) - x\tan^{-1}(x) + z\tanh^{-1}(x) + \frac{\tan^{-1}(x)}{x} + \frac{\tanh^{-1}(x)}{x}$$

$$\lim_{x\to 1}\left[\ln\left(1-x^4\right) - x\tan^{-1}(x) + z\tanh^{-1}(x) + \frac{\tan^{-1}(x)}{x} + \frac{\tanh^{-1}(x)}{x}\right] = 3\ln(2)$$

$$\lim_{x\to 0}\left[\ln\left(1-x^4\right) - x\tan^{-1}(x) + z\tanh^{-1}(x) + \frac{\tan^{-1}(x)}{x} + \frac{\tanh^{-1}(x)}{x}\right] = 2$$

Thus, combine with the other result we get

$$\sum_{n=1}^{\infty} \frac{1}{n(16n^2-1)^2} = 3 - 3\ln 2 - \mathbf{G}$$

□

Exercise 2.2.36: Evaluate:

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5} - \frac{1}{6n-1}\right)$$

Solution: We may logarithmically differentiate the Weierstrass product

$$\cos(\pi x) = \prod_{n=0}^{\infty}\left(1 - \frac{4x^2}{(2n+1)^2}\right)$$

to get

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2-(2x)^2}=\frac{\pi}{8x}\tan(\pi x),$$

and noticing

$$\frac{1}{6n+1}-\frac{1}{6n+5}=\frac{4}{(6n+1)(6n+5)}=\frac{4}{9}\frac{1}{(2n+1)^2-\left(\frac{2}{3}\right)^2},$$

we can conclude that

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}-\frac{1}{6n-1}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{6n+1}-\frac{1}{6n+5}\right)=\frac{4}{9}\frac{\pi}{8\cdot\frac{1}{3}}\tan\left(\frac{\pi}{3}\right)=\frac{\pi}{2\sqrt{3}}$$

□

Exercise 2.2.37: Evaluate:

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{n(6n+1)}$$

Solution: Let S denote the sum. Then

$$\begin{aligned}S=\sum_{n=1}^{\infty}\frac{H_{2n}}{n(6n+1)}&=\sum_{n=1}^{\infty}\frac{H_{2n}}{n}\int_0^1 x^{6n}\mathrm{d}x\\&=\int_0^1\left(\sum_{n=1}^{\infty}\frac{H_{2n}}{n}x^{6n}\right)\mathrm{d}x\end{aligned}\tag{1}$$

Let $f(x)=\sum_{n=1}^{\infty}\frac{H_n}{n}x^n$ where $|x|<1$. It can be shown that

$$f(x)=\mathrm{Li}_2(x)+\frac{1}{2}\ln^2(1-x)\tag{2}$$

Then, we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{2n}}{n}x^{6n}&=f\left(x^3\right)+f\left(-x^3\right)\\&=\mathrm{Li}_2\left(x^3\right)+\mathrm{Li}_2\left(-x^3\right)+\frac{\ln^2\left(1-x^3\right)+\ln^2\left(1+x^3\right)}{2}\end{aligned}\tag{3}$$

Substitute (3) into (1) to get

$$S=\int_0^1\left(\mathrm{Li}_2\left(x^3\right)+\mathrm{Li}_2\left(-x^3\right)+\frac{\ln^2\left(1-x^3\right)+\ln^2\left(1+x^3\right)}{2}\right)\mathrm{d}x\tag{4}$$

Note that

$$\begin{aligned}\int_0^1\left(\mathrm{Li}_2\left(x^3\right)+\mathrm{Li}_2\left(-x^3\right)\right)\mathrm{d}x&=\frac{1}{2}\int_0^1\sum_{n=1}^{\infty}\frac{x^{6n}}{n^2}\mathrm{d}x\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2(6n+1)}\\&=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{6}{n}+\frac{36}{1+6n}\right)\\&=\frac{1}{2}\left(\frac{\pi^2}{6}-6\psi_0\left(\frac{1}{6}\right)-6\gamma_0-36\right)\\&=\frac{\pi^2}{12}+\frac{3\pi\sqrt{3}}{2}+6\ln 2+\frac{9}{2}\ln 3-18\end{aligned}\tag{5}$$

$$\begin{aligned}\frac{1}{2}\int_0^1 \ln^3\left(1-x^3\right)\mathrm{d}x &= \frac{1}{6}\int_0^1 t^{-2/3}\ln^2(1-t)\mathrm{d}t \quad (t=x^3)\\ &= \frac{1}{6}\left[\frac{\partial^2}{\partial y^2}\mathrm{B}(x,y)\right]_{x=1/3,y=1}\\ &= \frac{\pi^2}{8}-\frac{\sqrt{3}\pi}{2}+\frac{9}{2}+\frac{9}{8}\ln^2 3-\frac{9\ln 3}{2}+\frac{1}{4}\sqrt{3}\pi\ln 3-\frac{\psi_1\left(\frac{4}{3}\right)}{2} \qquad (6)\end{aligned}$$

Substitute (5) and (6) into equation (4) to get

$$S=-\frac{27}{2}+\frac{5\pi^2}{24}+\frac{9}{8}\ln^2 3+\frac{\pi\sqrt{3}}{4}(4+\ln 3)+6\ln 2-\frac{1}{2}\psi_1\left(\frac{4}{3}\right)+\frac{1}{2}\int_0^1\ln^2(1+x^3)\mathrm{d}x$$

Now, it remains to calculate $\int_0^1 \ln^2(1+x^3)\mathrm{d}x$.
According to Mathematica, it equals

$$\begin{aligned}\int_0^1 \ln^2(1+x^3)\mathrm{d}x = 18-\frac{5}{36}\pi^2+\frac{\ln^2 3}{4}+3\ln^2 2-12\ln 2+\frac{\ln\frac{2187}{16}-12}{2\sqrt{3}}\pi+\mathrm{Li}_2\left(-\frac{1}{3}\right)\\ -\left(1+i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3-i\sqrt{3}}{6}\right)+\left(1-i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3-i\sqrt{3}}{4}\right)\\ -\left(1-i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3+i\sqrt{3}}{6}\right)+\left(1+i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3+i\sqrt{3}}{4}\right)\end{aligned}$$

Hence, the final result is

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{2n}}{n(6n+1)} = -\frac{9}{2}+\frac{5}{36}\pi^2+\frac{5}{4}\ln^2 3+\frac{3}{2}\ln^2 2-\frac{\ln 2}{\sqrt{3}}+\left(\frac{7}{4\sqrt{3}}+\frac{\sqrt{3}}{4}\right)\ln 3\\ +\frac{1}{2}\left\{\mathrm{Li}_2\left(-\frac{1}{3}\right)-\psi_1\left(\frac{4}{3}\right)-\left(1+i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3-i\sqrt{3}}{6}\right)\right.\\ +\left(1-i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3-i\sqrt{3}}{4}\right)-\left(1-i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3+i\sqrt{3}}{6}\right)\\ \left.+\left(1+i\sqrt{3}\right)\mathrm{Li}_2\left(\frac{3+i\sqrt{3}}{4}\right)\right\}\end{aligned}$$

□

Exercise 2.2.38: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{n^2 2^n}$$

Solution: Once again recall the Taylor Series of $-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$,

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2 2^n} &= -\int_0^{1/2}\frac{\ln(1-x)}{x}\mathrm{d}x\\ &= [\ln(x)\ln(1-x)]_0^{1/2} - \int_0^{1/2}\frac{\ln x}{1-x}\mathrm{d}x\\ &= -\ln^2 2 - \int_0^{1/2}\frac{\ln x}{1-x}\mathrm{d}x\\ &= -\ln^2 2 - \int_{1/2}^{1}\frac{\ln(1-x)}{x}\mathrm{d}x\\ &= -\frac{1}{2}\ln^2 2 - \frac{1}{2}\int_0^1\frac{\ln(1-x)}{x}\mathrm{d}x\\ &= -\frac{1}{2}\ln^2 2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\\ &= \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\end{aligned}$$

□

Exercise 2.2.39: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_{2\pi n}^{\infty}\frac{\sin(y)}{y}\mathrm{d}y$$

Solution: Since $\sin(t)=\sin(t+2\pi n),\ \ n\in\mathbb{Z}$, we can make the sub $y=t+2\pi n,\ \ \mathrm{d}y=\mathrm{d}t$. Make the sub and write the sum/integral as:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}\frac{\sin(t)}{t+2\pi n}\mathrm{d}t &= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}\sin(t)\mathrm{d}t\int_0^{\infty}e^{-(t+2\pi n)u}\mathrm{d}u\\ &= -\int_0^{\infty}\sin(t)\mathrm{d}t\int_0^{\infty}e^{-ut}\ln\left(1-e^{-2\pi u}\right)\mathrm{d}u\\ &= -\int_0^{\infty}\frac{\ln\left(1-e^{-2\pi u}\right)}{1+u^2}\mathrm{d}u\end{aligned}$$

Now, use parts: $u=\ln\left(1-e^{-2\pi u}\right),\ \mathrm{d}v=\dfrac{1}{1+u^2}\mathrm{d}u$, to obtain:

$$2\pi\int_0^{\infty}\frac{\tan^{-1}(u)}{e^{2\pi u}-1}\mathrm{d}u$$

This is a rather famous integral which can be tackled in various ways. A way I found that works pretty slick for this integral is to use **Hermite's Integral** for the Hurwitz Zeta function. There are various forms, but this one comes from diffing w.r.t s.

$$\begin{aligned}\zeta'(s) = \frac{-1}{(s-1)^2} &+ 2\int_0^{\infty}\frac{\cos(s\cdot\tan^{-1}(u))}{(1+u^2)^{s/2}}\cdot\frac{\tan^{-1}(u)}{(e^{2\pi u}-1)}\mathrm{d}u\\ &- \int_0^{\infty}\frac{\sin(s\cdot\tan^{-1}(u))}{(1+u^2)^{s/2}}\cdot\frac{\ln(1+u^2)}{(e^{2\pi u}-1)}\mathrm{d}u\end{aligned}$$

Now, let $s \to 0$ to obtain:

$$\begin{aligned}\zeta'(0) = \frac{-1}{2}\ln(2\pi) &= -1 + 2\int_0^\infty \frac{\tan^{-1}(u)}{e^{2\pi u}-1}\mathrm{d}u \\ \Rightarrow 2\pi\int_0^\infty \frac{\tan^{-1}(u)}{e^{2\pi u}-1}\mathrm{d}u &= \pi\left(1-\frac{1}{2}\ln(2\pi)\right) \\ &= \pi - \frac{\pi}{2}\ln(2\pi)\end{aligned}$$

□

Exercise 2.2.40: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{3n}}{3n+1}$$

Solution: Since

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n+1}z^n = -\frac{\ln^2(1+z)}{2z} = f(z)$$

Applying a discrete Fourier transform with $\omega = e^{2\pi i/3}$, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{3n}}{3n+1} = \Re\frac{f(1)+f(\omega)+f(\omega^2)}{3} = \frac{1}{3}\left(-\frac{\ln^2 2}{2}-\frac{\pi^2}{36}-\frac{\pi^2}{36}\right) = -\frac{\pi^2}{54}-\frac{\ln^2 2}{6}$$

□

Exercise 2.2.41: Evaluate:

$$\sum_{n=0}^{\infty}\left(\frac{1}{4n+1}+\frac{1}{4n+3}-\frac{1}{2n+2}\right)$$

Solution: **Method One:**

First, write

$$S = \sum_{n=0}^{\infty}\left(\frac{1}{4n+1}+\frac{1}{4n+3}-\frac{1}{2n+2}\right) = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{4n+1}+\frac{1}{4n+3}-\sum_{n=0}^{N}\frac{1}{2n+2}\right)$$

Now, all odd numbers are of the form $4n+1$ or $4n+3$. In other words, $\sum_{n=0}^{N}\frac{1}{4n+1}+\frac{1}{4n+3} = \sum_{n=0}^{2N+1}\frac{1}{2n+1}$. It's easy to write the sums of the reciprocals of the first odd numbers in terms of harmonic numbers:

$$\sum_{n=0}^{K}\frac{1}{2n+1} = \sum_{n=0}^{2K+1}\frac{1}{n}-\sum_{n=1}^{K}\frac{1}{2n} = H_{2K+1}-\frac{1}{2}H_K$$

So far,

$$S = \lim_{N\to\infty}\left(H_{4N+3}-\frac{1}{2}H_{2N+1}-\frac{1}{2}H_{N+1}\right)$$

Now employ $H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right)$, to get

$$\begin{aligned} S &= \lim_{N\to\infty} \left(\ln(4N+3) - \frac{1}{2}\ln(2N+1) - \frac{1}{2}\ln(N+1) \right) \\ &= \lim_{N\to\infty} \ln \frac{4N+3}{\sqrt{(2N+1)(N+1)}} = \lim_{N\to\infty} \ln \frac{4+\frac{3}{N}}{\sqrt{2+\frac{3}{N}+\frac{1}{N^2}}} \\ &= \ln \frac{4}{\sqrt{2}} = \frac{3}{2}\ln 2 \end{aligned}$$

Method Two:

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) &= \sum_{k=1}^{\infty} \int_0^1 \left(x^{4k-4} + x^{4k-2} - 2x^{4k-1} \right) \mathrm{d}x \\ &= \int_0^1 \left(\frac{1}{1-x^4} + \frac{x^2}{1-x^4} - \frac{2x^3}{1-x^4} \right) \mathrm{d}x \\ &= \int_0^1 \left(\frac{1}{1+x} + \frac{x}{1+x^2} \right) \mathrm{d}x \\ &= \frac{3\ln 2}{2} \end{aligned}$$

□

Exercise 2.2.42: Evaluate:

$$\sum_{n=0}^{\infty} \frac{H_n}{(n+1)(2n+1)\binom{2n}{n}}$$

Solution:

$$\sum_{n=0}^{\infty} \frac{x^{2n+2}}{(n+1)(2n+1)\binom{2n}{n}} = 4\left(\arcsin\left(\frac{x}{2}\right)\right)^2$$

Differentiation followed by the substitution $x \to \sqrt{x}$ gives:

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)\binom{2n}{n}} = \frac{2\arcsin\left(\frac{\sqrt{x}}{2}\right)}{\sqrt{x}\sqrt{1-\left(\frac{\sqrt{x}}{2}\right)^2}}$$

Thus, we split the series as:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H_n}{(n+1)(2n+1)\binom{2n}{n}} &= \sum_{n=0}^{\infty} \frac{H_{n+1}}{(n+1)(2n+1)\binom{2n}{n}} \\ &\quad - \sum_{n=0}^{\infty} \frac{1}{(n+1)^2(2n+1)\binom{2n}{n}} \end{aligned}$$

The first series can be dealt with using, $\dfrac{H_{n+1}}{n+1} = -\int_0^1 x^n \ln(1-x)\,\mathrm{d}x$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{H_{n+1}}{(n+1)(2n+1)\binom{2n}{n}} &= -\sum_{n=0}^{\infty} \int_0^1 \frac{x^n \ln(1-x)}{(2n+1)\binom{2n}{n}} \mathrm{d}x \\
&= -2\int_0^1 \frac{\arcsin(\sqrt{x}/2)\ln(1-x)}{\sqrt{x}\sqrt{1-(\sqrt{x}/2)^2}} \mathrm{d}x \\
&= -8\int_0^{1/2} \frac{\arcsin x \cdot \ln(1-4x^2)}{\sqrt{1-x^2}} \mathrm{d}x \\
&= -8\int_0^{\pi/6} \theta \ln(1-4\sin^2\theta) \mathrm{d}\theta \\
&= -8\int_0^{\pi/6} \theta \ln\left(4\sin\left(\theta+\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}-\theta\right)\right) \mathrm{d}\theta
\end{aligned}$$

Using the Fourier Series, $\ln(2\sin\theta) = -\sum\limits_{n=1}^{\infty} \dfrac{\cos 2n\theta}{n}$ we get:

$$\begin{aligned}
&\int_0^{\pi/6} \theta \ln\left(2\sin\left(\frac{\pi}{6}+\theta\right)\right) \mathrm{d}\theta \\
&= -\sum_{n=1}^{\infty} \int_0^{\pi/6} \frac{\theta\cos\left(\frac{n\pi}{3}+2n\theta\right)}{n} \mathrm{d}\theta \\
&= -\frac{\pi}{12}\sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)}{n^3} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^3}
\end{aligned}$$

and,

$$\begin{aligned}
&\int_0^{\pi/6} \theta \ln\left(2\sin\left(\frac{\pi}{6}-\theta\right)\right) \mathrm{d}\theta \\
&= -\sum_{n=1}^{\infty} \int_0^{\pi/6} \frac{(\pi/6-\theta)\cos(2n\theta)}{n} \mathrm{d}\theta \\
&= -\frac{1}{4}\zeta(3) + \frac{1}{4}\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^3}
\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \frac{H_{n+1}}{(n+1)(2n+1)\binom{2n}{n}} = -\frac{2}{9}\zeta(3) + \frac{2\pi}{3}\sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)}{n^2}$$

Similarly we may deal with the second series:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(n+1)^2(2n+1)\binom{2n}{n}} &= 8\int_0^{1/2} \frac{\arcsin^2(x)}{x} \mathrm{d}x \\
&= -4\zeta(3) + 4\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^3} + \frac{4\pi}{3}\sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)}{n^2}
\end{aligned}$$

Combining the results we get:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(2n+1)\binom{2n}{n}} = \frac{22}{9}\zeta(3) - \frac{2\pi}{3}\sum_{n=1}^{\infty}\frac{\sin(2n\pi/3)}{n^2}$$
$$= \frac{22}{9}\zeta(3) - \frac{\pi}{9\sqrt{3}}\left(\psi'\left(\frac{1}{3}\right) - \psi'\left(\frac{2}{3}\right)\right)$$

□

Exercise 2.2.43: Evaluate:

$$\sum_{n=1}^{\infty} H_n \left(\sum_{k=n+1}^{\infty}\frac{1}{k^3}\right)$$

Solution:

$$\sum_{n=1}^{\infty} H_n \sum_{k=n+1}^{\infty}\frac{1}{k^3} = \sum_{n=1}^{\infty}\frac{1}{(n+1)^3}\left(\sum_{k=1}^{n} H_k\right)$$
$$= \sum_{n=1}^{\infty}\frac{(n+1)H_{n+1} - (n+1)}{(n+1)^3}$$
$$= \sum_{n=1}^{\infty}\frac{(n+1)H_{n+1}}{(n+1)^3} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$$
$$= \sum_{n=1}^{\infty}\frac{H_n}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Since,

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3)$$

Hence,

$$\sum_{n=1}^{\infty} H_n \left(\sum_{k=n+1}^{\infty}\frac{1}{k^3}\right) = 2\zeta(3) - \zeta(2)$$

□

Exercise 2.2.44: Evaluate:

$$A(p,q) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k^{(p)}}{k^q}$$

Solution: Note that

$$\frac{(-1)^{k-1}}{k} = \int_0^1 (-x)^{k-1}\mathrm{d}x$$

and

$$\frac{1}{n} = \int_0^1 y^{n-1}\mathrm{d}y$$

For the first one,

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{n=1}^{k}\frac{1}{n}=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_0^1(-x)^{k-1}\mathrm{d}x\int_0^1 y^{n-1}\mathrm{d}y=\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\int_0^1(-x)^{k-1}\mathrm{d}x\int_0^1 y^{n-1}\mathrm{d}y\\
&=\sum_{n=1}^{\infty}\int_0^1\frac{(-x)^{n-1}}{1+x}\mathrm{d}x\int_0^1 y^{n-1}\mathrm{d}y=\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{(-xy)^{n-1}}{1+x}\mathrm{d}x\mathrm{d}y\\
&=\int_0^1\int_0^1\frac{1}{(1+x)(1+xy)}\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^1\frac{1}{(1+x)(1+xy)}\mathrm{d}y\mathrm{d}x\\
&=\int_0^1\frac{\ln(1+x)}{x(1+x)}\mathrm{d}x=\int_0^1\frac{\ln(1+x)}{x}\mathrm{d}x-\int_0^1\frac{\ln(1+x)}{(1+x)}\mathrm{d}x\\
&=\frac{\zeta(2)}{2}-\frac{\ln^2 2}{2}
\end{aligned}$$

For the second one,

$$A(1,2)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\sum_{n=1}^{k}\frac{1}{n}$$

$$\frac{(-1)^{k-1}}{k^2}=\int_0^1(-x)^{k-1}\mathrm{d}x\int_0^1 z^{k-1}\mathrm{d}z=(-1)^{k-1}\int_0^1\int_0^1(xz)^{k-1}\mathrm{d}x\mathrm{d}z$$

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\sum_{n=1}^{k}\frac{1}{n}=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_0^1\int_0^1(-1)^{k-1}(xz)^{k-1}\mathrm{d}x\mathrm{d}z\int_0^1 y^{n-1}\mathrm{d}y\\
&=\int_0^1\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{(-xyz)^{n-1}}{1+xz}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1\int_0^1\int_0^1\frac{1}{(1+xz)(1+xyz)}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&=\int_0^1\int_0^1\frac{\ln(1+xz)}{xz(1+xz)}\mathrm{d}x\mathrm{d}z=\int_0^1\int_0^1\frac{\ln(1+xz)}{xz}\mathrm{d}x\mathrm{d}z-\int_0^1\int_0^1\frac{\ln(1+xz)}{1+xz}\mathrm{d}x\mathrm{d}z\\
&=\int_0^1\int_0^1\frac{\ln(1+xz)}{xz}\mathrm{d}x\mathrm{d}z-\int_0^1\frac{\ln^2(1+z)}{2z}\mathrm{d}z\\
&=\frac{3}{4}\zeta(3)-\frac{\zeta(3)}{8}=\frac{5}{8}\zeta(3)
\end{aligned}$$

$$\int_0^1\int_0^1\frac{\ln(1+xz)}{xz}\mathrm{d}x\mathrm{d}z=\sum_{k=0}^{\infty}\int_0^1\int_0^1\frac{(-1)^k(xz)^k}{k+1}\mathrm{d}x\mathrm{d}z=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3}=\frac{3}{4}\zeta(3)$$

For the third one,

$$A(2,1)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{n=1}^{k}\frac{1}{n^2}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{n=1}^{k}\frac{1}{n^2}=\int_0^1\int_0^1\int_0^1\sum_{k=1}^{\infty}\sum_{n=1}^{k}(-1)^{k-1}x^{k-1}(yz)^{n-1}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^1\int_0^1\int_0^1\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}(-1)^{k-1}x^{k-1}(yz)^{n-1}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^1\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{(-xyz)^{n-1}}{1+x}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^1\int_0^1\int_0^1\frac{1}{(1+x)(1+xyz)}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^1\int_0^1\frac{\ln(1+xy)}{(1+x)(xy)}\mathrm{d}x\mathrm{d}y$$
$$=\zeta(3)-\frac{\zeta(2)\ln 2}{2}$$

In general, if I have not made any mistake, this can be extended to $A(p,q)$.

$$A(p,q)=\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{p+q\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q}}{(1+x_1x_2\cdots x_q)(1+x_1x_2\cdots x_{p+q})}$$

Proceeding along similar lines, we also get that

$$B(p,q)=\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}=\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{p+q\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q}}{(1-x_1x_2\cdots x_q)(1-x_1x_2\cdots x_{p+q})}$$

We also get that

$$C(p,q)=\sum_{k=1}^{\infty}\frac{1}{k^q}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{i^p}=\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{p+q\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q}}{(1-x_1x_2\cdots x_q)(1+x_1x_2\cdots x_{p+q})}$$

$$D(p,q)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^q}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{i^p}=\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{p+q\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q}}{(1+x_1x_2\cdots x_q)(1-x_1x_2\cdots x_{p+q})}$$

By the same argument as above, in general, nested sums like

$$\sum_{k=1}^{\infty}\frac{(\pm1)^{k-1}}{k^q}\sum_{n=1}^{k}\frac{(\pm1)^{n-1}}{n^p}\sum_{m=1}^{n}\frac{(\pm1)^{m-1}}{m^r}\cdots$$

equals

$$\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{p+q+r+\cdots\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q+r+\cdots}}{(1\mp x_1\cdots x_q)(1(\mp)(\pm)x_1\cdots x_{p+q})\cdots(1(\mp)(\pm)\cdots(\pm)x_1\cdots x_{p+q+r+\cdots})}$$

For instance,

$$\sum_{k=1}^{\infty}\frac{1}{k^q}\sum_{n=1}^{k}\frac{1}{n^p}\sum_{m=1}^{n}\frac{1}{m^r}=\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{p+q+r\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q+r}}{(1-x_1\cdots x_q)(1-x_1\cdots x_{p+q})\cdots(1-x_1\cdots x_{p+q+r})}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^q}\sum_{n=1}^{k}\frac{1}{n^p}\sum_{m=1}^{n}\frac{1}{m^r}=\underbrace{\int_0^1\cdots\int_0^1}_{p+q+r\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q+r}}{(1+x_1\cdots x_q)(1+x_1\cdots x_{p+q})\cdots(1+x_1\cdots x_{p+q+r})}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^q}\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^p}\sum_{m=1}^{n}\frac{1}{m^r}$$
$$=\underbrace{\int_0^1\cdots\int_0^1}_{p+q+r\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q+r}}{(1+x_1\cdots x_q)(1-x_1\cdots x_{p+q})\cdots(1-x_1\cdots x_{p+q+r})}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^q}\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^p}\sum_{m=1}^{n}\frac{1}{m^r}=\underbrace{\int_0^1\cdots\int_0^1}_{p+q+r\text{ times}}\frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{p+q+r}}{(1-x_1\cdots x_q)(1+x_1\cdots x_{p+q})\cdots(1+x_1\cdots x_{p+q+r})}$$

Similarly, for negative p,q r etc, we can replace the integrals $\int_0^1$ by the appropriate differentiation operator evaluated at 1. □

Exercise 2.2.45: Evaluate:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n}{n^3}$$

Solution: Using contour integration and the kernel $\pi\cot(\pi z)\Big(\psi(-z)+\gamma\Big)\psi_1(-z)$.

Let

$$f(z)=\frac{\pi\cot(\pi z)\Big(\psi(-z)+\gamma\Big)\psi_1(-z)}{z^3}$$

At the positive integers,

$$f(z)=\frac{1}{z^3}\Bigg[\left(\frac{1}{z+n}-2\zeta(2)(z-n)+\mathcal{O}\Big((z-n)^3\Big)\right)\left(\frac{1}{z-n}+H_n-\Big(H_n^{(2)}+\zeta(2)\Big)(z-n)\right.$$
$$+\Big(H_n^{(3)}-\zeta(3)\Big)\cdot(z-n)^2+\mathcal{O}\Big((z-n)^3\Big)\left(\frac{1}{(z-n)^2}+\Big(H_n^2+\zeta(2)\Big)\right.$$
$$\left.\left.-2\Big(H_n^3-\zeta(3)\Big)(z-n)+\mathcal{O}\Big((z-n)^2\Big)\right)\right]$$
$$=\frac{1}{z^3}\left(\frac{1}{(z-n)^4}+\frac{H_n}{(z-n)^2}-\frac{2\zeta(2)}{(z-n)^2}+\frac{H_n^{(2)}H_n}{z-n}-\frac{\zeta(2)H_n}{z-n}+\frac{\zeta(3)}{z-n}-\frac{H_n^{(3)}}{z-n}+\mathcal{O}(1)\right)$$

Therefore,

$$\operatorname{Res}[f(z),n]=\operatorname{Res}\left[\frac{1}{z^3}\frac{1}{(z-n)^4},n\right]+\operatorname{Res}\left[\frac{1}{z^3}\frac{H_n}{(z-n)^3},n\right]+\operatorname{Res}\left[\frac{-2\zeta(2)}{z^3}\frac{1}{(z-n)^2},n\right]$$
$$+\operatorname{Res}\left[\frac{H_n^{(2)}H_n}{z^3}\frac{1}{z-n},n\right]+\operatorname{Res}\left[\frac{-\zeta(2)H_n}{z^3}\frac{1}{z-n},n\right]+\operatorname{Res}\left[\frac{\zeta(3)}{z^3}\frac{1}{z-n},n\right]$$
$$+\operatorname{Res}\left[\frac{-H_n^{(3)}}{z^3}\frac{1}{z-n}\right]$$
$$=-\frac{10}{n^5}+\frac{6H_n}{n^5}+\frac{6\zeta(2)}{n^4}+\frac{H_n^{(2)}H_n}{n^3}-\frac{\zeta(2)H_n}{n^3}+\frac{\zeta(3)}{n^3}-\frac{H_n^{(3)}}{n^3}$$

At the negative integers,

$$\begin{aligned}\text{Res}[f(z),-n] &= -\frac{\psi_1(n)\Big(\psi(n)+\gamma\Big)}{n^3} = -\frac{\Big(\zeta(2)-H_{n-1}^{(2)}\Big)H_{n-1}}{n^3}\\ &= -\frac{\left(\zeta(2)-H_n^{(2)}+\frac{1}{n^2}\right)\left(H_n-\frac{1}{n}\right)}{n^3}\\ &= -\frac{\zeta(2)H_n}{n^3}+\frac{\zeta(2)}{n^4}+\frac{H_n^{(2)}H_n}{n^3}-\frac{H_n^{(2)}}{n^4}-\frac{H_n}{n^5}+\frac{1}{n^6}\end{aligned}$$

And at the origin,

$$\begin{aligned}f(z) &= \frac{\pi\cot(\pi z)\Big(\psi(-z)+\gamma)\Big)\psi_1(-z)}{n^3} = \frac{1}{z^3}\left(\frac{1}{z}-2\zeta(2)z-2\zeta(4)z^3-2\zeta(6)z^5+\mathcal{O}(z^7)\right)\\ &\cdot\left(\frac{1}{z}-\zeta(2)z-\zeta(3)z^2-\zeta(4)z^3-\zeta(5)z^4-\zeta(6)z^5+\mathcal{O}(z^6)\right)\left(\frac{1}{z^2}+\zeta(2)+2\zeta(3)z\right.\\ &\left.+3\zeta(4)z^2+4\zeta(5)z^3+5\zeta(6)z^4+\mathcal{O}(z^5)\right)\\ &= \frac{1}{z^7}-\frac{2\zeta(2)}{z^5}+\frac{\zeta(3)}{2z^4}-\frac{\zeta^2(2)}{z^3}+\frac{3\zeta(5)}{z^2}-\frac{5\zeta(2)\zeta(3)}{z^2}-\frac{2\zeta^2(3)}{z}+\frac{2\zeta(6)}{z}\\ &-\frac{8\zeta(2)\zeta(4)}{z}+\frac{2\zeta^3(2)}{z}+\mathcal{O}(1)\end{aligned}$$

So

$$\text{Res}[f(z),0] = -2\zeta^2(3)+2\zeta(6)-8\zeta(2)\zeta(4)+2\zeta^3(2)$$

Summing all the residues we have

$$\begin{aligned}&-10\sum_{n=1}^{\infty}\frac{1}{n^6}+6\sum_{n=1}^{\infty}\frac{H_n}{n^5}+6\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n}{n^3}-\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\zeta(3)\sum_{n=1}^{\infty}\frac{1}{n^3}\\ &-\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}-\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n}{n^3}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{H_n}{n^5}+\sum_{n=1}^{\infty}\frac{1}{n^6}\\ &-2\zeta^2(3)+2\zeta(6)-8\zeta(2)\zeta(4)+2\zeta^3(2)=0\end{aligned}$$

$$\Longrightarrow 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n}{n^3} = -10\zeta(6)+\frac{17}{2}\zeta(2)\zeta(4)+4\zeta^2(3)-2\zeta^3(2)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}$$

The Euler sum on the right evaluates to $\zeta^2(3)-\dfrac{\zeta(6)}{3}$.
And we end up with

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n}{n^3} = -\frac{31}{6}\zeta(6)+\frac{17}{4}\zeta(2)\zeta(4)+\frac{5}{2}\zeta^2(3)-\zeta^3(2)$$

□

Exercise 2.2.46: Evaluate:

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\psi_0(k)-\psi_0(1)}{k^3}$$

Solution: We will use the formula

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\psi_0(n)-\psi_0(1)}{n^{2q+1}}=\frac{1}{(2q)!}\int_0^1\frac{\ln^{2q}(z)\ln(1+z)}{z(1+z)}\mathrm{d}z$$

Putting $q=1$ and using partial fractions, we obtain

$$\begin{aligned}S&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\psi_0(n)-\psi_0(1)}{n^3}\\&=\frac{1}{2}\int_0^1\frac{\ln(1+z)\ln^2(z)}{z(1+z)}\mathrm{d}z\\&=\frac{1}{2}\int_0^1\frac{\ln(1+z)\ln^2(z)}{z}\mathrm{d}z-\frac{1}{2}\int_0^1\frac{\ln(1+z)\ln^2(z)}{1+z}\mathrm{d}z\\&=\frac{7\pi^4}{720}-\frac{1}{2}\int_0^1\frac{\ln(1+z)\ln^2(z)}{1+z}\mathrm{d}z\end{aligned}$$

Here, I exploited the fact that

$$\int\frac{\ln(1+x)\ln^2(x)}{x}\mathrm{d}x=-\ln^2(x)\mathrm{Li}_2(-x)+2\ln(x)\mathrm{Li}_3(-x)-2\mathrm{Li}_4(-x)$$

The evaluation of the other integral is much more challenging.

$$\begin{aligned}&\int_0^1\frac{\ln(1+z)\ln^2(z)}{1+z}\mathrm{d}z\\&=\int_1^2\frac{\ln(z)\ln^2(z-1)}{z}\mathrm{d}z\quad z\mapsto z-1\\&=-\int_{\frac{1}{2}}^1\frac{\ln z\ln^2(1-z)}{z}\mathrm{d}z-\int_{\frac{1}{2}}^1\frac{\ln^3(z)}{z}\mathrm{d}z+2\int_{\frac{1}{2}}^1\frac{\ln(1-z)\ln^2(z)}{z}\mathrm{d}z\quad z\mapsto\frac{1}{z}\\&=-\int_0^{\frac{1}{2}}\frac{\ln(1-z)\ln^2(z)}{1-z}\mathrm{d}z+2\int_{\frac{1}{2}}^1\frac{\ln(1-z)\ln^2(z)}{z}\mathrm{d}z+\frac{1}{4}\ln^4(2)\end{aligned}$$

These two integrals can be evaluated by equations (10) and (12) on Page 310 of this book. It turns out that

$$\int_0^{\frac{1}{2}}\frac{\ln(1-z)\ln^2(z)}{1-z}\mathrm{d}z=-\frac{\pi^4}{360}-\frac{1}{4}\ln^4(2)$$

$$\int_{\frac{1}{2}}^1\frac{\ln^2(z)\ln(1-z)}{z}\mathrm{d}z=2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{\pi^4}{45}+\frac{7}{4}\ln(2)\zeta(3)+\frac{\pi^2\ln^2(2)}{12}-\frac{1}{12}\ln^4(2)$$

Combining all these efforts we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\psi_0(n)-\psi_0(1)}{n^3}=\frac{\pi^4}{48}-2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7}{4}\ln(2)\zeta(3)+\frac{\pi^2}{12}\ln^2(2)-\frac{1}{12}\ln^4(2)$$

□

Exercise 2.2.47: Evaluate:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{2k-1}$$

✎ Solution:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{2k-1}&=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{2k}\right)\\&=\sum_{n=1}^{\infty}\frac{H_{2n}}{n^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3}\\&=8\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3}\\&=4\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^3}H_n-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3}\\&=4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}H_n+\frac{7}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3}\end{aligned}$$

From **Exercise 3.2.7** we have

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=-\frac{11\pi^4}{360}+\frac{\ln^4 2-\pi^2\ln^2 2}{12}+2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{7\ln 2}{4}\zeta(3)$$

The other sum can be evaluated by noting that(Proved at **Exercise 1.1.30** and **Exercise 2.1.21**)

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q}=\left(1+\frac{q}{2}\right)\zeta(q+1)-\frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k)$$

Putting $q=3$ gives us

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{5}{2}\zeta(4)-\frac{1}{2}\zeta^2(2)=\frac{\pi^4}{72}$$

The final answer is

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{2k-1}=-\frac{53}{720}\pi^4+\frac{\ln^4 2-\pi^2\ln^2 2}{3}+8\mathrm{Li}_4\left(\frac{1}{2}\right)+7\ln 2\zeta(3)$$

□

➽ Exercise 2.2.48: Prove:

$$\sum_{k=1}^{\infty}\frac{\left(\psi_0(k)-\psi_0(1)\right)^2}{k^2}=\frac{11}{360}\pi^4$$

☞ Proof: Method One:

$$\sum_{k=1}^{\infty}\frac{\left(\psi_0(k)-\psi_0(1)\right)^2}{k^2}=\sum_{k=1}^{\infty}\frac{H_{k-1}^2}{k^2}=\sum_{k=1}^{\infty}\frac{H_k^2}{k^2}-2\sum_{k=1}^{\infty}\frac{H_k}{k^3}+\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{11\pi^4}{360}$$

Method Two:

The trick here was to consider the integral:

$$\int_0^1\frac{\ln^3(1-x)}{x}\mathrm{d}x$$

Now we use the series

$$\frac{\ln^2(1-x)}{2}=\sum_{n=1}^{\infty}\frac{\psi_0(k)-\psi_0(1)}{k}x^k$$

We get the relation:

$$\begin{aligned}\int_0^1 \frac{\ln^3(1-x)}{x}\mathrm{d}x &= 2\sum_{n=1}^{\infty}\frac{\psi_0(n)-\psi_0(1)}{k}\int_0^1 x^k\ln(1-x)\mathrm{d}x \\ &= -2\sum_{n=1}^{\infty}\frac{\left(\psi_0(k)-\psi_0(1)\right)^2}{k^2} - 2\sum_{n=1}^{\infty}\frac{\psi_0(k)-\psi_0(1)}{k^3}\end{aligned}$$

From our previous analysis we find

$$\sum_{n=1}^{\infty}\frac{\psi_0(k)-\psi_0(1)}{k^3} = \frac{\pi^4}{360}$$

and the integral on the right is

$$\int_0^1 \frac{\ln^3(1-x)}{x}\mathrm{d}x = -\frac{\pi^4}{15}$$

Therefore we have

$$\sum_{n=1}^{\infty}\frac{\left(\psi_0(k)-\psi_0(1)\right)^2}{k^2} = \frac{11\pi^4}{360}$$

□

Exercise 2.2.49: Evaluate:

$$\sum_{n=1}^{\infty}\frac{H_n}{2^n n^3}$$

Solution: Firstly, note that

$$\sum_{n=1}^{\infty}\frac{x^k\left(\psi_0(k)-\psi_0(1)\right)}{k^3} = \frac{\ln x}{2}\int_0^x\frac{\ln^2(1-t)}{t}\mathrm{d}t - \frac{1}{2}\int_0^x\frac{\ln t\ln^2(1-t)}{t}\mathrm{d}t$$

You may verify this by simple differentiation. Both these integrals are known and you may find them in **Lewin's book on page 310 (eq 6 and 12)**.

$$\int_0^{\frac{1}{2}}\frac{\ln^2(1-t)}{t}\mathrm{d}t = -\frac{\ln^3 2}{3} + \frac{\zeta(3)}{4}$$

$$\int_0^{\frac{1}{2}}\frac{\ln t\ln^2(1-t)}{t}\mathrm{d}t = -\frac{\pi^4}{360} + \frac{\ln^4 2}{4}$$

So we get

$$\sum_{n=1}^{\infty}\frac{\psi_0(n)-\psi_0(1)}{2^n n^3} = \frac{1}{24}\ln^4(2) - \frac{\zeta(3)\ln(2)}{8} + \frac{\pi^4}{720}$$

This immediately yields

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{2^n n^3} &= \sum_{n=1}^{\infty}\frac{\psi_0(n)-\psi_0(1)}{2^n n^3} + \sum_{k=1}^{\infty}\frac{1}{2^k k^4} \\ &= \frac{1}{24}\ln^4(2) - \frac{\zeta(3)\ln(2)}{8} + \frac{\pi^4}{720} + \mathrm{Li}_4\left(\frac{1}{2}\right)\end{aligned}$$

□

Exercise 2.2.50: Evaluate:

$$\sum_{n=1}^{\infty}\frac{H_n'}{n^3}$$

where $H_n^{'}$ is the alternating harmonic number.

✎ **Solution:** Namely,

$$H_n^{'} = \ln(2) + (-1)^{n-1} \int_0^1 \frac{x^n}{1+x} \mathrm{d}x$$

Using that representation,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^{'}}{n^3} &= \ln(2) \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \int_0^1 \frac{x^n}{1+x} \mathrm{d}x \\
&= \zeta(3)\ln(2) - \int_0^1 \frac{1}{1+x} \sum_{n=1}^{\infty} \frac{(-x)^n}{n^3} \mathrm{d}x \\
&= \zeta(3)\ln(2) - \int_0^1 \frac{\mathrm{Li}_3(-x)}{1+x} \mathrm{d}x \\
&= \zeta(3)\ln(2) - \mathrm{Li}_3(-x)\ln(1+x)\Big|_0^1 + \int_0^1 \frac{\mathrm{Li}_2(-x)\ln(1+x)}{x} \mathrm{d}x \\
&= \zeta(3)\ln(2) + \frac{3}{4}\zeta(3)\ln(2) - \frac{1}{2}\Big(\mathrm{Li}_2(-1)\Big)^2 \\
&= \frac{7}{4}\zeta(3)\ln(2) - \frac{\pi^4}{288}
\end{aligned}$$

This also can be Evaluated by using the fact that

$$\sum_{n=1}^{\infty} \frac{H_n^{'}}{n^q} = \zeta(q)\ln(2) - \frac{q}{2}\zeta(q+1) + 2\eta(z) + \sum_{k=1}^{q} \eta(k)\eta(q-k+1)$$

where $\eta(z)$ is the **Dirichlet Eta Function** and $H_n^{'} = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j}$. □

Exercise 2.2.51: Evaluate:

$$\sum_{k\geq 0} \frac{\Gamma(t+2k)\Gamma(k+1)}{\Gamma(t)\Gamma(2k+1)} \frac{(-x^2)^k}{k!}$$

✎ **Solution:** The series

$$S = \sum_{k=0}^{\infty} \frac{\Gamma(2k+\alpha)\Gamma(k+1)}{k!\ \Gamma(\alpha)\ \Gamma(2k+1)} (-x^2)^k$$

can be expressed in a closed form as follows.

$$\begin{aligned}
S &= \sum_{k=0}^{\infty} \frac{\Gamma(2k+\alpha)\Gamma(k+1)}{k!\ \Gamma(\alpha)\Gamma(2k+1)} (-x^2)^k \\
&= \sum_{k=0}^{\infty} \frac{(\alpha)_{2k}(1)_k}{k!\ (1)_{2k}} (-x^2)^k \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+1}{2}\right)_k}{k!\ \left(\frac{1}{2}\right)_k} (-x^2)^k \\
&= {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}; -x^2\right)
\end{aligned}$$

Now using

$$
\begin{aligned}
{}_2F_1\left(a, a+\frac{1}{2}; \frac{1}{2}; z\right) &= \frac{1}{2}\left[(1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right] \\
&= (1-z)^{-a}\ \cosh\left(2a\tanh^{-1}(\sqrt{z})\right)
\end{aligned}
$$

then is it seen that the series in question becomes

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{\Gamma(2k+\alpha)\Gamma(k+1)}{k!\ \Gamma(\alpha)\ \Gamma(2k+1)}\ (-x^2)^k &= \frac{(1+ix)^{\alpha} + (1-ix)^{\alpha}}{2(1+x)^{\alpha}} \\
&= (1+x)^{-\alpha/2}\ \cos\left(t\ \tan^{-1}(x)\right)
\end{aligned}
$$

□

Exercise 2.2.52: Evaluate:

$$
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^2}
$$

Solution: Consider the series form of $\sin^{-1}(x)$ which is given by

$$
\sin^{-1}(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n+1}}{4^n(2n+1)}
$$

Dividing both sides by x and then integrating yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n+1}}{4^n(2n+1)^2} = \int_0^x \frac{\sin^{-1}(t)}{t}\ \mathrm{d}t \\
&= \sin^{-1}(x)\ \ln\left(1-e^{2i\sin^{-1}(x)}\right) - \frac{i}{2}\left(\sin^{-1}(x)\right)^2 - \frac{i}{2}\mathrm{Li}_2\left(e^{2i\sin^{-1}(x)}\right) + \frac{i}{2}\mathrm{Li}_2(1)
\end{aligned}
$$

When $x = 1/2$ this reduces to

$$
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^2} = \frac{\pi^2 i}{36} - i\mathrm{Li}_2\left(e^{\pi i/3}\right)
$$

Using the known value

$$
\mathrm{Li}_2\left(e^{\pi i/3}\right) = \frac{\pi^2}{36} + i\ \left[\frac{\sqrt{3}}{16}\psi'\left(\frac{1}{3}\right) - \frac{1}{16\sqrt{3}}\psi'\left(\frac{5}{6}\right)\right]
$$

then the desired result is obtained, namely,

$$
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^2} = \frac{\sqrt{3}}{16}\psi'\left(\frac{1}{3}\right) - \frac{1}{16\sqrt{3}}\psi'\left(\frac{5}{6}\right)
$$

□

Exercise 2.2.53: Prove:

$$
\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+1}\right) = \tan^{-1}\left\{\frac{\tan\left(\pi\sqrt{\frac{\sqrt{2}-1}{2}}\right)}{\tanh\left(\pi\sqrt{\frac{\sqrt{2}+1}{2}}\right)}\right\} - \frac{\pi}{8}
$$

☞ **Proof:** We know that

$$\Gamma(z)=\lim_{n\to+\infty}\frac{n!n^z}{z(z+1)(z+2)...(z+n)}=\lim_{n\to+\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)}$$

$$\Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin\pi z}$$

$$\arctan\theta=\frac{i}{2}\ln\left(\frac{i+\theta}{i-\theta}\right)$$

Hence,

$$\begin{aligned}
S&=\sum_{k=1}^{+\infty}\arctan\frac{1}{1+k^2}=\lim_{n\to+\infty}\sum_{k=1}^{n}\arctan\frac{1}{1+k^2}=\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{i}{2}\ln\left(\frac{i+\frac{1}{1+k^2}}{i-\frac{1}{1+k^2}}\right)\\
&=\frac{i}{2}\lim_{n\to+\infty}\ln\prod_{k=1}^{n}\left(\frac{i+\frac{1}{1+k^2}}{i-\frac{1}{1+k^2}}\right)=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\prod_{k=1}^{n}\left(\frac{i+\frac{1}{1+k^2}}{i-\frac{1}{1+k^2}}\right)\right)\\
&=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\prod_{k=1}^{n}\left(\frac{i\left(1+k^2\right)+1}{i\left(1+k^2\right)-1}\right)\right)=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\prod_{k=1}^{n}\left(\frac{1+k^2-i}{1+k^2+i}\right)\right)\\
&=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\prod_{k=1}^{n}\left(\frac{k^2+1-i}{k^2+1+i}\right)\right)=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\prod_{k=0}^{n}\left(\frac{(k+1)^2+1-i}{(k+1)^2+1+i}\right)\right)
\end{aligned}$$

Note that

$$(k+1)^2+1\pm i=\left(k+1+i\sqrt{1\pm i}\right)\left(k+1-i\sqrt{1\pm i}\right)$$

$$\begin{aligned}
&\Rightarrow\frac{i}{2}\ln\left(\lim_{n\to+\infty}\frac{\prod_{k=0}^{n}\left(k+1+i\sqrt{1-i}\right)}{\prod_{k=0}^{n}\left(k+1+i\sqrt{1+i}\right)}\cdot\frac{\prod_{k=0}^{n}\left(k+1-i\sqrt{1-i}\right)}{\prod_{k=0}^{n}\left(k+1-i\sqrt{1+i}\right)}\right)\\
&=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\frac{\frac{1}{n!n^{1+i\sqrt{1-i}}}\prod_{k=0}^{n}\left(k+1+i\sqrt{1-i}\right)}{\frac{1}{n!n^{1+i\sqrt{1+i}}}\prod_{k=0}^{n}\left(k+1+i\sqrt{1+i}\right)}\cdot\frac{\frac{1}{n!n^{1-i\sqrt{1-i}}}\prod_{k=0}^{n}\left(k+1-i\sqrt{1-i}\right)}{\frac{1}{n!n^{1-i\sqrt{1+i}}}\prod_{k=0}^{n}\left(k+1-i\sqrt{1+i}\right)}\right)\\
&=\frac{i}{2}\ln\left(\lim_{n\to+\infty}\frac{\prod_{k=0}^{n}\left(k+1+i\sqrt{1-i}\right)}{n!n^{1+i\sqrt{1-i}}}\cdot\frac{n!n^{1+i\sqrt{1+i}}}{\prod_{k=0}^{n}\left(k+1+i\sqrt{1+i}\right)}\cdot\frac{\prod_{k=0}^{n}\left(k+1-i\sqrt{1-i}\right)}{n!n^{1-i\sqrt{1-i}}}\right.\\
&\qquad\left.\cdot\frac{n!n^{1-i\sqrt{1+i}}}{\prod_{k=0}^{n}\left(k+1-i\sqrt{1+i}\right)}\right)\\
&=\frac{i}{2}\ln\left(\frac{\Gamma\left(1+i\sqrt{1+i}\right)\cdot\Gamma\left(1-i\sqrt{1+i}\right)}{\Gamma\left(1+i\sqrt{1-i}\right)\cdot\Gamma\left(1+i\sqrt{1-i}\right)}\right)
\end{aligned}$$

But

$$i\sqrt{1\pm i}=i\sqrt{\sqrt{2}\left(\cos\frac{\pi}{4}\pm i\sin\frac{\pi}{4}\right)}=\sqrt[4]{2}i\exp\pm\frac{\pi}{8}=\sqrt[4]{2}i\left(\cos\pm\frac{\pi}{8}+i\sin\pm\frac{\pi}{8}\right)$$

$$= \sqrt[4]{2}i\left(\cos\frac{\pi}{8} \pm i\sin\frac{\pi}{8}\right) = \sqrt[4]{2}\left(\mp\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}\right)$$

Then

$$\begin{aligned}
&\Rightarrow \frac{i}{2}\ln\left(\frac{\Gamma\left(1+\sqrt[4]{2}\left(-\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)\cdot\Gamma\left(1-\sqrt[4]{2}\left(-\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}{\Gamma\left(1+\sqrt[4]{2}\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)\cdot\Gamma\left(1-\sqrt[4]{2}\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}\right)\\
&= \frac{i}{2}\ln\left(\frac{\pi\left(-\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}\right)}{\sin\left(\pi\left(-\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}\right)\right)}\cdot\frac{\sin\left(\pi\left(\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}\right)\right)}{\pi\left(\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}\right)}\right)\\
&= \frac{i}{2}\ln\left(\frac{-\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}}{\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}}\cdot\frac{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}{\sin\left(\sqrt[4]{2}\pi\left(-\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{-\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}}{\sqrt[4]{2}\sin\frac{\pi}{8}+\sqrt[4]{2}i\cos\frac{\pi}{8}} &= \frac{-\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}\\
&= -\frac{\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}\cdot\frac{\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\\
&= -\left(\sin^2\frac{\pi}{8}-\cos^2\frac{\pi}{8}-2i\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)\\
&= \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} = \exp\left(\frac{\pi i}{4}\right)
\end{aligned}$$

Then

$$\begin{aligned}
&\Rightarrow \frac{i}{2}\ln\left(\exp\left(\frac{\pi i}{4}\right)\cdot\frac{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}{\sin\left(\sqrt[4]{2}\pi\left(-\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}\right)\\
&= \frac{i}{2}\ln\left(-\exp\left(\frac{\pi i}{4}\right)\cdot\frac{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}\right)\right)}\right)
\end{aligned}$$

On the other hand, Working expression

$$\begin{aligned}
\frac{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}\right)\right)} &= \frac{e^{\sqrt[4]{2}\pi i\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)}-e^{-\sqrt[4]{2}\pi i\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)}}{e^{\sqrt[4]{2}\pi i\left(\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}\right)}-e^{-\sqrt[4]{2}\pi i\left(\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}\right)}} \\
&= \frac{e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}}{e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}} \\
&= \frac{2e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-2e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}}{2e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-2e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}} \\
&= \frac{\begin{array}{l}\left(e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}\right)\left(e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}\right)\\ -\left(e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}\right)\left(e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}\right)\end{array}}{\begin{array}{l}\left(e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}\right)\left(e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}\right)\\ +\left(e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}\right)\left(e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}\right)\end{array}}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{-\frac{1}{i}\cdot\left(-i\frac{e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}}{e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}}\right)-\frac{e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}}{e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}}}{-\frac{1}{i}\cdot\left(-i\frac{e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}}{e^{\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi i\sin\frac{\pi}{8}}}\right)+\frac{e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}-e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}}{e^{\sqrt[4]{2}\pi\cos\frac{\pi}{8}}+e^{-\sqrt[4]{2}\pi\cos\frac{\pi}{8}}}} \\
&= \frac{-\frac{1}{i}\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)-\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)}{-\frac{1}{i}\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)+\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)} = -\frac{i\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)+\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)}{i\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)-\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)} \\
&= -\frac{i+\frac{\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)}{\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)}}{i-\frac{\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)}{\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)}}
\end{aligned}$$

Then the sum is equivalent to

$$S=\frac{i}{2}\ln\left(\exp\left(\frac{\pi i}{4}\right)\cdot-\frac{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)\right)}{\sin\left(\sqrt[4]{2}\pi\left(\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}\right)\right)}\right)$$

$$=-\frac{\pi}{8}+\frac{i}{2}\ln\left(\frac{i+\dfrac{\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)}{\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)}}{i-\dfrac{\tan\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)}{\tanh\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)}}\right)=\tan^{-1}\left(\frac{\tan\left(\sqrt[4]{2}\pi\cos\frac{\pi}{8}\right)}{\tanh\left(\sqrt[4]{2}\pi\sin\frac{\pi}{8}\right)}\right)-\frac{\pi}{8}$$

$$=\tan^{-1}\left(\frac{\tan\left(\sqrt[4]{2}\pi\dfrac{\sqrt{2+\sqrt{2}}}{2}\right)}{\tanh\left(\sqrt[4]{2}\pi\dfrac{\sqrt{2-\sqrt{2}}}{2}\right)}\right)-\frac{\pi}{8}=\tan^{-1}\left(\frac{\tan\left(\pi\dfrac{\sqrt{\sqrt{2}+1}}{2}\right)}{\tanh\left(\pi\dfrac{\sqrt{\sqrt{2}-1}}{2}\right)}\right)-\frac{\pi}{8}$$

In addition,

$$\sum_{n=2}^{\infty}\arctan\left(\frac{x}{n^2-1}\right)=\frac{\arctan x}{2}+\arctan\left(\frac{\tan\left\{\pi(1+x^2)^{\frac{1}{4}}\cos\left(\frac{\arctan x}{2}\right)\right\}}{\tanh\left\{\pi(1+x^2)^{\frac{1}{4}}\sin\left(\frac{\arctan x}{2}\right)\right\}}\right)$$

$$\sum_{n=0}^{\infty}\arctan\left(\frac{x}{n^2+1}\right)=\frac{\arctan x}{2}+\arctan\left(\frac{\tan\left\{\pi(1+x^2)^{\frac{1}{4}}\sin\left(\frac{\arctan x}{2}\right)\right\}}{\tanh\left\{\pi(1+x^2)^{\frac{1}{4}}\cos\left(\frac{\arctan x}{2}\right)\right\}}\right)$$

$$\sum_{n=1}^{\infty}\arctan\left(\frac{(-1)^n}{n^2+1}\right)=\arctan\left(\frac{\sin\left(2^{\frac{1}{4}}\pi\sin\left(\frac{\pi}{8}\right)\right)}{\sinh\left(2^{\frac{1}{4}}\pi\cos\left(\frac{\pi}{8}\right)\right)}\right)-\frac{\pi}{8}$$

□

Exercise 2.2.54: Prove:

$$\sum_{n=1}^{\infty}\frac{2^n}{C_n}=4+\frac{3}{2}\pi$$

where $C_n=\dfrac{1}{n+1}\binom{2n}{n}$ the nth Catalan number.

Proof: The sum is equivalent to

$$\sum_{n=1}^{\infty}\frac{(n+1)2^n}{\binom{2n}{n}}$$

Begin with the classic

$$\sum_{n=1}^{\infty}\frac{x^n}{\binom{2n}{n}}=\frac{4\sqrt{x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)}{(4-x)^{3/2}}-\frac{x}{x-4}\tag{1}$$

which converges when $x < 4$.
the above sum in (1) can be derived from the already known, and popular,

$$\frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}}$$

and letting $x \to \frac{\sqrt{x}}{2}$, diffing and what not to hammer it into the required form.
so, we will use $x = 2$. multiply (1) by x:

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{\binom{2n}{n}} = \frac{4x^{3/2}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)}{(4-x)^{3/2}} - \frac{x^2}{x-4}$$

Diff:

$$\sum_{n=1}^{\infty} \frac{(n+1)x^n}{\binom{2n}{n}} = \frac{24\sqrt{x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)}{(x-4)^2\sqrt{4-x}} - \frac{x(x-10)}{(x-4)^2}$$

Now, let $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(n+1)2^n}{\binom{2n}{n}} = \frac{24\sqrt{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)}{(2-4)^2\sqrt{4-2}} - \frac{2(2-10)}{(2-4)^2}$$

note that $\sin^{-1}(1/\sqrt{2}) = \frac{\pi}{4}$, and arrive at

$$\sum_{n=1}^{\infty} \frac{2^n}{C_n} = 4 + \frac{3}{2}\pi$$

□

Exercise 2.2.55: Evaluate:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

Solution: This a Euler Sum. Let S denote the sum, then

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} \right) \\
&= \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \right)^2 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} \\
&= \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \right)^2 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} T_n \qquad (1)
\end{aligned}$$

where $T_n = \sum_{k=1}^{2n} \frac{\chi_2(k)}{k} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}$.
On the other hand,

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \left\{ \sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{2k-1} - \frac{(-1)^{n-1}}{2n-1} \right\} \\ &= -\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{2k-1} \\ &= -\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{(-1)^{n-1}}{2n-1} \frac{(-1)^{k-1}}{2k-1} \\ &= -\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} T_n \end{aligned} \tag{2}$$

Add (1) and (2):

$$\begin{aligned} 2S &= -\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \right)^2 \\ \Rightarrow S &= -\frac{1}{2} \left(\frac{\pi^2}{8} \right) + \frac{1}{2} \left(\frac{\pi}{4} \right)^2 \\ \Rightarrow S &= -\frac{\pi^2}{32} \end{aligned}$$

From this evaluation, we also get the byproduct

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} T_n = \frac{3\pi^2}{32}$$

□

Exercise 2.2.56: Prove:

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right)^{k^2} \cdot \left(\frac{2k+1}{2k-1}\right)^k = \frac{\pi\sqrt{2}}{6}$$

Proof: Taking logarithms and expanding out the series results in cancellation of many terms.

$$\begin{aligned} \ln \prod_{k=2}^{N} \left(\frac{2k+1}{2k-1}\right)^k &= \sum_{k=2}^{N} k \ln \left(\frac{2k-1}{2k+1}\right) \\ &= -\ln(3) + N \ln(2N+1) - \sum_{k=1}^{N-1} \ln(2k-1) \\ &= -\ln(3) + N \ln(2N+1) - \ln \left(\frac{(2N)!}{2^N N!}\right) \\ &= \ln \left(\frac{N! 2^N}{3 \cdot (2N)!} (2N+1)^N\right) \end{aligned}$$

Therefore

$$\prod_{k=2}^{N} \left(\frac{2k+1}{2k-1}\right)^k = \frac{N! 2^N}{3 \cdot (2N)!} (2N+1)^N$$

Similarly, we obtain

$$\begin{aligned}\ln\prod_{k=2}^{N}\left(1-\frac{1}{k^2}\right)^{k^2} &= \sum_{k=2}^{N}\left[k^2\ln(k+1)+k^2\ln(k-1)-2k^2\ln(k)\right]\\ &= \ln(2)+N^2\ln\left(1+\frac{1}{N}\right)-(2N-1)\ln(N)+2\sum_{k=3}^{N-1}\ln(k)\\ &= \ln(2)+N^2\ln\left(1+\frac{1}{N}\right)-(2N-1)\ln(N)+2\ln\left(\frac{N!}{2N}\right)\\ &= \ln\left(\frac{(N!)^2}{2N^2}\cdot\frac{\left(1+\frac{1}{N}\right)^{N^2}}{N^{2N-1}}\right)\end{aligned}$$

Therefore

$$\prod_{k=2}^{N}\left(1-\frac{1}{k^2}\right)^{k^2} = \frac{(N!)^2}{2}\cdot\frac{\left(1+\frac{1}{N}\right)^{N^2}}{N^{2N+1}}$$

Multiply both products:

$$\begin{aligned}\prod_{k=2}^{N}\left(\frac{2k+1}{2k-1}\right)^{k}\left(1-\frac{1}{k^2}\right)^{k^2} &= \frac{N!2^N}{3\cdot(2N)!}(2N+1)^N\frac{(N!)^2}{2}\frac{\left(1+\frac{1}{N}\right)^{N^2}}{N^{2N+1}}\\ &= \frac{1}{6}\frac{(N!)^3}{(2N)!}\frac{2^N(2N+1)^N(N+1)^{N^2}}{N^{N^2+2N+1}}\end{aligned}$$

Let $N\to\infty$ and use **Stirling's Approximation**:

$$\begin{aligned}\prod_{k=2}^{\infty}\left(\frac{2k+1}{2k-1}\right)^{k}\left(1-\frac{1}{k^2}\right)^{k^2} &= \frac{1}{6}\lim_{N\to\infty}\frac{(N!)^3}{(2N)!}\frac{2^N(2N+1)^N(N+1)^{N^2}}{N^{N^2+2N+1}}\\ &= \frac{\sqrt{2}\pi}{6}\lim_{N\to\infty}\frac{(N+1)^{N^2}(2N+1)^N}{e^N N^{N^2+N}2^N}\\ &= \frac{\sqrt{2}}{6}\pi\end{aligned}$$

□

Exercise 2.2.57: Prove:

$$\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}k}{\binom{2n}{k}} = \frac{n}{n+1}$$

Proof: Rewrite the general term as

$$\begin{aligned}\frac{k}{\binom{2n}{k}} &= \frac{k\cdot k!(2n-k)!}{(2n)!}\\ &= \frac{(k+1)!(2n-k)!}{(2n)!}-\frac{k!(2n-k)!}{(2n)!}\\ &= (2n+2)(2n+1)\frac{(k+1)!(2n-k)!}{(2n+2)!}-(2n+1)\frac{k!(2n-k)!}{(2n+1)!}\\ &= (2n+2)(2n+1)\mathrm{B}(k+2,2n-k+1)-(2n+1)\mathrm{B}(k+1,2n-k+1)\end{aligned}$$

The sum is thus just equivalent to

$$(2n+2)(2n+1)\sum_{k=1}^{2n-1}(-1)^{k+1}\mathrm{B}(k+2,2n-k+1)$$
$$-(2n+1)\sum_{k=1}^{2n-1}(-1)^{k+1}\mathrm{B}(k+1,2n-k+1)$$

We have

$$\sum_{k=1}^{2n-1}(-1)^{k+1}\mathrm{B}(k+2,2n-k+1)=\sum_{k=1}^{2n-1}\int_0^1(-x)^{k+1}(1-x)^{2n-k}\,\mathrm{d}x$$
$$=\int_0^1\frac{x^2(1-x)^{2n-1}+x^{2n+1}}{1+\dfrac{x}{1-x}}\,\mathrm{d}x=\int_0^1x^2(1-x)^{2n}\,\mathrm{d}x+\int_0^1x^{2n+1}(1-x)\,\mathrm{d}x$$
$$=\mathrm{B}(3,2n+1)+\mathrm{B}(2n+2,2)=\frac{2!(2n)!}{(2n+3)!}+\frac{(2n+1)!1!}{(2n+3)!}$$
$$=\frac{2}{(2n+1)(2n+2)(2n+3)}+\frac{2n+1}{(2n+1)(2n+2)(2n+3)}$$
$$=\frac{1}{(2n+1)(2n+2)}$$

and similarly

$$\sum_{k=1}^{2n-1}(-1)^{k+1}\mathrm{B}(k+1,2n-k+1)=\sum_{k=1}^{2n-1}\int_0^1(-x)^{k}(1-x)^{2n-k}\,\mathrm{d}x$$
$$=\int_0^1\frac{x^1(1-x)^{2n-1}+x^{2n}}{1+\dfrac{x}{1-x}}\,\mathrm{d}x=\int_0^1x(1-x)^{2n}\,\mathrm{d}x+\int_0^1x^{2n}(1-x)\,\mathrm{d}x$$
$$=\mathrm{B}(2,2n+1)+\mathrm{B}(2n+1,2)=\frac{1!(2n)!}{(2n+2)!}+\frac{(2n)!1!}{(2n+2)!}$$
$$=\frac{1}{(2n+1)(2n+2)}+\frac{1}{(2n+1)(2n+2)}$$
$$=\frac{1}{(2n+1)(n+1)}$$

Putting everything together:

$$\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}k}{\binom{2n}{k}}=(2n+2)(2n+1)\sum_{k=1}^{2n-1}(-1)^{k+1}\mathrm{B}(k+2,2n-k+1)$$
$$-(2n+1)\sum_{k=1}^{2n-1}(-1)^{k+1}\mathrm{B}(k+1,2n-k+1)$$
$$=(2n+2)(2n+1)\cdot\frac{1}{(2n+1)(2n+2)}-(2n+1)\cdot\frac{1}{(2n+1)(n+1)}$$
$$=1-\frac{1}{n+1}=\frac{n}{n+1}$$

□

Exercise 2.2.58: Prove:

$$\sum_{k=0}^{n-1}\frac{1}{1+8\sin^2\left(\dfrac{k\pi}{n}\right)}=\frac{n(2^n+1)}{3(2^n-1)}$$

☞ **Proof:** I'm going to use the fact that $\dfrac{n/z}{z^n-1}$ has simple poles at the nth roots of unity. Also notice that

$$\begin{aligned}1+8\sin^2\left(\frac{k\pi}{n}\right) &= 1+4\left[1-\cos\left(\frac{2k\pi}{n}\right)\right] = 1+2\left(2-e^{2\pi ki/n}-e^{-2\pi ki/n}\right)\\ &= 1+2\left(2-z-\frac{1}{z}\right) \quad \text{where } z=e^{2\pi ki/n}\\ &= 1-\frac{2(z-1)^2}{z} = -\frac{(2z-1)(z-2)}{z}\end{aligned}$$

Now Let

$$f(z)=\frac{-z}{(2z-1)(z-2)}\frac{n/z}{z^n-1}=\frac{-1}{(2z-1)(z-2)}\frac{n}{z^n-1}$$

Then since $f(z)$ has a finite number of isolated singularities,

$$\sum_{k=0}^{n-1}\operatorname{Res}\left[f(z),e^{2\pi ki/n}\right]+\operatorname{Res}[f(z),1/2]+\operatorname{Res}[f(z),2]+\operatorname{Res}[f(z),\infty]=0$$

$$\operatorname{Res}\left[f(z),e^{2\pi ki/n}\right]=\frac{1}{1+8\sin^2\left(\frac{k\pi}{n}\right)}$$

$$\operatorname{Res}[f(z),1/2]=\frac{1}{2}\lim_{z\to 1/2}\frac{-1}{z-2}\frac{n}{z^{n-1}}=\frac{n}{3}\frac{1}{(1/2)^n-1}=-\frac{n}{3}\frac{2^n}{2^n-1}$$

$$\operatorname{Res}[f(z),2]=\lim_{z\to 2}\frac{-1}{2z-1}\frac{n}{z^n-1}=-\frac{n}{3}\frac{1}{2^n-1}$$

$$\operatorname{Res}[f(z),\infty]=0$$

So we have

$$\sum_{k=0}^{n-1}\frac{1}{1+8\sin^2\left(\frac{k\pi}{n}\right)}-\frac{n}{3}\frac{2^n+1}{2^n-1}=0$$

$$\Longrightarrow \sum_{k=0}^{n-1}\frac{1}{1+8\sin^2\left(\frac{k\pi}{n}\right)}=\frac{n}{3}\frac{2^n+1}{2^n-1}$$

□

Exercise 2.2.59: Prove:

$$\sum_{n\geq 1}H_n^{(v)}z^n=\frac{\operatorname{Li}_v(z)}{1-z}$$

☞ **Proof:**

$$\begin{aligned}\sum_{n=1}^{\infty}x^nH_n^{(\nu)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{x^n}{k^\nu}=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{x^n}{k^\nu}=\sum_{k=1}^{\infty}\frac{1}{k^\nu}\frac{x^k}{1-x}\\ &= \frac{1}{1-x}\sum_{k=1}^{\infty}\frac{x^k}{k^\nu}=\frac{\operatorname{Li}_\nu(x)}{1-x}\end{aligned}$$

□

Exercise 2.2.60: Prove:

$$\sum_{n=1}^{\infty}\frac{5n-6}{\binom{3n}{n}2^n}=\frac{1}{625}\left(105-86\pi+162\ln 2\right)$$

$$\sum_{n=0}^{\infty}\frac{50n-6}{\binom{3n}{n}2^n}=\pi$$

Proof: By using

$$\frac{1}{\binom{3n}{n}}=n\int_0^1 x^{n-1}(1-x)^{2n}\,\mathrm{d}x$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{5n-6}{\binom{3n}{n}2^n}&=\int_0^1\frac{1}{x}\sum_{n=1}^{\infty}(5n^2-6n)\left(\frac{x(1-x)^2}{2}\right)^n\,\mathrm{d}x\\&=2\int_0^1\frac{(1-x)^2(11x^3-22x^2+11x-2)}{(1+x^2)^3(2-x)^3}\mathrm{d}x\\&=2\left[\frac{1}{625}\left(\frac{50}{(-2+x)^2}+\frac{235}{-2+x}+\frac{50(2+11x)}{(1+x^2)^2}-\frac{655+515x}{1+x^2}\right.\right.\\&\quad\left.\left.-172\arctan(x)-54\ln(2-x)+27\ln\left(1+x^2\right)\right)\right]_{x=0}^{x=1}\\&=\frac{1}{625}\left(105-86\pi+162\ln 2\right)\end{aligned}$$

The second sum can be done in the same way. For e.g.

$$\sum_{n=1}^{\infty}\frac{50n-6}{\binom{3n}{n}2^n}=16\int_0^1\frac{(1-x)^2(11+7x-14x^2+7x^3)}{(1+x^2)^3(2-x)^3}\mathrm{d}x$$

□

Exercise 2.2.61: Prove:

$$\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{(-1)^n}{4^{3n}}=\frac{\Gamma\left(\frac{1}{8}\right)^2\Gamma\left(\frac{3}{8}\right)^2}{2^{7/2}\pi^3}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{1}{4^{3n}}=\frac{\pi}{\Gamma\left(\frac{3}{4}\right)^4}$$

☞ **Proof:**

$$\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{(-1)^n}{4^{3n}}=\frac{4}{\pi^2}\mathbf{K}\left(\frac{1-\sqrt{2}}{2}\right)^2=\frac{4}{\pi^2}\left(1-\frac{1-\sqrt{2}}{2}\right)\mathbf{K}\left(\frac{\frac{1-\sqrt{2}}{2}}{\frac{1-\sqrt{2}}{2}-1}\right)^2$$
$$=\frac{8(\sqrt{2}-1)}{\pi^2}\mathbf{K}\left(3-2\sqrt{2}\right)^2$$
$$=\frac{8(\sqrt{2}-1)}{\pi^2}\cdot\frac{(\sqrt{2}+1)\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{\pi 2^{13/2}}$$
$$=\frac{\Gamma\left(\frac{1}{8}\right)^2\Gamma\left(\frac{3}{8}\right)^2}{2^{7/2}\pi^3}$$

The second sum can also be calculated in the same way.

$$\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{1}{4^{3n}}=\frac{4}{\pi^2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)^2=\frac{\pi}{\Gamma\left(\frac{3}{4}\right)^4}$$

□

Exercise 2.2.62: Prove:

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-\zeta(3n)}{n}=\ln\left(\frac{2\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}\right)$$

☞ **Proof:**

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-\zeta(3n)}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{1}{k^{2n}}-\frac{1}{k^{3n}}\right)=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=2}^{\infty}\left(\frac{1}{k^{2n}}-\frac{1}{k^{3n}}\right)$$
$$=\sum_{k=2}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{k^2}\right)^n-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{k^3}\right)^n\right)=\sum_{k=2}^{\infty}\left[-\ln\left(1-\frac{1}{k^2}\right)+\ln\left(1-\frac{1}{k^3}\right)\right]$$
$$=-\ln\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)+\ln\prod_{k=2}^{\infty}\left(1-\frac{1}{k^3}\right)=-\ln\left(\frac{1}{2}\right)+\ln\left(\frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}\right)$$
$$=\ln\left(\frac{2\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}\right)$$

□

Exercise 2.2.63: Prove:

$$\sum_{n=1}^{\infty}\frac{H_n}{(4n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n}{(4n+3)^2}=\frac{7}{2}\zeta(3)-\frac{3\pi^2}{8}\ln 2-\frac{\pi}{2}\mathbf{G}$$

☞ **Proof:** Let

$$f(z)=\frac{\pi\left(\psi_0(-z)+\gamma\right)\cot(\pi z)}{(4z+3)^2}$$

$f(z)$ has a number of poles in the complex plane. The residue of the pole of order 2 at $z=-\frac{3}{4}$ is found to be

$$\frac{\pi^2}{8}\left(3\ln 2-\frac{\pi}{2}\right)-\frac{\pi}{16}\psi_1\left(\frac{3}{4}\right)$$

At a negative integer n, the pole is simple and the residue is

$$\frac{H_{n-1}}{(4n-3)^2}$$

At a positive integer $n(\geq 0)$, the pole has order 2 and the residue is

$$\frac{H_n}{(4n+3)^2}-\frac{8}{(4n+1)^3}$$

The sum of all the residues is 0, so

$$\frac{\pi^2}{8}\left(3\ln 2-\frac{\pi}{2}\right)-\frac{\pi}{16}\psi_1\left(\frac{3}{4}\right)+\sum_{n=1}^{\infty}\frac{H_n}{(4n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n}{(4n+3)^2}-8\sum_{n=0}^{\infty}\frac{1}{(4n+3)^3}=0$$

$$\Longrightarrow \sum_{n=1}^{\infty}\frac{H_n}{(4n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n}{(4n+3)^2}=\frac{7}{2}\zeta(3)-\frac{3\pi^2}{8}\ln 2-\frac{\pi}{2}\mathbf{G}$$

□

Exercise 2.2.64: Evaluate:

$$\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{3}\right)}{e^{\pi k}+1}$$

✎ **Solution:** Start by showing that

$$4\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{3}\right)}{e^{\pi k}+1}=\cot\left(\frac{\pi}{6}\right)-\sqrt{3}\varphi\left(-e^{-\pi}\right)\varphi\left(-e^{-3\pi}\right)$$

where $\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}$ is a **Theta Function**.

We have

$$\varphi\left(-e^{-\pi}\right)=\frac{\sqrt[4]{\pi}}{\sqrt[4]{2}\Gamma\left(\frac{3}{4}\right)}$$

$$\varphi\left(-e^{-3\pi}\right)=\frac{\sqrt[4]{\pi}}{6\Gamma\left(\frac{3}{4}\right)}\left(3^{1/4}-6\sqrt{2}+4\sqrt{6}\right)\sqrt[4]{9+6\sqrt{3}}$$

Putting these, I arrived at the amazing result

$$\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{3}\right)}{e^{\pi k}+1}=\frac{\sqrt{3}}{4}\left[1-\frac{\sqrt{\pi}}{6\Gamma^2\left(\frac{3}{4}\right)}\left(3^{1/4}-6\sqrt{2}+4\sqrt{6}\right)\sqrt[4]{\frac{9+6\sqrt{3}}{2}}\right]$$

□

Exercise 2.2.65: Evaluate:

$$\sum_{k=1}^{\infty} \frac{(2^{2k-1}-2)(4^{2k+1}-3^{2k+1})}{144^k\, k\,(2k+1)}\zeta(2k)$$

Solution: Within the interval $0 < x < \pi/2$, the logtangent function has the series representation

$$\ln(\tan x) = \ln x - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(2^{2k-1}-1)B_{2k}}{k\,(2k)!}x^{2k}$$

So, let's integrate this series over a suitable interval, such as:

$$\int_{\pi/4}^{\pi/3} \ln(\tan x)\,\mathrm{d}x = \int_{\pi/4}^{\pi/3} \ln x\,\mathrm{d}x - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(2^{2k-1}-1)B_{2k}}{k\,(2k)!}\int_{\pi/4}^{\pi/3} x^{2k}\,\mathrm{d}x$$

$$= (x\ln x - x)\Big|_{\pi/4}^{\pi/3} - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(2^{2k-1}-1)B_{2k}}{k\,(2k)!}\left(\frac{x^{2k+1}}{2k+1}\right)\Bigg|_{\pi/4}^{\pi/3}$$

$$= \frac{\pi}{3}\ln\frac{\pi}{3} - \frac{\pi}{4}\ln\frac{\pi}{4} - \frac{\pi}{12} - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(2^{2k-1}-1)B_{2k}\pi^{2k+1}}{k\,(2k+1)\,(2k)!}\left(\frac{1}{3^{2k+1}} - \frac{1}{4^{2k+1}}\right)$$

$$= \pi\ln\left(\frac{\sqrt{2}\,\pi^{1/12}}{3^{1/3}}\right) - \frac{\pi}{12} - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(2^{2k-1}-1)B_{2k}\pi^{2k+1}}{k\,(2k+1)\,(2k)!}\left(\frac{4^{2k+1}-3^{2k+1}}{12^{2k+1}}\right)$$

$$= \pi\ln\left(\frac{\sqrt{2}\,\pi^{1/12}}{3^{1/3}\,e^{1/12}}\right) - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(2^{2k-1}-1)B_{2k}\pi^{2k+1}}{k\,(2k+1)\,(2k)!}\left(\frac{4^{2k+1}-3^{2k+1}}{12^{2k+1}}\right)$$

Next, we use the classic Zeta function result:

$$\zeta(2n) = (-1)^{n+1}\,\frac{B_{2n}\,\pi^{2n}2^{2n-1}}{(2n)!}$$

In the re-arranged form:

$$B_{2n} = (-1)^{n+1}\frac{(2n)!}{\pi^{2n}\,2^{2n-1}}\zeta(2n)$$

to obtain

$$\pi\ln\left(\frac{\sqrt{2}\,\pi^{1/12}}{3^{1/3}\,e^{1/12}}\right) + \frac{\pi}{6}\sum_{k=1}^{\infty}\frac{(2^{2k-1}-2)(4^{2k+1}-3^{2k+1})}{144^k\,k\,(2k+1)}\zeta(2k)$$

The logtangent integral is also solvable in terms of the **Clausen Function**:

$$\int_0^{\phi}\ln(\tan x)\,\mathrm{d}x = -\frac{1}{2}\mathrm{Cl}_2(2\phi) - \frac{1}{2}\mathrm{Cl}_2(\pi - 2\phi)$$

So the previous integral is equivalent to

$$\int_{\pi/4}^{\pi/3}\ln(\tan x)\mathrm{d}x = -\frac{1}{2}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \mathrm{Cl}_2\left(\frac{\pi}{2}\right) = \mathbf{G} - \frac{5}{6}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

Since

$$\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) = \frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ ,\ \mathrm{Cl}_2\left(\frac{\pi}{2}\right) = \mathbf{G}$$

Equating the two different evaluations of the logtangent integral, we get the final result:

$$\sum_{k=1}^{\infty}\frac{(2^{2k-1}-2)(4^{2k+1}-3^{2k+1})}{144^k\,k\,(2k+1)}\zeta(2k) = \frac{6\mathbf{G}}{\pi} - \frac{5}{\pi}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + 6\ln\left(\frac{3^{1/3}\,e^{1/12}}{\sqrt{2}\,\pi^{1/12}}\right)$$

If expressed in terms of the Eta function (alternating Zeta function), since the logtangent series then reduces to:

$$\ln(\tan x) = \ln x + \sum_{k=1}^{\infty} \frac{2^{2k}\,\eta(2k)}{\pi^{2k}\,k} x^{2k}$$

So, for $0 < p < q < 1/2$ and $p\,, q \in \mathbb{Q}$, we have

$$\begin{aligned}\int_{p\pi}^{q\pi} \ln(\tan x)\,\mathrm{d}x &= \frac{1}{2}\left[\mathrm{Cl}_2(2p\pi) + \mathrm{Cl}_2(\pi - 2p\pi) - \mathrm{Cl}_2(2q\pi) - \mathrm{Cl}_2(\pi - 2q\pi)\right] \\ &= (x\ln x - x)\Big|_{p\pi}^{q\pi} + \pi \sum_{k=1}^{\infty} \frac{2^{2k}(q^{2k+1} - p^{2k+1})}{k\,(2k+1)}\eta(2k)\end{aligned}$$

Or

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{2^{2k}(q^{2k+1} - p^{2k+1})}{k\,(2k+1)}\eta(2k) \\ &= \frac{1}{2\pi}\left[\mathrm{Cl}_2(2p\pi) + \mathrm{Cl}_2(\pi - 2p\pi) - \mathrm{Cl}_2(2q\pi) - \mathrm{Cl}_2(\pi - 2q\pi)\right] - \frac{1}{\pi}(x\ln x - x)\Big|_{p\pi}^{q\pi}\end{aligned}$$

setting $p = 1/4$ and $q = 3/10$ gives the series:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(6^{2k+1} - 5^{2k+1})}{100^k\,k\,(2k+1)}\eta(2k) &= \frac{20\mathbf{G}}{\pi} - \frac{10}{\pi}\left[\mathrm{Cl}_2\left(\frac{2\pi}{5}\right) + \mathrm{Cl}_2\left(\frac{3\pi}{5}\right)\right] + 20\ln\left(\frac{5^{3/10}\,e^{1/20}}{2^{1/5}\,3^{3/10}\,\pi^{1/20}}\right) \\ &= \frac{20\mathbf{G}}{\pi} - \frac{10}{\pi}\left[\mathrm{Cl}_2\left(\frac{2\pi}{5}\right) + \mathrm{Cl}_2\left(\frac{3\pi}{5}\right)\right] + 10\ln\left(\frac{125}{108}\sqrt{\frac{e}{\pi}}\right)\end{aligned}$$

□

Exercise 2.2.66: Prove:

$$\sum_{k=1}^{\infty} \frac{\cos^2(k)}{4k^2 - 1} = \frac{1}{2} - \frac{\pi}{8}\sin(1)$$

Proof:

$$\sum_{n=1}^{\infty} \frac{n\sin(nx)}{a^2 + n^2} = \frac{\pi}{2}\frac{\sinh(a(\pi - x))}{\sinh(a\pi)}$$

Integrate with respect to x and apply half angle formula:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n}{a^2 + n^2}\int_0^{\phi} \sin(nx)\mathrm{d}x &= \frac{\pi}{2}\frac{1}{\sinh(a\pi)}\int_0^{\phi} \sinh(a(\pi - x))\mathrm{d}x \\ -\sum_{n=1}^{\infty} \frac{\cos(\phi n) - 1}{n^2 + a^2} &= \frac{\pi}{2}\frac{\cosh(a\pi) - \cosh(a(\pi - \phi))}{a\sinh(a\pi)} \\ -2\sum_{n=1}^{\infty} \frac{\cos^2\left(\frac{\phi n}{2}\right) - 1}{n^2 + a^2} &= \frac{\pi}{2}\frac{\cosh(a\pi) - \cosh(a(\pi - \phi))}{a\sinh(a\pi)}\end{aligned}$$

Put $a = \dfrac{i}{2}$:

$$8\sum_{n=1}^{\infty} \frac{\cos^2\left(\frac{\phi n}{2}\right) - 1}{4n^2 - 1} = -\pi\sin\left(\frac{\phi}{2}\right)$$

Finally, $\phi = 2$ gives

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{4n^2-1} = \frac{1}{2} - \frac{\pi}{8}\sin(1)$$

The General Formula is

$$\sum_{n=1}^{\infty} \frac{\cos^2(\phi n)}{a^2n^2-1} = \frac{a - \pi\cot\left(\frac{\pi}{a}\right)}{2a} + \frac{\pi}{4a}\left\{\frac{\cos\left(\frac{\pi}{a}\right) - \cos\left(\frac{\pi-2\phi}{a}\right)}{\sin\left(\frac{\pi}{a}\right)}\right\}$$

□

Exercise 2.2.67: Prove:

$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}$$

Proof: Method One:

Let

$$P(x) = \prod_{n=1}^{\infty}\left(1 - e^{-2\pi n x}\right)$$

Then using the Poisson summation formula, we can prove that

$$\frac{P(x)}{P\left(\frac{1}{x}\right)} = \frac{\exp\left(\frac{\pi}{12}\left(x - \frac{1}{x}\right)\right)}{\sqrt{x}}, \quad x > 0$$

Thus if we let $Q(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln P(x)$, it follows that

$$Q(x) + \frac{1}{x^2}Q\left(\frac{1}{x}\right) = -\frac{1}{2x} + \frac{\pi}{12}\left(1 + \frac{1}{x^2}\right)$$

Plugging $x = 1$, we obtain $Q(1) = \frac{\pi}{12} - \frac{1}{4}$. Finally, it is easy to find that our sum is equal to

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n}-1} = \frac{Q(1)}{2\pi} = \frac{1}{24} - \frac{1}{8\pi}$$

Method Two:

There is another method using elliptic integrals.

$$\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}=-\frac{q}{2}\frac{\mathrm{d}}{\mathrm{d}q}\ln\prod_{n=1}^{\infty}\left(1-q^{2n}\right)=-\frac{q}{12}\frac{\mathrm{d}}{\mathrm{d}q}\ln\left(\frac{2}{\pi^3\sqrt{q}}kk'\mathbf{K}^3\right)=\frac{1}{24}-\frac{q}{12}\frac{\mathrm{d}}{\mathrm{d}q}\ln\left(kk'\mathbf{K}^3\right)$$

$$=\frac{1}{24}-\frac{q}{12}\frac{\mathrm{d}\left(\frac{\mathbf{K}'}{\mathbf{K}}\right)}{\mathrm{d}q}\frac{1}{\frac{\mathrm{d}\left(\frac{\mathbf{K}'}{\mathbf{K}}\right)}{\mathrm{d}k}}\frac{\mathrm{d}\left(\ln\left(kk'\mathbf{K}^3\right)\right)}{\mathrm{d}k}$$

$$=\frac{1}{24}+\frac{1}{12\pi}\frac{\mathbf{K}^2}{\frac{\mathrm{d}\mathbf{K}'}{\mathrm{d}k}\mathbf{K}-\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}k}\mathbf{K}'}\left(\frac{1}{k}-\frac{k}{k'^2}+\frac{3}{\mathbf{K}}\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}k}\right)$$

$$=\frac{1}{24}+\frac{1}{12\pi}\frac{\mathbf{K}^2kk'^2}{k^2\mathbf{K}'\mathbf{K}-\mathbf{K}\mathbf{E}'-\mathbf{K}'\mathbf{E}+k'^2\mathbf{K}\mathbf{K}'}\left(\frac{k'^2-k^2}{kk'^2}+\frac{3}{\mathbf{K}}\frac{\mathbf{E}-k'^2\mathbf{K}}{kk'^2}\right)$$

$$=\frac{1}{24}-\frac{1}{6\pi^2}\left(\mathbf{K}^2(1-2k^2)+3\mathbf{E}\mathbf{K}-3k'^2\mathbf{K}^2\right)$$

$$=\frac{1}{24}-\frac{1}{6\pi^2}\left(\mathbf{K}^2(k^2-2)+3\mathbf{K}\mathbf{E}\right)=\frac{1}{24}+\frac{1}{6}\left(\frac{\mathbf{K}}{\pi}\right)^2(2-k^2)-\frac{\mathbf{K}\mathbf{E}}{2\pi^2}$$

Here, $\mathbf{K}$ and $\mathbf{E}$ represent Complete Elliptic Integrals of the first and second kind respectively. Now it is well known that when $q=e^{-\pi}$, the values of $k,\mathbf{K}$ and $\mathbf{E}$ are given by

$$k=\frac{1}{\sqrt{2}}\ ,\ \mathbf{K}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}\ ,\ \mathbf{E}=\frac{4\Gamma^2\left(\frac{3}{4}\right)+\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}}$$

Thus, we obtain

$$\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1}=\frac{1}{24}-\frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma^2\left(\frac{3}{4}\right)}{16\pi^3}=\frac{1}{24}-\frac{1}{8\pi}$$

Use the relation

$$\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n\frac{\mathbf{K}'}{\mathbf{K}}}-1}=\frac{1}{24}+\frac{1}{6}\left(\frac{\mathbf{K}}{\pi}\right)^2(2-k^2)-\frac{\mathbf{K}\mathbf{E}}{2\pi^2}$$

we can prove that

$$\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n\sqrt{2}}-1}=\frac{1}{24}-\frac{1}{8\pi\sqrt{2}}-\frac{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{768\pi^3}$$

□

Exercise 2.2.68: Prove:

$$\sum_{n=1}^{\infty}\operatorname{sech}(\pi n)=\frac{1}{2}\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2}-1\right)$$

Proof: We can write the series as

$$2\sum_{n=1}^{\infty}\frac{e^{\pi n}}{1+e^{2\pi n}}=2\sum_{n=1}^{\infty}\frac{e^{-\pi n}}{1+e^{-2\pi n}}$$

Which, considering the Jacobi identity, we can write as:

$$2\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}=\frac{\theta^2-1}{2}$$

Since

$$q=e^{-\pi}$$

and

$$\theta^2(e^{-\pi})=\frac{\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}$$

we have

$$\sum_{n=1}^{\infty}\operatorname{sech}(\pi n)=\frac{1}{2}\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2}-1\right)$$

□

Exercise 2.2.69: Prove:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k(1+e^{\pi k})}=\frac{\pi}{8}-\frac{5\ln(2)}{8}$$

Proof:

$$\begin{aligned}
S=\sum_{k=1}^{\infty}\frac{(-1)^k}{k(1+e^{\pi k})}&=-\sum_{j=1}^{\infty}(-1)^{j+1}\ln\left(1+e^{-\pi j}\right)\\
&=-\ln\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1+q^{2n}}\quad\text{where } q=e^{-\pi}\\
&=-\frac{1}{6}\ln\left(\frac{2q^{\frac{1}{4}}(kk')^{-\frac{1}{2}}}{\frac{1}{4}q^{-\frac{1}{2}}kk'^{-\frac{1}{2}}}\right)=-\frac{1}{6}\ln\left(\frac{8q^{\frac{3}{4}}}{k^{\frac{3}{2}}}\right)\\
&=-\frac{1}{2}\ln(2)+\frac{\pi}{8}-\frac{1}{8}\ln(2)\quad\text{as } k=\frac{1}{\sqrt{2}}\\
&=\frac{\pi}{8}-\frac{5}{8}\ln(2)
\end{aligned}$$

Where I used the identities

$$\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)^6=2q^{\frac{1}{4}}(kk')^{-\frac{1}{2}}$$

$$\prod_{n=1}^{\infty}\left(1+q^{2n}\right)^6=\frac{1}{4}q^{-\frac{1}{2}}kk'^{-\frac{1}{2}}$$

□

Exercise 2.2.70: Evaluate:

$$\sum_{n\geq 0}\frac{x^n}{(n+k+1)\,n!}\quad k\in\mathbb{N}$$

✎ **Solution:**

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{(n+k+1)n!} &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\int_0^1 t^{n+k}\mathrm{d}t\\ &= \int_0^1 t^k\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\,\mathrm{d}t\\ &= \int_0^1 t^k e^{xt}\mathrm{d}t\\ &= (-x)^{-k-1}\left(\Gamma(1+k)-\Gamma(1+k,-x)\right)\end{aligned}$$

□

➻ **Exercise 2.2.71:** Evaluate:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{n+2k+1}-\sum_{k=1}^{\infty}\frac{1}{n+2k}\right)^2$$

✎ **Solution:**

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2 &= \sum_{n=0}^{\infty}\left(\int_0^1 x^n\sum_{k=1}^{\infty}(-x)^{k-1}\mathrm{d}x\right)\left(\int_0^1 y^n\sum_{k=1}^{\infty}(-y)^{k-1}\mathrm{d}y\right)\\ &= \sum_{n=0}^{\infty}\left(\int_0^1\frac{x^n}{1+x}\mathrm{d}x\right)\left(\int_0^1\frac{y^n}{1+y}\mathrm{d}y\right)\\ &= \sum_{n=0}^{\infty}\int_0^1\int_0^1\frac{(xy)^n}{(1+x)(1+y)}\mathrm{d}x\mathrm{d}y\\ &= \int_0^1\int_0^1\frac{\mathrm{d}x\mathrm{d}y}{(1+x)(1+y)(1-xy)} = \ln 2\end{aligned}$$

□

➻ **Exercise 2.2.72:** Prove:

$$\begin{aligned}P &= \left(\frac{2}{1}\right)^{\frac{1}{8}}\cdot\left(\frac{2\cdot 2}{1\cdot 3}\right)^{\frac{3}{16}}\cdot\left(\frac{2\cdot 2\cdot 2\cdot 4}{1\cdot 3\cdot 3\cdot 3}\right)^{\frac{6}{32}}\cdot\left(\frac{2\cdot 2\cdot 2\cdot 2\cdot 4\cdot 4\cdot 4\cdot 4}{1\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 5}\right)^{\frac{10}{64}}\cdots\\ &= \exp\left(\frac{7\zeta(3)}{24\zeta(2)}\right)\end{aligned}$$

☞ **Proof:** It is clear that

$$\ln P = -\sum_{n=1}^{\infty}\frac{n(n+1)}{2^{n+3}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\ln(k+1)$$

To simplify this formula, we observe that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\ln(k+1) &= \lim_{s\to 0^+}\Gamma(s+1)\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(k+1)^{-s}-1}{(-s)}\\ &= -\lim_{s\to 0^+}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\Gamma(s)}{(k+1)^s}\\ &= -\lim_{s\to 0^+}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^{\infty}x^{s-1}e^{-(k+1)x}\,\mathrm{d}x\\ &= -\lim_{s\to 0^+}\int_0^{\infty}x^{s-1}e^{-x}(1-e^{-x})^n\,\mathrm{d}x\\ &= -\int_0^{\infty}e^{-x}(1-e^{-x})^n\,\frac{\mathrm{d}x}{x}\end{aligned}$$

This gives

$$\begin{aligned}\ln P &= \sum_{n=1}^{\infty} \frac{n(n+1)}{2^{n+3}} \int_0^{\infty} e^{-x}(1-e^{-x})^n \frac{\mathrm{d}x}{x} \\ &= \frac{1}{8}\int_0^{\infty} e^{-x} \sum_{n=1}^{\infty} n(n+1)\left(\frac{1-e^{-x}}{2}\right)^n \frac{\mathrm{d}x}{x} \\ &= \int_0^{\infty} x^{-1} \frac{e^{-x}(1-e^{-x})}{(1+e^{-x})^3}\,\mathrm{d}x\end{aligned}$$

Now we introduce the function

$$I(s) = \int_0^{\infty} x^{s-1} \frac{e^{-x}(1-e^{-x})}{(1+e^{-x})^3}\,\mathrm{d}x$$

It is clear that $I(s)$ is analytic for $\Re s > -1$ and $\ln P = \lim\limits_{s\to 0^+} I(s)$. But for large s, we have

$$\begin{aligned}I(s) &= \int_0^{\infty} x^{s-1} \frac{e^{-x}(1-e^{-x})}{(1+e^{-x})^3}\,\mathrm{d}x \\ &= \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} (-1)^{n-1} n^2 e^{-nx}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} n^2 \int_0^{\infty} x^{s-1} e^{-nx}\,\mathrm{d}x \\ &= \Gamma(s)\eta(s-2) = \Gamma(s)(1-2^{3-s})\zeta(s-2)\end{aligned}$$

By the Identity Theorem, this remains valid on any open subset of $\mathbb{C}$ where $I(s)$ is analytic. In particular, this holds for $\Re s > -1$. Thus taking $s \to 0^+$, we obtain $\ln P = -7\zeta'(-2)$. Finally, differentiating both sides of the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

and plugging $s = 2$, we obtain

$$\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2} = -\frac{\zeta(3)}{24\zeta(2)}$$

Hence we have

$$P = \exp\left(\frac{7\zeta(3)}{24\zeta(2)}\right)$$

□

Exercise 2.2.73: Prove:

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\left(\coth(\pi) + \pi\,\mathrm{csch}^2(\pi) - \frac{2}{\pi}\right)$$

Proof: Here is the approach using residues.

$$\sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2)^2} = -\mathrm{Res}_{z=i} \frac{\pi\cot(\pi z)}{(1+z^2)^2} - \mathrm{Res}_{z=-i} \frac{\pi\cot(\pi z)}{(1+z^2)^2}$$

The Residues are found to be

$$\mathrm{Res}_{z=i} \frac{\pi\cot(\pi z)}{(1+z^2)^2} = \frac{\pi}{4}\left(-\pi\,\mathrm{csch}^2(\pi) - \coth(\pi)\right)$$

$$\text{Res}_{z=-i}\frac{\pi\cot(\pi z)}{(1+z^2)^2}=\frac{\pi}{4}\left(-\pi\text{csch}^2(\pi)-\coth(\pi)\right)$$

So we have

$$\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)^2}=\frac{\pi}{2}\left(\pi\text{csch}^2(\pi)+\coth(\pi)\right)$$

$$1+2\sum_{n=1}^{\infty}\frac{1}{(1+n^2)^2}=\frac{\pi}{2}\left(\pi\text{csch}^2(\pi)+\coth(\pi)\right)$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{(1+n^2)^2}=\frac{1}{4}(\pi^2\text{csch}^2(\pi)+\pi\coth(\pi)-2)$$

□

Exercise 2.2.74: Prove:

$$\sum_{k=1}^{\infty}\ln\left(\tanh\frac{\pi k}{2}\right)=\ln\left(\frac{1}{\Gamma\left(\frac{3}{4}\right)}\sqrt[4]{\frac{\pi}{2}}\right)$$

Proof:

$$\sum_{k=1}^{\infty}\ln\left(\tanh\frac{\pi k}{2}\right)=\ln\left(\prod_{k=1}^{\infty}\tanh\frac{\pi k}{2}\right)=\ln\left(\prod_{k=1}^{\infty}\frac{1-e^{-\pi k}}{1+e^{-\pi k}}\right)$$

Now

$$\prod_{k=1}^{\infty}\frac{1-q^n}{1+q^n}=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}=\vartheta_4(0,q)$$

Here $\vartheta_4(0,q)$ is the **Jacobi Theta Function**. So putting $q=e^{-\pi}$ gives

$$\prod_{k=1}^{\infty}\tanh\frac{\pi k}{2}=\vartheta_4\left(0,e^{-\pi}\right)$$

The value of $\vartheta_4(0,e^{-\pi})$ is known.

$$\vartheta_4(0,e^{-\pi})=\frac{1}{\Gamma\left(\frac{3}{4}\right)}\sqrt[4]{\frac{\pi}{2}}$$

This immediately proves our result:

$$\sum_{k=1}^{\infty}\ln\left(\tanh\frac{\pi k}{2}\right)=\ln\left(\frac{1}{\Gamma\left(\frac{3}{4}\right)}\sqrt[4]{\frac{\pi}{2}}\right)$$

□

Exercise 2.2.75: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^3}$$

Solution: Let $f(z)=\dfrac{\pi\csc(\pi z)\psi_1(-z)}{z^3}$. Since

$$\pi\csc(\pi z)=\frac{(-1)^n}{z-n}+(-1)^n\zeta(2)(z-n)+\mathcal{O}\left((z-n)^3\right)$$

$$\psi_1(-z)=\frac{1}{(z-n)^2}+\left(H_n^{(2)}+\zeta(2)\right)+\mathcal{O}\left((z-n)\right)$$

We have

$$z^3 f(z) = \frac{(-1)^n}{(z-n)^3} + \frac{(-1)^n \left(H_n^{(2)} + 2\zeta(2)\right)}{z-n} + \mathcal{O}(1)$$

Then at $z = n$,

$$\begin{aligned}\sum_{n=1}^{\infty} \text{Res}(f,n) =& 6\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5} + \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3} + 2\zeta(2)\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \\ =& -\frac{45}{8}\zeta(5) - \frac{\pi^2}{4}\zeta(3) + \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3}\end{aligned}$$

At $z = n$,

$$\begin{aligned}\sum_{n=1}^{\infty} \text{Res}(f,-n) =& \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\zeta(2) - H_{n-1}^{(2)}\right)}{n^3} \\ =& \frac{15}{16}\zeta(5) + \frac{\pi^2}{8}\zeta(3) + \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3}\end{aligned}$$

At $z = 0$,

$$\text{Res}(f,0) = \frac{\pi^2}{3}\zeta(3) + 4\zeta(5)$$

Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2)}}{n^3} = \frac{11}{32}\zeta(5) - \frac{5\pi^2}{48}\zeta(3)$$

□

Exercise 2.2.76: Prove:

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\left(\coth(\pi) + \pi\text{csch}^2(\pi) - \frac{2}{\pi}\right)$$

Proof: Here is the approach using residues.

$$\sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2)^2} = -\text{Res}_{z=i} \frac{\pi\cot(\pi z)}{(1+z^2)^2} - \text{Res}_{z=-i} \frac{\pi\cot(\pi z)}{(1+z^2)^2}$$

The Residues are found to be

$$\text{Res}_{z=i} \frac{\pi\cot(\pi z)}{(1+z^2)^2} = \frac{\pi}{4}\left(-\pi\text{csch}^2(\pi) - \coth(\pi)\right)$$

$$\text{Res}_{z=-i} \frac{\pi\cot(\pi z)}{(1+z^2)^2} = \frac{\pi}{4}\left(-\pi\text{csch}^2(\pi) - \coth(\pi)\right)$$

So we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2)^2} = \frac{\pi}{2}\left(\pi\text{csch}^2(\pi) + \coth(\pi)\right)$$

$$1 + 2\sum_{n=1}^{\infty} \frac{1}{(1+n^2)^2} = \frac{\pi}{2}\left(\pi\text{csch}^2(\pi) + \coth(\pi)\right)$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(1+n^2)^2} = \frac{1}{4}(\pi^2\text{csch}^2(\pi) + \pi\coth(\pi) - 2)$$

□

Exercise 2.2.77: Evaluate:

$$\sum_{n=1}^{\infty}\frac{\psi_1(n)}{2^n n^2}$$

Solution:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\psi_1(n)}{2^n n^2}&=-\sum_{n=1}^{\infty}\psi_1(n)\left(\frac{\ln(2)}{2^n n}+\int_0^{\frac12}x^{n-1}\ln(x)\mathrm{d}x\right)\\&=-\ln(2)\sum_{n=1}^{\infty}\frac{\psi_1(n)}{2^n n}-\int_0^{\frac12}\frac{\ln(x)}{1-x}\left(\zeta(2)-\mathrm{Li}_2(x)\right)\mathrm{d}x\end{aligned}\tag{1}$$

Evaluation of $\int_0^{\frac12}\frac{\ln(x)}{1-x}\left(\zeta(2)-\mathrm{Li}_2(x)\right)\mathrm{d}x$

Using the identity $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-\ln(x)\ln(1-x)$, we have

$$\begin{aligned}\int_0^{\frac12}\frac{\ln(x)}{1-x}\left(\zeta(2)-\mathrm{Li}_2(x)\right)\mathrm{d}x&=\int_0^{\frac12}\frac{\ln^2(x)\ln(1-x)}{1-x}\mathrm{d}x+\int_0^{\frac12}\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x}\mathrm{d}x\\&=-\frac{\pi^4}{360}-\frac{\ln^4(2)}{4}+\left(\frac12\mathrm{Li}_2^2(1-x)\right)\Big|_0^{\frac12}\\&=-\frac{19\pi^4}{1440}-\frac{\pi^2}{24}\ln^2(2)-\frac{\ln^4(2)}{8}\end{aligned}\tag{2}$$

The red integral was evaluated using equation 12 on page 310 of *Leonard Lewin - Polylogarithms and associated functions*. The proof is discussed on page 203 and 204.

Evaluation of $\sum_{n=1}^{\infty}\frac{\psi_1(n)}{2^n n}$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\psi_1(n)}{2^n n}=\sum_{n=1}^{\infty}\psi_1(n)\int_0^{\frac12}x^{n-1}\mathrm{d}x=\int_0^{\frac12}\frac{\zeta(2)-\mathrm{Li}_2(x)}{1-x}\mathrm{d}x=\int_0^{\frac12}\frac{\ln(x)\ln(1-x)}{1-x}\mathrm{d}x\\&+\int_0^{\frac12}\frac{\mathrm{Li}_2(1-x)}{1-x}\mathrm{d}x=\frac{\ln^3(2)}{2}+\frac12\int_0^{\frac12}\frac{\ln^2(1-x)}{x}\mathrm{d}x+\zeta(3)-\mathrm{Li}_3\left(\frac12\right)\\&=\frac{\ln^3(2)}{2}+\frac12\left(-\ln^3(2)-2\ln(2)\mathrm{Li}_2\left(\frac12\right)+2\zeta(3)-2\mathrm{Li}_3\left(\frac12\right)\right)+\zeta(3)-\mathrm{Li}_3\left(\frac12\right)\\&=\frac{\pi^2}{12}\ln(2)+\frac{\ln^3(2)}{6}+\frac{\zeta(3)}{4}\end{aligned}\tag{3}$$

The brown integral was evaluated using the generalized result found in this thread.

The Final Answer

Substitute (2) and (3) into (1) to get

$$\sum_{n=1}^{\infty}\frac{\psi_1(n)}{2^n n^2}=\frac{19\pi^4}{1440}-\frac{\pi^2}{24}\ln^2(2)-\frac{\ln^4(2)}{24}-\frac{\zeta(3)\ln(2)}{4}$$

□

Exercise 2.2.78: Prove that:

$$\sum_{n=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+1}=1-\gamma-\frac12\ln(2)$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2k+1)-1}{2k+3} = \frac{13}{12} - \frac{1}{3}\gamma - 2\ln(2) - 2\ln(\mathbf{A})$$

☞ **Proof:** By logarithimically differentiating the product representation of gamma, then repeatedly differentiating more, we know that for a non-negative integer n,

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta^*(n+1)$$

where $\zeta^*(n) = \begin{cases} \zeta(n) & n \neq 1 \\ \gamma & n = 1 \end{cases}$, it follows that (for $|x| < 1$)

$$\sum_{n=1}^{\infty} \zeta^*(n) x^n = -x\psi^{(0)}(1-x)$$

Dividing by x and integrating (from 0 to some variable, which we replace with x after integrating), we get

$$\sum_{n=1}^{\infty} \frac{\zeta^*(n)}{n} x^n = \log\Gamma(1-x)$$

Replacing x with $-x$ and finding the difference between the 2 series (which converge absolutely of course), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\zeta^*(2n+1)}{2n+1} x^{2n+1} = \sum_{n=1}^{\infty} \frac{1-(-1)^n}{2} \frac{\zeta^*(n)}{n} x^n &= \frac{1}{2}\left(\ln\Gamma(1-x) - \ln\Gamma(1+x)\right) \\ &= \frac{1}{2}\ln\left(\frac{\Gamma(1-x)}{\Gamma(1+x)}\right) \end{aligned}$$

We also know that

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \operatorname{artanh}(x)$$

So using absolute convergence again, we have for $|x| < 1$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\zeta(2n+1)-1}{2n+1} x^{2n+1} &= (1-\gamma)x - \operatorname{artanh}(x) + \frac{1}{2}\ln\left(\frac{\Gamma(1-x)}{\Gamma(1+x)}\right) \\ &= (1-\gamma)x + \frac{1}{2}\ln\left(\frac{\Gamma(2-x)}{\Gamma(2+x)}\right) \end{aligned}$$

Then letting $x \to 1$ gives the original series.

The second series,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{2k+3} = \sum_{k=1}^{\infty}\sum_{n=2}^{\infty} \frac{1}{(2k+3)n^{2k+1}} &= \sum_{n=2}^{\infty} n^2 \left(\sum_{k=4}^{\infty} \frac{1}{kn^k} - \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{kn^{2k}}\right) \\ &= \sum_{n=2}^{\infty} n^2 \left(-\ln\left(1-\frac{1}{n}\right) - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{2n^2} + \frac{1}{2}\ln\left(1-\frac{1}{n^2}\right)\right) \\ &= \sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\ln\left(\frac{n+1}{n-1}\right) - \frac{1}{n} - \frac{1}{3n^3}\right) \end{aligned}$$

Now, consider the partial sum:

$$\begin{aligned}S_N=\sum_{n=2}^{N}n^2\ln\left(\frac{n+1}{n-1}\right)&=\sum_{n=2}^{N}((n+1)^2-2(n+1)+1)\ln(n+1)\\&\quad-\sum_{n=2}^{N}((n-1)^2+2(n-1)+1)\ln(n-1)\\&=(N+1)^2\ln(N+1)+N^2\ln N-2\ln 2+\ln\left(\frac{N(N+1)}{2}\right)\\&\quad+2N\ln N-2\sum_{n=2}^{N+1}n\ln n-2\sum_{n=2}^{N}n\ln n\end{aligned}$$

Now, from the definition of Glaisher-Kinkelin constant,

$$\sum_{n=2}^{N}n\ln n=\left(\frac{N^2}{2}+\frac{N}{2}+\frac{1}{12}\right)\ln N-\frac{N^2}{4}+\ln\mathbf{A}+o(1)$$

A simple calculation leads to,

$$\begin{aligned}\sum_{n=2}^{N}n^2\left(\frac{1}{2}\ln\left(\frac{n+1}{n-1}\right)-\frac{1}{n}-\frac{1}{3n^3}\right)&=\frac{1}{3}\ln N+\frac{13}{12}-2\ln 2-2\ln\mathbf{A}-\frac{1}{3}H_N+o(1)\\&=\frac{13}{12}-2\ln 2-2\ln\mathbf{A}-\frac{\gamma}{3}+o(1)\end{aligned}$$

i.e.,

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+3}=\frac{13}{12}-2\ln 2-2\ln\mathbf{A}-\frac{\gamma}{3}$$

□

Exercise 2.2.79: Prove:

$$\sum_{n=1}^{\infty}\frac{H_{n+2}}{n(n+2)}=\frac{7}{4}$$

Proof: Using $\dfrac{H_m}{m}=\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+m)}$, we may rewrite the summation as:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{H_{n+2}}{n(n+2)}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(n+k+2)}=\sum_{r=2}^{\infty}\frac{1}{r+2}\sum_{k=1}^{r-1}\frac{1}{k(r-k)}=2\sum_{r=2}^{\infty}\frac{H_{r-1}}{r(r+2)}\\&=2\sum_{r=2}^{\infty}\frac{1}{r(r+2)}\left(H_{r+2}-\frac{1}{r}-\frac{1}{r+1}-\frac{1}{r+2}\right)=2S-2\sum_{r=1}^{\infty}\frac{1}{r(r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)\\&=2\sum_{r=1}^{\infty}\frac{1}{r(r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)=\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+2}\right)\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)\\&=\sum_{r=1}^{\infty}\left(\frac{1}{r^2}-\frac{1}{(r+2)^2}\right)+\sum_{r=1}^{\infty}\left(\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right)=1+\frac{1}{4}+\frac{1}{2}=\frac{7}{4}\end{aligned}$$

we recall: $H_a=\displaystyle\int_0^1\frac{1-x^a}{1-x}\,\mathrm{d}x=\gamma+\psi_0(1+a)$, and on integration by parts once we have,

$$\int_0^1 x^{a-1}\ln(1-x)\,\mathrm{d}x=-\frac{H_a}{a}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{n+a}}{n(n+a)} &= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n+a-1}\ln(1-x)\,\mathrm{d}x = \int_0^1 x^{a-1}\ln^2(1-x)\,\mathrm{d}x\\ &= \lim_{b\to 1}\frac{\partial^2}{\partial b^2}\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \lim_{b\to 1}\frac{\partial^2}{\partial b^2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\\ &= \lim_{b\to 1}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left(\left(\psi_0(b)-\psi_0(a+b)\right)^2+\psi_1(b)-\psi_1(a+b)\right)\\ &= \frac{1}{a}\left(\left(\gamma+\psi_0(a+1)\right)^2+\frac{\pi^2}{6}-\psi_1(a+1)\right)\end{aligned}$$

□

Exercise 2.2.80: Evaluate:

$$\sum_{n=1}^{\infty}\cot^{-1}(n^2)$$

Solution:

$$\begin{aligned}\sum_{n=1}^{\infty}\operatorname{arccot}\left(n^2\right) &= \sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2}\right) = -\arg\prod_{n=1}^{\infty}\left(1-\frac{i}{n^2}\right) = \left[-\arg\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right]_{z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}\\ &= \left[-\arg\left(\frac{\sin\pi z}{\pi z}\right)\right]_{z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} = \frac{\pi}{4}-\arctan\left(\cot\frac{\pi\sqrt{2}}{2}\tanh\frac{\pi\sqrt{2}}{2}\right)\end{aligned}$$

□

Exercise 2.2.81: Prove:

$$\sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^2} = \frac{19}{24}\zeta(6)+\zeta^2(3)$$

Proof: We use the Abel's rearrangement over the N-th partial sum of the series,

$$\begin{aligned}\sum_{n=1}^{N}\frac{\left(H_n^{(2)}\right)^2}{n^2} &= \sum_{n=1}^{N-1}\left[\left(H_n^{(2)}\right)^2-\left(H_{n+1}^{(2)}\right)^2\right]\sum_{k=1}^{n}\frac{1}{k^2}+\left(H_N^{(2)}\right)^2\sum_{k=1}^{N}\frac{1}{k^2}\\ &= \left(H_N^{(2)}\right)^3-\sum_{n=0}^{N-1}\frac{\left(H_n^{(2)}+H_{n+1}^{(2)}\right)H_n^{(2)}}{(n+1)^2} = \left(H_N^{(2)}\right)^3-\sum_{n=1}^{N}\frac{\left(2H_n^{(2)}-\frac{1}{n^2}\right)\left(H_n^{(2)}-\frac{1}{n^2}\right)}{n^2}\\ &= \left(H_N^{(2)}\right)^3-\sum_{n=1}^{N}\frac{1}{n^2}\left(2\left(H_n^{(2)}\right)^2-3\frac{H_n^{(2)}}{n^2}+\frac{1}{n^4}\right) = \frac{1}{3}\left(H_N^{(2)}\right)^3+\sum_{n=1}^{N}\frac{H_n^{(2)}}{n^4}-\frac{1}{3}\sum_{n=1}^{N}\frac{1}{n^6}\end{aligned}$$

I.e., $\displaystyle\sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^2} = \frac{1}{3}\zeta(2)^3+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}-\frac{1}{3}\zeta(6)$

M.N.C.E. showed in this answer one way of dealing with $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} = \zeta(3)^2-\frac{1}{3}\zeta(6)$. Combining the results lead to,

$$\sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^2} = \zeta(3)^2+\frac{19}{24}\zeta(6)$$

□

Exercise 2.2.82:

Solution:

□

Exercise 2.2.83:

Solution:

□

Exercise 2.2.84:

Solution:

□

Exercise 2.2.85:

Solution:

□

Exercise 2.2.86:

Solution:

□

Exercise 2.2.87:

Solution:

□

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2.3 贴吧

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2.4 QQ 群

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Exercise 2.4.1: 计算级数:

$$\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n}-1}$$

Solution: 由 Lambert 级数

$$Q\left(q\right)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1+q^n}$$

其中 $q=e^{-\pi\frac{\mathrm{K}'}{\mathrm{K}}}$

可知

$$\sum_{n=1}^{\infty}\frac{n^3}{q^{-2n}-1}=\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}=\frac{1-Q(q^2)}{240}$$

在 Paramanands 的这篇 博客中，我们有

$$Q(q^2)=\left(\frac{2\mathbf{K}}{\pi}\right)^4\left(1-k^2+k^4\right)$$

在所求级数中，$q=e^{-\pi}\Rightarrow\frac{\mathbf{K}'}{\mathbf{K}}=1$，即

$$\frac{\mathbf{K}'}{\mathbf{K}}=\frac{\displaystyle\int_0^1\frac{\mathrm{d}x}{\sqrt{\left(1-x^2\right)\left(1-\left(1-k^2\right)x^2\right)}}}{\displaystyle\int_0^1\frac{\mathrm{d}x}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}}=1\Rightarrow k=\frac{1}{\sqrt{2}}$$

所以

$$\left(1-k^2+k^4\right)=1-\frac{1}{2}+\frac{1}{4}=\frac{3}{4}\ ,\ \mathbf{K}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2$$

由此可得

$$Q\left(e^{-2\pi}\right)=\frac{3}{4}\left(\frac{2\mathbf{K}}{\pi}\right)^4=\frac{3}{64\pi^6}\Gamma\left(\frac{1}{4}\right)^8$$

所以

$$\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n}-1}=\frac{1}{240}-\frac{1}{5120\pi^6}\Gamma\left(\frac{1}{4}\right)^8$$

□

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2.5 其他

✎ *

Exercise 2.5.1: 计算级数:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{j=1}^{2k}\frac{(-1)^j}{j}$$

✎ **Solution:** 利用

$$\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j}=\ln 2+(-1)^{n-1}\int_0^1\frac{x^n}{1+x}\,\mathrm{d}x$$

我们有

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{j=1}^{2k}\frac{(-1)^j}{j}&=\ln 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^1\frac{x^{2k}}{1+x}\,\mathrm{d}x\\&=\ln^2 2+\int_0^1\frac{1}{1+x}\sum_{k=1}^{\infty}\frac{\left(-x^2\right)^k}{k}\,\mathrm{d}x=\ln^2 2-\int_0^1\frac{\ln\left(1+x^2\right)}{1+x}\,\mathrm{d}x\\&=\ln^2 2+\frac{\pi^2}{48}-\frac{3}{4}\ln^2 2=\frac{\pi^2}{48}+\frac{1}{4}\ln^2 2\end{aligned}$$

□

Exercise 2.5.2: 计算级数:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(2k)}{k} x^{2k}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(2k)}{k} x^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k}}{k\Gamma(2k)} \int_0^{\infty} \frac{u^{2k-1}}{e^u - 1} \mathrm{d}u$$

$$= \int_0^{\infty} \frac{x}{e^u - 1} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (ux)^{2k}}{(2k-1)!} \mathrm{d}u = \int_0^{\infty} \frac{x \sin(ux)}{e^u - 1} \mathrm{d}u = \frac{\pi x \coth(\pi x) - 1}{2}$$

□

Exercise 2.5.3:

Solution:

□

Exercise 2.5.4:

Solution:

□

Exercise 2.5.5:

Solution:

□

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第 3 章 Unsolved Problems
未解决问题

3.1 Integrals 积分

✎ *

Exercise 3.1.1: Prove:

$$\int_0^\infty \left(\mathrm{si}(x)\cos(x) - \mathrm{ci}(x)\sin(x)\right)^2 \mathrm{d}x = \frac{\pi}{2}$$

where

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\mathrm{d}t \quad , \quad \mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\mathrm{d}t$$

Exercise 3.1.2: Evaluate:

$$\int_{-\infty}^{\infty} \frac{\sin\left(\frac{a}{b}\pi x^2\right)}{\sinh^2(\pi x)}\mathrm{d}x$$

Exercise 3.1.3: Prove:

$$\int_{\frac{25\pi}{4}}^{\frac{53\pi}{4}} \frac{1}{(1+2^{\sin(x)})(1+2^{\cos(x)})}\mathrm{d}x = \frac{7\pi}{4}$$

Exercise 3.1.4: Evaluate:

$$\int_0^1 \ln(1-\tan^2(x))\mathrm{d}x$$

Exercise 3.1.5: Prove:

$$\int_0^\infty \left(\frac{1}{(x+1)^2\ln(x+1)} - \frac{\ln(x+1)\tan^{-1}(x)}{x^3}\right)\mathrm{d}x = \mathbf{G} - \gamma + \frac{1}{4}\pi\ln 2 - \frac{3}{2}$$

Exercise 3.1.6: Prove:

$$\int_0^{\pi/2} \frac{\cos[(1-a)x]}{(\cos x)^{a-1}} \frac{1}{\cosh 2b - \cos 2x}\,\mathrm{d}x = \frac{\pi}{4}\frac{e^{(a-1)b}}{(\cosh b)^a \sinh b}$$

Exercise 3.1.7: Prove:

$$\int_0^\infty \frac{\ln(x^2+1)}{\cosh^2(x)}\mathrm{d}x = 2\left(\ln(\pi) + \psi\left(\frac{1}{2}+\frac{1}{\pi}\right)\right)$$

Exercise 3.1.8: Prove:

$$\int_0^\infty \frac{\ln\left(x^2+a^2\right)}{\cosh x+\cos t}\mathrm{d}x=\frac{2\pi}{\sin t}\ln\frac{\Gamma\left(\frac{a}{2\pi}+\frac{\pi+t}{2\pi}\right)}{\Gamma\left(\frac{a}{2\pi}+\frac{\pi-t}{2\pi}\right)}+\frac{2t}{\sin t}\ln 2\pi$$

Exercise 3.1.9: Evaluate:

$$\int_0^\infty \frac{2\arctan\left(\frac{x}{a}\right)\sin\left(p\arctan\frac{x}{b}\right)-\ln(a^2+x^2)\cos\left(p\arctan\frac{x}{b}\right)}{(b^2+x^2)^{p/2}\left(\frac{1}{4}\ln^2(a^2+x^2)+\arctan^2\left(\frac{x}{a}\right)\right)}\,\mathrm{d}x$$

Exercise 3.1.10: Evaluate:

$$\int_0^1 \ln\left(\frac{x^4-2x^2+5}{5x^4-2x^2+1}\right)\frac{\mathrm{d}x}{1-x^2}$$

Exercise 3.1.11: Prove:

$$\int_0^\infty e^{-x^2}\,\frac{a\cos(2ax)+x\sin(2ax)}{x^2+a^2}\,\mathrm{d}x=\frac{\pi}{2}e^{-a^2}$$

Exercise 3.1.12: Prove:

$$\int_0^\infty \frac{a\cos\left(4ax(x^2-a^2)\right)+x\sin\left(4ax(x^2-a^2)\right)}{e^{x^2(x^2-6a^2)}\left(x^2+a^2\right)}\,\mathrm{d}x=\frac{\pi}{2}e^{a^4}$$

Exercise 3.1.13: Prove:

$$\int_0^\infty e^{-x^2(x^2-6a^2)}\cos\left(4ax(x^2-a^2)\right)\,\mathrm{d}x=\frac{e^{a^4}}{4}\Gamma\left(\frac{1}{4}\right)$$

Exercise 3.1.14: Prove:

$$\int_0^\infty \frac{\frac{1}{2}(a^2+b^2-t^2)\ln(a^2+t^2)+2at\arctan\left(\frac{t}{a}\right)}{(a^2+b^2-t^2)^2+4a^2t^2}\,\mathrm{d}t=0$$

Exercise 3.1.15: Evaluate:

$$\int_0^1 x\ln(1+x^2)\ln^3\left(\frac{1-x}{1+x}\right)\mathrm{d}x$$

Exercise 3.1.16: Prove:

$$\int_0^\infty x^{-x}e^{-x}\Gamma(x)\sin\pi x\mathrm{d}x=\frac{\pi}{2}$$

Exercise 3.1.17: Prove:

$$\int_0^1 \frac{x\ln(1+x)\ln(1+x^2)}{x^2+1}\mathrm{d}x=\frac{\pi}{4}\mathbf{G}+\frac{1}{12}\ln^3(2)+\frac{\pi^2}{96}\ln(2)-\frac{5}{8}\zeta(3)$$

Exercise 3.1.18: Prove:

$$\int_0^1 \frac{\ln(1+x)\ln(1+x^2)}{x}\mathrm{d}x=\frac{\pi}{2}\mathbf{G}-\frac{33}{32}\zeta(3)$$

Exercise 3.1.19: Prove:

$$\int_0^1 \frac{\mathrm{Li}_2\left(-x^2\right)}{1+x}\mathrm{d}x = -\frac{\pi^2}{12}\ln(2) + \pi\mathbf{G} - \frac{33}{16}\zeta(3)$$

Exercise 3.1.20: Prove:

$$\int_0^1 \ln\left(\cos\left(\frac{\pi}{2}\sqrt{1-x^2}\right)\right)\mathrm{d}x = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathbf{H}_1(\pi n)$$

where $\mathbf{H}_\nu(z)$ is the Struve function.

Exercise 3.1.21: Prove:

$$\int_0^\infty \frac{\ln(1+x)\arctan(\sqrt{x})}{4+x^2}\,\mathrm{d}x = \frac{\pi}{2}\arctan\left(\frac{1}{2}\right)\ln 5$$

Exercise 3.1.22: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{\arctan\left(\dfrac{2\sin\theta}{2\cos\theta-1}\right)\sin\dfrac{\theta}{2}}{\sqrt{\cos\theta}}\,\mathrm{d}\theta$$

Exercise 3.1.23: Evaluate:

$$\int_0^1 \frac{\ln(ax^2+2bx+a)}{x^2+1}\mathrm{d}x, \quad a>b>0$$

Exercise 3.1.24: Prove:

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x)^3\left((1+\ln x)^2+\pi^2\right)} = \frac{3}{2} - \frac{e^2}{(e-1)^3}$$

Exercise 3.1.25: Prove:

$$\int_0^{\pi/2} \frac{\cos(\tan x - x)}{\cos x}\,\mathrm{d}x = \frac{\pi}{e}$$

Exercise 3.1.26: Prove:

$$\int_0^{\pi/2} \frac{\cos\left(b\tan x - (1-a)x\right)}{\cos^{a+1}(x)}\,\mathrm{d}x = \frac{\sin(\pi a)\Gamma(a)}{e^b b^a}\ , \ 0<a<1,\ b>0$$

Exercise 3.1.27: Evaluate:

$$\int_0^1 \ln(x^2+x+1)\ln(x^2-x+1)\mathrm{d}x$$

Exercise 3.1.28: Evaluate:

$$\int_0^1 \ln^2(x^4+x^2+1)\mathrm{d}x$$

Exercise 3.1.29: Prove:

$$\int_0^\infty \frac{\cos(x)}{x}\left(\int_0^x \frac{\sin t}{t}dt\right)^2\mathrm{d}x = -\frac{7}{6}\zeta(3)$$

Exercise 3.1.30: Prove:

$$\int_0^1 \frac{\ln(x)\ln(x^2-2x\cos(a)+1)}{x}\mathrm{d}x = 2\mathrm{Cl}_3(a)$$

Exercise 3.1.53: Prove:

$$\int_0^1 \mathbf{K}'(k)^3 \mathrm{d}k = \frac{\Gamma^8\left(\frac{1}{4}\right)}{128\pi^2}$$

Exercise 3.1.54: Prove:

$$\int_0^1 \frac{\ln x \ln\left(1-x^4\right)}{1+x^2}\mathrm{d}x = \frac{\pi^3}{16} - 3\mathbf{G}\ln 2$$

Exercise 3.1.55: Prove:

$$\int_0^1 \frac{\mathrm{Li}_2\left(\frac{x}{4}\right)}{4-x} \ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right) \mathrm{d}x = \frac{\pi^4}{1944}$$

Exercise 3.1.56: Prove:

$$\int_0^1 \frac{1+x^2+x^4+x^5}{1-x^6} \ln^{2n-1}\left(\frac{1}{x}\right) \mathrm{d}x = \frac{1}{4n}\left(1 - \frac{1}{2^{2n}} + \frac{1}{6^{2n}}\right)(2\pi)^{2n} B_{2n-1}$$

Exercise 3.1.57: Evaluate:

$$\int_0^\infty x^2 e^{-x^2+x} \ln\left(1 - 2t\cos(x) + t^2\right) \mathrm{d}x$$

Exercise 3.1.58: Prove:

$$\int_0^1 \frac{x \tan^{-1}\left(\frac{x(x^4-1)}{x^6+1}\right)}{x^4-1}\mathrm{d}x = \frac{\pi}{16}\ln\left(\frac{4\sqrt{5}+9}{5}\right)$$

Exercise 3.1.59: Prove:

$$\int_0^1 \frac{\ln^2 x \ln^2(1-x)}{\sqrt{x}\,(1-x)}\mathrm{d}x = -\pi^4\ln(2) - \frac{56\pi^2}{3}\zeta(3) + 14\ln^2(4)\zeta(3) + 248\zeta(5)$$

Exercise 3.1.60: Prove:

$$\int_0^\infty \frac{\mathrm{arcsinh}^4 x}{x\sqrt{x}}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\Gamma^2\left(\frac{1}{4}\right)\left(5\pi^2 + 48\mathbf{G}\right)$$

Exercise 3.1.61: Prove:

$$\int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{7\zeta(6)}{48} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta^2(3)}{18} + \frac{\zeta(3)\zeta(4)}{12}$$

Exercise 3.1.62: Prove:

$$\int_0^{\frac{\pi}{2}} \left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right]^2 \mathrm{d}x = \frac{\pi}{4} - \frac{24}{\pi^3}$$

Exercise 3.1.63: Prove:

$$\int_0^1 \ln(1+x^3)\ln(1-x)\mathrm{d}x = -\frac{1}{2}\psi_1\left(\frac{1}{3}\right) + \ln^2(2) - 2\ln(2) + \frac{5\pi^2}{36} - \frac{\pi}{\sqrt{3}} + 6$$

Exercise 3.1.64: Prove:

$$\int_0^1 x\ln(\sqrt{1+x}-\sqrt{1-x})\ln(\sqrt{1+x}+\sqrt{1-x})\mathrm{d}x=\frac{1}{8}\ln^2(2)-\frac{3}{8}\ln(2)+\frac{1}{16}$$

Exercise 3.1.65: Prove:

$$\int_0^\infty \frac{\ln(1+x)\ln x}{1+x^3}\mathrm{d}x=\frac{5\pi^3}{81\sqrt{3}}-\frac{\pi^2}{27}\ln 3-\frac{13}{27}\zeta(3)$$

$$\int_0^\infty \frac{\ln(1+x)\ln x}{1+x^4}\mathrm{d}x$$
$$=\frac{\pi^3}{64}+\frac{9\pi^3}{128\sqrt{2}}-\frac{\pi^2}{32\sqrt{2}}\ln 2-\frac{5\pi^2}{64\sqrt{2}}\ln\left(1+\sqrt{2}\right)+\frac{7}{16}\zeta(3)+\frac{1}{512}\psi_2\left(\frac{1}{8}\right)$$

$$\int_0^\infty \frac{\ln(1+x)\ln x}{1+x^6}\mathrm{d}x$$
$$=\frac{7\pi^3}{144}+\frac{\pi^3}{36\sqrt{3}}+\frac{13\pi^2}{144\sqrt{3}}\ln\left(2-\sqrt{3}\right)+\frac{91}{216}\zeta(3)+\frac{1}{1728}\psi_2\left(\frac{1}{12}\right)$$

Exercise 3.1.66: Evaluate:

$$\int_0^5\sqrt{1+\left(\frac{\pi}{2}\cos(10\pi x)\right)^2}\mathrm{d}x$$

Exercise 3.1.67: Prove:

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\mathrm{d}^3k}{1-\frac{1}{3}\left(\cos k_1+\cos k_2+\cos k_3\right)}=\frac{\sqrt{6}}{4}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)$$

Exercise 3.1.68: Evaluate:

$$\int_0^{+\infty}e^{-x}\left(\frac{1}{x\left(e^{-x}-1\right)}+\frac{1}{x^2}+\frac{1}{2x}\right)\mathrm{d}x$$

Exercise 3.1.69: Prove:

$$\int_0^1\frac{\ln(1-x)\ \mathrm{Li}_3\left(\frac{1+x}{2}\right)}{x}\mathrm{d}x=\frac{29\,\zeta(5)}{16}-\frac{19\pi^2}{96}\zeta(3)+\frac{5\,\zeta(3)}{16}\ln^2 2+\frac{\ln^5 2}{40}$$
$$-\frac{5\pi^2}{72}\ln^3 2+\frac{11\pi^4}{1440}\ln 2-3\,\mathrm{Li}_5\left(\frac{1}{2}\right)$$

Exercise 3.1.70: Prove:

$$\int_0^1\frac{\ln^2(1-x)\ \mathrm{Li}_2\left(\frac{1+x}{2}\right)}{x}\mathrm{d}x=\frac{81\,\zeta(5)}{32}+\frac{5\pi^2}{16}\zeta(3)-\frac{\zeta(3)}{8}\ln^2 2+\frac{1}{15}\ln^5 2$$
$$-\frac{\pi^2}{18}\ln^3 2-\frac{\pi^4}{15}\ln 2+2\,\mathrm{Li}_5\left(\frac{1}{2}\right)+2\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2$$

Exercise 3.1.71: Prove:

$$\int_0^1\frac{\ln^2 x\,\ln(1+x)\,\ln(1-x)}{1+x}\mathrm{d}x=\frac{213}{16}\zeta(5)-\frac{5\pi^2}{12}\zeta(3)+\frac{7\ln^2 2}{4}\zeta(3)-\frac{49\pi^4}{720}\ln 2$$
$$+\frac{\pi^2}{18}\ln^3 2-\frac{\ln^5 2}{10}-8\,\mathrm{Li}_5\left(\frac{1}{2}\right)-4\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2$$

Exercise 3.1.72: Prove:

$$\int_0^1 \frac{\ln^2 x\,\ln(1+x)\,\ln(1-x)}{1-x}\mathrm{d}x = \frac{\zeta(5)}{4} - \frac{29\pi^2}{48}\zeta(3) + \frac{7\ln^2 2}{4}\zeta(3) + \frac{\pi^4}{180}\ln 2 - \frac{\pi^2}{18}\ln^3 2 + \frac{\ln^5 2}{10} + 8\,\mathrm{Li}_5\left(\frac{1}{2}\right) + 4\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2$$

Exercise 3.1.73: Prove:

$$\int_0^1 \frac{\ln^3(1-t)\,\ln(1+t)}{t}\,\mathrm{d}t = \frac{\ln^5 2}{5} - \frac{\pi^2\ln^3 2}{6} - \frac{7\,\zeta(3)}{16}\left(\pi^2 - 6\ln^2 2\right) - \frac{81\,\zeta(5)}{16} + 6\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 + 6\,\mathrm{Li}_5\left(\frac{1}{2}\right)$$

Exercise 3.1.74: Prove:

$$\int_0^1 \frac{\ln^3(1+t)\,\ln(1-t)}{t}\,\mathrm{d}t = \frac{\pi^2\ln^3 2}{6} - \frac{\ln^5 2}{5} + \frac{7\,\zeta(3)}{16}\left(\pi^2 - 6\ln^2 2\right) + \frac{3\,\zeta(5)}{4} - 6\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 - 6\,\mathrm{Li}_5\left(\frac{1}{2}\right)$$

Exercise 3.1.75: Prove:

$$\int_0^\infty \frac{x^2\,\ln\cosh(x)}{\cosh(2x)}\mathrm{d}x = \frac{\pi^2}{24}\mathbf{G} - \frac{\pi^3}{128}\ln 2 - \frac{\pi^4}{384} + \frac{1}{3072}\psi_3\left(\frac{1}{4}\right)$$

Exercise 3.1.76: Prove:

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}\,\sqrt{1-x}\,\sqrt[3]{x+4}} = \frac{3}{2^{4/3}\,5^{5/6}\,\pi}\Gamma^3\left(\frac{1}{3}\right)$$

Exercise 3.1.77: Prove:

$$\int_0^1 \frac{\ln(x+1)\ln x\ln(1-x)}{x}\mathrm{d}x$$

Exercise 3.1.78: Prove:

$$\int_0^\infty \frac{\ln(x^2+1)\,\arctan x}{e^{\pi x}-1}\mathrm{d}x = \frac{\ln^2 2}{2} + \ln 2\cdot\ln\pi - 1$$

Exercise 3.1.79: Prove:

$$\int_0^1 \frac{x\ln\ln\frac{1}{x}}{x^4-\sqrt{2}x^2+1}\,\mathrm{d}x = \frac{\pi}{8\sqrt{2}}\left(7\ln\pi - 4\ln\sin\frac{\pi}{8} - 8\ln\Gamma\left(\frac{1}{8}\right)\right)$$

Exercise 3.1.80: Prove:

$$\int_0^1 \frac{x^2\,\ln(-\ln x)}{x^2+x+1}\mathrm{d}x = \frac{\pi}{\sqrt{3}}\ln\Gamma\left(\frac{1}{3}\right) - \frac{2\pi}{3\sqrt{3}}\ln(2\pi) + \frac{\pi}{4\sqrt{3}}\ln 3 + \frac{\ln^2 3}{4} - \gamma$$

Exercise 3.1.81: Prove:

$$\int_0^1 \left(\frac{\sin^{-1}(x)}{x^3\sqrt{1-x^2}} - \frac{1}{x^2}\right)\mathrm{d}x = \mathbf{G} + \frac{1}{2}$$

Exercise 3.1.82: Prove:

$$\int_0^1 \ln^2(1+x)\,\ln^2(1-x)\,\mathrm{d}x = 24 - \frac{4\pi^2}{3} - \frac{\pi^4}{90} - 24\ln 2 + 12\ln^2 2 - 4\ln^3 2 + \ln^4 2 + \frac{4\pi^2}{3}\ln 2 - \frac{2\pi^2}{3}\ln^2 2 - 8\zeta(3) + 8\zeta(3)\ln 2$$

Exercise 3.1.83: Prove:

$$\int_0^{\pi/2} \frac{\mathrm{d}\varphi}{4+\ln^2\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)} = \frac{\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)}{8}$$

Exercise 3.1.84: Evaluate:

$$\int_0^1 \frac{1}{\Gamma(x+1)\Gamma(n-x+1)}\mathrm{d}x$$

Exercise 3.1.85: Prove:

$$\int_0^1 \frac{\mathrm{d}x}{(1-x)^{7/10}\,(1+\phi\,x)^{2/5}\,\sqrt{x}} = \frac{1}{2^{4/5}\,\pi\,\phi}\Gamma^2\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio.

Exercise 3.1.86: Prove:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_{2n-1}} \prod_{j=1}^{n} \cos\left(s_{2j-1}^2 - s_{2j}^2\right)\,\mathrm{d}s_{2n}\cdots\mathrm{d}s_1 = \frac{\pi^n}{2^{2n-1}n!}$$

Exercise 3.1.87: Prove:

$$\int_0^{\infty} \frac{\sin x}{\cosh ax + \cos x}\frac{x}{x^2-\pi^2}\mathrm{d}x = \tan^{-1}\left(\frac{1}{a}\right) - \frac{1}{a}$$

Exercise 3.1.88: Prove:

$$\int_1^{\infty} \frac{x^{n-2}\ln(\ln x)}{1+x^2+x^4+\cdots+x^{2n-2}}\mathrm{d}x = \begin{cases} \dfrac{\pi}{2n}\tan\left(\dfrac{\pi}{2n}\right)\ln(2\pi) + \dfrac{\pi}{n}\displaystyle\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{k\pi}{n}\right)\ln\left(\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)}\right) & n \text{ even} \\ \dfrac{\pi}{2n}\tan\left(\dfrac{\pi}{2n}\right)\ln(\pi) + \dfrac{\pi}{n}\displaystyle\sum_{k=1}^{(n-1)/2}(-1)^{k-1}\sin\left(\frac{k\pi}{n}\right)\ln\left(\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)}\right) & n \text{ odd} \end{cases}$$

Exercise 3.1.89: Prove:

$$\int_0^{\pi} \ln^3\left[\left(2\sin\left(\frac{t}{2}\right)\right)\left(2\cos\left(\frac{t}{2}\right)\right)^p\right]\mathrm{d}t = -2\pi\zeta(3)\left[\frac{3}{4}(1+p^3) - \frac{3}{8}(p+p^2)\right]$$

Exercise 3.1.90: Prove:

$$\int_0^{\infty} \frac{e^{-xt}(\cos(ax)-\cos(bx))}{x}\mathrm{d}x = \frac{1}{2}\ln\left(\frac{t^2+b^2}{t^2+a^2}\right)$$

Exercise 3.1.91: Prove:

$$\int_0^{\infty} \frac{\arctan x \ln x}{(1+x^2)^2}\left(1-x^2\right) \mathrm{d}x = -\frac{\pi^2}{8}$$

Exercise 3.1.92: Prove:

$$\int_0^{\infty} \frac{x}{\left(e^{2\pi x}-1\right)\left(x^4+4\alpha^4\right)}\mathrm{d}x = \frac{\pi}{8\alpha^2}\frac{\cos(2\pi\alpha)-e^{2\pi\alpha}}{\cosh(2\pi\alpha)-\cos(2\pi\alpha)} + \frac{\pi}{16\alpha^2} - \frac{1}{8\alpha}\sum_{n=1}^{2\alpha}\frac{1}{(n-\alpha)^2+\alpha^2}$$

Exercise 3.1.93: Prove:

$$\int_0^1 \frac{\ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x = -\frac{\pi^4}{144} - \frac{\pi^2}{12}\ln^2 2 + \frac{1}{12}\ln^4 2 + 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\zeta(3)\ln 2$$

Exercise 3.1.94: Prove:

$$\int_0^1 \frac{\ln^2(1+x)\ln(1-x)}{x}\mathrm{d}x = -\frac{\pi^4}{240}$$

Exercise 3.1.95: Prove:

$$\int_0^1 \frac{\ln x \ln^2(1+x)\ln(1-x)}{x}\mathrm{d}x = \frac{7}{48}\pi^2\zeta(3) - \frac{25}{16}\zeta(5)$$

Exercise 3.1.96: Prove:

$$\int_0^1 \frac{\ln x \ln^2(1-x)\ln(1+x)}{x}\mathrm{d}x = -\frac{1}{9}\pi^2\ln^3 2 + \frac{2}{15}\ln^5 2 + 4\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 + 4\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{1}{16}\pi^2\zeta(3) + \frac{7}{4}\zeta(3)\ln^2 2 - \frac{7}{2}\zeta(5)$$

Exercise 3.1.97: Prove:

$$\int_0^1 \frac{\ln x \ln^3(1+x)}{x}\mathrm{d}x = \frac{1}{3}\pi^2\ln^3 2 - \frac{2}{5}\ln^5 2 - 12\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 - 12\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{1}{2}\pi^2\zeta(3) - \frac{21}{4}\zeta(3)\ln^2 2 + \frac{99}{16}\zeta(5)$$

Exercise 3.1.98: Prove:

$$\int_0^1 \frac{\ln(1+x)\ln(1-x)\,\mathrm{Li}_2(-x)}{x}\mathrm{d}x = \frac{1}{9}\pi^2\ln^3 2 - \frac{2}{15}\ln^5 2 - 4\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2 - 4\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{5}{96}\pi^2\zeta(3) - \frac{7}{4}\zeta(3)\ln^2 2 + \frac{123}{32}\zeta(5)$$

Exercise 3.1.99: Prove:

$$\int_0^1\int_0^1 \ln\Gamma(x+\pi y)\mathrm{d}x\mathrm{d}y = -\frac{3\pi}{4} + \frac{1}{2}\ln 2 + \frac{1}{2}\ln\pi + \frac{1}{2}\pi\ln\pi$$

Exercise 3.1.100: Prove:

$$\int_0^1\int_0^1 \ln\Gamma\left(\frac{x}{2}+\frac{3y}{2}\right)\mathrm{d}x\mathrm{d}y = -\frac{3}{8} + \frac{2}{3}\ln 2 + \frac{1}{2}\ln\pi - \frac{7\zeta(3)}{12\pi^2}$$

Exercise 3.1.101: Prove:

$$\int_0^1\int_0^1 \ln\Gamma\left(x+y^3\right)\mathrm{d}x\mathrm{d}y = -\frac{7}{16} + \frac{1}{2}\ln 2\pi$$

Exercise 3.1.102: Prove:

$$\int_0^1 \int_0^1 \ln\Gamma\left(x+\frac{y^2}{2}\right)\mathrm{d}x\mathrm{d}y = -\frac{5}{18}+\frac{1}{3}\ln 2+\frac{1}{2}\ln\pi$$

Exercise 3.1.103: Prove:

$$\int_0^1 \frac{\ln(x)}{\sqrt{1-x^2}\sqrt{x^2+2+2\sqrt{2}}}\mathrm{d}x = \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\sqrt{1+\sqrt{2}}}{8\sqrt[4]{2}\sqrt{\pi}}\left\{\frac{\sqrt{2}-1}{4}\ln\left(2+2\sqrt{2}\right)-\frac{2-\sqrt{2}}{4}\pi\right\}$$

Exercise 3.1.104: Prove:

$$\int_0^{\frac{\pi}{2}} \frac{x^{2p}}{1+\cos^2(x)}\mathrm{d}x = \frac{\sqrt{2}}{2^{2p+2}}\left[\frac{\pi^{2p+1}}{2p+1}+2(2p)!\sum_{n=0}^{p-1}\frac{(-1)^n\pi^{2p-2n-1}}{(2p-2n-1)!}\mathrm{Li}_{2n+2}(3-2\sqrt{2})\right]$$

Exercise 3.1.105: Prove:

$$\int_0^\infty x^{s-1}\left(\psi(x+1)+\gamma\right)\mathrm{d}x = \zeta(1-s)\frac{\pi}{\sin(\pi s)}$$

Exercise 3.1.106: Prove:

$$\int_0^\infty \frac{x^{s-1}\sin\left(\frac{\pi s}{2}-x\right)}{x^2+b^2}\mathrm{d}x = \frac{\pi}{2}e^{-b}b^{a-2},\quad 0<s<3,\quad b>0$$

Exercise 3.1.107: Prove:

$$\int_0^\infty \frac{\ln(1+x)\ln(1+x^2)}{x^3}\mathrm{d}x = \frac{\pi}{2}$$

Exercise 3.1.108: Let $n\in\mathbb{Z}^+,\ 0<\theta<\frac{\pi}{2}$, prove

$$\int_0^\pi \sin(n\alpha)\arctan\left(\frac{\tan\left(\frac{\alpha}{2}\right)}{\tan\left(\frac{\theta}{2}\right)}\right)\mathrm{d}\alpha = \frac{\pi}{2n}\left((-1)^{n+1}+(\sec\theta-\tan\theta)^n\right)$$

Exercise 3.1.109: Prove:

$$\int_0^\pi \sin^p(t)e^{iqt}\mathrm{d}t = \frac{\pi}{2^p}\left[\frac{\Gamma(1+p)e^{\pi iq/2}}{\Gamma\left(1+\frac{p+q}{2}\right)\Gamma\left(1+\frac{p-q}{2}\right)}\right]$$

Exercise 3.1.110: Let $\Re(a)>\frac{3}{4}$, prove:

$$\int_0^{+\infty}\frac{\mathrm{d}x}{\Gamma^2(a+x)\Gamma^2(a-x)} = \frac{1}{2}\cdot\frac{\Gamma(4a-3)}{\Gamma^4(2a-1)}$$

Exercise 3.1.111: Prove:

$$\int_0^\omega \frac{1}{\sqrt{1+x^4}}\mathrm{d}x = \frac{16\pi^{3/2}}{3\Gamma\left(-\frac{1}{4}\right)^2}$$

where $\omega=\sqrt[4]{\dfrac{\sqrt{3}}{2-\sqrt{3}}}$.

- **Exercise 3.1.112:** Prove:

$$\int_0^2 \frac{1}{\sqrt{1+x^3}}\mathrm{d}x=\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)}{6\sqrt{\pi}}$$

- **Exercise 3.1.113:** Prove:

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(-t)\ln(1-t)}{1+t}\,\mathrm{d}t&=\sum_{n=1}^{\infty}\frac{H_{n-1}^{(2)}H_n}{n}(-1)^n\\&=-4\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{21}{8}\zeta(3)\ln(2)+\frac{\pi^4}{24}-\frac{\ln^4(2)}{6}+\frac{1}{8}\pi^2\ln^2(2)\end{aligned}$$

- **Exercise 3.1.114:** Prove:

$$\int_0^1 \frac{\ln\left(x+\sqrt{2}\right)}{\sqrt{2-x}\sqrt{1-x}\sqrt{x}}\mathrm{d}x=\frac{\pi^{3/2}}{8\sqrt{2}}\cdot\frac{7\ln 2-\ln\left(17-12\sqrt{2}\right)-\pi}{\Gamma\left(\frac{3}{4}\right)^2}$$

- **Exercise 3.1.115:** Prove:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\tan^{-1}\left(\sqrt{3}\tan\theta\right)\mathrm{d}\theta-\int_0^{\frac{\pi}{6}}\tan^{-1}\left(\sqrt{3}\tan\theta\right)\mathrm{d}\theta=\frac{\pi^2}{12}$$

- **Exercise 3.1.116:** Prove:

$$\int_0^{\infty}\frac{\mathrm{d}x}{1+\left(\sum\limits_{k=1}^{2013}\left(\frac{k}{x+k}\right)-x\right)^{2014}}=\frac{\pi}{1007}\csc\left(\frac{\pi}{2014}\right)$$

- **Exercise 3.1.117:** Prove:

$$\int_a^b \cos^{-1}\left(\frac{x}{\sqrt{(a+b)x-ab}}\right)\mathrm{d}x=\frac{\pi(b-a)^2}{4(a+b)}$$

- **Exercise 3.1.118:** Prove:

$$\int_0^{\infty}\frac{\ln(x^2+2\sin ax+1)}{x^2+1}\mathrm{d}x=\pi\ln\left(2\cos\left(\frac{a}{2}\right)\right)+a\ln\left(\tan\left(\frac{a}{2}\right)\right)+2\sum_{k=0}^{\infty}\frac{\sin(2k+1)a}{(2k+1)^2}$$

- **Exercise 3.1.119:** Prove:

$$\int_0^1\frac{\sqrt{x}}{(1+x)^3}\ln\ln\left(\frac{1}{x}\right)\mathrm{d}x=-\frac{\mathbf{G}}{2\pi}+\frac{\pi}{8}\ln\left(\frac{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$$

- **Exercise 3.1.120:** Prove:

$$\int_0^1\int_0^1\frac{1-x}{1-xy}\ln\ln\left(\frac{1}{xy}\right)\mathrm{d}x\mathrm{d}y=\gamma+\frac{\pi^2}{6}\left(\ln(2\pi)+1\right)-2\pi^2\ln(\mathbf{A})$$

Exercise 3.1.121: Evaluate:

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy(1-x)(1-y)}$$

Exercise 3.1.122: Evaluate:

$$\int_{-\infty}^{\infty} \frac{\tan^{-1}(x)}{x^2-2x\sin(a)+1}\mathrm{d}x$$

Exercise 3.1.123: Prove:

$$\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{t(t^2+21t+112)}} = \frac{1}{4\pi\sqrt{7}}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)$$

Exercise 3.1.124: Prove:

$$\int_0^{\infty} \frac{e^{-t}\sin t\ln^2 t}{t}\mathrm{d}t = \frac{1}{192}\pi\left[7\pi^2+12(2\gamma+\ln 2)^2\right]$$

Exercise 3.1.125: Prove:

$$\int_0^{\infty} \frac{\ln x}{\left(\sqrt{2}x+\sqrt{2x^2+1}\right)\sqrt{1+x^2}}\mathrm{d}x = \frac{1+\ln\sqrt[4]{2}}{\sqrt{2\pi}}\Gamma\left(\frac{3}{4}\right)^2 - \frac{\sqrt{2\pi^3}}{8}\Gamma\left(\frac{3}{4}\right)^{-2} + (\ln 2-1)\sqrt{2}$$

Exercise 3.1.126: Prove:

$$\int_0^{\infty} \frac{\sqrt{x^2+k'^2}-x}{\sqrt{1+x^2}}\ln x\mathrm{d}x = \mathbf{E}(k) - \left(\frac{1+k'^2}{2}\right)\mathbf{K}(k) + \left(k'^2\mathbf{K}(k)-\mathbf{E}(k)\right)\frac{\ln k'}{2} + \ln 2 - 1$$

Exercise 3.1.127: Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{x\cos(x)}{1+\sin^4(x)}\mathrm{d}x$$

Exercise 3.1.128: Let $p,q,r>0$ be such that $pqr=1$, and define its complementary parameters as

$$\tilde{p} = r(q^2+1)^{1/2}, \quad \tilde{q} = p(r^2+1)^{1/2}, \quad \text{and} \quad \tilde{r} = q(p^2+1)^{1/2}$$

Prove:

$$\int_0^1 \frac{\arctan q\sqrt{p^2x^2+1}}{q\sqrt{p^2x^2+1}}\frac{\mathrm{d}x}{(r^2+1)p^2x^2+1} = \frac{\pi^2}{8} + \frac{1}{2}\left\{\arctan^2\left(\frac{1}{\tilde{p}}\right) - \arctan^2(\tilde{q}) - \arctan^2(\tilde{r})\right\}$$

Exercise 3.1.129: Prove:

$$\int_0^{\frac{\pi}{2}} \arctan^2\left(2\phi\sqrt{\phi}\sin(x)\right)\mathrm{d}x = \frac{\pi^3}{12} - \frac{3\pi}{4}\ln^2\phi$$

where ϕ is the Golden Ratio.

Exercise 3.1.130: Prove:

$$\int_0^{\infty} \frac{\tan^{-1}x}{\sinh\left(\frac{\pi x}{2}\right)}\mathrm{d}x = 4\ln\Gamma\left(\frac{1}{4}\right) - 2\ln\pi - 3\ln 2$$

Exercise 3.1.131: Prove:

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt[4]{x}\sqrt[4]{2-x\sqrt{3}}} = \frac{2\sqrt{2}}{3\sqrt[8]{3}}\pi$$

Exercise 3.1.132: Prove:

$$\int_0^1 \frac{\mathrm{d}x}{(x-2)\sqrt[5]{x^2(1-x)^3}} = -\frac{2^{11/10}\pi}{\sqrt{5+\sqrt{5}}}$$

Exercise 3.1.133: Prove:

$$\int_0^{\frac{\pi}{2}} \theta \ln^3(\tan\theta)\mathrm{d}\theta = \frac{7\pi^2}{32}\zeta(3) + \frac{93}{16}\zeta(5)$$

Exercise 3.1.134: Prove:

$$\int_0^1 \frac{x\arctan x \ln\left(1-x^2\right)}{1+x^2}\mathrm{d}x = -\frac{\pi^3}{48} - \frac{\pi\ln^2 2}{8} + \mathbf{G}\ln 2$$

Exercise 3.1.135: Prove:

$$\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x^2}\ln\left(\frac{1+x^2}{2}\right)\mathrm{d}x = \frac{\pi}{4}\ln^2 2 + \frac{\pi^3}{32} - 2\mathbf{G}\ln 2$$

Exercise 3.1.136: Prove:

$$\int_0^\infty \frac{\ln\cos(x)}{e^{2x}+1}\mathrm{d}x = -\frac{1}{4}\ln^2(2)$$

Exercise 3.1.137: Evaluate:

$$\int_{-\infty}^{\infty}\left(1 - \frac{\tanh(x+y)}{\tanh(x-y)}\right)\mathrm{d}x$$

Exercise 3.1.138: Prove:

$$\int_0^1 \frac{\arctan\left(x^{3+\sqrt{8}}\right)}{1+x^2}\mathrm{d}x = \frac{\ln(2)\ln\left(3+\sqrt{8}\right)}{16}$$

Exercise 3.1.139: Prove:

$$\int_0^\infty e^{-x^2}\cos^n(x)\mathrm{d}x = \frac{\sqrt{\pi}}{2^{n+1}}\sum_{r=0}^{n}\binom{n}{r}\exp\left(\frac{-(2r-n)^2}{4}\right)$$

Exercise 3.1.140: Prove:

$$\int_{-i\infty+\gamma}^{i\infty+\gamma}\frac{\Gamma(s)}{\Gamma\left(s-\frac{3}{2}\right)}\left(\frac{\pi}{4}\right)^{1-2s}\mathrm{d}s = 2i\sqrt{\pi}$$

where $0 < \gamma < 1$.

Exercise 3.1.141: Prove:

$$\int_0^\infty \frac{t^{n-1}e^{-(a+1)t}}{1-e^{-t}}\mathrm{d}t = \Gamma(n)\left(\zeta(n) - \sum_{k=1}^{a}\frac{1}{k^n}\right)$$

Exercise 3.1.142: Prove:

$$\int_0^\infty \frac{\ln\left(1+x^3\right)}{(1+x^2)^2}\mathrm{d}x = -\frac{\mathbf{G}}{6}+\frac{\pi}{24}\left(3\ln(2)+8\ln\left(2+\sqrt{3}\right)-9\right)$$

Exercise 3.1.143: Prove:

$$\int_0^\infty \frac{\ln\left(1+x^2\right)\sqrt{x}}{1+x^2}\arctan x\mathrm{d}x = \frac{\pi}{6\sqrt{2}}\left(12\mathbf{G}+9\ln^2 2+3\pi\ln 2+\pi^2\right)$$

Exercise 3.1.144: Prove:

$$\int_0^\infty \frac{\ln\left(1+x^3\right)}{1+x^2}\mathrm{d}x = -\frac{\mathbf{G}}{3}+\frac{\pi}{4}\ln(2)+\frac{2\pi}{3}\ln\left(2+\sqrt{3}\right)$$

Exercise 3.1.145: Prove:

$$\int_0^\infty \frac{\ln^2(x^3+1)}{x^3+1}\mathrm{d}x = \frac{\sqrt{3}\pi}{18}\left(9\ln^2(3)+4\psi'\left(\frac{2}{3}\right)\right)-\frac{\pi^3\sqrt{3}}{54}-\frac{\pi}{3}\ln(3)$$

Exercise 3.1.146: Evaluate:

$$\int_0^1 x^n\ln(1-x)\ln x\ln(1+x)\mathrm{d}x$$

Exercise 3.1.147: Prove:

$$\int_0^1 \tan^{-1}\left(\frac{\tanh^{-1}(x)-\tan^{-1}(x)}{\pi+\tanh^{-1}(x)-\tan^{-1}(x)}\right)\cdot\frac{\mathrm{d}x}{x} = \frac{\pi}{8}\ln\left(\frac{\pi^2}{8}\right)$$

Exercise 3.1.148: Evaluate:

$$\int_0^a \frac{\mathrm{Li}_3(1-x)}{x}\,\mathrm{d}x$$

Exercise 3.1.149: Prove:

$$\int_0^\infty \frac{\ln(x^3+1)}{x^3+1}\mathrm{d}x = \frac{\pi\ln(3)}{\sqrt{3}}-\frac{\pi^2}{9}$$

Exercise 3.1.150: Prove:

$$\int_0^\infty \frac{\tan^{-1}(x)\tan^{-1}\left(\frac{x}{2}\right)}{x^2+1}\mathrm{d}x = \frac{\pi^3}{24}-\frac{\pi}{4}\mathrm{Li}_2\left(\frac{1}{3}\right)$$

Exercise 3.1.151: Prove:

$$\int_0^\infty \frac{x^{2n}}{(x^2+1)^{2n+1}}\frac{\ln(x^2-x+1)}{\ln(x)}\mathrm{d}x = \frac{\pi n\Gamma(2n)}{2^{4n}\Gamma^2(n+1)} = \frac{\pi}{2^{4n+1}}\binom{2n}{n}$$

Exercise 3.1.152: Prove:

$$\int_0^\infty \prod_{n=1}^{7}\frac{\sin\left(\frac{x}{2n-1}\right)}{\frac{x}{2n-1}}\mathrm{d}x = \frac{\pi}{2}$$

Exercise 3.1.153: Evaluate:

$$\int_0^{\frac{\pi}{2}} (\cos x)^2 \ln\left(1+(\tan x)^4\right) \mathrm{d}x$$

Exercise 3.1.154: Prove:

$$\int_0^{\infty} \mathbf{K}\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) \frac{\sqrt{\sqrt{1+x^2}-1}}{\sqrt{1+x^2}+1} \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \frac{\pi^3}{4\left(\Gamma\left(\frac{5}{8}\right)\cdot\Gamma\left(\frac{7}{8}\right)\right)^2}$$

Exercise 3.1.155: Prove that for any real $s>0$ and $\alpha\in(0,1)$,

$$\int_0^{\infty} \frac{\mathrm{Li}_s(-x)}{x^{1+\alpha}}\,\mathrm{d}x = -\frac{\pi}{\alpha^s \sin \pi\alpha}$$

Exercise 3.1.156: Prove:

$$\int_{-1}^{1} \frac{\arctan(tx)}{x+1} \ln\left(\frac{1+x^2t^2}{1+t^2}\right) \mathrm{d}x = \frac{2}{3}\arctan^3 t\ , \quad t\in\mathbb{R}$$

Exercise 3.1.157: Prove:

$$\int_0^{s} \frac{\mathrm{Li}_2(x)}{\sqrt{s-x}(1-x)}\,\mathrm{d}x = \frac{4}{3\sqrt{1-s}}\arcsin^3\sqrt{s}, \quad 0\le s<1$$

Exercise 3.1.158: Prove:

$$\int_0^{\infty} \frac{\arctan(x^3)}{e^{2\pi x}-1}\mathrm{d}x = \frac{1}{4}\ln(2\pi) - \frac{\pi}{4\sqrt{3}} - \frac{1}{2}\ln\left(1+e^{-\pi\sqrt{3}}\right)$$
$$\int_0^{\infty} \frac{\arctan(x^3)}{e^{4\pi x}-1}\mathrm{d}x = \frac{1}{8}\ln(12\pi) - \frac{\pi}{4\sqrt{3}} - \frac{1}{4}\ln\left(1+e^{-2\pi\sqrt{3}}\right)$$

Exercise 3.1.159: Prove:

$$\int_0^{\infty} \sin(x)\sin(\sqrt{x})\mathrm{d}x = \frac{\sqrt{\pi}}{2}\sin\left(\frac{3\pi-1}{4}\right)$$

Exercise 3.1.160: Prove:

$$\int_0^1 \ln(\operatorname{artanh}(x))\ \mathrm{d}x = -\gamma + \ln\left(\frac{\pi}{4}\right)$$
$$\int_0^1 \ln(\operatorname{arsech}(x))\ \mathrm{d}x = -\gamma - 2\ln(2) - 2\ln\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$$

Exercise 3.1.161: Prove:

$$\int_0^{\infty} \frac{t^{s-1}\cos(xt)}{t^2}\mathrm{d}t = -x^{2-s}\Gamma(s-2)\cos\left(\frac{\pi s}{2}\right)$$

Exercise 3.1.162: Evaluate:

$$\int_0^1 \left(\frac{1}{1-x}+\frac{1}{\ln x}\right)^3 \mathrm{d}x$$

Exercise 3.1.163: Prove:

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+e^{2a}} \frac{1}{x^2+\pi^2} \mathrm{d}x = \frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}}$$

Exercise 3.1.164: Evaluate:

$$\int_0^{\infty} \frac{\exp\left(\dfrac{x}{x^2+b^2}\right)\sin\left(\dfrac{b}{x^2+b^2}\right) - \exp\left(\dfrac{x\cos a}{x^2+2bx\sin a+b^2}\right)\sin\left(\dfrac{x\sin a+b}{x^2+2bx\sin a+b^2}\right)}{x} \mathrm{d}x$$

$$\int_0^{\infty} \frac{\exp\left(\dfrac{x}{x^2+b^2}\right)\cos\left(\dfrac{b}{x^2+b^2}\right) - \exp\left(\dfrac{x\cos a}{x^2+2bx\sin a+b^2}\right)\sin\left(\dfrac{x\sin a+b}{x^2+2bx\sin a+b^2}\right)}{x} \mathrm{d}x$$

Exercise 3.1.165: For $|a| \geq n$, prove:

$$\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\left(\frac{\sin x}{x}\right)^n \mathrm{d}x = \frac{\pi e^{-|a|b}}{2b^{n+1}} \sinh^n(b)$$

Exercise 3.1.166: Prove:

$$\int_0^{\infty} \frac{\cos(ax)\cosh(px)}{\cos(2ax)+\cosh(2px)} \mathrm{d}x = \frac{\pi}{4}\frac{p}{p^2+a^2}$$

$$\int_0^{\infty} \frac{\sin(ax)\sinh(px)}{\cos(2ax)+\cosh(2px)} \mathrm{d}x = \frac{\pi}{4}\frac{a}{p^2+a^2}$$

Exercise 3.1.167: Prove:

$$\int_0^{\frac{\pi}{2}} \sin^{\mu}(\theta)\cos^{2t-\mu}(\theta)\cos(2t\theta)\mathrm{d}\theta = \frac{\pi\Gamma(2t-\mu-1)}{2\sin\left(\dfrac{\pi\mu}{2}\right)\Gamma(2t)\Gamma(-\mu)}$$

$$\int_0^{\frac{\pi}{2}} \sin^{\mu}(\theta)\cos^{2t-\mu}(\theta)\sin(2t\theta)\mathrm{d}\theta = \frac{\pi\Gamma(2t-\mu)}{2\sin\left(\dfrac{\pi\mu}{2}\right)\Gamma(2t)\Gamma(1-\mu)}$$

Exercise 3.1.168: Evaluate:

$$\int_0^{\infty} \frac{(1-x^2)\arctan x^2}{1+4x^2+x^4} \mathrm{d}x$$

Exercise 3.1.169: Prove:

$$I_{n+1} = (-1)^{n+1}\int_0^1 B_{2n+1}(x)\cot(\pi x)\mathrm{d}x \sim \frac{2(2n+1)!}{(2\pi)^{2n+1}}$$

Exercise 3.1.170: Prove:

$${}_2F_1(a,b;2b;z) = (1-z)^{-a/2}\,{}_2F_1\left(\frac{a}{2}, b-\frac{a}{2}; b+\frac{1}{2}; \frac{z^2}{4z-4}\right)$$

Exercise 3.1.171: Prove:

$$\int_0^1 k^3\mathbf{K}(k)^2\mathrm{d}k = \frac{2+7\zeta(3)}{8}$$

- **Exercise 3.1.172:** Prove:

$$\int_0^\infty \frac{\sin\left(\frac{r\pi}{2}+ax\right)}{x^{n-r}}\mathrm{d}x=\frac{(n-1)(n-2)\cdots(n-r)}{(n-1)!}\cdot\frac{\pi a^{r-n-1}}{2\sin\left(\frac{\pi n}{2}\right)}$$

- **Exercise 3.1.173:**
- **Exercise 3.1.174:**
- **Exercise 3.1.175:**
- **Exercise 3.1.176:**
- **Exercise 3.1.177:**
- **Exercise 3.1.178:**
- **Exercise 3.1.179:**
- **Exercise 3.1.180:**
- **Exercise 3.1.181:**
- **Exercise 3.1.182:**
- **Exercise 3.1.183:**
- **Exercise 3.1.184:**
- **Exercise 3.1.185:**
- **Exercise 3.1.186:**

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3.2 Series 级数

✎ *

- **Exercise 3.2.1:** Prove:

$$\sum_{k=1}^{\infty}\left(k^2\pi-\frac{1}{4}\right)e^{-\pi k^2}=\frac{1}{8}$$

- **Exercise 3.2.2:** Prove:

$$\sum_{n=-\infty}^{\infty}e^{-5\pi n^2}=\frac{\sqrt[4]{\pi}}{5\Gamma\left(\frac{3}{4}\right)}\sqrt{10+5\sqrt{5}}$$

- **Exercise 3.2.3:** Prove:

$$24\pi^5\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^5}{\cosh\left(\frac{2n+1}{2}\pi\sqrt{3}\right)\left[(2n+1)^6\pi^6-(2z)^6\right]}=\frac{1}{\cos z\cos(\omega z)\cos(\omega^2 z)}$$

where $\omega=e^{2i\pi/3}$.

- **Exercise 3.2.4:** Prove:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^7\cosh\left(\frac{2n+1}{2}\pi\sqrt{3}\right)}=\frac{\pi^7}{23040}$$

Exercise 3.2.5: Prove:

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}+1}=\frac{1}{2}\sqrt{5}$$

where F_n is a Fibonacci Number.

Exercise 3.2.6: Evaluate:

$$\sum_{x=0}^{\infty}\frac{x}{6x^3+13x^2+9x+2}$$

Exercise 3.2.7: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}=2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{11\pi^4}{360}+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{\pi^2}{12}\ln^2 2$$

Exercise 3.2.8: Evaluate:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(H_n\right)^2}{n}$$

Exercise 3.2.9: Evaluate:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{2^n n^2}$$

Exercise 3.2.10: Prove:

$$\sum_{n\geq 1}(-1)^{n+1}\frac{H_{\lfloor n/2\rfloor}}{n^3}=\frac{1}{2}\zeta(2)^2-\frac{7}{4}\zeta(3)\ln(2)$$

Exercise 3.2.11: Prove:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\psi_2\left(\frac{k}{10}+\frac{1}{2}\right)=-7\zeta(3)+\frac{50}{3}\pi^2-\frac{36\pi^3}{5}\sqrt{5-2\sqrt{5}}$$

Exercise 3.2.12: Prove:

$$\frac{\pi z}{2}\coth\left(\frac{\pi z}{2}\right)=1+\cfrac{z^2}{1+\cfrac{1^2(z^2+1^2)}{3+\cfrac{2^2(z^2+2^2)}{5+\cfrac{3^2(z^2+3^2)}{7+\ddots}}}}$$

Exercise 3.2.13: Prove:

$$\sum_{n=1}^{\infty}\frac{\left(H_n\right)^3(-1)^{n+1}}{n}=-\frac{9}{8}\zeta(3)\ln(2)+\frac{\pi^4}{144}-\frac{\ln^4(2)}{4}+\frac{1}{8}\pi^2\ln^2(2)$$

Exercise 3.2.14: Prove:

$$\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\cos\left(\frac{m\pi}{3}\right)\cos\left(\frac{n\pi}{3}\right)}{n^2+m^2}=\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)$$

Exercise 3.2.15: Prove:

$$\sum_{n=0}^{\infty}\binom{4n}{n}\frac{1}{(3n+1)2^{4n+1}}=\frac{\sqrt[3]{17+3\sqrt{33}}}{3}-\frac{2}{3\sqrt[3]{17+3\sqrt{33}}}-\frac{1}{3}$$

Exercise 3.2.16: Prove:

$$\sum_{k=1}^{\infty}\frac{H_k}{(4k+1)^2}=-\mathbf{G}\left(\frac{\pi}{4}+\frac{\ln 8}{2}\right)+\frac{7}{4}\zeta(3)+\frac{\pi^3}{32}-\frac{\pi^2}{16}\ln 8$$

Exercise 3.2.17: Prove:

$$\sum_{k=1}^{\infty}\frac{k}{(-1)^{k-1}e^{k\pi\sqrt{3}}+1}=-\frac{1}{24}+\frac{1}{4\pi\sqrt{3}}$$

Exercise 3.2.18: Prove:

$$\sum_{n=1}^{\infty}\frac{n^3}{\sinh(\pi n)}=\frac{\pi^2}{16\Gamma\left(\frac{3}{4}\right)^8}$$

Exercise 3.2.19: Prove:

$$\sum_{k=0}^{\infty}(-1)^k\arctan\left\{\exp\left(-\frac{(2k+1)\pi}{\sqrt{2}}\right)\right\}=\frac{1}{4}\arcsin\left(\sqrt{2}-1\right)$$

Exercise 3.2.20: Prove:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{e^{\pi k}+1}=\frac{1}{8}-\frac{\pi}{16\Gamma^4\left(\frac{3}{4}\right)}$$

Exercise 3.2.21: Prove:

$$\sum_{n=0}^{\infty}\frac{(-1)^k}{(2k+1)\cosh\left(\frac{(2k+1)\sqrt{3}\pi}{2}\right)}=\frac{\pi}{24}$$

Exercise 3.2.22: Prove:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-2k)^{n+2}=\frac{2^n n(n+2)!}{6}$$

Exercise 3.2.23: Prove:

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{n(n+1)\cdots(n+k)}=\frac{2^k}{k!}\left(\ln 2-\sum_{i=1}^{k}\frac{(1/2)^i}{i}\right)$$

Exercise 3.2.24: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)=\frac{7}{4}\zeta(3)\ln 2-\frac{\pi^4}{288}$$

Exercise 3.2.25: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)=\frac{\pi^4}{60}+\frac{\pi^2}{12}\ln^2 2-\frac{1}{12}\ln^4 2-2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

Exercise 3.2.26: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)=-\frac{17\pi^4}{480}-\frac{\pi^2}{6}\ln^2 2+\frac{1}{6}\ln^4 2+4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{7}{2}\zeta(3)\ln 2$$

Exercise 3.2.27: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)^2=-\frac{61\pi^4}{1440}+\frac{\pi^2}{3}\ln^2 2+\frac{1}{6}\ln^4 2+4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{7}{4}\zeta(3)\ln 2$$

Exercise 3.2.28: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)=\frac{29\pi^4}{1440}+\frac{\pi^2}{8}\ln^2 2-\frac{1}{8}\ln^4 2-3\mathrm{Li}_4\left(\frac{1}{2}\right)$$

Exercise 3.2.29: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2=-\frac{1}{9}\pi^2\ln^3 2+\frac{2}{15}\ln^5 2+4\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2+4\mathrm{Li}_5\left(\frac{1}{2}\right)$$
$$-\frac{11}{48}\pi^2\zeta(3)+\frac{7}{2}\zeta(3)\ln^2 2-\frac{19}{32}\zeta(5)$$

Exercise 3.2.30: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)=\frac{1}{9}\pi^2\ln^3 2-\frac{2}{15}\ln^5 2-4\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2-4\mathrm{Li}_5\left(\frac{1}{2}\right)$$
$$+\frac{5}{32}\pi^2\zeta(3)-\frac{7}{4}\zeta(3)\ln^2 2+\frac{23}{8}\zeta(5)$$

Exercise 3.2.31: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\right)=-\frac{13}{48}\pi^2\zeta(3)+\frac{125}{32}\zeta(5)$$

Exercise 3.2.32: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)=\frac{2}{45}\pi^4\ln 2+\frac{1}{36}\pi^2\ln^3 2-\frac{1}{60}\ln^5 2+2\mathrm{Li}_5\left(\frac{1}{2}\right)$$
$$-\frac{1}{48}\pi^2\zeta(3)-\frac{7}{8}\zeta(3)\ln^2 2-\frac{37}{16}\zeta(5)$$

Exercise 3.2.33: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)=\frac{2}{45}\pi^4\ln 2+\frac{1}{36}\pi^2\ln^3 2-\frac{1}{60}\ln^5 2+2\mathrm{Li}_5\left(\frac{1}{2}\right)$$
$$+\frac{1}{16}\pi^2\zeta(3)-\frac{7}{8}\zeta(3)\ln^2 2-\frac{193}{64}\zeta(5)$$

Exercise 3.2.34: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^3=-\frac{1}{6}\pi^2\ln^3 2+\frac{1}{5}\ln^5 2+6\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2+6\mathrm{Li}_5\left(\frac{1}{2}\right)$$
$$-\frac{9}{32}\pi^2\zeta(3)+\frac{21}{8}\zeta(3)\ln^2 2-\frac{9}{4}\zeta(5)$$

Exercise 3.2.35: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)^3=-\frac{29}{160}\pi^4\ln 2+\frac{11}{12}\pi^2\ln^3 2-\frac{1}{20}\ln^5 2-6\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2$$
$$-24\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{1}{12}\pi^2\zeta(3)+\frac{367}{16}\zeta(5)$$

Exercise 3.2.36: Prove:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\right)$$
$$=\frac{49}{720}\pi^4\ln 2-\frac{1}{18}\pi^2\ln^3 2+\frac{1}{10}\ln^5 2+4\mathrm{Li}_4\left(\frac{1}{2}\right)\ln 2+8\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{1}{96}\pi^2\zeta(3)-\frac{35}{4}\zeta(5)$$

Exercise 3.2.37: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)=-\frac{101}{45360}\pi^6+\frac{5}{2}\zeta^2(3)$$

Exercise 3.2.38: Prove:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^3=\frac{31}{5040}\pi^6-\frac{5}{2}\zeta^2(3)$$

Exercise 3.2.39: Prove:

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_n}{4^n n}=\frac{\pi^2}{3}$$

Exercise 3.2.40: Prove:

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n}x^n=\mathrm{Li}_4(x)-\frac{1}{2}\mathrm{Li}_2^2(x)-\ln(1-x)\,\mathrm{Li}_3(x)$$

Exercise 3.2.41: Evaluate:

$$\sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^3}{n^2}$$

Exercise 3.2.42: Prove:

$$\sum_{n=1}^{\infty}\mathrm{Ci}(2\pi nx)=\frac{1}{2}\left(H_x-\ln x-\gamma-\frac{1}{2x}\right)$$

Exercise 3.2.43: Prove:

$$\prod_{n=1}^{\infty}\left(1-\sin^2\left(\frac{1}{6^n}\right)\right)=\frac{4}{7}\mathbf{G}+\frac{\pi^2}{7}-\frac{4}{5}\zeta(3)$$
$$\prod_{n=1}^{\infty}\cos^2\left(\left(\frac{1}{4}\right)^n\right)=\frac{30\cdot 3^{4/5}}{(e^{4\pi}\ln 5)^{1/3}}\quad,\quad\prod_{n=1}^{\infty}\cos^2\left(\left(\frac{1}{5}\right)^n\right)=\frac{10\sqrt{7\mathbf{G}}}{49\gamma^{9/8}}$$
$$\prod_{n=1}^{\infty}\cos^2\left(\left(\frac{1}{8}\right)^n\right)=\frac{10(1280)^{1/9}\cdot 7^{2/5}}{49}$$

- **Exercise 3.2.44:** Evaluate:

$$\lim_{x\to 1^-}\left\{\sum_{n=1}^{\infty}\frac{\widetilde{H_n^3}}{n+1}x^{n+1}+\ln^3 2\ln(1-x)\right\}$$

where $\widetilde{H_n}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$.

- **Exercise 3.2.45:** Prove:

$$\lim_{n\to\infty}\sqrt{n}\frac{1}{2^{(n-1)}(n-1)!}\sum_{k=0}^{n-1}\frac{1}{2^k k!}\frac{\Gamma(n+k)}{\Gamma(n-k)}=\frac{e}{\sqrt{\pi}}$$

- **Exercise 3.2.46:** Prove:

$$\psi_0\left(\frac{p}{q}\right)+\psi_0\left(1-\frac{p}{q}\right)=-2\gamma-2\ln q+\sum_{n=1}^{q-1}\cos\left(\frac{2\pi np}{q}\right)\ln\left(2-2\cos\left(\frac{2\pi n}{q}\right)\right)$$

- **Exercise 3.2.47:** Prove:

$$\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)=\frac{\sqrt{2\pi}^{\,n-1}}{\sqrt{n}}$$

- **Exercise 3.2.48:** Prove:

$$\mathrm{G}(z+1)=(2\pi)^{z/2}\exp\left(-\frac{z+z^2(1+\gamma)}{2}\right)\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\exp\left(\frac{z^2}{2k}-z\right)$$

- **Exercise 3.2.49:**
- **Exercise 3.2.50:**
- **Exercise 3.2.51:**
- **Exercise 3.2.52:**
- **Exercise 3.2.53:**
- **Exercise 3.2.54:**

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第 4 章 Fun Stuff 有趣的东西

4.1 Equalities Involving $\zeta(3)$ and Catalan's Constant

✎ * **Proposition 1.**The following formulas hold:

$$e^{\frac{7\zeta(3)}{4\pi^2}+\frac{1}{4}} = \lim_{m\to\infty}\prod_{n=1}^{2m+1}\frac{1}{\sqrt[4]{e}}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n} \tag{1}$$

$$e^{\frac{7\zeta(3)}{4\pi^2}-\frac{1}{4}} = \lim_{m\to\infty}\prod_{n=1}^{2m}\sqrt[4]{e}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n} \tag{2}$$

$$e^{\frac{7\zeta(3)}{\pi^2}} = \lim_{m\to\infty}\left(\frac{2^{2^2}\cdot 4^{4^2}\cdot 6^{6^2}\cdots(2m)^{(2m)^2}}{1^{1^2}\cdot 3^{3^2}\cdot 5^{5^2}\cdots(2m-1)^{(2m-1)^2}}\right)^4\left(\frac{(2m+2)^{4m+5}}{(2m+1)^{12m+9}}\right)^m \tag{3}$$

Proposition 2. The following formulas hold:

$$e^{\frac{2\mathbf{G}}{\pi}-\frac{1}{2}} = \lim_{m\to\infty}\prod_{n=1}^{2m}\left(1-\frac{2}{2n+1}\right)^{n(-1)^n} \tag{4}$$

$$e^{\frac{2\mathbf{G}}{\pi}+\frac{1}{2}} = \lim_{m\to\infty}\prod_{n=1}^{2m+1}\left(1-\frac{2}{2n+1}\right)^{n(-1)^n} \tag{5}$$

$$e^{\frac{4\mathbf{G}}{\pi}} = \lim_{m\to\infty}\left(\frac{3^3\cdot 7^7\cdot 11^{11}\cdots(4m-1)^{4m-1}}{1^1\cdot 5^5\cdot 9^9\cdots(4m-3)^{4m-3}}\right)^2\frac{(4m+3)^{2m+1}}{(4m+1)^{6m+1}} \tag{6}$$

Fact 3. Formula (3) is equivalent to the following formula:

$$\begin{aligned} e^{\frac{7\zeta(3)}{4\pi^2}} &= e^{\sum_{n=1}^{\infty}\frac{n(n+1)}{2^{n+3}}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\ln(k+1)} \\ &= \prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}\left(k+1\right)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n(n+1)}{2^{n+3}}} \\ &= \left(\frac{2^1}{1^1}\right)^{\frac{1\cdot 2}{2^4}}\left(\frac{2^2}{1^1\cdot 3^1}\right)^{\frac{2\cdot 3}{2^5}}\left(\frac{2^3\cdot 4^1}{1^1\cdot 3^3}\right)^{\frac{3\cdot 4}{2^6}}\left(\frac{2^4\cdot 4^4}{1^1\cdot 3^6\cdot\ 5^1}\right)^{\frac{4\cdot 5}{2^7}}\cdots \end{aligned}$$

Fact 4. Formula (4) follows from rearranging the factors of the following formula:

$$\begin{aligned} e^{\frac{\mathbf{G}}{\pi}} &= e^{\sum_{n=1}^{\infty}\frac{n}{2^{n+2}}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\ln(2k+1)} \\ &= \prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}\left(2k+1\right)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n}{2^{n+2}}} \\ &= \left(\frac{3^1}{1^1}\right)^{\frac{1}{2^3}}\left(\frac{3^2}{1^1\cdot 5^1}\right)^{\frac{2}{2^4}}\left(\frac{3^3\cdot 7^1}{1^1\cdot 5^3}\right)^{\frac{3}{2^5}}\left(\frac{3^4\cdot 7^4}{1^1\cdot 5^6\cdot 9^1}\right)^{\frac{4}{2^6}}\cdots \end{aligned}$$

which in turn is equivalent to formula (6).

1. Proof of Proposition 2

We begin with the following formula which is a classically known Fourier expansion:

Formula 5. Let $\sigma \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}$. Then

$$\sum_{m=0}^{\infty} \frac{\cos\Big(\pi\big(2m+1\big)\sigma\Big)}{2m+1} = \frac{1}{2}\ln\left|\cot\left(\frac{\pi}{2}\sigma\right)\right|.$$

The following formula, which follows directly from Formula 5 by integrating both sides over the interval $\left[0, \frac{1}{2}\right]$, is also well-known:

Formula 6.

$$\mathbf{G} = \int_0^{\pi/4} \ln(\cot\theta)\mathrm{d}\theta.$$

By applying integration by parts to the latter integral, we obtain

Corollary 7.

$$\mathbf{G} = \frac{1}{2}\int_0^{1/2} \frac{\pi^2\alpha}{\sin(\pi\alpha)}\mathrm{d}\alpha.$$

The following formula is also well-known: **Formula 8.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then

$$\cos\Big(2\pi\alpha s\Big) = \frac{\sin\Big(\pi\alpha\Big)}{\pi}\left(\frac{1}{\alpha} + 2\alpha\sum_{m=1}^{\infty}\frac{\Big(-1\Big)^m}{\alpha^2-m^2}\cos\Big(2\pi m s\Big)\right).$$

Setting $s = 0$ in Formula 8 gives:

Corollary 9. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Then

$$1 = \frac{\sin\Big(\pi\alpha\Big)}{\pi}\left(\frac{1}{\alpha} + 2\alpha\sum_{m=1}^{\infty}\frac{\Big(-1\Big)^m}{\alpha^2-m^2}\right).$$

Lemma 10. Let $m \in \mathbb{Z}$, $m \geq 1$. Then

$$\int_0^{1/2}\frac{\alpha^2}{\alpha^2-m^2}\mathrm{d}\alpha = \frac{1}{2} + \frac{m}{2}\ln\frac{2m-1}{2m+1}.$$

We now proceed with the proof of formula (4). By Corollary 9, we have

$$\frac{\pi^2\alpha}{\sin\Big(\pi\alpha\Big)} = \pi + 2\pi\alpha^2\sum_{m=1}^{\infty}\frac{\Big(-1\Big)^m}{\alpha^2-m^2}.$$

Integrating both sides with respect to α over the interval $\left[0, \frac{1}{2}\right]$ gives

$$\int_0^{1/2}\frac{\pi^2\alpha}{\sin\Big(\pi\alpha\Big)}\,\mathrm{d}\alpha = \frac{\pi}{2} + 2\pi\int_0^{1/2}\left(\sum_{m=1}^{\infty}\frac{(-1)^m\,\alpha^2}{\alpha^2-m^2}\right)\mathrm{d}\alpha. \tag{7}$$

Consider the sequence of functions

$$f_m\Big(\alpha\Big) = \frac{\Big(-1\Big)^m\alpha^2}{\alpha^2-m^2}$$

on the interval$I = \left[0, \frac{1}{2}\right]$,where $m = 1, 2, \ldots$.Since $\alpha \in I$, we clearly have

$$\left|f_m\big(\alpha\big)\right| = \frac{\alpha^2}{\left|\alpha^2 - m^2\right|} \leq \frac{\frac{1}{4}}{m^2 - \frac{1}{4}} = \frac{1}{4m^2 - 1} \leq \frac{1}{2m^2},$$

for all m. Since

$$\sum_{m=1}^{\infty} \frac{1}{2m^2}$$

converges, it follows from the Weierstrass M-test that the series

$$\sum_{m=1}^{\infty} f_m\big(\alpha\big)$$

converges uniformly on I, and, by well-known principles can therefore be integrated term by term. In other words,if we set

$$a_m = \big(-1\big)^m \left(1 + m \ln \frac{2m-1}{2m+1}\right),$$

then (7) and Lemma 10 imply that

$$\int_0^{1/2} \frac{\pi^2 \alpha}{\sin\big(\pi\alpha\big)} \mathrm{d}\alpha = \frac{\pi}{2} + \pi \sum_{m=1}^{\infty} a_m. \tag{8}$$

The left-hand side of (8) is a definite integral of the continuous function $\frac{\pi^2\alpha}{\sin(\pi\alpha)}$ over the interval $\left[0, \frac{1}{2}\right]$. Hence the left-hand side of (8) is a real number which implies that

$$\lim_{m\to\infty} a_m = 0.$$

Keeping this in mind, define

$$A_n = \sum_{m=1}^{n} a_m.$$

Then, by (8), we have

$$\begin{aligned}
\int_0^{1/2} \frac{\pi^2 \alpha}{\sin\big(\pi\alpha\big)} \mathrm{d}\alpha &= \frac{\pi}{2} + \pi \lim_{n\to\infty} A_n \\
&= \frac{\pi}{2} + \pi \lim_{N\to\infty} A_{2N} \\
&= \frac{\pi}{2} + \pi \lim_{N\to\infty} \sum_{m=1}^{N} \left(a_{2m-1} + a_{2m}\right) \\
&= \pi \left(\frac{1}{2} + \lim_{N\to\infty} \sum_{m=1}^{N} \left(-\left(2m-1\right) \ln \frac{4m-3}{4m-1} + 2m \ln \frac{4m-1}{4m+1}\right)\right) \\
&= \pi \left(\frac{1}{2} + \lim_{N\to\infty} \ln \prod_{m=1}^{N} \frac{\big(4m-1\big)^{4m-1}}{\big(4m-3\big)^{2m-1}\big(4m+1\big)^{2m}}\right) \\
&= \pi \left(\frac{1}{2} + \ln \prod_{m=1}^{\infty} \frac{\big(4m-1\big)^{4m-1}}{\big(4m-3\big)^{2m-1}\big(4m+1\big)^{2m}}\right).
\end{aligned}$$

By Corollary 7, the left-hand side equals $2\mathbf{G}$, therefore

$$\frac{2\mathbf{G}}{\pi}-\frac{1}{2}=\ln\prod_{m=1}^{\infty}\frac{\Big(4m-1\Big)^{4m-1}}{\Big(4m-3\Big)^{2m-1}\Big(4m+1\Big)^{2m}}.$$

Therefore,

$$\begin{aligned}
e^{\frac{2\mathbf{G}}{\pi}-\frac{1}{2}} &= \lim_{m\to\infty}\frac{3^3}{1^1\cdot 5^2}\cdot\frac{7^7}{5^3\cdot 9^4}\cdot\frac{11^{11}}{9^5\cdot 13^6}\cdots\frac{(4m-1)^{4m-1}}{(4m-3)^{2m-1}(4m+1)^{2m}}\\
&= \lim_{m\to\infty}\frac{3^3\cdot 7^7\cdot 11^{11}\cdots(4m-1)^{4m-1}}{5^5\cdot 9^9\cdot 13^{13}\cdots(4m-3)^{4m-3}\cdot(4m+1)^{2m}}\\
&= \lim_{m\to\infty}\left(\frac{1}{3}\right)^{-1}\left(\frac{3}{5}\right)^{2}\left(\frac{5}{7}\right)^{-3}\cdots\left(\frac{4m-1}{4m+1}\right)^{2m}\\
&= \lim_{m\to\infty}\prod_{n=1}^{2m}\left(1-\frac{2}{2n+1}\right)^{n(-1)^n},
\end{aligned}$$

and this completes the proof of formula (4). Multiplying both sides of the latter formula by e and using the fact that

$$e=\lim_{m\to\infty}\left(1-\frac{2}{4m+3}\right)^{-(2m+1)}$$

gives formula (5). Finally, multiplying formulas (4) and (5) together and expanding gives formula (6).

2. Proof of Proposition 1

We will first prove formula (1).

Lemma 11. Let $m\in\mathbb{N}$ and $\delta\in\left(0,\frac{1}{2}\right)$. Then

$$\begin{aligned}
&\pi^2\int_{\delta}^{1/2}\left(\frac{1}{2}-\sigma\right)\frac{\cos\Big(\pi\Big(2m+1\Big)\sigma\Big)}{2m+1}\,\mathrm{d}\sigma\\
&=\frac{\cos\Big(\pi(2m+1)\delta\Big)}{\Big(2m+1\Big)^3}+\frac{\pi\left(\delta-\frac{1}{2}\right)\sin\Big(\pi\Big(2m+1\Big)\delta\Big)}{\Big(2m+1\Big)^2}.
\end{aligned}$$

Corollary 12. Let $m\in\mathbb{N}$. Then

$$\pi^2\int_{0}^{1/2}\left(\frac{1}{2}-\sigma\right)\frac{\cos\Big(\pi\Big(2m+1\Big)\sigma\Big)}{2m+1}\mathrm{d}\sigma=\frac{1}{\Big(2m+1\Big)^3}.$$

We now recall the following basic formula:

$$\sum_{m=0}^{\infty}\frac{1}{\Big(2m+1\Big)^3}=\frac{7}{8}\zeta\Big(3\Big).\tag{9}$$

We will establish the following: **Formula 13.**

$$\zeta\Big(3\Big)=\frac{4}{7}\pi\mathbf{G}-\frac{2}{7}\pi^2\int_{0}^{1/2}\frac{\pi\sigma^2}{\sin\Big(\pi\sigma\Big)}\mathrm{d}\sigma.$$

The following statement is similar to Lemma 10.

Lemma 14. Let $m \in \mathbb{Z}$, $m \geq 1$. Then

$$\int_0^{1/2} \frac{\sigma^3}{\sigma^2 - m^2} \mathrm{d}\sigma = \frac{1}{8} + \frac{m^2}{2} \ln \frac{4m^2 - 1}{4m^2}.$$

Lemma 15.

$$\lim_{n\to\infty} \frac{e^n}{\left(1 + \frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \frac{e^{-n}}{\left(1 - \frac{1}{n}\right)^{n^2}} = \sqrt{e}\ .$$

We now proceed with the proof of formula (1). By Corollary 9, we have

$$\frac{\pi\sigma^2}{\sin\left(\pi\sigma\right)} = \sigma + 2\sigma^3 \sum_{m=1}^{\infty} \frac{\left(-1\right)^m}{\sigma^2 - m^2}\ .$$

Integrating both sides with respect to σ over the interval $\left[0, \frac{1}{2}\right]$ and using formula (4) and Lemma 14 gives

$$\begin{aligned}
&\int_0^{1/2} \frac{\pi\sigma^2}{\sin\left(\pi\sigma\right)} \mathrm{d}\sigma = \int_0^{1/2} \left(\sigma + 2\sigma^3 \sum_{m=1}^{\infty} \frac{\left(-1\right)^m}{\sigma^2 - m^2}\right) \mathrm{d}\sigma \\
&= \int_0^{1/2} \sigma \mathrm{d}\sigma + 2 \int_0^{1/2} \left(\sum_{m=1}^{\infty} \left(-1\right)^m \frac{\sigma^3}{\sigma^2 - m^2}\right) \mathrm{d}\sigma \\
&= \frac{1}{8} + 2 \sum_{m=1}^{\infty} \left(-1\right)^m \int_0^{1/2} \frac{\sigma^3}{\sigma^2 - m^2} \mathrm{d}\sigma \\
&= \frac{1}{8} + 2 \sum_{m=1}^{\infty} \left(-1\right)^m \left(\frac{1}{8} + \frac{m^2}{2} \ln \frac{4m^2 - 1}{4m^2}\right) \\
&= \frac{1}{8} + \sum_{m=1}^{\infty} \left(-1\right)^m \left(\frac{1}{4} + m^2 \ln \frac{4m^2 - 1}{4m^2}\right),
\end{aligned}$$

which equals

$$\begin{aligned}
&\frac{1}{8} + \sum_{\ell=1}^{\infty} \left(- \left(\frac{1}{4} + \left(2\ell - 1\right)^2 \ln \frac{4\left(2\ell - 1\right)^2 - 1}{4\left(2\ell - 1\right)^2}\right) \right. \\
&\quad \left. + \left(\frac{1}{4} + \left(2\ell\right)^2 \ln \frac{4\left(2\ell\right)^2 - 1}{4\left(2\ell\right)^2}\right)\right) \\
&= \frac{1}{8} + \sum_{\ell=1}^{\infty} \left(- \left(2\ell - 1\right)^2 \ln \frac{4\left(2\ell - 1\right)^2 - 1}{4\left(2\ell - 1\right)^2} \right. \\
&\quad \left. + \left(2\ell\right)^2 \ln \frac{4\left(2\ell\right)^2 - 1}{4\left(2\ell\right)^2}\right) \\
&= \frac{1}{8} + \sum_{\ell=1}^{\infty} \ln \left(\frac{\left(4\ell - 1\right)^{4\ell - 1} \left(4\ell + 1\right)^{(2\ell)^2} \left(4\ell - 2\right)^{2(2\ell - 1)^2}}{\left(4\ell\right)^{2(2\ell)^2} \left(4\ell - 3\right)^{(2\ell - 1)^2}} \right)
\end{aligned}$$

$$= \frac{1}{8} + \sum_{\ell=1}^{\infty} \ln\left(\frac{\left(4\ell-1\right)^{4\ell-1}}{\left(4\ell-3\right)^{2\ell-1}\left(4\ell+1\right)^{2\ell}} \right)$$
$$+ \sum_{\ell=1}^{\infty} \ln\left(\frac{\left(4\ell+1\right)^{4\ell^2+2\ell}\left(4\ell-2\right)^{2(2\ell-1)^2}}{\left(4\ell\right)^{2(2\ell)^2}\left(4\ell-3\right)^{4\ell^2-6\ell+2}} \right)$$
$$= \frac{2\mathbf{G}}{\pi} - \frac{3}{8} + 2\sum_{\ell=1}^{\infty} \ln\left(\frac{\left(4\ell+1\right)^{2\ell^2+\ell}\left(4\ell-2\right)^{(2\ell-1)^2}}{\left(4\ell\right)^{(2\ell)^2}\left(4\ell-3\right)^{2\ell^2-3\ell+1}} \right).$$

Therefore, by Formula 13, it follows that

$$\frac{7}{4\pi^2}\zeta\left(3\right) = \frac{3}{16} + \ln \prod_{\ell=1}^{\infty} \frac{\left(4\ell\right)^{(2\ell)^2}\left(4\ell-3\right)^{2\ell^2-3\ell+1}}{\left(4\ell+1\right)^{2\ell^2+\ell}\left(4\ell-2\right)^{(2\ell-1)^2}}. \tag{10}$$

Now the latter infinite product can be written as

$$\lim_{N\to\infty} \prod_{\ell=1}^{N} 2^{4\ell-1} \frac{\left(2\ell\right)^{(2\ell)^2}}{\left(2\ell-1\right)^{(2\ell-1)^2}} \frac{\left(4\left(\ell-1\right)+1\right)^{2(\ell-1)^2+(\ell-1)}}{\left(4\ell+1\right)^{2\ell^2+\ell}},$$

which equals

$$\lim_{N\to\infty} \left(\frac{2}{4N+1}\right)^{2N^2+N} \prod_{\ell=1}^{N} \frac{\left(2\ell\right)^{(2\ell)^2}}{\left(2\ell-1\right)^{(2\ell-1)^2}}$$
$$= \lim_{N\to\infty} \frac{2^{2N^2+N}\left(2N+1\right)^{(2N+1)^2}}{\left(4N+1\right)^{2N^2+N}} \prod_{\ell=1}^{N} \frac{\left(2\ell\right)^{(2\ell)^2}}{\left(2\ell+1\right)^{(2\ell+1)^2}}$$
$$= \lim_{N\to\infty} \left(\frac{2^{2N^2+N}\left(2N+1\right)^{(2N+1)^2}}{\left(4N+1\right)^{2N^2+N}} \frac{2}{\left(2N+2\right)^{(2N+1)(N+1)}} \right.$$
$$\left. \times \prod_{\ell=1}^{N} \frac{\left(2\ell\right)^{(2\ell+1)\ell}\left(2\ell+2\right)^{(2\ell+1)(\ell+1)}}{\left(2\ell+1\right)^{(2\ell+1)^2}} \right)$$

$$= \lim_{N\to\infty}\left(e^{\frac{N+1}{2}}\left(\frac{4N+2}{4N+1}\right)^{2N^2+N}\left(\frac{2N+1}{2N+2}\right)^{(2N+1)(N+1)}\frac{2}{\sqrt{e}}\right.$$
$$\left.\times\prod_{\ell=1}^{N}\frac{\Big(2\ell\Big)^{(2\ell+1)\ell}\Big(2\ell+2\Big)^{(2\ell+1(\ell+1)}}{\sqrt{e}\Big(2\ell+1\Big)^{(2\ell+1)^2}}\right).$$

We claim that

$$\lim_{N\to\infty}\frac{e^{\frac{N+1}{2}}}{\left(1-\frac{1}{4N+2}\right)^{(2N+1)N}\left(1+\frac{1}{2N+1}\right)^{(2N+1)(N+1)}} = e^{-\frac{3}{16}}. \tag{11}$$

Indeed, by Lemma 15, we have

$$\lim_{N\to\infty}\frac{e^{N+\frac{1}{4}}}{\left(1+\frac{1}{2N+1}\right)^{\frac{(2N+1)^2}{2}}} = 1 = \lim_{N\to\infty}\frac{e^{\frac{1}{2}}}{\left(1+\frac{1}{2N+1}\right)^{\frac{2N+1}{2}}}$$

and

$$\lim_{N\to\infty}\frac{e^{-\frac{N}{2}-\frac{5}{16}}}{\left(1-\frac{1}{4N+2}\right)^{\frac{(4N+2)^2}{8}}} = 1 = \lim_{N\to\infty}\frac{e^{\frac{1}{4}}}{\left(1-\frac{1}{4N+2}\right)^{-\frac{4N+2}{4}}},$$

hence (11) follows. Combining (10) with (11) gives

$$e^{\frac{7\zeta(3)}{4\pi^2}} = \frac{2}{\sqrt{e}}\prod_{l=1}^{\infty}\frac{(2l)^{(2l+1)l}(2l+2)^{(2l+2)(l+1)}}{\sqrt{e}(2l+1)^{(2l+1)^2}}.$$

Therefore,

$$\begin{aligned} e^{\frac{7\zeta(3)}{4\pi^2}} &= \frac{2}{\sqrt{e}}\lim_{m\to\infty}\prod_{n=1}^{m}\frac{(2n)^{(2n+1)n}(2n+2)^{(2n+2)(n+1)}}{\sqrt{e}(2n+1)^{(2n+1)^2}} \\ &= \frac{(2m+2)^{(2m+1)(m+1)}}{e^{\frac{m+1}{2}}}\frac{2^{2^2}\cdot 4^{4^2}\cdot 6^{6^2}\cdots(2m)^{(2m)^2}}{3^{3^2}\cdot 5^{5^2}\cdot 7^{7^2}\cdots(2m+1)^{(2m+1)^2}} \\ &= e^{-\frac{1}{4}}\lim_{m\to\infty}\prod_{n=1}^{2m+1}\frac{1}{\sqrt[4]{e}}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n}. \end{aligned}$$

and this completes the proof of formula (1).

It remains to prove formulas (2) and (3). Note that

$$\begin{aligned} &\prod_{n=1}^{2m}\sqrt[4]{e}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n} \\ &= \frac{\left(1-\frac{1}{2m+2}\right)^{(2m+1)(m+1)}}{e^{-(m+\frac{1}{4})}}\prod_{n=1}^{2m+1}\frac{1}{\sqrt[4]{e}}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n}. \end{aligned}$$

Therefore, formula (2) will follow from formula (1) once we show that

$$\lim_{m\to\infty}\frac{e^{-(2m+\frac{1}{2})}}{\left(1-\frac{1}{2m+2}\right)^{(2m+2)(2m+1)}} = e.$$

This follows by writing $(2m+2)(2m+1)$ as $(2m+2)^2-(2m+2)$ and using Lemma 15. Now multiplying formulas (1) and (2) together and squaring gives

$$e^{\frac{7\zeta(3)}{\pi^2}} = \lim_{m\to\infty}\frac{1}{\sqrt{e}}\left(\frac{2m+1}{2m+2}\right)^{-(2m+1)(2m+2)}\prod_{n=1}^{2m}\left(1-\frac{1}{n+1}\right)^{2n(n+1)(-1)^n}$$
$$= \lim_{m\to\infty}\frac{1}{\sqrt{e}}\left(\frac{(2m+2)^{2m+2}}{(2m+1)^{6m+2}}\right)^{2m+1}\left(\frac{2^{2^2}\cdot 4^{4^2}\cdot 6^{6^2}\cdots(2m)^{(2m)^2}}{1^{1^2}\cdot 3^{3^2}\cdot 5^{5^2}\cdots(2m-1)^{(2m-1)^2}}\right)^4$$

Formula (3) is now a consequence of the equality

$$\frac{1}{\sqrt{e}} = \lim_{m\to\infty}\left(\frac{2m+1}{2m+2}\right)^{m+2}.$$

*

4.2 Tables of Generalized Logarithmic Integrals

✎ * We denote the **Generalized Logarithmic Integral** by

$$L\begin{bmatrix}a,b,c\\d,e,f\end{bmatrix};z = \int_0^z \frac{\ln^a x\ln^b(1-x)\ln^c(1+x)}{x^d(1-x)^e(1+x)^f}\mathrm{d}x \tag{1}$$

(1) has been evaluated for certain a,b,c,d,e,f and z. Here is a list of few such values:

1. $L\begin{bmatrix}0,n,0\\1,0,0\end{bmatrix};z = (-1)^n n!\left(\zeta(n+1)-\mathrm{Li}_{n+1}(1-z)\right)+n!\sum_{j=1}^{n}\frac{(-1)^j\ln^{n-j+1}(1-z)}{(n-j+1)!}\mathrm{Li}_j(1-z)$

2. $L\begin{bmatrix}0,0,n\\1,0,0\end{bmatrix};z$
$= \frac{\ln^{n+1}(1+z)}{1+n}-n!\sum_{j=1}^{n}\frac{\ln^{n-j+1}(1+z)}{(n-j+1)!}\mathrm{Li}_j\left(\frac{1}{1+z}\right)+n!\zeta(n+1)-n!\mathrm{Li}_{n+1}\left(\frac{1}{1+z}\right)$

3. $L\begin{bmatrix}0,1,1\\1,0,0\end{bmatrix};1 = -\frac{5\zeta(3)}{8}$

4. $L\begin{bmatrix}0,1,1\\2,0,0\end{bmatrix};1 = -\frac{\pi^2}{12}-\ln^2(2)$

5. $L\begin{bmatrix}0,2,1\\0,0,0\end{bmatrix};1 = \frac{7\zeta(3)}{2}-6+\frac{2}{3}(\ln(2)-3)\ln^2(2)-\frac{1}{6}\pi^2(\ln(4)-2)+\ln(16)$

6. $L\begin{bmatrix}1,0,1\\0,1,0\end{bmatrix};1 = \zeta(3)-\frac{\pi^2}{4}\ln(2)$

7. $L\begin{bmatrix}1,1,1\\0,0,0\end{bmatrix};1 = -6+4\ln 2-\ln^2 2+\frac{5}{2}\zeta(2)-3\zeta(2)\ln 2+\frac{21}{8}\zeta(3)$

8. $L\begin{bmatrix}1,1,1\\1,0,0\end{bmatrix};1 = -\frac{3\pi^4}{160}+\frac{7\ln(2)}{4}\zeta(3)-\frac{\pi^2\ln^2(2)}{12}+\frac{\ln^4(2)}{12}+2\mathrm{Li}_4\left(\frac{1}{2}\right)$

9. $L\begin{bmatrix}1,1,1\\0,1,0\end{bmatrix};1\Big] = \frac{17\pi^4}{1440} - \frac{\pi^2}{24}\ln^2(2) - \frac{\ln^4(2)}{12} + \frac{7}{8}\zeta(3)\ln(2) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)$

10. $L\begin{bmatrix}1,1,2\\1,0,0\end{bmatrix};1\Big] = \frac{7}{8}\zeta(2)\zeta(3) - \frac{25}{16}\zeta(5)$

11. $L\begin{bmatrix}2,1,0\\1,0,0\end{bmatrix};z\Big] = 2\ln(z)\mathrm{Li}_3(z) - \ln^2(z)\mathrm{Li}_2(z) - 2\mathrm{Li}_4(z)$

12. $L\begin{bmatrix}2,1,0\\0,1,0\end{bmatrix};x\Big] = -2\left[\mathrm{Li}_4\left(\frac{-x}{1-x}\right) + \mathrm{Li}_4(x) - \mathrm{Li}_4(1-x) + \mathrm{Li}_4(1)\right] + 2\Big[\ln(1-x)\mathrm{Li}_3(x)$
$-\ln x\mathrm{Li}_3(1-x)\Big] + 2\ln x\ln(1-x)\mathrm{Li}_2(1-x) - \frac{\pi^2}{6}\ln^2(1-x) + \frac{1}{2}\ln^2(1-x)\ln^2(x)$
$+\frac{1}{3}\ln(x)\ln^3(1-x) - \frac{1}{12}\ln^4(1-x) + 2\zeta(3)\ln\left(\frac{x}{1-x}\right)$

13. $L\begin{bmatrix}1,0,2\\0,1,0\end{bmatrix};1\Big] = -\frac{7\pi^4}{144} - \frac{5}{12}\pi^2\ln^2(2) + \frac{\ln^4(2)}{2} + \frac{21}{4}\zeta(3)\ln(2) + 4\mathrm{Li}_4\left(\frac{1}{2}\right)$

14. $L\begin{bmatrix}m,0,0\\0,1,0\end{bmatrix};z\Big] = m!\sum_{j=0}^{m}(-1)^j\frac{(\ln z)^{m-j}}{(m-j)!}\mathrm{Li}_{j+1}(z)$

15. $L\begin{bmatrix}m,0,0\\0,0,1\end{bmatrix};z\Big] = m!\sum_{j=0}^{m}(-1)^{j+1}\frac{(\ln z)^{m-j}}{(m-j)!}\mathrm{Li}_{j+1}(-z)$

16. $L\begin{bmatrix}m,0,0\\1,1,0\end{bmatrix};z\Big] = m!\sum_{j=0}^{m}(-1)^j\frac{(\ln z)^{m-j}}{(m-j)!}\chi_{j+1}(z)$

where $\chi_\nu(z)$ is the **Legendre's Chi Function**.

17. $L\begin{bmatrix}m,0,0\\\frac{1}{2},0,1\end{bmatrix};z^2\Big] = 2^{m+1}m!\sum_{j=0}^{m}(-1)^j\frac{(\ln z)^{m-j}}{(m-j)!}\mathrm{Ti}_{j+1}(z)$

where $\mathrm{Ti}_\nu(z)$ is the **Inverse Tangent Integral**.

18. $L\begin{bmatrix}0,m,0\\0,0,1\end{bmatrix};z\Big] = \ln^m(1-z)\ln\left(\frac{1+z}{2}\right) + (-1)^{m+1}m!\,\mathrm{Li}_{m+1}\left(\frac{1}{2}\right)$
$+m!\sum_{j=1}^{m}(-1)^{j+1}\frac{\ln^{m-j}(1-z)}{(m-j)!}\mathrm{Li}_{j+1}\left(\frac{1-z}{2}\right)$

19. $L\begin{bmatrix}0,0,m\\0,1,0\end{bmatrix};z\Big] = -\ln^m(1+z)\ln\left(\frac{1-z}{2}\right) + (-1)^{m+1}m!\,\mathrm{Li}_{m+1}\left(\frac{1}{2}\right)$
$+m!\sum_{j=1}^{m}(-1)^{j}\frac{\ln^{m-j}(1+z)}{(m-j)!}\mathrm{Li}_{j+1}\left(\frac{1+z}{2}\right)$

20. $L\begin{bmatrix}2,3,0\\1,0,0\end{bmatrix};1\Big] = -\frac{23}{1260}\pi^6 + 12\zeta^2(3)$

21. $L\begin{bmatrix}4,3,0\\1,0,0\end{bmatrix};1\Big] = -\frac{61\pi^8}{1575} - 12\pi^2\zeta^2(3) + 432\zeta(3)\zeta(5)$

22. $L\begin{bmatrix}0,2,1\\0,0,1\end{bmatrix};1\Big] = 2\zeta(3)\ln 2 - \frac{\pi^4}{360} + \frac{\ln^4 2}{4} - \frac{1}{6}\pi^2\ln^2 2$

Some proofs of the above formula can be seen here. *

4.3 A Class of Trigonometric Integrals

✎ * **Lemma 1**

For suitable x, we have

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{x^{n+1}} = \frac{x - \cos\theta}{x^2 - 2x\cos\theta + 1} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{x^{n+1}} = \frac{\sin\theta}{x^2 - 2x\cos\theta + 1} \tag{2}$$

Proof : (1) and (2) are the real and imaginary parts of the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{x}\right)^n = \frac{x}{x - e^{i\theta}}$$

Theorem 1

For suitable a, we have

$$\int_0^{\infty} \frac{a - \cos(x)}{(a^2 - 2a\cos x + 1)(1 + x^2)} \mathrm{d}x = \frac{\pi e}{2(ae - 1)} \tag{3}$$

Proof: By Lemma 1,

$$\begin{aligned}
\int_0^{\infty} \frac{a - \cos(x)}{(a^2 - 2a\cos x + 1)(1 + x^2)} \mathrm{d}x &= \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} \int_0^{\infty} \frac{\cos(nx)}{1 + x^2} \mathrm{d}x \\
&= \frac{\pi}{2a} \sum_{n=0}^{\infty} \left(\frac{1}{ae}\right)^n \\
&= \frac{\pi}{2a} \left(\frac{1}{1 - \frac{1}{ae}}\right) \\
&= \frac{\pi e}{2(ae - 1)}
\end{aligned}$$

Theorem 2

We have

$$\begin{aligned}
&\int_0^{\infty} \frac{a - \cos(x)}{a^2 - 2a\cos x + 1} \cdot \frac{1}{1 + x^4} \mathrm{d}x \\
&= \frac{\pi}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{2}}\right) \frac{a\exp\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right) - \cos\left(\frac{1}{\sqrt{2}}\right)}{1 - 2a\exp\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right) + a^2\exp\left(\sqrt{2}\right)}
\end{aligned} \tag{4}$$

Proof: Again by Lemma 1,

$$\begin{aligned}
\int_0^{\infty} \frac{a - \cos(x)}{a^2 - 2a\cos x + 1} \cdot \frac{1}{1 + x^4} \mathrm{d}x &= \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} \int_0^{\infty} \frac{\cos(nx)}{1 + x^4} \mathrm{d}x \\
&= \frac{\pi}{2\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} \left\{ e^{-n/\sqrt{2}} \left(\cos\frac{n}{\sqrt{2}} + \sin\frac{n}{\sqrt{2}}\right)\right\} \\
&= \frac{\pi}{2\sqrt{2}} \left[\Re \sum_{n=0}^{\infty} \frac{e^{-n(1+i)/\sqrt{2}}}{a^{n+1}} + \Im \sum_{n=0}^{\infty} \frac{e^{-n(1+i)/\sqrt{2}}}{a^{n+1}} \right]
\end{aligned}$$

$$= \frac{\pi}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{2}}\right) \frac{a\exp\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right) - \cos\left(\frac{1}{\sqrt{2}}\right)}{1 - 2a\exp\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right) + a^2\exp\left(\sqrt{2}\right)}$$

Example:

Putting $a = 2$ in theorem 2, we arrive at

$$\int_0^\infty \frac{2-\cos x}{5-4\cos x} \times \frac{1}{1+x^4}\mathrm{d}x$$
$$= \frac{\pi}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{2}}\right) \frac{2\exp\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right) - \cos\left(\frac{1}{\sqrt{2}}\right)}{1 - 4\exp\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right) + 4\exp\left(\sqrt{2}\right)}$$

Lemma 2

$$\sum_{n=1}^{\infty} \frac{n\cos(n\theta)}{x^{n+1}} = \frac{\cos\theta(1+x^2) - 2x}{(x^2 - 2x\cos\theta + 1)^2}$$

Theorem 3

$$\int_0^\infty \frac{1}{(1-2a\cos x + a^2)(1+x^2)}\mathrm{d}x = \frac{\pi(e^2-1)}{2(e-a)(ae-1)}\left\{\frac{a}{a^2-1} - \frac{e}{e^2-1}\right\} \tag{5}$$

Proof:

$$\int_0^\infty \frac{1}{(1-2a\cos x + a^2)(1+x^2)}\mathrm{d}x = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\int_{2\pi n}^{2\pi(n+1)} \frac{1}{(1-2a\cos x + a^2)(1+x^2)}\mathrm{d}x$$
$$= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\int_0^{2\pi} \frac{1}{(1-2a\cos x + a^2)(1+(x+2\pi n)^2)}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{1-2a\cos x + a^2}\sum_{n=-\infty}^{+\infty}\frac{1}{1+(x+2\pi n)^2}\mathrm{d}x$$
$$= \frac{1}{4}\tanh\left(\frac{1}{2}\right)\int_0^{2\pi} \frac{\sec^2\frac{x}{2}}{(a^2-2a\cos x+1)\left(\tan^2\frac{x}{2} + \tanh^2\frac{1}{2}\right)}\mathrm{d}x$$

This integral can be evaluated by elementary techniques of integration. The final answer is

$$\frac{\pi(1+e)^2}{2(e-a)(ae-1)}\tanh\left(\frac{1}{2}\right)\left\{\frac{a}{a^2-1} - \frac{e}{e^2-1}\right\}$$

Theorem 4

$$\int_0^\infty \frac{\cos x}{1-2a\cos x + a^2} \times \frac{1}{1+x^2}\mathrm{d}x$$
$$= \frac{a\pi(e^2-1)}{2(e-a)(ae-1)}\left\{\frac{a}{a^2-1} - \frac{e}{e^2-1}\right\} - \frac{\pi e}{2(ae-1)} \tag{6}$$

Using this theorem we obtain the mysterious result

$$\int_0^\infty \frac{\cos x}{5-4\cos x} \times \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{2e-1}\left[\frac{e^2-1}{e-2}\left\{\frac{2}{3} - \frac{e}{e^2-1}\right\} - \frac{e}{2}\right]$$

I'm going to evaluate integrals (5) and (6) using the identities above, and two additional integrals.

For $|a| < 1$,

$$\frac{1}{1-a^2} + \frac{2}{1-a^2}\sum_{k=1}^{\infty} a^k \cos(kx) = \frac{1}{1-2a\cos x + a^2}$$

$$\frac{a}{1-a^2} + \frac{1+a^2}{1-a^2}\sum_{k=1}^{\infty} a^{k-1} \cos(kx) = \frac{\cos x}{1-2a\cos x + a^2}$$

$$\frac{1}{1-a^2}\sum_{k=1}^{\infty} k a^{k-1} \sin(kx) = \frac{\sin x}{(1-2a\cos x + a^2)^2}$$

Hence

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(1-2a\cos x + a^2)} = \frac{1}{1-a^2}\int_0^\infty \left(1 + 2\sum_{k=1}^{\infty} a^k \cos(kx)\right)\mathrm{d}x$$
$$= \frac{\pi}{2}\frac{1}{1-a^2} + \frac{2}{1-a^2}\sum_{k=1}^{\infty} a^k \int_0^\infty \frac{\cos(kx)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}\frac{1}{1-a^2} + \frac{\pi}{1-a^2}\sum_{k=1}^{\infty}\left(\frac{a}{e}\right)^k$$
$$= \frac{\pi}{2}\frac{1}{1-a^2} + \frac{\pi}{1-a^2}\frac{a/e}{1-a/e} = \frac{\pi}{2}\frac{1}{1-a^2}\left(1 + \frac{2a}{e-a}\right) = \frac{\pi}{2}\frac{1}{1-a^2}\frac{e+a}{e-a}$$

$$\int_0^\infty \frac{x\sin x}{(1+x^2)(1-2a\cos x + a^2)^2}\mathrm{d}x = \int_0^\infty \frac{x}{1+x^2}\frac{1}{1-a^2}\sum_{k=1}^{\infty} k a^{k-1}\sin(kx)\mathrm{d}x$$
$$= \frac{1}{1-a^2}\sum_{k=1}^{\infty} k a^{k-1}\int_0^\infty \frac{x\sin(kx)}{1+x^2}\mathrm{d}x = \frac{\pi}{2}\frac{1}{a(1-a^2)}\sum_{k=1}^{\infty} k\left(\frac{a}{e}\right)^k$$
$$= \frac{\pi}{2}\frac{1}{a(1-a^2)}\frac{a/e}{(1-a/e)^2} = \frac{\pi}{2}\frac{1}{1-a^2}\frac{e}{(e-a)^2}$$

$$\int_0^\infty \frac{\mathrm{d}x}{(x^4+4)(1-2a\cos x + a^2)} = \int_0^\infty \frac{1}{x^4+4}\frac{1}{1-a^2}\left(1 + 2\sum_{k=1}^{\infty} a^k\cos(kx)\right)\mathrm{d}x$$
$$= \frac{1}{1-a^2}\frac{\sqrt{2}}{4}\frac{\pi}{4}\csc\left(\frac{\pi}{4}\right) + \frac{2}{1-a^2}\sum_{k=1}^{\infty} a^k\int_0^\infty \frac{\cos(kx)}{x^4+4}\mathrm{d}x$$
$$= \frac{\pi}{8}\frac{1}{1-a^2} + \frac{\pi}{4}\frac{1}{1-a^2}\sum_{k=1}^{\infty} a^k e^{-k}(\cos k + \sin k)$$

$$\sum_{k=0}^{\infty}(ae^{-1}e^i)^k = \frac{1}{1-ae^{-1}e^i} = \frac{1-ae^{-1}\cos(1) + iae^{-1}\sin(1)}{1-2ae^{-1}\cos(1)+a^2e^{-2}}$$

Therefore,

$$\sum_{k=0}^{\infty} a^k e^{-k}\cos(k) = \frac{1-ae^{-1}\cos(1)}{1-2ae^{-1}\cos(1)+a^2e^{-2}} = \frac{e^2-ae\cos(1)}{e^2-2ae\cos(1)+a^2}$$
$$\Longrightarrow \sum_{k=1}^{\infty} a^k e^{-k}\cos(k) = \frac{e^2-ae\cos(1)}{e^2-2ae\cos(1)+a^2} - 1 = \frac{ae\cos(1)-a^2}{e^2-2ae\cos(1)+a^2}$$

And

$$\sum_{k=1}^{\infty} a^k e^{-k}\sin(k) = \sum_{k=0}^{\infty} a^k e^{-k}\sin(k) = \frac{ae\sin(1)}{e^2-2ae\cos(1)+a^2}$$

So

$$\int_0^\infty \frac{\mathrm{d}x}{(x^4+4)(1-2a\cos x+a^2)} = \frac{\pi}{8}\frac{1}{1-a^2} + \frac{\pi}{4}\frac{1}{1-a^2}\frac{ae\cos(1)+ae\sin(1)-a^2}{e^2-2ae\cos(1)+a^2}$$
$$= \frac{\pi}{8}\frac{1}{1-a^2}\frac{e^2+2ae\sin(1)-a^2}{e^2-2ae\cos(1)+a^2}$$

Differentiate the identity $\sum_{k=1}^{\infty} a^k \sin(kx) = \frac{a\sin x}{1-2a\cos x+a^2}$ $(|a|<1)$ with respect to x.

$$\sum_{k=1}^{\infty} ka^{k-1}\cos(kx) = \frac{\cos x(1-2a\cos x+a^2)-\sin x(2a\sin x)}{(1-2a\cos x+a^2)^2} = \frac{(1+a^2)\cos x-2a}{(1-2a\cos x+a^2)^2}$$

Now multiply both sides by $2a(1-a^2)$.

$$2(1-a^2)\sum_{k=1}^{\infty} ka^k\cos(kx) = \frac{2a(1-a^4)\cos(x)-4a^2(1-a^2)}{(1-2a\cos x+a^2)^2} \tag{7}$$

Next rewrite the identity $1+2\sum_{k=1}^{\infty} a^k\cos(kx) = \frac{1-a^2}{(1-2a\cos x+a^2)^2}$ $(|a|<1)$ in a different form.

$$1+2\sum_{k=1}^{\infty} a^k\cos(kx) = \frac{(1-a^2)(1-2a\cos x+a^2)}{(1-2a\cos x+a^2)^2}$$

And multiply both sides of that equation by $1+a^2$.

$$1+a^2+2(1+a^2)\sum_{k=1}^{\infty} a^k\cos(kx) = \frac{(1-a^4)(1-2a\cos x+a^2)}{(1-2a\cos x+a^2)^2} \tag{8}$$

Finally add equations (7) and (8).

$$1+a^2+2(1-a^2)\sum_{k=1}^{\infty} ka^k\cos(kx)+2(1+a^2)\sum_{k=1}^{\infty} a^k\cos(kx)$$
$$= \frac{1-3a^2+3a^4-a^6}{(1-2a\cos x+a^2)^2} = \frac{(1-a^2)^3}{(1-2a\cos x+a^2)^2}$$

That identity can then be used to evaluate something like $\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(1-2a\cos x+a^2)^2}$.
On the other hand

$$\tan^{-1}\left(\frac{p\sin qx}{1+p\cos qx}\right) = \Im\ln\left(1+i\frac{p\sin qx}{1+p\cos qx}\right) = \Im\ln\left(1+p\cos qx+ip\sin qx\right)$$
$$= \Im\ln\left(1+pe^{iqx}\right) = \Im\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}p^n e^{iqnx} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}p^n\sin(qnx)$$

Using this series we can create some more crazy results. We will assume $1>p>-e^q$ from now on.

1.

$$\int_0^\infty \frac{x}{1+x^2}\tan^{-1}\left(\frac{p\sin qx}{1+p\cos qx}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}p^n\int_0^\infty\frac{x\sin(qnx)}{1+x^2}dx$$
$$= \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}p^n e^{-qn}$$
$$= \frac{\pi}{2}\ln(1+pe^{-q})$$

2.

$$\begin{aligned}\int_0^\infty \frac{x}{(1+x^2)^2}\tan^{-1}\left(\frac{p\sin qx}{1+p\cos qx}\right)\,\mathrm{d}x &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}p^n\int_0^\infty \frac{x\sin(qnx)}{(1+x^2)^2}\mathrm{d}x\\ &= \frac{q\pi}{4}\sum_{n=1}^\infty (-1)^{n+1}\,p^n e^{-qn}\\ &= \frac{\pi}{4}\frac{pq}{e^q+p}\end{aligned}$$

3.

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^4}\tan^{-1}\left(\frac{p\sin qx}{1+p\cos qx}\right)\,\mathrm{d}x &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}p^n\int_0^\infty \frac{x\sin(qnx)}{1+x^4}\mathrm{d}x\\ &= \frac{\pi}{2}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}p^n e^{-qn/\sqrt{2}}\sin\left(\frac{qn}{\sqrt{2}}\right)\\ &= \frac{\pi}{2}\Im\log\left(1+pe^{-q/\sqrt{2}+iq/\sqrt{2}}\right)\\ &= \frac{\pi}{2}\tan^{-1}\left(\frac{p\sin\left(\frac{q}{\sqrt{2}}\right)}{e^{q/\sqrt{2}}+p\cos\left(\frac{q}{\sqrt{2}}\right)}\right)\end{aligned}$$

Differentiating 1,2,3 of the previous post respectively w.r.t p, we obtain

1'. $\displaystyle\int_0^\infty \frac{x}{1+x^2}\frac{\sin qx}{1+2p\cos(qx)+p^2}\mathrm{d}x = \frac{\pi}{2}\frac{1}{p+e^q}$

2'. $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\frac{\sin qx}{1+2p\cos(qx)+p^2}\mathrm{d}x = \frac{\pi}{4}\left(\frac{q}{e^q+p}-\frac{pq}{(e^q+p)^2}\right)$

3'. $\displaystyle\int_0^\infty \frac{x}{1+x^4}\frac{\sin qx}{1+2p\cos(qx)+p^2}\mathrm{d}x = \frac{\pi}{2}\frac{\sin\left(\frac{q}{\sqrt{2}}\right)}{e^{q/\sqrt{2}}+2p\cos\left(\frac{q}{\sqrt{2}}\right)+e^{-q/\sqrt{2}}p^2}$ *

4.4 Clausen Functions (and related series, functions, integrals)

✎ * Since the Clausen functions are intimately related to a number of other important special functions, such as Inverse Tangent Integrals, Polylogarithms, Polygamma Functions, Zeta Functions, and more besides - many of which are at the forefront of modern mathematical research.

Define the Clausen Functions $\mathrm{Cl}_m(\theta)$ and $\mathrm{Sl}_m(\theta)$ as follows:

$$\mathrm{Cl}_m(\theta) = \begin{cases}\displaystyle\sum_{k=1}^\infty \frac{\sin k\theta}{k^m} & \text{if } m \text{ is even}\\ \displaystyle\sum_{k=1}^\infty \frac{\cos k\theta}{k^m} & \text{if } m \text{ is odd}\end{cases},\quad \mathrm{Sl}_m(\theta) = \begin{cases}\displaystyle\sum_{k=1}^\infty \frac{\cos k\theta}{k^m} & \text{if } m \text{ is even}\\ \displaystyle\sum_{k=1}^\infty \frac{\sin k\theta}{k^m} & \text{if } m \text{ is odd}\end{cases}$$

As one would therefore expect, basic trigonometry can be used to develop all sorts of Clausen function properties. For example, consider the difference

$$\begin{aligned}
&\mathrm{Cl}_{2m}(\theta)-\mathrm{Cl}_{2m}(\pi-\theta)=\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2m}}-\sum_{k=1}^{\infty}\frac{\sin k(\pi-\theta)}{k^{2m}}\\
&=\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2m}}-\sum_{k=1}^{\infty}\frac{(\sin\pi k\cos k\theta-\cos\pi k\sin k\theta)}{k^{2m}}\\
&=\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2m}}+\sum_{k=1}^{\infty}\frac{(\cos\pi k\sin k\theta)}{k^{2m}}=\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2m}}+\sum_{k=1}^{\infty}(-1)^k\frac{\sin k\theta}{k^{2m}}\\
&=2\left(\frac{\sin 2\theta}{2^{2m}}+\frac{\sin 4\theta}{4^{2m}}+\frac{\sin 6\theta}{6^{2m}}+\cdots\right)\\
&=\frac{2}{2^{2m}}\left(\sin 2\theta+\frac{\sin 4\theta}{2^{2m}}+\frac{\sin 6\theta}{3^{2m}}+\cdots\right)=\frac{1}{2^{2m-1}}\mathrm{Cl}_{2m}(2\theta)
\end{aligned}$$

We now have the duplication formula for the CL-type Clausen function of even order:

Result 1:

$$\mathrm{Cl}_{2m}(2\theta)=2^{2m-1}\Big[\mathrm{Cl}_{2m}(\theta)-\mathrm{Cl}_{2m}(\pi-\theta)\Big]$$

Next, let's take the above duplication formula, replace θ with the variable x, and integrate both sides:

$$\int_0^{\varphi}\mathrm{Cl}_{2m}(2x)\,\mathrm{d}x=2^{2m-1}\left[\int_0^{\varphi}\mathrm{Cl}_{2m}(x)\,\mathrm{d}x-\int_0^{\varphi}\mathrm{Cl}_{2m}(\pi-x)\,\mathrm{d}x\right]$$

The L.H.S. is equal to

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\int_0^{\varphi}\sin 2kx\,\mathrm{d}x=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}\Big[\cos 2kx\Big]_0^{\varphi}\\
&=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos 2k\varphi}{k^{2m+1}}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}=\frac{1}{2}\Big[\zeta(2m+1)-\mathrm{Cl}_{2m+1}(2\varphi)\Big]
\end{aligned}$$

Whereas the difference of the two integrals on the R.H.S. is

$$\begin{aligned}
&2^{2m-1}\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\int_0^{\varphi}\left[\sin kx-\sin k(\pi-x)\right]\,\mathrm{d}x\\
=&2^{2m-1}\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}\Big[-\cos kx+\cos k(\pi-x)\Big]_0^{\varphi}\\
=&2^{2m-1}\sum_{k=1}^{\infty}\frac{[\cos k(\pi-\varphi)-\cos k\varphi]}{k^{2m+1}}=2^{2m-1}[\mathrm{Cl}_{2m+1}(\pi-\varphi)-\mathrm{Cl}_{2m+1}(\varphi)]
\end{aligned}$$

We now have the duplication formula for a CL-type Clausen function of odd order:

Result 2:

$$\mathrm{Cl}_{2m+1}(2\theta)=\zeta(2m+1)+2^{2m}\Big[\mathrm{Cl}_{2m+1}(\theta)-\mathrm{Cl}_{2m+1}(\pi-\theta)\Big]$$

Let's return to the first of the two duplication formulae, and derive a few particular values and correlations:

$$\mathrm{Cl}_{2m}(2\theta)=2^{2m-1}\Big[\mathrm{Cl}_{2m}(\theta)-\mathrm{Cl}_{2m}(\pi-\theta)\Big]$$

Let $\theta = 0$, to obtain the obvious results (n = integer):

$$\mathrm{Cl}_{2m}(0) = 0 \ , \ \mathrm{Cl}_{2m}(\pi n) = 0$$

Let $\theta = \pi/3$, to obtain

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{3}\right) = \frac{(1+2^{2m-1})}{2^{2m-1}}\mathrm{Cl}_{2m}\left(\frac{2\pi}{3}\right)$$

Similarly, let $\theta = \pi/4$, to obtain

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{2}\right) = 2^{2m-1}\mathrm{Cl}_{2m}\left(\frac{\pi}{4}\right) - 2^{2m-1}\mathrm{Cl}_{2m}\left(\frac{3\pi}{4}\right)$$

The leftmost Clausen function is expressible in terms of the Dirichlet Beta function:

$$\beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^x}$$

Since

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{k^{2m}} = 1 - \frac{1}{3^{2m}} + \frac{1}{5^{2m}} - \frac{1}{7^{2m}} + \cdots = \beta(2m)$$

So, in summation, thus far we have the following useful values/relations:

Result 3:

$$\mathrm{Cl}_{2m}(0) = 0 \ , \ \mathrm{Cl}_{2m}(\pi n) = 0 \ , \ \mathrm{Cl}_{2m}\left(\frac{\pi}{2}\right) = \beta(2m)$$

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{3}\right) = \frac{(1+2^{2m-1})}{2^{2m-1}}\mathrm{Cl}_{2m}\left(\frac{2\pi}{3}\right) \ , \ \mathrm{Cl}_{2m}\left(\frac{\pi}{4}\right) = \frac{\beta(2m)}{2^{2m-1}} + \mathrm{Cl}_{2m}\left(\frac{3\pi}{4}\right)$$

Continuing on with the first of the two duplication formulae, note that we can apply it to itself as follows:

$$\begin{aligned}
&\mathrm{Cl}_{2m}(2\theta) = 2^{2m-1}\left[\mathrm{Cl}_{2m}(\theta) - \mathrm{Cl}_{2m}(\pi-\theta)\right] \\
\Rightarrow &\mathrm{Cl}_{2m}(4\theta) = 2^{2m-1}\left[\mathrm{Cl}_{2m}(2\theta) - \mathrm{Cl}_{2m}(\pi-2\theta)\right] \\
= &2^{2m-1}\left[2^{2m-1}\left(\mathrm{Cl}_{2m}(\theta) - \mathrm{Cl}_{2m}(\pi-\theta)\right)\right] - 2^{2m-1}\mathrm{Cl}_{2m}(2(\pi/2-\theta)) \\
= &4^{2m-1}\left[\mathrm{Cl}_{2m}(\theta) - \mathrm{Cl}_{2m}(\pi-\theta)\right] - 4^{2m-1}\left[\mathrm{Cl}_{2m}(\pi/2-\theta) - \mathrm{Cl}_{2m}(\pi/2+\theta)\right]
\end{aligned}$$

Result 4:

$$\mathrm{Cl}_{2m}(4\theta) = 4^{2m-1}\left[\mathrm{Cl}_{2m}(\theta) - \mathrm{Cl}_{2m}(\pi-\theta) - \mathrm{Cl}_{2m}\left(\frac{\pi}{2}-\theta\right) + \mathrm{Cl}_{2m}\left(\frac{\pi}{2}+\theta\right)\right]$$

By applying the exact same process to the second of the two duplication formulae, then for CL-type Clausen function of odd index we have:

Result 5:

$$\begin{aligned}
&\mathrm{Cl}_{2m+1}(4\theta) \\
&= \zeta(2m+1) + 4^{2m}\left[\mathrm{Cl}_{2m+1}(\theta) - \mathrm{Cl}_{2m+1}(\pi-\theta) - \mathrm{Cl}_{2m+1}\left(\frac{\pi}{2}-\theta\right) + \mathrm{Cl}_{2m+1}\left(\frac{\pi}{2}+\theta\right)\right]
\end{aligned}$$

In Result 4, set $\theta = \pi/8$ to obtain:

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{2}\right) = 4^{2m-1}\left[\mathrm{Cl}_{2m}\left(\frac{\pi}{8}\right) - \mathrm{Cl}_{2m}\left(\frac{7\pi}{8}\right) - \mathrm{Cl}_{2m}\left(\frac{3\pi}{8}\right) + \mathrm{Cl}_{2m}\left(\frac{5\pi}{8}\right)\right]$$

We've already seen that the leftmost Clausen function is expressible in terms of the Dirichlet Beta function, so we quickly conclude that:

Result 6:

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{8}\right) - \mathrm{Cl}_{2m}\left(\frac{3\pi}{8}\right) + \mathrm{Cl}_{2m}\left(\frac{5\pi}{8}\right) - \mathrm{Cl}_{2m}\left(\frac{7\pi}{8}\right) = \frac{\beta(2m)}{4^{2m-1}}$$

And in particular:

$$\mathrm{Cl}_{2}\left(\frac{\pi}{8}\right) - \mathrm{Cl}_{2}\left(\frac{3\pi}{8}\right) + \mathrm{Cl}_{2}\left(\frac{5\pi}{8}\right) - \mathrm{Cl}_{2}\left(\frac{7\pi}{8}\right) = \frac{\beta(2)}{4} = \frac{\mathbf{G}}{4}$$

To express a CL-type Clausen function of rational argument and even order as a sum of polygamma functions, proceed as follows:

$$\mathrm{Cl}_{2m}\left(\frac{p\pi}{q}\right) = \sum_{k=1}^{\infty}\frac{\sin(p\pi/q)}{k^{2m}}$$

Split this into q parts

$$\sum_{k=0}^{\infty}\frac{\sin\left[(kq+1)\frac{p\pi}{q}\right]}{(kq+1)^{2m}} + \sum_{k=0}^{\infty}\frac{\sin\left[(kq+2)\frac{p\pi}{q}\right]}{(kq+2)^{2m}} + \sum_{k=0}^{\infty}\frac{\sin\left[(kq+3)\frac{p\pi}{q}\right]}{(kq+3)^{2m}} + \cdots +$$
$$\sum_{k=0}^{\infty}\frac{\sin\left[(kq+(q-1))\frac{p\pi}{q}\right]}{(kq+(q-1))^{2m}} + \sum_{k=0}^{\infty}\frac{\sin\left[(kq+q)\frac{p\pi}{q}\right]}{(kq+q)^{2m}}$$

Convert into a finite sum:

$$\sum_{j=1}^{j=q}\sum_{k=0}^{\infty}\frac{\sin\left(kp\pi + \frac{jp\pi}{q}\right)}{(kq+j)^{2m}} = \frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sum_{k=0}^{\infty}\frac{\sin\left(kp\pi + \frac{jp\pi}{q}\right)}{(k+j/q)^{2m}} = \frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sum_{k=0}^{\infty}\frac{\sin\left(kp\pi + \frac{jp\pi}{q}\right)}{(k+j/q)^{2m}}$$
$$= \frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sum_{k=0}^{\infty}\frac{\cos(kp\pi)\sin\left(\frac{jp\pi}{q}\right)}{(k+j/q)^{2m}} = \frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\sum_{k=0}^{\infty}\frac{(-1)^{kp}}{(k+j/q)^{2m}}$$

We now have two distinct cases to consider, since

$$(-1)^{kp} = \begin{cases} 1 & \text{if } p \text{ is even} \\ (-1)^k & \text{if } p \text{ is odd} \end{cases}$$

Case 1. when p is even:

This is, naturally, the simpler of the two, since the result above reduces to:

$$\frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\sum_{k=0}^{\infty}\frac{1}{(k+j/q)^{2m}}$$

We then apply the series definition for the polygamma function

$$\psi_n(z) = (-1)^{n+1}\,n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}$$

to obtain

$$\frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\frac{\psi_{2m-1}\left(\frac{j}{q}\right)}{(2m-1)!}$$

This, in turn, can be expressed in the infinitely more elegant form:

Result 7A:

$$(2m-1)!\,q^{2m}\,\mathrm{Cl}_{2m}\left(\frac{p\pi}{q}\right)=\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\psi_{2m-1}\left(\frac{j}{q}\right)$$

Case 2. when p is odd:

When p is odd, we are faced with an alternating series, which must be split in two:

$$\begin{aligned}
&\frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+j/q)^{2m}}\\
=&\frac{1}{q^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\left[\sum_{k=0}^{\infty}\frac{1}{(2k+j/q)^{2m}}-\sum_{k=0}^{\infty}\frac{1}{(2k+1+j/q)^{2m}}\right]\\
=&\frac{1}{(2q)^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\left[\sum_{k=0}^{\infty}\frac{1}{[k+(j/2q)]^{2m}}-\sum_{k=0}^{\infty}\frac{1}{[k+((j+q)/2q)]^{2m}}\right]\\
=&\frac{1}{(2q)^{2m}}\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\left[\frac{\psi_{2m-1}\left(\frac{j}{2q}\right)-\psi_{2m-1}\left(\frac{j+q}{2q}\right)}{(2m-1)!}\right]
\end{aligned}$$

From which we obtain the final result:

Result 7B:

$$(2m-1)!\,(2q)^{2m}\,\mathrm{Cl}_{2m}\left(\frac{p\pi}{q}\right)=\sum_{j=1}^{j=q}\sin\left(\frac{jp\pi}{q}\right)\left[\psi_{2m-1}\left(\frac{j}{2q}\right)-\psi_{2m-1}\left(\frac{j+q}{2q}\right)\right]$$

Here's an extremely elementary application of trigonometry to express a couple of trig series in terms of Clausen functions. Start of with a CL-type Clausen function of odd order and double argument:

$$\mathrm{Cl}_{2m+1}(2\theta)=\sum_{k=1}^{\infty}\frac{\cos 2k\theta}{k^{2m+1}}$$

Next, split the sum using the double angle formula for the cosine:

$$\mathrm{Cl}_{2m+1}(2\theta)=\sum_{k=1}^{\infty}\frac{(\cos^2 k\theta-\sin^2 k\theta)}{k^{2m+1}}=\sum_{k=1}^{\infty}\frac{\cos^2 k\theta}{k^{2m+1}}-\sum_{k=1}^{\infty}\frac{\sin^2 k\theta}{k^{2m+1}}$$

Again, using nothing but the most basic level of trigonometry, ie $\sin^2 x+\cos^2 x=1$, we see that the sum of the last two series must be

$$\sum_{k=1}^{\infty}\frac{\cos^2 k\theta}{k^{2m+1}}+\sum_{k=1}^{\infty}\frac{\sin^2 k\theta}{k^{2m+1}}=\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}=\zeta(2m+1)$$

Taking the sum/difference of these two results gives:

Result 8:

$$\sum_{k=1}^{\infty}\frac{\sin^2 k\theta}{k^{2m+1}}=\frac{1}{2}\Big[\zeta(2m+1)-\mathrm{Cl}_{2m+1}(2\theta)\Big]$$

$$\sum_{k=1}^{\infty}\frac{\cos^2 k\theta}{k^{2m+1}} = \frac{1}{2}\Big[\zeta(2m+1) + \mathrm{Cl}_{2m+1}(2\theta)\Big]$$

As mentioned right at the start, there are deep connections between the Clausen functions and - amongst others - polylogarithms. Here's a simple example.

For $|z| \le 1$, the Polylogarithm of order m has the series expansion:

$$\mathrm{Li}_m(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^m}$$

Setting $z = e^{i\theta}$ in the series above gives:

$$\begin{aligned}\mathrm{Li}_m(e^{i\theta}) &= \sum_{k=1}^{\infty}\frac{(e^{i\theta})^k}{k^m} = \sum_{k=1}^{\infty}\frac{(\cos\theta + i\sin\theta)^k}{k^m}\\ &= \sum_{k=1}^{\infty}\frac{(\cos k\theta + i\sin k\theta)}{k^m} = \sum_{k=1}^{\infty}\frac{\cos k\theta}{k^m} + i\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^m}\end{aligned}$$

So, for a polylogarithm of even order, we have:

$$\mathrm{Li}_{2m}(e^{i\theta}) = \sum_{k=1}^{\infty}\frac{\cos k\theta}{k^{2m}} + i\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2m}} = \mathrm{Sl}_{2m}(\theta) + i\,\mathrm{Cl}_{2m}(\theta)$$

And similarly, for a polylogarithm of odd order:

$$\mathrm{Li}_{2m+1}(e^{i\theta}) = \sum_{k=1}^{\infty}\frac{\cos k\theta}{k^{2m+1}} + i\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2m+1}} = \mathrm{Cl}_{2m+1}(\theta) + i\,\mathrm{Sl}_{2m+1}(\theta)$$

Result 9:

$$\begin{aligned}&\mathrm{Li}_{2m}(e^{i\theta}) = \mathrm{Sl}_{2m}(\theta) + i\,\mathrm{Cl}_{2m}(\theta)\ ,\ \mathrm{Li}_{2m+1}(e^{i\theta}) = \mathrm{Cl}_{2m+1}(\theta) + i\,\mathrm{Sl}_{2m+1}(\theta)\\ &\mathrm{Li}_{2m}(e^{-i\theta}) = \mathrm{Sl}_{2m}(\theta) - i\,\mathrm{Cl}_{2m}(\theta)\ ,\ \mathrm{Li}_{2m+1}(e^{-i\theta}) = \mathrm{Cl}_{2m+1}(\theta) - i\,\mathrm{Sl}_{2m+1}(\theta)\end{aligned}$$

As later posts in this thread will demonstrate, the (definite integral) moments of Clausen functions have a number of important applications, so it would, therefore, be quite useful to find a closed form for these moments:

$$\mathbf{Cl}_{(m,n)}(\theta) = \int_0^{\theta} x^n\,\mathrm{Cl}_m(x)\,\mathrm{d}x$$

In order to find these, we start off with the integral definition of $\mathrm{Cl}_2(\theta)$:

$$\mathrm{Cl}_2(\theta) = -\int_0^{\theta}\ln\left|2\sin\frac{x}{2}\right|\mathrm{d}x$$

We could just as easily use the series definition, but it important to observe that, within the range $0 < \theta \le 2\pi$, the absolute value of the logsine term can be omitted. Hence while within this range, we can write

$$\mathrm{Cl}_2(\theta) = -\int_0^{\theta}\ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x\ ,\ \mathrm{Cl}_1(\theta) = -\ln\left(2\sin\frac{\theta}{2}\right)$$

with no loss of generality. Note that the omission of $\theta = 0$ was to avoid the divergence of $\mathrm{Cl}_1(\theta)$ at this point (although, in the integrals presented below, this singularity is of no

consequence).

The generalized moments of Clausen functions will be dealt with shortly, but for the moment, let's consider the special - and perhaps most important - case of the moments of $\mathrm{Cl}_1(\theta)$.

$$\mathbf{Cl}_{(1,n)}(\theta) = \int_0^\theta x^n \,\mathrm{Cl}_1(x)\,\mathrm{d}x = -\int_0^\theta x^n \ln\left(2\sin\frac{x}{2}\right)\,\mathrm{d}x$$

Applying the series definition for $\mathrm{Cl}_1(\theta)$ this can be written as

$$\mathbf{Cl}_{(1,n)}(\theta) = -\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\theta x^n \cos kx\,\mathrm{d}x$$

Although tedious to derive, the closed form of that last trigonometric integral is easily deduced. Define

$$\mathcal{I}_{(p)} = \int_0^\theta x^p \cos kx\,\mathrm{d}x$$

Then we note that, for $p > 2$:

$$\mathcal{I}_{(p)} = \frac{x^p}{k}\sin kx\Big|_0^\theta + \frac{px^{p-1}}{k^2}\cos kx\Big|_0^\theta - \frac{p(p-1)}{k^2}\mathcal{I}_{(p-2)}$$

A similarly, for $p > 4$:

$$\mathcal{I}_{(p)} = \frac{x^p}{k}\sin kx\Big|_0^\theta + \frac{px^{p-1}}{k^2}\cos kx\Big|_0^\theta - \frac{p(p-1)x^{p-2}}{k^3}\sin kx\Big|_0^\theta$$
$$- \frac{p(p-1)(p-2)x^{p-3}}{k^4}\cos kx\Big|_0^\theta + \frac{p(p-1)(p-2)(p-3)}{k^4}\mathcal{I}_{(p-4)}$$

Using the iteration process above, and creating two sums - one for sine terms and the other for cosine terms - we get the desired result:

$$\mathcal{I}_{(p)} = p!\left[\sum_{j=0}^{\lfloor p/2\rfloor}(-1)^j\frac{x^{p-2j}\sin kx}{k^{2j+1}(p-2j)!}\Big|_0^\theta + \sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\frac{x^{p-2j-1}\cos kx}{k^{2j+2}(p-2j-1)!)}\Big|_0^\theta\right]$$

Inserting this into our moment integral gives:

$$\mathbf{Cl}_{(1,p)}(\theta) = \int_0^\theta x^p \,\mathrm{Cl}_1(x)\,\mathrm{d}x = -\int_0^\theta x^p \ln\left(2\sin\frac{x}{2}\right)\,\mathrm{d}x = -\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\theta x^p\cos kx\,\mathrm{d}x$$
$$= -p!\sum_{k=1}^{\infty}\frac{1}{k}\left[\sum_{j=0}^{\lfloor p/2\rfloor}(-1)^j\frac{x^{p-2j}\sin kx}{k^{2j+1}(p-2j)!}\Big|_0^\theta + \sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\frac{x^{p-2j-1}\cos kx}{k^{2j+2}(p-2j-1)!)}\Big|_0^\theta\right]$$

For the lower bound - $x = 0$ - the terms in the (finite!) sine series vanish. The same is true for all the lower bound term in the (finite) cosine series, except for the final term - containing x_0, which is present only when $p = 2m+1$ is odd, in which case this final cosine term is $(-1)^{\lfloor (p-1)/2\rfloor+1}/k^{p+1}$. To account for this term, we introduce the function

$$\frac{[1+(-1)^{p+1}]}{2} = \begin{cases} 0, & \text{if } p \text{ is even} \\ 1, & \text{if } p \text{ is odd} \end{cases}$$

Finally, expressing our sum in terms of Clausen functions, we arrive at:

Result 10:

$$\begin{aligned}&\int_0^\theta x^p \operatorname{Cl}_1(x)\mathrm{d}x \\ &= -p!\left[\sum_{j=0}^{\lfloor p/2\rfloor}(-1)^j\frac{\theta^{p-2j}}{(p-2j)!}\operatorname{Cl}_{2j+2}(\theta)+\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\frac{\theta^{p-2j-1}}{(p-2j-1)!)}\operatorname{Cl}_{2j+3}(\theta)\right] \\ &\quad + p!(-1)^{\lfloor (p-1)/2\rfloor}\frac{[1+(-1)^{p+1}]}{2}\zeta(p+2)\end{aligned}$$

A number of simple - but nonetheless important - trigonometric integrals follow immediately from the previous evaluation. For example

$$\begin{aligned}&\int_0^\theta x^p \operatorname{Cl}_1(x)\,\mathrm{d}x = -\int_0^\theta x^p \ln\left(2\sin\frac{x}{2}\right)\,\mathrm{d}x \\ &= -\frac{\theta^{p+1}}{p+1}\ln\left(2\sin\frac{\theta}{2}\right)+\frac{1}{2(p+1)}\int_0^\theta x^{p+1}\cot\frac{x}{2}\,\mathrm{d}x\end{aligned}$$

Which can be re-written in the more convenient form:

Result 11:

$$\begin{aligned}&\int_0^\phi x^{p+1}\cot x\,\mathrm{d}x \\ &= (p+1)!\frac{(-1)^{\lfloor (p-1)/2\rfloor}[1+(-1)^{p+1}]}{2^{p+2}}\zeta(p+2)+\phi^{p+1}\ln(2\sin\phi) \\ &\quad -(p+1)!\sum_{j=0}^{\lfloor p/2\rfloor}(-1)^j\frac{\phi^{p-2j}}{2^{2j+1}(p-2j)!}\operatorname{Cl}_{2j+2}(\phi) \\ &\quad -(p+1)!\sum_{j=0}^{\lfloor (p-1)/2\rfloor}(-1)^j\frac{\phi^{p-2j-1}}{2^{2j+2}(p-2j-1)!}\operatorname{Cl}_{2j+3}(\phi)\end{aligned}$$

Logcosine moments - part 1:

Following on from the logsine moments above, we find that the equivalent logcosine moments are slightly trickier - which was to be expected - although they are far richer, since the complex parts also yield useful information.

By analogy, we start off with:

$$\begin{aligned}&\int_0^\theta x^m \ln\left(2\cos\frac{x}{2}\right)\,\mathrm{d}x = \int_0^\theta x^m\ln\left(\frac{1+e^{-ix}}{e^{-ix/2}}\right)\,\mathrm{d}x \\ &= \int_0^\theta x^m\ln(1+e^{-ix})\,\mathrm{d}x + i\,\frac{\theta^{m+2}}{2(m+2)}\end{aligned}$$

We'll ignore that final complex function of θ for now, and continue with the evaluation of the complex logarithmic integral part:

$$\int_0^\theta x^m\ln(1+e^{-ix})\,\mathrm{d}x = \sum_{k=1}^\infty\frac{(-1)^{k+1}}{k}\int_0^\theta x^m(\cos kx - i\,\sin kx)\,\mathrm{d}x$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^\theta x^m \cos kx \, \mathrm{d}x - i \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^\theta x^m \sin kx \, \mathrm{d}x$$
$$= \int_0^\theta x^m \left[\mathrm{Cl}_1(x) - \frac{1}{2}\mathrm{Cl}_1(2x) \right] \mathrm{d}x - i \int_0^\theta x^m \left[\mathrm{Sl}_1(x) - \frac{1}{2}\mathrm{Sl}_1(2x) \right] \mathrm{d}x$$
$$\sum_{k=1}^{\infty} \frac{1}{k} \int_0^\theta x^m \left(\cos kx - \frac{\cos 2kx}{2} \right) \mathrm{d}x - i \sum_{k=1}^{\infty} \frac{1}{k} \int_0^\theta x^m \left(\sin kx - \frac{\sin 2kx}{2} \right) \mathrm{d}x$$

We already have the closed form for the two leftmost CL-type integrals, so it remains to find the remaining two SL-types. [The real logcosine moments are omitted below, since they are easily deduced from the previous result]. Let

$$\mathcal{I}_{(p)} = \int_0^\theta x^p \sin kx \, \mathrm{d}x = -\frac{1}{k} \cos kx \Big|_0^\theta + \frac{px^{p-1}}{k^2} \sin kx \Big|_0^\theta - \frac{p(p-1)}{k^2} \mathcal{I}_{(p-2)} = \cdots$$

As before, a similar iteration process leads to the general result:

$$\int_0^\theta x^p \sin kx \, \mathrm{d}x =$$
$$p! \left[\sum_{j=0}^{\lfloor p/2 \rfloor} (-1)^{j+1} \frac{x^{p-2j}}{k^{2j+1}(p-2j)!} \cos kx \Big|_0^\theta + \sum_{j=0}^{\lfloor (p-1)2 \rfloor} (-1)^{j+1} \frac{x^{p-2j-1}}{k^{2j+2}(p-2j-1)!} \sin kx \Big|_0^\theta \right]$$

So the imaginary part of $\int_0^\theta x^p \ln\left(2\cos\frac{x}{2}\right) \mathrm{d}x$ yields

$$p! \sum_{k=0}^{\infty} \frac{1}{k} \sum_{j=0}^{\lfloor p/2 \rfloor} (-1)^{j+1} \frac{x^{p-2j}}{k^{2j+1}(p-2j)!} \cos kx \Big|_0^\theta$$
$$+ p! \sum_{k=0}^{\infty} \frac{1}{k} \sum_{j=0}^{\lfloor (p-1)2 \rfloor} (-1)^{j+1} \frac{x^{p-2j-1}}{k^{2j+2}(p-2j-1)!} \sin kx \Big|_0^\theta$$
$$+ \frac{p!}{2} \sum_{k=0}^{\infty} \frac{1}{k} \sum_{j=0}^{\lfloor p/2 \rfloor} (-1)^{j+1} \frac{x^{p-2j}}{(2k)^{2j+1}(p-2j)!} \cos 2kx \Big|_0^\theta$$
$$- \frac{p!}{2} \sum_{k=0}^{\infty} \frac{1}{k} \sum_{j=0}^{\lfloor (p-1)2 \rfloor} (-1)^{j+1} \frac{x^{p-2j-1}}{(2k)^{2j+2}(p-2j-1)!} \sin 2kx \Big|_0^\theta + \frac{\theta^{m+2}}{2(m+2)} = 0$$

$$p! \sum_{j=0}^{\lfloor p/2 \rfloor} (-1)^{j+1} \frac{x^{p-2j}}{(p-2j)!} \mathrm{Sl}_{2j+2}(x) \Big|_0^\theta + p! \sum_{j=0}^{\lfloor (p-1)2 \rfloor} (-1)^{j+1} \frac{x^{p-2j-1}}{(p-2j-1)!} \mathrm{Sl}_{2j+3}(x) \Big|_0^\theta$$
$$- \frac{p!}{2} \sum_{j=0}^{\lfloor p/2 \rfloor} (-1)^{j+1} \frac{x^{p-2j}}{2^{2j+1}(p-2j)!} \mathrm{Sl}_{2j+2}(2x) \Big|_0^\theta$$
$$- \frac{p!}{2} \sum_{j=0}^{\lfloor (p-1)2 \rfloor} (-1)^{j+1} \frac{x^{p-2j-1}}{2^{2j+2}(p-2j-1)!} \mathrm{Sl}_{2j+3}(2x) \Big|_0^\theta + \frac{\theta^{m+2}}{2(m+2)} = 0$$

As promised before - on other threads, and indeed other forums - I'll start to find closed form expressions for various polygamma functions, at the rational arguments $1/2, 1/3, 2/3, 1/4, 3/4, 1/6,$

and 5/6. This might take a while, and be posted in stages.

To start with, let's consider the following particular Clausen function of (arbitrary) odd order:

$$\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)=\sum_{k=1}^{\infty}\frac{\cos(\pi k/3)}{k^{2m+1}}$$

We want to split this into six sums, where the first sum contains the first of every six terms, the second contains the second of every six terms, and so on. We also change summation index so our new series start at $k=0$, rather than $k=1$ above.

$$\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)=\sum_{k=1}^{\infty}\frac{\cos(\pi k/3)}{k^{2m+1}}=$$
$$\sum_{k=0}^{\infty}\frac{\cos\frac{\pi}{3}(6k+1)}{(6k+1)^{2m+1}}+\sum_{k=0}^{\infty}\frac{\cos\frac{\pi}{3}(6k+2)}{(6k+2)^{2m+1}}+\sum_{k=0}^{\infty}\frac{\cos\frac{\pi}{3}(6k+3)}{(6k+3)^{2m+1}}+$$
$$\sum_{k=0}^{\infty}\frac{\cos\frac{\pi}{3}(6k+4)}{(6k+4)^{2m+1}}+\sum_{k=0}^{\infty}\frac{\cos\frac{\pi}{3}(6k+5)}{(6k+5)^{2m+1}}+\sum_{k=0}^{\infty}\frac{\cos\frac{\pi}{3}(6k+6)}{(6k+6)^{2m+1}}$$

Simplify the trig term in each series:

$$\cos\frac{\pi}{3}(6k+n)=\cos\left(2\pi k+\frac{\pi n}{3}\right)$$
$$=\cos 2\pi k\cos\frac{\pi n}{3}-\sin 2\pi k\sin\frac{\pi n}{3}\equiv\cos\frac{\pi n}{3}$$

Our new sextet of series is thus

$$\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+1)^{2m+1}}+\cos\left(\frac{2\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+2)^{2m+1}}$$
$$+\cos(\pi)\sum_{k=0}^{\infty}\frac{1}{(6k+3)^{2m+1}}+\cos\left(\frac{4\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+4)^{2m+1}}$$
$$+\cos\left(\frac{5\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+5)^{2m+1}}+\cos(2\pi)\sum_{k=0}^{\infty}\frac{1}{(6k+6)^{2m+1}}$$

Multiply both sides by 6^{2m+1}, and then subtract the third and sixth series on the RHS from the Clausen term on the LHS (using $\cos\pi=-1$ and $\cos 2\pi=1$) to obtain:

$$6^{2m+1}\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)+\sum_{k=0}^{\infty}\frac{1}{(k+1/2)^{2m+1}}-\sum_{k=0}^{\infty}\frac{1}{(k+1)^{2m+1}}$$
$$=\cos\left(\frac{\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(k+1/6)^{2m+1}}+\cos\left(\frac{2\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(k+1/3)^{2m+1}}$$
$$+\cos\left(\frac{4\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(k+2/3)^{2m+1}}+\cos\left(\frac{5\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(k+5/6)^{2m+1}}$$

Express the cosine terms on the RHS in real/rational form to make the RHS

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1/6)^{2m+1}}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1/3)^{2m+1}}$$
$$-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+2/3)^{2m+1}}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+5/6)^{2m+1}}$$

Now use

$$\psi_{n\geq 1}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+x)^{n+1}}$$

to re-write the RHS as:

$$\begin{aligned}
&\frac{1}{2}\left(\frac{(-1)^{2m}}{(2m)!}\right)\left\{\psi_{2m}\left(\frac{1}{6}\right) - \psi_{2m}\left(\frac{1}{3}\right) - \psi_{2m}\left(\frac{2}{3}\right) + \psi_{2m}\left(\frac{5}{6}\right)\right\} \\
=&\frac{1}{2\,(2m)!}\left\{\psi_{2m}\left(\frac{1}{6}\right) - \psi_{2m}\left(\frac{1}{3}\right) - \psi_{2m}\left(\frac{2}{3}\right) + \psi_{2m}\left(\frac{5}{6}\right)\right\}
\end{aligned}$$

Next, apply the same process to the two series on the LHS (with the Clausen term):

$$\begin{aligned}
&6^{2m+1}\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right) + \sum_{k=0}^{\infty}\frac{1}{(k+1/2)^{2m+1}} - \sum_{k=0}^{\infty}\frac{1}{(k+1)^{2m+1}} \\
=&6^{2m+1}\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right) + \frac{1}{(2m)!}\left\{\psi_{2m}\left(\frac{1}{2}\right) - \psi_{2m}(1)\right\}
\end{aligned}$$

Multiplying both sides by $2(2m)!$ then gives the identity

$$\begin{aligned}
&2\,(2m)!\,6^{2m+1}\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right) + 2\psi_{2m}\left(\frac{1}{2}\right) - 2\psi_{2m}(1) \\
=&\psi_{2m}\left(\frac{1}{6}\right) - \psi_{2m}\left(\frac{1}{3}\right) - \psi_{2m}\left(\frac{2}{3}\right) + \psi_{2m}\left(\frac{5}{6}\right)
\end{aligned}$$

Next, repeat all of the above, but this time in terms of the Clasuen function with argument $2\pi/3$

$$\begin{aligned}
&\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right) = \sum_{k=1}^{\infty}\frac{\cos(2\pi k/3)}{k^{2m+1}} \\
&= \cos\left(\frac{2\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+1)^{2m+1}} + \cos\left(\frac{4\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+2)^{2m+1}} + \\
&\cos(2\pi)\sum_{k=0}^{\infty}\frac{1}{(6k+3)^{2m+1}} + \cos\left(\frac{8\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+4)^{2m+1}} + \\
&\cos\left(\frac{10\pi}{3}\right)\sum_{k=0}^{\infty}\frac{1}{(6k+5)^{2m+1}} + \cos(4\pi)\sum_{k=0}^{\infty}\frac{1}{(6k+6)^{2m+1}} \\
&= -\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(6k+1)^{2m+1}} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(6k+2)^{2m+1}} + \sum_{k=0}^{\infty}\frac{1}{(6k+3)^{2m+1}} \\
&-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(6k+4)^{2m+1}} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(6k+5)^{2m+1}} + \sum_{k=0}^{\infty}\frac{1}{(6k+6)^{2m+1}}
\end{aligned}$$

Continue exactly as before, and you get the second relation

$$\begin{aligned}
&2\,(2m)!\,6^{2m+1}\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right) - 2\psi_{2m}\left(\frac{1}{2}\right) - 2\psi_{2m}(1) \\
&= -\psi_{2m}\left(\frac{1}{6}\right) - \psi_{2m}\left(\frac{1}{3}\right) - \psi_{2m}\left(\frac{2}{3}\right) - \psi_{2m}\left(\frac{5}{6}\right)
\end{aligned}$$

Relative to the arguments $1/3, 2/3, 1/6$, and $5/6$, the arguments 1 and $1/2$ are pretty straightforward, so I'll simply state them now and prove them later.

$$\psi_{2m}(1) = -(2m)!\zeta(2m+1) \ , \ \psi_{2m}\left(\frac{1}{2}\right) = -(2m)!\left(2^{2m+1}-1\right)\zeta(2m+1)$$

Now, if you add the final forms or relation 1 and relation 2 you get:

$$\begin{aligned}
&2\,(2m)!\,6^{2m+1}\left[\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)+\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right)\right]-4\psi_{2m}(1)\\
=&2\,(2m)!\,6^{2m+1}\left[\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)+\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right)\right]+4\,(2m)!\zeta(2m+1)\\
=&2\left\{\psi_{2m}\left(\frac{1}{3}\right)+\psi_{2m}\left(\frac{2}{3}\right)\right\}
\end{aligned}$$

Or

$$\begin{aligned}
&(2m)!\,6^{2m+1}\left[\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)+\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right)\right]+2\,(2m)!\zeta(2m+1)\\
=&\psi_{2m}\left(\frac{1}{3}\right)+\psi_{2m}\left(\frac{2}{3}\right)
\end{aligned}$$

On the other hand, the reflection formula for the polygamma function gives:

$$\begin{aligned}
&\psi_{2m}(x)-\psi_{2m}(1-x)=\pi\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\cot\pi x\\
\Rightarrow&\psi_{2m}\left(\frac{1}{3}\right)-\psi_{2m}\left(\frac{2}{3}\right)=\pi\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\cot\pi x\Bigg|_{x=1/3}
\end{aligned}$$

So

$$\begin{aligned}
&\psi_{2m}\left(\frac{1}{3}\right)\\
=&\frac{(2m)!\,6^{2m+1}}{2}\left[\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)+\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right)\right]+\,(2m)!\zeta(2m+1)+\pi\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\cot\pi x\Bigg|_{x=1/3}
\end{aligned}$$

and

$$\begin{aligned}
&\psi_{2m}\left(\frac{2}{3}\right)\\
=&\frac{(2m)!\,6^{2m+1}}{2}\left[\mathrm{Cl}_{2m+1}\left(\frac{\pi}{3}\right)+\mathrm{Cl}_{2m+1}\left(\frac{2\pi}{3}\right)\right]+\,(2m)!\zeta(2m+1)-\pi\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\cot\pi x\Bigg|_{x=1/3}
\end{aligned}$$

*

4.5 Theta Functions Tutorial

✎ * 1. The Theta Functions

The four theta functions are defined as

$$\vartheta_1(z,q) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1/2)^2}e^{i(2n+1)z} = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z \tag{1}$$

$$\vartheta_2(z,q) = \sum_{n=-\infty}^{\infty} q^{n(n+1/2)^2}e^{i(2n+1)z} = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z \tag{2}$$

$$\vartheta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2inz} = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \tag{3}$$

$$\vartheta_4(z,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2inz} = 1+2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos(2nz) \tag{4}$$

where z and q are complex numbers. To ensure convergence, we let $|q|<1$. An alternative notation is to write

$$q = e^{i\pi\tau}$$

q is called the nome of the Theta Function and τ is it's parameter.

1.1 Identities involving products of Theta Functions

A lot of interesting identities of the theta functions can be obtained by the multiplication of two of their series, followed by rearrangement of terms in the product series.

Problem

Show that

$$\vartheta_1(x,q)\vartheta_1(y,q) = \vartheta_3(x+y,q^2)\vartheta_2(x-y,q^2) - \vartheta_2(x+y,q^2)\vartheta_3(x-y,q^2)$$

Solution

We have

$$\vartheta_1(x,q)\vartheta_1(y,q) = -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{m+n}q^{(m+1/2)^2+(n+1/2)^2}e^{i(2m+1)x+i(2n+1)y}$$

We now change the integer variables of summation from (m,n) to (r,s) by the following transformation equations

$$m+n=r, \quad m-n=s$$

If (m,n) are both even or odd, (r,s) will both be even. If, however, (m,n) have opposite parity, (r,s) will both be odd. Hence, by permitting (r,s) to range over pairs of even integers and over all pairs of odd integers, every pair of integers (m,n) will be obtained exactly once. Thus, we can write

$$\left(m+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)^2 = \frac{1}{2}(r+1)^2+\frac{s^2}{2}$$

$$(2m+1)x+(2n+1)y = (r+1)(x+y)+s(x-y)$$

So

$$\vartheta_1(x,q)\vartheta_1(y,q) = -\sum(-1)^r q^{(1/2)(r+1)^2+(1/2)s^2} e^{i(r+1)(x+y)+is(x-y)}$$

The summation is taken over all even and odd pairs of (r,s), as just explained. It follows that

$$\begin{aligned}\vartheta_1(x,q)\vartheta_1(y,q) = &-\sum_r\sum_s q^{2(r+1/2)^2+2s^2} e^{i(2r+1)(x+y)+2is(x-y)} \\ &+\sum_r\sum_s q^{2r^2+2(s+1/2)^2} e^{2ir(x+y)+i(2s+1)(x-y)}\end{aligned}$$

where the summation are now to be taken over all pairs of integers. Now, each of the double series can be written as products of two single series, as follows

$$\begin{aligned}\vartheta_1(x,q)\vartheta_1(y,q) &= -\sum_r q^{2(r+1/2)^2+2s^2} e^{i(2r+1)(x+y)} \sum_s q^{2s^2} e^{2is(x-y)} \\ &\quad +\sum_r q^{2r^2} e^{2ir(x+y)} \sum_s q^{2(s+1/2)^2} e^{i(2s+1)(x-y)} \\ &= \vartheta_3(x+y,q^2)\vartheta_2(x-y,q^2) - \vartheta_2(x+y,q^2)\vartheta_3(x-y,q^2) \qquad (5)\end{aligned}$$

Similarly, the following identities can be established.

$$\vartheta_1(x,q)\vartheta_2(y,q) = \vartheta_1(x+y,q^2)\vartheta_4(x-y,q^2) + \vartheta_4(x+y,q^2)\vartheta_1(x-y,q^2) \qquad (6)$$

$$\vartheta_2(x,q)\vartheta_2(y,q) = \vartheta_2(x+y,q^2)\vartheta_3(x-y,q^2) + \vartheta_3(x+y,q^2)\vartheta_4(x-y,q^2) \qquad (7)$$

$$\vartheta_3(x,q)\vartheta_3(y,q) = \vartheta_3(x+y,q^2)\vartheta_3(x-y,q^2) + \vartheta_2(x+y,q^2)\vartheta_2(x-y,q^2) \qquad (8)$$

$$\vartheta_3(x,q)\vartheta_4(y,q) = \vartheta_4(x+y,q^2)\vartheta_4(x-y,q^2) - \vartheta_1(x+y,q^2)\vartheta_1(x-y,q^2) \qquad (9)$$

$$\vartheta_4(x,q)\vartheta_4(y,q) = \vartheta_3(x+y,q^2)\vartheta_3(x-y,q^2) - \vartheta_2(x+y,q^2)\vartheta_2(x-y,q^2) \qquad (10)$$

These are left as an **exercise** for you.

1.2 Identities of the Basic Theta Functions

Define

$$\vartheta_i(q) \equiv \vartheta_i(z=0,q)$$

Now, by equations (2), (3) and (4) we see that

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \qquad (11)$$

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \qquad (12)$$

$$\vartheta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \qquad (13)$$

Adding equations (12) and (13) we obtain

$$\vartheta_3(q) + \vartheta_4(q) = 2\sum_{n \text{ even}} q^{n^2} = 2\vartheta_3(q^4) \qquad (15)$$

Also, by squaring equations (12) and (13) we see that

$$\vartheta_3^2(q) = \sum_{n=0}^{\infty} r_2(n) q^n \tag{16}$$

$$\vartheta_4^2(q) = \sum_{n=0}^{\infty} (-1)^n r_2(n) q^n \tag{17}$$

where $r_2(n)$ counts the number of ways of writing $n = j^2 + k^2$. Here we distinguish sign and permutation so that for example, $r_2(5) = 8$ since $(\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2$ and set $r_2(0) = 1$.

Note that $r_2(2n) = r_2(n)$. This may be proved elementarily by knowing that $2(a^2 + b^2) = (a+b)^2 + (a-b)^2$. Thus by adding equations (16) and (17) we obtain the beautiful identity

$$\vartheta_3^2(q) + \vartheta_4^2(q) = 2\sum_{n=0}^{\infty} r_2(2n) q^{2n} = 2\sum_{n=0}^{\infty} r_2(n) q^{2n} = 2\vartheta_3^2(q^2) \tag{18}$$

Three more important relations are obtained by setting $x, y = 0$ in equations (8),(9) and (10):

$$\vartheta_3^2(q^2) + \vartheta_2^2(q^2) = \vartheta_3^2(q) \tag{19}$$

$$\vartheta_3(q)\vartheta_4(q) = \vartheta_4^2(q^2) \tag{20}$$

$$\vartheta_3^2(q^2) - \vartheta_2^2(q^2) = \vartheta_4^2(q) \tag{21}$$

Problem

Prove the Jacobi's Identity:

$$\vartheta_3^4(q) = \vartheta_4^4(q) + \vartheta_2^4(q) \tag{22}$$

Solution

Multiply equations (19) and (21) and then use identity (20).

1.3 Periodicity Properties

Sometimes, the alternate notation

$$q = e^{i\pi\tau}$$

is used to express the theta function. Thus, by (1) we can write

$$i\vartheta_1(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{(n+1/2)^2 i\pi\tau + (2n+1)iz}$$

Now, increment the argument z by $\pi\tau$.

$$\begin{aligned}
\vartheta_1(z+\pi\tau|\tau) &= -i\sum_{n=-\infty}^{\infty} (-1)^n e^{(n+1/2)^2 i\pi\tau + i(2n+1)(z+\pi\tau)} \\
&= ie^{-i\pi\tau - 2iz}\sum_{n=-\infty}^{\infty} (-1)^{n+1} e^{(n+3/2)^2 i\pi\tau + i(2n+3)z} \\
&= -(qe^{2iz})^{-1}\vartheta_1(z|\tau)
\end{aligned}$$

$\pi\tau$ is termed as a **quasi-period** of $\vartheta_1(z|\tau)$ with periodicity factor $-(qe^{biz})^{-1}$. Writing $\lambda = qe^{2iz}$, the following identities may be obtained similarly:

$$\vartheta_1(z|\tau) = -\vartheta_1(z+\pi|\tau) = -\lambda\vartheta_1(z+\pi\tau|\tau) = \lambda\vartheta_1(z+\pi+\pi\tau|\tau) \tag{23}$$
$$\vartheta_2(z|\tau) = -\vartheta_2(z+\pi|\tau) = \lambda\vartheta_2(z+\pi\tau|\tau) = -\lambda\vartheta_2(z+\pi+\pi\tau|\tau) \tag{24}$$
$$\vartheta_3(z|\tau) = \vartheta_3(z+\pi|\tau) = \lambda\vartheta_3(z+\pi\tau|\tau) = \lambda\vartheta_3(z+\pi+\pi\tau|\tau) \tag{25}$$
$$\vartheta_4(z|\tau) = \vartheta_4(z+\pi|\tau) = -\lambda\vartheta_4(z+\pi\tau|\tau) = -\lambda\vartheta_4(z+\pi+\pi\tau|\tau) \tag{26}$$

Exercise
By incrementing z by the half periods $\frac{\pi}{2}, \frac{1}{2}\pi\tau$ and $\frac{1}{2}(\pi+\pi\tau)$, verify the following identities:

$$\vartheta_1(z|\tau) = -\vartheta_2(z+\pi/2|\tau) = -i\mu\vartheta_4(z+\pi\tau/2|\tau) = -i\mu\vartheta_3(z+(\pi+\pi\tau)/2|\tau) \tag{27}$$
$$\vartheta_2(z|\tau) = \vartheta_1(z+\pi/2|\tau) = \mu\vartheta_3(z+\pi\tau/2|\tau) = \mu\vartheta_4(z+(\pi+\pi\tau)/2|\tau) \tag{28}$$
$$\vartheta_3(z|\tau) = \vartheta_4(z+\pi/2|\tau) = \mu\vartheta_2(z+\pi\tau/2|\tau) = \mu\vartheta_1(z+(\pi+\pi\tau)/2|\tau) \tag{29}$$
$$\vartheta_4(z|\tau) = \vartheta_3(z+\pi/2|\tau) = -i\mu\vartheta_1(z+\pi\tau/2|\tau) = i\mu\vartheta_2(z+(\pi+\pi\tau)/2|\tau) \tag{27}$$

where $\mu = q^{1/4}e^{iz}$.

1.4 Further Identities of Theta Functions

Now, we are going to deduce a large number of identities involving products of four theta functions, all having the same nome.
For example, squaring and subtracting (5) and (10) we get

$$\vartheta_4^2(x,q)\vartheta_4^2(y,q) = [\vartheta_3^2(x+y,q^2) - \vartheta_2^2(x+y,q^2)] \times [\vartheta_3^2(x-y,q^2) - \vartheta_2^2(x+y,q^2)] \tag{28}$$

Putting $y = 0$, gives

$$\vartheta_3^2(x,q^2) - \vartheta_2^2(x,q^2) = \vartheta_4(x,q)\vartheta_4(0,q) \tag{29}$$

We now deduce from (28) that

$$\vartheta_4(x+y,q)\vartheta_4(x-y,q)\vartheta_4^2(0,q) = \vartheta_4^2(x,q)\vartheta_4^2(y,q) - \vartheta_1^2(x,q)\vartheta_1^2(y,q)$$

In a similar manner, we obtain the following set of identities:

$$\vartheta_1(x+y,q)\vartheta_1(x-y,q)\vartheta_3^2(0,q) = \vartheta_1^2(x,q)\vartheta_3^2(y,q) - \vartheta_3^2(x,q)\vartheta_3^2(y,q) \tag{30}$$
$$\vartheta_2(x+y,q)\vartheta_2(x-y,q)\vartheta_3^2(0,q) = \vartheta_2^2(x,q)\vartheta_3^2(y,q) - \vartheta_4^2(x,q)\vartheta_1^2(y,q) \tag{31}$$
$$\vartheta_3(x+y,q)\vartheta_3(x-y,q)\vartheta_3^2(0,q) = \vartheta_1^2(x,q)\vartheta_1^2(y,q) + \vartheta_3^2(x,q)\vartheta_3^2(y,q) \tag{32}$$
$$\vartheta_4(x+y,q)\vartheta_4(x-y,q)\vartheta_3^2(0,q) = \vartheta_1^2(x,q)\vartheta_2^2(y,q) + \vartheta_3^2(x,q)\vartheta_4^2(y,q) \tag{33}$$

Exercise

Show that

$$\vartheta_3^2(x,q)\vartheta_3^2(0,q) = \vartheta_4^2(x,q)\vartheta_4^2(0,q) + \vartheta_2^2(x,q)\vartheta_2^2(0,q) \tag{34}$$
$$\vartheta_4^2(x,q)\vartheta_3^2(0,q) = \vartheta_1^2(x,q)\vartheta_2^2(0,q) + \vartheta_3^2(x,q)\vartheta_4^2(0,q) \tag{35}$$
$$\vartheta_4^2(x,q)\vartheta_2^2(0,q) = \vartheta_1^2(x,q)\vartheta_3^2(0,q) + \vartheta_2^2(x,q)\vartheta_4^2(0,q) \tag{36}$$
$$\vartheta_3^2(x,q)\vartheta_2^2(0,q) = \vartheta_1^2(x,q)\vartheta_4^2(0,q) + \vartheta_2^2(x,q)\vartheta_3^2(0,q) \tag{37}$$

1.5 The identity $\vartheta_1'(0,q) = \vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)$

Differentiating equation (6) with respect to xx and then putting $x = y = 0$, we have that

$$\vartheta_1'(0,q)\vartheta_2(0,q) = 2\vartheta_1'(0,q^2)\vartheta_4(0,q^2) \tag{*}$$

Next, putting $x = y = 0$ in (7) and (9) we show that

$$\vartheta_2^2(0,q) = 2\vartheta_2(0,q^2)\vartheta_4(0,q^2)\ ,\ \vartheta_3(0,q)\vartheta_4(0,q) = \vartheta_4^2(0,q^2)$$

Dividing $(*)$ by these two equations

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)} = \frac{\vartheta_1'(0,q^2)}{\vartheta_2(0,q^2)\vartheta_3(0,q^2)\vartheta_4(0,q^2)}$$

By repeated application

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)} = \frac{\vartheta_1'(0,q^{2^n})}{\vartheta_2(0,q^{2^n})\vartheta_3(0,q^{2^n})\vartheta_4(0,q^{2^n})}$$

Letting $n \to \infty$ in this identity, $q^{2^n} \to 0$. Hence

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)} = \lim_{q\to 0} \frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)} \tag{**}$$

Now, from equations (1)-(4), we find that

$$\vartheta_1'(0,q) = 2q^{1/4} + \mathcal{O}(q^{9/4})$$
$$\vartheta_2(0,q) = 2q^{1/4} + \mathcal{O}(q^{9/4})$$
$$\vartheta_3(0,q) = 1 + \mathcal{O}(q)$$
$$\vartheta_4(0,q) = 1 + \mathcal{O}(q)$$

Substituting these in $(**)$ and simplifying the limit we get

$$\vartheta_1'(0,q) = \vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q) \tag{38}$$

1.6 Infinite Product Representation of Theta Functions

We have

$$\vartheta_1(z,q) = 2q^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos(2z)+q^{4n}) \tag{39}$$

$$\vartheta_2(z,q) = 2q^{\frac{1}{4}}\cos z\prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n}\cos(2z)+q^{4n}) \tag{40}$$

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n-1}\cos(2z)+q^{4n-2}) \tag{41}$$

$$\vartheta_4(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n-1}\cos(2z)+q^{4n-2}) \tag{42}$$

These can be proved using the **Jacobi Triple Product** identity:

$$\sum_{n=-\infty}^{\infty}x^{n^2}y^n = \prod_{m=1}^{\infty}\left(1-x^{2m}\right)\left(1-x^{2m-1}y\right)\left(1-x^{2m-1}y^{-1}\right) \tag{43}$$

There are many methods to prove Jacobi Triple Product but we shall not discuss it's proof in this tutorial. *

4.6 Logarithmic-Trigonometric Integrals

✎ * **Problem 1:**

Without residues - and without using Lewin's book! - evaluate

$$S_m(\theta) = \int_0^{\theta} x^m \ln(\sin x)\,\mathrm{d}x$$

in terms of Clausen functions and transcendental constants, where the Clausen functions are defined by:

$$\mathrm{Cl}_2(\theta) = -\int_0^{\theta}\ln\left|2\sin\frac{x}{2}\right|\mathrm{d}x$$

$$\mathrm{Cl}_n(\theta) = \begin{cases}\displaystyle\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^n} & \text{if } n \text{ is even}\\ \displaystyle\sum_{k=1}^{\infty}\frac{\cos k\theta}{k^n} & \text{if } n \text{ is odd}\end{cases},\quad \mathrm{Sl}_n(\theta) = \begin{cases}\displaystyle\sum_{k=1}^{\infty}\frac{\cos k\theta}{k^n} & \text{if } n \text{ is even}\\ \displaystyle\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^n} & \text{if } n \text{ is odd}\end{cases}$$

Hint: at some point you might split the finite series into odd resp. even terms, using

$$\sum_{k=0}^{m}f(k) = \sum_{j=0}^{\lfloor m/2\rfloor}f(2j) + \sum_{j=0}^{\lfloor (m-1)/2\rfloor}f(2j+1)$$

This is not a closed form, but this is how I would evaluate the integral for specific values of m.

Using the fact that

$$\ln(1-e^{2ix}) = \ln(2\sin x) + i\left(x-\frac{\pi}{2}\right)$$

we have

$$\ln 2\int_0^\theta x^m\mathrm{d}x+\int_0^\theta x^m\ln(\sin x)\mathrm{d}x=\int_0^\theta x^m\ln(1-e^{2ix})\mathrm{d}x-i\int_0^\theta x^m\left(x-\frac{\pi}{2}\right)\mathrm{d}x$$

Then If we impose the restriction $0<\theta<\pi$,

$$\begin{aligned}\int_0^\theta x^m\ln(\sin x)\ \mathrm{d}x&=-\ln 2\int_0^\theta x^m\ \mathrm{d}x+\Re\int_0^\theta x^m\ln(1-e^{2ix})\ \mathrm{d}x\\&=-\ln(2)\frac{\theta^{m+1}}{m+1}-\Re\int_0^\theta x^m\sum_{n=1}^\infty\frac{e^{2inx}}{n}\ \mathrm{d}x\\&=-\ln(2)\frac{\theta^{m+1}}{m+1}-\sum_{n=1}^\infty\frac{1}{n}\int_0^\theta x^m\cos(2nx)\ \mathrm{d}x\end{aligned}$$

and

$$\int x^{2n}\cos x\,\mathrm{d}x=(2n)!\left[\sum_{k=0}^n(-1)^k\frac{x^{2n-2k}}{(2n-2k)!}\sin x+\sum_{k=0}^{n-1}(-1)^k\frac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\right]$$

$$\int x^{2n+1}\cos x\,\mathrm{d}x=(2n+1)!\left[\sum_{k=0}^n(-1)^k\frac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x+\sum_{k=0}^n(-1)^k\frac{x^{2n-2k}}{(2n-2k)!}\cos x\right]$$

Let's show a few easy ones

Type 1:

$$\int_0^{\pi/4}\ln^m(\tan x)\,\mathrm{d}x$$

Solution:

$$\begin{aligned}&\int_0^{\pi/4}\ln^m(\tan x)\,\mathrm{d}x=\int_0^1\frac{(\ln x)^m}{1+x^2}\,\mathrm{d}x\\&=\sum_{k=0}^\infty(-1)^k\int_0^1x^{2k}(\ln x)^m\,\mathrm{d}x=(-1)^m m!\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{m+1}}=(-1)^m m!\,\beta(m+1)\end{aligned}$$

Where $\beta(x)=\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^x}$ is the **Dirichlet Beta Function**.

The case $x=2$ is Catalan's constant: $\beta(2)=\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^2}=\mathbf{G}$.

Type 2:

$$\int_0^{\pi/2}\ln^m(\tan x)\,\mathrm{d}x$$

Solution:

$$\int_0^{\pi/2}\ln^m(\tan x)\,\mathrm{d}x=\int_0^{\pi/4}\ln^m(\tan x)\,\mathrm{d}x+\int_{\pi/4}^{\pi/2}\ln^m(\tan x)\,\mathrm{d}x$$

Make the sub $x=\pi/2-y$ on the second integral to obtain

$$\begin{aligned}&\int_0^{\pi/2}\ln^m(\tan x)\,\mathrm{d}x=\int_0^{\pi/4}\ln^m(\tan x)\,\mathrm{d}x+\int_0^{\pi/4}\ln^m(\cot x)\,\mathrm{d}x\\&=\int_0^{\pi/4}\ln^m(\tan x)\mathrm{d}x+(-1)^m\int_0^{\pi/4}\ln^m(\tan x)\mathrm{d}x=[1+(-1)^m]\int_0^{\pi/4}\ln^m(\tan x)\mathrm{d}x\end{aligned}$$

Hence we have:

$$\int_0^{\pi/2}\ln^m(\tan x)\mathrm{d}x=[1+(-1)^m]m!\,\beta(m+1)$$

Type 3:

$$\mathcal{T}_m(\theta) = \int_0^\theta \ln^m(\tan x)\mathrm{d}x$$

Solution:

$$\int_0^\theta \ln^m(\tan x)\mathrm{d}x = \int_0^{z=\tan\theta} \frac{(\ln x)^m}{1+x^2}\mathrm{d}x = \sum_{k=0}^{\infty}(-1)^k \int_0^z x^{2k}(\ln x)^m \mathrm{d}x$$

Let $x \to z\,y$ to get

$$\sum_{k=0}^{\infty}(-1)^k \int_0^z x^{2k}(\ln x)^m \mathrm{d}x = \sum_{k=0}^{\infty}(-1)^k\, z^{2k+1} \int_0^1 y^{2k}(\ln z + \ln y)^m \mathrm{d}y$$

Use the finite binomial theorem on the logarithmic part

$$\sum_{j=0}^{m}\binom{m}{j}(\ln z)^{m-j} \sum_{k=0}^{\infty}(-1)^k\, z^{2k+1} \int_0^1 y^{2k}(\ln y)^j \mathrm{d}y$$

and the classic - easily provable - result

$$\int_0^1 x^n (\ln x)^m \,\mathrm{d}x = \frac{(-1)^m\, m!}{(n+1)^{m+1}}$$

to obtain

$$\begin{aligned}
&\sum_{j=0}^{m}(-1)^j j!\binom{m}{j}(\ln z)^{m-j} \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)^{j+1}} \\
&= m!\sum_{j=0}^{m}(-1)^j \frac{(\ln z)^{m-j}}{(m-j)!} \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)^{j+1}}
\end{aligned}$$

Finally, observe that for $0 \le z \le 1$ the Inverse Tangent Integrals

$$\mathrm{Ti}_1(z) = \frac{\tan^{-1} z}{z}\ ,\ \mathrm{Ti}_{m+1}(z) = \int_0^z \frac{\mathrm{Ti}_m(x)}{x}\mathrm{d}x$$

have the series representation

$$\mathrm{Ti}_m(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)^k}$$

and the proof is complete:

$$\mathcal{T}_m(\theta) = m!\sum_{j=0}^{m}(-1)^j \frac{\ln^{m-j}(\tan\theta)}{(m-j)!}\,\mathrm{Ti}_{j+1}(\tan\theta)$$

Type 4:

$$\mathcal{S}(\theta) = \int_0^\theta \ln(\sin x)\mathrm{d}x\ ,\ \mathcal{C}(\theta) = \int_0^\theta \ln(\cos x)\mathrm{d}x$$

Proof:

$$\begin{aligned}
\mathcal{S}(\theta) &= \int_0^\theta \ln(\sin x)\mathrm{d}x = \int_0^\theta \ln(2\sin x)\mathrm{d}x - \ln 2\int_0^\theta \mathrm{d}x \\
&= \frac{1}{2}\int_0^{2\theta} \ln\left[2\sin\left(\frac{y}{2}\right)\right]\mathrm{d}x - \theta\ln 2
\end{aligned}$$

In terms of the Clausen function

$$\mathrm{Cl}_2(\phi) = -\int_0^\phi \left|\ln\left(2\sin\left(\frac{x}{2}\right)\right)\right| \mathrm{d}x = \sum_{k=1}^{\infty} \frac{\sin k\phi}{k^2}$$

the answer is, therefore

$$\mathcal{S}(\theta) = -\frac{1}{2}\mathrm{Cl}_2(2\theta) + \frac{1}{2}\mathrm{Cl}_2(0) - \theta\ln 2 = -\frac{1}{2}\mathrm{Cl}_2(2\theta) - \theta\ln 2$$

Next, make the sub $x = \pi/2 - y$ in $\mathcal{C}(\theta)$ to get

$$\begin{aligned}\mathcal{C}(\theta) &= -\int_{\pi/2}^{\pi/2-\theta} \ln\left[\cos\left(\frac{\pi}{2} - y\right)\right] \mathrm{d}y = -\int_{\pi/2}^{\pi/2-\theta} \ln(\sin y)\mathrm{d}y \\ &= \mathcal{S}\left(\frac{\pi}{2}\right) - \mathcal{S}\left(\frac{\pi}{2} - \theta\right) = -\frac{1}{2}\mathrm{Cl}_2(\pi) + \frac{1}{2}\mathrm{Cl}_2(\pi - 2\theta) - \theta\ln 2 \\ &= \frac{1}{2}\mathrm{Cl}_2(\pi - 2\theta) - \theta\ln 2\end{aligned}$$

The proof is complete:

$$\begin{aligned}\mathcal{S}(\theta) &= \int_0^\theta \ln(\sin x)\mathrm{d}x = -\frac{1}{2}\mathrm{Cl}_2(2\theta) - \theta\ln 2 \\ \mathcal{C}(\theta) &= \int_0^\theta \ln(\cos x)\mathrm{d}x = \frac{1}{2}\mathrm{Cl}_2(\pi - 2\theta) - \theta\ln 2\end{aligned}$$

setting $\theta = \pi/4$, and then using the (easily proven) result $\mathrm{Cl}_2(\pi/2) = \mathbf{G}$ (Catalan's constant), we obtain the classic results:

$$\int_0^{\pi/4} \ln(\sin x)\mathrm{d}x = -\frac{\mathbf{G}}{2} - \frac{\pi}{4}\ln 2\ , \quad \int_0^{\pi/4} \ln(\cos x)\mathrm{d}x = \frac{\mathbf{G}}{2} - \frac{\pi}{4}\ln 2$$

This is exactly as we would expect, since

$$\int_0^{\pi/4} \ln(\tan x)\mathrm{d}x = -\mathbf{G} = \int_0^{\pi/4} \ln(\sin x)\mathrm{d}x - \int_0^{\pi/4} \ln(\cos x)\mathrm{d}x$$

Type 5:

$$\mathrm{Lc}_\pm(\theta) = \int_0^\theta \ln(1 \pm \cos x)\mathrm{d}x$$

Proof:

For

$$\mathrm{Lc}_+(\theta) = \int_0^\theta \ln(1 + \cos x)\mathrm{d}x$$

use

$$\cos\frac{x}{2} = \left(\frac{1+\cos x}{2}\right)^{1/2}$$

for $|\theta| \le \pi$, to obtain

$$\begin{aligned}\mathrm{Lc}_+(\theta) &= \int_0^\theta \ln\left(2\cos^2\frac{x}{2}\right)\mathrm{d}x = 2\int_0^\theta \ln\left(2\cos\frac{x}{2}\right)\mathrm{d}x - \ln 2\int_0^\theta \mathrm{d}x \\ &= -2\int_\pi^{\pi-\theta} \ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x - \theta\ln 2 = 2\mathrm{Cl}_2(\pi - \theta) - 2\mathrm{Cl}_2(\pi) - \theta\ln 2\end{aligned}$$

Hence:

$$\mathrm{Lc}_+(\theta) = 2\mathrm{Cl}_2(\pi-\theta) - \theta\ln 2$$

Similarly, for

$$\mathrm{Lc}_-(\theta) = \int_0^\theta \ln(1-\cos x)\mathrm{d}x$$

use

$$\sin\frac{x}{2} = \left(\frac{1-\cos x}{2}\right)^{1/2}$$

for $0 \le \theta \le 2\pi$, to obtain

$$\mathrm{Lc}_-(\theta) = \int_0^\theta \ln\left(2\sin^2\frac{x}{2}\right)\mathrm{d}x = 2\int_0^\theta \ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x - \ln 2\int_0^\theta \mathrm{d}x$$
$$= -2\mathrm{Cl}_2(\theta) - \theta\ln 2$$

Hence:

$$\mathrm{Lc}_+(\theta) = \int_0^\theta \ln(1+\cos x)\mathrm{d}x = 2\mathrm{Cl}_2(\pi-\theta) - \theta\ln 2$$
$$\mathrm{Lc}_-(\theta) = \int_0^\theta \ln(1-\cos x)\mathrm{d}x = -2\mathrm{Cl}_2(\theta) - \theta\ln 2$$

Setting $\theta = \pi/2$ in those last two eqns gives the classic results

$$\int_0^{\pi/2} \ln(1+\cos x)\mathrm{d}x = 2\mathbf{G} - \frac{\pi}{2}\ln 2\ ,\quad \int_0^{\pi/2} \ln(1-\cos x)\mathrm{d}x = -2\mathbf{G} - \frac{\pi}{2}\ln 2$$

Type 6:

$$\int_0^\theta \ln|\cos x + \sin x|\mathrm{d}x$$

Proof:

$$\int_0^\theta \ln|\cos x+\sin x|\,\mathrm{d}x = \int_0^\theta \ln\left|\sqrt{2}\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)\right|\mathrm{d}x$$
$$= -\frac{\theta}{2}\ln 2 + \int_0^\theta \ln\left|2\sin\left(x+\frac{\pi}{4}\right)\right|\mathrm{d}x = -\frac{\theta}{2}\ln 2 + \frac{1}{2}\int_{\pi/2}^{2\theta+\pi/2}\ln\left|2\sin\frac{y}{2}\right|\mathrm{d}y$$
$$= -\frac{\theta}{2}\ln 2 - \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{2}+2\theta\right) + \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = -\frac{\theta}{2}\ln 2 - \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{2}+2\theta\right) + \frac{\mathbf{G}}{2}$$

Thus

$$\int_0^\theta \ln|\cos x+\sin x|\mathrm{d}x = -\frac{\theta}{2}\ln 2 - \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{2}+2\theta\right) + \frac{\mathbf{G}}{2}$$

Type 7:

$$\int_0^\theta x\ln(\sin x)\mathrm{d}x$$

Proof:

$$\int_0^\theta x\ln(\sin x)\mathrm{d}x = \int_0^\theta x\ln(2\sin x)\mathrm{d}x - \frac{\theta^2}{2}\ln 2 = \frac{1}{4}\int_0^{2\theta} y\ln\left(2\sin\frac{y}{2}\right)\mathrm{d}y - \frac{\theta^2}{2}\ln 2$$
$$= \frac{1}{4}\int_0^{2\theta} x\left[-\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Cl}_2(x)\right]\mathrm{d}x - \frac{\theta^2}{2}\ln 2 = -\frac{1}{4}x\mathrm{Cl}_2(x)\Big|_0^{2\theta} + \frac{1}{4}\int_0^{2\theta}\mathrm{Cl}_2(x)\mathrm{d}x - \frac{\theta^2}{2}\ln 2$$
$$= -\frac{\theta}{2}\mathrm{Cl}_2(2\theta) - \frac{\theta^2}{2}\ln 2 + \frac{1}{4}\int_0^{2\theta}\mathrm{Cl}_2(x)\mathrm{d}x = -\frac{\theta}{2}\mathrm{Cl}_2(2\theta) - \frac{\theta^2}{2}\ln 2 + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{2\theta}\sin kx\mathrm{d}x$$
$$= -\frac{\theta}{2}\mathrm{Cl}_2(2\theta) - \frac{\theta^2}{2}\ln 2 - \frac{1}{4}\sum_{k=1}^{\infty}\frac{\cos kx}{k^3}\Big|_0^{2\theta} = -\frac{\theta}{2}\mathrm{Cl}_2(2\theta) - \frac{\theta^2}{2}\ln 2 - \frac{1}{4}\sum_{k=1}^{\infty}\frac{\cos 2k\theta}{k^3} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^3}$$
$$= -\frac{\theta}{2}\mathrm{Cl}_2(2\theta) - \frac{\theta^2}{2}\ln 2 - \frac{1}{4}\mathrm{Cl}_3(2\theta) + \frac{1}{4}\zeta(3)$$

since

$$\mathrm{Cl}_{2m+1}(\varphi) = \sum_{k=1}^{\infty}\frac{\cos k\varphi}{k^{2m+1}}$$

Thus

$$\int_0^\theta x\ln(\sin x)\mathrm{d}x = \frac{1}{4}\left[\zeta(3) - 2\theta^2\ln 2 - 2\theta\mathrm{Cl}_2(2\theta) - \mathrm{Cl}_3(2\theta)\right]$$

Type 8:

$$\int_0^\theta x\ln(\cos x)\mathrm{d}x$$

Proof:

Make the sub $x = \pi/2 - y$ to obtain

$$\int_0^\theta x\ln(\cos x)\mathrm{d}x = -\int_{\pi/2}^{\pi/2-\theta}\left(y - \frac{\pi}{2}\right)\ln(\sin y)\mathrm{d}y$$
$$= -\int_{\pi/2}^{\pi/2-\theta} x\ln(\sin x)\mathrm{d}x + \frac{\pi}{2}\int_{\pi/2}^{\pi/2-\theta}\ln(\sin x)\mathrm{d}x$$

Which, in terms of the solutions to Type 4 & 7, is:

$$-\frac{1}{4}\left[\zeta(3) - 2\theta^2\ln 2 - 2\theta\mathrm{Cl}_2(2\theta) - \mathrm{Cl}_3(2\theta)\right]_{\pi/2}^{\pi/2-\theta} + \frac{\pi}{2}\left[-\frac{1}{2}\mathrm{Cl}_2(2\theta) - \theta\ln 2\right]_{\pi/2}^{\pi/2-\theta}$$

Which - unless I've made some hideously daft mistake - boils down to

$$\frac{\theta^2}{2}\ln 2 - \frac{\theta}{2}\mathrm{Cl}_2(\pi - 2\theta) + \frac{1}{4}\mathrm{Cl}_3(\pi - 2\theta) - \frac{1}{4}\mathrm{Cl}_3(\pi)$$

And yet,

$$\mathrm{Cl}_3(\pi) = \sum_{k=1}^{\infty}\frac{\cos\pi k}{k^3} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^3} = -\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3} = -\eta(3)$$
$$= -\left(1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \cdots\right) = -\zeta(3)\left(1 - \frac{2}{2^3}\right) = -\frac{3}{4}\zeta(3)$$

The proof is now complete, and the final result is

$$\int_0^\theta x\ln(\cos x)\mathrm{d}x = \frac{3}{16}\zeta(3) + \frac{\theta^2}{2}\ln 2 - \frac{\theta}{2}\mathrm{Cl}_2(\pi - 2\theta) + \frac{1}{4}\mathrm{Cl}_3(\pi - 2\theta)$$

Type 9:

$$\mathcal{I}(a,b) = \int_0^\theta \ln(a\cos x + b\sin x)\mathrm{d}x$$

Proof:

For non-zero a and b, and within the first quadrant $0 \le \theta \le \pi/2$, the integrand is always positive, and hence converges. And so, to solve integrals of this type, we appeal to the auxiliary angle theorems of trigonometry, namely:

(1) $a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(x \pm \varphi)$; $\varphi = \tan^{-1}(\mp b/a)$

(2) $a\cos x + b\sin x = \sqrt{a^2+b^2}\sin(x \pm \varphi)$; $\varphi = \tan^{-1}(\pm a/b)$

Thus $\mathcal{I}(a,b)$ can be expressed as, say,

$$\frac{\theta}{2}\ln(a^2+b^2) + \int_0^\theta \ln\left[\sin(x-\varphi)\right]\mathrm{d}x$$

for $\varphi = \tan^{-1}(-a/b) = -\tan^{-1}(a/b)$.

$\int_0^\theta \ln\left[\sin(x-\varphi)\right]\mathrm{d}x$ is then evaluated as follows

$$\int_0^\theta \ln\left[\sin(x-\varphi)\right]\mathrm{d}x = \int_0^\theta \ln\left[2\sin(x-\varphi)\right]\mathrm{d}x - \theta\ln 2$$

Now substitute $x - \varphi = y/2$

$$\frac{1}{2}\int_{-2\varphi}^{2\theta-2\varphi} \ln\left(2\sin\frac{y}{2}\right)\mathrm{d}y - \theta\ln 2 = \frac{1}{2}\left[\mathrm{Cl}_2(-2\varphi) - \mathrm{Cl}_2(2\theta-2\varphi)\right] - \theta\ln 2$$

However, the Clausen function of order 2 is an odd function (a property it inherits from the sine function contained therein), hence

$$\mathrm{Cl}_2(-\phi) = \sum_{k=1}^{\infty}\frac{\sin(-k\phi)}{k^2} = -\sum_{k=1}^{\infty}\frac{\sin(k\phi)}{k^2} = -\mathrm{Cl}_2(\phi)$$

Thus

$$\int_0^\theta \ln\left[\sin(x-\varphi)\right]\mathrm{d}x = -\frac{1}{2}\left[\mathrm{Cl}_2(2\varphi) + \mathrm{Cl}_2(2\theta-2\varphi)\right] - \theta\ln 2$$

and the final evaluation is

$$\int_0^\theta \ln(a\cos x + b\sin x)\mathrm{d}x = \frac{\theta}{2}\ln(a^2+b^2) - \frac{1}{2}\left[\mathrm{Cl}_2(2\varphi) + \mathrm{Cl}_2(2\theta-2\varphi)\right] - \theta\ln 2$$

Type 10:

$$I_0 = \int_0^1 \ln\Gamma(x)\mathrm{d}x$$

Not strictly a log-trig integral this one, but a classic Gamma function integral that involves log-trig integration in the solution.

Proof:

Start with Euler's reflection formula for the Gamma Function:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

Take the logarithm of both sides, and then integrate from 0 to 1 to obtain

$$\int_0^1 \ln\Gamma(x)\mathrm{d}x + \int_0^1 \ln\Gamma(1-x)\mathrm{d}x = \ln\pi - \int_0^1 \ln(\sin\pi x)\mathrm{d}x$$

Substitute $x \to 1-y$ in the second loggamma integral to get:

$$2\int_0^1 \ln\Gamma(x)\mathrm{d}x = 2I_0 = \ln\pi - \int_0^1 \ln(\sin\pi x)\mathrm{d}x$$

where

$$\begin{aligned}\int_0^1 \ln(\sin\pi x)\mathrm{d}x &= \int_0^1 \ln(2\sin\pi x)\mathrm{d}x - \ln 2\int_0^1 \mathrm{d}x \\ &= \int_0^1 \ln(2\sin\pi x)\mathrm{d}x - \ln 2\end{aligned}$$

Next, set $\pi x = y/2$ in the logsine integral to obtain

$$\int_0^1 \ln(2\sin\pi x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} \ln\left(2\sin\frac{y}{2}\right)\mathrm{d}x = -\frac{1}{2\pi}\mathrm{Cl}_2(2\pi) = -\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{\sin 2\pi k}{k^2} = 0$$

Hence

$$2\int_0^1 \ln\Gamma(x)\mathrm{d}x = \ln\pi + \ln 2 \quad \text{or} \quad \int_0^1 \ln\Gamma(x)\mathrm{d}x = \ln\sqrt{2\pi}$$

For integrals similar to the last one - although far more advanced - I'd highly recommend the following paper:

http://arxiv.org/pdf/math/0012078.pdf

Here's but one example of the many, many beautiful integrals in the aforementioned paper (due to **Olivier Espinosa** and **Victor H. Moll**):

$$\int_0^1 \ln^2\Gamma(x)\mathrm{d}x = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma}{3}\ln\sqrt{2\pi} + \frac{4}{3}(\ln\sqrt{2\pi})^2 - (\gamma + 2\ln\sqrt{2\pi})\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}$$

Type 11:

$$\mathbb{T}(m,n) = \int_0^{\pi/4} \ln^n(\tan x)\ln^m[-\ln(\tan x)]\mathrm{d}x$$

Which has the equivalent algebraic form

$$\mathbb{T}(m,n) = \int_0^1 \frac{(\ln x)^n \ln^m(-\ln x)}{1+x^2}\mathrm{d}x$$

Proof:

Actually, this one's not anywhere near as scary as it looks... Start with the hyperbolic integral

$$\mathcal{I}(p) = \int_0^{\infty}\frac{x^{p-1}}{\cosh x}\mathrm{d}x = 2\int_0^{\infty}\frac{x^{p-1}\,e^{-x}}{(1+e^{-2x})}\mathrm{d}x = 2\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}x^{p-1}e^{-(2k+1)x}\mathrm{d}x$$

for the parameter $p \in \mathbb{Q}^+$.

Make the substitution $(2k+1)x = y$ to obtain

$$\begin{aligned}&2\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}x^{p-1}e^{-(2k+1)x}\mathrm{d}x = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\int_0^{\infty}\left(\frac{y}{2k+1}\right)^{p-1}e^{-y}\mathrm{d}y \\ &= 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^p}\int_0^{\infty}x^{p-1}e^{-x}dy = 2\,\Gamma(p)\beta(p)\end{aligned}$$

On the other hand, returning to

$$\mathcal{I}(p) = 2\int_0^\infty \frac{x^{p-1}e^{-x}}{(1+e^{-2x})}\mathrm{d}x = 2\Gamma(p)\beta(p)$$

and setting $y = e^{-x}$ yields

$$\int_0^\infty \frac{x^{p-1}\,e^{-x}}{(1+e^{-2x})}\mathrm{d}x = \Gamma(p)\,\beta(p) = \int_0^1 \frac{(-\ln y)^{p-1}}{(1+y^2)}\mathrm{d}y$$

Differentiating both sides m-times w.r.t. the parameter p gives

$$\sum_{j=0}^{m}\binom{m}{j}\Gamma^{(m-j)}(p)\beta^{(j)}(p) = \int_0^1 \frac{(-\ln y)^{p-1}\ln^m(-\ln y)}{(1+y^2)}\mathrm{d}y$$
$$= \int_0^{\pi/4} \ln^{p-1}(\tan x)\ln^m[-\ln(\tan x)]\mathrm{d}x = \mathbb{T}(m,p-1)$$

And thus we obtain the final result, a binomial sum of derivatives of the Gamma and Dirichlet Beta Functions.

$$\mathbb{T}(m,n) = \int_0^{\pi/4} \ln^n(\tan x)\ln^m[-\ln(\tan x)]\mathrm{d}x = \sum_{j=0}^{m}\binom{m}{j}\Gamma^{(m-j)}(n+1)\,\beta^{(j)}(n+1)$$

One last point before moving on from **Type 11**. Apply the substitution $x = \pi/2 - y$ to

$$\mathbb{T}(m,n) = \int_0^{\pi/4} \ln^n(\tan x)\ln^m[-\ln(\tan x)]\mathrm{d}x$$

to obtain

$$(-1)^n\int_{\pi/4}^{\pi/2} \ln^n(\tan x)\ln^m[\ln(\tan x)]\mathrm{d}x = \sum_{j=0}^{m}\binom{m}{j}\Gamma^{(m-j)}(n+1)\beta^{(j)}(n+1)$$

Case 1:

Vardi gave the beautiful, classic integral:

$$\int_{\pi/4}^{\pi/2} \ln\ln(\tan x)\mathrm{d}x = \frac{\pi}{2}\ln\left[\sqrt{2\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

By our generalized formula, this is $\Gamma'(1)\beta(1) + \Gamma(1)\beta'(1)$. Now we work out the derivative of the Dirichlet Beta function at $x = 1$. Using $\beta(1) = \pi/4$, $\Gamma'(1) = -\gamma$, and Vardi's result, we easily deduce that:

$$\beta'(1) = \sum_{k=0}^{\infty}(-1)^k\frac{\ln(2k+1)}{(2k+1)} = \frac{\pi\gamma}{4} + \frac{\pi}{2}\ln\left[\sqrt{2\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

Case 2:

Now let's find an evaluation of the second order Vardi-type integral:

$$\int_{\pi/4}^{\pi/2} \ln^2[\ln(\tan x)]\mathrm{d}x$$

Our formula for **Type 11** gives

$$\int_{\pi/4}^{\pi/2} \ln^2[\ln(\tan x)]\mathrm{d}x = \Gamma''(1)\beta(1) + 2\Gamma'(1)\beta'(1) + \Gamma(1)\beta''(1)$$

So we want the second order derivative of the Gamma function at $x = 1$. This is easily found in terms of the digamma and trigamma functions:

$$\psi_1(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\Gamma'(x)}{\Gamma(x)} = \frac{\Gamma(x)\Gamma''(x) - \Gamma'(x)^2}{\Gamma(x)^2} = \frac{\Gamma''(x)}{\Gamma(x)} - \psi_0(x)^2$$

Hence:

$$\Gamma''(1) = \Gamma(1)\left[\psi_1(1) + \psi_0(1)^2\right] = \psi_1(1) + \gamma^2 = \zeta(2) + \gamma^2 = \frac{\pi^2}{6} + \gamma^2$$

Plugging this - as well as our earlier evaluation of the first order beta derivative - into our **Type 11** formula, we then obtain the the following beautiful result for the second order Vardi-type integral:

$$\int_{\pi/4}^{\pi/2} \ln^2[\ln(\tan x)]\mathrm{d}x = \beta''(1) + \frac{\pi^3}{24} - \frac{\pi\gamma^2}{4} - \pi\gamma\ln\left[\sqrt{2\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

Case 3:

Following the same method as above, a little further calculation yields the third order Vardi-type integral to be:

$$\int_{\pi/4}^{\pi/2} \ln^3[\ln(\tan x)]\mathrm{d}x = \beta'''(1) - 3\gamma\,\beta''(1) + \frac{\pi\gamma^3}{2} - \frac{\pi}{2}\zeta(3) + \left(\frac{\pi^3}{4} + \frac{3\pi\gamma^2}{2}\right)\ln\left[\sqrt{2\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

*

4.7 关于 Euler-Poisson 积分的几种解法

✎ * 方法 1：因为积分值只与被积函数和积分域有关，与积分变量无关，所以

$$I^2 = \left(\int_0^\infty e^{-x^2}\mathrm{d}x\right)^2 = \int_0^\infty e^{-x^2}\mathrm{d}x \cdot \int_0^\infty e^{-y^2}\mathrm{d}y = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\mathrm{d}x\mathrm{d}y$$

用极坐标系下二重积分的计算法

$$I = \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^\infty e^{-r^2}r\mathrm{d}r = \frac{\pi}{2}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^\infty = \frac{\pi}{4}$$

而 $e^{-r^2} \geq 0$, 则 $I > 0$. 即

$$I = \int_0^\infty e^{-x^2}\mathrm{d}x = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

方法 2： 因为 $\left(1+\dfrac{x^2}{n}\right)^{-n}$ 当 $n\to+\infty$ 时一致收敛于 e^{-x^2}, 利用积分号下取极限，则有

$$I=\int_0^{\infty}e^{-x^2}\mathrm{d}x=\int_0^{\infty}\left[\lim_{n\to\infty}\left(1+\frac{x^2}{n}\right)^{-n}\right]\mathrm{d}x=\lim_{n\to\infty}\int_0^{\infty}\left(1+\frac{x^2}{n}\right)^{-n}\mathrm{d}x$$

令 $x=\sqrt{n}t$, 则

$$I=\lim_{n\to\infty}\sqrt{n}\int_0^{\infty}\frac{1}{\left(1+t^2\right)^n}\mathrm{d}t=\sqrt{n}I_n$$

由于

$$\begin{aligned}I_{n-1}&=\int_0^{\infty}\frac{1}{\left(1+t^2\right)^{n-1}}\mathrm{d}t=\frac{t}{\left(1+t^2\right)^{n-1}}\bigg|_0^{\infty}+2\left(n-1\right)\int_0^{\infty}\frac{1}{\left(1+t^2\right)}\mathrm{d}t\\&=2\left(n-1\right)I_{n-1}-2\left(n-1\right)I_n\end{aligned}$$

所以 $I_n=\dfrac{2n-3}{2n-2}I_{n-1}$, 而 $I_1=\displaystyle\int_0^{\infty}\frac{1}{1+t^2}\mathrm{d}t=\frac{\pi}{2}$, 递推得

$$I_n=\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}$$

因此 $\displaystyle\int_0^{\infty}e^{-x^2}\mathrm{d}x=\lim_{n\to\infty}\frac{\sqrt{n}\,(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}$. 根据 Wallis 公式，有

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{\left[(2n)!!\right]^2}{(2n+1)\left[(2n-1)!!\right]^2}=\lim_{n\to\infty}\frac{\left[(2n-2)!!\right]^2}{(2n-1)\left[(2n-1)!!\right]^2}$$

所以

$$\begin{aligned}I&=\int_0^{\infty}e^{-x^2}\mathrm{d}x=\frac{\pi}{2}\lim_{n\to\infty}\frac{\sqrt{n}\,(2n-3)!!}{(2n-2)!!}=\frac{\pi}{2}\lim_{n\to\infty}\frac{(2n-3)!!\sqrt{2n-1}}{(2n-2)!!}\cdot\sqrt{\frac{n}{2n-1}}\\&=\frac{\pi}{2}\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{\pi}}{2}\end{aligned}$$

方法 3： 考虑两个含参变量积分

$$f\left(x\right)=\left(\int_0^{x}e^{-t^2}\mathrm{d}t\right)^2\quad,\quad g\left(x\right)=\int_0^{1}\frac{e^{-x^2\left(1+u^2\right)}}{1+u^2}\mathrm{d}u$$

利用积分号下微分法，得

$$\begin{aligned}f'\left(x\right)&=2e^{-x^2}\int_0^{x}e^{-t^2}\mathrm{d}t\\g'\left(x\right)&=\int_0^{1}\frac{\partial}{\partial x}\left[\frac{e^{-x^2\left(1+u^2\right)}}{1+u^2}\right]\mathrm{d}u=-2xe^{-x^2}\int_0^{1}e^{-x^2u^2}\mathrm{d}u\end{aligned}$$

对后一积分，令 $xu=t$, 则

$$g'\left(x\right)=-2xe^{-x^2}\int_0^{x}e^{-t^2}\mathrm{d}t=-f'\left(x\right)\quad\left(x\geq0\right)$$

于是

$$f\left(x\right)+g\left(x\right)=c\quad\left(x\geq0\right)\tag{1}$$

由于 $f(0)=0, g(0)=\dfrac{\pi}{4}$, 故 $c=\dfrac{\pi}{4}$, 即

$$f(x)+g(x)=\frac{\pi}{4} \quad (x\geq 0)$$

当 $u\in[0,1]$, 有

$$0\leq \frac{e^{-x^2\left(1+u^2\right)}}{1+u^2}\leq e^{-x^2u^2}\leq e^{-x^2} \quad (x\geq 0)$$

因此，当 $x\to\infty$ 时，函数 $\dfrac{e^{-x^2\left(1+u^2\right)}}{1+u^2}$ 关于 $u\in[0,1]$ 一致的趋于 0.

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\int_0^1 \frac{e^{-x^2\left(1+u^2\right)}}{1+u^2}\mathrm{d}u=\int_0^1 \lim_{x\to\infty}\frac{e^{-x^2\left(1+u^2\right)}}{1+u^2}\mathrm{d}u=0$$

从而，由 $f(x)$ 的定义及 (1)，得

$$I=\int_0^\infty e^{-x^2}\mathrm{d}x=\sqrt{\lim_{x\to\infty} f(x)}=\sqrt{\lim_{x\to\infty}\frac{\pi}{4}-g(x)}=\sqrt{\frac{\pi}{4}}=\frac{\sqrt{\pi}}{2}$$

方法 4：设 $f(t)=\displaystyle\int_0^\infty e^{-tx^2}\mathrm{d}x$, 对 $f(t)$ 取拉普拉斯变换，得

$$\mathcal{L}\left(\int_0^\infty e^{-tx^2}\mathrm{d}x\right)=\int_0^\infty\int_0^\infty e^{-tx^2}e^{-st}\mathrm{d}t\mathrm{d}x=\int_0^\infty \mathcal{L}\left(e^{-tx^2}\right)\mathrm{d}x=\int_0^\infty\frac{\mathrm{d}x}{s+x^2}=\frac{\pi}{2\sqrt{s}}$$

再取拉普拉斯逆变换，有 $f(t)=\displaystyle\int_0^\infty e^{-tx^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2\sqrt{t}}$, 在上式中，令 $t=1$, 则

$$I=f(1)=\int_0^\infty e^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}$$

方法 5：这种利用伽马函数的方法应该是高数中第一次接触的，出现在同济高数上册第五章最后，不过教材中打了星号，所以多数人都不了解，首先我们引入伽马函数的定义

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$$

所以，我们令 $x^2=t$, 有

$$I=\int_0^\infty e^{-x^2}\mathrm{d}x=\frac{1}{2}\int_0^\infty t^{-\frac{1}{2}}e^{-t}\mathrm{d}t=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$$

其中 $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$ 可利用余元公式求得，这里不做证明.

方法 6：不难证明，函数 $(1+t)e^{-t}$ 在 $t=0$ 时达到它的最大值 1. 因此当 $t\neq 0$ 时，$(1+t)e^{-t}<1$, 令 $t=\pm x^2$, 即得

$$\left(1-x^2\right)e^{x^2}<1\ ,\quad \left(1+x^2\right)e^{-x^2}<1$$

或

$$\left(1-x^2\right)<e^{-x^2}<\frac{1}{1+x^2} \quad (x>0)$$

假设限定第一个不等式中的 x 在 (0,1) 内变化，而第二个不等式中 x 则看作是任意的，把上式同 n 次方，有

$$\left(1-x^2\right)^n<e^{-nx^2} \quad (0<x<1)$$

$$e^{-nx^2}<\frac{1}{(1+x^2)^n}\quad(x>0)$$

第一个不等式即从 0 到 1 积分，第二个不等式取从 0 到 $+\infty$ 的积分，得

$$\int_0^1\left(1-x^2\right)^n\mathrm{d}x<\int_0^1 e^{-nx^2}\mathrm{d}x<\int_0^\infty e^{-nx^2}\mathrm{d}x<\int_0^\infty\frac{1}{(1+x^2)^n}\mathrm{d}x$$

在 $\int_0^1\left(1-x^2\right)^n\mathrm{d}x$ **中，令** $x=\cos t$, **则**

$$\int_0^1\left(1-x^2\right)^n\mathrm{d}x=\int_0^{\frac{\pi}{2}}\sin^{2n+1}t\mathrm{d}t=\frac{(2n)!!}{(2n+1)!!}$$

在 $\int_0^\infty\frac{1}{(1+x^2)^n}\mathrm{d}x$ **中，令** $x=\cot t$, **则**

$$\int_0^\infty\frac{1}{(1+x^2)^n}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\sin^{2n-2}t\mathrm{d}t=\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}$$

在 $\int_0^\infty e^{-nx^2}\mathrm{d}x$ **中，令** $x=\dfrac{t}{\sqrt{n}}$, **则**

$$\int_0^\infty e^{-nx^2}\mathrm{d}x=\frac{1}{\sqrt{n}}\int_0^\infty e^{-t^2}\mathrm{d}t=\frac{1}{\sqrt{n}}I$$

综上所述

$$\sqrt{n}\cdot\frac{(2n)!!}{(2n+1)!!}<I<\sqrt{n}\cdot\frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{\pi}{2}$$

取平方得

$$\frac{n}{2n+1}\cdot\frac{[(2n)!!]^2}{(2n+1)[(2n-1)!!]^2}<I^2<\frac{n}{2n-1}\cdot\frac{[(2n-3)!!]^2}{[(2n-2)!!]^2}\cdot\left(\frac{\pi}{2}\right)^2$$

根据 Wallis 公式

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{[(2n)!!]^2}{(2n+1)[(2n-1)!!]^2}$$

不等式两边当 $n\to\infty$ **时的极限都是** $\dfrac{\pi}{4}$, **所以**

$$I^2=\frac{\pi}{4}\Rightarrow I=\frac{\sqrt{\pi}}{2}$$

方法 7：当然也可以利用三重积分

$$8I^3=\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2-z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2-z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\Rightarrow 8I^3=4\pi\int_0^\infty\rho^2e^{-\rho^2}\,\mathrm{d}\rho=2\pi\int_0^\infty e^{-\rho^2}\,\mathrm{d}\rho=2\pi\cdot I\Rightarrow 8I^3=2\pi I\Rightarrow I=\frac{\sqrt{\pi}}{2}$$

方法 8：注意到

$$n!=\int_0^\infty e^{-\sqrt[n]{x}}\mathrm{d}x\iff\frac{1}{n}!=\int_0^\infty e^{-x^n}\mathrm{d}x\to\frac{1}{2}!=\int_0^\infty e^{-x^2}\mathrm{d}x$$

$$\int_0^1\left(1-\sqrt[n]{x}\right)^m\mathrm{d}x=\int_0^1\left(1-\sqrt[m]{x}\right)^n\mathrm{d}x=\frac{1}{C_{m+n}^n}=\frac{1}{C_{m+n}^m}=\frac{m!\,n!}{(m+n)!}$$

所以我们有

$$\frac{\pi}{4}=\int_0^1\sqrt{1-x^2}\,\mathrm{d}x=\frac{\left(\frac{1}{2}!\right)^2}{\left(\frac{1}{2}+\frac{1}{2}\right)!}=\left(\frac{1}{2}!\right)^2$$

所以

$$I=\int_0^\infty e^{-x^2}\mathrm{d}x=\frac{1}{2}!=\sqrt{\frac{\pi}{4}}=\frac{\sqrt{\pi}}{2}$$

方法 9： 利用

$$\int_0^\infty e^{-x^2}\mathrm{d}x=\sqrt{\pi}\int_0^\infty\frac{1}{\sqrt{\pi}}e^{-x^2}\mathrm{d}x=\sqrt{\pi}P(X\geq 0)$$

其中

$$X\sim N\left(0,\frac{1}{2}\right)\quad,\quad P(X>0)=P(X>E(X))=\frac{1}{2}$$

所以

$$I=\int_0^\infty e^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}$$

方法 10： 利用

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(\frac{-t^2}{2}\right)\exp(-i\omega t)\mathrm{d}t$$

所以

$$F(\omega)=\frac{1}{\pi}\int_0^{+\infty}\exp\left(\frac{-t^2}{2}\right)\cos(\omega t)\mathrm{d}t$$

所以我们有

$$F'(\omega)=-\omega F(\omega)\Rightarrow F(\omega)=C\exp\left(\frac{-\omega^2}{2}\right)$$

由

$$\exp\left(\frac{-x^2}{2}\right)=\int_{-\infty}^{+\infty}F(\omega)\exp(i\omega x)\mathrm{d}\omega$$

可得 $C=\frac{1}{\sqrt{2\pi}}$. 令 $\omega=0$, 有

$$F(0)=C=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(\frac{-t^2}{2}\right)\mathrm{d}t$$

所以

$$\sqrt{2}\int_{-\infty}^{+\infty}\exp\left(-t^2\right)\mathrm{d}t=\sqrt{2\pi}\Rightarrow I=\frac{\sqrt{\pi}}{2}$$

*

4.8 A General Form for Log-Gamma Integrals

✎ * **Evaluate:**

$$\int_0^z x^t\ln\Gamma\left(1+x\right)\mathrm{d}x$$

Solution: Notice that

$$\begin{aligned}\int_0^z x^t \ln\Gamma(1+x)\,\mathrm{d}x &= \int_0^z x^t \ln\Big[x\Gamma(x)\Big]\mathrm{d}x = \int_0^z x^t \ln x\mathrm{d}x + \int_0^z x^t \ln\Gamma(x)\,\mathrm{d}x \\ &= \frac{z^{1+t}\Big[(1+t)\ln z - 1\Big]}{(1+t)^2} + \int_0^z x^t \ln\Gamma(x)\,\mathrm{d}x\end{aligned}$$

Using the **Kummer's Fourier Series of** $\ln\Gamma(x)$ and

$$\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{2n} = -\frac{1}{2}\ln(2\sin\pi x)\ ,\ \sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{2n} = \frac{1}{2} - x$$

we have

$$\ln\Gamma(x) = \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln(2\sin\pi x) + (\gamma + \ln 2\pi)\left(\frac{1}{2} - x\right) + \sum_{n=1}^{\infty}\frac{\ln n}{n\pi}\sin(2\pi nx)$$

Hence we have

$$\begin{aligned}\int_0^z x^t \ln\Gamma(x)\,\mathrm{d}x = &\frac{z^{t+1}\ln 2\pi}{2(t+1)} + \frac{z^{t+1}(\gamma+\ln 2\pi)(t-2zt-2z+2)}{2(t+1)(t+2)} \\ &- \frac{1}{2}\int_0^z x^t \ln(2\sin\pi x)\,\mathrm{d}x + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n}\int_0^z x^t\sin(2\pi nx)\,\mathrm{d}x\end{aligned}$$

where

$$\begin{aligned}\int_0^z x^t \ln(2\sin\pi x)\,\mathrm{d}x &= \frac{1}{(2\pi)^{t+1}}\int_0^{2\pi z} x^t \ln\left(2\sin\frac{x}{2}\right)\mathrm{d}x \\ &= -\frac{x^t}{(2\pi)^{t+1}}\mathrm{Cl}_2(x)\Big|_0^{2\pi z} + \frac{t}{(2\pi)^{t+1}}\int_0^{2\pi z} x^{t-1}\mathrm{Cl}_2(x)\,\mathrm{d}x \\ &= -\left(\frac{z}{2\pi}\right)^t \mathrm{Cl}_2(2\pi z) + \frac{t}{(2\pi)^{t+1}}\int_0^{2\pi z} x^{t-1}\sum_{k=1}^{\infty}\frac{\sin kx}{k^2}\mathrm{d}x \\ &= -\left(\frac{z}{2\pi}\right)^t \mathrm{Cl}_2(2\pi z) + \frac{t}{(2\pi)^{t+1}}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{2\pi z} x^{t-1}\sin kx\mathrm{d}x\end{aligned}$$

for the last integral,it's not hard to see that

$$\begin{aligned}\int_0^{2\pi z} x^{t-1}\sin kx\,\mathrm{d}x = (t-1)!\Bigg[&\sum_{j=0}^{\lfloor t-1/2\rfloor}(-1)^{j+1}\frac{x^{t-1-2j}}{k^{2j+1}(t-1-2j)!}\cos kx\Bigg|_0^{2\pi z} \\ &+ \sum_{j=0}^{\lfloor (t-2)2\rfloor}(-1)^{j+1}\frac{x^{t-2j-2}}{k^{2j+2}(t-2j-2)!}\sin kx\Bigg|_0^{2\pi z}\Bigg]\end{aligned}$$

let $t-1=t$ and $k=2\pi n$ we can evaluate $\displaystyle\int_0^z x^t\sin(2\pi nx)\,\mathrm{d}x$.

Now we obtain the result for the initial integral

$$\int_0^z x^t \ln\Gamma(1+x)\,\mathrm{d}x = \frac{z^{1+t}\left[(1+t)\ln z - 1\right]}{(1+t)^2} + \frac{z^{t+1}\ln 2\pi}{2(t+1)} + \frac{z^{t+1}(\gamma+\ln 2\pi)(t-2zt-2z+2)}{2(t+1)(t+2)}$$
$$+\frac{1}{2}\left(\frac{z}{2\pi}\right)^t \mathrm{Cl}_2(2\pi z) - \frac{t(t-1)!}{2(2\pi)^{t+1}}\sum_{k=1}^{\infty}\frac{1}{k^2}\left\{\sum_{j=0}^{\lfloor t-1/2\rfloor}\frac{(-1)^{j+1}x^{t-1-2j}}{k^{2j+1}(t-1-2j)!}\cos kx\Bigg|_0^{2\pi z}\right.$$
$$\left.+\sum_{j=0}^{\lfloor (t-2)2\rfloor}\frac{(-1)^{j+1}x^{t-2j-2}}{k^{2j+2}(t-2j-2)!}\sin kx\Bigg|_0^{2\pi z}\right\} + \frac{t!}{\pi}\sum_{n=1}^{\infty}\frac{\ln n}{n}\left\{\sum_{j=0}^{\lfloor t/2\rfloor}\frac{(-1)^{j+1}x^{t-2j}}{(2\pi n)^{2j+1}(t-2j)!}\cos(2\pi nx)\Bigg|_0^{z}\right.$$
$$\left.+\sum_{j=0}^{\lfloor (t-1)2\rfloor}\frac{(-1)^{j+1}x^{t-2j-1}}{(2\pi n)^{2j+2}(t-2j-1)!}\sin(2\pi nx)\Bigg|_0^{z}\right\}$$

For example

$$\int_0^1 x\ln\Gamma(1+x)\,\mathrm{d}x = \ln\left(\frac{2^{\frac{1}{4}}\pi^{\frac{1}{4}}}{\mathbf{A}e^{\frac{1}{4}}}\right)$$
$$\int_0^2 x^2\ln\Gamma(1+x)\,\mathrm{d}x = \frac{4}{3}\ln\left(\frac{8\pi}{\mathbf{A}^3}\right) + \frac{\zeta(3)}{2\pi^2} - \frac{5}{2}$$
$$\int_0^{\frac{1}{2}} x^2\ln\Gamma(1+x)\,\mathrm{d}x = \ln\left(\frac{\mathbf{A}^{\frac{1}{8}}\pi^{\frac{1}{48}}}{2^{\frac{89}{2880}}}\right) - \frac{5}{8}\zeta'(3) - \frac{3\zeta(3)}{32\pi^2} - \frac{3}{128}$$

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4.9 Proving Identities of Special Functions

✎ * **Problem 1:** Show that

$$\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} - \frac{\ln^3 3}{6}$$

Proof: We will need two functional equations of the dilogarithm.

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2)$$
$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-x^{-1}) = -\frac{1}{2}\ln^2(x)$$

Hence

$$\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = \mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\left[2\mathrm{Li}_2\left(\frac{1}{3}\right) + 2\mathrm{Li}_2\left(-\frac{1}{3}\right)\right]$$
$$= \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right) = \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{3}\left[-\mathrm{Li}_2\left(1-\frac{3}{4}\right) - \frac{1}{2}\ln^2\left(\frac{4}{3}\right)\right]$$
$$= \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{1}{6}\ln^2\left(\frac{4}{3}\right) + \frac{1}{3}\mathrm{Li}_2\left(\frac{1}{4}\right) = \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{1}{6}\ln^2\left(\frac{4}{3}\right)$$
$$+\frac{1}{3}\left[2\mathrm{Li}_2\left(\frac{1}{2}\right) + 2\mathrm{Li}_2\left(-\frac{1}{2}\right)\right] = \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{6}\ln^2\left(\frac{4}{3}\right)$$
$$+\frac{2}{3}\mathrm{Li}_2\left(-\frac{1}{2}\right) = \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{6}\ln^2\left(\frac{4}{3}\right) + \frac{2}{3}\left[-\mathrm{Li}_2\left(1-\frac{2}{3}\right)\right.$$
$$\left.-\frac{1}{2}\ln^2\left(\frac{3}{2}\right)\right] = \frac{2}{3}\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{6}\ln^2\left(\frac{4}{3}\right) - \frac{1}{3}\ln^2\left(\frac{3}{2}\right) = \frac{2}{3}\left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)\right)$$

$$+\frac{1}{6}\ln^2\left(\frac{4}{3}\right)-\frac{1}{3}\ln^2\left(\frac{3}{2}\right)=\frac{\pi^2}{18}-\frac{1}{3}\ln^2(2)+\frac{1}{6}\ln^2\left(\frac{4}{3}\right)-\frac{1}{3}\ln^2\left(\frac{3}{2}\right)$$
$$=\frac{\pi^2}{18}-\frac{1}{3}\ln^2(2)+\frac{1}{6}(\ln 4-\ln 3)^2-\frac{1}{3}(\ln 3-\ln 2)^2=\frac{\pi^2}{18}-\frac{1}{3}\ln^2(2)$$
$$+\frac{1}{6}\Big(4\ln^2(2)-4\ln(2)\ln(3)-\ln^2(3)\Big)-\frac{1}{3}\Big(\ln^2(3)-2\ln(3)\ln(3)+\ln^2(2)\Big)$$
$$=\frac{\pi^2}{18}-\frac{1}{6}\ln^2(3)$$

Problem 2: Prove that

$$\sum_{k\geq 0}\frac{(-1)^k}{zk+1}=\frac{1}{2z}\left(\psi_0\left(\frac{z+1}{2z}\right)-\psi_0\left(\frac{1}{2z}\right)\right)$$

Proof: Note that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{zk+1}=\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{zk}\,\mathrm{d}x=\int_0^1\left(\sum_{k=0}^{\infty}(-1)^k x^{zk}\right)\mathrm{d}x$$
$$=\int_0^1\left(\sum_{k=0}^{\infty}x^{2kz}-\sum_{k=0}^{\infty}x^{(2k+1)z}\right)\mathrm{d}x=\int_0^1\left(\frac{1}{1-x^{2z}}-\frac{x^z}{1-x^{2z}}\right)\mathrm{d}x$$
$$=\frac{1}{2z}\int_0^1\frac{(1-\sqrt{x})x^{\frac{1}{2z}-1}}{1-x}\mathrm{d}x=\frac{1}{2z}\int_0^1\frac{x^{\frac{1}{2z}-1}-x^{\frac{1}{2z}-\frac{1}{2}}}{1-x}\mathrm{d}x$$
$$=\frac{1}{2z}\int_0^1\frac{\left(1-x^{\frac{1}{2z}-\frac{1}{2}}\right)-\left(1-x^{\frac{1}{2z}-1}\right)}{1-x}\mathrm{d}x=\frac{1}{2z}\left\{\psi_0\left(\frac{z+1}{2z}\right)-\psi_0\left(\frac{1}{2z}\right)\right\}$$

In the last step, I used that

$$\psi_0(s+1)=-\gamma+\int_0^1\frac{1-x^s}{1-x}\mathrm{d}x$$

Problem 3: Show that $\zeta'(0)=-\frac{1}{2}\ln(2\pi)$.

Proof: From the functional equational equation of Reimann zeta function and the fact that

$$\Gamma\left(\frac{1-s}{2}\right)=\frac{2}{1-s}\Gamma\left(\frac{3-s}{2}\right)$$

we have

$$-\zeta(1-s)=\pi^{-s+1/2}\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{3-s}{2}\right)}(s-1)\zeta(s)\tag{1}$$

Now we know that,

$$(s-1)\zeta(s)=1+(s-1)\gamma-(s-1)^2\gamma_1+\cdots\tag{2}$$

where γ_n are the **Stieltjes constants**.

Taking logarithmic derivative of (1) and using (2), we obtain

$$\frac{\zeta'(1-s)}{\zeta(1-s)}=\ln(\pi)-\frac{1}{2}\psi_0\left(\frac{s}{2}\right)-\frac{1}{2}\psi_0\left(\frac{3-s}{2}\right)-\frac{\gamma-2\gamma_1(s-1)+\cdots}{1+\gamma(s-1)+\cdots}\tag{3}$$

Set $s = 1$ in (3):

$$\zeta'(0) = \zeta(0)\left(\ln(\pi) - \frac{1}{2}\psi_0\left(\frac{1}{2}\right) - \frac{1}{2}\psi_0(1) - \gamma\right) = -\frac{1}{2}\left(\ln(\pi) + \ln(2)\right) = -\frac{1}{2}\ln(2\pi)$$

Problem 4: Prove that

$$\sum_{p=0}^{q-1} \psi\left(a + \frac{p}{q}\right) = q(\psi(qa) - \ln(q))$$

Proof: Using the Multiplication theorem of Gamma function, we have

$$\sum_{p=0}^{q-1} \ln\Gamma\left(a + \frac{p}{q}\right) = \ln\prod_{p=0}^{q-1}\Gamma\left(a + \frac{p}{q}\right) = \ln\left((2\pi)^{\frac{q-1}{2}} q^{\frac{1}{2}-qa}\Gamma(qa)\right)$$

Now, take derivative with respect to a:

$$\sum_{p=0}^{q-1} \psi_0\left(a + \frac{p}{q}\right) = -q\ln q + q\psi_0(qa)$$

Problem 5: For $\Re(c) > \Re(b) > 0$, prove

$${}_2F_1(a,\, b\,;\, c\,;\, x) = (1-x)^{-a}\, {}_2F_1\left(a,\, c-b\,;\, c\,;\, \frac{x}{x-1}\right)$$

Proof:

$$\begin{aligned}
{}_2F_1(a,b;c;x) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,\mathrm{d}t \\
&= \frac{1}{B(b,c-b)}\int_0^1 (1-u)^{b-1}u^{c-b-1}(1-x+xu)^{-a}\,\mathrm{d}u \\
&= (1-x)^{-a}\frac{1}{B(b,c-b)}\int_0^1 u^{c-b-1}(1-u)^{b-1}\left(1+u\,\frac{x}{1-x}\right)^{-a}\,\mathrm{d}u \\
&= (1-x)^{-a}\,\frac{B(c-b,b)}{B(b,c-b)}\,{}_2F_1\left(a,c-b;c;-\frac{x}{1-x}\right) \\
&= (1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)
\end{aligned}$$

Problem 6: Show that for $a \neq b$ and a and b not negative integers,

$$\sum_{n=0}^{\infty}\frac{1}{(n+a)(n+b)} = \frac{\psi(a)-\psi(b)}{a-b}$$

Proof:

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{(k+a)(k+b)} &= \sum_{k=0}^{\infty}\frac{1}{(k+a)(k+b)}\frac{(k+a)-(k+b)}{(a-b)} \\
&= \frac{1}{(a-b)}\left\{\sum_{k=0}^{\infty}\frac{1}{k+b} - \sum_{k=0}^{\infty}\frac{1}{k+a}\right\}
\end{aligned}$$

On the other hand,

$$\psi_0(x) = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k-1}\right) - \gamma = \zeta(1) - \gamma - \sum_{k=0}^{\infty}\frac{1}{k+x}$$

$$\Rightarrow \frac{1}{(a-b)}\left\{\sum_{k=0}^{\infty}\frac{1}{k+b}-\sum_{k=0}^{\infty}\frac{1}{k+a}\right\}=\frac{1}{(a-b)}\left\{-\psi_0(b)+\psi_0(a)\right\}=\frac{\psi_0(a)-\psi_0(b)}{(a-b)}$$

Problem 7: Prove Weierstrauss' canonical product for the Gamma Function

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\prod_{k=1}^{\infty}\left\{e^{-x/k}\left(1+\frac{x}{k}\right)\right\}$$

Proof: I'm starting with Euler's limit definition of the Gamma function.

$$\begin{aligned}\Gamma(x)&=\lim_{n\to\infty}\frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)}=\lim_{n\to\infty}n^x\frac{1}{x}\frac{1}{x+1}\frac{2}{x+2}\cdots\frac{n}{x+n}\\&=\lim_{n\to\infty}e^{x\ln n}\frac{1}{x}\frac{1}{1+x}\frac{1}{1+\frac{x}{2}}\cdots\frac{1}{1+\frac{x}{n}}=\lim_{n\to\infty}e^{x\ln x}\frac{1}{x}\prod_{k=1}^{n}\frac{1}{1+\frac{x}{k}}\\&=\lim_{n\to\infty}\frac{e^{x\ln n-x-\frac{x}{2}-\cdots-\frac{x}{n}}}{x}\prod_{k=1}^{n}\frac{e^{\frac{x}{k}}}{1+\frac{x}{k}}=\lim_{n\to\infty}\frac{e^{-x(H_n-\ln n)}}{x}\prod_{k=1}^{n}\frac{e^{\frac{x}{k}}}{1+\frac{x}{k}}\\&=\frac{e^{-\gamma x}}{x}\prod_{k=1}^{\infty}\frac{e^{\frac{x}{k}}}{1+\frac{x}{k}}\end{aligned}$$

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第 5 章 一些链接 Some Links

5.1 特殊函数 Special Functions

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https://en.wikipedia.org/wiki/Arithmetic-geometric_mean

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https://en.wikipedia.org/wiki/Bessel_function

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